Teichmuller Mapping and Beltrami Coefficient

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1 Quasiconformal Map in the Continuous Space

Given two Riemann surfaces M and N, a map $f : M \to N$ is conformal if it preserves the surface metric with a constant multiplicative factor. Conformal mappings preserve angles of local geometry. It is worth noticing that any analytic function is conformal at any point where its derivative is non-zero. Also, any conformal maps with continuous partial derivatives is non zero.

By generalization of conformal maps, we introduce the quasi-conformal maps, which are orientation preserving homeomorphisms between Riemann surfaces with bounded conformality distortion. By definition a map $f : \mathbb{C} \to \mathbb{C}$ is quasi-conformal if the Beltrami equation is satisfied:

$$\frac{\partial f}{\partial \bar{z}} = \mu(z)\frac{\partial f}{\partial z}$$

for complex valued function μ which satisfies the condition that $\|\mu\|_\infty < 1$. μ is called the Beltrami coefficient(BC). Note that the map is conformal at a point p when $\mu(p) = 0$, which means that when $\mu(p) = 0$, the map f is holomorphic at point p.

$Proof$:

If $\mu(p) = 0$ then $\frac{\partial f}{\partial \bar{z}}(p) = 0$

If we denote $f(z) = u(x, y) + iv(x, y)$, where u, v are real functions, then by definition, we have:

$$(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})(u + iv) = 0$$

$$\begin{cases} \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0 \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = 0 \end{cases}$$

which means that the Cauchy-Riemann Equations are satisfied.

From $\mu(p)$, we can determiine the angles and amount of the directions of maximal magnification and shrinking. The angle of maximal magnification is $\frac{arg(\mu(p))}{2}$ with amount $1 + |\mu|$, while the angle of maximal shrinking is $\frac{arg(\mu(p)-\pi}{2}$ with amount $1 + |\mu|$.

$Proof$:

Since f is a diffeomorphism, it is well approximated by a linear map df. Let the mapping be $df = u+iv = Az+B\bar{z}$, where A, B are complex constants, then

$$u + iv = (A + B)x + i(A - B)y$$

$$J = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} = \begin{bmatrix} Real(A+B) & Imag(B-A) \\ Imag(A+B) & Real(A-B) \end{bmatrix}$$

where J is the Jacobian matrix of the function

Hence we would obtain that

$$\begin{cases} A+B = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \\ A-B = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \end{cases}$$

$$\begin{cases} A = \frac{1}{2}[(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) + i(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y})] = \frac{\partial f}{\partial z} \\ B = \frac{1}{2}[(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}) + i(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y})] = \frac{\partial f}{\partial \bar{z}} \end{cases}$$

Hence the map is essentially $df = \frac{\partial f}{\partial z} z + \frac{\partial f}{\partial \bar{z}} \bar{z}$

Since the map is orientation preserving, we would require the determinant of the Jacobian

$$det(J) = |A|^2 - |B|^2 = |\frac{\partial f}{\partial z}|^2 - |\frac{\partial f}{\partial \bar{z}}|^2 > 0$$

Also, the linear map changes into $df = A(z + \frac{B}{A}\bar{z}) = A(z + \mu\bar{z})$

Note that the distortion is caused by the μ factor in the function $S(z) = z + \mu\bar{z}$. Denote all the points on the unit circle by $z = e^{i\theta}$, then $S(z) = z + \mu\bar{z} = e^{i\theta} + |\mu|e^{i(arg(\mu)-\theta)}$, the function $|S(z)|$ obtains its maximum at $\theta = arg(\mu)/2$ and minimum at $\theta = (arg(\mu) - \pi)/2$. The corresponding maximum and minimum are $1 + |\mu|$ and $1 - |\mu|$.

The Beltrami Coeffiecient $\mu(f)$ is actually related to the Jacobian $J(f)$ of f by the following formula:

$$det(J) = |\frac{\partial f}{\partial z}|^2(1 - |\mu(f)|^2)$$

Hence the map f is a diffeomorphism if $|\mu(f)|$ is everywhere less than 1.

