



An expression of the form

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots$$

is called an infinite series based on the sequence  $(a_n)$ .

We'll use limits to make sense of infinite series.

First, some important classes of examples.

1) Given  $c, r \in \mathbb{R}$  where  $c \neq 0$ ,

$$\sum_{n=0}^{\infty} cr^n = c + cr + cr^2 + cr^3 + \dots \text{ is a geometric series.}$$

2)  $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$  is the harmonic series.

3) More generally, if  $p \in \mathbb{R}$ , then

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \dots \text{ is a Dirichlet p-series.}$$

Remark The  $\sum$  notation is useful for writing compactly and precisely. But if you get lost in the notation, it might help you to write out a few terms w/ a "+" as we've done above.

Of course, we can only actually add up finitely many numbers. We must give precise meaning to these infinite "sums", with limits.

Definition: Given the infinite sum  $\sum_{n=1}^{\infty} a_n$ , we define the sequence of partial sums as

$$S_N := \sum_{n=1}^N a_n = a_1 + a_2 + \dots + a_N.$$

That is,  $S_N$  is the truncation of the infinite series to the first  $N$  terms.

Eg:  $\sum_{n=1}^{\infty} \frac{1}{n^2}$  has  $S_1 = 1$ ,  $S_2 = 1\frac{1}{4}$ ,  $S_3 = 1\frac{13}{36}$ , etc.

- It's also sometimes helpful to consider the remainder

$$R_N := \sum_{n=N+1}^{\infty} a_n = a_{N+1} + a_{N+2} + \dots$$

Thus, for any value of  $N$ ,

$$\sum_{n=1}^{\infty} a_n = S_N + R_N.$$

- If  $\lim_{N \rightarrow \infty} S_N$  converges to  $S$ , then  
we say the series converges to  $S$   
and write  $\sum_{n=1}^{\infty} a_n = S$ .

Unsurprisingly, if  $S_N$  diverges, we say  $\sum_{n=1}^{\infty} a_n$  diverges  
and if  $\lim_{N \rightarrow \infty} S_N = \pm\infty$   
we write  $\sum_{n=1}^{\infty} a_n = \pm\infty$  (and say it diverges to  $\pm\infty$ ).

### Examples:

1)  $\sum_{n=1}^{\infty} 1 = \infty$ . The partial sum  $S_N = N$ ,  
and  $\lim_{N \rightarrow \infty} N = \infty$ .

2)  $\sum_{n=0}^{\infty} (-1)^n$  diverges (not to  $\pm\infty$ ).

The partial sums  $S_N$  are 1 for  $N$  even  
and 0 for  $N$  odd

so  $\limsup S_N = 1$  but  $\liminf S_N = 0$ .

3) Let  $d_n$  be the  $n$ th digit of the decimal expansion  
of  $\sqrt{2} = 1.414213\dots$

so  $d_0=1, d_1=4, d_2=1, d_3=4, d_4=2, d_5=1, d_6=3, \dots$

Now

$\sum_{n=0}^{\infty} \frac{d_n}{10^n}$  has  $S_N$  the  $N$ th partial decimal expansion!

$S_0=1, S_1=1.4, S_2=1.41, \dots$

By our previous discussion, we see that  $\sum_{n=0}^{\infty} \frac{dn}{10^n} = \sqrt{2}$ .

4) (\*\*\* Important example \*\*\*)

Consider the geometric series  $\sum_{n=0}^{\infty} c \cdot a^n$ , where  $c \neq 0$ .

By a homework problem,

$$(1-a)(1+a+a^2+\dots+a^N) = 1-a^{N+1}$$

Thus,  $S_N = c \cdot \frac{1-a^{N+1}}{1-a}$ .

By another homework problem,

$$\lim_{N \rightarrow \infty} a^{N+1} = \begin{cases} 0 & \text{if } -1 < a < 1 \\ \infty & \text{if } a > 1 \\ 1 & \text{if } a = 1 \end{cases}$$

and does not exist if  $a \leq -1$ .

AoL gives us  $\lim_{N \rightarrow \infty} S_N$  for  $a \neq 1$ :

$$\sum_{n=0}^{\infty} ca^n = \lim_{N \rightarrow \infty} S_N = \begin{cases} c \cdot \frac{1}{1-a} & \text{if } -1 < a < 1 \\ \infty & \text{if } a > 1 \end{cases}$$

and diverges, not to  $\infty$ , for  $a \leq -1$ .

For  $a=1$ , we proceed directly:

$$\sum_{n=0}^{\infty} c \cdot 1^n = \sum_{n=0}^{\infty} c = c + c + c + \dots = \begin{cases} \infty & \text{if } c > 0 \\ -\infty & \text{if } c < 0 \end{cases}$$

Concrete example  $\sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots = 2$   
 where  $S_N = \frac{1 - \frac{1}{2^{N+1}}}{1 - \frac{1}{2}} = 2 - \frac{1}{2^N}$ .

$$5) \sum_{n=0}^{\infty} \frac{n}{n+1} = 0 + \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \dots$$

Since every term past  $n=0$

$$\text{satisfies } \frac{n}{n+1} \geq \frac{1}{2}$$

$$\text{we see that } S_N \geq \frac{1}{2}N$$

$$\text{so } \liminf S_N \geq \liminf \frac{1}{2}N = \infty$$

$$\text{so } \lim_{N \rightarrow \infty} S_N = \infty, \text{ and the series diverges to } \infty.$$

Let's go through our limit "toolbox" and apply the tools to  $\lim_{N \rightarrow \infty} S_N$ , the limit of partial sums.

Arithmetic of (Infinite) Limits is helpful

Theorem (Arithmetic of Series)

If  $\sum_{n=1}^{\infty} a_n = A$  and  $\sum_{n=1}^{\infty} b_n = B$  where  $A, B$  are real numbers or  $\pm\infty$

then

$$\text{i) } \sum_{n=1}^{\infty} c a_n = c A \quad \text{for any } c \neq 0.$$

$$\text{ii) } \sum_{n=1}^{\infty} (a_n + b_n) = A + B$$

(But as usual, we give no definition to the symbol  $\infty - \infty$ .)

Proof Arithmetic of Limits!

$$\text{i) } \sum_{n=1}^{\infty} c a_n = \lim_{N \rightarrow \infty} c a_1 + c a_2 + \dots + c a_N = c \cdot \lim_{N \rightarrow \infty} a_1 + a_2 + \dots + a_N = c \cdot A.$$

$$\begin{aligned} \text{ii) } \sum_{n=1}^{\infty} (a_n + b_n) &= \lim_{N \rightarrow \infty} a_1 + b_1 + a_2 + b_2 + \dots + a_N + b_N \\ &= \lim_{N \rightarrow \infty} (a_1 + a_2 + \dots + a_N) + (b_1 + b_2 + \dots + b_N) = A + B. \quad \square \end{aligned}$$

Less obviously, AoL gives us an easy (though "incomplete") way to show that some series diverge.

Lemma: If  $\sum_{n=1}^{\infty} a_n$  converges (to any real number)  
then  $\lim_{n \rightarrow \infty} a_n = 0$ .

Pf:

Since  $\sum_{n=1}^{\infty} a_n = \lim_{N \rightarrow \infty} S_N$  converges, by A.o.L., say to  $A$ ,

$$\begin{aligned} \lim_{N \rightarrow \infty} S_N - S_{N-1} &= \lim_{N \rightarrow \infty} (a_1 + \dots + a_N) - (a_1 + \dots + a_{N-1}) \\ &= \lim_{N \rightarrow \infty} a_N = A - A = 0. \quad \square \end{aligned}$$

Eg:  $\sum_{n=0}^{\infty} \frac{1}{2^n} = 2$ , so we must (and do) have  $\lim_{n \rightarrow \infty} \frac{1}{2^n} = 0$ .

Corollary (the  $n$ th Term Test for divergence)

If  $\lim_{n \rightarrow \infty} a_n \neq 0$ , then the series  $\sum_{n=1}^{\infty} a_n$  diverges.

