Analysis I Homework 7

A. Show that any sequence of positive real numbers either has a subsequence that converges, or else a subsequence that diverges to ∞.

We have to proof that $(a_n)_{n=1}^{\infty}$, $a_n > 0$

$\Rightarrow$ $\exists$ subsequence that converges to L or has a subsequence that converges to ∞

$\Rightarrow$ Sequence $a_n \geq 0$ which does not converges to L or ∞ does not converges at all

If there are at least two stacking points then $L_1 \neq L_2$

$\Rightarrow$ $\exists$ two subsequences that converges to L_1 and L_2

B. For each of the following series, determine whether the series converges or diverges. (Sum from 1 to infinity.) For those that converge, find the value that they converge to.

i) $\sum (4^n + 2^n)/3^n$

ii) $\sum \sin^n(3)$

iii) $\sum 1/(n^2+2n)$

iv) $\sum (\sqrt{n+2} - \sqrt{n})$

i) $\sum_{n=1}^{\infty} (4^n+2^n)/3^n = \sum_{n=1}^{\infty} \left[\left(\frac{4}{3}\right)^n + \left(\frac{2}{3}\right)^n\right] = \sum_{n=1}^{\infty}\left(\frac{4}{3}\right)^n + \sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n$

If $\sum a_n$ convergent then $\lim a_n = 0$.

So $\sum_{n=1}^{\infty}\left(\frac{4}{3}\right)^n \quad a_n = \left(\frac{4}{3}\right)^n$

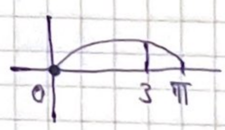
$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{4}{3}\right)^n = \infty \neq 0$

$\Rightarrow \sum_{n=1}^{\infty}\left(\frac{4}{3}\right)^n$ divergent then $\sum_{n=1}^{\infty}(4^n+2^n)/3^n$ divergent

ii) $\sum_{n=1}^{\infty} \sin^n(3)$

$a_n = \sin^n(3) = q^n \qquad 0 < q = \sin 3 < 1$

$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} q^n \qquad |q| = |\sin 3| < 1$



$\Rightarrow$ geometric series, convergent if $|q| < 1$

iii) $\sum_{n=1}^{\infty} 1/(n^2+2n)$

$a_n = 1/(n^2+2n)$

$b_n = 1/n^2$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{n^2+2n}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{n^2+2n} = \lim_{n\to\infty} \frac{\cancel{n^2}(1)}{\cancel{n^2}(1+\frac{2}{n})} = 1 \in \mathbb{R}^+$

$\Rightarrow \sum a_n$ and $\sum b_n$ have the same nature and because $\sum 1/n^2 = \sum b_n$

$\sum a_n$ is convergent

iv) $\sum_{n=1}^{\infty} = \left(\sqrt{n+2} - \sqrt{n}\right)$

$$a_n = \sqrt{n+2} - \sqrt{n} = \left(\sqrt{n+2} - \sqrt{n}\right) \frac{\sqrt{n+2} + \sqrt{n}}{\sqrt{n+2} + \sqrt{n}} = \frac{n+2-n}{\sqrt{n+2}+\sqrt{n}} = \frac{2}{\sqrt{n+2}+\sqrt{n}} \to 0$$

$$a_n = \frac{2}{\sqrt{n+2}+\sqrt{n}} \qquad b_n = \frac{1}{n} = \frac{1}{n^{1/2}}$$

$\sum b_n = \sum \frac{1}{n^{1/2}} \Rightarrow$ divergent

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2\sqrt{n}}{(\sqrt{n+2}+\sqrt{n})} = \lim_{n\to\infty} \frac{\cancel{\sqrt{n}} \cdot 2}{\cancel{\sqrt{n}}\left(\sqrt{1+\underbrace{\tfrac{2}{n}}_{0}} + 1\right)} = \frac{2}{1+1} = 1 \in \mathbb{R}^+$$

$\Rightarrow \sum a_n$ and $\sum b_n$ have same nature

$\Rightarrow \sum a_n$ divergent

C. For each of the following series, determine whether the series converges, diverges to $\pm\infty$, or diverges, not to $\pm\infty$. Don't try to find what the series converges to.

i) $\sum \sin^2(3n+2)/(2n^2-3)$

ii) $\sum 1/(n-2\pi)$

iii) $\sum (n+1)/(n^3+2)$

i) $\sum_{n=1}^{\infty} \sin^2(3n+2)/(2n^2-3)$

$a_n = \sin^2 \frac{3n+2}{2n^2-3}$

$b_n = 1/n^2$, $\sum b_n$ convergent

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sin^2 \frac{3n+2}{2n^2-3}}{\frac{1}{n^2}\left(\frac{3n+2}{2n^2-3}\right)^2} \cdot \left(\frac{3n+2}{2n^2-3}\right)^2 = \lim \frac{n^2(9n^2+12n+4)}{4n^4-12n^2+9} = \frac{9}{4} \in \mathbb{R}^+$$

$\Rightarrow \sum a_n$ and $\sum b_n$ have same nature

$\Rightarrow \sum a_n$ convergent

ii) $a_n = 1/n-2\pi \sim b_n = 1/n \Rightarrow$ this comes from $\sum_{n=1}^{\infty} 1/(n-2\pi)$

