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A modal Udwadia-Kalaba formulation for vibro-impact modelling of continuous flexible systems with intermittent contacts

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Abstract

Most systems consist on dynamical substructures connected at a number of constraining points. Moreover, constraints often display intermittent contact phenomena, such as arising from clearance supports. A significant difficulty when computing time-domain responses is the manner to enforce such coupling constraints. Here, we explore the Udwadia-Kalaba (U-K) formulation, which has been very seldom used in this context. By extending the basic U-K analytical framework, we address continuous flexible subsystems modelled by their unconstrained modes and coupled through the highly nonlinear intermittent point-constraints. For continuous flexible systems, a modal U-K formulation is implemented such that the constraint is applied when contact is detected at the clearance location. A crucial aspect is that constraint violations must be prevented, not only at the acceleration level, but also at the velocity and displacement levels, in order to avoid computational drift. This is achieved through a constraint violation correction method. For single gap-constraints, a convenient formulation is obtained, in which the constraint matrix is pre-computed prior to the simulation time-loop and applied whenever an intermittent contact is detected, leading to an efficient computation of vibro-impact responses. For systems with several intermittent constraints, an essential difficulty within the context of the proposed formulation is that every possible combination of contact/non-contact conditions is expressed by a different constraint matrix. A pragmatic solution is to keep track of the current system contact configuration and rebuild the constraint matrix whenever a change in the constraint state is detected. We formulate and illustrate such computational strategy, as applied to random-excited multi-supported beams with a significant number of clearance supports. Results are compared with dynamical computations performed using a classic penalty technique for enforcing the nonlinear support constraints, emphasizing the viability of the proposed technique for performing predictive analysis of flexible structures with multiple clearance supports.

Keywords: Vibrations; Vibro-impacts; Time-domain simulations; Udwadia-Kalaba.

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1. Introduction

This paper deals with flexible constrained systems, and their effective dynamical modelling and computation when dealing with intermittent constraints. Most systems of fundamental and industrial interest consist on dynamical substructures connected at a number of constraining points. Coupling is therefore an essential feature in many systems and most modelling and computational difficulties are connected with the manner in which the coupling constraints are enforced. Typically, these are modelled using standard techniques such as Lagrange multipliers or penalty methods, each one with specific merits and drawbacks. In the present paper we explore a different approach, the Udwadia-Kalaba (U-K) formulation [1,2], originally proposed in the early 90s for discrete constrained systems, which is anchored on analytical dynamics but avoids the explicit use of Lagrange multipliers. It leads to constrained formulations in terms of standard Ordinary Differential Equation (ODE) systems, even when a redundant set of coordinates is used, instead of numerically challenging mixed Differential-Algebraic Equations (DAE).

Up to now, this general and elegant formulation has been nearly exclusively used for conceptual systems of discrete masses or articulated rigid bodies. To the authors' best knowledge, the single exception in the literature is the investigation [3], which addressed a flexible slider-crank mechanism modelled using a Finite Element beam formulation. However, in spite of the possible natural extension of the U-K formulation to deal with flexible systems modelled through their unconstrained modes, such promising approach seems surprisingly absent from the literature. Recently we have extended the basic U-K analytical framework in order to deal with continuous flexible subsystems modelled by their unconstrained modes and coupled through linear point-constraints [4]. This modal U-K approach was then applied to compute the coupled dynamics of a complex constrained structure, showing that the proposed technique provides reliable results, with significant computational efficiency.

For many systems of mechanical interest, constraints often display intermittent contact phenomena, such as arising from clearance supports, see for instance [5,6]. Very scarce work has been published on the application of the U-K formulation to systems with intermittent contacts [7,8], typically addressing near-rigid mechanism devices with few contact locations. In the present paper we further extend our previous work [4], in order to deal with intermittent point-constraints, which are by nature highly nonlinear. For continuous systems displaying a single clearance support, the modal U-K formulation is developed such that a support constraint is applied only when contact is detected at the corresponding clearance location. In the convenient formulation obtained, constraint matrices may be precomputed prior to the simulation time-loop, leading to an efficient computation of the highly nonlinear responses.

Then, the difficult case consisting on systems with several intermittent constraints is addressed. In the framework of our extended modal U-K formulation, a difficulty posed by structures with multiple intermittent constraints is that every possible combination of contact/non-contact conditions is expressed by a different constraint matrix. When the number of intermittent contact locations increases, the huge set of possible constraint states renders precomputation of all possible constraint matrices impractical. Therefore, a different computational strategy is implemented, by updating the constraint matrix within the time-loop whenever a change of state is detected. Another crucial aspect with intermittent constraints is that constraint violations must be prevented, not only at the acceleration level (the standard U-K procedure), but also at the velocity and displacement levels, in order to prevent unacceptable motion drifts. This is achieved through a constraint violation correction method based on geometric projection, see [9,10].