Example:  $\sum_{n=1}^{\infty} \sqrt{1 - \frac{1}{n^2}}$  diverges, since  $\lim_{n \rightarrow \infty} \sqrt{1 - \frac{1}{n^2}} = 1$  (by A.o.L.).

\*\*\* Caution: If  $\lim_{n \rightarrow \infty} a_n = 0$ , then the series  $\sum_{n=1}^{\infty} a_n$   
may converge or  
may diverge,  
The  $n$ th Term Test gives us no information in this case. \*\*\*

Remark: Now you see clearly the difference between "if"  
and "if and only if".

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Example: Although  $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$ , the harmonic series  $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$ .

We can see this by collecting terms:

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \underbrace{\frac{1}{2}}_{\rightarrow \frac{1}{2}} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{\rightarrow \frac{1}{2}} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}_{\rightarrow \frac{1}{2}} + \dots$$

(In general,  $\sum_{n=2^k}^{2^{k+1}-1} \frac{1}{n} > \frac{1}{2}$ .)

$$\text{Thus, } \lim_{N \rightarrow \infty} S_N \geq \lim_{N \rightarrow \infty} \frac{1}{2} + \frac{1}{2} + \dots + \frac{1}{2} = \left( \lim_{N \rightarrow \infty} \lfloor \log_2 N \rfloor \cdot \frac{1}{2} \right) = \infty$$


the series diverges to  $\infty$ . ✓

The Cauchy Completeness Theorem lets us give an "if and only if" extension of the  $n$ th Term Test, at the cost of a considerable amount of complexity.

Theorem (Cauchy Convergence for Series)

Let  $\sum_{k=1}^{\infty} a_k$  have sequence of partial sums  $S_n = \left( \sum_{k=1}^n a_k \right)$ .

(\*) The series converges if and only if  
 $\forall \epsilon > 0, \exists N \text{ s.t. } [n, m > N] \Rightarrow [|S_n - S_m| < \epsilon]$

Proof: Follows immediately by specializing the Cauchy Completeness Theorem. 

What does Cauchy Convergence have to do with the  $n$ th Term Test?

If  $n > m$ , then in the above situation,

$$S_n - S_m = a_{m+1} + a_{m+2} + \dots + a_n.$$

So (\*) says that

$$\forall \epsilon > 0, \exists N \text{ s.t. } [n > m > N] \Rightarrow [|a_{m+1} + a_{m+2} + \dots + a_n| < \epsilon].$$

Thus, the  $n$ th Term Test may be seen as the special situation  $m=n-1$  in Cauchy Convergence for series. Obviously, if Cauchy fails in this special case, then it fails (but the converse may not be true).

Example (Harmonic series, with Cauchy convergence).

Since for any  $N$ , we can find  $n > m > N$

$$\text{s.t.} \quad \frac{1}{m+1} + \frac{1}{m+2} + \dots + \frac{1}{n} > \frac{1}{2}$$

the Cauchy Convergence Criterion fails for  $\sum_{n=1}^{\infty} \frac{1}{n}$  with  $\varepsilon = \frac{1}{2}$ .

So the series diverges.

The Monotone Sequence Theorem is also useful, as follows:

Proposition: If  $a_n$  is a nonnegative real sequence

then either  $\sum_{n=1}^{\infty} a_n$  converges (and is bounded)

or else  $\sum_{n=1}^{\infty} a_n = \infty$

Proof: Since  $\overset{(\text{fund'n})}{a_n} \geq 0$  and  $S_{m+1} = S_m + a_{m+1}$ ,

the sequence  $S_n$  of partial sums is an increasing sequence.

By the MST, it is either bounded and convergent

or else unbounded and diverges to  $\infty$ .  $\blacksquare$

Key point: The Proposition tells us that checking convergence of a series w/ nonnegative terms is equivalent to finding an upper bound on its sequence of partial sums!

We'll develop this in detail, but let's first compute one more example.

Example: (Technique of Telescoping Sums)Find  $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ , or explain why it diverges.Solution:We first notice that  $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$ (Use the technique of partial fractions:

Set  $\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}$

$$\Rightarrow 1 = A(n+1) + B \cdot n \Rightarrow 0n + 1 = (A+B)n + A$$

and solve to find  $A=1, B=-1$ .)

So we're finding

$$\sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n+1} \right), \quad \text{Here } S_N = \left( 1 - \frac{1}{2} \right) + \left( \frac{1}{2} - \frac{1}{3} \right) + \left( \frac{1}{3} - \frac{1}{4} \right) + \dots + \left( \frac{1}{N} - \frac{1}{N+1} \right)$$

We notice that the  $\frac{1}{n+1}$  from the  $n$ th term cancels w/ the  $\frac{1}{n+1}$  " "  $(n+1)$ th term.This leaves  $S_N = 1 - \frac{1}{N+1}$ . (Notice that 1 has nothing to cancel with, nor  $\frac{1}{N+1}$  yet.)

$$\text{Now } \lim_{N \rightarrow \infty} S_N = \lim_{N \rightarrow \infty} \left( 1 - \frac{1}{N+1} \right) = 1 - 0 = 1 = \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

Self-check/exercise: Find  $\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$  (using a similar technique).B. Tests for nonnegative series

For now, all the series we look at will have nonnegative terms.

We saw that the MST tells us such a series converges if and only if its partial sums are bounded.

Bounding partial sums is much easier than working with limits directly! (But a drawback is that we rarely will be able to calculate series' limits, only to say whether the limit converges.)

A basic tool compares two series directly.

### Theorem (Comparison Test)

Let  $a_n, b_n$  be real sequences with  $0 \leq a_n \leq b_n$  (for all  $n$ ).

Then 1)  $\sum_{n=1}^{\infty} b_n$  converges  $\Rightarrow \sum_{n=1}^{\infty} a_n$  converges

2)  $\sum_{n=1}^{\infty} a_n = \infty \Rightarrow \sum_{n=1}^{\infty} b_n = \infty$

Proof: 1) It suffices by previous discussion to bound the partial sums. But if  $\sum_{n=1}^{\infty} b_n = B$ , then the partial sums of  $\sum_{n=1}^{\infty} b_n$  are bounded by  $B$ , while the partial sums for  $\sum_{n=1}^{\infty} a_n$  are bounded by those for  $\sum b_n$ .

In symbols,


$$\sum_{n=1}^N a_n \leq \sum_{n=1}^N b_n \leq B.$$

Thus  $\lim_{n \rightarrow \infty} \sum_{n=1}^n a_n$  converges. ✓

2) Similarly,

$$\sum_{n=1}^N a_n \leq \sum_{n=1}^N b_n$$

and since the LHS is unbounded, the RHS is also.

By MST / Proposition,  $\lim_{N \rightarrow \infty} \sum_{n=1}^N b_n = \infty$ . 

### Important examples:

1) Since  $0 \leq \frac{1}{n^2} \leq \frac{2}{n(n+1)}$  for all  $n \geq 1$  (as  $n^2 + n \leq 2n^2$ )

we have

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \leq \sum_{n=1}^{\infty} \frac{2}{n(n+1)} = 2 \quad (\text{by a previous example}).$$

By the Comparison Test,  $\sum_{n=1}^{\infty} \frac{1}{n^2}$  converges to some value  $S$ .

The argument shows that  $S \leq 2$ .

You can take partial sums to compute as many lower bounds as you like.

Fact:  $S = \frac{\pi^2}{6}$

2) If  $p \geq 2$ , then  $0 \leq \frac{1}{n^p} \leq \frac{1}{n^2}$  for all  $n \geq 1$   
 so  $\sum_{n=1}^{\infty} \frac{1}{n^p}$  converges by the Comparison Test  
 (Comparison w/  $\sum_{n=1}^{\infty} \frac{1}{n^2}$ .)

3) If  $p < 1$ , then  $0 \leq \frac{1}{n} = \frac{1}{n^2} \leq \frac{1}{n^p}$ .  
 Since we showed that  $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$ ,  
 also  $\sum_{n=1}^{\infty} \frac{1}{n^p} = \infty$ , by the Comparison Test,

Remarks: Like the  $n$ th Term Test, the Comparison Test  
 will sometimes (often?) give no information.