$a_n = \frac{1}{n-2\pi} > \frac{1}{n} = b_n$

$\sum b_n = \sum 1/n$ divergent

$\Rightarrow \sum a_n = \infty$ divergent

$a_n = 1/n-2\pi \quad b_n = 1/n$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{n-2\pi} = 1 \in \mathbb{R}^+ \Rightarrow \sum a_n$ and $\sum b_n$ have same nature

$\Rightarrow \sum a_n$ divergent

iii) $\sum_{n=1}^{\infty} (n+1)/(n^3+2) \quad a_n = n+1/n^3+2 \quad b_n = n/n^3 = 1/n^2$

$\Rightarrow \sum b_n$ convergent $(\alpha = 2 > 1)$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n+1}{n^3+2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^3+n^2}{n^3+n^2} = 1 \in \mathbb{R}^+$$

$\Rightarrow \sum a_n$ and $\sum b_n$ have same nature so $\sum a_n$ converges

D. Determine whether each of the following series are absolutely convergent, conditionally convergent, or divergent.

i) $\sum (-2)^{n^2} / (n+2)!$

ii) $\sum (-3)^n / (3n^2+6n)$

iii) $\sum (-4)^n / (4^n \cdot \sqrt{2n+1})$

i) $\sum (-2)^{n^2} / (n+2)!$

$$\sum \frac{(-1)^{n^2} \cdot 2^{n^2}}{(n+2)!}$$

$$a_n = \frac{2^{n^2}}{(n+2)!}$$

$$\frac{a_{n+1}}{a_n} = \frac{\frac{2^{(n+1)^2}}{(n+3)!}}{\frac{2^{n^2}}{(n+2)!}} = \frac{2^{n^2+2n+1}}{(n+3)\,2^{n^2}} = \frac{2^{2n+1}}{n+3} > 1$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2^{n^2}}{(n+2)!} \neq 0$$

$\Rightarrow a_n \not\to 0$ diverges

ii) $\sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{3n^2+6^n}$

$$\lim_{n\to\infty} \frac{3^n}{3n^2+6^n} = \lim_{n\to\infty} \frac{\cancel{6^n}\left(\left(\frac{3}{6}\right)^n\right)}{\cancel{6^n}\left(3\frac{n^2}{6^n}+1\right)} = \frac{0}{1} = 0$$

$$\frac{a_{n+1}}{a_n} = \frac{\frac{3^{n+1}}{3(n+1)^2+6^{n+1}}}{\frac{3^n}{3n^2+6^n}} = \frac{(3n^2+6^n)\cdot 3}{3n^2+6n+3+6^n\cdot 6} = \frac{9n^2+3\cdot 6^n}{3n^2+6n+3+6\cdot 6^n} < 1$$

Whether $\frac{9n^2+3\cdot 6^n}{3n^2+6n+3+6\cdot 6^n} < 1$

$\Leftrightarrow 9n^2+3\cdot 6^n < 3n^2+6n+3+6\cdot 6^n \Leftrightarrow 6n^2 < 3\cdot 6^n + 6n+3 \;/ : 6$

$\Leftrightarrow n^2 < 3\cdot 6^{n-1} + n + 1/2$

$\Rightarrow \sum (-1)^n a_n$ is convergent

D. iii) $\sum_{n=1}^{\infty} \frac{(-4)^n}{4n\sqrt{2n+1}} = \sum_{n=1}^{\infty} \frac{(-1)^n \cancel{4^n}}{\cancel{4^n}\sqrt{2n+1}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{2n+1}}$

$\sum \frac{1}{\sqrt{2n+1}} \sim \sum b_n$

$b_n = 1/\sqrt{n}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{\sqrt{2n+1}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{2n+1}} = \lim_{n\to\infty} \sqrt{\frac{n}{2n+1}} = \sqrt{\lim_{n\to\infty} \frac{n}{n(2+\underbrace{\frac{1}{n}}_{0})}} = \frac{1}{\sqrt{2}} \in \mathbb{R}^+$

$\sum a_n$ and $\sum b_n$ have same nature

$\Rightarrow$ divergent

$\sum \frac{(-1)^n 4^n}{4^n\sqrt{2n+1}} = \sum \frac{(-1)^n}{\sqrt{2n+1}}$

$a_n = \frac{1}{\sqrt{2n+1}}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} 1/\sqrt{2n+1} = 0$

$a_{n+1} = 1/\sqrt{2n+3} < a_n \; 1/\sqrt{2n+1} \quad (a_n) \downarrow$

$\Rightarrow \sum (-1)^n \; a_n$ convergent

E. Show that if $a_n = 5n^3 - 14n^2 + 7n - 3$, then $a_n = \Theta(n^3)$.

$\exists \alpha > 0 \quad \exists \beta > 0 \quad \exists N \quad \forall n > N$

$\alpha n^3 \leq 5n^3 - 14n^2 + 7n - 3 \leq \beta n^3$

$1 \cdot n^3 \leq 5n^3 - 14n^2 + 7n - 3 \leq 5n^3 + 14n^3 + 7n^3 + 3n^3 = 29n^3$

$\Leftrightarrow 0 \leq 4n^3 - 14n^2 + 7n - 3$

$14n^2 - 7n + 3 \leq 4n^3$

$\exists N = 10 \quad \forall n > N$ It holds

$\exists \alpha = 1$

$\exists \beta = 29$

$\exists N = 10 \; \forall n > N$

$\alpha \cdot \beta_n \leq a_n \leq \beta \cdot b_n$