Let $f = u + iv$. From the Beltrami Equation, we would have

$$\mu(f) = \frac{u_x - v_y + i(v_x + u_y)}{u_x + v_y + i(v_x - u_y)}$$

Let $\mu = \rho + i\tau$. We can write v_x, v_y as linear combinations of u_x and u_y

Proof

$$\begin{bmatrix} 1-\rho & \tau \\ -\tau & -(1+\rho) \end{bmatrix} \begin{bmatrix} u_x \\ -u_y \end{bmatrix} = \begin{bmatrix} \tau & (1+\rho) \\ 1-\rho & \tau \end{bmatrix} \begin{bmatrix} -v_x \\ v_y \end{bmatrix}$$

The determinant of the two matrices are both $\rho^2 + \tau^2 - 1 < 0$ since we require that $|\mu|$ is strictly smaller than 1

By inversing the two matrices we can obtain that

$$\begin{cases} v_y = \alpha_1 u_x + \alpha_2 u_y \\ -v_x = \alpha_2 u_y + \alpha_3 u_y \end{cases}$$

and

$$\begin{cases} -u_y = \alpha_1 v_x + \alpha_2 v_y \\ u_x = \alpha_2 v_y + \alpha_3 v_y \end{cases}$$

where $\alpha_1 = \frac{((\rho-1)^2+\tau^2}{1-\rho^2-\tau^2}, \alpha_2 = -\frac{2\tau}{1-\rho^2-\tau^2}, \alpha_3 = \frac{((\rho+1)^2+\tau^2}{1-\rho^2-\tau^2}$

Note that $\nabla \cdot (\begin{bmatrix} v_y \\ -v_x \end{bmatrix}) = 0$ and $\nabla \cdot (\begin{bmatrix} -u_y \\ u_x \end{bmatrix}) = 0$ we have

$$\nabla \cdot (A \begin{bmatrix} u_x \\ u_y \end{bmatrix}) = 0$$

and

$$\nabla \cdot (A \begin{bmatrix} v_x \\ v_y \end{bmatrix}) = 0$$

are the partial differential equation system that we intend to solve, where

$$A = \begin{bmatrix} \alpha_1 & \alpha_2 \\ \alpha_2 & \alpha_3 \end{bmatrix}$$

is a symmetric positive definite matrix

Proof

It is obvious that A is symmetric, to prove that it is positive definite

Note that $det(A - xI) = x^2 - (\alpha_1 + \alpha_3)x + (\alpha_1\alpha_3 - \alpha_2^2)$

After simplification the eigenvalues of A are $p_1 = \frac{(\sqrt{\rho^2+\tau^2}+1)^2}{1-\rho^2-\tau^2}$ and $p_2 = \frac{(\sqrt{\rho^2+\tau^2}-1)^2}{1-\rho^2-\tau^2}$ which are both positive

Therefore the matrix is symmetric positive definite.

2 Quasiconformal Map in the Discrete Space

In the discrete case, if we are given a quasi-conformal mapping, we can easily compute its associated Beltrami Coefficient, which is a conplex-valued function defined on each triangular face of the mesh. On each face T, denote $u_x = a_T, u_y = b_T, v_x = c_T, v_y = d_T$. The restriction of f on each face can be written as

$$f|_T(x, y) = \begin{bmatrix} a_T x + b_T y + r_T \\ c_T x + d_T y + s_T \end{bmatrix}$$

Therefore, the gradient of f, $\nabla_T f$, on each face T can be computed by:

$$\begin{bmatrix} v_1 - v_0 \\ v_2 - v_0 \end{bmatrix} \nabla_T f = \begin{bmatrix} f(v_1) - f(v_0) \\ f(v_2) - f(v_0) \end{bmatrix}$$

where $[v_0, v_1]$ and $[v_0, v_2]$ are two edges on T, we can therefore obtain the corresponding a_T, b_T, c_T, d_T and the Beltrami Coefficient on this triangular face T can be computed similarly by:

$$\mu(f) = \frac{a_T - d_T + i(c_T + b_T)}{a_T + d_T + i(c_T - b_T)}$$

3 Linear Beltrami Solver

When we are given the Beltrami Coefficient on each face T, we want to compute the quasiconformal mapping based on the Beltrami Coefficient. Let the Beltrami coefficient be $\mu_f(T) = \rho_T + i\tau_T$The discrete version of the linear system is given by:

