1 Pumping lemma for context-free languages

The pumping lemma for context-free languages (Bar-Hillel lemma): Let L be a CFL, then there exists $p > 0$ such that if $z \in L$ and $|z| \geq p$, then z can be written as $uvwxy$ where

1. $|vwx| \leq p$
2. $vx \neq \varepsilon$
3. $uv^kwx^ky \in L \quad \forall k \geq 0$

Pumping Lemma for CFL can be used to prove that some languages are not context free.

Exercise 1:
$\Sigma = \{a, b, c\}$
$L = \{a^nb^nc^n \mid n \geq 0\}$
Solution: Consider string $z = a^pb^pc^p$. Let's assume L is a CFL, and because $|z| \geq p$, the pumping lemma for CFL should hold, meaning for $\forall u, v, w, x, y \in \{a, b, c\}^*$ with $|vwx| \leq p$ and $vx \neq \varepsilon$, any string after pumping should still be in the language: $z' = uv^kwx^ky \in L$ should be true for $\forall k \geq 0$. However, $uwy \notin L$ (this is the case where $k = 0$): vx will not contain at least one of the symbols in $\{a, b, c\}$, and as a result, the new string cannot contain the exact same number of each after pumping. This leads to a contradiction with our original assumptions, so the language is not a CFL.

Exercise 2:
$\Sigma = \{a, b, c\}$
$L = \{a^nb^mc^n \mid n \geq m\}$
Solution: Consider string $z = a^pb^p$. Let's assume L is a CFL, and because $|z| \geq p$, the pumping lemma for CFL should hold, meaning for $\forall u, v, w, x, y \in \{a, b, c\}^*$ with $|vwx| \leq p$ and $vx \neq \varepsilon$, any string after pumping should still be in the language. This can lead to two different cases: because of $|vwx| \leq p$, vx (the part that we pump) can either contain only a-s or only c-s, or it can contain b-s as well (either exclusively, or mixed with a-s or c-s).
If $\mid vx \mid_b = 0$, then for $k = 0$ the string $z' = uwy \notin L$, there will be either less a-s or c-s than p because of pumping with 0.
If $\mid vx \mid_b > 0$, so the part that we pump contains at least one b, then $k = 2$ (or any $k > 1$) will result in a z' string where the number of b-s is greater than p (the number of a-s and c-s), and this $z' \notin L$.
Because we came to a contradiction with our original assumption all possible cases, the language is not a CFL.

Exercise 3:
$\Sigma = \{a, b, c\}$

$L = \{a^i b^j c^k \mid i \leq j \leq k\}$
Solution hint: Try to choose $z = a^p b^p c^p$, and showing the contradiction should be similar to Exercise 2 (except this time your cases will not depend on b) .

Exercise 4:
$\Sigma = \{a, b\}$
$L = \{ss \mid s \in \{a, b\}^*\}$
Solution hint: Try to choose $s = a^p b^p a^p b^p$, and showing the contradiction should be manageable from here.

Exercise 5:
$\Sigma = \{a\}$
$L = \{a^i \mid i \text{ prime }\}$
Solution hint: Let $n \geq p$ be a prime, and choose string $z = a^n$. Set $\mid uwy \mid= m$ (the length of the fixed part of your string that's not affected by pumping), choose $k = n + 1$ as your pumping number, and from here you can show that you can factorize the length of your new string (with the help of n and m), which is a contradiction, as primes cannot be factorized any further.