Eg:  $0 \leq \frac{1}{n^2} \leq \frac{1}{n}$ , but  $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$   
 while  $\sum_{n=1}^{\infty} \frac{1}{n^2}$  converges.

The Comparison Test would give no information,  
 since the "small" series converges  
 and the "large" series diverges.

Summary of Last "Important Example":

The Dirichlet  $p$ -series  $\sum_{n=1}^{\infty} \frac{1}{n^p}$   
 converges if  $p \geq 2$   
 but diverges if  $p \leq 1$ .

We'll need another technique for  $1 < p < 2$ ,  
 but first, let's do more examples of Comparison.

Example: Either show  $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$  converges, or that it diverges to  $\infty$ .

Solution:

Clearly,  $0 \leq \frac{1}{n^2+1} \leq \frac{1}{n^2}$  for all  $n \geq 1$ ,

so  $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$  converges by Comparison w/  $\sum \frac{1}{n^2}$ .

Example Either show  $\sum_{n=1}^{\infty} \frac{1}{n^2-5}$  converges, or that it diverges to  $\infty$ .

Solution (w/ full discussion)

Although the 1st two terms of  $\frac{1}{n^2-5}$  are  $< 0$ ,  
it is equivalent to consider convergence of  
 $\sum_{n=3}^{\infty} \frac{1}{n^2-5}$  or indeed  $\sum_{n=1000000}^{\infty} \frac{1}{n^2-5}$ .

We try comparison w/  $\frac{1}{n^2}$ : for  $n \geq 3$ , we have  
 $0 \leq \frac{1}{n^2} \leq \frac{1}{n^2-5}$

but unfortunately this is the "no information" situation.

Finally, we repeat the trick we used w/  $\frac{1}{n^2}$  and  $\frac{1}{n^2+n}$   
and instead compare

$$\frac{1}{n^2-5} \text{ with } \frac{2}{n^2}.$$

$$\text{Since } n^2 \leq 2(n^2 - 5)$$

$$\Leftrightarrow 10 \leq n^2, \quad \text{as holds for } n \geq 4,$$

$$\text{we have } 0 \leq \frac{1}{n^2-5} \leq \frac{2}{n^2} \quad \text{for } n \geq 4,$$

$$\text{and as } \sum \frac{2}{n^2} \text{ converges, so does } \sum_{n=1}^{\infty} \frac{1}{n^2-5}. \quad \checkmark$$

Solution (short):

$$\text{Notice that } 0 \leq \frac{1}{n^2-5} \leq \frac{2}{n^2}$$

$$\Leftrightarrow n^2 \leq 2n^2 - 10$$

$$\Leftrightarrow 10 \leq n^2 \quad \text{holds for } n \geq 4.$$

By comparison, since  $\sum \frac{1}{n^2}$  converges  
 $\sum_{n=1}^{\infty} \frac{1}{n^2-5}$  also converges.  $\checkmark$

Self-check: Try the same w/  $\sum_{n=1}^{\infty} \frac{1}{n+\pi}$  and  $\sum_{n=1}^{\infty} \frac{1}{n-\pi}$ .  
(Hints: both diverge.)

Example: Either show  $\sum_{n=0}^{\infty} \frac{\sin^2 n}{2^n}$  converges, or that it diverges to  $\infty$ .

Solution: We will use only that  $\sin^2 n$  is a bounded, positive sequence (and no other properties of  $\sin^2 n$ ). Indeed,

$$0 \leq \sin^2 n \leq 1$$

Thus,  $0 \leq \frac{\sin^2 n}{2^n} \leq \frac{1}{2^n}$ . Since  $\sum_{n=0}^{\infty} \frac{1}{2^n}$  is a convergent geometric series (as  $\frac{1}{2} < 1$ ),  
 $\sum_{n=0}^{\infty} \frac{\sin^2 n}{2^n}$  also converges. ✓

Example: Either show  $\sum_{n=1}^{\infty} \frac{n-2}{n^2}$  converges, or that it diverges to  $\infty$ .

Solution 1: Direct comparison. Since  $\frac{n-2}{n^2} \geq \frac{1}{2n}$   
 $\Leftrightarrow 2n^2 - 4n \geq n^2$   
 $\Leftrightarrow n^2 \geq 4n$ , as happens for  $n \geq 4$   
 and as  $\sum_{n=1}^{\infty} \frac{1}{2n} = \infty$ , also  $\sum_{n=1}^{\infty} \frac{n-2}{n^2} = \infty$ . ✓

Selfcheck: Why didn't we compare w/  $\sum \frac{1}{n}$ ?

Solution 2: We have from prior examples that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = A \quad \text{for some real number } A, \text{ while}$$

$$\sum_{n=1}^{\infty} \frac{1}{n} = \infty.$$

Apply Arithmetic of Series.

$$\sum_{n=1}^{\infty} \frac{n-2}{n^2} = \sum_{n=1}^{\infty} \frac{n}{n^2} - 2 \cdot \sum_{n=1}^{\infty} \frac{1}{n^2} = \infty - 2A = \infty. \quad \checkmark$$

We return to consider the Dirichlet  $p$ -series  $\sum_{n=1}^{\infty} \frac{1}{n^p}$  for  $1 < p < 2$ .

We use a trick similar to the one we used for  $p=1$ ,  
that is, for  $\sum_{n=1}^{\infty} \frac{1}{n}$ .

Lemma: If  $1 < p$ , then  $\sum_{n=1}^{\infty} \frac{1}{n^p}$  converges.

Proof: We collect 2 terms, then 4 terms, then 8 terms, then...

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \underbrace{\frac{1}{2^p} + \frac{1}{3^p}}_{< 2 \cdot \frac{1}{2^p}} + \underbrace{\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p} + \dots}_{< 4 \cdot \frac{1}{4^p}} + \dots$$

$$< 1 + 2 \cdot \frac{1}{2^p} + 4 \cdot \frac{1}{4^p} + 8 \cdot \frac{1}{8^p} + \dots$$

$$= \sum_{k=0}^{\infty} \frac{2^k}{(2^k)^p} = \sum_{k=0}^{\infty} \frac{1}{2^{kp-k}} = \sum_{k=0}^{\infty} \frac{1}{(2^{p-1})^k}.$$

Thus,  $\sum_{n=1}^{\infty} \frac{1}{n^p}$  is bounded above by the geometric series  
with "ratio"  $\frac{1}{2^{p-1}}$ .

Now, since  $p > 1$ , we see that  $2^{p-1} > 1$   
so that  $\frac{1}{2^{p-1}} < 1$

so the geometric series converges,

forcing  $\sum_{n=1}^{\infty} \frac{1}{n^p}$  to also converge (since bounded). ■

Let's collect what we know about Dirichlet  $p$ -series. These will  
be an excellent source of series to compare with.

Corollary (Convergence/Divergence of Dirichlet  $p$ -series)

For a real number  $p$ , the series  $\sum_{n=1}^{\infty} \frac{1}{n^p}$

1) converges if  $p > 1$ , and

2) diverges to  $\infty$  if  $p \leq 1$ .

Example:  $\sum_{n=1}^{\infty} \frac{1}{n^2}$ ,  $\sum_{n=1}^{\infty} \frac{1}{n^3}$ ,  $\sum_{n=1}^{\infty} \frac{1}{n^{1.0001}}$  all converge,

while  $\sum_{n=1}^{\infty} \frac{1}{n} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{0.99}} = \infty$ .

Big-Theta notation

Definition: If  $a_n$  and  $b_n$  are sequences so that,  
for some positive constants  $\alpha, \beta$   
and some  $N$

we have that for all  $n > N$ ,  $\alpha b_n \leq a_n \leq \beta b_n$   
then we say  $a_n$  is big theta of  $b_n$   
and write  $a_n = \Theta(b_n)$ .

Notice: If  $\alpha b_n \leq a_n \leq \beta b_n$ , then also  $\frac{1}{\beta} a_n \leq b_n \leq \frac{1}{\alpha} a_n$ .  
Thus,  $a_n = \Theta(b_n) \Leftrightarrow b_n = \Theta(a_n)$ . ✓

Remarks: Big theta notation is extremely important in algorithm analysis, where it is the "gold standard" of complexity for an algorithm.


