Application: (from the Interlude)

Recall that we wanted to prove the following Lemma, in connection w/ a cover of $[0,3]$ by open intervals with lengths forming a geometric series.

Lemma *: If the interval $[0,a]$ is contained in the union of open intervals $(a_0,b_0), (a_1,b_1), \ldots$ then $a \leq \sum_{i=0}^{\infty} b_i - a_i$.

We proved the analogue of Lemma * for a finite cover with a straightforward induction argument.

Now Lemma * follows, since $[0,a]$ is compact, so the open cover has a finite subcover. Since an infinite sum of positive numbers is $\geq$ any finite subsum, we have the desired. ✓

There are many similar-flavored applications of compactness all through Analysis and Topology.

V Limits and continuity of real functions:

So far, we've mainly looked at sequences.

These are functions with a domain of $\mathbb{N}$. (defined on $\mathbb{N}$).

That is, functions $\mathbb{N} \to \mathbb{R}$, or $\mathbb{N} \to \mathbb{C}$, or $\mathbb{N} \to M$

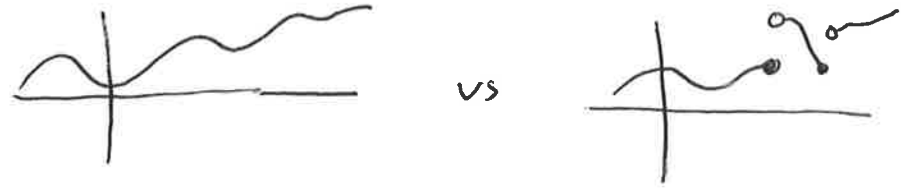
(real sequences) (complex sequences) (metric space sequences)

where we map n to a_n.

In this section, we'll switch focus to look at functions $\mathbb{R} \to \mathbb{R}$. These are the functions that you are probably most familiar with from high school.

A main goal will be a "rigorous and careful" definition of a "continuous" function. That is, a rigorous definition of the idea of functions we can draw "without lifting our pen."

Eg:



Of course, it should also apply to hard-to-sketch functions like $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$!!!

A. Limits of real functions:

To start with, for a function $f: \mathbb{R} \to \mathbb{R}$, it is straightforward to translate the definition of limits of sequences to a definition of the limit "at" ∞.

Throughout, $f: \mathbb{R} \to \mathbb{R}$ will be a real function.

Definition 1: We say the limit as x goes to ∞ of $f(x)$ is L (and write $\lim_{x\to\infty} f(x) = L$) if

$$\forall \varepsilon > 0, \ \exists N \text{ s.t. } [x > N] \Rightarrow [|f(x) - L| < \varepsilon].$$

Note that in the real function situation, we allow any real $x > N$ as opposed to sequences, where we consider only naturals.

Self-check: Write down definitions for $\lim_{x \to -\infty} f(x) = L$

and for a limit as x goes to ∞

and f is a function $f : \mathbb{R} \to M$. metric space

The meaning of Definition 1 should be familiar from sequences.

Similarly,

Definition 2: We say $f(x)$ diverges to ∞ as x goes to ∞,

and write $\lim_{x \to \infty} f(x) = \infty$

if $\forall M, \exists N$ s.t. $[x > N] \Rightarrow [f(x) > M]$.

By comparing and contrasting these and similar definitions

it becomes apparent that to find the definition of

the limit as x approaches a real number c,

we should replace the "$\exists N, x > N$" with something

more similar to the "$\forall \varepsilon > 0$ ___ $|f(x) - L| < L$"

part of Definition 1.

Which brings us to:

Definition 3: We say that $\lim_{x \to c} f(x) = L$

(the limit as x goes to c of $f(x)$ is L)

if $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $[0 < |x - c| < \delta] \Rightarrow [|f(x) - L| < \varepsilon]$.

Thus, the definitions says that in order for $f(x)$ to be in

an ε-neighborhood of L

it is enough to have x in a δ-neighborhood of c.

(We allow $f(c)$ to be any value, by the $0 < |x - c|$ part.)

We do some examples directly from definition.

Example: $\lim_{x\to 0} x^2 = 0$ since $|x^2| = x^2 < \varepsilon$

$\iff |x| < \sqrt{\varepsilon}$

so for any $\varepsilon > 0$ we take $\delta = \sqrt{\varepsilon}$

and then the definition is satisfied.

Example: $\lim_{x\to\infty} \sqrt{x} = \infty$, as $\sqrt{x} > M \iff x > M^2$

so we take $N = M^2$ to satisfy the definition.

Notice the similarity to and (small) differences from $\lim_{n\to\infty} \sqrt{n}$!!

Example: $\lim_{x\to 2} x^2 = 4$.

We need to find upper-bounds on $|x^2 - 4| = |x+2| \cdot |x-2|$

in terms of an upper-bound on $|x-2|$.

First, if $|x-2| < \delta$, then $|x+2| = |x-2+4| < \delta + 4$

by the $\triangle$ inequality

also, now $|x+2| \cdot |x-2| < \delta^2 + 4\delta < 5\delta$ when $\delta < 1$.

So now take $\delta = \min\{\underbrace{\tfrac{1}{2}}_{<1}, \tfrac{\varepsilon}{5}\}$

so $|x+2| \cdot |x-2| < 5 \cdot \delta \leq 5 \cdot \frac{\varepsilon}{5} = \varepsilon$

as desired ✓

As with sequences, we'll soon have theorems that let us (usually) invoke detailed ε-δ calculations like that!

Example: Let $f(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$

Then $\lim_{x\to 0} f(x) = 0$. If $x < 0$, then $|f(x)| < \varepsilon$

is trivial!

while if $x > 0$, we can take $\delta = \varepsilon$.

So we overall take $\delta = \varepsilon$ to satisfy the limit definition. ✓

<u>Example:</u> Consider the two functions

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases} \quad \text{and} \quad g(x) = x \cdot f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Then for any $c \in \mathbb{R}$,

$\lim_{x \to c} f(x)$ does not converge,

because for any $\delta > 0$, the interval $(c-\delta, c+\delta)$ contains both rational + irrational points

so that $f(x) = 1$ on some points, 0 on others.

Taking $\varepsilon = 1/3$, we see that a limit L must be both $> 2/3$ and $< 1/3$, a contradiction. ✓

However,

$$\lim_{x \to 0} g(x) = 0$$

since $\forall \varepsilon > 0$ we can take $\delta = \varepsilon$

and then $|x| < \delta = \varepsilon \implies |g(x)| < \varepsilon$

as $g(x)$ is either x or 0. ✓

Let's build up some theorems, so we only need ε-δ arguments in unusual circumstances.

<u>Proposition:</u> For any $c \in \mathbb{R}$,

1) $\lim_{x \to c} x = c$
2) $\lim_{x \to c} d = d$ for any constant $d \in \mathbb{R}$.

<u>Proof:</u> <u>Self-check</u>! $\square$

<u>Theorem</u> (Arithmetic of Limits, real function version)

Let $f(x)$, $g(x)$ be real functions, $c, \alpha \in \mathbb{R}$

and suppose $\lim_{x \to c} f(x) = L_f$, $\lim_{x \to c} g(x) = L_g$.

Then i) $\lim_{x \to c} f(x) + g(x) = L_f + L_g$

ii) $\lim_{x \to c} f(x) - g(x) = L_f - L_g$

iii) $\lim_{x \to c} \alpha \cdot f(x) = \alpha \cdot L_f$

iv) $\lim_{x \to c} f(x) \cdot g(x) = L_f \cdot L_g$

v) $\lim_{x \to c} f(x)/g(x) = L_f/L_g$ so long as $L_g \neq 0$.