$$\begin{bmatrix} 1 - \rho_T & \tau_T \\ -\tau_T & -(1 + \rho_T) \end{bmatrix} \begin{bmatrix} a_T \\ -b_T \end{bmatrix} = \begin{bmatrix} \tau_T & (1 + \rho_T) \\ 1 - \rho_T & \tau_T \end{bmatrix} \begin{bmatrix} -c_T \\ d_T \end{bmatrix}$$

we can solve for $\alpha_1 = \frac{((\rho_T-1)^2+\tau_T^2}{1-\rho_T^2-\tau_T^2}, \alpha_2 = -\frac{2\tau_T}{1-\rho_T^2-\tau_T^2}, \alpha_3 = \frac{((\rho_T+1)^2+\tau_T^2}{1-\rho_T^2-\tau_T^2}$

Note that the gradient of f on the face T, $\nabla_T f$ can be solved by inversing the matrix $(v_1 - v_0, v_2 - v_0)^T$. Let $T = [v_i, v_j, v_k]$ and $w_I = f(v_I)$, where $I = i, j, k$. Suppose$v_I = g_I + \sqrt{-1}h_I$ and $w_I = s_I + \sqrt{-1}t_I$, then a_T, b_T, c_T, d_T can be written as:

$$\begin{aligned} a_T &= A_i^T s_i + A_j^T s_j + A_k^T s_k \quad & b_T &= B_i^T s_i + B_j^T s_j + B_k^T s_k \\ c_T &= A_i^T t_i + A_j^T t_j + A_k^T t_k \quad & d_T &= B_i^T t_i + B_j^T t_j + B_k^T t_k \end{aligned}$$

where

$$\begin{aligned} A_i^T &= (h_j - h_k)/2Area(T) \quad & A_j^T &= (h_k - h_i)/2Area(T) \quad & A_k^T &= (h_i - h_j)/2Area(T) \\ B_i^T &= (g_k - g_j)/2Area(T) \quad & B_j^T &= (g_i - g_k)/2Area(T) \quad & B_k^T &= (g_j - g_i)/2Area(T) \end{aligned}$$

Proof

The linear system is essentially

$$\begin{bmatrix} g_i - g_k & h_i - h_k \\ g_j - g_k & h_j - h_k \end{bmatrix} \begin{bmatrix} a_T & b_T \\ c_T & d_T \end{bmatrix} = \begin{bmatrix} s_i - s_k & t_i - t_k \\ s_j - s_k & t_j - t_k \end{bmatrix}$$

$$det(\begin{bmatrix} g_i - g_k & h_j - h_k \\ g_j - g_k & h_i - h_k \end{bmatrix}) = 2Area(T)$$

Hence the solution is

$$\begin{bmatrix} a_T & b_T \\ c_T & d_T \end{bmatrix} = \frac{1}{2Area(T)} \begin{bmatrix} h_j - h_k & -(h_i - h_k) \\ -(g_j - g_k) & g_i - g_k \end{bmatrix} \begin{bmatrix} s_i - s_k & t_i - t_k \\ s_j - s_k & t_j - t_k \end{bmatrix}$$

$$= \frac{1}{2Area(T)} \begin{bmatrix} (h_j - h_k)s_i + (h_k - h_i)s_j + (h_i - h_j)s_k & (h_j - h_k)t_i + (h_k - h_i)t_j + (h_i - h_j)t_k \\ (g_k - g_j)s_i + (g_i - g_k)s_j + (g_j - g_i)s_k & (g_k - g_j)t_i + (g_i - g_k)t_j + (g_j - g_i)t_k \end{bmatrix}$$

$A_i^T, A_j^T, A_k^T, B_i^T, B_j^T, B_k^T$ can be calculated directly from the Beltrami Coefficients.

In order to discretize the partial differential equation system and solve for the quasiconformal mapping, we need the definition of discrete divergence:

Let $V = (V_1, V_2)$ is a discrete vector field. For each vertex v_i, let N_i be the collection of neighborhood triangles attached to v_i. If we assume that the areas of the triangles are the same, the definition of discrete divergence Div of V is:

$$Div(V)(v_i) = \sum_{T \in N_i} A_i^T V_1(T) + B_i^T V_2(T)$$

By careful checking, we can prove that

$$\textstyle\sum_{T \in N_i} A_i^T b_T = \sum_{T \in N_i} B_i^T a_T \quad \sum_{T \in N_i} A_i^T d_T = \sum_{T \in N_i} B_i^T c_T$$

Proof

We prove the first one and the second one can be similarly proved.

