Analysis I Homework 2

A Give the Dedekind cuts in $\mathbb{Q} \geq 0$ corresponding to the following. Your definition should not refer to the elements themselves.

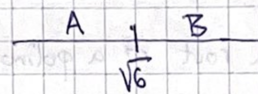
(i) $\sqrt{6}$

(ii) $\sqrt[3]{5}$

(iii) $\sqrt{2}+\sqrt{5}$

(i) $A = \{r \in \mathbb{Q} : r^2 < 6\}$

$B = \{r \in \mathbb{Q} : r^2 \geq 6\}$



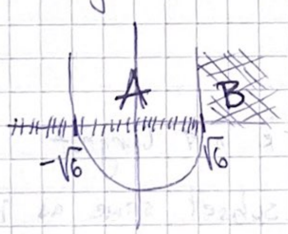

I want $\sqrt{6}$ to be a root of polinominal $P(x)$.

$(x-\sqrt{6})$ is a factor of $P(x)$.

$\Rightarrow$ From this we know that $A_{\sqrt{6}} \cup B_{\sqrt{6}} = \mathbb{Q}_{\geq 0}$

$P(x) = (x-\sqrt{6})(x+\sqrt{6}) = x^2 - 6$

With this we can state that the A subset will have not largest element or it will never reach 6. In the B subset same as in A subset there will be no Largest element, which proves that B subset is not empty.




This proves the Dedekind cut for $A_{\sqrt{6}} | B_{\sqrt{6}}$.

(ii) $A = \{r \in \mathbb{Q} : r^3 < 5\}$

$B = \{r \in \mathbb{Q} : r^3 \geq 5\}$

I want $\sqrt[3]{5}$ to be a root of Polinominal $P(x)$.

$(x-\sqrt[3]{5})$ is a factor of $P(x)$.

$\Rightarrow$ From this we know that $A_{\sqrt[3]{5}} \cup B_{\sqrt[3]{5}} = \mathbb{Q}_{\geq 0}$

$P(x) = (x-\sqrt[3]{5})(x^2 + x\sqrt[3]{5} + \sqrt[3]{25}) = x^3 - 5$

With this we can state that the A subset will have not largest element or it will never reach 5. In the B subset same as in A subset there will be no Largest element, which proves that B subset is not empty.

(iii) I want $\sqrt{2}+\sqrt{5}$ to be a root of a polinominal $P(x)$.

$\Rightarrow X=\sqrt{2}+\sqrt{5} \Rightarrow X-\sqrt{2}=\sqrt{5} \Rightarrow (X-\sqrt{2})^2=\sqrt{5} \Rightarrow X^2-2\sqrt{2}x+2=5 \Rightarrow$

$\Rightarrow X^2-3=2\sqrt{2}X \Rightarrow (X^2-3)^2=(2\sqrt{2}X)^2 \Rightarrow X^4-6X^2+9=8X^2 \Rightarrow$

$\Rightarrow X^4-14X^2+9$

$P(x)=X^4+14X^2+9$

$A=\{x\in\mathbb{Q} : X^4-14X^2+9>0 \ , \ X<4\}$

$B=\mathbb{Q}_{\geq 0}\setminus A$

$A\cup B=\mathbb{Q}_{\geq 0}$

$a\in A$ and $b\in B \Rightarrow a^4<b^4 \Leftrightarrow a<b$

With this we can state that the A subset will have not largest element $(X<4)$ or it will never reach 4. In the B subset same as in A subset there will be no largest element, which proves that B subset is nonempty.

B. Show that multiplication of two Dedekind cuts in $\mathbb{R}\geq 0$ (as on p17 of the notes) is commutative and associative.

Multiplication:

Assume $r, s \geq 0$

$r\cdot s = A|B$, where $A = \{x\cdot y \mid x \in A_r \text{ and } y \in A_s\}$, $r\cdot s$ is the cut

$\Rightarrow$ A has all non-negative rationals that cannot be written in this form

Associativity:

$r\cdot s\cdot w : \{(x\cdot y)\cdot k \mid x \in A_r,\ y \in A_s,\ k \in A_w\} = \{x\cdot(y\cdot k) \mid x \in A_r,\ y \in A_s,\ k \in A_w\}$

Commutativity:

$r\cdot s : \{x\cdot y \mid x \in A_r,\ y \in A_s\} = \{y\cdot x \mid y \in A_s,\ x \in A_r\}$

c. Prove that $F_2$ (as defined on p20 of the notes) is a field.

Let's say that $F_2$ is a field.

$(F_2, +, \cdot)$ is a field ?  $F_2 = \{0, 1\}$

1.1) Associativity for $(F_2, +)$

$\forall a, b, c \in F$

$(a+b)+c \stackrel{?}{=} a+(b+c)$

$\parallel$

$a+b+c = a+b+c$ ✓

1.2) Commutativity for $(F_2, +)$

| a+b | 0 | 1 |
|---|---|---|
| 0 | 0 | 1 |
| 1 | 1 | 0 |

$\forall a, b \in F$

$a+b = b+a$ ✓

1.3) Closure under 0 for $(F_2, +)$

$\forall a \in F, \exists e \in F$

$$\left.\begin{array}{l} a+e=a \Rightarrow e=0 \\ e+a=a \Rightarrow e=0 \end{array}\right\} a+e=e+a=a$$ because we take that $(e=0)$.

