Analysis I   Homework 5+

A. Prove Arithmetic of Infinity Limits, part (iii).

$(r_n),(s_n) \Rightarrow$ real sequences

$\lim_{n\to\infty} r_n = r$

$\lim_{n\to\infty} s_n = \infty$

For $r>0 \Rightarrow \lim_{n\to\infty}(r_n \cdot s_n) = +\infty$, and for $r<0 \Rightarrow \lim_{n\to\infty}(r_n \cdot s_n) = -\infty$.

If $\lim_{n\to\infty} r_n = r \Leftrightarrow \forall \varepsilon>0 \ \exists n_\varepsilon \in \mathbb{N} \ \forall n > n_\varepsilon \ |r_n - r| < \varepsilon \Leftrightarrow r-\varepsilon < r_n < r+\varepsilon$,

$\lim_{n\to\infty} s_n = \infty \Leftrightarrow \forall N>0 \ \exists n_2 \ \forall n > n_2 \ s_n > N$ and $r>0$. Then $M>0$

$\exists n_M = \max\{n_\varepsilon, n_2\}$ and $n > n_M$

$\Rightarrow \lim_{n\to\infty}(r_n \cdot s_n) = +\infty$ ■

B. Explain why if a_n converges, then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1}$

If $\exists \lim_{n\to\infty} a_n = L \Rightarrow \lim_{n\to\infty} a_{n+1} = L$

For $\lim\limits_{n\to\infty} a_n = L \iff \forall \varepsilon > 0, \exists n_\varepsilon \; \forall n > n_\varepsilon \quad |a_n - L| < \varepsilon$

$C_n = a_{n+1}$ where $n = 1, 2, \dots$

$\varepsilon > 0 \quad \exists n_0 = n_\varepsilon - 1$ and $n > n_0$

$|c_n - L| = |a_{n+1} - L| < \varepsilon$

$\iff n+1 > n_\varepsilon$

$\iff n > n_\varepsilon - 1$

$\Rightarrow \lim\limits_{n\to\infty} C_n = L = \lim\limits_{n\to\infty} a_{n+1}$ $\blacksquare$

C. Let $r>0$ be a positive real number. This problem will give a simple approach to $\lim_{n\to\infty} r^n$.

(i) Explain briefly why $a_n = r^n$ can be written recursively as $a_0 = 1$, $a_n = r \cdot a_{n-1}$ for $n>0$.

(ii) Using problem A and (i), solve for the possible limits of a_n if the sequence converges.

(iii) Show that a_n is monotone. (It may be increasing or decreasing, depending on r.)

(iv) Combine (ii) and (iii) to show $\lim_{n\to\infty} r^n$ converges to 0 for $r<1$, converges to 1 for $r=1$, and diverges to ∞ for $r>1$.

(i) $a_n = r^n$, $r>0$ Recursively written $a_0 = 1$, $a_n = r \cdot a_{n-1}$, $n>0$

Proof: 1) $n=0$ $a_0 = r^0 = a_0 = 1$

2) If it's true that $n=k$, $a_k = r^k$, $a_k = r \cdot a_{k-1}$

3) We should prove $n=k+1$, $a_{k+1} = r^{k+1}$ $a_{k+1} = r \cdot a_k$

$\Rightarrow a_{k+1} = r \cdot a_k = r \cdot r^k = r^{k+1}$ $\blacksquare$

(ii) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} (r \cdot a_{n-1})$

$r=1$ $\lim_{n\to\infty} a_n = \lim_{n\to\infty} 1^n = 1$

$0<r<1$ $\lim_{n\to\infty} a_n = \lim_{n\to\infty} r \cdot a_{n-1} = r \cdot \lim_{n\to\infty} a_{n-1}$

$L = r \cdot L$

$(1-r) \neq 0$ $L=0$

$\Rightarrow L = 0$ if converges

(iii) $0<r<1$

$a_n = r^n$

$$\frac{a_{n+1}}{a_n} = \frac{r^{n+1}}{r^n} = r < 1 \quad (a_n) \searrow$$

If $r>1$

$$\frac{a_{n+1}}{a_n} = \frac{r^{n+1}}{r^n} = r > 1 \quad (a_n) \nearrow$$

(iv) $a_n = r^n$ $(a_n) \searrow$

$0 < r < 1 \Rightarrow 0 < r^n$ $\Bigg\} \Rightarrow \exists \lim_{n \to \infty} a_n = L \Rightarrow a_n = r \cdot a_{n-1} \,/\, \exists \lim_{n \to \infty}$

$$\lim_{n \to \infty} a_n = r \lim_{n \to \infty} a_{n-1}$$

$$L = r \cdot L$$

$$(1 - r) L = 0$$

$$\Rightarrow L = 0$$

$r > 1$ $(a_n) \nearrow$ it is not bounded

$\Rightarrow \lim_{n \to \infty} a_n = \infty$

D. Consider the recursively defined sequence $S_1 = 1$ and $S_{n+1} = 1/(4-S_n)$ for $n \geq 1$.

(i) Prove that S_n converges. (Hint: is the sequence monotone?)

(ii) Solve to find the limit.