Occasionally we have a function which is only defined on some domain $A \subseteq \mathbb{R}$, rather than on all reals.

In this case, we require only that

$$\forall \varepsilon > 0, \ \exists \ \delta \text{ s.t. } \left[x \in A \text{ and } 0 < |x - c| < \delta\right] \Rightarrow \left[\, |f(x) - L| < \varepsilon \right]$$

that is, we only consider values of x where f is defined.

For example, in AoL $^{(v)}$ above, we need to restrict the domain of $f(x)/g(x)$ to values of x where $g(x) \neq 0$.

The proof is entirely similar to AoL for sequences

- reduce to (i), (iv), (v)
- reduce (v) to the case where $f(x) = 1$.
- make ε-δ arguments similar to the ε-N arguments we made back then.

__Example:__ $\lim_{x\to 2} \frac{x^2-4}{2x-4} = \lim_{x\to 2} \frac{(x+2)(x-2)}{2(x-2)}$.

Since $\frac{(x+2)(x-2)}{2\cdot(x-2)} = \frac{x+2}{2}$ for $x \neq 2$ (and $x=2$ is irrelevant to limit definition as we require $0 \underset{\equiv}{<} |x-2| < \delta$)

we have

$$\lim_{x\to 2} \frac{(x+2)(x-2)}{2(x-2)} = \lim_{x\to 2} \frac{x+2}{2} = \frac{2+2}{2} = 2.$$

__Note:__ As usual, if you're ever tempted to write "$\frac{0}{0}$" as your final answer, it is likely that you have "missed the point"!

We also have

__Theorem__ (Arithmetic of Infinite Limits, real function version)

Let $f(x)$, $g(x)$, $h(x)$ be real functions, $c \in \mathbb{R}$ and suppose $\lim_{x\to c} f(x) = L_f$ but $\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = \infty$.

Then

i) $\lim_{x\to c} f(x) + g(x) = \infty$

$\lim_{x\to c} g(x) + h(x) = \infty$

ii) $\lim_{x\to c} g(x) \cdot h(x) = \infty$

iii) $\lim_{x\to c} f(x) \cdot g(x) = \begin{cases} \infty & \text{if } L_f > 0 \\ -\infty & \text{if } L_f < 0 \end{cases}$

iv) $\lim_{x\to c} \frac{f(x)}{g(x)} = 0$.

The proofs of all parts are entirely similar to the sequence case.

As before, this gives us an Arithmetic of Signed Infinities with the same indeterminate forms!

A more efficient way to get from Theorems like AoL for sequences to a result for series is by the following useful result:

<u>Theorem:</u> (Relating Sequences to Functions)

For a real function $f(x)$ and $c \in \mathbb{R}$,

we have $\lim_{x \to c} f(x) = L$

$\iff$ for any sequence a_n w/ a_n in the domain of f and $\neq c$

with $\lim_{n\to\infty} a_n = c$

we have $\lim_{n\to\infty} f(a_n) = L$.

We'll prove this RStF Theorem shortly, but first let's describe a typical application: We apply RStF once to translate a function problem to a sequence problem, use a sequence theorem, then apply RStF again to translate back to functions.

<u>Example:</u> Using RStF, prove AoL (i) for real functions.

<u>Solution:</u> As $\lim_{x\to c} f(x) = L_f$, for any sequence a_n in domain of f and avoiding c, w/ $\lim_{n\to\infty} a_n = c$

we have $\lim_{n\to\infty} f(a_n) = L_f$.

Similarly, $\lim_{n\to\infty} g(a_n) = L_g$

Now by AoL, sequence version, for any such sequence

$\lim_{n\to\infty} f(a_n) + g(a_n) = L_f + L_g$.

Now a 2nd application of RStF implies that

$\lim_{x\to c} f(x) + g(x) = L_f + L_g$.

<u>Proof (of RStF)</u>: As usual, there are 2 directions.

<u>($\Longrightarrow$)</u>: We suppose that $\lim_{x \to c} f(x) = L$

and $\lim_{n \to \infty} a_n = c$, as in the hypothesis of the Theorem.

Now we "chain" the 2 definitions together, in a usual "ε-machine" type argument:

As $\forall \varepsilon > 0,\ \exists \delta > 0$ s.t. $[0 < |x - c| < \delta] \Rightarrow [\,|f(x) - L| < \varepsilon]$, and

$\forall \varepsilon^* > 0,\ \exists N$ s.t. $[n > N] \Rightarrow [\,|a_n - c| < \varepsilon^*]$

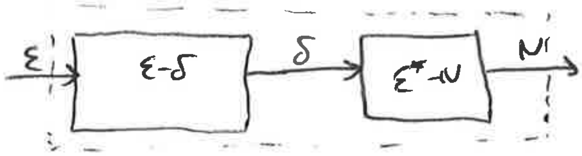
Given ε, the 1st definition produces δ. We take $\varepsilon^* = \delta$

and feed it into the 2nd definition to produce N,

Now $[n > N] \Rightarrow [\,|a_n - c| < \delta = \varepsilon^*] \Rightarrow [\,|f(a_n) - L| < \varepsilon]$

as required for $\lim_{n \to \infty} f(a_n) = L$.

<u>Picture</u>:



<u>($\Longleftarrow$)</u> We suppose that every sequence a_n w/ $\lim_{n \to \infty} a_n = c$

yields $\lim_{n \to \infty} f(a_n) = L$ (by hypothesis)

but (for contradiction) that $\lim_{x \to c} f(x) \neq L$.

Since $\lim_{x \to c} f(x) \neq L$,

there is <u>some</u> fixed value ε_0 so that <u>no</u> δ

satisfies $[0 < |x - c| < \delta] \Rightarrow [\,|f(x) - L| < \varepsilon_0]$.

This tells us that for each n,

we can find a value a_n so that $0 < |a_n - c| < \frac{1}{n}$

but $|f(a_n) - L| \geq \varepsilon_0$.

But now since $|a_n - c| < \frac{1}{n}$, the Sandwich Theorem $\Rightarrow \lim_{n \to \infty} a_n = c$

hence that $\lim_{n \to \infty} f(a_n) = L$

hence that for large enough n, $|f(a_n) - L| < \varepsilon_0$ #

a contradiction. ∎

We examine a few more consequences.

Example: Show that $\lim_{x\to c} \sqrt{x} = \sqrt{c}$ for any $c \geq 0$.

Solution: By an earlier problem, for any nonnegative sequence s_n with $\lim_{n\to\infty} s_n = s$, we have $\lim_{n\to\infty} \sqrt{s_n} = \sqrt{s}$.
The result now follows directly from RStF. ✓

Similarly, we have Uniqueness of Limits:

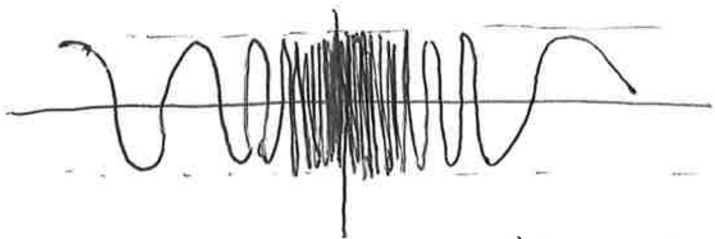
Proposition: For a real function $f(x)$ and $c \in \mathbb{R}$,
if $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} f(x) = M$
then $L = M$.

