1 Linear Dependence

Column's of B are linearly dependent $\vec{b_1}, \vec{b_2}, ..., \vec{b_p}$.

There are scalars $c_1, ..., c_p$ not all zero, such that $c_1\vec{b_1}+c_2\vec{b_2}+...+c_p\vec{b_p} = \vec{0}$

Proof. $A(BC) = (AB)C$

$$C = [\vec{c_1}...\vec{c_p}]$$

Then,

$$BC = [Bc_1...Bc_p]$$

so,

$$\begin{aligned} A(BC) &= [ABc_1...ABc_p] \\ &= [(AB)\vec{c_1}...(AB)\vec{c_p}] \\ &= ABC \end{aligned}$$

□

In general, $AB \neq BA$.
If $AB = AC$ it does generally not follow that $B = C$.
If $AB = BA$ we say A and B commute. In the product AB, we say
A is right-multiplied by B, and B is left-multiplied by A.

2 Powers of a Matrix if it is by Square or $m \times m$

By convention $A^0 = I$

$$A^1 = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} (1+2) & 1 \\ (2+0) & 2 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 6 & 2 \end{bmatrix}$$

3 The Transpose of a Matrix

If A is a $m \times n$ matrix, the transpose of A, denoted A^T is the $n \times m$ matrix whose columns are the rows of A, in the same order.

example -

1.

$$A = \begin{bmatrix} 1 & -1 & 5 \\ 2 & 0 & 6 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 \\ -1 & 0 \\ 5 & 6 \end{bmatrix}$$

2.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

3.

$$A = I_{12} \Rightarrow A^T = A$$

4.

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 4 & 3 \\ 2 & 3 & 2 \end{bmatrix} \Rightarrow A^T = A$$

note - Symmetric about the diagonal

Properties

1. $(A^T)^T = A$
2. $(A + B)^T = A^T + B^T$
3. $(rA)^T = r(A^T)$
4. $(AB)^T = B^T A^T$

Example -

$$A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} B = \begin{bmatrix} -1 & 2 \\ 1 & -3 \end{bmatrix}$$

$$AB = \begin{bmatrix} 2 & -7 \\ 1 & -3 \end{bmatrix}$$

$$(AB)^T = \begin{bmatrix} 2 & 1 \\ -7 & -3 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}, B^T = \begin{bmatrix} -1 & 1 \\ 2 & -3 \end{bmatrix}$$

$$A^T B^T = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} \neq (AB)^T$$

$$B^T A^T = \begin{bmatrix} 2 & 1 \\ -7 & 3 \end{bmatrix} = (AB)^T$$

4 The Inverse of a Matrix

A $n \times n$ matrix A is invertible if there exists another $n \times n$ matrix C such that $AC = CA = I$.

In this case, we call C the inverse of A, denoted A^{-1}.

The inverse is unique:

If B is an inverse of A, then $B = BI = B(AC) = (BA)C = IC = C$. So, $AA^{-1} = A^{-1}A = I$. A matrix that is not invertible is called singular; an invertible matrix is called non-singular.

Example

$$A = \begin{bmatrix} -1 & 1 \\ 5 & -4 \end{bmatrix}, C = \begin{bmatrix} 4 & 1 \\ 5 & 1 \end{bmatrix}$$

Show that $C = A^{-1}$ We must show that $AC = I$ and $CA = I$ A must be square to have an inverse.

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}, B^T = \begin{bmatrix} 3 & 2 \\ -2 & 0 \\ -1 & -1 \end{bmatrix}$$

Then,

$$AC = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, CA \neq I_3$$

Inverse of a 2×2 matrix. Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

- If $ad - bc \neq 0$ then A is invertible, and

$$A^{-1} = \frac{1}{ad - bc} \cdot \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

- If $ad - bc = 0$, then A is not invertible

$$A^{-1} = \frac{1}{ad - bc} \cdot \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = I$$

Thus a 2×2 matrix A is invertible iff $detA \neq 0$.

Let A be an invertible $n \times n$ matrix. For every $b^{\rightharpoonup} \epsilon \mathbb{R}^n$, the equation $Ax^{\rightharpoonup} = b^{\rightharpoonup}$ has a unique solution, $x^{\rightharpoonup} = A^{-1}b^{\rightharpoonup}$.

Proof. $x^{\rightharpoonup} = A^{-1}b^{\rightharpoonup}$ is a solution since $Ax^{\rightharpoonup} = A(A^{-1}b^{\rightharpoonup} = (AA^{-1})b^{\rightharpoonup} = Ib^{\rightharpoonup} = b^{\rightharpoonup}$.

If $u^{\rightharpoonup}$ is a solution, then

$$Au^{\rightharpoonup} = b^{\rightharpoonup} Then A^{-1}(Au^{\rightharpoonup}) \quad = A^{-1}b^{\rightharpoonup}(A^{-1}A)u^{\rightharpoonup} = A^{-1}b^{\rightharpoonup}Iu^{\rightharpoonup} \quad = x^{\rightharpoonup}u^{\rightharpoonup} = x^{\rightharpoonup}$$

Thus the solution is unique $\square$

example -

$$A = \begin{bmatrix} 5 & 3 \\ 2 & 1 \end{bmatrix}$$

Use the inverse of A to solve the system

$$\begin{aligned} 5x_1 + 3x_2 &= 2 \\ 2x_1 + x_2 &= -6 \end{aligned}$$

$$A^{-1} = \frac{1}{5 \cdot 1 - 3 \cdot 2} \cdot \begin{bmatrix} 1 & -3 \\ -2 & 5 \end{bmatrix} = -1 \begin{bmatrix} 1 & -3 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 2 & -5 \end{bmatrix}$$

$$Then, x^{\rightharpoonup} = A^{-1} \cdot \begin{bmatrix} 2 \\ -6 \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 2 & -5 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -6 \end{bmatrix} = \begin{bmatrix} -20 \\ 34 \end{bmatrix}$$

$$x_1 = -20, x_2 = 34$$

HOMEWORK
§2.1 : $27, 28, 33$
§2.2 : $1, 3, 4, 5, 6, 7, 8, 13$