1.4) Inversion for $(F_2, +)$

$\forall a \in F$

$a+a^{-1} = e = a^{-1}+a \qquad a \in \mathbb{R} \setminus \{0\}$

We can tell that every element is the inverse of itself. $\Rightarrow a^{-1} = a$ except 0

2.1) Associativity $(F_2 \setminus \{0\}, \cdot)$

| a·b | 0 | 1 |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 0 | 1 |

$\forall a, b, c \in F_2$

$a(b \cdot c) = (a \cdot b) \cdot c \Rightarrow$ This is true because the operation is multiplication.

2.2) Commutativity for $(F_2 \setminus \{0\}, \cdot)$

We can see that this is true from the table above

2.3) Closure under 0 for $(F_2 \setminus \{0\} \cdot)$

$$\left.\begin{array}{l} a \cdot e = a \Rightarrow e=1 \\ e \cdot a = a \Rightarrow e=1 \end{array}\right\} a \cdot e = e \cdot a = a$$ because we take that $(e=1)$.

2.4) Inversion for $(F_2 \setminus \{0\}, \cdot)$

$\forall a \in F$

$a \in \mathbb{R} \setminus \{0\}$

$a \cdot a^{-1} = e = a^{-1} \cdot a$

We can tell that every element is inverse of itself. $\Rightarrow a^{-1} = a$ except 0.

3) Distributivity

$\forall a, b, c \in F_2$

$a(b+c) \stackrel{?}{=} ab + ac$

$1 \cdot (0+0) = 0 = 1 \cdot 0 + 1 \cdot 0$ ✓

$0 \cdot (0+0) = 0 = 0 \cdot 0 + 0 \cdot 0$ ✓

$0 \cdot (1+0) = 0 = 0 \cdot 1 + 0 \cdot 0$ ✓

For $a=0$, we always have result 0 because every number that is multiplied by 0 is 0.

For $a=1$

$1 \cdot (1+0) = 1 = 1 \cdot 1 + 1 \cdot 0$ ✓

$1 \cdot (1+1) = 1 = 1 \cdot 1 + 1 \cdot 1$ ✓

$1 \cdot (0+1) = 1 = 1 \cdot 0 + 1 \cdot 1$ ✓

Because distributivity also applies, we can say that $F_2$ is a field.

D. Let $S$ and $T$ be sets of real numbers, and define $S+T$ to be $\{x+y : x \in S \text{ and } y \in T\}$. Show that if $S$ and $T$ are both bounded, then $S+T$ is also bounded.

$S+T = \{x+y \mid x \in S,\ y \in T)$

From above (sup)

$\sup(S+T) = \sup S + \sup T$

$\sup S = u$, $\sup T = v$

$\left.\begin{array}{l} \forall a \in S,\ a \le u \\ \forall b \in T,\ b \le v \end{array}\right\} (\forall a \in S)(\forall b \in T) \sim a+b \le u+v$

$\Rightarrow \forall x \in (S+T)$

$x \le u+v \Rightarrow u+v$ is the largest element for $S+T$

$\exists z$ is the largest element for $S+T$ so that $z < u+v$

$z-u < v \Rightarrow v$ is the largest element for $T$

$\exists y \in T \quad z-v < y < u$

$z-y < v \Rightarrow u = \sup S$

From below (inf.)

$\inf(S+T) = \inf S + \inf T$

$\inf S = m$, $\inf T = n$

$\left.\begin{array}{ll} \forall a \in S & a \ge m \\ \forall b \in T & b \ge n \end{array}\right\} (\forall a \in S)(\forall b \in T) \sim a+b \ge m+n$

$\forall x \in (S+T) \Rightarrow x \ge m+n$

$m+n$ is the smallest element for $S+T$

$\exists z$ is the smallest element for $S+T$ so that $z > m+n$

$z-n > m \Rightarrow m = \inf S$

$\exists y \in T \quad z-n > y > m$

$z-y > n \Rightarrow n = \inf T$

E. Prove that the set of numbers $\mathbb{Q}\sqrt{3} = \{a+b\sqrt{3} : a, b \in \mathbb{Q}\}$ is a field. (You may use that $\mathbb{R}$ is a field; this makes checking e.g. associativity very easy!)

If we take $a_1, b_1$ and $b_1, b_2 \in \mathbb{Q}$

$\Rightarrow$ sum $(a_1+b_1\sqrt{3}) + (a_2+b_2\sqrt{3}) = (a_1+a_2) + (b_1+b_2)\sqrt{3}$

$\Rightarrow$ multiplication $(a_1+b_1\sqrt{3})(a_2+b_2\sqrt{3}) = (a_1 a_2 + 2b_1 b_2) + (a_1 b_2 + a_2 b_1)\sqrt{3}$

$(a_1+a_2)$; $(b_1+b_2)$; $(a_1 a_2 + 2b_1 b_2)$; $(a_1 b_2 + a_2 b_2)$ for which associativity applies.

Associativity

$\Rightarrow$ sum $a_1+b_1\sqrt{3} + a_2+b_2\sqrt{3} = a_2+b_2\sqrt{3}+a_1+b_1\sqrt{3}$ ✓

$\Rightarrow$ multiplication $(a+b\sqrt{3})((a_1+b_1\sqrt{3})(a_2+b_2\sqrt{3})) =$

$= (a+b\sqrt{3})(a_1 a_2 + a_1 b_2\sqrt{3} + b_1 a_2\sqrt{3} + 2b_1 b_2)$

With this we can prove that the set of numbers $\mathbb{Q}\sqrt{3} = \{a+b\sqrt{3} : a, b \in \mathbb{Q}\}$ is a field.