Analysis I Homework 8

A. Let H be the set of all points (x,y) in $\mathbb{R}^2$ such that $x^2+xy+3y^2=3$. Show that H is a closed subset of $\mathbb{R}^2$ (considered with the Euclidean metric). Is A bounded?

$H=\{(x,y) \mid x^2+xy+3y^2=3\}$

Let $(x_n,y_n)\to(x,y)$ and $(x_n,y_n)\in H \Leftrightarrow x_n^2+x_ny_n+3y_n^2=3 \;/\lim_{n\to\infty}$

$\lim_{n\to\infty}(x_n^2+x_ny_n+3y_n^2)=3 \Leftrightarrow x^2+xy+3y^2=3 \Rightarrow (x,y)\in H$

Bounded:

$x^2+xy+3y^2=3$

$\Leftrightarrow \frac{1}{2}x^2+\frac{1}{2}(x+y)^2+\frac{5}{2}y^2=3 \;/\cdot 2$

$\Leftrightarrow x^2+(x+y)^2+5y^2=6$

$\Rightarrow x^2\le 6 \wedge 5y^2\le 6$

$\Leftrightarrow |x|\le\sqrt{6} \wedge |y|\le\sqrt{\frac{6}{5}}$

$\Rightarrow (x,y)\in \underbrace{[-\sqrt{6},\sqrt{6}]\times\left[-\sqrt{\frac{6}{5}},\sqrt{\frac{6}{5}}\right]}_{M}$

$\Rightarrow H\subseteq M$

$(x_n,y_n)\to(x,y) \Leftrightarrow x_n\to x$ and $y_n\to y$

Proof:

$(\Leftarrow)$ Let $x_n\to x$ and $y_n\to y$

$\forall \varepsilon_1\ \exists n_1\ \forall n>n_1\ |x_n-x|<\varepsilon_1=\frac{\varepsilon}{\sqrt{2}}$

$\forall \varepsilon_1\ \exists n_2\ \forall n>n_2\ |y_n-y|<\varepsilon_1=\frac{\varepsilon}{\sqrt{2}}$

Let $\varepsilon>0\ \exists n\ge\max\{n_1,n_2\}$

$\sqrt{(x_n-x)^2+(y_n-y)^2}\le\sqrt{\varepsilon_1^2+\varepsilon_1^2}=\sqrt{2}\cdot\varepsilon_1=\varepsilon$

$(\Rightarrow)$ Let $\forall\varepsilon_1>0,\ \exists n_1\ \forall n>n_1\ \sqrt{(x_n-x)^2+(y_n-y)^2}<\varepsilon$.

Let $\varepsilon>0\ \exists n_0=n_1\ \forall n>n_0$

$|x_n-x|\le\sqrt{(x_n-x)^2+(y_n-y)^2}<\varepsilon \Rightarrow \lim_{n\to\infty}x_n=x$

B. Given a metric space M with metric d, verify that any ε-ball is an open set.

$B_\varepsilon(x_0) = \{x \in M \mid d(x_0, x) < \varepsilon\}$

B_ε is open set $\Leftrightarrow \forall y \in B_\varepsilon(x_0)\ \exists \varepsilon_1$

$B_{\varepsilon_1}(y) \subseteq B_\varepsilon(x_0)$

Let $\varepsilon_1 = \varepsilon - d(x_0, y) > 0$, $(y \in B_\varepsilon(x_0))$

$B_{\varepsilon_1}(y) \subseteq B_\varepsilon(x_0)$

Let $z \in B_{\varepsilon_1}(y) \Leftrightarrow d(z, y) < \varepsilon_1$

$d(z, x_0) \leq d(z, y) + d(y, x_0) < \varepsilon_1 + d(y, x_0) = \varepsilon - \cancel{d(x_0, y)} + \cancel{d(y, x_0)} = \varepsilon$

$\Rightarrow z \in B_\varepsilon(x_0)$

$B_{\varepsilon_1}(y) \subseteq B_\varepsilon(x_0)$

C. Show that a set A in $\mathbb{R}^2$ is open in the Eucledian metric $\Leftrightarrow$ it is open in the max metric. Hint: As usual, there are two directions to prove in an $\Leftrightarrow$. The picture on p73 of the notes may be somewhat helpful.

($\Rightarrow$) Let $A \subseteq \mathbb{R}^2$ is open set in the Eucledian metric.

We have to prove that A is open set in $\mathbb{R}^2$ in the max metric.