Sketch: Apply RStF to produce a_n w/ $\lim_{n\to\infty} f(a_n) = L = M$;
apply Uniqueness of Limits of sequences. $\square$.

In a slightly different direction, RStF can be useful for showing the limit of a function does not converge.

Corollary (to RStF):
If f is a real function, c a real number,
and a_n, b_n are real sequences w/ $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = c$
but $\lim_{n\to\infty} f(a_n) \neq \lim_{n\to\infty} f(b_n)$
then $\lim_{x\to c} f(x)$ diverges.
(Proof is immediate!)

Example of Corollary: ("The Topologist's Sin Curve")
Consider the function $\sin\frac{1}{x}$



(drawing not to scale)

The graph suggests that $\lim_{x\to 0} \sin\frac{1}{x}$ does not converge.

We can verify using the Corollary.

Consider $a_n = \frac{1}{2\pi n}$, so $\sin\frac{1}{a_n} = \sin 2\pi n = 0$ (for all n)

and $b_n = \frac{1}{2\pi n + \frac{\pi}{2}}$, so $\sin\frac{1}{b_n} = \sin(2\pi n + \frac{\pi}{2}) = 1$ " " " .

Thus, $\lim_{n\to\infty} \sin\frac{1}{a_n} = 0$

but $\lim_{n\to\infty} \sin\frac{1}{b_n} = 1$. As $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0$, but $0 \neq 1$,

$\lim_{x\to 0} \sin\frac{1}{x}$ does not converge. ✓

The graph may also suggest to you that eg. $\lim_{x\to 1/10} \sin\frac{1}{x}$ fails to converge, but this is <u>not</u> the case. Indeed, $\lim_{x\to c} \sin\frac{1}{x}$ converges for any $c \neq 0$.

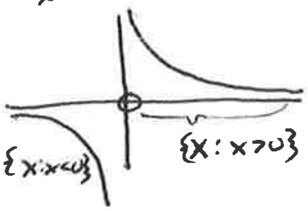
B. One-sided limits

The point c divides the real line into

$$\{a : a < c\} \quad \text{and} \quad \{b : b > c\}$$

For some functions that come up naturally the behavior on the two sides of c are quite different.

<u>Eg.</u>: $f(x) = \frac{1}{x}$ at $c = 0$



It's convenient to be able to describe different limit behavior on the two sides.

You can probably write down the definitions by now without help! (but let's do at least one side together).

<u>Definition</u> We say that $f(x)$ <u>converges to L as x approaches c from the right</u>

and write $\lim_{x\to c^+} f(x)=L$

if

$$\forall \varepsilon>0,\ \exists \delta>0 \text{ s.t. } [c<x<c+\delta] \Rightarrow [|f(x)-L|<\varepsilon].$$

Similarly, write $\lim_{x\to c^+} f(x)=\infty$ if

$$\forall M,\ \exists \delta>0 \text{ s.t. } [c<x<c+\delta] \Rightarrow [f(x)>M].$$

<u>Notice</u> The definition of $\lim_{x\to c^+} f(x)$ is the same as the limit $\lim_{x\to c} f_{right}(x)$,

where $f_{right}(x)$ is the restriction of f to the domain $\{x : x>c\}$,

ie, $f_{right}(x)=\begin{cases} f(x) & \text{if } x>c \\ \text{otherwise undefined.} \end{cases}$

Similar definitions hold for $\lim_{x\to c^-} f(x)=L$ and $\lim_{x\to c^-} f(x)=\infty$.

<u>Self-check:</u> Write down these definitions.

Relation w/ 2-sided limits:

<u>Theorem</u> For a real function $f(x)$ and $c, L\in\mathbb{R}$, we have

1) $\lim_{x\to c} f(x)=L \Leftrightarrow \lim_{x\to c^+} f(x)=L$ <u>and</u> $\lim_{x\to c^-} f(x)=L$.

2) $\lim_{x\to c} f(x)=\infty \Leftrightarrow \lim_{x\to c^+} f(x)=\infty$ <u>and</u> $\lim_{x\to c^-} f(x)=\infty$.

The proof follows from the definitions and by noting that

$$0<|x-c|<\delta \Leftrightarrow c-\delta<x<c+\delta \text{ and } x\neq c$$
$$\Leftrightarrow c-\delta<x<c \text{ or } c<x<c+\delta.$$

Let's write one part carefully.

Proof details: (Part (1), $\Leftarrow$ direction).

We're given that

$$\forall \varepsilon>0,\ \exists \delta_+>0 \text{ s.t. } [c<x<c+\delta_+] \Rightarrow [|f(x)-L|<\varepsilon]$$

$$\forall \varepsilon>0,\ \exists \delta_->0 \text{ s.t. } [c-\delta_-<x<c] \Rightarrow [|f(x)-L|<\varepsilon].$$

So for a given $\varepsilon>0$, take $\delta=\min\{\delta_+,\delta_-\}$.

Then $[c-\delta<x<c$ or $c<x<c+\delta]$

gives $[c-\delta_-<x<c$ or $c<x<c+\delta_+]$, hence the desired

for $|f(x)-L|$ in the 2-sided

limit definition. $\square$

Example: Let $f(x)=\begin{cases} x^2 & \text{for } x\geq 0 \\ -x & \text{for } x<0 \end{cases}$

As $\lim\limits_{x\to 0+} f(x) = \lim\limits_{x\to 0+} x^2 = 0$

$\lim\limits_{x\to 0-} f(x) = \lim\limits_{x\to 0-} -x = 0$ (by the theorem),

a 2nd application of the theorem says that

$\lim\limits_{x\to 0} f(x)=0.$ ✓

Indeed, the theorem gives us a method for "gluing" together two functions with a common limit at some point. ✓

Example: Let $\lfloor x\rfloor$ be the "floor" or "round down" function

so $\lfloor x\rfloor :=$ the greatest integer that is $\leq x$.

Then $\lim\limits_{x\to 1} \lfloor x\rfloor$ does not converge, since

$$\lim_{x\to 1+} \lfloor x\rfloor = \lim_{x\to 1+} 1 = 1$$

but

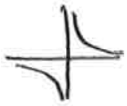
$$\lim_{x\to 1-} \lfloor x\rfloor = \lim_{x\to 1-} 0 = 0$$

and $0\neq 1$.

(since $\lfloor x\rfloor = 1$ for all x with $1<x<1+\delta$ for any $\delta<1$; similarly from left).

Self-check: Show that $\lim_{x\to\sqrt{2}} \lfloor x \rfloor = 1$

Example: $f(x) = \frac{1}{x}$ at $c=0$



Since we proved

$$\lim_{n\to\infty} \frac{1}{a_n} = \infty \quad \text{for any positive sequence } a_n \text{ with } \lim_{n\to\infty} a_n = 0$$

RStF yields that

$$\lim_{x\to 0} \frac{1}{|x|} = \infty \quad \text{or similarly} \quad \lim_{x\to 0^+} \frac{1}{x} = \infty.$$

Also,

$$\lim_{x\to 0^-} \frac{1}{x} = \lim_{x\to 0^-} \frac{-1}{|x|} = -\infty.$$

Thus,

$\lim_{x\to 0} \frac{1}{x}$ diverges, not to $+\infty$ or $-\infty$. ✓

C. Continuous Functions

We began this Chapter with a goal of giving a careful + rigorous description of functions whose "graph may be drawn without lifting your pen."

