IV A little topology of $\mathbb{R}$, and of other metric spaces.

We've discussed so far sequences. These are functions $\mathbb{N} \to \mathbb{R}$.
Shortly, we'll discuss limit behavior of functions $\mathbb{R} \to \mathbb{R}$.

But just as understanding a bit about complex sequences and metric space sequences helped us understand real sequences, so will some broader perspective help us understand real functions.

A. Open and closed sets

Recall that the definition of limit ($\lim_{n\to\infty} s_n = s$) says that, for large enough values of n, s_n is within a distance ε of the limit s.

For $\mathbb{R}$, this reads as

$$\forall \varepsilon > 0, \ \exists N \text{ s.t. } \underbrace{[n > N]}_{\substack{n \text{ large enough} \\ (\text{depends on } \varepsilon)}} \implies \underbrace{\left[|s_n - s| < \varepsilon \right]}_{\substack{s_n \text{ close to } s \\ (\text{within } \varepsilon)}}$$

For an arbitrary metric space (M, d), the definition reads as

$$\forall \varepsilon > 0, \ \exists N \text{ s.t. } [n > N] \implies [d(s_n, s) < \varepsilon].$$

We're going to focus on the last part of this definition, and introduce notation.

<u>Definition</u> For a point $a \in \mathbb{R}$ and $\varepsilon > 0$, the <u>ε-ball centered at a</u> or <u>ε-neighborhood of a</u> is the set $\{x \in \mathbb{R} : |x - a| < \varepsilon\}$.

Thus, the ε-neighborhood of a is all points within ε of a.
Write $B_\varepsilon(a)$ for this set, and notice that $B_\varepsilon(a) = (a - \varepsilon, a + \varepsilon)$, an open interval.

(Of course, the interval statement is special to $\mathbb{R}$!!)

With this notation, the limit definition (of $\lim_{n\to\infty} s_n = s$) becomes:

$$\forall \varepsilon>0,\ \exists N \text{ s.t. } [n>N] \Rightarrow [s_n \in B_\varepsilon(s)],$$

Since all of the ingredients for ε-balls make sense in arbitrary metric spaces, it is hard to resist generalizing them!

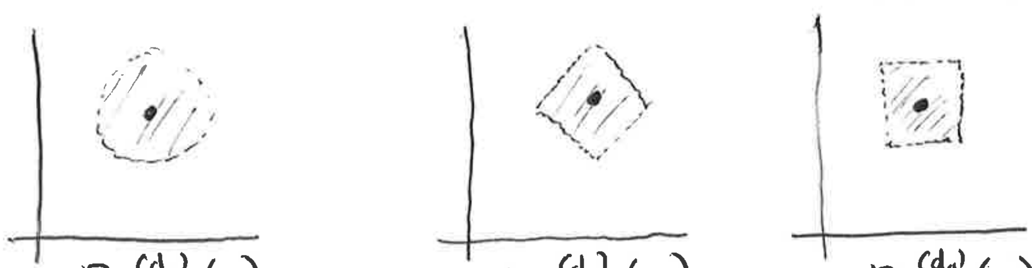
<u>Definition</u> For a metric space (M,d), a point $a\in M$ and a real number $\varepsilon>0$ the <u>ε-ball centered at a</u> or <u>ε-neighborhood of a</u> in M is $B_\varepsilon(a) := \{x\in M : d(x,a)<\varepsilon\}$, the set of all points properly within distance ε of a.

The restatement of the definition of limit is exactly the same for an arbitrary metric space as for $\mathbb{R}$.

<u>Notation</u> Sometimes, you may want to emphasize the metric that you are using. In this case, write $B_\varepsilon^{(d)}(a)$ for the ε-ball with metric d.

<u>Examples</u> We previously examined 3 metrics in $\mathbb{R}^2$: the Euclidean metric d_2, the Manhattan metric d_1, and the max metric d_0.

The ε-balls look like the following. (Compare w/ unit circles earlier considered).



