Homework 4

A. (I) Using Aritmetic of Limits, find $\lim_{n\to\infty} 2n/(3n+5)$ and nbsp.

(ii) Working directly from the definition of limits, give a direct verification that your answer in (i) is correct. (Your answer should involve the letter ε).

(i) $\lim_{n\to\infty} \frac{2n}{3n+5} = \lim_{n\to\infty} \frac{\cancel{n}\cdot 2}{\cancel{n}(3+\underbrace{\frac{5}{n}}_{0})} = \frac{2}{3}$

(ii) Directly from the definition of limits, we will have that:

$\lim_{n\to\infty} \frac{2n}{3n+5} = \frac{2}{3}$

$\forall \varepsilon > 0 \ \exists n_0 \in \mathbb{N} \ \ \forall n > n_0 \quad \left|\frac{2n}{3n+5} - \frac{2}{3}\right| < \varepsilon$ => We should prove this!

If there is none $\varepsilon > 0$ (arbitrary positive real number), then

$\exists n_0 > \frac{1}{3}\left[\frac{10}{3\varepsilon} - 5\right]$ and there is none $n > n_0$, we will have

$\Rightarrow |S_n - L| = \left|\frac{2n}{3n+5} - \frac{2}{3}\right| = \left|\frac{\cancel{6n} - \cancel{6n} - 10}{3(3n+5)}\right| = \frac{10}{3(3n+5)} < \frac{10}{3(3n_0+5)} \leq \varepsilon$

For n_0 it it's valid:

$\frac{10}{3(3n_0+5)} \leq \frac{\varepsilon}{1} \ / \cdot \frac{3n_0+5}{\varepsilon}$

$\Leftrightarrow \frac{10}{3\varepsilon} \leq 3n_0 + 5$

$\Leftrightarrow \frac{1}{3}\left[\frac{10}{3\varepsilon} - 5\right] \leq n_0$

For $n_0 > \frac{1}{3}\left[\frac{10}{3\varepsilon} - 5\right]$ ∎

B. Using the definitions of limit and/or infinite limit, show that if $\lim_{n\to\infty} S_n = \infty$, then also $\lim_{n\to\infty} \sqrt[3]{S_n} = \infty$.

If $\lim_{n\to\infty} S_n = \infty \Rightarrow \lim_{n\to\infty} \sqrt[3]{S_n} = \infty$

$\lim_{n\to\infty} S_n = \infty \Leftrightarrow \forall M>0 \quad \exists n_0 \in \mathbb{N} \quad \forall n > n_0 \quad S_n > M$

Let $\lim_{n\to\infty} S_n = \infty$ so that $\forall M>0 \ \exists n_0 = n_0(M) \quad \forall n > n_0 \quad S_n > M \Rightarrow$

$\Rightarrow$ from the definition $\lim_{n\to\infty} \sqrt[3]{S_n} = \infty$ so $\forall K>0 \quad \exists n_0 = n_0(K)$

$\forall n > n_0 \quad \sqrt[3]{S_n} > K$

If $K > 0$

$\Rightarrow \exists n_0 \quad \forall n > n_0 \qquad S_n > M = K^3$

$\Leftrightarrow S_n > K^3 / \sqrt[3]{\ }$

$\Leftrightarrow \sqrt[3]{S_n} > K = \sqrt[3]{K^3}$

$\Rightarrow \lim_{n\to\infty} \sqrt[3]{S_n} = \infty \quad \blacksquare$

C. Find the following limits. You may use any theorem we have proved, such as Arithmetic of limits. (Please do not use any theorem not discussed in class!) Indicate clearly what result you are using. The right answer without a justifiable reason will be given zero credit.

(i) $\lim_{n\to\infty} \frac{3n^2+1}{2n^2+2n-1}$

(ii) $\lim_{n\to\infty} \frac{\sqrt{5n^2+2n}}{n+2}$

(iii) $\lim_{n\to\infty} \frac{-3n^3+4}{3n^3+2n^2-1n}$

(i) $\lim_{n\to\infty} \frac{3n^2+1}{2n^2+2n-1} = \lim_{n\to\infty} \frac{\cancel{n^2}(3+\underbrace{\frac{1}{n^2}}_{0})}{\cancel{n^2}(2+\underbrace{\frac{2n}{n}}_{0}-\underbrace{\frac{1}{n^2}}_{0})} = \frac{3}{2}$

$\Rightarrow$ Here I am using: that if the highest degrees are the same, then the solution is the quotient of the numbers in front of the highest degrees.

(ii) $\lim_{n\to\infty} \sqrt{\frac{5n^2+2n}{n+2}} = \lim_{n\to\infty} \frac{\sqrt{n^2(5+\frac{2n}{n^2})}}{n(1+\frac{2}{n})} = \lim_{n\to\infty} \frac{\cancel{n}\sqrt{5+\underbrace{\frac{2n}{n^2}}_{0}}}{\cancel{n}(1+\underbrace{\frac{4}{n}}_{0})} = \sqrt{5}$

$\Rightarrow$ Here I am using: that if the biggest degrees are the same, then the solution is the quotient of the numbers in front of the highest degrees.