The developed computational approach is illustrated on a pinned-pinned beam structure with a clearance support at mid-length, subjected to a near-distributed random excitation. Although much better flow turbulence excitation modelling is provided in [11], this crude model of turbulence excitation is adequate for the present paper. Then, the computational strategy is applied to a random-excited multi-supported flexible beam with a significant number of clearance supports, typical of the tubular bundles found in steam generators and nuclear fuel assemblies [12,13]. This analysis significantly expands on the scarce results found in the literature focusing on the application of a U-K formulation to systems with intermittent contacts, which typically addressed mechanisms with rigid components and/or single clearance locations [7,8]. Comparison with dynamical computations using a classic penalty technique for enforcing the nonlinear support constraints emphasizes the viability of the proposed simulation technique.

2. Theoretical formulation

2.1. Udwadia-Kalaba formulation

The original Udwadia-Kalaba formulation was deduced from Gauss' Principle of Least Action. A simple and elegant approach for obtaining the U-K formulation for constrained systems, directly from the classical formulation with Lagrange multipliers, is offered in a paper by Arabyan and Wu [7]. A system of M particles with mass matrix $\mathbf{M}$ is subjected to an external force vector $\mathbf{F}_e(t)$ of constraint-independent forces and a set of $P = P_h + P_{nh}$ holonomic and non-holonomic constraints, depending on the system displacement $\mathbf{x}(t)$ and velocity $\dot{\mathbf{x}}(t)$, as well as explicitly on time, given by the general equations:

$$\boldsymbol{\varphi}_p(\mathbf{x},t) = 0 \quad , \quad p = 1,2,\cdots,P_h \tag{1}$$

$$\boldsymbol{\psi}_p(\mathbf{x},\dot{\mathbf{x}},t) = 0 \quad , \quad p = P_h + 1,\cdots,P \tag{2}$$

which, by time-differentiation, is written as a matrix-vector constraint system in terms of accelerations:

$$\mathbf{A}\ddot{\mathbf{x}} = \mathbf{b} \tag{3}$$

where the $P \times M$ matrix $\mathbf{A}\left(\mathbf{x}(t),\dot{\mathbf{x}}(t),t\right)$ and $P \times 1$ vector $\mathbf{b}\left(\mathbf{x}(t),\dot{\mathbf{x}}(t),t\right)$ are functions of the motion. The P constraints need not be independent, therefore the rank r of matrix $\mathbf{A}$ is $r \le P$. The dynamical solution $\mathbf{x}_u(t)$ of the unconstrained system is obviously given by:

$$\ddot{\mathbf{x}}_u = \mathbf{M}^{-1}\mathbf{F}_e \tag{4}$$

while, using the method of Lagrange multipliers, the response $\mathbf{x}(t)$ of the constrained system also depends on the vector $\mathbf{F}_c(t) = -\mathbf{A}^T\boldsymbol{\lambda}(t)$ of constraining forces:

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{A}^T\boldsymbol{\lambda} = \mathbf{F}_e \tag{5}$$

and, from (3) and (5), an augmented index-one DAE formulation is built for the dynamics of the constrained system:

$$\begin{bmatrix} \mathbf{M} & \mathbf{A}^T \\ \mathbf{A} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{x}} \\ \boldsymbol{\lambda} \end{Bmatrix} = \begin{Bmatrix} \mathbf{F}_e \\ \mathbf{b} \end{Bmatrix} \tag{6}$$

Then, assuming the matrix in (6) invertible and defining $\mathbf{B} = \mathbf{A}\mathbf{M}^{-1/2}$, one obtains the explicit solution [1,4,7]:

$$\ddot{\mathbf{x}} = \ddot{\mathbf{x}}_u + \mathbf{M}^{-1/2}\mathbf{B}^+\left(\mathbf{b} - \mathbf{A}\ddot{\mathbf{x}}_u\right) \tag{7}$$

where $\mathbf{B}^+$ is the Moore-Penrose pseudo-inverse of matrix $\mathbf{B}$, which shows the correction brought by the constraints to the unconstrained acceleration vector. On the other hand, the corresponding constraint forces are obtained as:

$$\mathbf{F}_c = \mathbf{M}^{1/2}\mathbf{B}^+\left(\mathbf{b} - \mathbf{A}\ddot{\mathbf{x}}_u\right) \tag{8}$$

Equations (7) and (8) are the basic results obtained by Udwadia and Kalaba, which may be applied to linear or nonlinear, conservative or dissipative systems. For a given excitation $\mathbf{F}_e(t)$, formulation (7) may be efficiently solved using a suitable time-step integration scheme. The superlative elegance of the U-K formulation lays in the fact that it encapsulates, in a single explicit equation, both the dynamical equations of the system and the constraints applied. The constraint forces (8) are additional information, not essential for building the system time-domain constrained solution. Moreover, due to the specific features of the Moore-Penrose pseudo-inverse, equation (7) always leads to constrained formulations in terms of standard ODE systems, even when non-independent or redundant constraints are used, avoiding numerically challenging mixed DEA systems.