In algorithm analysis, we want to show the number of steps taken to be  $\Theta(n^2)$  or  $\Theta(n^3)$  or  $\Theta(2^n)$  or similar.

For ANA-I, big theta notation will help us analyze series convergence. Here, we'll want to show the terms to be  $\Theta(\frac{1}{n})$  or  $\Theta(\frac{1}{n^2})$  or  $\Theta(\frac{1}{2^n})$  or similar.

Definition: Two series  $\sum_{n=1}^{\infty} a_n$  and  $\sum_{n=1}^{\infty} b_n$  are said to be equiconvergent if they both converge ~~or~~ both diverge.

Corollary (of Direct Comparison) If  $a_n = \Theta(b_n)$  for positive sequences  $a_n, b_n$   
then  $\sum_{n=1}^{\infty} a_n$  and  $\sum_{n=1}^{\infty} b_n$  are equiconvergent.

Pf: If  $\sum_{n=1}^{\infty} b_n = \infty$ , then also  $\sum_{n=1}^{\infty} \alpha b_n = \infty$  for any constant  $\alpha$ , hence  $\sum_{n=1}^{\infty} a_n = \infty$ .

Similarly, if  $\sum_{n=1}^{\infty} b_n$  converges, then so does  $\sum_{n=1}^{\infty} \beta b_n$ , (for any constant  $\beta$ )  
and hence so does  $\sum_{n=1}^{\infty} a_n$ . 

The following properties of  $\Theta$  notation follow easily:

Proposition: If  $a_n, b_n, c_n, d_n$  are positive sequences with  $a_n = \Theta(b_n)$ ,  $c_n = \Theta(d_n)$

then 1)  $\frac{1}{a_n} = \Theta(\frac{1}{b_n})$

2)  $a_n + c_n = \Theta(b_n + d_n)$

3)  $a_n \cdot c_n = \Theta(b_n \cdot d_n)$ .

4)  $\alpha \cdot a_n = \Theta(b_n)$  for any real  $\alpha > 0$ .

Self-check: Verify these!

Example:  $\sum_{n=1}^{\infty} \frac{n-2}{n^2+1}$

Solution:

Since  $\frac{n}{2} \leq n-2 \leq n$  for  $n \geq 4$  (so  $\Theta(n-2) = n$ )

and  $n^2 \leq n^2+1 \leq 2n^2$  for  $n \geq 1$ , (so  $\Theta(n^2+1) = n^2$ )

we have  $\Theta(\frac{n-2}{n^2+1}) = \Theta(\frac{n}{n^2}) = \Theta(\frac{1}{n})$ ,

so  $\sum_{n=1}^{\infty} \frac{n-2}{n^2+1}$  is equiconvergent to  $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$ . ✓

Another approach, having about the same power, uses limits:

Theorem (Limit Comparison Test)

If  $a_n$  and  $b_n$  are positive real-valued sequences so that

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = L \text{ for some real number } L \text{ with } 0 < L < \infty$$

then

$$\sum_{n=1}^{\infty} a_n \text{ and } \sum_{n=1}^{\infty} b_n \text{ are equiconvergent,}$$

Example:  $\sum_{n=1}^{\infty} \frac{n-2}{n^2+1}$ , again.

Since  $\lim_{n \rightarrow \infty} \frac{\frac{n-2}{n^2+1}}{\frac{1}{n}} = \lim_{n \rightarrow \infty} \frac{n^2-2n}{n^2+1} = \lim_{n \rightarrow \infty} \frac{1 - 2/n}{1 + 1/n^2} = \frac{1-0}{1+0} = 1$ ,

$\sum_{n=1}^{\infty} \frac{n-2}{n^2+1}$  is equiconvergent to  $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$ . ✓

### Proof (of Limit Comparison Test)

By definition of Limit

$$\forall \varepsilon > 0, \exists N \text{ s.t. } [n > N] \Rightarrow \left[ \left| \frac{a_n}{b_n} - L \right| < \varepsilon \right]$$

Thus, for  $n > N$  we have

$$-\varepsilon < \frac{a_n}{b_n} - L < \varepsilon$$

$$\Leftrightarrow (L - \varepsilon)b_n < a_n < (L + \varepsilon)b_n$$

Now, for small enough  $\varepsilon$ , we have  $L - \varepsilon > 0$  (as  $L > 0$ ),

which gives us  $\alpha = (L - \varepsilon)$

$$\beta = (L + \varepsilon) \quad (\text{for this fixed small } \varepsilon)$$

so that for  $n > N$  we have  $\alpha b_n < a_n < \beta b_n$

and in particular  $a_n = \Theta(b_n)$ .

Equiconvergence now follows by the previous Corollary. ■

Example Discuss convergence of  $\sum_{n=1}^{\infty} \frac{1}{2n^2 - 3n + 5}$

Solution 1: Since  $n^2$  is the highest power of  $n$  in the denominator, we try Limit Comparison w/  $\frac{1}{n^2}$ .

$$\lim_{n \rightarrow \infty} \frac{\frac{1}{2n^2 - 3n + 5}}{\frac{1}{n^2}} = \lim_{n \rightarrow \infty} \frac{1}{2 - 3/n + 5/n^2} = \frac{1}{2 - 0 + 0} = \frac{1}{2}$$

and since  $0 < 1/2 < \infty$ , and since  $\frac{1}{2n^2 - 3n + 5}$  is eventually positive the series is equiconvergent to  $\sum_{n=1}^{\infty} \frac{1}{n^2}$ , which converges. ✓

Solution 2 (sketch) Show  $\Theta(2n^2 - 3n + 5) = n^2$  directly, so that  $\Theta\left(\frac{1}{2n^2 - 3n + 5}\right) = \Theta\left(\frac{1}{n^2}\right)$

so that  $\sum \frac{1}{2n^2 - 3n + 5}$  equiconvergent to  $\sum \frac{1}{n^2}$  as in Solution 1. ✓

In series where an exponential or factorial term is present, it is often easier to use one of the following tests: the Root and Ratio Tests.

### Theorem (Ratio Test)

Let  $a_n$  be a positive real sequence.

- 1) If  $\limsup \frac{a_{n+1}}{a_n} < 1$ , then  $\sum_{n=1}^{\infty} a_n$  converges.
- 2) If  $\liminf \frac{a_{n+1}}{a_n} > 1$ , then  $\sum_{n=1}^{\infty} a_n = \infty$
- 3) Otherwise, no information.

We'll prove this a little later. First, an example:

Example Discuss convergence of  $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$

Solution Since  $2^n$  is an exponential, it is worthwhile to try the Ratio Test. We evaluate

$$\lim_{n \rightarrow \infty} \frac{\frac{(n+1)^2}{2^{n+1}}}{\frac{n^2}{2^n}} = \lim_{n \rightarrow \infty} \frac{2^n}{2^{n+1}} \cdot \left(\frac{n+1}{n}\right)^2 = \lim_{n \rightarrow \infty} \frac{2^n}{2^n \cdot 2} \cdot (1+0)^2 = \frac{1}{2}$$

Since  $\frac{1}{2} < 1$ , the series converges. ✓

The Root Test has a similar flavor, and is sometimes easier to apply.

### Theorem (Root Test)

Let  $a_n$  be a positive real sequence.

- 1) If  $\limsup (a_n)^{1/n} < 1$ , then  $\sum_{n=1}^{\infty} a_n$  converges
- 2) If  $\limsup (a_n)^{1/n} > 1$ , then  $\sum_{n=1}^{\infty} a_n = \infty$
- 3) Otherwise, no information.

Example Discuss convergence of  $\sum_{n=1}^{\infty} \left(\frac{n+3}{2n}\right)^n$

Solution Since we have an  $n$ th power, we try the Root Test.