$B_\varepsilon^{(d_2)}(\cdot)$ $\quad$ $B_\varepsilon^{(d_1)}(\cdot)$ $\quad$ $B_\varepsilon^{(d_0)}(\cdot)$

In all 3 cases, $B_\varepsilon(\bullet)$ is the inside of the shaded region, The boundary is <u>not</u> included — this is because B_ε is defined with $<$, and not with $\leq$.

In $\mathbb{R}$, any (bounded) open interval can be written as an ε-ball of its center, and the union of 2 overlapping open intervals is another open interval. (E.g. $(1;3) \cup (2,5)$.)
In metric spaces like $\mathbb{R}^2$, however, we may have other "shaded regions excluding boundary", such as:

How can we describe these?

<u>Definition</u> A set A contained in a metric space is <u>open</u> if $\forall a \in A$, $\exists \varepsilon > 0$ s.t. $B_\varepsilon(a) \subseteq A$.

<u>Picture</u>

One possible B_ε for one point in an open set

We'll mostly talk about open sets in $\mathbb{R}$ or occasionally $\mathbb{C}$, but sometimes it will be helpful to consider $\mathbb{R}^2$ or even more exotic metric spaces.

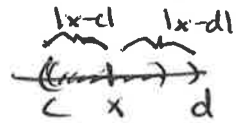
<u>Examples: open sets in $\mathbb{R}$</u>

1) $\mathbb{R}$ is an open set (since <u>any</u> ε-ball is contained in $\mathbb{R}$).

2) Any open interval (c,d) is an open set, since if $x \in (c,d)$, we can take

$$\varepsilon = \min\{|x-c|, |x-d|\}$$

and then $B_\varepsilon(x) \subseteq (c,d)$

Illustration: 

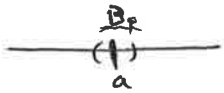
(~~~ is $B_\varepsilon(x)$.)

As the same construction works for any x (of course, with different ε's), (c,d) is open.

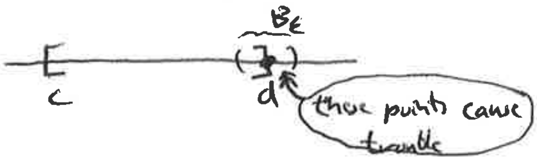
3) The empty set $\emptyset$ is open, vacuously. (there are no points in $\emptyset$, so the $\forall$ condition is automatic.)

<u>Antiexamples: Some subsets of $\mathbb{R}$ that are not open</u>

A) A singleton set $\{a\}$ is <u>not</u> open, since for any $\varepsilon > 0$ no matter how small, there are points in $B_\varepsilon(a) = (a-\varepsilon, a+\varepsilon)$ that are smaller than a. (and also of course larger " ").



B) For similar reasons, a closed interval $[c,d]$ is <u>not</u> open. The "interior points" of (c,d) cause no problem, but for any $\varepsilon > 0$, there are points of $B_\varepsilon(d)$ that are larger than d.



Examples in $\mathbb{R}^2$ of open sets:

Notation collision: open interval vs ordered pair

1) The open square

$S :=$ the set of ordered pairs (x,y) s.t. $0<x<1$ and $0<y<1$.

is an open set in the d_2 metric, because for any point (a,b) in S, we can take $\varepsilon = \min\{|a-0|, |a-1|, |b-0|, |b-1|\}$, that is, ε is the distance to the closest side.

E.g. 

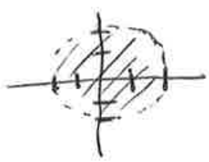
Now $B_\varepsilon(a,b)$ is contained in S, as required.

2) Any ε-ball in any metric space is open in that metric space (Homework: Use the Δ-inequality to check this!)

In particular, a disc such as

$\{(x,y) : x^2+y^2<4\}$

is open in the d_2 metric.



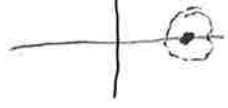
3) It appears that the shape pictured earlier is open (in the d_2 metric) since every point has a small open ball around it. (If we'd defined the shape carefully with equations, then we could check.)