2.2. Modal U-K formulation

We now adapt the U-K formulation to deal with continuous flexible systems whose dynamics are described in terms of their unconstrained modal coordinates. Formulation (7) is transformed to the modal space through:

$$\mathbf{x} = \boldsymbol{\Phi}\boldsymbol{q} \quad \Rightarrow \quad \dot{\mathbf{x}} = \boldsymbol{\Phi}\dot{\boldsymbol{q}} \quad \Rightarrow \quad \ddot{\mathbf{x}} = \boldsymbol{\Phi}\ddot{\boldsymbol{q}} \tag{9}$$

where, for $s = 1,2,\cdots,S$ constrained subsystems each consisting on N_s unconstrained modes, we define the vectors which assemble the corresponding physical responses $\mathbf{x}^s(t)$ and modal responses $\boldsymbol{q}^s(t)$, as well as the matrix which assembles the modeshapes $\boldsymbol{\Phi}^s$. Then, as discussed by Antunes and Debut [4], replacing (9) into (7) leads to the U-K formulation in terms of the subsystems unconstrained modes:

$$\ddot{\boldsymbol{q}} = \ddot{\boldsymbol{q}}_u + \boldsymbol{M}^{-1/2}\boldsymbol{B}^+\left(\mathbf{b} - \boldsymbol{A}\ddot{\boldsymbol{q}}_u\right) \tag{10}$$

with the modal quantities $\boldsymbol{M} = \boldsymbol{\Phi}^T\mathbf{M}\boldsymbol{\Phi}$, $\boldsymbol{A} = \mathbf{A}\boldsymbol{\Phi}$ and $\boldsymbol{B} = \boldsymbol{A}\boldsymbol{M}^{-1/2}$. Formulation (10) in terms of the modal quantities is quite similar to the U-K formulation (7) in physical coordinates, except for the introduced changes leading to the modified constraint matrices $\boldsymbol{A}(t)$ and $\boldsymbol{B}(t)$. The modal forces stemming from the constraints are:

$$\boldsymbol{F}_c = \boldsymbol{M}^{1/2}\boldsymbol{B}^{+}\left(\mathbf{b} - \boldsymbol{A}\ddot{\boldsymbol{q}}_u\right) \tag{11}$$

Finally, the physical motions are obtained from the modal responses (10) using (9), while from (11) the physical constraining forces are computed as:

$$\mathrm{F}_c = \left(\boldsymbol{\Phi}_c^T\right)^{+}\boldsymbol{M}^{1/2}\boldsymbol{B}^{+}\left(\mathbf{b} - \boldsymbol{A}\ddot{\boldsymbol{q}}_u\right) \tag{12}$$

where the modeshapes of matrix $\boldsymbol{\Phi}_c$ are taken at the constraint locations. For further details, see [4].

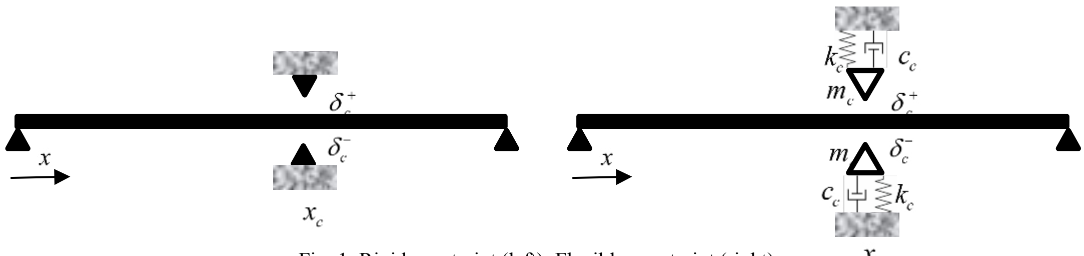


Fig. 1. Rigid constraint (left); Flexible constraint (right).

3. Intermittent constraints

3.1. Computational strategy

Fig. 1 (left) shows the basic scheme for imposing a single rigid intermittent constraint at a clearance support. When the beam is vibrating without contact, no constraint is imposed, whereas constraints are imposed whenever contact is detected. Defining the row vector $\boldsymbol{\Phi}_c^T$ containing the values of the beam modeshapes at the support location, contact tests are performed at each time step with respect to the beam local motion amplitude $X(x_c,t) = \boldsymbol{\Phi}_c^T\boldsymbol{q}(t)$ and the support gaps δ_c^- and δ_c^+ of the lower and upper stops:

$$\begin{cases} X(x_c,t) \le \delta_c^- & \boldsymbol{A} = \boldsymbol{\Phi}_c^T \quad \mathbf{b} = \delta_c^- \\ \delta_c^- < X(x_c,t) < \delta_c^+ & \boldsymbol{A} \equiv 0 \quad\;\; \mathbf{b} \equiv 0 \\ X(x_c,t) \ge \delta_c^+ & \boldsymbol{A} = \boldsymbol{\Phi}_c^T \quad \mathbf{b} = \delta_c^+ \end{cases} \tag{13}$$