Suppose v_i is the vertex of triangles $T_1, T_2, ..., T_n$

$$\sum_{T \in N_i} A_i^T b_T = \sum_{T \in N_i} A_i^T (B_i^T s_i^T + B_j^T s_j^T + B_k^T s_k^T)$$

$$\sum_{T\in N_i} B_i^T a_T = \sum_{T\in N_i} B_i^T (A_i^T s_i^T + A_j^T s_j^T + A_k^T s_k^T)$$

In a particular triangle T_1,

$$\begin{aligned}
A_i^{T_1} b_{T_1} - B_i^{T_1} a_{T_1} &= A_i^{T_1}(B_i^{T_1} s_i^{T_1} + B_j^{T_1} s_j^{T_1} + B_k^{T_1} s_k^{T_1}) - B_i^{T_1}(A_i^{T_1} s_i^{T_1} + A_j^{T_1} s_j^{T_1} + A_k^{T_1} s_k^{T_1}) \\
&= A_i^{T_1} B_j^{T_1} s_j^{T_1} + A_i^{T_1} B_k^{T_1} s_k^{T_1} - B_i^{T_1} A_j^{T_1} s_j^{T_1} - B_i^{T_1} A_k^{T_1} s_k^{T_1} \\
&= A_i^{T_1}(-B_i^{T_1} - B_k^{T_1}) s_j^{T_1} + A_i^{T_1} B_k^{T_1} s_k^{T_1} - B_i^{T_1}(-A_i^{T_1} - A_k^{T_1}) s_j^{T_1} - B_i^{T_1} A_k^{T_1} s_k^{T_1} \\
&= A_i^{T_1} B_k^{T_1}(s_k^{T_1} - s_j^{T_1}) + B_i^{T1} A_k^{T_1}(s_j^{T_1} - s_k^{T_1}) \\
&= (A_i^{T_1} B_k^{T_1} - B_i^{T_1} A_k^{T_1})(s_j^{T1} - s_k^{T_1}) \\
&= \frac{1}{2Area(T_1)}(s_j^{T_1} - s_k^{T_1})
\end{aligned} \tag{1}$$

The third equality is because of the fact that

$$\textstyle\sum_{m=i,j,k} A_m^T = 0, \quad \sum_{m=i,j,k} B_m^T = 0$$

When we sum up among all the $T_m, m = 1, 2, ...n$, all the $s_j^{T_m} - s_k^{T_m}$ will cancel out, which proves the first statement. Therefore

$$Div\begin{pmatrix} -b_T \\ a_T \end{pmatrix} = 0, Div\begin{pmatrix} c_T \\ -d_T \end{pmatrix} = 0$$

Hence the partial differential equation system can be discretized as

$$Div(A\begin{pmatrix} -b_T \\ a_T \end{pmatrix}) = 0, Div(A\begin{pmatrix} c_T \\ -d_T \end{pmatrix}) = 0$$

This is the same as:

$$\sum_{T\in N_i} A_i^T(\alpha_1(T)a_T + \alpha_2(T)b_T) + B_i^T(\alpha_2(T)a_T + \alpha_3(T)b_T) = 0$$

$$\sum_{T\in N_i} A_i^T(\alpha_1(T)c_T + \alpha_2(T)d_T) + B_i^T(\alpha_2(T)c_T + \alpha_3(T)d_T) = 0$$

for all vertices $v_i \in D$. Note that a_T, b_T, c_T, d_T can be written as linear combinations of the x and y coordinates of quasiconformal maps. Therefore, we can compute the quasiconformal maps by solving the linear system.