Antiexamples in $(\mathbb{R}^2, d_2)$ of non-open sets:

1) a single point $\{(x,y)\}$, by a similar argument as in $\mathbb{R}$.

2) Any subset of the form $\{(x,0) : x \in A\}$ (where A is any nonempty subset of $\mathbb{R}$).

This is because any ball around a point on the x-axis contains points off the x-axis.



3) The <u>closed square</u>

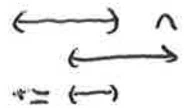
$\overline{S} :=$ the set of points (x,y)

s.t. $0 \le x \le 1$ and $0 \le y \le 1$.

<u>Self-check:</u> Why isn't $\overline{S}$ open?

<u>Notice:</u> In $\mathbb{R}$, the union of 2 open intervals, such as $(1,2) \cup (3,4)$ is an open set. (<u>Self-check:</u> Why does it satisfy the definition?)

Also, the intersection of 2 open intervals is either empty, or else a (smaller) open interval.



In any case, it is open.

Much more is true! Since the proof is the same, we'll do for metric spaces.

<u>Proposition:</u> Let (M,d) be a metric space. Then

1) The union of any family of open sets is open. (Even if there are an ∞ number of open sets!)

2) The intersection of any <u>finite</u> family of open sets is open.

<u>Examples:</u> Consider the infinite union of intervals in $\mathbb{R}$ given by

$$\bigcup_{n=1}^{\infty} (n, n+1/n) = (1,2) \cup (2, 2\tfrac{1}{2}) \cup (3, 3\tfrac{1}{3}) \cup \cdots$$

Since it is an (infinite) union of open sets, it is open. (Why? Each point in the union is contained in one of the open intervals, so in an ε-neighborhood inside the open interval and inside the set.)

2) The intersection of open intervals

$$(-\tfrac{1}{2}, \tfrac{1}{2}) \cap (-\tfrac{1}{3}, \tfrac{1}{3}) \cap (-\tfrac{1}{4}, \tfrac{1}{4}) = (-\tfrac{1}{4}, \tfrac{1}{4})$$

is open.

Antiexample: What about the intersection of an infinite family of open sets? This need not be open.

Eg: $\bigcap_{n=1}^{\infty} (-1/n, 1/n) = (-1,1) \cap (-1/2, 1/2) \cap (-1/3, 1/3) \cap \cdots$
$= \{0\}$

which we previously showed was not open.

Remark: We're being careful not to say "closed" in the place of not open. "Closed" will have another meaning.

Proof (of Proposition):

1) Suppose $\{A_\lambda : \lambda \in I\}$ is a family of open sets, with one open set for each λ in the "index set" I.
(Eg: if $I = \{1,2,3\}$, then we have A_1, A_2, and A_3;
if $I = \mathbb{N}$, then we have $A_0, A_1, A_2, A_3, \ldots$
I want to also allow for uncountable index sets.)

Now if $a \in \bigcup_{\lambda \in I} A_\lambda$, then there is some λ_0 so that $a \in A_{\lambda_0}$.

and since A_{λ_0} is open, $\exists\, \varepsilon > 0$ s.t.
$B_\varepsilon(a) \subseteq A_{\lambda_0} \subseteq \bigcup_{\lambda \in I} A_\lambda$. ✓

(Self-check: How does this proof translate to the infinite family of open intervals $(n, 1/n)$ that we looked at earlier?)