The following definition turns out to be a good model for this idea:

Definition: We say that a real function $f(x)$ is continuous at the point c

if $\lim_{x\to c} f(x) = f(c)$

and (to avoid certain degeneracies),

c is the limit of a sequence a_n in the domain of f minus the point c.

(The 2nd part just says that

f is defined for ∞-ly many points inside any ε-ball of c, and is mostly to avoid the case where f is only defined at some isolated point.)

Examples: Since for any c, $\lim_{x\to c} x = c$,
the function $f(x) = x$ is continuous at every $c \in \mathbb{R}$.

Definition: If $f(x)$ is continuous at every point c in a set A, then we say f is <u>continuous on A</u>.

Thus, $f(x) = x$ is continuous on all of $\mathbb{R}$.

Propositions: Any polynomial function is continuous on $\mathbb{R}$.
Any rational function is continuous on its domain.

Proof: Combine the $f(x) = x$ example,
and $f(x) = L$ example (also continuous, as $\lim_{x\to c} L = L = L$),
with Arithmetic of Limits, applied repeatedly. $\blacksquare$

Eg: $x^2+2 = x\cdot x + 2$ is obtained by multiplying, then summing continuous functions, so is continuous. ✓

Indeed, Arithmetic of Limits tells us that if f, g are continuous at c, then so are $f(x)\cdot g(x)$ and $f(x)+g(x)$.

This leads us to the following definition:

Definition: An <u>algebra of (real) functions</u> is a family $\mathcal{F}$ of functions so that
i) The constant function $1 \in \mathcal{F}$
ii) If $f(x) \in \mathcal{F}$, then $c\cdot f(x) \in \mathcal{F}$ (for any $c \in \mathbb{R}$)
iii) If $f(x), g(x) \in \mathcal{F}$, then $f(x)+g(x) \in \mathcal{F}$
iv) If $f(x), g(x) \in \mathcal{F}$, then $f(x)\cdot g(x) \in \mathcal{F}$.

Parts (ii) and (iii) of the definition may remind you of previous experience with "vectors".

<u>Example:</u> The family $\mathcal{P}(\mathbb{R})$ of all polynomials with real coefficients form an algebra of functions
1 is a polynomial (degenerately)
and sums, products, constant multiples of polynomials are polynomials.

<u>Example:</u> The family $\mathcal{C}(\mathbb{R})$ of real functions that are continuous on all of $\mathbb{R}$ is an algebra of functions.
We've seen that constant functions are continuous, and properties (ii), (iii), (iv) follow immediately by Arithmetic of Limits.

<u>Example:</u> The family $\mathcal{B}$ of bounded real functions is an algebra of functions.

<u>Since:</u> i) $|1| \leq 1$

ii) $|f(x)| \leq M \Rightarrow |cf(x)| \leq |c| \cdot M$

iii) $|f(x)| \leq M_f$ and $|g(x)| \leq M_g$

$$\Rightarrow |f(x)+g(x)| \stackrel{\triangle \text{ ineq.}}{\leq} |f(x)|+|g(x)| \leq M_f + M_g$$

and iv) $|f(x) \cdot g(x)| = |f(x)| \cdot |g(x)| \leq M_f \cdot M_g$

(when $|f| \leq M_f$, $|g| \leq M_g$ as above.)

Do you know any other algebras of functions?

That $\mathcal{C}(\mathbb{R})$ is an algebra of functions follows from (part of) AoL. Our other limit theorems likewise give us consequences for continuity.

For example, RStF yields

<u>Theorem</u> (Sequential Characterization of Continuity)

Let $f(x)$ be a real function and $c \in \mathbb{R}$; suppose that $f(x)$ is defined in some open ball around c.

Then f is continuous at c

$\Leftrightarrow$ for every sequence a_n w/ values in domain f and $\lim_{n\to\infty} a_n = c$ we have $\lim_{n\to\infty} f(a_n) = f(c)$.

<u>Proof:</u> Immediate from RStF!! ∎

As before, this gives us a method to show that a function is <u>not</u> continuous by "sampling" points in a sequence.

<u>Corollary:</u> Let $f(x)$ be a real function defined in some open ball around point c.

If there is a sequence a_n of points in domain of f

so that $\lim_{n\to\infty} a_n = c$

but $\lim_{n\to\infty} f(a_n) \neq f(c)$

then f is not continuous at c.

<u>Example:</u> (Modified Topologist's Sin Curve).

Let $$f(x) = \begin{cases} \sin \frac{1}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0. \end{cases}$$

Then $f(x)$ is not continuous at 0 (by the Corollary)

since $a_n = \frac{1}{2\pi n}$ has $\lim_{n\to\infty} \frac{1}{2\pi n} = 0$

but

$$\lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} \sin 2\pi n = \lim_{n\to\infty} 0 = 0$$

while $f(0) = 1$. ✓

<u>Self-check:</u> Verify that replacing $f(0)=1$ in the above example w/ $f(0)=0$ or $f(0)=$ any other value will still not yield a continuous function.

<u>Exercise:</u> Using the Sandwich Theorem, (+ RSEF) show that $\lim_{x\to 0} x \sin \frac{1}{x} = 0$

Thus, the function

$$g(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is continuous at 0, and indeed on the entire real line.

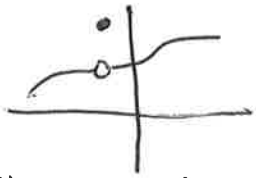
Some examples of how a function can fail to be continuous.

"jump"
Different right + left limits

"Hole"

$f(c)$ is not defined, but $\lim_{x\to c} f(x)$ is.

Limits can help us fill such holes!



"removeable discontinuity"
$\lim_{x\to c} f(x)$ converges to a number other than $f(c)$.

We can "remove" the discontinuity by filling in the right value at c.

Of course, functions can be discontinuous at more than 1 pt.

<u>Example:</u>

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

diverges at every real number c, so is discontinuous at every real number c.

Example: $g(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$

has $\lim_{x \to 0} g(x) = 0 = g(0)$

so $g(x)$ is continuous at $x=0$.

Self-check: Verify that $g(x)$ is not continuous at any other point c.

Thus, $g(x)$ is an example of a function that is continuous at a single real number.

The situation can get much wilder.

Example: Consider $h(x) = \begin{cases} 1/10^n & \text{if } x \text{ has finite decimal expansion ending in } 1/10^n \text{ place} \\ 0 & \text{otherwise} \end{cases}$

So e.g. $h(1/3) = 0$

but $h(0.33) = 1/100$.

Since any irrational has a $\delta = 1/10^{n+1}$ ball around it with $h(x) \le 1/10^n$ in this ball, we see that
$$\lim_{x \to d} h(x) = 0 \text{ for any irrational number } d.$$

Since any δ-ball contains irrationals,
$\lim_{x \to c} h(x)$ is 0 when it converges.

It follows that

$h(x)$ is continuous at every irrational number,

but discontinuous at any x with finite decimal expansion.

So $h(x)$ is continuous at ∞-ly many points

but also is discontinuous at ∞-ly many points (countably many.)

on any open interval!!!

Left- and right-sided limits give a method for gluing together continuous functions.