Besides, f has to satisfy certain constraints on the boundary. One common situation is to give the Dirichlet condition on the whole boundary. That is, for any $v_b \in \partial K_1$

$$f(v_b) = w_b \in \partial K_2$$

Note that the Dirichlet condition is not required to be enforced on the whole boundary. The proposed algorithm also allows a free boundary condition. In the case that K_1 and K_2 are rectangles, the desired quasi-conformal map should satisfy

$$f(0) = 0; f(1) = 1; f(i) = i; f(1 + i) = 1 + i$$

$$Re(f) = 0, [0, i], \qquad Re(f) = 1, [1, 1+i]$$
$$Imag(f) = 0, [0, 1] \quad Imag(f) = 1, [i, 1+i]$$

When $K_i (i = 1, 2)$ is a unit disk, we can parameterize it onto a domain D_i, which is a triangle with boundary vertices p_0^i, p_1^i, and p_2^i. p_0^i is on the y-axis, whereas p_1^i and p_2^i are on the x-axis. This can be done by removing a triangular face at the point 1 and map K_i to the upper half plane using a Mobius transformation $\psi(z) = \sqrt{-1}\frac{1+z}{1-z}$. In this case, the desired quasiconformal map f should satisfy

$$f(p_0^1) = p_0^2; f(p_1^1) = p_1^2; Imag(f) = 0; [p_0^1, p_1^1]$$

When K_i is a genus-0 closed surface mesh, we can again parameterize it onto a domain D_i, which is a triangular with boundary vertices p_0^i, p_1^i, p_2^i. This can be done by removing a triangular face on the surface and mapping K_i to the 2D plane using stereographic projection. In this case, the desired quasi-conformal map $\bar{f}$ should satisfy

$$f(p_0^1) = p_0^2; f(p_1^1) = p_1^2; f(p_2^1) = p_2^2$$

Suppose interior landmark correspondences $\{p_i\}_{i=1}^n \to \{q_i\}_{i=1}^n$ are also enforced; one should add this constraint to the linear system.

4 Teichmuller Map in the Continuous Space

We can define the Extremal quasi-conformal mapping can be defined as follows.

Def. Let $f : S_1 \to S_2$ be a quasi-conformal mapping between S_1 and S_2. f is said to be an extremal mapping if for any quasi-conformal mapping $h : S_1 \to S_2$ isotopic to f relative to the boundary,

$$K(f) \leq K(h)$$

It is uniquely extremal if the inequality is strict when $h \neq f$

A Teichmuller mapping (T-Map) is closely related to the extremal mapping. T-Map is defined as follows.

Def. Let $f : S_1 \to S_2$ be a quasi-conformal mapping. f is said to be a T-Map associated to the quadratic differential $q = \varphi dz^2$, where $\varphi : S_1 \to \mathbb{C}$ is a holomorphic function, if its associated Beltrami differential is of the form

$$\mu(f) = k\frac{\bar{\varphi}}{|\varphi|}$$

for some constant $k < 1$ and quadratic differential $q \neq 0$ with $\|q\|_1 = \int_{S_1} |\varphi| < \infty$

It means that a T-Map is a quasi-conformal mapping whose BC has a constant norm. Thus is has a uniform conformality distortion over the whole domain.

Def. Suppose S_1 and S_2 are open Riemann surfaces with the same topology. The boundary dilation $K_1[f]$ of f is defined as

$$K_1[f] = inf\{K(h|_{S_1 - C} : h \in \Omega, C \subset S_1\}$$

where C is compact and Ω is the family of quasi-conformal homeomoprhisms of S_1 onto S_2 which are homotopic to f modulo the boundary.

Strebel Theorem Let f be an extremal quasi-conformal mapping with $K(f) > 1$. If $K_1[f] < K(f)$, then f is T-Map associated with an integrable holomorphic quadratic differential on S_1. Hence, f is also an unique extremal mapping.

In other words, an extremal mapping between S_1 and S_2 witha suitable boundary condition is a T-Map. For simply connected open surfaces, the required boundary conditions are some properties on the derivatives of the boundary coresopndence. More specifically, the first derivative of the boundary coresopndence should be non-zero everywhere and the second derivative should be bounded.

Thm. Let $h : \partial\mathbb{D} \to \partial\mathbb{D}$ be an orientation -preserving diffeomorphism of $\partial\mathbb{D}$. Suppose further that $h'(e^{i\theta}) \neq 0$ and $h''(e^{i\theta})$ is bounded. Then there is a T-Map f that is the unique extremal extension of h to $\mathbb{D}$.

Thus, if the boundary correspondence satisfies certain conditions on its derivatives, the extremal map of the unit disk must be a T-Map.

Now, in the case when interior landmark constraints are enforced, the existence of a unique T-Map can be guaranteed if the boundary and landmark correspondence satisfy suitable conditions. The unique T-Map is extremal, which minimizes the maximal conformality distortion.

