Math 2700 - Review for Exam 2 **P. Allaart**

SHOW ALL YOUR WORK! NO WORK=NO CREDIT!!

No Calculators Allowed! - But you shouldn't need any.

1. Compute the following matrix products. Or, if the product is not defined, explain why not.

a) $\begin{bmatrix} 3 & 0 & -1 \\ 5 & 1 & 2 \end{bmatrix} \begin{bmatrix} 4 & -1 \\ 1 & 1 \\ -2 & 3 \end{bmatrix} =$

b) $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 3 & 2 & 1 \end{bmatrix} =$

c) $\begin{bmatrix} 1 & 1 \\ -1 & 2 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} 3 & 1 & -1 \\ 0 & 4 & 2 \\ -2 & 5 & 1 \end{bmatrix} =$

2. Determine if the matrix A is invertible, and if it is, find its inverse.

$$A = \begin{bmatrix} 1 & 0 & 5 \\ 3 & 2 & -1 \\ 3 & 2 & 1 \end{bmatrix}$$

3. Suppose the last column of AB is entirely zero but B itself has no column of zeros. What can you say about the columns of A? Explain.

4. Calculate the determinant

$$\begin{vmatrix} 2 & 3 & -5 & 0 \\ -2 & 1 & 5 & -1 \\ 4 & -1 & 1 & 7 \\ 2 & 2 & 6 & 6 \end{vmatrix} =$$

5. Let T be the mapping $T(x_1, x_2, x_3) = (3x_1 + x_2 - x_3, x_1 + 2x_2)$.

a) Find the standard matrix of T.

b) Determine if T is one-to-one. Show all of the details!

c) Determine if T is onto. Show all of the details!

6. Let

$$A = \begin{bmatrix} 1 & 0 & 2 & 2 & -2 \\ -1 & 3 & -3 & -2 & 6 \\ 3 & 0 & 6 & 4 & 0 \\ 1 & 6 & 0 & 6 & -4 \end{bmatrix}, \qquad \text{and} \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ -2 \\ 12 \end{bmatrix}.$$

Use the LU factorization of A given below to solve the equation $A\mathbf{x} = \mathbf{b}$. That is, solve the two equations $L\mathbf{y} = \mathbf{b}$ and $U\mathbf{x} = \mathbf{y}$.

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 1 & 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 & 2 & -2 \\ 0 & 3 & -1 & 0 & 4 \\ 0 & 0 & 0 & -2 & 6 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} = LU.$$

7. Find an LU factorization of the matrix

$$A = \begin{bmatrix} 2 & 1 & 0 & 1 \\ 4 & 1 & 3 & 4 \\ -2 & 0 & -2 & -4 \\ 6 & -1 & 12 & 15 \end{bmatrix}$$

8. Suppose A, B, and X are $n \times n$ matrices such that A, X and $A+X$ are invertible, and

$$(A + X)^{-1} = X^{-1}B. \tag{1}$$

a) Explain why B is invertible.

b) Solve the matrix equation (1) for X. If you need to multiply by the inverse of a matrix, explain why that matrix is invertible.

9. **Extra credit!!** Let A be an $n \times n$ singular (=noninvertible) matrix. Describe how to construct an $n \times n$ nonzero matrix B such that $AB = 0$. Explain why your method works. (Hint: what can you say about the columns of A?)

Math 2700, Exam 2 - Solutions

1. a) $\begin{bmatrix} 3 & 0 & -1 \\ 5 & 1 & 2 \end{bmatrix}\begin{bmatrix} 4 & -1 \\ 1 & 1 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 14 & -6 \\ 17 & 2 \end{bmatrix}$.   b) $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\begin{bmatrix} 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 1 \\ 6 & 4 & 2 \\ 9 & 6 & 3 \end{bmatrix}$

c) undefined: the number of ~~row~~ columns in the first matrix $(=2)$ does not match the number of rows in the second $(=3)$.

2. $$[A \;\; I] = \left[\begin{array}{ccc|ccc} 1 & 0 & 5 & 1 & 0 & 0 \\ 3 & 2 & -1 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 5 & 1 & 0 & 0 \\ 0 & 2 & -16 & -3 & 1 & 0 \\ 0 & 2 & -14 & -3 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 5 & 1 & 0 & 0 \\ 0 & 2 & -16 & -3 & 1 & 0 \\ 0 & 0 & 2 & 0 & -1 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 5/2 & -5/2 \\ 0 & 2 & 0 & -3 & -7 & 8 \\ 0 & 0 & 2 & 0 & -1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 5/2 & -5/2 \\ 0 & 1 & 0 & -3/2 & -7/2 & 4 \\ 0 & 0 & 1 & 0 & -1/2 & 1/2 \end{array}\right] \quad \therefore A^{-1} = \begin{bmatrix} 1 & 5/2 & -5/2 \\ -3/2 & -7/2 & 4 \\ 0 & -1/2 & 1/2 \end{bmatrix}.$$

3. The last column of AB is a linear combination of the columns of A with weights from the last column of B. Since the weights are not all zero and the linear combination yields the zero vector, it follows that the columns of A are <u>linearly dependent</u>.
(but not necessarily equal, or multiples of each other!)