Formulation (13) applies the right kinematical constraints for rigid clearance supports. When addressing flexible supports, see Fig. 1 (right), these may be seen as additional oscillators with parameters m_c, c_c and k_c, whose dynamics are computed, similarly to the beam dynamics. The relevant contact tests then become:

$$\begin{cases} X(x_c,t) \le \delta_c^- + X_c^-(t) & \boldsymbol{A} = \left[\boldsymbol{\Phi}_c^T \;\; -1 \;\; 0\right] \quad \mathbf{b} = \delta_c^- \\ \delta_c^- + X_c^-(t) < X(x_c,t) < \delta_c^+ + X_c^+(t) & \boldsymbol{A} \equiv 0 \qquad\qquad\quad\; \mathbf{b} \equiv 0 \\ X(x_c,t) \ge \delta_c^+ + X_c^+(t) & \boldsymbol{A} = \left[\boldsymbol{\Phi}_c^T \;\; 0 \;\; -1\right] \quad \mathbf{b} = \delta_c^+ \end{cases} \tag{14}$$

where $X_c^+(t)$ and $X_c^-(t)$ are the dynamical responses of the beam stops, respectively at the positive and negative gaps (they are ordered after the vector of the beam modal responses, first the negative stop then the positive).

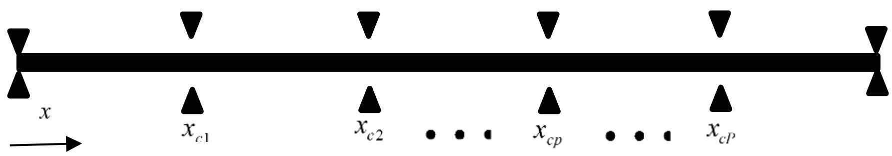


Fig. 2. Multi-supported beam with clearances.

Formulations (13) or (14) straightforwardly apply when there is no contact ($\boldsymbol{A} \equiv 0$, $\mathbf{b} \equiv 0$), as well as from the instant when contact is detected ($\boldsymbol{A} \neq 0$, $\mathbf{b} \neq 0$). However, a straight application of such formulae would induce a permanent constraint after the first contact with the upper or lower stop (whichever occurs first), as no release mechanism is embedded in (13) or (14). In order to allow for constraint release, as soon the as the contact force $\mathrm{F}_c(t)$ changes its sign, meaning that support compression becomes traction - which is incompatible with the physics of unilateral contacts - the constraint is released ($\boldsymbol{A}\equiv 0$, $\mathbf{b}\equiv 0$). When only a single clearance support exists, formulations (10)-(14) are extremely efficient, because the number of constraint matrices is limited. All possible $\boldsymbol{A}$ and $\boldsymbol{B}^{+}$ appearing in (13)-(14) can be precomputed prior to the time-loop and replaced in (10)-(12) as needed.

We now turn to systems with multiple intermittent constraints, as typified by the multi-supported beam with clearances illustrated in Fig. 2. Matrix $\mathbf{A}$ encapsulates all the active constraints at each time-step and, for such systems, the constraint matrix can change widely in time. When the number of intermittent contact locations becomes large, or even moderate, the number of possible constraint states increases dramatically. For instance, assuming a system with planar motions and P clearance supports, each one enabling 3 possible dynamical states (no contact, contact at stop δ_{cp}^{-} or contact at stop δ_{cp}^{+}), the number of possible constraint matrices is $N_c = 3^P$ (10 supports lead to about 59000 different contact configurations). Under such circumstances, precomputing all the possible constraint matrices prior to the time-loop is impractical. A pragmatic solution is to keep track of the current system contact configuration, and rebuild the constraint matrix whenever a change in the constraint state is detected.

Defining the row vectors $\mathbf{\Phi}_{cp}^{T}$ containing the values of the beam modeshapes at the $p=1,2,\cdots,P$ support locations, contact tests are performed at each time step with respect to the beam local motion amplitudes $X(x_{cp},t)=\mathbf{\Phi}_{cp}^{T}\boldsymbol{q}(t)$ and the support gaps δ_{cp}^{-} and δ_{cp}^{+} . Defining each line of $\mathbf{A}$ as $\mathbf{a}_p^T$ and also each element of $\mathbf{b}$ as b_p , the rigid constraints (13) are generalized as:

$$\begin{cases} X(x_{cp},t)\le\delta_{cp}^{-} & \mathbf{a}_p^T=\mathbf{\Phi}_c^T \quad b_p=\delta_{cp}^{-} \\ \delta_{cp}^{-}<X(x_{cp},t)<\delta_{cp}^{+} & \mathbf{a}_p^T\equiv 0 \quad\;\; b_p\equiv 0 \\ X(x_{cp},t)\ge\delta_{cp}^{+} & \mathbf{a}_p^T=\mathbf{\Phi}_c^T \quad b_p=\delta_{cp}^{+} \end{cases} \quad ; \quad p=1,2,\cdots,P \tag{15}$$