Thm. Let S_1 and S_2 be open Riemann surfaces with the same topology. Let $\{p_i\}_{i=1}^n \in S_1$ and $\{q_i\}_{i=1}^n \in S_2$ be the corresponding interior landmark constraints. Let $f : (S_1, \{p_i\}_{i=1}^n) \to (S_2, \{q_i\}_{i=1}^n)$ be the extremal quasi-conformal mapping, such that p_i corresponds to q_i for all $1 \leq i \leq n$. If $K_1[f] < K(f)$, then f is a T-Map associated with an integrable holomorphic quadratic differential on $(S_1, \{p_i\}_{i=1}^n)$. Hence f is a unique extremal mapping.

Thm. Let $h : \partial\mathbb{D} \to \partial\mathbb{D}$ be an orientation preserving diffeomorphism of $\partial\mathbb{D}$. Suppose further that $h'(e^{i\theta}) \neq 0$ and $h''(e^{i\theta})$ is bounded. Let $\{p_i\}_{i=1}^n \in \mathbb{D}$ and $\{q_i\}_{i=1}^n \in \mathbb{D}$ be the corresponding interior landmark constraints. Then there is a T-Map $f : (\mathbb{D}, \{p_i\}_{i=1}^n) \to (\mathbb{D}, \{q_i\}_{i=1}^n)$ matching the interior landmarks, which is the unique extremal extension of h to $\mathbb{D}$. That is, $f : (\mathbb{D}, \{p_i\}_{i=1}^n) \to (\mathbb{D}, \{q_i\}_{i=1}^n)$ is an extremal T-Map.

Hence, in most situations, an extremal quasi-conformal mappping is a T-Map, even for domains with nontrivial topologies. In soe rare situations when an extremal mapping is not exactly a T-Map, one can still get a T-Map whose dilation is arbitrarily close to the extremal dilation

5 QC-Iteration

With the Linear Beltrami Solver, we can obtain the quasi-confomal mapping associated with the given Beltrami Coefficients. In order to obtain the T-Map f we iteratively search for the unique BC of Teichmuller type associated to f.

The QC iteration starts with an initial map $f_0 : D_1 \to D_2$ satisfying the given boundary condition and landmark constraints. The initial map is chosen to be the quasiconformal mapping obtained from the LBS associated to the initial BC $\mu_0 = 0$:

$$f_0 = LBS(\mu_0 = 0)$$

Because of the landmark constraints and boundary conditions, the BC of f_0 may not be equal to $\mu_0 = 0$. Let v_0 be the BC calculated from the computation process in Section 2, then

$$v_0 = \mu(f_0)$$

In order to find the BC associated with Teichmuller mapping. We consider the two properties of T-Map.

1. $v = k\frac{\bar{\phi}}{|\phi|}$, where $0 \leq k \leq$ and $\phi : S_1 \to \mathbb{C}$ is a holomorphic function.
2. $v = argmin_{\mu : S_1 \to \mathbb{C}} \|v\|_\infty$,

Given the initial BC, v_0, we first apply the Laplace smoothing L on both the norm $|v_0|$ anf the argument $arg(v_0)$ to obtain a new BC, $\bar{\mu_1}$. The Laplace smoothing L is applied on the norm and the argument of v_0 independently.

$$L(v_0)(T) = L_0(T)e^{i\theta_0(T)}$$

where

$$L_0(T) = \sum_{T_i \in Nbhd(T)} \frac{|v_0|}{|Nbhd(T)|}, \theta_0(T) = \sum_{T_i \in Nbhd(T)} \frac{arg(v_0)}{|Nbhd(T)|}$$

where T is a triangular face of K_1, $Nbhd(T)$ is the set of neighborhood faces $|Nbhd(T)|$ is the number of neighborhood faces in the set $Nbhd(T)$

When v_0 is not a constant, the Laplace smoothing on $|v_0|$ diffuses the norm of v_0 and hence decreases the norm of v_0. When it is a constant, the norm will stop decreasing. We should stop the iteration when the difference is small enough.

Also, we can prove that upon convergence, the argument of the associated BC, v^* of the optimal map, f^*, shouuld be equal to the argument of a the conjugate of a holomorphic function φ. Therefore, this property implies that the argument of v^* should be harmonic.