<u>Proposition</u> If $h(x) = \begin{cases} L & \text{for } x = c \\ f(x) & \text{for } x > c \\ g(x) & \text{for } x < c \end{cases}$

and $\lim_{x\to c^+} f(x) = L = \lim_{x\to c^-} g(x)$

then $h(x)$ is continuous at c.

<u>Proof</u>: By a previous result, $\lim_{x\to c} h(x) = h(c) = L$

$\Longleftrightarrow \lim_{x\to c^+} h(x) = \lim_{x\to c^-} h(x) = L$. ∎

<u>Example</u>: $h(x) = \begin{cases} x & \text{for } x < 0 \\ x^2 & \text{for } x \geq 0 \end{cases}$

is continuous at all real points.

We've seen that $\lim_{x\to c} \sqrt{x} = \sqrt{c}$. for $c \geq 0$

(so $\sqrt{x}$ is continuous on its domain.)

A similar argument shows that $\lim_{x\to c} f(x) = L \Rightarrow \lim_{x\to c} \sqrt{f(x)} = \sqrt{L}$.

(for $L > 0$).

The following is a generalization replacing $\sqrt{\ }$ with other continuous functions.

<u>Theorem</u> Let f, g be real functions and $c \in \mathbb{R}$

be s.t. f is defined on an open ball around c

g is defined on an open ball around $f(c)$

If f is continuous at c, and g is continuous at $f(c)$

then $g(f(x))$ is continuous at c.

(That is, "composition of continuous functions is continuous.")

Eg: $\sqrt{x^2+1}$ is a continuous function on the entire real line.

Proof: Proceed similarly to RS&F. (Indeed, RS&F could yield another proof.)

By definition, we know that

$\forall \varepsilon^*>0, \exists \delta^*>0$ s.t. $[|x-c|<\delta^*] \Rightarrow [|f(x)-f(c)|<\varepsilon^*]$

and $\forall \varepsilon>0, \exists \delta>0$ s.t. $[|y-f(c)|<\delta] \Rightarrow [|g(y)-g(f(c))|<\varepsilon]$.

Now given $\varepsilon>0$, the 2nd definition produces δ so that $|g(y)-g(f(c))|<\varepsilon$ when $|y-f(c)|<\delta$.

Take $\varepsilon^*=\delta$ + plug into 1st definition to produce a δ^*

s.t. $|x-c|<\delta^* \Rightarrow |f(x)-f(c)|<\varepsilon^*=\delta$

$\Rightarrow |g(f(x))-g(f(c))|<\varepsilon$

as desired. ∎

Eg: Assuming that $g(x)=\sin x$ is continuous,

also $g(\frac{1}{x})=\sin\frac{1}{x}$ is continuous except at 0.

(limit behavior at 0 was a subject of previous examples.)

<u>Limits + continuity in metric spaces</u>

The definition of limit transfers straightforwardly to metric spaces.

If (M_1,d_1) and (M_2,d_2) are metric spaces

and $f: M_1 \to M_2$ is a function

then $\lim_{x\to c} f(x)=L$ for $c\in M_1$ and $L\in M_2$

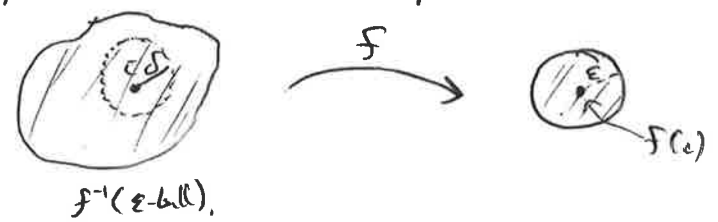
if $\forall\varepsilon>0\ \exists\delta>0$ s.t. $[0<d_1(x,c)<\delta] \Rightarrow [d_2(f(x),L)<\varepsilon]$

(Self-check: Why does d_1 "go with" δ, d_2 with ε?)

Likewise, we say that $f: M_1 \to M_2$ is continuous at c if $\lim_{x \to c} f(x) = f(c)$

(and $f(x)$ is defined on some open ball around c, or slightly weaker condition).

This yields an alternate picture of limits



and may make the next section a bit more interesting!

D. Continuous functions on compact sets

An essential feature of continuous functions is that they map (sequentially) compact sets to (sequentially) compact sets.

It's not difficult to do this on the level of metric spaces.

We start w/ a lemma mildly generalizing a previous result:

Lemma: If $f: M_1 \to M_2$ is a continuous metric space function and x_n is a sequence of points in M_1 (in domain of f) with $\lim_{n \to \infty} x_n = c$ then $\lim_{n \to \infty} f(x_n) = f(c)$.

Proof is entirely similar to RStF.

Theorem: If f is a continuous function from a metric space M_1 to a metric space M_2
and $K \subseteq M_1$ is sequentially compact
then $f(K) := \{ f(x) : x \in K \} \subseteq M_2$
is also sequentially compact.

Cor: A continuous real function sends closed bounded sets to " " ".

Eg: $f(x) = x^2+1$ sends the interval $[-1,3]$ to $[1,10]$.

Proof (of Theorem):
We must find a convergent subsequence from each sequence in $f(K)$.
Now, if y_n is a sequence in $f(K)$,
by definition, there is a sequence x_n in K
with $f(x_n) = y_n$.
Since K is sequentially compact,
x_n has a convergent subsequence x_{n_k}, w/ $\lim_{k\to\infty} x_{n_k} = c$.
But now by the lemma,
$$\lim_{k\to\infty} f(x_{n_k}) = \lim_{k\to\infty} y_{n_k} = f(c)$$
so y_{n_k} is a convergent subsequence of y_n, as desired. ∎

Another, nearly equivalent, way of stating this idea on the reals is with the Extreme Value theorem.
We go ahead and state generally:

Corollary (Extremal Value Theorem) (EVT)

If M is a metric space, $K \subseteq M$ is sequentially compact, and $f: M \to \mathbb{R}$ is a continuous function, then $f(K)$ has a minimum and maximum value.

(That is, there are points $x_{min}, x_{max} \in K$ s.t. for every $x \in K$ it holds that $f(x_{min}) \leq f(x) \leq f(x_{max})$.)

Proof: First, by the theorem, $f(K)$ is sequentially compact.

Thus, $f(K)$ is bounded, so that $\alpha = \sup f(K)$ is a real number.

If α is not a max, then by definition of sup,

$$\forall \varepsilon > 0 \quad \text{we can find } y \in f(K) \text{ s.t. } \alpha - \varepsilon < y < \alpha$$

By taking $\varepsilon = 1/n$, we produce a sequence y_n with

$$\alpha - \tfrac{1}{n} < y_n < \alpha \qquad \text{(for each } n\text{)}.$$

By the Sandwich Theorem, $\lim_{n \to \infty} y_n = \alpha$.

But by sequential compactness, we must have $\alpha \in f(K)$

so $\alpha = f(x_{max})$ as we wanted to show.

The construction for x_{min} is entirely similar. ∎

Examples The maps $\mathbb{R}^2 \to \mathbb{R}$, $(x,y) \mapsto x$ and $\mathbb{R}^2 \to \mathbb{R}$, $(x,y) \mapsto y$ are continuous. (check)

So if $K \subseteq \mathbb{R}^2$ is a compact region, then K has a least/greatest x/y coordinate.

The EVT tells us that we can find "optimal" values of continuous functions defined on sequentially compact sets, an observation of significant practical importance.
(See the Optimization section of your favorite Calculus textbook.)

Note that continuous functions on open intervals may or may not have a maximum or minimum value.