while, when addressing flexible supports with parameters m_c , c_c and k_c , constraints (14) become:

$$\begin{cases} X(x_{cp},t)\le\delta_{cp}^{-}+X_{cp}^{-}(t) & \mathbf{a}_p^T=\left[\mathbf{\Phi}_{cp}^T \;\; -1 \;\; 0\right] \quad b_p=\delta_{cp}^{-} \\ \delta_{cp}^{-}+X_{cp}^{-}(t)<X(x_{cp},t)<\delta_{cp}^{+}+X_{cp}^{+}(t) & \mathbf{a}_p^T\equiv 0 \quad\quad\quad\quad\quad\;\; b_p\equiv 0 \\ X(x_{cp},t)\ge\delta_{cp}^{+}+X_{cp}^{+}(t) & \mathbf{a}_p^T=\left[\mathbf{\Phi}_{cp}^T \;\; 0 \;\; -1\right] \quad b_p=\delta_{cp}^{+} \end{cases} \quad ; \quad p=1,2,\cdots,P \tag{16}$$

where $X_{cp}^{+}(t)$ and $X_{cp}^{-}(t)$ are the dynamical responses of the beam stops at location p, respectively at the positive and negative gaps (ordered after the vector of the beam modal responses, first the negative stop then the positive).

As discussed, a vector $\mathbf{s}(t)$ of size P describes the constraint state at each time-step, with the $p=1,2,\cdots,P$ terms taking one of 3 values, according to the test results performed in (15) or (16). Then, whenever $\mathbf{s}(t_i)\neq\mathbf{s}(t_{i-1})$, the constraint matrix $\boldsymbol{A}(t_i)$ is updated and $\boldsymbol{B}^{+}(t_i)$ is recomputed.

3.2. Enforcing displacement and velocity constraints

As for the classical formulation (6), the U-K formulations (7) or (10) also apply constraints at the acceleration level. Therefore, unless some complementary stabilization technique is used, numerical drift will cumulate during computations, while constraints become progressively violated at the displacement and velocity levels. For linear constraints, the initial conditions typically warrant reasonable simulation times without significant drift. However, vibro-impacting systems will typically experience immediate catastrophic drift, as illustrated in Fig. 3 (left), which shows a beam response at the single clearance-support location (gap-stops are plotted in magenta). Many strategies have been proposed for enforcing constraints at the displacement and velocity level [9,10]. Baumgarte's stabilization technique [14], which is quite popular, may be implemented with respect to the U-K formulation as follows [7]:

$$\ddot{\mathbf{x}}=\ddot{\mathbf{x}}_u+\mathbf{M}^{-1/2}\mathbf{B}^{+}\left(\mathbf{b}^{*}-\mathbf{A}\ddot{\mathbf{x}}_u\right) \quad ; \quad \mathbf{b}^{*}=\mathbf{b}-\alpha_1\boldsymbol{\psi}_p(\mathbf{x},\dot{\mathbf{x}},t)-\alpha_2\boldsymbol{\varphi}_p(\mathbf{x},t) \tag{17}$$

where $\varphi_p(\mathbf{x},t)$ and $\psi_p(\mathbf{x},\dot{\mathbf{x}},t)$ are the low-derivative constraint violations, while α_1 and α_2 are suitable coefficients. Unfortunately, these are case-dependent and no general criteria exist for choosing their values. Moreover, formulation (17) introduces an artificial time-scale connected with the constrain violation stabilization, which somewhat contradicts the advantages of equation (7). Therefore, in this work we implemented the violation elimination technique proposed by Yoon et al. [15], based on a geometric projection approach, which is applied after each integration time-step. First, displacement violations are eliminated by correcting the constrained displacements:

$$\mathbf{x}_c = \mathbf{x} + \Delta\mathbf{x} \quad \Rightarrow \quad \mathbf{x}_c = \mathbf{x} - \mathbf{A}^+ \boldsymbol{\varphi}_p(\mathbf{x},t) \quad \Rightarrow \quad \boldsymbol{q}_c = \boldsymbol{q} - \boldsymbol{A}^+ \boldsymbol{\varphi}_p(\boldsymbol{q},t) \tag{18}$$

then a similar procedure is applied to the velocity constraint violations:

$$\dot{\mathbf{x}}_c = \dot{\mathbf{x}} + \Delta\dot{\mathbf{x}} \quad \Rightarrow \quad \dot{\mathbf{x}}_c = \dot{\mathbf{x}} - \mathbf{A}^+ \boldsymbol{\psi}_p(\mathbf{x}_c,\dot{\mathbf{x}},t) \quad \Rightarrow \quad \dot{\boldsymbol{q}}_c = \dot{\boldsymbol{q}} - \boldsymbol{A}^+ \boldsymbol{\psi}_p(\boldsymbol{q}_c,\dot{\boldsymbol{q}},t) \tag{19}$$

leading, after using the corrections (18)-(19), to the much improved results shown in Fig. 3 (right).