Indeed,

$$\lim_{n \rightarrow \infty} \left( \left( \frac{n+3}{2n} \right)^n \right)^{1/n} = \lim_{n \rightarrow \infty} \left( \frac{n+3}{2n} \right) = \lim_{n \rightarrow \infty} \frac{1}{2} + \frac{3}{2n} = \frac{1}{2} + 0$$

and since  $\frac{1}{2} < 1$ , the series converges. ✓

Proof (of Ratio Test):

1) Let  $L = \limsup \frac{a_{n+1}}{a_n}$ . By definition,

$$\forall \varepsilon > 0, \exists N \text{ s.t. } [n > N] \Rightarrow \left[ \frac{a_{n+1}}{a_n} < L + \varepsilon \right]$$

Thus, in this situation, we have

$$a_{N+k} < a_{N+k-1} \cdot (L + \varepsilon) < a_{N+k-2} (L + \varepsilon)^2 < \dots < a_{N+1} (L + \varepsilon)^{k-1}$$

so the terms of the sequence are eventually bounded above

by a geometric series with ratio  $r = L + \varepsilon$ .

$$\left( \text{as } \sum_{n=N+1}^{\infty} a_n < \sum_{n=N+1}^{\infty} a_{N+1} \cdot (L + \varepsilon)^{n-(N+1)} = \sum_{n=0}^{\infty} a_{N+1} (L + \varepsilon)^n \right)$$


The result follows by taking  $\varepsilon$  small enough that  $r = L + \varepsilon$  is  $< 1$ , as we can do since  $L < 1$ . ✓

2) By definition <sup>of liminf</sup>  $\forall \varepsilon > 0, \exists N \text{ s.t. } [n > N] \Rightarrow \left[ \frac{a_{n+1}}{a_n} > L - \varepsilon \right]$ .

By taking  $\varepsilon$  s.t.  $L - \varepsilon > 1$ , as we can do since  $L > 1$ ,

we see that for large enough  $n$ , we have  $a_{n+1} > a_n$ .

Since  $a_n > 0$ , it follows that  $\lim_{n \rightarrow \infty} a_n > 0$ ,

hence that  $\sum_{n=1}^{\infty} a_n$  diverges. 

The proof of the Root Test follows a similar idea.

(Omitted — see the clear Wikipedia article.)

Note that the general idea of both the Ratio and Root Tests is that the series "acts like" a geometric series with the associated test statistic (like  $\limsup \frac{a_{n+1}}{a_n}$ ).

Anticexamples Consider the Ratio Test <sup>on</sup>  $\sum_{n=1}^{\infty} \frac{1}{n}$  and  $\sum_{n=1}^{\infty} \frac{1}{n^2}$ .

We calculate

$$\lim_{n \rightarrow \infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} = \lim_{n \rightarrow \infty} \frac{n}{n+1} = \lim_{n \rightarrow \infty} \frac{1}{1 + \frac{1}{n}} = 1$$

$$\lim_{n \rightarrow \infty} \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \lim_{n \rightarrow \infty} \left( \frac{n}{n+1} \right)^2 = 1^2 = 1.$$

since  $\sum \frac{1}{n} = \infty$  while  $\sum \frac{1}{n^2}$  converges, we see that the Ratio Test statistic cannot give us useful information when we have a limit of 1. (Or  $\limsup \geq 1$ ,  $\liminf \leq 1$ .)

Instead, we try another test, like Direct Comparison. ✓

We'll do more examples with these tests shortly, after we extend to series with both positive and negative terms.

### C. Absolute convergence vs conditional convergence.

For series with both positive and negative terms, the following allows us to apply the Comparison, Ratio, and Root tests to show convergence,

Theorem For a sequence  $a_n$  of real numbers, if  $\sum_{n=1}^{\infty} |a_n|$  converges, then also  $\sum_{n=1}^{\infty} a_n$  converges.

Proof: We add  $|a_n|$  to get from negative to nonnegative:

Since  $-|a_n| \leq a_n \leq |a_n|$  for any  $n$ ,

also  $0 \leq a_n + |a_n| \leq 2|a_n|$  for any  $n$ .

Now since  $\sum_{n=1}^{\infty} 2|a_n| = 2 \cdot \sum_{n=1}^{\infty} |a_n|$  converges,

by Direct Comparison so does  $\sum_{n=1}^{\infty} a_n + |a_n|$ .

and by Arithmetic of Series,

so does  $\sum_{n=1}^{\infty} a_n = \left( \sum_{n=1}^{\infty} a_n + |a_n| \right) - \left( \sum_{n=1}^{\infty} |a_n| \right)$ .  $\blacksquare$

Definition: If  $\sum_{n=1}^{\infty} |a_n|$  converges, then we say  $\sum_{n=1}^{\infty} a_n$  converges absolutely or absolutely converges.

(By the theorem, an absolutely converging series converges.)

Definition 2: If  $\sum_{n=1}^{\infty} a_n$  converges, but  $\sum_{n=1}^{\infty} |a_n| = \infty$ , then we say  $\sum_{n=1}^{\infty} a_n$  converges conditionally.

\*\*\* All of our tests for convergence of series w/ nonnegative terms are now tests for absolute convergence! \*\*\*

Example: Discuss convergence of  $\sum_{n=1}^{\infty} \frac{(-1)^n \cdot n^{100}}{3^n}$

Solution: Since we have an exponential, try Ratio Test on

$$\left| \frac{(-1)^n \cdot n^{100}}{3^n} \right| = \frac{n^{100}}{3^n}$$

$$\text{Here } \lim_{n \rightarrow \infty} \frac{\frac{(n+1)^{100}}{3^{n+1}}}{\frac{n^{100}}{3^n}} = \lim_{n \rightarrow \infty} \frac{3^n}{3^{n+1} \cdot 3} \cdot \left( \frac{n+1}{n} \right)^{100} = \frac{1}{3} \cdot 1^{100} = \frac{1}{3}$$

and since  $\frac{1}{3} < 1$ , the series converges absolutely.

(that is,  $\sum_{n=1}^{\infty} \left| \frac{(-1)^n \cdot n^{100}}{3^n} \right| = \sum_{n=1}^{\infty} \frac{n^{100}}{3^n}$  converges.)

Corollary:  $\lim_{n \rightarrow \infty} \frac{n^{100}}{3^n} = 0$ .

Recall that  $n!$  is recursively defined by  $0! = 1$ ,  $n! = n \cdot (n-1)!$ .

Example Discuss convergence of  $\sum_{n=0}^{\infty} \frac{(-1)^n \cdot 5^n \cdot (n+2)}{n!}$ .

Solution Since we have exponentials and factorials, the Ratio Test looks like a good test to try. We apply to  $\left| \frac{(-1)^n \cdot 5^n \cdot (n+2)}{n!} \right| = \frac{5^n \cdot (n+2)}{n!}$

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\frac{5^{n+1} \cdot (n+3)}{(n+1)!}}{\frac{5^n \cdot (n+2)}{n!}} &= \lim_{n \rightarrow \infty} \frac{5^{n+1} \cdot 5}{5^n} \cdot \frac{n+3}{n+2} \cdot \frac{n!}{(n+1)!} \\ &= \lim_{n \rightarrow \infty} 5 \cdot \frac{1+3/n}{1+2/n} \cdot \frac{1}{n+1} = 5 \cdot 1 \cdot \frac{1}{\infty} = 0. \end{aligned}$$

Since  $0 < 1$ , the series converges absolutely. ✓

Example Discuss convergence of  $\sum_{n=1}^{\infty} \frac{(-1)^n \sqrt{n+2}}{n^2+3}$

Solution We see clearly that  $\sqrt{n+2} = \Theta(\sqrt{n})$   
 (a)  $\sqrt{n} \leq \sqrt{n+2} \leq \sqrt{3n}$   
 and  $n^2+3 = \Theta(n^2)$   
 (a)  $n^2 \leq n^2+3 \leq 4n^2$

So  $\frac{\sqrt{n+2}}{n^2+3} = \Theta\left(\frac{\sqrt{n}}{n^2}\right) = \Theta\left(\frac{1}{n^{3/2}}\right)$ . Since  $3/2 > 1$ , the series is equiconvergent to a convergent series.

So  $\sum_{n=1}^{\infty} \frac{(-1)^n \cdot \sqrt{n+2}}{n^2+3}$  converges absolutely. ✓

Example Discuss convergence of  $\sum_{n=1}^{\infty} \frac{\sin n}{2^n}$ .