4. $$\begin{vmatrix} 2 & 3 & -5 & 0 \\ -2 & 1 & 5 & -1 \\ 4 & -1 & 1 & 7 \\ 2 & 2 & 6 & 6 \end{vmatrix} = \begin{vmatrix} 2 & 3 & -5 & 0 \\ 0 & 4 & 0 & -1 \\ 0 & -7 & 11 & 7 \\ 0 & -1 & 11 & 6 \end{vmatrix} = -\begin{vmatrix} 2 & 3 & -5 & 0 \\ 0 & -1 & 11 & 6 \\ 0 & -7 & 11 & 7 \\ 0 & 4 & 0 & -1 \end{vmatrix} = -\begin{vmatrix} 2 & 3 & -5 & 0 \\ 0 & -1 & 11 & 6 \\ 0 & 0 & -66 & -35 \\ 0 & 0 & 44 & 23 \end{vmatrix} = -2\cdot(-1)\begin{vmatrix} -66 & -35 \\ 44 & 23 \end{vmatrix}$$
$$= 2\cdot 22\begin{vmatrix} -3 & -35 \\ 2 & 23 \end{vmatrix} = \boxed{44}$$

5. a) $T(1,0,0) = (3,1)$, $T(0,1,0) = (1,2)$, $T(0,0,1) = (-1,0) \Rightarrow A = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 2 & 0 \end{bmatrix}$

b) Since A has more columns than rows, A does not have a pivot in each column $\therefore T$ is <u>not</u> one-to-one.

c) $A = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 3 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 2 & 0 \\ 0 & \boxed{-5} & -1 \end{bmatrix} \Rightarrow A$ has a pivot in each row $\therefore T$ <u>is</u> onto.

6. $$[L \;\; \vec{b}] = \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 & 1 \\ 3 & 0 & 1 & 0 & -2 \\ 1 & 2 & -2 & 1 & 12 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & -5 \\ 0 & 2 & -2 & 1 & 11 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & -5 \\ 0 & 0 & -2 & 1 & 7 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 1 & -3 \end{array}\right] \therefore \vec{y} = \begin{bmatrix} 1 \\ 2 \\ -5 \\ -3 \end{bmatrix}.$$

$$[U \;\; \vec{y}] = \left[\begin{array}{ccccc|c} 1 & 0 & 2 & 2 & -2 & 1 \\ 0 & 3 & -1 & 0 & 4 & 2 \\ 0 & 0 & 0 & -2 & 6 & -5 \\ 0 & 0 & 0 & 0 & 2 & -3 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 0 & 2 & 2 & 0 & -2 \\ 0 & 3 & -1 & 0 & 0 & 8 \\ 0 & 0 & 0 & -2 & 0 & 4 \\ 0 & 0 & 0 & 0 & 2 & -3 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 0 & 2 & 0 & 0 & 2 \\ 0 & 3 & -1 & 0 & 0 & 8 \\ 0 & 0 & 0 & -2 & 0 & 4 \\ 0 & 0 & 0 & 0 & 2 & -3 \end{array}\right]$$

$$\sim \left[\begin{array}{ccccc|c} 1 & 0 & 2 & 0 & 0 & 2 \\ 0 & 1 & -1/3 & 0 & 0 & 8/3 \\ 0 & 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 & -3/2 \end{array}\right] \quad \therefore \begin{cases} x_1 = 2 - 2x_3 \\ x_2 = 8/3 + \frac{1}{3}x_3 \\ x_3 \text{ free} \\ x_4 = -2 \\ x_5 = -3/2 \end{cases} \Rightarrow \vec{x} = x_3\begin{bmatrix} -2 \\ 1/3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 8/3 \\ 0 \\ -2 \\ -3/2 \end{bmatrix}.$$

7. $$\begin{bmatrix} \boxed{2} & 1 & 0 & 1 \\ 4 & 1 & 3 & 4 \\ -2 & 0 & -2 & -4 \\ 6 & -1 & 12 & 15 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & 0 & 1 \\ 0 & \boxed{-1} & 3 & 2 \\ 0 & 1 & -2 & -3 \\ 0 & -4 & 12 & 12 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & 0 & 1 \\ 0 & -1 & 3 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 4 \end{bmatrix} = U; \quad L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{bmatrix}$$

8. a) $(A+X)^{-1} = X^{-1}B \Rightarrow X(A+X)^{-1} = B$. Since X and $(A+X)^{-1}$ are invertible, their product, B, is invertible.

b) From (a): $X(A+X)^{-1} = B \Rightarrow X(A+X)^{-1}(A+X) = B(A+X) \Rightarrow XI = BA + BX \Rightarrow X = BA + BX$
$\Rightarrow X - BX = BA \Rightarrow IX - BX = BA \Rightarrow (I-B)X = BA \Rightarrow \boxed{X = (I-B)^{-1}BA.}$ (Other answers are possible).
$I - B$ is invertible since $(I-B)X = BA \Rightarrow I - B = BAX^{-1}$, a product of invertible matrices.

9. Since A is singular, the columns of A are linearly dependent. So the equation $A\vec{x} = \vec{0}$ has a nontrivial solution, say $A\vec{c} = \vec{0}$ where $\vec{c} \neq \vec{0}$. Construct an $n \times n$ matrix B whose every column is $\vec{c}$. Then $AB = 0$.