Analysis 1 Homework 9

A. Calculate each of the following limits, or explain why it does not converge. You may use any theorems that were stated in lecture.

i) $\lim_{x\to 1+} |x-2|/(x-2)$

ii) $\lim_{x\to 1+} |x-1|$

iii) $\lim_{x\to 1+} |x-1|/(x-1)^2$

i) $\lim_{x\to 1^+} |x-2|/(x-2) = \begin{bmatrix} x-2<0 \\ x\to 1^+ \\ |x-2| = -x+2 \end{bmatrix} \lim_{x\to 1^+} \frac{-x+2}{x-2} = \lim_{x\to 1^+} \frac{-(x-2)}{x-2} = -1$

ii) $\lim_{x\to 1^+} |x-1| = \left[\underset{x-1>0}{x\to 1^+} \Rightarrow |x-1| = x-1 \right] = \lim_{x\to 1^+} x-1 = 0$

iii) $\lim_{x\to 1^+} |x-1|/(x-1)^2 = \lim_{x\to 1^+} \frac{\cancel{x-1}}{(x-1)^2} = \lim_{x\to 1^+} \frac{1}{x-1} = \frac{1}{0^+} = +\infty$

B. i) Prove directly from the definitions that if $g: \mathbb{R} \to [1,\infty)$ is a function so that $\lim_{x\to c} g(x) = L$, then $\lim_{x\to c} 1/g(x) = 1/L$

ii) Prove the same result of the previous part, using Relating Sequences to Functions.

i) $g: \mathbb{R} \to [1,\infty)$

$\lim_{x\to c} g(x) = L$

$\lim_{x\to c} \frac{1}{g(x)} = \frac{1}{L}$

$\forall \varepsilon>0\ \exists \delta>0\ \forall x \neq c\ \ |x-c| < \delta_1 \Rightarrow |g(x) - L| < \varepsilon$

Let $\varepsilon > 0$ $\exists \delta = \delta_1$ and let $x \neq c$, $|x-c| < \delta$

$\Rightarrow \left| \frac{1}{g(x)} - \frac{1}{L} \right| = \frac{|L - g(x)|}{g(x) \cdot L} < \frac{\varepsilon}{1 \cdot 1} = \varepsilon$

ii) $\lim_{x\to c} g(x) = L$

$\Leftrightarrow \lim_{n\to\infty} g(x_n) = L \qquad \lim_{n\to\infty} x_n = c$

$\lim_{x\to c} \frac{1}{g(x)} = L \qquad \forall x_n \to c \Rightarrow \lim_{n\to\infty} \frac{1}{g(x_n)} = \frac{1}{L}$

$\Rightarrow \lim_{n\to\infty} \frac{1}{g(x_n)} = \lim_{n\to\infty} \frac{1}{L} = \frac{1}{L}$

C. Prove that $\lim_{x\to c} f(x) = \infty$ if and only if for every sequence a_n that is in the domain of f and converges to c (but never equal to c), we have $\lim_{n\to\infty} f(a_n) = \infty$. (That is, prove a version of Relating Sequences to Functions for infinite limits. Make sure you handle both directions of the if and only if!)

($\Rightarrow$) Let $\lim_{x\to c} f(x) = \infty \Leftrightarrow \forall N>0 \ \exists \delta \ \forall x \neq c \ |x-c| < \delta \Rightarrow$

$\Rightarrow f(x) > N$ and let (a_n), $a_n \in D_f$ and $\lim_{n\to\infty} a_n = c \Leftrightarrow$

$\Leftrightarrow \forall \varepsilon \ \exists n_0 \ \forall n > n_0 \ |a_n - c| < \varepsilon$

$\lim_{n\to\infty} f(a_n) = \infty \quad \forall N>0 \ \exists n_0 \ \forall n > n_0$

Let $N>0$ $\exists n_0$ and let $n > n_0$ $(|a_n - c| < \varepsilon)$ $x = a_n$ $|x - c| < \delta$

$\Rightarrow f(a_n) = f(x) > N$

($\Leftarrow$) If $\forall a_n \ \lim_{n\to\infty} a_n = c \quad \lim_{n\to\infty} f(a_n) = \infty \Rightarrow \lim_{x\to c} f(x) = \infty$

Let $\forall a_n \ \lim_{n\to\infty} a_n = c$, it holds $\lim_{n\to\infty} f(a_n) = \infty$, and if we assume that $\lim_{x\to c} f(x) \neq \infty$