Remark We'll need to know only that  $-1 \leq \sin n \leq 1$ .

Non-solution Although you may see the  $2^n$  and think "Ratio Test",  
 $\frac{|\sin(n+1)|}{|\sin n|}$  is not easy to deal with in limsup  
 and indeed,  $\frac{\sin 2}{\sin 1} \approx 1.08 > 1$ , while  $\frac{\sin 3}{\sin 2} \approx 0.16$ .

Solution 1 (easy): Direct comparison! Since  $-1 \leq \sin n \leq 1$ ,  
 also  $0 \leq |\sin n| \leq 1$ , so  $0 \leq \left| \frac{\sin n}{2^n} \right| \leq \frac{1}{2^n}$ .  
 As  $\sum_{n=1}^{\infty} \frac{1}{2^n}$  converges, so ~~this series converges~~ this series converges absolutely. ✓

Solution 2: The Ratio Test looks hopeless, but we try the Root Test.  
 Since  $0 \leq |\sin n| \leq 1$ , also  $0 \leq |\sin n|^{1/n} \leq 1$   
 and so  $0 \leq \left( \frac{|\sin n|}{2^n} \right)^{1/n} \leq \frac{1}{2}$ .

Thus,

$$\limsup \left( \frac{|\sin n|}{2^n} \right)^{1/n} \leq \frac{1}{2} < 1$$

and so by the Root Test

$\sum_{n=1}^{\infty} \frac{\sin n}{2^n}$  converges absolutely. ✓

General examples  $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$  converges absolutely for all  $p > 1$   
 but either converges conditionally  
 or diverges when  $p \leq 1$ .  
 (as  $\left| \frac{(-1)^n}{n^p} \right| = \frac{1}{n^p}$ ; and by our result on Dirichlet  $p$ -series.

Fact (to be shown later):  $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$  converges (conditionally).

In a later class, you may show  $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = \ln \frac{1}{2} \approx -0.69$ .

The trouble with conditionally convergent series such as  $\sum \frac{(-1)^n}{n}$  is that if we reorder the terms, we may change the sum of the series.

Although this is counter-intuitive — addition is commutative — it is a result of the observations that

$$\sum_{k=0}^{\infty} \frac{(-1)^{2k+1}}{2k+1} = \sum_{k=0}^{\infty} \frac{-1}{2k+1} = -\infty \quad \leftarrow \text{odd terms}$$

$$\sum_{k=1}^{\infty} \frac{(-1)^{2k}}{2k} = \sum_{k=1}^{\infty} \frac{1}{2k} = \infty \quad \leftarrow \text{even terms}$$

and that  $\infty - \infty$  is an indeterminate form.

Example Rearrange the terms of  $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$  to add to 2018.

Solution <sup>First:</sup> Take enough positive terms  $\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \dots$   
so that the sum is (just) over 2018.

Now take enough negative terms  
so that the sum is (just) under 2018.

Then take + terms over 2018 again

Then " - " under " "

Repeat in this process.

As the terms  $\frac{1}{n}$  go to 0, this process will converge at 2018.

Self-check: Why does this argument fail for  $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ ?

Of course, 2018 can be replaced here with any other number.

Moral: The order of summation is important for a conditionally convergent series!!

### D. Alternating series test and estimation

A series whose terms alternate in sign is called an alternating series.

$$\text{Eg) } \sum_{n=1}^{\infty} \frac{(-1)^n}{n} = -1 + 1/2 - 1/3 + 1/4 - \dots$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} = 1 - 1/2 + 1/4 - 1/8 + \dots$$

There is an easy test for convergence of alternating series, but it will not tell us about absolute convergence.

The test also provides an easy-to-apply estimation result, which will be very helpful for numerical computation.

### Theorem (Alternating Series Test and Estimation)

Let  $a_n$  be a positive, decreasing real-valued sequence with  $\lim_{n \rightarrow \infty} a_n = 0$

Then

$$1) \quad \sum_{n=1}^{\infty} (-1)^n \cdot a_n \text{ converges} \quad (\text{Test})$$

$$2) \quad \text{As usual, write } R_N \text{ for } \sum_{n=N+1}^{\infty} (-1)^n \cdot a_n$$

(so  $\sum (-1)^n \cdot a_n = S_N + R_N$ ) for the remainder from the  $n$ th partial sum.

$$\text{Then } |R_N| \leq a_{N+1} \quad (\text{Estimation}).$$

Example Since  $1/n$  is a positive decreasing sequence w/  $\lim_{n \rightarrow \infty} 1/n = 0$ ,  $\sum_{n=1}^{\infty} (-1)^n / n$  converges, (by (1)).

Also,  $-1 + 1/2 = -1/2$  is within  $1/3$  of the limit sum  
 $-1 + 1/2 - 1/3 = -5/6$  is within  $1/4$   
 $-1 + 1/2 - 1/3 + 1/4 = -7/12$  is within  $1/5$ , and so on.

Proof (of AST/E):

1) We split into even/odd subseqs for the partial sums  $S_n$ .

Even:  $S_{2k} = S_{2k-2} + \underbrace{a_{2k} - a_{2k-1}}_{\text{negative}} \leq S_{2k-2}$  (as  $a_{2k} \leq a_{2k-1}$ )

so the even subsequence  $S_{2k}$  is decreasing.

Odds:  $S_{2k+1} = S_{2k-1} + \underbrace{a_{2k} - a_{2k+1}}_{\text{positive}} \geq S_{2k-1}$  similarly

so the odd subsequence  $S_{2k+1}$  is increasing.

Also,  $-a_1 = S_1 \leq S_{2k+1} \leq S_{2k+2} \leq S_2 \leq -a_1 + a_2$

odd increasing

adding  $+a_{2k+2}$

even decreasing

so both are bounded between  $-a_1$  and  $-a_1 + a_2$ .

Now the MST tells us that  $S_{2k}$  and  $S_{2k+1}$  converge to  $L_{\text{even}}$  and  $L_{\text{odd}}$ , respectively.

But  $L_{\text{even}} - L_{\text{odd}} = \lim_{k \rightarrow \infty} S_{2k} - S_{2k+1} = \lim_{k \rightarrow \infty} a_{2k+1} = 0$   
by hypothesis.

So  $L_{\text{even}} = L_{\text{odd}} = L$ . By a result proved in tutorial, the series converges to  $L$ .

2) Using the results of (1), since  $S_{2k+1}$  is increasing, while  $S_{2k}$  is decreasing,

we have for any  $k$  that  $S_{2k+1} \leq L \leq S_{2k}$ .

Now  $0 \leq L - S_{2k+1} \leq S_{2k+2} - S_{2k+1} = a_{2k+2}$  (odd remainders)

and  $\underbrace{S_{2k+1} - S_{2k}}_{-a_{2k+1}} \leq L - S_{2k} \leq 0$  (even remainders)

In either case, we have  $|L - S_n| = |R_n| \leq a_{n+1}$ , as desired.  $\blacksquare$

In practice, Alternating Series Estimation is even more useful than the Alternating Series Test.

- Since the AST does not tell us about absolute convergence, it is generally our last resort, for possibly-conditionally-convergent series.
- The ASE is useful and easy to apply even when we prefer another test for convergence.

Example Discuss convergence of  $\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2+1}$ . If it converges, estimate the sum to within an accuracy of 0.1.

Solution We first check absolute convergence, using a Comparison Test.

Since  $0 \leq \left| \frac{(-1)^n}{n^2+1} \right| = \frac{1}{n^2+1} \leq \frac{1}{n^2}$ , and  $\sum \frac{1}{n^2}$  converges (as  $2 > 1$ ), we see that the series absolutely converges.

Now we apply ASE. The series is alternating for all  $n$ , and  $\frac{1}{(n+1)^2+1} \leq \frac{1}{n^2+1}$  (as  $(n+1)^2+1 \geq n^2+1$ ) for all  $n \geq 0$ .

Finally  $\lim_{n \rightarrow \infty} \frac{1}{n^2+1} = 0$  since the series absolutely converges. (or by AOL).