Proof

Note that $\varphi = |\varphi|e^{i\theta}$ We have $v^* = k\frac{\bar{\varphi}}{|\varphi|} = ke^{-i\theta}$ and $arg(v^*) = -\theta$

Note that φ is a holomorphic function, which means that $log(\varphi) = log(|\varphi|) + i\theta$ is holomorphic in one branch.

Therefore, θ is a harmonic function

At the optimal state, $\Delta\theta^* = 0$. We can then find the harmonic conjugate of $-\theta^*$, denoted by ζ, by

$$-\tfrac{\partial\theta^*}{\partial x} = \tfrac{\partial\zeta}{\partial y}, \quad -\tfrac{\partial\theta^*}{\partial y} = -\tfrac{\partial\zeta}{\partial x}$$

We get a holomorphic function $\zeta - i\theta^*$. Consider $\varphi = e^{\zeta - i\theta}$. Then we get the optimal BC $\mu^* = ke^{i\theta^*} = k\frac{\bar{\varphi}}{|\varphi|}$ and it is of Teichmuller type.

Besides, a BC of Teichmuller type must have a constant norm. We apply the averaging operator A on $\bar{\mu_1}$ to project $\bar{\mu_0}$ to a BC with constant norm. We first change the modulus of $\bar{\mu_0}$ to a positive constant k, which means $\mu_1 = karg(\bar{\mu_1})$. A good choice for k can be

$$k = \frac{\int_{S_1} |\bar{\mu_1}| dS_1}{Area(S_1)}$$

In other words, k is the mean of the norm of $\bar{\mu_1}$ over the whole domain. When $\bar{\mu_1}$ does not have a constant norm, $k < \|\bar{\mu_1}\|_\infty$. When $\bar{\mu_1}$ has a constant norm, the norm of $\bar{\mu_1}$ will be kept unchanged. The averaging operator A can be defined as follows:

$$\mu_1(T) = A(\bar{\mu_1})(T) = (\frac{\sum |\bar{\mu_1}(T)|}{No.of faces})\frac{\bar{\mu_1}(T)}{|\bar{\mu_1}(T)|}$$

with the Laplace smoothing and averaging on v_0, we obtain a new Beltrami Coefficient, $\mu_1 = A(L(v_n))$ An updated quasi-conformal map, f_1, can then be obtained by the LBS: $f_1 = LBS(\mu_1)$, and an updated BC, $v_1 = \mu(f_1)$

Therefore, we keep the iteration going until the algorithm converges to the desired T-Map. More specifically, given the pair (f_n, v_n), we obtain the new pair (f_{n+1}, v_{n+1}) by:

$$\mu_{n+1} = A(L(v_n))$$
$$f_{n+1} = LBS(\mu_{n+1})$$
$$v_{n+1} = \mu(f_{n+1})$$

Consequently, we get a sequence of pairs (f_n, v_n), which converges to the optimal BC associated to the T-Map. In practice, we stop the iteration when $\|v_{n+1} - v_n\| < \epsilon$

5.1 QC iteration for open surfaces

Input: Triangular meshes: K_1 and K_2 and the landmark constraints and/or boundary condition.

Output: Optimal BC v and the T-Map f

1. Obtain the initial mapping $f_0 = LBS(\mu_0 = 0, \text{set} v_0 = \mu(f)$
2. Given v_n, compute $\mu_{n+1} = A(L(v_n))$; Compute $f_{n+1} = LBS(\mu_{n+1})$ and $v_{n+1} = \mu(f_{n+1})$
3. If $\|v_{n+1} - v_n\| \geq \epsilon$, continue. Otherwise, stop the iteration.

5.2 QC iteration for genus-0 closed surfaces

The QC iteration can also be applied to the case when D_i(i=1,2) is a unit sphere. In other words, given a set of landmark constraints between the unit sphere, our goal is to look for the T-Map $f : D_1 \to D_2$. special attention need to be paid in this case.

We can assume that the north pole is fixed. If not, it can also be achieved by a Mobius transformation. The LBS can be applied to unit spheres by stereographically projecting D_i onto big triangles in $\mathbb{R}^2$

Input: Triangular meshes: K_1 and K_2 and the landmark constraints and/or boundary condition.

