Surface Signature

Abhijeet Avhale

January 2025

1 Cross Modules

We start with a crossed module of Lie groups:

$$H \xrightarrow{\tau} G \xrightarrow{\triangleright} Aut(H)$$

where τ and $\triangleright$ are Lie group morphisms, satisfying the identities

$$\tau \circ \triangleright_g = Ad_g \circ \tau \qquad g \in G$$
$$\triangleright_{\tau(h)} = Ad_h \qquad h \in H$$

τ is called the **feedback** and $\triangleright$ the **action**.

Which is the general linear crossed module of the 2-vector space

Now, we take a 2-vector space V of two classical vector spaces V_0 and V_1 and a linear transformation ϕ:

$$V = V_1(\in \mathbb{R}^{n+p}) \xrightarrow{\phi} V_0(\in \mathbb{R}^{n+q})$$

We choose basis for V_1 and V_0 such that linear transformation ϕ has the form:

$$\phi = \begin{pmatrix} I_n & 0_{n\times p} \\ 0_{q\times n} & 0_{q\times p} \end{pmatrix}$$

The group G (GL_0) contains the invertible chain maps (F_v, F_u) such that $\phi F_v = F_u \phi$. Writing F_v and F_u in block matrix form,

$$F_v = \begin{bmatrix} P & Q \\ R & S \end{bmatrix}$$

$$F_u = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

Therefore, using chain map condition:

$$\phi F_v = \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} P & Q \\ R & S \end{bmatrix} = \begin{bmatrix} P & Q \\ 0 & 0 \end{bmatrix}$$

$$F_u \phi = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix}.$$

$$\therefore \begin{bmatrix} P & Q \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix}$$

i.e. P = A, Q = 0 and C = 0, Hence, G (GL_0) is given by

$$GL^0_{n,p,q} := \left\{ \left(\begin{bmatrix} P & 0_{n\times p} \\ R & S \end{bmatrix}, \begin{bmatrix} P & B \\ 0_{q\times n} & D \end{bmatrix} \right) \,\middle|\, \begin{matrix} P \in GL_n(\mathbb{R}), & R \in \mathbb{R}^{p\times n}, & \\ S \in GL_p(\mathbb{R}), & B \in \mathbb{R}^{n\times q}, & D \in GL_q(\mathbb{R}) \end{matrix} \right\}.$$

This is a group via entrywise matrix multiplication.

Write a chain homotopy $s : (f_V, f_U) \sim (\mathrm{id}_V, \mathrm{id}_U)$ as

$$s = \begin{bmatrix} K & L \\ M & N \end{bmatrix}.$$

Then the homotopy conditions

$$f_V - \mathrm{id}_V = \varphi \circ s, \quad f_U - \mathrm{id}_U = s \circ \varphi,$$

give

$$K = P - \mathrm{id}_n, \quad M = R, \quad L = B.$$

Moreover, the target of the homotopy must satisfy

$$S - \mathrm{id}_p = 0, \quad D - \mathrm{id}_q = 0.$$

Then $GL_1(V)$ is given by

$$GL^{-1}_{n,p,q} := \left\{ \begin{bmatrix} P - \mathrm{id}_n & B \\ R & N \end{bmatrix} \,\middle|\, P \in GL_n(\mathbb{R}),\ R \in \mathbb{R}^{p\times n},\ N \in \mathbb{R}^{p\times q},\ B \in \mathbb{R}^{n\times q} \right\}.$$

The group multiplication is given by horizontal composition:

$$\begin{pmatrix} P - id_n & B \\ R & N \end{pmatrix} \bullet_h \begin{pmatrix} P' - id_n & B' \\ R' & N' \end{pmatrix} = \begin{pmatrix} P'P - id_n & P'B + B' \\ R'P + R & R'B + N + N' \end{pmatrix}.$$

The feedback sends a homotopy to its target:

$$\begin{pmatrix} P - id_n & B \\ R & N \end{pmatrix} \mapsto \left(\begin{pmatrix} P & 0_{n\times p} \\ R & \mathrm{id}_p \end{pmatrix}, \begin{pmatrix} P & B \\ 0_{q\times n} & \mathrm{id}_q \end{pmatrix} \right).$$

We hence have

$$\ker \tau = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & N \end{pmatrix} \mid N \in \mathbb{R}_{p\times q} \right\}.$$