| Scores | |
|---|---|
| 1. | |
| 2. | |
| 3. | |
| 4. | |
| 5. | |
| 6. | |
| 7. | |
| Total: | |

ANA-I Foundations of Analysis
Final Examination A – 31 Jan 2022

Name ________________________________

**General Instructions:** Please answer the following, showing all your work and writing neatly. You may have 1 handwritten A4-sized sheet of paper, but no other notes, books, or calculators.
110 total points.

1. (6 points each) Calculate the following limits, or explain why they diverge. You may use any theorems we have proved in class or on homework.
   (a) $\displaystyle\lim_{x\to 5-} \frac{x+2}{x+5}$
   (b) $\displaystyle\lim_{x\to 5-} \frac{x-2}{x-5}$
   (c) $\displaystyle\lim_{n\to\infty} \frac{1-n-n^2}{1-2n-4n^2}$
   (d) $\displaystyle\lim_{n\to\infty} \sin\left(\frac{n+1}{n^2+1}\right)$
   (e) $\displaystyle\lim_{n\to\infty} \frac{\sin(n+1)}{n^2+1}$

2. (6 points each) Series
   (a) Find the exact value that $\displaystyle\sum_{n=1}^{\infty} \sqrt{n} - \sqrt{n+2}$ converges to, or else conclude that the series diverges.
   (b) Determine whether $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \cdot n^2 + 2}{(n-1)!}$ converges absolutely, converges conditionally, or diverges.

3. (6 points each) Examples. Justify your answers briefly.
   (a) Is $\displaystyle\sum_{n=1}^{\infty}(\frac{1}{n} - \frac{1}{n+1})$ conditionally convergent or absolutely convergent? Explain your answer!
   (b) Find the Dedekind cut for $1+\sqrt[3]{2}$. Your answer should not refer directly to any irrational numbers.
   (c) Let $S$ be the set of points $(x, y)$ in $\mathbb{R}^2$ so that $x^2 + 4y^2 = 1$. Explain why $S$ is sequentially compact.
   (d) Let $T = \{(0, y) : y \in \mathbb{Q}\}$. Explain why $T$ is not an open subset of $\mathbb{R}^2$.

(-see reverse side-)

4. (10 points) Let $a_n$ be recursively defined by $a_0 = 3$, $a_{n+1} = a_n^2 - a_n$ for $n \geq 0$. Find $\lim_{n\to\infty} a_n$.

5. (12 points) Show directly from definition that if $\lim_{x\to 1-} f(x) = 2$ and $\lim_{x\to 1-} g(x) = 3$, then $\lim_{x\to 1-} f(x) + g(x) = 5$.

6. (12 points) Estimate the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ to within an accuracy of 0.5.

   Hint: Find an upper bound on the remainders of partial sums of the series. One method begins by recalling that $\dfrac{1}{n^2} \leq \left(\dfrac{2}{n} - \dfrac{2}{n+1}\right)$ for $n \geq 1$.

7. (10 points) Show that $f(x) = \dfrac{1}{x^2+1}$ is uniformly continuous on the entire real line.