Output: Optimal BC v and the T-Map f

1. Add vertices around the north pole as landmarks and fix their positions. Obtain the initial mapping $f_0 = LBS(\mu_0 = 0, \text{set} v_0 = \mu(f)$

2. Given v_n, f_n, when n is even, add vertices ${s_j}_{j=1}^m$ around the south pole as landmarks. Set the correspondence as $s_j \to f_n(s_j)$. Rotate the south pole of D_i to the north pole. When n is odd, add vertices $\{s_j\}_{j=1}^m$ around the south pole as landmarks. Set the correspondence as $s_j \to f_n(s_j)$. compute $\mu_{n+1} = A(L(v_n))$; Compute $f_{n+1} = LBS(\mu_{n+1})$ and $v_{n+1} = \mu(f_{n+1})$

3. If $\|v_{n+1} - v_n\| \geq \epsilon$, continue. Otherwise, stop the iteration.

5.3 QC iteration for soft landmark constraints

Input: Triangular meshes: K_1 and K_2 and the landmark constraints and/or boundary condition. landmark constraint tolerance δ

Output: Optimal BC v and the T-Map f

1. obtain an initial guess of landmark marching T-Map g_0, set the stopping criteria to be $\|v_{n+1} - v_n\| < 100\epsilon$

2. Given μ_n, g_n. Starting from μ_n, compute the T-Map g_{n+1} using the algorithm without setting the interior landmark constraints. Set the stopping criteria to be $\|v_{n+1} - v_n\| < \epsilon$. Let $\mu_{n+1} = BC$ of g_{n+1}.

3. If $\|p_i - q_i\| \geq \delta$ for some $i = 1, 2, ..., n$, continue. Otherwise, stop the iteration.

6 Spherical Parameterization

Thm Let $f : M_1 \to M_2$ and $g : M_2 \to M_3$ be quasi-conformal maps. Suppose the Beltrami differentials of f^{-1} and g are the same. Then the Beltrami differential of $g \circ f$ is equal to 0. Hence, $g \circ f : M_1 \to M_2$ is conformal.

Proof.

We first prove the composite formula for Beltrami-Coefficient

$$
\begin{aligned}
\mu_{g\circ f} &= \frac{\partial(g\circ f)/\partial\bar{z}}{\partial(g\circ f)/\partial z} \\
&= \frac{\frac{\partial g}{\partial f}\frac{\partial f}{\partial\bar{z}} + \frac{\partial g}{\partial\bar{f}}\frac{\partial\bar{f}}{\partial\bar{z}}}{\frac{\partial g}{\partial f}\frac{\partial f}{\partial z} + \frac{\partial g}{\partial\bar{f}}\frac{\partial\bar{f}}{\partial z}} \\
&= \frac{\frac{\partial f}{\partial\bar{z}}/\frac{\partial f}{\partial z} + \frac{\partial g}{\partial\bar{f}}/\frac{\partial g}{\partial f} * \frac{\partial\bar{f}}{\partial\bar{z}}/\frac{\partial f}{\partial z}}{1 + \frac{\partial g}{\partial\bar{f}}/\frac{\partial g}{\partial f} * \frac{\partial\bar{f}}{\partial z}/\frac{\partial f}{\partial z}} \\
&= \frac{\mu_f + \frac{\bar{f}_z}{f_z}(\mu_g\circ f)}{1 + \frac{\bar{f}_z}{f_z}(\mu_g\circ f)} \\
&= \frac{\mu_f + \bar{f}_z/f_z(\mu_g\circ f)}{1 + \bar{\mu_f}(\mu_g\circ f)}
\end{aligned} \tag{2}
$$

where we used that

$$\tfrac{\partial\bar{f}}{\partial\bar{z}} = \bar{f}_z \quad \tfrac{\partial\bar{f}}{\partial z}/\tfrac{\partial f}{\partial z} = \bar{\mu_f}$$

Note that $g\circ f = \mathbf{id}$, when $g = f^{-1}$
Therefore, $\mu_f + \frac{\bar{f}_z}{f_z}(\mu_{f^{-1}}\circ f) = 0$
Then, because $\mu_{f^{-1}} == \mu_g$, we have

$$\mu_f + \frac{\bar{f}_z}{f_z}(\mu_g\circ f) = \mu_f + \frac{\bar{f}_z}{f_z}(\mu_{f^{-1}}\circ f) = 0$$

Hence, $\mu_{g\circ f} = 0$