(Anti) examples

- The continuous function $f(x)=x$ has neither max nor min on any open interval.
- The continuous function $f(x)=x^2$ has minimum value of 0 on $(-1,3)$, but no maximum value.

A function which fails to be continuous, even at a single point, may also fail to have max or min. Consider $1/x$ on $[-1,1]$!

E. The Intermediate Value Theorem

Our motivation to look at continuous functions was to describe those functions that we can sketch without lifting our pen.

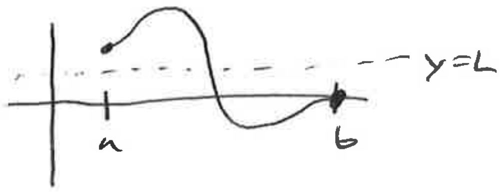
The definition, after your experience with limits, might convince you that we have the right description.

The following theorem is even more convincing along these lines:

Theorem: (Intermediate Value Theorem)

If f is a real function, continuous on a closed interval $[a,b]$ and L is a number w/ $f(a)<L<f(b)$ then $\exists\, c \in [a,b]$ with $f(c)=L$.

That is, in drawing f on the closed interval $[a,b]$, our pen must pass through every horizontal line $y=L$ between $y=f(a)$ and $y=f(b)$.



The IVT has a number of useful and important consequences.

One is that continuous functions must pass through 0 in order to pass from negative to positive!

Corollary: (Trichotomy for Continuous Functions)

If f is continuous on the closed interval $[a,b]$ and $f(a)<0$ but $f(b)>0$ then there is some $c\in[a,b]$ s.t. $f(c)=0$.

(Proof is immediate by specializing.)

A second (related) consequence is a general method to find zeros of a continuous function, such as roots of a high-degree polynomial.

Method: (Bisection method to find 0 of continuous function on closed interval.)

Proceed recursively as follows.

Start with continuous function f on closed interval $[a,b]$ s.t. $f(a)<0$ and $f(b)>0$.

Set $a_0=a$, $b_0=b$.

Repeat:

Given $a_i<b_i$ s.t. $f(a_i)<0$, $f(b_i)>0$

test $f(\underbrace{\tfrac{a_i+b_i}{2}}_{\text{average}})$.

If $f(\frac{a_i+b_i}{2})=0$, then we lucked out and found a zero of f!!

If $f(\frac{a_i+b_i}{2})<0$, then set $a_{i+1}=\frac{a_i+b_i}{2}$, $b_{i+1}:=b_i$.

If $f(\frac{a_i+b_i}{2})>0$, - - - $a_{i+1}:=a_i$, $b_{i+1}:\frac{a_i+b_i}{2}$

Notice a_{i+1}, b_{i+1} satisfy requirements of "repeat".

Until $|a_i-b_i|$ is as small as desired.

(Notice $|a_i-b_i|=\frac{|a-b|}{2^i}$, since we chose midpoint at each step.)

The bisection method usually will not give an exact value c where $f(c)=0$, but it approximates w/ arbitrary accuracy. Since the length of $[a_i, b_i]$ halves with each iteration, a relatively small number of iterations gives good accuracy.

Example: Use the method of bisection to approximate $\sqrt{2}$ (solution to $x^2-2=0$) within 0.01.

Solution: Since $1^2-2<0$ while $2^2-2>0$, we may take $a_0=1$, $b_0=2$.

Note that x^2-2 is continuous everywhere (as continuous functions form an algebra of functions.)

i	a_i	b_i	avg	$f(\text{avg})$	
0	1	2	$3/2$	$1/4>0$,	so $b_1:=\text{avg}$.
1	1	$3/2$	$5/4$	$-7/16<0$,	so $a_2:=\text{avg}$
2	$5/4$	$3/2$	$11/8$	$-7/64$	- - -
3	$11/8$	$3/2$	$23/16$	$\frac{23^2}{16^2}-2=\frac{17}{16^2}$	
4	$11/8$	$23/16$	$45/32$	$\frac{45^2}{32^2}-2=\frac{-23}{32^2}$	
5	$45/32$	$23/16$	$91/64$	$\frac{91^2}{64^2}-2=\frac{89}{64^2}$	
6	$45/32$	$91/64$	$181/128$		

← accurate to within $\frac{1}{128}<\frac{1}{100}=0.01$ since root lies on $[45/32,\ 181/128]$ or $[181/128,\ 91/64]$ both of length $\frac{|a_0-b_0|}{128}=\frac{1}{128}$.

(And Only another 3 applications would be required to be accurate to within 0.001.)

There are several proofs of the IVT.
My favorite essentially uses the method of bisection.

Proof (of IVT):

Construct a sequence of intervals as in the method of bisection:
Given $[a,b]$ interval, let $a_0 := a$ and $b_0 := b$.
Recursively take

$$[a_{n+1}, b_{n+1}] := \begin{cases} \left[\frac{a_n+b_n}{2}, b_n\right] & \text{if } f\left(\frac{a_n+b_n}{2}\right) < L \\ \left[a_n, \frac{a_n+b_n}{2}\right] & \text{otherwise.} \end{cases}$$

Now (a_n) and (b_n) are real sequences
bounded between a and b
and monotonic by construction.
So by MST, both (a_n) and (b_n) are convergent sequences.
Also, since $|b_n - a_n| = \frac{|b-a|}{2^n}$ (just as in method of bisection),
we have $\lim_{n\to\infty} b_n - a_n = 0$
hence $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$. Call this limit c.
By the Sequential Characterization of Continuity,
we know that $\lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} f(b_n) = f(c)$.
But since $\forall n$, $f(a_n) < L$ (by construction)
and $\forall n$, $f(b_n) \geq L$
we get from a hw problem that

$$f(c) = \lim_{n\to\infty} f(a_n) \leq L \qquad \text{while}$$

$$f(c) = \lim_{n\to\infty} f(b_n) \geq L.$$

It follows that $f(c) = L$, as desired! ∎

Remark: See Ross's book for a proof based on lim sup and order completeness. (It's similar to the above, but not as connected to the method of bisection).
Abbott has a different proof idea based on "connectedness" which I like a lot, but which requires some additional topology.

Corollary (to EVT + IVT)

A continuous real function sends closed bounded intervals to closed bounded intervals.

Proof: If f is continuous on $[a,b]$,

then the EVT says that the image of $[a,b]$ under f has a minimum value m and maximum value M.

Hence $\underbrace{f([a,b])}_{\text{image of } [a,b] \text{ under } f} \subseteq [m, M]$, and $\exists a_0, b_0$ w/ $f(a_0)=m$, $f(b_0)=M$.

But if $L \in [m,M]$, then

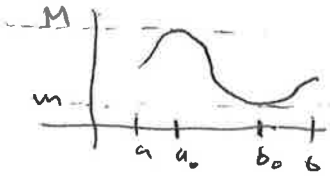
assume wlog that $a_0 < b_0$,

and IVT $\Rightarrow \exists c \in [a_0, b_0] \subseteq [a,b]$ w/ $f(c)=L$.

So $[m,M] \subseteq f([a,b])$

and so $[m,M] = f([a,b])$. ∎

Self-check / hw: We assumed "wlog" that $a_0 < b_0$. What if $a_0 > b_0$?



Examples Show that if f is continuous (on the entire real line) s.t. $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to-\infty} f(x) = -\infty$ then there is a point c s.t. $f(c)=0$.

Solution: Since $\lim_{x\to\infty} f(x)=\infty$,

$\forall M, \exists N$ st. $[x>N] \Rightarrow [f(x)>M]$.