So ASE applies.

Also,  $\frac{1}{n^2+1} \leq 0.1 = \frac{1}{10}$  first at  $n=3$ .

Thus,  $\sum_{n=0}^3 \frac{(-1)^n}{n^2+1} = 1 - \frac{1}{2} + \frac{1}{5} = 0.7$  is accurate within 0.1 ✓

Example Discuss convergence of  $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ . If it converges, estimate the sum to within an accuracy of 0.1.

Solution Since  $\left| \frac{(-1)^n}{n} \right| = \frac{1}{n}$  and  $\sum_{n=1}^{\infty} \frac{1}{n}$  is a main example of an interestingly divergent series,  $\sum \frac{(-1)^n}{n}$  does not converge absolutely.

However,  $\frac{1}{n}$  is clearly a positive decreasing sequence  
with  $\lim_{n \rightarrow \infty} \frac{1}{n} = 0!!$

Thus, the AST applies, and

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$  converges conditionally.

To estimate to within  $0.1 = 1/10$ , we stop just before  
the  $n=10$  term of  $1/n$ .

$$\text{Our estimate is } \sum_{n=1}^9 \frac{(-1)^n}{n} = -1 + 1/2 - 1/3 + 1/4 - 1/5 + 1/6 - 1/7 + 1/8 - 1/9 \\ = \frac{-1879}{2520} \approx -0.75 \quad \checkmark$$

Example Discuss convergence of  $\sum_{n=0}^{\infty} \frac{(-1)^n \cdot 4}{2n+1}$ . If it converges, describe how many terms are required to estimate the series within 0.1.

Solution Since (by the homework)  $2n+1 = \Theta(n)$

and  $4 = \Theta(1)$ , we have  $\Theta\left(\frac{4}{2n+1}\right) = \Theta\left(\frac{1}{n}\right)$ ,

and as  $\sum_{n=1}^{\infty} \frac{1}{n}$  diverges, the series does not  
converge absolutely.

But as  $\frac{4}{2(n+1)+1} \leq \frac{4}{2n+1}$  (since  $2(n+1)+1 \geq 2n+1$ )  
the ~~terms are~~ decreasing. Clearly  $\frac{4}{2n+1}$  is positive for all  $n$ .

And  $\lim_{n \rightarrow \infty} \frac{4}{2n+1} = 0$  by A.o.L.

Thus, the AST applies and  $\sum_{n=0}^{\infty} \frac{(-1)^n \cdot 4}{2n+1}$  converges.

To estimate within 0.1, we use the ASE. As

$$\frac{4}{2n+1} \leq 0.1 \Leftrightarrow 40 \leq 2n+1 \Leftrightarrow 19.5 \leq n$$

So our estimate is with  $\sum_{n=0}^{19} \frac{(-1)^n \cdot 4}{2n+1}$ . ✓

Fact:  $\sum_{n=0}^{\infty} \frac{(-1)^n \cdot 4}{2n+1} = \pi$ , although this might not be so easy  
to see with the first few partial sums!

N	0	1	2	3	4	5	...
$S_N$	4	$2\frac{2}{3}$	$3\frac{7}{5}$	$2\frac{99}{105}$	$3\frac{107}{315}$	$2\frac{3382}{3465}$	...
$a_{n+1}$	$4/3$	$4/5$	$4/7$	$4/9$	$4/11$	$4/13$	

There's also a series that turns out to converge to  $1/e$ .

Example Discuss convergence of  $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ . If it converges, estimate the sum to within accuracy of 0.01.

Solution To test absolute convergence, we use the Ratio Test:

$$\lim_{n \rightarrow \infty} \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \lim_{n \rightarrow \infty} \frac{n!}{(n+1)!} = \lim_{n \rightarrow \infty} \frac{1}{n+1} = 0 < 1$$

So the series converges absolutely.

To estimate, we notice that  $\frac{1}{n!} > 0$  for all  $n$

and that, since  $n!$  is increasing,  $\frac{1}{n!}$  is decreasing.

That  $\lim_{n \rightarrow \infty} \frac{1}{n!} = 0$  follows from absolute convergence.

So the AST applies.

To get accuracy within  $0.01 = \frac{1}{100}$ , we notice that

$$\frac{1}{5!} = \frac{1}{120} < \frac{1}{100}. \text{ So summing through } n=4 \text{ suffices!}$$

$$\sum_{n=0}^4 \frac{(-1)^n}{n!} = 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} = \frac{3}{8} = 0.375. \checkmark$$

Self-check: How many terms would you need for accuracy within  $\frac{1}{1000} = 0.001$ ?

Fact:  $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = 1/e \approx 0.367879$

Example Discuss convergence of  $\sum_{n=1}^{\infty} (-1 - \frac{1}{n})^n$

Solution Notice that  $(-1 - \frac{1}{n})^n = (-1)^n \cdot (1 + \frac{1}{n})^n$ , so the series is alternating. But since  $\liminf (1 + \frac{1}{n})^n \geq 1$

(as the terms are  $\geq 1$  in absolute value),

the AST does not apply.

Indeed, since the even terms are  $\geq 1$   
and the odd " "  $\leq -1$ ,

we see that  $\lim_{n \rightarrow \infty} (-1)^n \cdot (1 + \frac{1}{n})^n$  does not converge, and so by the  $n$ th term test, the series does not converge either. ✓

Remark The AST is almost the converse of the  $n$ th term test that you might have wished for at some points. We need also that the series is alternating and that the terms are (eventually) decreasing; and of course, it doesn't tell us anything about absolute convergence.

### Strategies for testing series convergence:

There is no algorithm to determine whether a series converges or diverges, but there are strategies for using our tests. Let's summarize (For series  $\sum_{n=0}^{\infty} a_n$ .)

- The  $n$ th term test says if  $\lim_{n \rightarrow \infty} a_n \neq 0$  then the series diverges. "Usually" unhelpful, but can save a lot of work when it helps.  
Easy to check if you're fluid w/ limits.
- The Ratio (or Root) Test will only help if there's some kind of exponential/factorial/similar in the series terms.  
It's easy to check: if  $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} < 1$  then series converges  
 $> 1$  " " diverges  
and is good to try if you have some exponential/factorial and the test statistic is not difficult to compute.

(Recall also limsup / lim inf version!)

- Direct / Limit Comparison Tests are powerful, but difficult to use in that you must "tune" the test by finding a series to compare with. The "beg, then" approach will often yield insight to this problem (and occasionally will answer the problem entirely..).
- The Alternating Series Test, since it does not prove or disprove absolute convergence, is the test of last resort for an alternating series, to be used after you have shown the series not to converge absolutely. (It is generally easy to use in this case.)

### Interlude: Countable + Uncountable sets

Infinite sets can have surprising properties.

For instance, although

$$\mathbb{N} \subsetneq \mathbb{Z} \subsetneq \mathbb{Q} \subsetneq \mathbb{R} \subsetneq \mathbb{C},$$

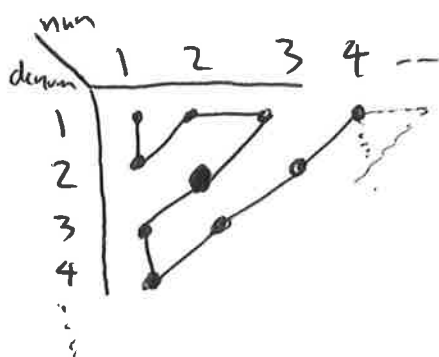
we have exhibited a sequence  $a_n$  which

takes on every integer value exactly once

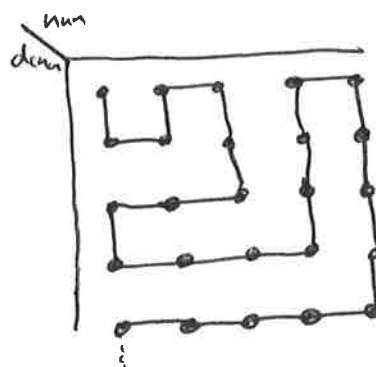
$$(a_n) = (0, 1, -1, 2, -2, 3, -3, 4, -4, \dots)$$

and sketched a sequence  $r_n$  that takes on every rational value exactly once.

