Programming 2
Exercises

Lambda Calculus

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Lambda Calculus – parentheses

- Whole expression
  - $x = (x)$
- Abstraction body extends as far right as possible:

$$\lambda x.\, abc \ldots z = \lambda x.\, (abc \ldots z)$$

- Abstraction - association to the *right*:

$$\lambda x_1.\lambda x_2. \ldots .\lambda x_n. f(x_1, x_2, \ldots x_n) = \lambda x_1.\left(\lambda x_2.\left(\ldots .\lambda x_n. f(x_1, x_2, \ldots x_n)\right)\right)$$

- Application - association to the *left*:

$$x_1 x_2 \ldots x_n = \Big(\big((x_1)x_2\big) \ldots\Big) x_n$$

Parentheses - exercises

In the following λ-expressions, make all parentheses explicit:

- x
- xy
- $\lambda x.x$
- $\lambda xz.x\ y$
- $(\lambda x.x\ y)5$
- abc
- $\lambda x.xz\ \lambda y.ab$

Write with as little parentheses as possible:

- $((xy)(\lambda y.(\lambda z.(z(xy)))))$

Parentheses - exercises

In the following λ-expressions, make all parentheses explicit:

- $x = (x)$
- $xy = ((x)y)$
- $\lambda x. x = (\lambda x. (x))$
- $\lambda xz. x\ y = (\lambda xz. ((x)y))$
- $(\lambda x. x\ y)5 = ((\lambda x. ((x)\ y))5)$
- $abc = (((a)b)c)$
- $\lambda x. xz\ \lambda y. ab = (\lambda x. (((x)z)\lambda y. ((a)b))$

Write with as few parentheses as possible:

- $((xy)(\lambda y. (\lambda z. (z(xy))))) = xy\lambda y. \lambda z. z(xy)$

Free and bound variables

Find the free and bound variables in the following expressions:

- $\lambda x. xy$
- $(\lambda x. x)z$
- x
- $(\lambda x. xz)xz$
- $(\lambda x. x)(\lambda y. yx)x$
- $\lambda x. zy\lambda y. yx$

Free and bound variables

Find the free and bound variables in the following expressions:

- $\lambda x.xy$
- $(\lambda x.x)z$
- x
- $(\lambda x.xz)xz$
- $(\lambda x.x)(\lambda y.yx)x$
- $\lambda x.zy\lambda y.yx$

α-conversion

Problem:

- $(\lambda x. \lambda y. x + y)z \sim \lambda y. z + y$
- $(\lambda x. \lambda y. x + y)y \neq \lambda y. y + y$

α-conversion: Renaming *bound* variables in λ-expression yields equivalent λ-expression

- Example: $\lambda a. a \ \equiv \lambda b. b$

Solution:

- $(\lambda x. \lambda y. x + y)y = (\lambda x. \lambda z. x + z)y \sim \lambda z. y + z$

α-conversion

Simplify the expressions :

- $\big(\lambda f.\lambda x.f(fx)\big)(\lambda y.y+x)$

- $(\lambda xy.xyy)(\lambda y.y)y$

α-conversion

Simplify the expressions:

- $\big(\lambda f.\lambda x.f(fx)\big)(\lambda y.y+x) = \big(\lambda f.\lambda z.f(fz)\big)(\lambda y.y+x)$
$$\sim \lambda z.\Big((\lambda y.y+x)\big((\lambda y.y+x)z\big)\Big)$$
$$\sim \lambda z.z+x+x$$
- $(\lambda xy.xyy)(\lambda y.y)y = (\lambda xy_1.xy_1y_1)(\lambda y_2.y_2)y$
$$\sim (\lambda y_1.(\lambda y_2.y_2)y_1y_1)y$$
$$\sim (\lambda y_2.y_2)yy$$
$$\sim yy$$

Logical values and operators

- true = $\lambda t.\lambda f.t$
- false = $\lambda t.\lambda f.f$
- if = $\lambda p.\lambda m.\lambda n(p\ m\ n)$
- and = $\lambda p.\lambda q.(p\ q\ p)$
- or = $\lambda p.\lambda q.(p\ p\ q)$
- not = $\lambda p.(p \text{ false true})$

Examples in lecture!

Logical values and operators

Prove the “and” expression yields the logical And table:

p	q	(p & q)
True	True	True
True	False	False
False	True	False
False	False	False

Logical values and operators

Prove the “and” expression yields the logical And table:

p	q	(p & q)
True	True	True
True	False	False
False	True	False
False	False	False

- $\text{and t t} = (\lambda p.\lambda q.pqp)\ \text{t t} \sim (\lambda q.\text{t}\ q\ \text{t})\ \text{t} \sim \text{t t t}$
  $= (\lambda t.\lambda f.t)\ \text{t t} \sim \text{t}$
- $\text{and t f} = (\lambda p.\lambda q.pqp)\ \text{t f} \sim \text{t f t} = (\lambda t.\lambda f.t)\ \text{f t} \sim \text{f}$
- $\text{and f t} = (\lambda p.\lambda q.pqp)\ \text{f t} \sim \text{f t f} = (\lambda t.\lambda f.f)\ \text{t f} \sim \text{f}$
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