Take $M=1$, to produce $b := N+1$ w/ $f(b)>1$, (wlog $b>0$)

Similarly, we can produce a w/ $f(a)<-1$ and $a<0$ from definition of $\lim_{x\to-\infty} f(x) = -\infty$.

Now IVT on $[a,b]$ yields a value c w/ $f(c)=0$. ✓

Fixed point theorems for closed bounded sets

Definition: A fixed point of a function f is a point x s.t. $f(x) = x$.

[So if you think of a function $\mathbb{R} \to \mathbb{R}$ as moving points around, a fixed point is a point that stays fixed or "nailed down" under this movement.]

Examples:

- $f(x) = x$ fixes every point
- $g(x) = x^2$ has fixed points $0, 1$.
- $h(x) = x+1$ has no fixed points.

There are many theorems guaranteeing fixed points in various circumstances.
Fixed point theorems often have applications to equilibria or "stable points".

The IVT easily gives us a first fixed point theorem:

Theorem: If $f : [0,1] \to [0,1]$ is a continuous function, then f has at least one fixed point.

Proof: Consider $g(x) = x - f(x)$. By AoL, g is continuous.
Now,

$$g(0) = 0 - f(0) \le 0 \qquad (\text{as } f(0) \ge 0), \text{ while}$$

$$g(1) = 1 - f(1) \ge 0 \qquad (\text{as } f(1) \le 1)$$

So the IVT tells us there is a $c \in [0,1]$ with $g(c) = 0$, i.e. w/ $c - f(c) = 0$
i.e. w/ $c = f(c)$, as desired ∎

Self-check/exercise: Show that if $[a,b]$ is any closed bounded interval and f is a continuous function from $[a,b]$ to $[a,b]$, then f has a fixed point.

Notice: There is something special about closed bounded intervals here. A fixed point result may fail to hold for other domains.

E.g.: $h(x)=x+1$ is a continuous function from $\mathbb{R}$ to $\mathbb{R}$ having no fixed point. (It doesn't even fix any closed interval!!)

F. Uniform continuity

A function f is continuous on a domain A (such as $\mathbb{R}$ or $[a,b]$ or (a,b)) if f is continuous at each point in A.

That is, if $\forall c \in A$, $\lim_{x\to c} f(x)=f(c)$.

Expanded in ε and δ, this says

$$\forall c \in A,\ \forall \varepsilon>0,\ \exists \delta>0 \text{ s.t. } [\,|x-c|<\delta \text{ and } x\in A\,] \Rightarrow [\,|f(x)-f(c)|<\varepsilon\,].$$

Notice here that δ may depend on c!!

Example: Consider $f(x)=1/x$ on the interval $(0,1)$.

Given $\varepsilon>0$, we want to bound $|1/x-1/c|$ to show f is continuous.

But

$$\left|\frac{1}{x}-\frac{1}{c}\right| = \left|\frac{c-x}{xc}\right| < \varepsilon$$

$$\Leftrightarrow \quad |x-c| < \varepsilon\cdot|xc|.$$

It's easy to see from this that our choice of δ will need to depend on c, and will need to be quite small for c close to 0.

Self-check: Verify that $\delta=\min\left\{\frac{c}{2}, \frac{\varepsilon\cdot c^2}{2}\right\}$ "works".

Sometimes, however, we can choose δ independently of c.

<u>Example:</u> Consider $f(x) = 1/x$ on the interval $[1/2, 1]$.

By the same calculation as in the previous example, $\delta = \min\{1/4, \varepsilon/8\}$ will satisfy the definition of continuity for <u>every value of c on $[1/2, 1]$</u>.

Although the latter situation (where δ is not dependent on c) does not always occur, it is useful when it does, and we give a name:

<u>Definition:</u> We say that a function f is <u>uniformly continuous</u> on a domain $A \subseteq \mathbb{R}$

if

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } \forall c \in A, \quad [|x-c| < \delta \text{ and } x \in A] \Rightarrow [|f(x) - f(c)| < \varepsilon].$$

That is, if the δ we produce for a given ε may be produced without dependence on c.

Note the difference in order of the $\forall$, $\exists$ symbols!!

<u>Example:</u> $f(x) = 1/x$ is uniformly continuous on $[1/2, 1]$ (but not on $(0,1)$).

It is an important, useful, and surprising fact that any function that is continuous on a closed interval $[a,b]$ (or sequentially compact domain) is uniformly continuous on the same.

<u>Theorem:</u> Let A be a sequentially compact (closed + bounded) domain.
If f is continuous on A, then f is uniformly continuous on A.

<u>Proof:</u> Suppose not for contradiction.
Then there is some $\varepsilon > 0$
so that $\forall \delta > 0$
we can find $x, c \in A$ where $|x-c| < \delta$ but $|f(x)-f(c)| \geq \varepsilon$.
For this fixed ε and any $n \in \mathbb{N}^+$, take $\delta = 1/n$
and $x_n, c_n \in A$ to be points so that
$|x_n - c_n| < 1/n$ but $|f(x_n) - f(c_n)| \geq \varepsilon$.
But since A is sequentially compact,
c_n has a convergent subsequence c_{n_k}.
Let $c := \lim_{k\to\infty} c_{n_k}$.
Also, since $|x_{n_k} - c_{n_k}| < 1/n_k$, by the Sandwich Theorem
$\lim_{k\to\infty} x_{n_k} - c_{n_k} = 0$, hence $\lim_{k\to\infty} x_{n_k} = c$ as well.

Since f is continuous, $\lim_{k\to\infty} |f(x_{n_k}) - f(c_{n_k})| = |f(c) - f(c)| = 0$.
But since $|f(x_{n_k}) - f(c_{n_k})| \geq \varepsilon > 0$ for every k, this is a contradiction! # ■

<u>Example:</u> $\dfrac{x^2+4}{x^2-4}$ is continuous on $[-1,1]$, by AoL.
Since $[-1,1]$ is sequentially compact,
it is also uniformly continuous on this interval.

<u>Example/self-check:</u> $f(x) = x$ is uniformly continuous on $\mathbb{R}$
(even though $\mathbb{R}$ is not sequentially compact.)

G. Monotone functions

Recall that a sequence s_n is (weakly) increasing if $\forall n$, $s_n \le s_{n+1}$
and strictly increasing if $\forall n$, $s_n < s_{n+1}$.

We'll now define and examine increasing functions (real functions).

Throughout, let I be an interval, such as $[-3, 1)$, $[0, 2]$, $(-12, \pi)$, or $[0, \infty)$.

Definition A real function f is increasing on an interval I
if $\forall x, y \in I$ we have $[x < y] \Rightarrow [f(x) \le f(y)]$
and f is strictly increasing
if $\forall x, y \in I$, $[x < y] \Rightarrow f(x) < f(y)$.

Self-check Write down the entirely-similar definitions of decreasing / strictly decreasing.

Definition A real function f is monotone on an interval I
is it is increasing on I or decreasing on I.
(and strictly monotone if strictly increasing or strictly decreasing).

Example $f(x) = x^2$ is monotone on $[-2, 0]$ and on $[0, 2]$.
It is decreasing on $[-2, 0]$ and increasing on $[0, 2]$.
It is not monotone on $[-2, 2]$.

Notice that strictly monotone functions are injective (or one-to-one),
meaning that when $x \ne y$, also $f(x) \ne f(y)$
(that is, "distinct values are sent by f to distinct values").