2) Suppose $A_1, A_2, \ldots, A_n$ are open sets, and $a \in A_1 \cap A_2 \cap \cdots \cap A_n$.
Then for each A_i, there is an ε_i s.t. $a \in B_{\varepsilon_i}(a) \subseteq A_i$.
(since A_i is open.)
Take $\varepsilon = \min\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n\}$. Now for each i we have
$B_\varepsilon(a) \subseteq B_{\varepsilon_i}(a) \subseteq A_i$, so that $B_\varepsilon(a) \subseteq \bigcap_{i=1}^{n} A_i$. ∎

Observe: We can see clearly why the proof of (2) fails if we replace the finite family of closed sets w/ an infinite family. For we take $\varepsilon = \min\{\varepsilon_1, \varepsilon_2, \dots \varepsilon_n\}$; but infinite families are not guaranteed to have a minimum. The infimum of an ∞ set of positive numbers may be 0. Indeed, this is exactly what happened in our earlier antiexample $\bigcap_{n=1}^{\infty}\left(\frac{-1}{n}, \frac{1}{n}\right)$.

Closed sets: Recall that the complement of a set A in $\mathbb{R}$ consists of all real numbers not in A.

Write as A^c.

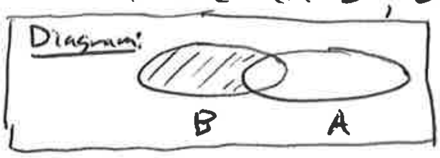
Similarly, the complement of a set A in a metric space M (such as $M = \mathbb{C}$) consist of all points in M that are not in A.

We use the same A^c notation.

Alternate notation:

Recall that $B \setminus A$ (read as "B minus A") is the set of all points that are in B, but not in A.

So $A^c = M \setminus A$.



(Complements in $\mathbb{Q}^{\geq 0}$ were useful for us back when we were discussing Dedekind cuts!)

Examples: Complements of real intervals in $\mathbb{R}$.

i) $(1,2)^c = (-\infty, 1] \cup [2, \infty)$ while

$[1,2]^c = (-\infty, 1) \cup (2, \infty)$

ii) $\mathbb{R}^c = \emptyset$, while $\emptyset^c = \mathbb{R}$

Notice that $(A^c)^c = A$. (As "not not in A" means "in A").

I'll give two definitions that make the word "closed".

Definitions:

1) A set A in a metric space M (such as $M=\mathbb{R}, \mathbb{C}, \ldots$) is <u>closed</u> if A^c is open.

2) A set A in a metric space M is <u>sequentially closed</u> if every accumulation point of any sequence of points in A is also in A.

<u>Examples</u> Consider the closed interval $A=[0,1]$ in $\mathbb{R}$.

- This set is closed, since $A^c = [0,1]^c = (-\infty,0)\cup(1,\infty)$ is the union of two open intervals, hence open.
- The set is also sequentially closed. Every accumulation point of a sequence in A is a limit of a (sub) sequence. Since $\mathbb{R}$ is complete, the limit is a real number and since the limit of a sequence of points that are ≥ 0 and ≤ 1 is also ≥ 0 and ≤ 1, the limit is on the same interval $[0,1]$.

<u>Antiexamples</u> Consider $B=(0,1)$, an open interval in $\mathbb{R}$.

- B is <u>not</u> closed, since $B^c = (-\infty,0]\cup[1,\infty)$ is not open. (<u>Self-check</u>: Why not?)
- B is not sequentially closed, either.
  $a_n = 1/2n$ is a sequence of points in $(0,1)=B$.
  but $\lim_{n\to\infty} a_n = \lim_{n\to\infty} 1/2n = 0$, which is not in B.

<u>Example:</u> $\mathbb{R}$ is <u>both</u> open, <u>and</u> (since $\mathbb{R}^c = \emptyset$) is also closed.
(such sets are sometimes called <u>clopen</u>.)
Obviously, any limit of any (sub)sequence in $\mathbb{R}$ is also in $\mathbb{R}$!! So $\mathbb{R}$ is sequentially closed.

<u>Antiexample:</u> The interval $[0,1)$ is neither open, nor closed, nor sequentially closed.

- not open, as ε-neighborhoods of 0 contain points <0
- not closed, " " " " 1 " " <1
- not sequentially closed as sequences such as $1-\frac{1}{n+1}$ have values in $[0,1)$ for each n, but limit of 1.

