Analysis I   Homework 1

A. Prove that for all positive natural numbers n,

$1\cdot 2+2\cdot 3+\dots+n\cdot(n+1)=n\cdot(n+1)\cdot(n+2)/3$

Proof.

Base step.

$n=1$

$n\cdot(n+1)\cdot(n+2)/3=\frac{1(1+1)\cdot(1+2)}{3}=\frac{2\cdot\cancel{3}}{\cancel{3}}=2$, so it holds.

Inductive step.

Assume that the formula holds for $k\in\mathbb{N}$.

$n=k$

$1\cdot 2+2\cdot 3+\dots+k(k+1)=k(k+1)(k+2)/3$

$n=k+1$

$1\cdot 2+2\cdot 3+\dots+k(k+1)+[(k+1)(k+1+1)]=(k+1)[(k+1+1)(k+1+2)]/3$

$k(k+1)(k+2)/3+[(k+1)(k+1+1)]=(k+1)[(k+1+1)(k+1+2)]/3$

$(k^2+k)(k+2)/3+[(k+1)(k+2)]=(k+1)[(k+2)(k+3)]/3$

$(k^3+2k^2+k^2+2k)/3+[k^2+2k+k+2]=(k+1)[(k^2+3k+2k+6)]/3$

$(k^3+3k^2+2k)/3+[k^2+3k+2]=(k+1)[(k^2+5k+6)]/3 \quad /\cdot 3$

$(k^3+3k^2+2k)+(3k^2+9k+6)=(k^3+5k^2+6k+k^2+5k+6)$

$k^3+3k^2+2k+3k^2+9k+6=k^3+6k^2+11k+6$

$k^3+6k^2+11k+6=k^3+6k^2+11k+6$ ■

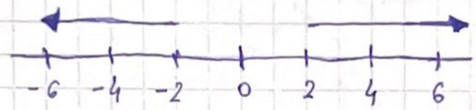
B. For each of the following subsets of $\mathbb{Z}$, explain whether the subset is well-ordered or not (by the usual ordering on $\mathbb{Z}$).

(i) even numbers

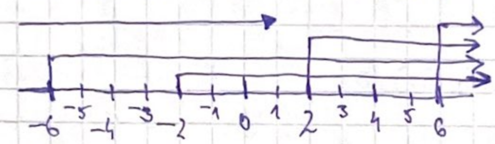
(ii) perfect squares

(iii) the integers that are stricly greater than -5

(i) Even numbers $\Rightarrow$ This subsets of $\mathbb{Z}$ won't be well-ordered simply by the fact that in order to be well-ordered there needs to be a least element to be qualified for it.

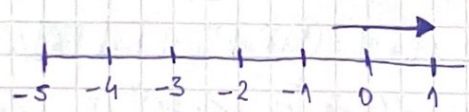


(ii) Perfect squares $\Rightarrow$ This subsets of $\mathbb{Z}$ will be well-ordered because it's linear in order.



$(-6)^2=36$, $(-2)^2=4$, $2^2=4$, $6^2=36$

(iii) The integers that are stricly greater than -5 $\Rightarrow$ This subsets of $\mathbb{Z}$ will be well-ordered because it's linear in order and contains a least element.



C. Using the ordered pairs definition of the integers $\mathbb{Z}$, verify the associativity property of $\mathbb{Z}$. (You may use the properties of the natural $\mathbb{N}$, including associativity, as stated in the notes)

$\forall\ a, b, c \in \mathbb{Z}$

What we need to prove is: $(a+b)+c = a+(b+c) \Rightarrow$ addition and

$(a\cdot b)\cdot c = a\cdot(b\cdot c) \Rightarrow$ multiplication

Pairs: $a=(m,n)$, $b=(x,y)$ and $c=(i,j)$

Proof:

$\Rightarrow$ Addition:

$[(m,n)+(x,y)]+(i,j) = (m+x, n+y)+(i,j) = [(m+x)+i, (n+y)+j] = [m+(x+i), n+(y+j)] =$

$= (m+n)+(x+i, y+j) \Rightarrow$ So we prove that $(m+n)+(x+i, y+j) = a+(b+c)$ ■

$\Rightarrow$ Multiplication:

$[(m,n)\cdot(x,y)]\cdot(i,j) = [(m\cdot x, n\cdot y)]\cdot(i,j) = [(m\cdot x)\cdot i, (n\cdot y)\cdot j] = [(m\cdot i)\cdot x, (n\cdot j)\cdot y] =$

$= (m\cdot n)\cdot(x\cdot i, y\cdot j) \Rightarrow$ So we prove that $(m\cdot n)\cdot(x\cdot i, y\cdot j) = a\cdot(b\cdot c)$ ■

$\Rightarrow$ With this we prove that the associativity in addition and multiplication are a property ordered pairs for the integers $\mathbb{Z}$.

D. Let G be set {0, ♥}. Find a binary operation ⊕ so that G is a group. Prove that your operation really yields a group! (One efficient way to specify ⊕ is to write out an addition table.

What we need to prove is that the set {0, ♥} is a group that is called G.

① (G_1) (0, ♥) is an associative groupoid (half group)

② (G_2) (0, ♥) to have a neutral element in the group

③ (G_3) (0, ♥) each element in the group (0, ♥) has an inverse element

④ (G_4) (0, ♥) if the groupoid (0, ♥) is commutative then (0, ♥) is a commutative group

·	0	♥
0	♥	0
♥	0	♥

0, ♥ ∈ G → 0 ⊕ ♥ ∈ G This holds

0 ⊕ ♥ = ♥ ∈ G This holds

♥ ⊕ 0 = ♥ This holds