Continuous monotone functions are "nice", and there are
useful relations between continuity and monotonicity.

Recall from high school that the defining property of an interval I is that if $a, b \in I$ and $a < c < b$, then also $c \in I$.

__Proposition:__ If f is a continuous monotone function on interval I then $f(I)$ is also an interval.

__Proof:__ Suppose wlog that f is increasing.
If $a = f(a_0)$ and $b = f(b_0)$ are in $f(I)$ then by IVT, for any L w/ $a < L < b$, there is a c_0 between a_0 and b_0 s.t. $f(c_0) = L$ as required to show $f(I)$ is an interval. ∎

__Example:__ $f(x) = x^2$ sends $[0,3)$ to $[0,9)$.

__Proposition:__ If f is a real-valued function that is continuous and increasing on the closed (bounded) interval $[a,b]$ then $f([a,b]) = [f(a), f(b)]$.

__Proof:__ By prev. results, $f([a,b])$ is a closed bounded interval, and by definition of increasing function, the minimum is $f(a)$ and maximum $f(b)$. ∎

__Example:__ Let $f(x) = x^2$. Then $f([1,3]) = [1,9]$.
(But notice that $f([-1,3]) = f([-1,0]) \cup f([0,3]) = [0,9]$, even though $0 \neq f(-1)$.)

__Example:__ The function $f(x) = 3$ is increasing (as well as decreasing) and $f([0,1]) = \{3\} = [3,3]$.

We noticed that strictly increasing (or monotone) functions are injective, so take on each value of their range exactly once.

We can use this to define an <u>inverse function</u>

$f^{-1}(b) := a$ so that a is the (unique) value where $f(a)=b$.

f^{-1} is defined on the range of f.

<u>Example:</u> Consider $f(x)=x^2$. As we've discussed,

f is strictly increasing on $[0,\infty)$ (and decreasing on $(-\infty,0]$).

By restricting f to $[0,\infty)$ we get an inverse function sending $[0,\infty)$ to $[0,\infty)$

This inverse function is also known as $\sqrt{x}$!!

Our next goal will be to show that the inverse function of a strictly monotone, continuous function is also continuous (under mild conditions).

<u>Definition:</u> A function f has the <u>Intermediate Value Property</u> on A if whenever a,b are in A and L is between $f(a)$ and $f(b)$ then there is a value c in A s.t. $f(c)=L$.

<u>Example:</u> By the IVT, any continuous function on a closed interval has the IVP on that interval.

<u>Warning:</u>

Not every function with the IVP is continuous.

However:

Theorem: If $f(x)$ is increasing and has the IVP on $[a,b]$ then f is continuous on $[a,b]$.

Proof: We want to show f is continuous at each $c \in [a,b]$.

First consider the case where $f(a) < f(c) < f(b)$:

Given a small enough value of ε, we'll have

$$f(a) < f(c) - \tfrac{\varepsilon}{2} < f(c) < f(c) + \tfrac{\varepsilon}{2} < f(b).$$

Using the IVP twice, we find c_- and c_+ s.t. $f(c_-) = f(c) - \frac{\varepsilon}{2}$ and $f(c_+) = f(c) + \frac{\varepsilon}{2}$.

But now, since f is increasing, whenever

x is between c_- and c_+,

also $f(x)$ " " $f(c_-)$ " $f(c_+)$

ie, between $f(c) - \frac{\varepsilon}{2}$ and $f(c) + \frac{\varepsilon}{2}$.

Thus, $\delta = \min\{ |c - c_-|, |c - c_+| \}$ suffices to force $|f(x) - f(c)| \leq \frac{\varepsilon}{2} < \varepsilon$ (for x s.t. $|x - c| < \delta$, of course!).

The cases where $f(a) = f(c)$ or $f(c) = f(b)$ are entirely similar, except that we need only bound from one side.

(Self-check: Write down some details!)

This theorem is especially useful when combined with:

Proposition: If f is strictly increasing on an interval I then the inverse function f^{-1} has the IVP on $f(I)$.

Proof: Suppose $\alpha = f(a)$ and $\beta = f(b)$ are in $f(I)$.

Now if L is between a and b,

then L is on I (by definition of interval)

so $f(L)$ is on $f(I)$, and now $L = f^{-1}(f(L))$.

Corollary: If f is continuous and strictly increasing on $[a,b]$ then f^{-1} is continuous on $[f(a), f(b)]$.

__Example:__ Since $f(x)=x^3$ is continuous and strictly increasing on $\mathbb{R}$, the inverse function $\sqrt[3]{x}$ is continuous on any closed interval (hence on $\mathbb{R}$).

H. Continuous functions and topology

As we've seen already, Since we have limits in any metric space, and since the definition of continuous function is based on limits, we can define continuous functions on any metric space:

$f: (M_1,d_1) \longrightarrow (M_2,d_2)$ is continuous at $c \in M$, if $\lim\limits_{x\to c} f(x) = f(c)$.

__Example:__ A function $f: \mathbb{C} \longrightarrow \mathbb{C}$ is continuous if

$$\forall z_0 \in \mathbb{C} \ \forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } [\,|z-z_0| < \delta\,] \Rightarrow [\,|f(z)-f(z_0)| < \varepsilon\,].$$

complex norm

Recall that the topology of a metric space is the family of open sets for that metric space.

It turns out that continuity can be expressed purely in terms of the topologies!

__Theorem:__ Let (M_1,d_1) and (M_2,d_2) be metric spaces and $f: M_1 \to M_2$ be a function.

Then f is continuous

$\Longleftrightarrow$

$\forall$ open set $B \subseteq M_2$,
$f^{-1}(B) := \{x \in M_1 : f(x) \in B\}$
is an open set in M_1.

That is, a function is continuous

$\Leftrightarrow$ "inverse images of open sets are open".

Proof: ($\Rightarrow$) If f is continuous and a is a point in $f^{-1}(B)$

then $f(a) = b \in B$.

Since B is open, there is an $\varepsilon > 0$ s.t.

$B_\varepsilon(b) \subseteq B$ (all in M_2).

Since f is continuous, there is $\delta > 0$ s.t.

$[d_1(x, a) < \delta] \Rightarrow [d_2(f(x), b) < \varepsilon]$

i.e., s.t. $B_\delta(a) \subseteq f^{-1}(B_\varepsilon(b)) \subseteq f^{-1}(B)$.

As we've found a δ-neighborhood around each pt in $f^{-1}(B)$

that is contained in $f^{-1}(B)$, we get that $f^{-1}(B)$ is open. ✓

($\Leftarrow$): Given $\varepsilon > 0$ and $a \in M_1$, let $b = f(a)$. We recall from homework

that $B_\varepsilon(b)$ is an open set.

Thus, $f^{-1}(B_\varepsilon(b))$ is an open set,

so there is some δ for each a w/ $f(a) = b$

where $B_\delta(a) \subseteq f^{-1}(B_\varepsilon(b))$.

Translating to distance inequalities, this says that

$d_1(x, a) < \delta \Rightarrow d_2(f(x), b) < \varepsilon$

which is as required for continuity of f. ∎

Warning: Although the inverse image of an open set under a continuous

function is open,

the image of an open set may be open, closed, or neither.

Example: $f(x) = x^2$ sends the open interval $(-1, 1)$

to the interval $[0, 1)$ not open.

(But you can easily verify that the inverse image

of any open set is open with a direct

argument.)