(iii) $\lim_{n\to\infty} \frac{-3n^3+4}{3n^3+2n^2-n} = \lim_{n\to\infty} \frac{\cancel{n^3}(-3+\underbrace{\frac{4}{n^3}}_{0})}{\cancel{n^3}(3+\underbrace{\frac{2n^2}{n^3}}_{0}-\underbrace{\frac{n}{n^3}}_{0})} = \frac{-3}{3} = -1$

$\Rightarrow$ Here I am using: that if the highest degrees are the same, then the solution is the quotient of the numbers in front of the highest degrees.

$\Rightarrow \lim_{n\to\infty} \frac{1}{n^\alpha} = 0, \ \alpha > 0$ This is the theorem that I'm using for all of the previous limits.

D. Find the following limits. You may use any theorem we have proved, such as Aritmetic of Limits. (Please do not use any theorem not discussed in class!) Indicate clearly what results you are using. The right answer without a justifiable reason will be given zero credit.

(i) $\lim_{n\to\infty} \dfrac{n^2-4}{2n^3-3n+1}$

(ii) $\lim_{n\to\infty} \dfrac{\sqrt{2n^3-1n}}{2n-3}$

(iii) $\lim_{n\to\infty} \dfrac{6n^2-2n^4}{12n^3+2n^2-17n}$

(i) $$\lim_{n\to\infty} \frac{n^2-4}{2n^3-3n+1} = \lim_{n\to\infty} \frac{\cancel{n^3}\left(\frac{n^2}{n^3}-\frac{4}{n^3}\right)}{\cancel{n^3}\left(2-\frac{3n}{n^3}+\frac{1}{n^3}\right)} = \frac{0}{2} = 0$$

$\Rightarrow$ Here I am using: that if the maximum degree below is greater than the maximum degree above, the solution is 0.

(ii) $$\lim_{n\to\infty} \frac{\sqrt{2n^3-1n}}{2n-3} = \lim_{n\to\infty} \frac{\sqrt{n^3\left(2-\frac{n}{n^3}\right)}}{n\left(2-\frac{3}{n}\right)} = \lim_{n\to\infty} \frac{\cancel{n}\cdot n^{1/2}\sqrt{2-\frac{n}{n^3}}}{\cancel{n}\left(2-\frac{3}{n}\right)} =$$

$$= \lim_{n\to\infty} \frac{\sqrt{n}\sqrt{2}}{2} = \pm\infty$$

$\Rightarrow$ Here I am using: that if the maximum degree above is greater than the maximum degree below, the solution is either $+\infty$ or $-\infty$.

(iii) $$\lim_{n\to\infty} \frac{6n^2-2n^4}{12n^3+2n^2-17n} = \lim_{n\to\infty} \frac{\cancel{n^4}\left(\frac{6n^2}{n^4}-2\right)}{\cancel{n^4}\left(\frac{12n^3}{n^4}+\frac{2n^2}{n^4}-\frac{17n}{n^4}\right)} =$$

$$= \frac{-2}{0} = -\infty$$

$\Rightarrow$ Here I am using: that if the maximum degree above is greater than the maximum degree below, the solution is either $+\infty$ or $-\infty$.

E. (i) Suppose that the S_n satisfies both $\lim_{n\to\infty} S_{2n} = 3$ and $\lim_{n\to\infty} S_{2n+1} = 3$

(That is, the sequence given by the even terms of S_n and that given by the odd terms of S_n both converge to 3). Show that also $\lim_{n\to\infty} S_n = 3$.

(ii) Give an example of a sequence where the sequences given by the even and by the odd terms both converge, but where the entire sequence does not converge.

(i) If S_n is sequence where $\lim_{n\to\infty} S_{2n} = 3$ and $\lim_{n\to\infty} S_{2n+1} = 3$ both converge to 3, then:

Proof:

$$n > \frac{n_1}{2}$$
$$\Uparrow$$
$$\lim_{n\to\infty} S_{2n} = 3 \iff \forall \varepsilon \ \exists n_1 \ \forall n \ \ 2n > n_1 \quad |S_{2n} - 3| < \varepsilon$$

$$\lim_{n\to\infty} S_{2n+1} = 3 \iff \forall \varepsilon \ \exists n_2 \ \forall n \ \ 2n+1 > n_2 \quad |S_{2n+1} - 3| < \varepsilon$$
$$\Downarrow$$
$$n > \frac{n_2 - 1}{2}$$

Let $\varepsilon > 0$ $\exists n_0 = \max\left\{\frac{n_1}{2}, \frac{n_2 - 1}{2}\right\}$ and there is none $n > n_0$

$\Rightarrow |S_n - 3| < \varepsilon$

$\Rightarrow \lim_{n\to\infty} S_n = 3$ ■

(ii) $S_n = (-1)^n$

$S_{2n} = 1$

$S_{2n+1} = -1$

There are two convergent sequences:

$\Rightarrow \lim_{n\to\infty} S_{2n} = L_1 = 1$ and $\lim_{n\to\infty} S_{2n+1} = L_2 = -1$

but it does not exist $\lim_{n\to\infty} S_n = \lim_{n\to\infty} (-1)^n$ ■