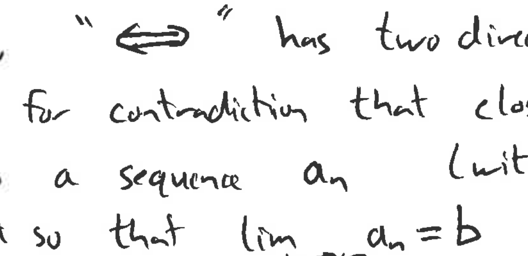
The first substantial theorem in this section tells us that "sequentially closed" is a redundant definition.

<u>Theorem:</u> Let M be a metric space (such as $\mathbb{R}$ or $\mathbb{C}$) and let $A \subseteq M$ be a subset.
Then A is closed $\Longleftrightarrow$ A is sequentially closed.

<u>Proof:</u> As usual, "$\Longleftrightarrow$" has two directions.
<u>($\Rightarrow$)</u>: Suppose for contradiction that closed set A has a sequence a_n (with $a_n \in A$ for all n) but so that $\lim_{n\to\infty} a_n = b$ where $b \notin A$.

Thus, $b \in A^c$.
Now, since A^c is open, $\exists \varepsilon^* > 0$ with $B_{\varepsilon^*}(b) \subseteq A^c$.
But the limit definition says that
$$\forall \varepsilon > 0,\ \exists N \text{ s.t. } [n > N] \Rightarrow [a_n \in B_\varepsilon(b)]$$
and as <u>no</u> points of a_n are in A^c, there cannot be such an N for $\varepsilon = \varepsilon^*$. Contradiction of limit definition. #

<u>Picture:</u>



($\Leftarrow$): Suppose for contradiction that A is sequentially closed but that A^c is not open.

As A^c is not open, there is some $b_0 \in A^c$ so that every ε-neighborhood of b_0 contains at least one point of A. $(= (A^c)^c)$

Now construct a sequence by letting a_n be a point in $B_{1/n}(b_0)$ that is in A. $\varepsilon = 1/n$

as $d(b_0, a_n) < 1/n$, so $\lim_{n\to\infty} d(b_0, a_n) = 0$

so $\lim_{n\to\infty} a_n = b_0$.

Since a_n is a sequence of points in A that converges to $b_0 \in A^c$, this contradicts sequentially closed for A. # ∎

Henceforth, we'll say "closed" instead of "sequentially closed", since the two definitions are equivalent.

Our union/intersection theorem for open sets translates over to closed sets:

Corollary: Let (M,d) be a metric space. Then

1) The intersection of any family of closed sets is closed. (Even if the family is infinite!)
2) The union of any finite family of closed sets is closed.

Sketch: If B_λ is closed for each λ in index set I, then $B_\lambda{}^c$ is open, so (check that)

$$\left(\bigcup_{\lambda\in I} B_\lambda\right)^c = \bigcap_{\lambda\in I} B_\lambda{}^c$$

while

$$\left(\bigcap_{\lambda\in I} B_\lambda\right)^c = \bigcup_{\lambda\in I} B_\lambda{}^c$$

(so complement interchanges $\cap, \cup$). ∎

<u>Aside:</u> For a metric space M, the collection of all open sets of M is called the <u>topology</u> of M.

<u>E.g.</u> The topology of $\mathbb{R}$ is all unions of open intervals.

Much of what we're talking about can be done w/o the metric – see the Topology class.

Arbitrary intersections of closed sets (and arb. unions of open sets) lead us to the following notion of "smallest closed set with..."

<u>Definition:</u> The <u>closure</u> of a subset A of a metric space is the set $\bar{A} := \bigcap_{\substack{C \text{ closed} \\ C \text{ contains } A}} C$

(the intersection of all closed sets containing A.)

Thus, $\bar{A}$ is a closed set containing A

so $\bar{A}$ is the smallest closed set containing A.

<u>Self-check:</u> Why is there no smallest open set containing A? What breaks down in the above?

