Analysis I Homework 3

A. Using the definition of limit, show that $\lim_{n\to\infty} 1/n^5 = 0$ and that $\lim_{n\to\infty} 1/n^{1/5} = 0$.

$\lim_{n\to\infty} 1/n^5 = 0$

$\forall \varepsilon > 0 \quad N^5 = 1/\varepsilon$

$n^5 > N^5 > 0 \Rightarrow 0 < 1/n^5 < 1/N^5 = 1/1/\varepsilon = \varepsilon$

$[n^5 > N^5] \Rightarrow [|1/n^5 - 0| < \varepsilon]$

$\lim_{n\to\infty} 1/n^{1/5} = 0$

$\forall \varepsilon > 0 \quad n^{1/5} > 1/\varepsilon$

$n^{1/5} > N^{1/5} > 0 \Rightarrow 0 < 1/n^{1/5} < 1/N^{1/5} = 1/1/\varepsilon = \varepsilon$

$[n^{1/5} > N^{1/5}] = [|1/n^{1/5} - 0| < \varepsilon]$

B. Using the definition of limit (so, without using Arithmetic of Limits), show that:

(i) $\lim_{n\to\infty} (4+n)/2n = 1/2$

(ii) $\lim_{n\to\infty} 2/n + 3/(n+1) = 0$

(i) $\left|\frac{4+n}{2n} - \frac{1}{2}\right| < \varepsilon$

$\left|\frac{4+\cancel{n}-\cancel{n}}{2n}\right| < \varepsilon$

$\left|\frac{4}{2n}\right| < \varepsilon$

$\frac{2}{n} < \varepsilon$

$n > \frac{2}{\varepsilon}$

$\forall \varepsilon > 0, \ |s_n - s| = |2/n| < \varepsilon$

(ii) $\lim_{n\to\infty} 2/n + 3/(n+1) = 0$

$\left|\frac{2}{n} + \frac{3}{n+1} - 0\right| < \varepsilon$

$\left|\frac{2n+2+3n}{n(n+1)} - 0\right| < \varepsilon$

$\left|\frac{5n+2}{n(n+1)}\right| < \varepsilon$

$\frac{5n+2}{n(n+1)} < \varepsilon$

$\frac{5n+2}{n^2+n} < \varepsilon \Rightarrow n_0 \in \mathbb{N}, \ \frac{1}{n_0} < \varepsilon$

$\forall n > n_0 \quad \frac{1}{n} < \frac{1}{n_0} < \varepsilon$

$\Rightarrow \lim_{n\to\infty} \left(\frac{2}{n} + \frac{3}{n+1}\right) = 0$

C. Suppose that (S_n) and (t_n) are sequences so that $S_n = t_n$ except for finitely many values of n. Using the definition of limit, explain why if $\lim_{n\to\infty} S_n = s$, then also $\lim_{n\to\infty} t_n = s$.

Let (S_n) and (t_n) be the last different elements.

So we have that: $\forall \varepsilon > 0,\ n > N_{s_n} \Rightarrow |S_n - s| < \varepsilon$

$\forall \varepsilon > 0,\ n > N_{t_n} \Rightarrow |t_n - s| < \varepsilon$

We will take that $N = \text{suprinum}(N_{s_n}, N_{t_n})$

$$\left.\begin{array}{l} |S_n - s| < \varepsilon \\ |t_n - s| < \varepsilon \end{array}\right\} n > N$$

$$\left.\begin{array}{l} |S_n - s| < \varepsilon \\ |t_n - s| < \varepsilon \end{array}\right\} \text{for which } N_0 = N+1 \text{ so then we will have}$$

$|S_n - s| < \varepsilon \Rightarrow n > N_0$

$|t_n - s| < \varepsilon \Rightarrow n > N_0$

From this we can prove that $S_n = t_n$ so:

$n > N_0 \Rightarrow |t_n - s| < \varepsilon$ for $N_0 = N+1$

D. Suppose real sequences (s_n) and (t_n) are bounded. (That is, that their ranges are bounded sets.
(i) Show the sequence given by $(s_n + t_n)$ is bounded
(ii) For any real number α, show that the sequence $(\alpha \cdot s_n)$ is bounded.

(i) $|s_n + t_n| \leq |s_n| + |t_n|$

$-(|s_n| + |t_n|) \leq |s_n + t_n| \leq |s_n| + |t_n|$

Every element of $|s_n + t_n|$ will have a bigger number in $|s_n| + |t_n|$
Every element of $-(|s_n| + |t_n|)$ will have least element that always have a smaller or equal numbers.

(ii) $\alpha \in \mathbb{R}$

If we say the set is bounded by x and y, than we will have:

$s_n = (x, y)$

$x \leq s_n \leq y$

$x \leq s_n$
$s_n \leq y$

$\alpha \cdot x \leq \alpha \cdot s_n$

$\alpha \cdot s_n \leq \alpha \cdot y$

This means that αx is a lower bound and αy is upperbound.
With this we can prove that $(\alpha \cdot s_n)$ is bounded.

5. (i) For a convergent real sequences s_n and a real number a, show that if $s_n \geq a$, for all but finitely many values of n, then $\lim_{n\to\infty} s_n \geq a$.
(ii) For each value of $a \in \mathbb{R}$, give an example of a convergent sequence s_n with $s_n > a$ for all n, but where $\lim_{n\to\infty} s_n = a$.

(i) $\lim_{n\to\infty} s_n - s$

$a \in \mathbb{R}$

We should prove that $s \geq a$ and assume that $s \leq a$.

If we take that s_m is the biggest element that $s_m < a$ then $n > m \Rightarrow$
$\Rightarrow s_n \geq s_m$.

$|s_n - s| < \varepsilon$ $\qquad$ $s_m \leq a$

$s_n > s - \varepsilon$ $\qquad$ $s_n \geq a$

$s_n < s + \varepsilon$

$s - \varepsilon < s_n < s + \varepsilon$

Because $\varepsilon > 0$ and $a > s$ we can take ε to be as small as $a - s$, so from here we will have that $s_n < s + a - s = a$. But because $s_n > a$ we can state otherwise so that makes s_m not the biggest element less than a.

(ii) $a \in \mathbb{R}$

If we take that s_n is sequence $\frac{1}{n}$, so $\lim_{n\to\infty} \frac{1}{n} = 0$ then we will have:

$(\forall \varepsilon > 0)$ $\quad n > N \Rightarrow |\frac{1}{n} - 0| < \varepsilon$

$\frac{1}{n} < \varepsilon \qquad 1 < \varepsilon n$

$n > \frac{1}{\varepsilon}$

$\Rightarrow N = \frac{1}{\varepsilon}$ and from here we know that the $\lim_{n\to\infty} \frac{1}{n} = 0$.