Scores	
1.	
2.	
3.	
4.	
5.	
6.	
7.	
Total:	

ANA-I Foundations of Analysis
Final Examination A – 23 January 2023

Name ________________________________

General Instructions: Please answer the following, showing all your work and writing neatly. You may have 1 handwritten A4-sized sheet of paper, but no other notes, books, or calculators.
110 total points.

1. (6 points each) Calculate the following limits, or explain why they diverge. You may use any theorems we have proved in class or on homework.

   (a) $\displaystyle\lim_{n\to\infty} \frac{3n^3+2n-1}{(2n-1)(n^2+1)}$

   (b) $\displaystyle\lim_{x\to 2} \frac{x^2-x-2}{x^2-4}$

   (c) $\displaystyle\lim_{x\to 1} \frac{x^2-4}{x^2-2x+1}$

   (d) $\displaystyle\lim_{x\to 1+} \frac{x^2-4}{x^2-1}$

   (e) $\displaystyle\lim_{x\to 0} x\cos(1/x^2)$

2. (6 points each) Series

   (a) Find the exact value that $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^n+4^n}{5^n}$ converges to, or else conclude that the series diverges.

   (b) Determine whether $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n}{2n^2+1}$ converges absolutely, converges conditionally, or diverges.

3. (6 points each) Examples. Justify your answers briefly.

   (a) Describe the Dedekind cut corresponding to $\sqrt{11}$. Your description should not refer directly to any irrational numbers.

   (b) Give an example of a closed subset of $\mathbb{R}$ that is not sequentially compact.

   (c) Give an example of a real function f, defined at all points, such that $f(0) = -1$ and $f(1) = 3$, but where $f(x)$ is never 0.

   (d) Give an example of a continuous function $f : [0,1] \to [0,1]$ having exactly one fixed point.

(-see reverse side-)

4. (10 points) Let a_n be recursively defined by $a_0 = 2$, $a_{n+1} = \sqrt{5a_n}$ for $n \geq 0$. Either show that a_n converges and find its limit, or else show that a_n diverges to $\pm\infty$.

5. (12 points) Prove directly from definition that if $\lim_{x \to 1+} f(x) = \infty$ and $\lim_{x \to 1-} f(x) = \infty$, then also $\lim_{x \to 1} f(x) = \infty$.

6. (12 points) Using the method of bisection, and starting by observing that $\sqrt{11}$ lies between 3 and 4, give a fraction that approximates $\sqrt{11}$ to within an accuracy of 0.1.

7. (10 points) Prove that if $f : \mathbb{R} \to \mathbb{R}$ is continuous, then the image of a bounded set over f is bounded.