$\exists B'_\varepsilon(x,y) = \{(x',y') \mid \sqrt{(x'-x)^2+(y'-y)^2} < \varepsilon\} \subseteq A$

Let $B''(x,y) = \{(x',y') \mid \max\{|x-x'|, |y-y'|\} < \varepsilon\}$

Let $(x',y') \in B'' \Rightarrow \max\{|x-x'|, |y-y'|\} \leq \sqrt{(x-x')^2+(y-y')^2} < \varepsilon$

$\Rightarrow (x',y') \in A \Rightarrow B'' \subseteq A$

($\Leftarrow$) Let $A \subseteq \mathbb{R}^2$ is open in the max metric

Let $(x,y) \in A$ and $(x',y') \in B'' \Rightarrow \max\{|x-x'|, |y-y'|\} < \varepsilon_1 = \frac{\varepsilon}{2}$

$\Rightarrow \sqrt{(x-x')^2+(y-y')^2} \leq 2\max\{|x-x'|, |y-y'|\} < 2 \cdot \varepsilon_1 = \varepsilon$

$B'_\varepsilon(x,y) \subseteq A$

B. By showing that any sequence in $A \cup L$ has the same limit as some sequence in A, prove that $\bar{A} \subseteq A \cup L$, where L is the set of accumulation points of sequences in A.

Let L be accumulation point of a sequence in A.

$(\forall x \in L \ \exists (x_n) \ x_n \neq x$ and $\lim_{n \to \infty} x_n = x)$

With this we will prove that every sequence in $A \cup L$ has same limit in every sequence in A. $(x \in \bar{A} \Leftrightarrow \forall B_\varepsilon(x) \cap A \neq \emptyset)$.

Let $x_0 \in \bar{A}$ and $\varepsilon = \frac{1}{n}$ $\quad B_{1/n}(x_0) = \{x \mid d(x, x_0) < x_n\}$

$B_{1/n}(x_0) \cap A \neq \emptyset$ $\quad \exists x_n \in A$ and $d(x_n, x_0) < \frac{1}{n}$

$\Rightarrow$ Sequence x_n converges to x_0

Two cases are possible:

1) $\exists \varepsilon' \ B_{\varepsilon'}(x_0) \cap A = \{x_0\}$

$\Rightarrow$ Sequence x_0 lean toward the sequence x_0

$\Rightarrow x_0 \in A$

2) $\forall \varepsilon \ B_\varepsilon(x_0) \cap A \setminus \{x_0\} \neq \emptyset$

$\forall 1/n > 0 \quad B_{1/n}(x_0) \cap A \setminus \{x_0\} \neq \emptyset$

$\exists x_n \in B_{1/n}(x_0) \cap A$ and $x_n \neq x_0$

$\Rightarrow (x_n) \to x_0$ and $x_0 \in L$

E. Show that both $\mathbb{R}$ and $\mathbb{R}^2$ may be covered by countably many open balls.

$\mathbb{Q} = \bigcup_{n=1}^{\infty} \{r_n\}$

$J_n = (r_n - 1, r_n + 1)$, $r_n \in \mathbb{Q} \}$

$\bigcup_{n=1}^{\infty} I_n = \mathbb{R}$ $\Rightarrow I_n \subseteq \mathbb{R}$ $\quad \bigcup I_n \subseteq \mathbb{R}$

$(\Leftarrow)$ $x = \mathbb{R} \Rightarrow (x - \frac{1}{3}, x + \frac{1}{3}) \cap \mathbb{Q} \neq 0 \Leftrightarrow$ (r_x is the cross section)

$\Rightarrow \exists r_x \in \mathbb{Q}$ and $r_x \in (x - \frac{1}{3}, x + \frac{1}{3})$

$\Rightarrow x \in (r_x - 1, r_x + 1)$ for some r_x

Because $|r_x - x| < \frac{1}{3} < 1$

$\Rightarrow \mathbb{R} \subseteq \bigcup I_n$

$\mathbb{Q}^2 = \bigcup_{n=1}^{\infty} \{(x_n, y_n)\}$

$J_n = B(x_n, y_n)$

$(\Rightarrow) (1) = \{(x, y) \mid \sqrt{(x - x_n)^2 + (y - y_n)^2} < 1\}$

$\bigcup J_n = \mathbb{R}^2$

$(\Leftarrow)$ Let $(x, y) \in \mathbb{R}^2$

$\Rightarrow B_{(x,y)}(\frac{1}{3}) = \{(x', y') \mid d((x', y'), (x, y)) < 1/3\}$

$B_{(x,y)}(1/3) \cap \mathbb{Q}^2 \neq 0$

$(z_1, z_2) \in \mathbb{Q}^2$ and $\sqrt{(x - z_1)^2 + (y - z_2)^2} < 1/3 < 1$

$d((x, y), (z_1, z_2)) < 1 \Rightarrow (x, y) \in J_n$