Recall this was constructed by alternating positive/negative as we did w/  $\mathbb{Z}$ , and "walking through" a table of numerator/denominator pairs in a manner such as



or



etc.

Thus, although  $\mathbb{N} \subsetneq \mathbb{Z} \subsetneq \mathbb{Q}$ ,

we can find a 1-1 correspondence (a "bijection")  
between  $\mathbb{N}$  and  $\mathbb{Z}$   
and between  $\mathbb{N}$  and  $\mathbb{Q}$ ,

Self-check: Can you find a 1-1 correspondence between  $\mathbb{Z}$  and  $\mathbb{Q}$ ?

Definition: If a set  $S$  is the range of some sequence  
then we say  $S$  is countable.

Prop:  $\mathbb{N}$ ,  $\mathbb{Z}$ ,  $\mathbb{Q}$  are countable, as is any finite set.

Pf: We've seen  $\mathbb{Z}$ ,  $\mathbb{Q}$  already.

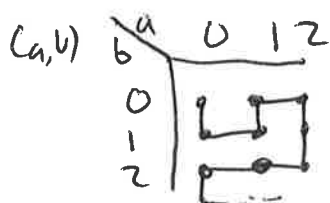
$\mathbb{N}$  follows from the sequence  $a_n = n$ .

A finite set is the range of the sequence which  
enumerates the set in some order, then repeats  
the last element infinitely often

e.g.  $\{0, 2, 3, 5\}$  and the sequence  $(0, 2, 3, 5, 5, 5, \dots)$   $\square$

Prop: The set  $\mathbb{N}^2 =$  all ordered pairs of natural numbers  
is countable.

Pf: Apply the same argument as for  $\mathbb{Q}$ !

 $\square$

Remark: It is easy to see that a set is countable and infinite (countably infinite) if and only if it is in 1-1 correspondence with  $\mathbb{N}$ .

A set that is not countable is (unsurprisingly) called uncountable. The main thing I want to tell you in this Introduce is:

Theorem:  $\mathbb{R}$  is uncountable.

Thus, although  $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$  are all of the "same size",  $\mathbb{R}$  is much bigger. I'll give you 2 proofs.

Proof 1: (By Cantor Diagonalization).

Suppose for contradiction that  $(a_n)$  is a sequence whose range is  $\mathbb{R}$ .

Then for each  $n$ , the number  $a_n$  has a decimal expansion ~~(following the 0)~~

Let  $d_n$  be the  $n$ th digit of  $a_n$ , so that if e.g.

$a_0 = 1. \textcircled{4} 1 3 2 \dots$	$d_0 = 4$
$a_1 = 0. 2 \textcircled{3} 5 6 \dots$	$d_1 = 3$
$a_2 = 7. 7 1 \textcircled{7} 1 7 \dots$	$d_2 = 7$
$\vdots$	$\vdots$

If  $d$  is a number 0-9, then let  $\bar{d} = \begin{cases} 0 & \text{if } d \neq 0 \\ 1 & \text{if } d = 0 \end{cases}$

Finally, let  $a$  be the number  $\textcircled{0. \dots}$  whose  $n$ th digit following the  $.$  is  $\bar{d}_n$ . So in the above e.g., we'd have

$$a = 0. \bar{4} \bar{3} \bar{7} \dots = 0.000\dots$$

Now we ask: where is  $a$  in our sequence?


$a \neq a_0$ , since  $a$  differs at 0th decimal place.

$a \neq a_1$  — — — — — 1st — — —

$a \neq a_2$  — — — — — 2nd — — —

$\vdots$

Since  $a$  differs from each  $a_n$  in (at least) the  $n$ th decimal place,  $a$  is not in the range of our sequence.

As  $a$  is a real number, and the range of  $a_n$  was  $\mathbb{R}$ , this is a contradiction!  $\#$  

Remarks: A similar proof approach w/ Cantor Diagonalization is used in high-level Computer Science classes to show that no computer program can be written to check if another computer program will terminate.

While Cantor Diagonalization yields a nice proof, the other proof that I'll show you ties in better to ANA-I. I'll start with a Lemma, whose proof I'll defer.

Lemma\*: If the interval  $[0, 3]$  is contained in the union of a sequence  $(a_i, b_i)$  of open intervals (that is,  $[0, 3] \subseteq \bigcup_{i=0}^{\infty} (a_i, b_i)$ )

then

$$3 \leq \sum_{i=0}^{\infty} b_i - a_i$$

(that is, length of  $[0, 3]$  is  $\leq$  the sum of lengths of covering intervals.)

Proof: (that  $\mathbb{R}$  is uncountable, based on the Lemma).

Suppose that  $c_n$  is a sequence having range of  $\mathbb{R}$ .  
Recall that  $\sum_{n=0}^{\infty} \frac{1}{2^n} = \frac{1}{1-1/2} = 2$ .

So let  $d_n$  be the subsequence of  $c_n$  having values in  $[0, 3]$   
and for each  $n$ , let  $(a_n, b_n)$  be an interval of  
length  $1/2^n$ , so that  $b_n - a_n = 1/2^n$ ,  
and so that  $d_n$  lies on the interval.

(E.g., we could take  $a_n = d_n - 1/2^{n+1}$ ,  $b_n = d_n + 1/2^{n+1}$ .)

But now each element of  $[0, 3]$  is some  $d_n$ , so lies on some interval.

Thus,  $[0, 3] \subseteq \bigcup_{n=0}^{\infty} (a_n, b_n)$

so the Lemma tells us that  $3 \leq \sum_{n=0}^{\infty} b_n - a_n = \sum_{n=0}^{\infty} \frac{1}{2^n} = 2$ ,

yielding our desired contradiction.  $\blacksquare$

To prove the Lemma, I'll first show that it holds if,  
instead of an infinite # of intervals, I have some finite number  
of them.

Lemma \*: If the interval  $[0, a]$  is contained in the union  
of open intervals  $(a_0, b_0), (a_1, b_1), \dots, (a_n, b_n)$   
then  $a \leq \sum_{i=0}^n b_i - a_i$ .

Proof: We proceed by induction on  $n$ .

Base case  $n=0$ . Here we have  $[0, a] \subseteq (a_0, b_0)$

so  $a_0 < 0$  and  $b_0 > a$ ,

so  $b_0 - a_0 > a - 0$  ✓

Induction step: We assume the result for  $n$  intervals (and any  $a$ ) and prove it for  $n+1$  intervals.

$$\text{Since } [0, a] \subseteq \bigcup_{i=0}^{n+1} (a_i, b_i),$$

in particular  $a$  is in one of the intervals.

Wlog, let  $a$  be in  $(a_{n+1}, b_{n+1})$

(otherwise, renumber the intervals.)

$$\text{Notice that } [0, a] \setminus (a_{n+1}, b_{n+1}) = [0, a_{n+1}].$$

Thus,

$$[0, a_{n+1}] \subseteq \bigcup_{i=0}^n (a_i, b_i)$$

so by induction,

$$a_{n+1} \leq \sum_{i=0}^n b_i - a_i. \quad \text{As also } a \leq b_{n+1},$$

we get

$$a \leq \left( \sum_{i=0}^n b_i - a_i \right) + b_{n+1} - a_{n+1}, \text{ as desired. } \square$$

What about an infinite collection of intervals?

Lemma \* will follow from Lemma \*' and the following theorem (which we defer further):

Theorem If a closed interval  $[a, b]$  is contained in a union of open intervals  $(a_i, b_i)$  (that is,  $[a, b] \subseteq \bigcup_{i=0}^{\infty} (a_i, b_i)$ ) then we can select some finite set of indices  $S$  so that  $[a, b] \subseteq \bigcup_{i \in S} (a_i, b_i)$

Example:  $[0, 2] \subseteq (-1, 1/10) \cup \bigcup_{i=1}^{\infty} (1/2^i, 3).$

Here  $(-1, 1/10)$  and  $(1/16, 3)$

will suffice. ✓

This Theorem and its proof will be a main topic of the next section.