Minimum Spanning Tree

Let $G = (V, E)$ be a graph. Assume that we have a positive cost $C(i, j)$ for every $ij \in E$. Let (S, S') be a cut, and let $A \subseteq E$ so that A respects the cut. Finally, assume that A is contained in some minimum spanning tree. Prove that if e is a light edge in (S, S'), then $A \cup e$ is contained in some minimum spanning tree.

Proof. Let T be a minimum spanning tree containing A. If $e \in T$ we are done. So assume that e is not in T. There must be a path p that connects u and v via a light edge(x, y). Inclusion of e to p will create exactly one cycle. Let $c = p \cup \{e\}$ be a cycle. Observe that p has an edge $e' = ab$ so that $a \in S$ and $b \in S'$. Note that $T' = (T \cup \{e\}) - e'$ is a spanning tree.

Furthermore, $w(e) < w(e')$, this $w(T') \leq w(T)$. Therefore T' is a minimum spanning tree. Note that $A \subseteq T'$. Then $A \cup \{e\} \subseteq T'$.

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