Real Analysis Reference Notes

Wenjie Li, Mathonline

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1 Lebesgue Measure

Def 1. Lesbegue Outer Measure

Thm 1. Outer Measure of Intervals

Thm 2. Countable Subadditivity of Lebesgue Measure

Def 2. Lesbegue Measurable Sets

Thm 3. Finite Union

Thm 4. Countable Union

Thm 5. Finite Additivity

Thm 6. Measurable sets is an σ-algebra containing all subsets of measure 0

Thm 7. Borel sets are measurable

Thm 8. Regularity of outer measure (Littlewood's first principal)

Thm 9. Littlewood's first principal (vi)

2 Measurable Functions

Def 1. Step Function

Thm 1. Let ϕ be a step function on [a,b] and let $\epsilon > 0$. Then $\exists$ a continuous function g on $[a, b]$ such that $\phi = g$ on $[a, b]$ except on a set of measure less than ϵ

Def 2. Simple Function

Thm 2. Let f be a simple function on [a,b] and let $\epsilon > 0$. Then $\exists$ a step function $\phi : [a, b] \to \mathbb{R}$ such that $f(x) = \phi(x)$ except on a set of measure less than ϵ, and also $\exists$ a continuous function g on $[a, b]$ such that $f = g$ on $[a, b]$ except on a set of measure less than ϵ

Def 3. Measurable Function

Thm 3. Linearity of Lebesgue measurable function

Thm 4. Product of Lebesgue measurable function

Thm 5. Let $f : E \to [-\infty, +\infty]$ be measurable with $E \in M$. Let $f = g$ a.e. on E. Then g is also measurable on E.

Thm 6. Littlewood's second principle

Thm 7. Littlewood's third principle (Egoroff's Thm)

3 Lebesgue Integral

Def 1. Lesbegue Integral of Simple Functions

Thm 1. Canonical property of Lebesgue Integral of Simple Functions

Thm 2. Linearity of Lebesgue Integral of Simple Functions (Not Rigorous)

Def 2. Lebesgue Integral of Bounded Functions

Thm 2. Bounded Integrable $\iff$ Measurable (Half of it, TO BE CONTINUED)

Thm 3. Riemann Integrable $\Rightarrow$ Lebesgue Integrables

Thm 4. Bounded Convergence Thm

Def 3. Lebesgue Integral of Non-Negative Functions

Thm 5. Fatou's Lemma

Thm 6. Monotone Convergence Thm

Thm 7. Countable Additivity (TO BE CONTINUED)

Thm 8. Absolute Continuity of Non-negative Functions (TO BE CONTINUED)

Def 4. General Lebesgue Integral

Thm 9. Lebesgue Dominate Convergence Theorem

4 Differentiation and Integration

Thm 1. Vilati Lemma and Proof (Not Complete, should add the below part),

Let J_n denote the interval whose center is the same as that of I_n such that $l(J_n) = 5l(I_n)$

Let $x \in E \setminus \cup_{n=1}^{N} I_n$, by Vitali property and since each I_n is closed, $\exists I$ such that I is disjoint from $I_1, ..., I_N, I$ cannot be disjoint from each of $I_i's$, so $\exists n > N$ such that I intersects I_n; by Archimedean, take smallest such n; I is disjoint from $I_1, ..., I_{n-1}$ and hence

$$l(I) \leq \alpha_{n-1} < 2l(I_n)$$

Since $l(I) < 2l(I_n)$ and since I inersects I_n, one can verify easily that $x \in I \subset J_n$, hence $(E \setminus \cup_{n=1}^{N} I_n) \subset \cup_{n=N+1}^{N} J_n$

Def 1. Upper and Lower Derivatives

Thm 2.1. Differentiability of Monotone Functions

Thm 2.2. Integration of Derivative of Monotone Function

Rmk: Monotone functions are always measurable because they are continuous a.e..

Thm 2.3. Change of Variable. Let f be integrable on $(a+\alpha, b+\alpha)$. Then:

$$\int_a^b f(x+\alpha)dx = \int_{a+\alpha}^{b+\alpha} f(y)dy$$

- If $f = X_E$, then the results hold because $f(x+\alpha) = X_{E-\alpha}(x)$ and $m((E-\alpha)\cap[a,b]) = m(E\cap[a+\alpha, b+\alpha])$

- For non-negative function f, take $0 \leq \phi_n$ such that ϕ_n monotonely increases to f and apply the Monotone Convergence Thm.

- Take $f = f^+ - f^-$

Def 2. Bounded Variation Functions

Thm 3.1. A function is of b.v. on $[a,b]$ iff it is the difference of 2 increasing functions

Thm 3.2. Bounded variation functions are differentiable and the derivatives are integrable

Thm 4 Absolute Continuity $\rightarrow$ Bounded Variation

Thm 5 Let $f \in L[a,b]$.. Then $x \rightarrow F(x)$ is of b.v. so is integrable over $[a,b]$ where

$$F(x) = \int_a^x f$$

Thm 6 Integral Criteria for Functions to be Zero Almost Everywhere

Thm 7 Fundamental Theorem of Calculus I

Thm 8 Fundamental Theorem of Calculus II (Integration of Derivative) (Not Complete)

5 Banach Space

Def 1. Banach Space

Thm 1. Absolute Summability Criterion for Completeness

Thm 2. Rieze -Fischer Theorem (L1 is complete)

Thm 3. Minkowski Inequality

Thm 4. Holder Inequality