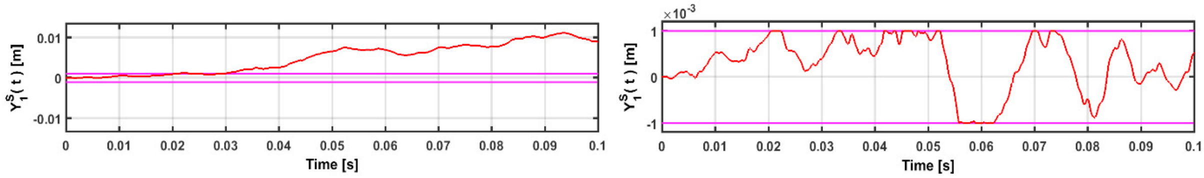


Fig. 3. Vibro-impact simulation with rigid clearance supports using the modal U-K algorithm: Computation without displacement and velocity constraint enforcing (left); Displacement and velocity constraint enforcing through geometric projection elimination (right).

4. Intermittent constraints using the penalty approach

For comparison purposes, results obtained using the U-K approach will be confronted with similar computations performed using a penalty method. Here a simple constraint force was implemented, following the Hunt and Crossley [16] model. For the general case of multi-supported beams, one has:

$$\begin{cases} X(x_{cp},t) \le \delta_{cp}^- + X_{cp}^-(t) & \mathbf{F}_{cp} = -k_p\left[X(x_{cp},t) - X_{cp}^-(t) - \delta_{cp}^-\right]\left(1+\mu_p\left[\dot{X}(x_{cp},t) - \dot{X}_{cp}^-(t)\right]\right) \\ \delta_{cp}^- + X_{cp}^-(t) < X(x_{cp},t) < \delta_{cp}^+ + X_{cp}^+(t) & \mathbf{F}_c \equiv 0 \\ X(x_{cp},t) \ge \delta_{cp}^+ + X_{cp}^+(t) & \mathbf{F}_{cp} = -k_p\left[X(x_{cp},t) - X_{cp}^+(t) - \delta_{cp}^+\right]\left(1+\mu_p\left[\dot{X}(x_{cp},t) - \dot{X}_{cp}^+(t)\right]\right) \end{cases} \quad ; \quad p = 1,2,\cdots,P \tag{20}$$

where k_p and μ_p are stiffness and damping penalty parameters, respectively. Notice that, opposite to the U-K approach, contact interpenetration is essential in the penalty method. Moreover, the interaction force will depend on the penalty parameters, therefore the resulting dynamics and interaction forces should display differences. Forces (20) are projected on the beam modes and, in the case of flexible supports, also applied to the clearance stops.

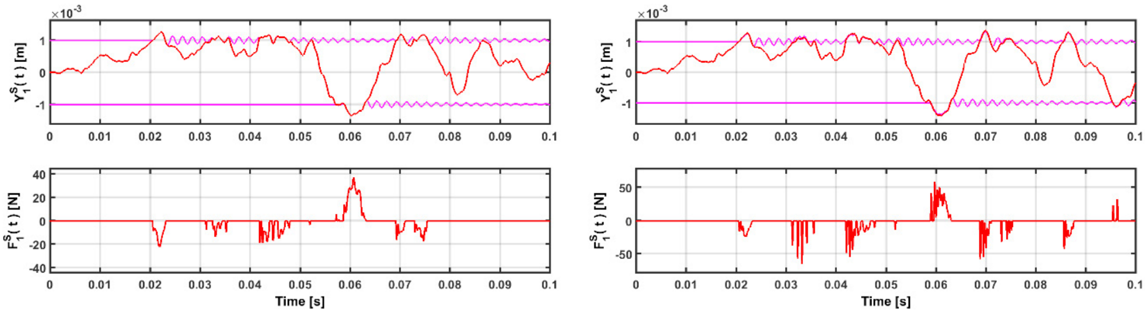


Fig. 4. Vibro-impact simulation with flexible clearance supports: Using the modal U-K algorithm with flexible clearance supports and constraint enforcing through geometric projection elimination (left); Using a penalty formulation (right).