<u>Examples</u> For bounded intervals in $\mathbb{R}$, we have e.g.

$$\overline{(0,1)} = \overline{[0,1)} = \overline{(0,1]} = \overline{[0,1]} = [0,1].$$

<u>Observe:</u> A is closed $\Leftrightarrow A = \bar{A}$.

(<u>Self-check:</u> Why?)

There's another good characterization of $\bar{A}$:

<u>Proposition:</u> If A is any subset of a metric space M then $\bar{A} = A \cup \{\lim_{n\to\infty} a_n : a_n \text{ a sequence in } A\}$

That is, $\bar{A}$ is the union of A with the set of all accumulation points of all sequences in A.

<u>Proof:</u> We show that $\bar{A} \subseteq A \cup \{\lim a_n : a_n \text{ a sequence in } A\}$ and also
" $\supseteq$ " " ----------- " -

(<u>Self-check</u>: how is this similar to our $\Leftrightarrow$ proofs?)

<u>$\supseteq$</u>: We showed that $\bar{A}$ is a closed set containing A.
As closed sets are sequentially closed, we have that $\bar{A}$
contains all limit points of sequences in $\bar{A}$,
hence all " " " " A. ✓

<u>$\subseteq$</u>: It is enough to show that RHS is closed, as
$\bar{A}$ is by definition contained in any closed set containing A.
Using sequential closure, we'll show that any limit
of a sequence b_n in the RHS is in RHS.
(That is, if $b_n \in RHS$ for all n, then $\lim_{n\to\infty} b_n \in RHS$.)
We'll do this by finding a sequence a_n where
$d(a_n, b_n) < \frac{1}{n}$.
<u>Details</u> are similar to those in the proof that
sequentially closed $\equiv$ closed, and will be on homework! ∎

<u>Examples</u> We previously saw that $\overline{(0,1)} = [0,1]$. The above gives
a new approach: both $0, 1$ are limits of sequences in $(0,1)$,
and since limits of such sequences are ≥ 0 and ≤ 1,
we get the desired.

<u>Examples</u> Recall that there is a sequence whose range is $\mathbb{Q}$.
Since every interval $(s-\varepsilon, s+\varepsilon)$ contains infinitely many
rational numbers, every real number s is an acc. pt
of this sequence.
The Proposition thus tells us that $\overline{\mathbb{Q}} = \mathbb{R}$.

Examples: If A is the range of the convergent sequence a_n,
then $\overline{A} = A \cup \{\lim_{n\to\infty} a_n\}$.

Eg: $\overline{\{\frac{1}{n} : n \in \mathbb{N}^+\}} = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}^+\}$.

B. Compact sets

In the last section, we considered closed and sequentially closed sets.

I'll similarly give two definitions here.

Definitions:

1) A subset K of a metric space M is sequentially compact
if any sequence a_n of points in K has a subsequence
a_{n_k} that converges to a limit in K.

2) A subset K of a metric space M is compact
if whenever O_λ is a family of open sets (one for each $\lambda \in I$, some index set I) so that $K \subseteq \bigcup_{\lambda \in I} O_\lambda$
then there is a finite subfamily (ie, a finite $I_0 \subseteq I$)
so that $K \subseteq \bigcup_{\lambda \in I_0} O_\lambda$.

That is to say, whenever K is contained in some union of open sets,
we can select a finite number of the open sets
so that K is contained in the union of this finite
subfamily.

Anti Eg: $(0,1)$ is not compact, since $(0,1) = \bigcup_{n \in \mathbb{N}^+} (\frac{1}{n}, 1)$
and any finite subfamily of the intervals
$(\frac{1}{n}, 1)$ will leave out sufficiently small numbers.

Nor is it sequentially compact, as every subsequence
of $1/n$ converges to 0.

When $K \subseteq \bigcup_{\lambda \in I} O_\lambda$, we say that K is <u>covered</u> by this family of open sets, or that the family is an <u>open cover</u>.

* Thus, Definition 2 says that a set is compact when every open cover admits a finite subcover. *

<u>Example</u>: Any finite set of points in M is both compact and seq. compact.

