Cumulative Distribution Function

For each of the following functions let's determine the values of k for which p is a probability mass function for a discrete variable.
(i) $p(x) = kx$, x=1,2,...,n
(ii) $p(x) = kx^2$, x=1,2,...,n

(i) p(x) is a probability function iff

$$\begin{aligned}
1 &= \sum_{x_i} p(x_i) \\
&= \sum_{x=1}^{n} p(x) \\
&= \sum_{x=1}^{n} kx \\
&= k\sum_{x=1}^{n} x \\
1 &= k\left(\frac{n(n+1)}{2}\right) \\
k &= \frac{2}{n(n+1)}
\end{aligned}$$

p(x) is a probability function iff the sum of the probabilities from 1 to n is equal to 1. The sum of all numbers from 1 to n is $\frac{n(n+1)}{2}$ so for a probability function $k = \frac{2}{n(n+1)}$.

(ii)

$$\begin{aligned}
1 &= \sum_{x_i} p(x_i) \\
&= \sum_{x=1}^{n} kx^2 \\
&= k\sum_{x=1}^{n} x^2 \\
&= k\left(\frac{n(n+1)(2n+1)}{6}\right) \\
k &= \frac{6}{n(n+1)(2n+1)}
\end{aligned}$$

Expected Value

definition - X discrete random variable with values $\{x_1, x_2, x_3, \ldots\}$ and function p. Then the expected value (mathermatical expectation) is given by

$$E[X] = \sum_{x_i} p(x_i)$$

example - Toss a fair coin once. Let X be 0 if it comes out heads, 1 if tails. Calculate E[X].

What are the possible values of X? X={0,1}

$P(X = 0) = P(X = 1) = \frac{1}{2} \Rightarrow P(0) = P(1) = \frac{1}{2}$

$$\begin{aligned} E[X] &= \sum_{x_i} x_i p(x_i) \\ &= 0 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = \frac{1}{2} \end{aligned}$$

example - Suppose that we place the numbers 3, 5, 7, 9, 11, & 13 on otherwise identical tickets. The tickets are placed in a hat and one ticket is selected at random. Let X be the number of the ticket drawn. Find E[X].

$$\begin{aligned} E[X] &= \sum_{x_i} p(x_i) \\ &= 3\frac{1}{6} + 5\frac{1}{6} + 7\frac{1}{6} + 9\frac{1}{6} + 11\frac{1}{6} + 13\frac{1}{6} \\ &= \frac{1}{6}(3 + 5 + 7 + 9 + 11 + 13) \\ &= \frac{48}{6} = 8 \end{aligned}$$

It can be considered a weighted average, although in this case it is evenly weighted.

example - Six numbers are selected at random from the positive integers 1-49 for a winning number.

- Match all 6 numbers and win a grand prize of 1,200,000 dollars.
- Match 5 numbers and win a second prize of 800 dollars.
- Match 5 number and win a third prize of 35 dollars.

Find the expected number of dollars that a player wins in a single play.

Let X be the dollar amount won by the player in a single play. Possible values of X={0, 35, 800, 1.2mil}

$$P(1200000) = P(X = 1200000) = \frac{1}{\binom{49}{6}} \approx 0.00000072$$

$$P(800) = P(X = 800) = \frac{\binom{6}{5}\binom{43}{1}}{\binom{49}{6}} \approx 0.000018$$

$$P(35) = P(X = 35) = \frac{\binom{6}{4}\binom{43}{2}}{\binom{49}{6}} \approx 0.00097$$

$$P(0) = P(X = 0) = 1 - (0.00000072 + 0.000018 + 0.00097) = 0.999011928$$

$$E[X] = 1200000(0.00000072) + 800(0.000018) + 35(0.00097) + 0 \approx 0.13$$

Theorem 3.1. *Let X be a discrete random variable with possible values* $\{x_1, x_2, x_3, \ldots\}$ *and probability function p. Let g be a real valued function whose domain contains* $\{x_1, x_2, x_3, \ldots\}$*. Then g(X) is a discrete random variable and*

$$E[g(X)] = \sum_{x_i} g(x)p(x_i)$$

example - The probability function p of a discrete random variable X is given below

$$p(x) = \begin{cases} \frac{x}{15}, x = 1, 2, 3, 4, 5 \\ 0, otherwise \end{cases}$$

Calculate the expected value of X(6-X). Let g(x)=x((6-x).

$$\begin{aligned} E[g(X)] &= \sum_{x_i} g(x_i)p(x_i) \\ &= \sum_{x=1}^{5} x(6-x)\frac{x}{15} \\ &= 5 \cdot \frac{1}{15} + 8\frac{2}{15} + 9\frac{3}{15} + 8\frac{4}{15} + 5\frac{5}{15} \\ &= \frac{1}{15}(5 + 6 + 27 + 32 + 25) = \frac{105}{15} = \frac{21}{3} = 7 \end{aligned}$$

definition - Let X be a discrete random variable with possible values $\{x_1, x_2, \ldots\}$, probability function p, an expected value E[X]. Then the variance of X is defined by

$$var[X] = E[(X - E[X])^2]$$

and the standard deviation of X

$$\sigma_X = \sqrt{var[X]}$$

remark - common in the literature to make the substitution $\mu = E[X]$.

$$var[X] = E[(X - \mu)^2]$$
$$\sigma_X = \sqrt{E[(X - \mu)^2]}$$

example Calculate the variance of the random variable X, where X denotes the number of spots obtained in a single roll of a fair die. $\{1, 2, 3, 4, 5, 6\}$ = possible values of X

$$p(x) = \begin{cases} \frac{1}{6}, x = 1, 2, 3, 4, 5, 6 \\ 0, otherwise \end{cases}$$

$$E[X] = \frac{1}{6}(1 + 2 + 3 + 4 + 5 + 6) = \frac{21}{6} = \frac{7}{2}$$

$$\begin{aligned} var[X] &= E[(X - E[X])^2] \\ &= E[(X - \frac{7}{2})^2] \\ &= \sum_{x_i} (x_i - \frac{7}{2})^2 \ p(x_i) = \frac{1}{6}\left[2\frac{25}{4} + 2\frac{9}{4} + 2\frac{1}{4}\right] \\ &= \frac{1}{12}[25 + 9 + 1] = \frac{35}{12} \end{aligned}$$

Theorem 3.1. *Let X be a discrete random variable and a, b are constants. Then*
(i) $E[aX+b] = aE[X]+b$
(ii) $var[aX+b] = a^2 var[X]$
(iii) $\sigma a_X + b = |a|\sigma_X$

Proof.

$$\begin{aligned}
E[aX+b] &= \sum_{x_i}(ax_i+b)p(x_i) \\
&= \sum_{x_i} ax_i p(x_i) + \sum_{x_i} bp(x_i) \\
&= a\sum_{x_i} x_i p(x_i) + b\sum_{x_i} p(x_i) \\
&= aE[X]+b
\end{aligned}$$

□

HOMEWORK - 3.4, 3.6, 3.7, 3.10*