5. Illustrative results

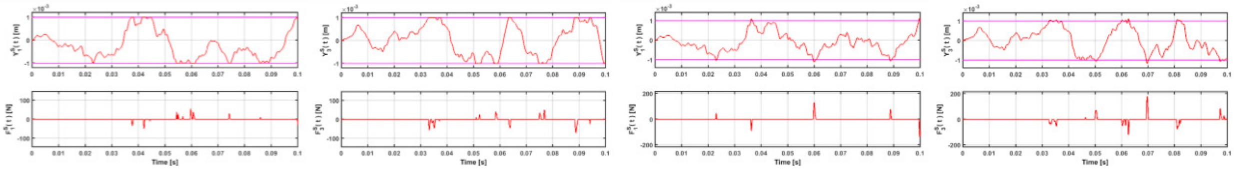
5.1. Single nonlinear constraint

As for the simulations presented in Fig. 3, the results shown in Fig. 4 are computed with respect to a beam with length $L = 2$ m with a clearance support at mid-span. The clearances are $\delta_c^+ = -\delta_c^- = 0.001$ m and the stops are either rigid or else flexible with parameters $m_c = 0.01$ Kg, $c_c = 1$ Ns/m and $k_c = 10^5$ N/m. The beam is excited by 18 equidistant random point-forces with constant excitation spectra in the range 0~1000 Hz and amplitude 5 N_{RMS}. Dynamics are modelled using $N = 48$ beam modes, unconstrained at the clearance supports, with frequencies $f_n = f_1 n^2$ in the range 1~2300 Hz (with modal stiffnesses up to $k_N = 10^8$ N/m), modal damping $\zeta_n = 0.01\,(\forall n)$, modal masses $m_n = 0.5\ \text{Kg}\ (\forall n)$ and modeshapes $\phi_n(x) = \sin(n\pi x / L)$.

Time-domain integrations are performed using a Velocity-Verlet explicit scheme, with a time-step 10^{-6} sec. Results in Fig. 3 (right) illustrated the response dynamics obtained using the U-K formulation with rigid supports. The more complex case with flexible vibrating stops is illustrated in Fig. 4, with the left and right plots pertaining to the U-K and the penalty formulation (using $k_p = 10^6$ N/m and $\mu_p = 0.1$), respectively. These are compatible, although differences in the response and contact force are visible, as might be expected.

5.2. Multiple nonlinear constraints

As previously, the following results were computed for a beam with $L = 2$ m, but now provided with 6 clearance supports at locations $x_{cp} = 0.286, 0.571, 0.857, 1.143, 1.429$ and 1.714 m. The beam and support data was reported above, as was the excitation and integration parameters. Figures 5 to 7 summarize the results obtained.



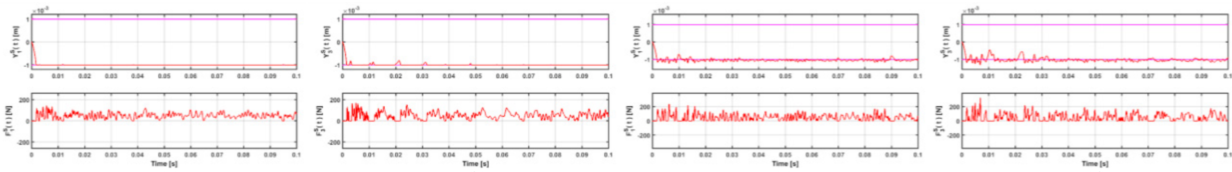
(a) (b) (c) (d)

Fig. 5. Vibro-impact simulation with rigid clearance supports (shown are results at supports 1 and 3): Using the modal U-K algorithm with flexible clearance supports and constraint enforcing through geometric projection elimination (a,b); Using a penalty formulation (c,d).

The plots in Fig. 5 (left) illustrate the response dynamics and contact forces obtained using the U-K formulation with rigid supports (for space reasons only the responses at clearance supports 1 and 3 are shown). The results of Fig. 5 (right) pertain to the penalty formulation. Notice that, although compatible, the results display visible differences, which for highly nonlinear multi-supported vibro-impacting systems are unavoidable. Also, for the illustrated value of the penalty parameter k_p, the impact forces computed using the penalty approach are higher than those stemming from the U-K method. The dependence of contact forces on k_p is understandable, however it is well known that the integral of the contact forces, $I_{cg} = \int_0^{T_{cg}} F_{cg}(t)dt$, is much less dependent on the penalty stiffness.

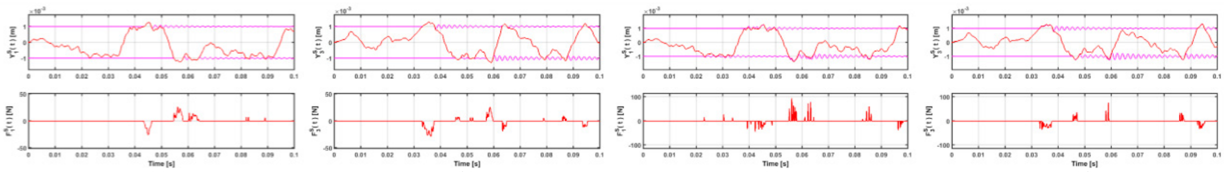
The computations in Fig. 6 pertain to vibro-impact regimes with near permanent contact, obtained by imposing to the beam point-loads of 50 N at each support location. These computations seem again compatible, showing that - beyond being able to deal with spiky impact forces - both modelling techniques are adapted to address continuous contact constraints as well. Also notice that the kinematic constraint enforcing of the U-K approach (left) prevent any interpenetration at the supports. On the contrary, by construction, the penalty approach (right) implies interpenetration at the constraint locations, a fact that is clearly highlighted in the beam response plots.