- Compact, as we can take an open set covering each point — as there are finitely many points, this yields finite subcover.
- Seq. compact, as any sequence in the finite set takes on some value infinitely often, yielding a constant subsequence.

Compactness and sequential compactness are sometimes called "finite-type conditions", as they say an infinite set has some properties similar to those of a finite set.

<u>Proposition</u>: Any closed, bounded subset of $\mathbb{R}$ or of $\mathbb{C}$ is sequentially compact.

<u>Proof</u>: Let K be a closed bounded subset of $\mathbb{R}$ or of $\mathbb{C}$.
By Bolzano-Weierstrass, any sequence a_n of numbers in K has a convergent subsequence, (as K is bounded).
And since K is closed, it is also seq. closed, so the limit of the convergent subsequence is in K. ∎

<u>Example</u>: $[0,1]$ is a seq. compact subset of $\mathbb{R}$.
Every sequence in $[0,1]$ has an accumulation point in $[0,1]$.

<u>Observe:</u> If K is seq. compact, then K is closed (in any metric space)

<u>Proof:</u> If K is not closed, then it is not seq. closed, so there is a sequence a_n of points in K that converges to a point outside of K. As every subsequence of a convergent sequence has the same limit, no subsequence converges to a point in K.

<u>Observe:</u> If K is seq. compact subset of $\mathbb{R}$ or of $\mathbb{C}$, then K is bounded.

<u>Self-check:</u> verify this. (by taking a sequence diverging to ∞).

The Proposition and the Observations yield:

<u>Corollary:</u> A subset K of $\mathbb{R}$ or of $\mathbb{C}$ is sequentially compact if and only if K is closed and bounded.

<u>Examples:</u>
- $(0,1]$ is not seq. compact.
  (not closed; consider sequence $1/n$.)
- $[0,\infty)$ is not seq. compact.
  (not bounded; consider sequence n.)
- $[0,1] \cup [3,5] \cup [9,13]$ is closed and bounded, so is sequentially compact.

So what about compactness?
- As we saw before, $(0,1)$ is not compact.
  $\bigcup_{n=1}^{\infty} (1/n, 1)$ admits no finite subcover.
- $[0,\infty)$ is also not compact.
  $\bigcup_{n=0}^{\infty} (n-1, n+1) = (-1,1) \cup (0,2) \cup (1,3) \cup \cdots$
  covers $[0,\infty)$, but admits no subcover as each $n \in \mathbb{N}$ is in a unique interval.

These examples suggest a relationship between compactness and sequential compactness. Indeed, it's not hard to show:

Proposition: If a subset K of a metric space M is compact, then also K is sequentially compact.

Proof: Suppose for contradiction that K is compact, but not sequentially compact. Then there is a sequence a_n in K having no accumulation point in K.

By our 2nd characterization of accumulation points, then for each point $b \in K$, there is an ε_b so that a_n is in $B_{\varepsilon_b}(b)$ only finitely often.

But now $K \subseteq \bigcup_{b \in K} B_{\varepsilon_b}(b)$ is an open cover.

By compactness, there is a finite subcover $B_{\varepsilon_1}(b_1), B_{\varepsilon_2}(b_2), \ldots, B_{\varepsilon_k}(b_k)$ but now since a_n has infinitely many points, at least one $B_{\varepsilon_i}(b_i)$ must also have infinitely many points, a contradiction. #

So a_n has an acc. pt and K is seq. compact. ∎

Fact: (Harder) Any seq. compact subset K of a metric space M is also compact.

* Thus, compactness and seq. compactness are equivalent in metric spaces. *

Proving this would take us further into metric spaces than is allowed by the scope of ANA-I.

Instead, we'll show equivalence in $\mathbb{R}$.

We already showed that a compact subset of $\mathbb{R}$ is seq. compact, hence closed and bounded.

We'll complete the characterization by showing:

Theorem: Any closed, bounded subset of $\mathbb{R}$ is compact.