$\neg(\forall N>0 \ \exists \delta > 0 \ \forall x \neq c \ |x-c| < \delta \Rightarrow f(x) > N)$

$\exists N>0 \ \forall \delta > 0 \ \exists x \neq c \ |x-c| < \delta \wedge f(x) \leq N$

$\exists N>0 \ \forall \delta = \frac{1}{N} \ \exists x_n = a_n \neq c \ |a_n - c| < \delta \wedge f(a_n) \leq N$

This is in contradiction with $\lim_{n\to\infty} f(a_n) = \infty$

D. Which of the following families of real functions are algebras of functions? (For each, either show it is an algebra of functions, or else identify a necessary property that is not satisfied)

i) Polynomials of degree at most 2. (That is polynomials of the form ax^2+bx+c, where any or all of a, b, c may be 0.

ii) Functions f such that $f(x) \leq 6$

iii) Functions f that are everywhere defined, and that are continuous at the point $c=2$.

i) f_1 and $f_2 \in \mathcal{P}_2 = \{ f \mid f$ is polinom with degree at most $2 \}$

$$\mathcal{P}_2 = \{ f \mid f = ax^2+bx+c,\ a, b, c \in \mathbb{R} \}$$

$f_1(x),\ f_2(x) \in \mathcal{P}_2$

$\Rightarrow f_1(x) = a_1 x^2 + b_1 x + c_1 \qquad f_2(x) = a_2 x^2 + b_2 x + c_2$

$\Rightarrow (f_1 + f_2)(x) = f_1(x) + f_2(x) = a_1 x^2 + b_1 x + c_1 + a_2 x^2 + b_2 x + c_2 =$

$$= \underbrace{(a_1 + a_2)}_{a} x^2 + \underbrace{(b_1 + b_2)}_{b} x + \underbrace{(c_1 + c_2)}_{c} =$$

$$= ax^2 + bx + c \in \mathcal{P}_2$$

ii) $\mathcal{P}_6 = \{ f \mid f(x) \leq 6 \}$

If $f_1(x), f_2(x) \in \mathcal{P}_6$

$f_1(x) + f_2(x) \in \mathcal{P}_6$

Doesn't hold, because:

for example $f_1(x) = 5 \leq 6 \Rightarrow f_1 \in \mathcal{P}_6$

$f_2(x) = 5 \leq 6 \Rightarrow f_2 \in \mathcal{P}_6$

but $(f_1 + f_2)(x) = f_1(x) + f_2(x) = 5 + 5 = 10 > 6$

$\Rightarrow f_1 + f_2 \notin \mathcal{P}_6$

iii) $N_2 = \{ f \mid f : \mathbb{R} \to \mathbb{R}$ and f in $x_0 = 2 \}$

$f_1, f_2 \in N_2$

$\Rightarrow (f_1 + f_2)(x) = f_1(x) + f_2(x)$

$f_1 + f_2 : \mathbb{R} \to \mathbb{R}$

E. Prove that $f(x) = x \cdot |x|$ is continuous at all points c in $\mathbb{R}$.

$\lim_{x \to x_0} f(x) = f(x_0)$

$\lim_{x \to c} f(x) = f(c)$

$\lim_{x \to c} x \cdot |x| = \lim_{x \to c} x \cdot x = \lim_{x \to c} x^2 = c^2 = c \cdot c = c \cdot |c| = f(c)$

def: $\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x \ |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$

$\varepsilon > 0 \ \exists \delta > 0 \ \forall x \ |x - x_0| < \delta$

$\lim_{x \to c} x|x| = \lim_{x \to c} x(-x) = -\lim_{x \to c} x^2 = -c^2 = c(-c) = c \cdot |c| = f(c)$

$\Rightarrow |f(x) - f(x_0)| = |x - x_0| < \delta = \varepsilon$

$\Rightarrow f(x) = x$

$\lim_{x \to 0^-} x|x| = \lim_{x \to 0^-} x(-x) = -\lim_{x \to 0^-} x^2 = -0^2 = 0 = c \cdot |c| = f(0)$ for $c = 0$

$\lim_{x \to 0^+} x|x| = \lim_{x \to 0^+} x \cdot x = \lim_{x \to 0^+} x^2 = 0^2 = 0 = c \cdot |c| = f(c) = f(0)$