The last computations in Fig. 7 illustrate the dynamics of the multi-supported beam with compliant clearance supports. Again the computations using both approaches are compatible, although the beam responses and impact forces display some visible differences. In particular, beyond the characteristic time-scales of the beam modes and the flexible stops, the contact forces also display a characteristic time-scale stemming from the penalty parameter.



(a) (b) (c) (d)

Fig. 6. Vibro-impact simulation with rigid clearance supports when preloads of 50N are applied at support locations (shown are results at supports 1 and 3): Using the modal U-K algorithm with flexible clearance supports and constraint enforcing through geometric projection elimination (a,b); Using a penalty formulation (c,d).



(a) (b) (c) (d)

Fig. 7. Vibro-impact simulation with flexible clearance supports (shown are results at supports 1 and 3): Using the modal U-K algorithm with flexible clearance supports and constraint enforcing through geometric projection elimination (left); Using a penalty formulation (right).

Conclusion

The dynamical modelling of complex systems with intermittent contact remains a challenging problem. In this paper we developed a modal Udwadia-Kalaba formulation suitable for continuous flexible systems with single or multiple intermittent contacts, with either rigid or flexible clearance supports, enabling the dynamical computation of many systems of practical interest. For such problems, lower-derivative constraint enforcing becomes mandatory, here achieved using a geometric projection elimination technique. Implementation of U-K for multiple intermittent constraints was performed by keeping track of the system contact configuration and rebuilding the constraint matrix at every detected change of the constraint state. After formulating the computational strategy, it was then applied to random-excited multi-supported beams with single and multiple clearance supports. The results thus obtained compare reasonably well with dynamical computations performed using a classic penalty technique for enforcing the nonlinear support constraints.

References

[1] F.E. Udwadia, R.E. Kalaba, A new perspective on constrained motion, Proceedings of the Royal Society A439 (1992) 407-410.

[2] F.E. Udwadia, R.E. Kalaba, Analytical dynamics: A new approach, Cambridge University Press, USA, 1996.

[3] E. Pennestri, P.P. Valentini, D. de Falco, An application of the Udwadia-Kalaba dynamic formulation to flexible multibody systems, J. Franklin Institute 347 (2010), 173-194.

[4] J. Antunes, V. Debut, Dynamical computation of constrained flexible systems using a modal Udwadia-Kalaba formulation: Application to musical instruments, J. Acoustical Society of America 141 (2017) 764-778.

[5] J.H. Bickford, Mechanisms for intermittent motion, Industrial Press, New York, USA, 1972.

[6] C. Studer, Numerics of unilateral contacts and friction: Modeling and numerical time integration in non-smooth dynamics, Springer-Verlag, Berlin Heidelberg, Germany, 2009.

[7] A. Arabyan, F. Wu, An improved formulation for constrained mechanical systems, Multibody System dynamics 2 (1998), 49-69.

[8] E. Pennestri, P.P. Valentini, L. Vita, Dynamic analysis of intermittent-motion mechanisms through the combined use of Gauss principle and logical functions, in Multiscale Problems in Multibody System Contacts (2007), 195-204.

[9] A. Laulusa, O.A. Bauchau, Review of classical approaches for constraint enforcement in multibody systems, J. Computational and Nonlinear Dynamics 3 (2008), 011004.

[10] O.A. Bauchau, A. Laulusa, Review of contemporary approaches for constraint enforcement in multibody systems, J. Computational and Nonlinear Dynamics 3 (2008), 011005.

[11] J. Antunes, X. Delaune, P. Piteau, L. Borsoi, A new method for the generation of representative time-domain turbulence excitations, J. Fluids and Structures 58 (2015), 1-19.

[12] P. Piteau, X. Delaune, J. Antunes, L. Borsoi, Experiments and computations of a loosely supported tube in a rigid bundle subjected to single-phase flow, J. Fluids and Structures 28 (2012), 56-71.

[13] J. Antunes, L. Borsoi, X. Delaune, P. Piteau, Identification of random excitation fields from vibratory responses with application to multi-supported tubes excited by flow turbulence, ASME Journal of Pressure Vessel Technology 136 (2014), Paper PVT-13-1074.

[14] J.W. Baumgarte, Stabilization of constraints and integrals of motion in dynamic systems, Computer Methods in Applied Mechanics and Engineering 1 (1972), 1-16.

[15] S. Yoon, R.M. Howe, D.T. Greenwood, Geometric elimination of constraint violations in numerical simulation of Lagrangian equations, J. Mechanical Design 116 (1994), 1058-1064.

[16] K.H. Hunt, F.R. Crossley, Coefficient of restitution interpreted as damping in vibroimpact, J. Applied Mechanics 42 (1975), 440-445.