(i) $S_1 = 1$, $S_2 = \frac{1}{4-S_1} = \frac{1}{4-1} = \frac{1}{3}$

$S_3 = \frac{1}{4-S_2} = \frac{1}{4-\frac{1}{3}} = \frac{3}{11}$, $S_4 = \frac{1}{4-S_3} = \frac{1}{4-\frac{3}{11}} = \frac{11}{41}$

$S_1 = 1 > S_2 = \frac{1}{3} > S_3 = \frac{3}{11} > S_4 = \frac{11}{41}$

I will prove that (S_n) is bounded from below

$S_n \geq 2-\sqrt{3} \quad \forall n \in \mathbb{N}$

1° $n=1 \quad S_1 = 1 \geq 2-\sqrt{3} = 0,..$

2° $n=k \quad S_k \geq 2-\sqrt{3}$

3° For $n=k+1 \Rightarrow S_{k+1} \geq 2-\sqrt{3}$

$S_{k+1} \geq 2-\sqrt{3} \iff \frac{1}{4-S_k} \geq 2-\sqrt{3} \iff 1 \geq 8-4\sqrt{3}-(2-\sqrt{3})S_k \iff$

$\iff (2-\sqrt{3})S_k \geq 7-4\sqrt{3} \;/\cdot 2-\sqrt{3} \iff S_k \geq \frac{7-4\sqrt{3}}{2-\sqrt{3}} \cdot \frac{2+\sqrt{3}}{2+\sqrt{3}} \iff$

$\iff S_k \geq \frac{14-8\sqrt{3}+7\sqrt{3}-12}{4-3} = 2-\sqrt{3}$

(ii) $(S_n)\searrow \quad \forall n \quad S_{n+1} \leq S_n$

1° $S_2 = \frac{1}{3} \leq S_1 = 1$

2° $n=k$, $S_{k+1} \leq S_k$

3° $n=k+1 \quad S_{k+2} \leq S_{k+1}$

$\iff \frac{1}{4-S_{k+1}} \leq \frac{1}{4-S_k} \iff 4-S_k \leq 4-S_{k+1} \;/-4 \iff -S_k \leq -S_{k+1} \;/\cdot(-1)$

$\iff S_k \geq S_{k+1}$

From (i)+(ii) $\Rightarrow \exists\, L = \lim_{n\to\infty} S_n$

$S_{n+1} = \frac{1}{4-S_n} \;/ \lim_{n\to\infty}$

$\iff L = \frac{1}{4-L} \iff 4L - L^2 = 1 \iff 0 = L^2 - 4L + 1 \iff$

$\iff L_{1/2} = \frac{4 \pm \sqrt{16-4}}{2} = 2 \pm \sqrt{3}$

$L_1 = 2-\sqrt{3} \qquad \cancel{L_2 = 2+\sqrt{3}}$

E. Working directly from the definition (and without using Cauchy Completeness Theorem), show any Cauchy sequence of real numbers is bounded.

(a_n) is Cauchy sequence

$\forall \varepsilon > 0 \quad \exists n_0 \quad \forall n, m > n_0 \quad |a_n - a_m| < \varepsilon$

$\exists n_1 \quad \forall n, m > n_1 \quad |a_n - a_m| < 1$

$m = n_1 + 1 \quad |a_n - a_{n_1+1}| < 1$

If $n > n_1$

$|a_n| = |a_n - a_{n_1+1} + a_{n_1+1}| \leq |a_n - a_{n_1+1}| + |a_{n_1+1}| < 1 + |a_{n_1+1}|$

$M = \max\{|a_1|, |a_2|, \dots, |a_{n_1}|, |a_{n_1+1}| + 1\}$

$\Rightarrow \forall n \in \mathbb{N} \quad |a_n| < M$

$\Rightarrow (a_n)$ is bounded

F. Make an educated guess as to the limit of the complex sequence $z_n = i^n/(2n+6)$. Working directly from the definition of complex limit, show that your answer is correct.

$$z_n = \frac{i^n}{2n+6} = \begin{cases} \frac{1}{2n+6}, & n = 4k \\ \frac{i}{2n+6}, & n = 4k+1 \\ \frac{-1}{2n+6}, & n = 4k+2 \\ \frac{-i}{2n+6}, & n = 4k+3 \end{cases}$$

$$\Rightarrow \lim_{n \to \infty} z_n = 0$$

6. Let S_n be bounded sequence, and let t_n converge to L.
(i) Show that if $L=0$, then the product $S_n \cdot t_n$ also converges to 0.
(ii) Give an example where S_n is bounded and t_n converges, but where the product does not converge.
Hint: One approach to (i) uses Sandwich Theorem.

(i) $|S_n| < M \quad \forall n \in \mathbb{N} \quad (-M < S_n < M)$

$\lim_{n\to\infty} t_n = 0$

$\lim_{n\to\infty} S_n \cdot t_n = 0$

$a_n = -M \cdot t_n \le S_n \cdot t_n \le M \cdot t_n = b_n$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (-M \cdot t_n) = -M \lim_{n\to\infty} t_n = -M \cdot 0 = 0$

$\lim_{n\to\infty} b_n = \lim_{n\to\infty} (M \cdot t_n) = M \lim_{n\to\infty} t_n = M \cdot 0 = 0$

$\Rightarrow \lim_{n\to\infty} (S_n \cdot t_n) = 0$

(ii) If $S_n = \cos n\pi$

$|S_n| \le 1 \quad \forall n \in \mathbb{N}$

$t_n = \frac{n}{n+1} \qquad \lim_{n\to\infty} t_n = 1$

$a_n = S_n \cdot t_n = \cos n\pi \cdot \frac{n}{n+1} = \begin{cases} \frac{n}{n+1}, & n = 2k \\ \frac{-n}{n+1}, & n = 2k+1 \end{cases}$

$a_{2n} = \frac{n}{n+1} \Rightarrow 1$

$a_{2n+1} = \frac{-n}{n+1} \to -1$

$\Rightarrow \nexists \lim_{n\to\infty} a_n$