Example: $[0,2]$ is compact subset of $\mathbb{R}$.

The proof of the theorem will be easier if we break it up into several lemmas.

Lemma 1: (Nested Intervals Theorem)

If $[a_1, b_1] \supseteq [a_2, b_2] \supseteq \cdots$ is a sequence of "nested" closed intervals,

then $\bigcap_{n=1}^{\infty} [a_n, b_n]$ is nonempty. (That is, there is a number that is in every $[a_n, b_n]$.)

Proof: The conditions say that a_n is an increasing sequence,
b_n " " decreasing "
and that for each n, $a_n \le b_n$.

Thus, $a = \lim_{n\to\infty} a_n$ and $b = \lim_{n\to\infty} b_n$ both converge by MST and $a \le b$, (as $b_n - a_n \ge 0$, so $b - a \ge 0$.)

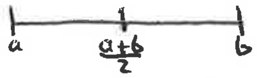
Now $\emptyset \ne [a,b] \subseteq \bigcap_{n=1}^{\infty} [a_n, b_n]$, as desired. ∎

Now we show the theorem holds for the special case of a closed bounded interval in $\mathbb{R}$.

Lemma 2: If $[a,b]$ is any closed (and bounded) interval in $\mathbb{R}$, then $[a,b]$ is compact.

<u>Proof:</u> Suppose that $[a,b]$ is not compact, so that some open cover O_λ (for $\lambda \in I$ index set) admits no finite subcover.

Divide $[a,b]$ into 2 equal-lengths parts



Now each part is covered by O_λ. If both parts admit finite subcovers, then the union of these subcovers would be a finite subcover of $[a,b]$.

So at least one half has no finite subcover.

Call this half of the interval T_1.

Now divide T_1 into 2 equal-length parts.

By the same argument, at least one half T_2 has no finite subcover.

Now divide T_2 into 2 equal-length parts:

...

Continuing in this manner, we get a nested sequence of intervals $T_1 \supseteq T_2 \supseteq T_3 \supseteq \ldots$,

None of which admits a finite subcover.

But the Nested Interval Lemma tells us that $\bigcap_{n=1}^{\infty} T_n$ contains some point c. Now $c \in O_{\lambda_0}$ for some $\lambda_0 \in I$

(since the open cover covers also c !!)

Also, since O_{λ_0} is open, $\exists\, \varepsilon$ so that $B_\varepsilon(c) \subseteq O_{\lambda_0}$.

But since the length of T_n is $\frac{1}{2^n}\cdot(b-a)$, for a large enough value of N, $T_N \subseteq O_{\lambda_0}$.

Since 1 is finite, and since O_{λ_0} covers T_N w/ 1 open set, this contradicts that T_N has no finite subcover! #

We conclude that $[a,b]$ is compact. ∎

What about closed bounded sets other than intervals?

Lemma 3: If K is compact and A is closed in a metric space M, then $K \cap A$ is also compact.

Proof: If O_λ is an open cover of $K \cap A$, then O_λ together with A^c is an open cover of K and a subcover of the latter yields a subcover of the former (by leaving out A^c). ∎

If K is any closed subset of $\mathbb{R}$, bounded by $\heartsuit$
(so each $x \in K$ satisfies $|x| \leq \heartsuit$)
then $K \subseteq [-\heartsuit, \heartsuit]$, so $K = K \cap [-\heartsuit, \heartsuit]$
so K is compact. (by Lemma 3). ✓

The Theorem that $K \subseteq \mathbb{R}$ is compact $\iff$ K is closed and bounded now follows.

So we have:

Corollary: For a subset $K \subseteq \mathbb{R}$, TFAE:
1) K is compact
2) K is seq. compact.
3) K is closed and bounded.

(Since we've shown $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (1)$.)

That closed and bounded intervals are compact (and seq. compact) is the essential topological property of these intervals. This is surprisingly useful.

The basic approach this opens is to replace an infinite "gadget" with a finite one, which we can then apply techniques like induction to.