Real Analysis

MATH 305, YALE UNIVERSITY, SPRING 2019

These are lecture notes for MATH 305b, "Real Analysis," taught by Hee Oh at Yale University during the spring of 2019. These notes are not official, and have not been proofread by the instructor for the course. They live in my lecture notes respository at

`https://github.com/jopetty/lecture-notes/tree/master/math-305`.

If you find any errors, please open a bug report describing the error and label it with the course identifier, or open a pull request so I can correct it.

Contents

1 January 14, 2019

Given an interval $(a, b) \subset \mathbb{R}$, we know that the size of this interval is $b - a$. The focus of this course will be the study of the generalization of this idea using the *Lebesgue measure* on $\mathbb{R}$. Equipped with this, we can talk of the *Lebesgue integral* of "nice" functions, which is more powerful than the Riemannian equivalent.

The Metric Space

Definition (Metric Space)**.** Given a set X, a metric function d is a function $d : X \times X \to \mathbb{R}$ obeying the following three properties. *Metric Space*

1. **Positivity:** $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$;
2. **Symmetry:** $d(x, y) = d(y, x)$ for all $x, y \in X$;
3. **Triangle Inequality:** $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

A metric space is a pair (X, d) where d is a metric function on X.

Example 1.1 (Metric Spaces)**.**

(a) In $\mathbb{R}$, we have the traditional $d(x, y) = |x - y|$.

(b) In $\mathbb{R}^2$, we have $d((x_1, x_2), (y_1, y_2)) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$.

(c) In $\mathbb{R}^2$, we also have $d((x_1, x_2), (y_1, y_2)) = \max\{|x_1 - y_2|, |x_2 - y_2|\}$.

(d) The discrete metric on a set X is defined by

$$d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{otherwise.} \end{cases}$$

(e) Given a metric space (X, d) and $Y \subset X$ then (Y, d) is also a metric space where d is restricted to $Y \times Y$.

Definition (Neighborhood)**.** Fix a metric space (X, d). For some $r \geq 0$, the r-neighborhood of x is $B(x, r)$, the set $\{y \in X \mid d(x, y) < r\}$. Notice that this depends on the metric! In $\mathbb{R}$ with the discrete metric, $B(0, 1) = \{0\}$ while $B(0, 2) = \mathbb{R}$ which is not what we expect from the traditional metric. *Neighborhood*

Definition (Interior Points)**.** Let $A \subset X$. A point $x \in A$ is an interior point of A if there exists some $r > 0$ such that $B(x, r) \subset A$. That is, we can draw a ball around x which lies entirely in A. *Interior Points*

Example 1.2. If $A = [0, 1)$, then the interior points of A are $(0, 1)$ but 0 is not an interior point.

Definition (Open Sets)**.** A subset $A \subset X$ is open if every point in A is interior. The empty set is vacuously open. *Open Sets*

Proposition 1.1. *For any $x \in X$ the r-neighborhood of x is an open subset of X.*

Proof. Let $y \in B(x, r)$. Let $r_0 = r - d(x, y)$. Then $r_0 > 0$ and $B(y, r_0) \subset B(x, r)$ regardless of which y is chosen since for any $z \in B(y, r_0)$ we know that $d(x, z) \leq d(x, y) + d(r, z) < d(x, y) + r_0 < r$. Then every point of $B(x, r)$ is interior and so it is open. ■

Using this, we can now call $B(x, r)$ the open ball of radius r centered at x.

Example 1.3. In $\mathbb{R}^2$ with the standard metric, an open ball looks like an open disc. With the maximum metric, it looks like an open square. In $\mathbb{R}$, we can look at the set of all rational numbers $\mathbb{Q}$. This set is not open since for all $q \in \mathbb{Q}$ and all $r > 0$ there exists an $x \in B(q, r)$ where $x \notin \mathbb{Q}$.

Proposition 1.2. *The intersection of finitely many open sets is open. The union of any open sets is open.*

Example 1.4. The intersection of infinitely many open sets is not necessarily open. Consider $\bigcap (0, 1/n)$ as $n \to \infty$. The intersection is simply $\{0\}$ which is not an open set.

Proof of Proposition 1.2. Let $A_1, \cdots, A_k$ be open subsets of X. Let $x \in A_1 \cap \cdots \cap A_k$. Since each A_i is open we know that x is an interior point of A_i, so there exists some r_i such that $B(x, r_i) \subset A_i$. Let r be the minimum of all such r_i. Then $B(x, r) \subset A_i$ for all i, so this open ball is contained in the intersection.

Now let $\{A_\alpha \mid \alpha \in I\}$ be a collection of open subsets. Let $x \in \bigcup A_\alpha$. Then x is contained in some open A_α, and so there exists an r_α such that $B(x, r_\alpha) \subset A_\alpha$, so $B(x, r_\alpha) \subset \bigcup A_\alpha$. ■

Definition (Interior of a Set)**.** For $A \subset X$, the set of all interior points of A is called the interior of A. This is usually written as $\mathrm{Int}(A)$ or A°. *Interior of a Set*

Example 1.5. If $A = [a, b]$ then $A^\circ = (a, b)$. If $A = \mathbb{Q}$ then $\mathbb{Q}^\circ = \emptyset$.

Proposition 1.3. *For all A, the interior of A is open. Furthermore, A° is the largest open subset of A in the sense that it contains all other open subsets of A.*

Proof. It's just the definitions. ∎

Proposition 1.4. *If $A \subset B$ then $A^\circ \subset B^\circ$.*

Corollary 1.1. *A set A is open if and only if $A = A^\circ$.*

Definition (Limit Point)**.** Let $A \subset X$. A point $x \in X$ is a limit point of A if for any $r > 0$ we know that $B(x, r) \cap A \neq \emptyset$. Notice that every point $a \in A$ is a limit point of A. *Limit Point*

Example 1.6. Let $A = [0, 1)$. Then 0 is a limit point of A since every open ball centered centered at 0 interesects A. Furthermore, 1 is also a limit point for the same reason. If $A = \mathbb{Q}$, then the set of limit points of $\mathbb{Q}$ is all of $\mathbb{R}$.

Definition (Closed Set)**.** A set $A \subset X$ is called closed if every limit point of A is contained in A. *Closed Set*

Example 1.7. The interval $[0, 1]$ is closed but $[0, 1)$ is not. To show that something isn't a limit point, use the minimum distance between this point and the interval. This must be positive since otherwise it would be in the interval. Then let your r be smaller than this, and the open ball with this radius centered at this point will not intersect the original interval. Generalize to higher dimensions as needed.

Corollary 1.2. *Given any metric space X, we know that $\emptyset$ is closed. Furthermore, $\bar{B}(y, r) = B[y, r] = \{y \mid d(x, y) \leq r\}$ is closed for any r.*

Proposition 1.5. *Let $A \subset X$. We know that A is open if and only if $A^\complement$ is closed.*