Real Analysis

math 305, yale university, spring 2019

These are lecture notes for math 305b, "Real Analysis," taught by Hee Oh at Yale University during the spring of 2019. These notes are not official, and have not been proofread by the instructor for the course. They live in my lecture notes respository at

https://github.com/jopetty/lecture-notes/tree/master/MATH-305.

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Given an interval $(a, b) \subset \mathbb{R}$, we know that the size of this interval is $b - a$. The focus of this course will be the study of the generalization of this idea using the *Lebesgue measure* on $\mathbb{R}$. Equipped with this, we can talk of the *Lebesgue integral* of "nice" functions, which is more powerful than the Riemannian equivalent.

1.1 The Metric Space

Definition (Metric Space)**.** Given a set X, a metric function d is a function $d : X \times X \to \mathbb{R}$ obeying the following three properties. *Metric Space*

1. **Positivity:** $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$;
2. **Symmetry:** $d(x, y) = d(y, x)$ for all $x, y \in X$;
3. **Triangle Inequality:** $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

A metric space is a pair (X, d) where d is a metric function on X.

Example 1.1 (Metric Spaces)**.**

(a) In $\mathbb{R}$, we have the traditional $d(x, y) = |x - y|$.
(b) In $\mathbb{R}^2$, we have $d((x_1, x_2), (y_1, y_2)) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$.
(c) In $\mathbb{R}^2$, we also have $d((x_1, x_2), (y_1, y_2)) = \max\{|x_1 - y_2|, |x_2 - y_2|\}$.
(d) The discrete metric on a set X is defined by

$$d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{otherwise.} \end{cases}$$

(e) Given a metric space (X, d) and $Y \subset X$ then (Y, d) is also a metric space where d is restricted to $Y \times Y$.

Definition (Neighborhood)**.** Fix a metric space (X, d). For some $r \geq 0$, the r-neighborhood of x is $B(x, r)$, the set $\{y \in X \mid d(x, y) < r\}$. Notice that this depends on the metric! In $\mathbb{R}$ with the discrete metric, $B(0, 1) = \{0\}$ while $B(0, 2) = \mathbb{R}$ which is not what we expect from the traditional metric. *Neighborhood*

Definition (Interior Points)**.** Let $A \subset X$. A point $x \in A$ is an interior point of A if there exists some $r > 0$ such that $B(x, r) \subset A$. That is, we can draw a ball around x which lies entirely in A. *Interior Points*

Example 1.2. If $A = [0, 1)$, then the interior points of A are $(0, 1)$ but 0 is not an interior point.

Definition (Open Sets)**.** A subset $A \subset X$ is open if every point in A is interior. The empty set is vacuously open.

Open Sets

Proposition 1.1. *For any $x \in X$ the r-neighborhood of x is an open subset of X.*

Proof. Let $y \in B(x, r)$. Let $r_0 = r - d(x, y)$. Then $r_0 > 0$ and $B(y, r_0) \subset B(x, r)$ regardless of which y is chosen since for any $z \in B(y, r_0)$ we know that $d(x, z) \leq d(x, y) + d(r, z) < d(x, y) + r_0 < r$. Then every point of $B(x, r)$ is interior and so it is open. ■

Using this, we can now call $B(x, r)$ the open ball of radius r centered at x.

Example 1.3. In $\mathbb{R}^2$ with the standard metric, an open ball looks like an open disc. With the maximum metric, it looks like an open square. In $\mathbb{R}$, we can look at the set of all rational numbers $\mathbb{Q}$. This set is not open since for all $q \in \mathbb{Q}$ and all $r > 0$ there exists an $x \in B(q, r)$ where $x \notin \mathbb{Q}$.

Proposition 1.2. *The intersection of finitely many open sets is open. The union of any open sets is open.*

Example 1.4. The intersection of infinitely many open sets is not necessarily open. Consider $\bigcap (0, 1/n)$ as $n \to \infty$. The intersection is simply $\{0\}$ which is not an open set.

Proof of Proposition 1.2. Let $A_1, \cdots, A_k$ be open subsets of X. Let $x \in A_1 \cap \cdots \cap A_k$. Since each A_i is open we know that x is an interior point of A_i, so there exists some r_i such that $B(x, r_i) \subset A_i$. Let r be the minimum of all such r_i. Then $B(x, r) \subset A_i$ for all i, so this open ball is contained in the intersection.

Now let $\{A_\alpha \mid \alpha \in I\}$ be a collection of open subsets. Let $x \in \bigcup A_\alpha$. Then x is contained in some open A_α, and so there exists an r_α such that $B(x, r_\alpha) \subset A_\alpha$, so $B(x, r_\alpha) \subset \bigcup A_\alpha$. ■

Definition (Interior of a Set)**.** For $A \subset X$, the set of all interior points of A is called the interior of A. This is usually written as $\mathrm{Int}(A)$ or A°.

Interior of a Set

Example 1.5. If $A = [a, b]$ then $A^\circ = (a, b)$. If $A = \mathbb{Q}$ then $\mathbb{Q}^\circ = \emptyset$.

Proposition 1.3. *For all A, the interior of A is open. Furthermore, A° is the largest open subset of A in the sense that it contains all other open subsets of A.*

Proof. It's just the definitions. ■

Proposition 1.4. *If $A \subset B$ then $A^\circ \subset B^\circ$.*

Corollary 1.1. *A set A is open if and only if $A = A^\circ$.*

Definition (Limit Point)**.** Let $A \subset X$. A point $x \in X$ is a limit point of A if for any $r > 0$ we know that $B(x, r) \cap A \neq \emptyset$. Notice that every point $a \in A$ is a limit point of A.

Limit Point

Example 1.6. Let $A = [0, 1)$. Then 0 is a limit point of A since every open ball centered centered at 0 interesects A. Furthermore, 1 is also a limit point for the same reason. If $A = \mathbb{Q}$, then the set of limit points of $\mathbb{Q}$ is all of $\mathbb{R}$.

Definition (Closed Set)**.** A set $A \subset X$ is called closed if every limit point of A is contained in A.

Closed Set

Example 1.7. The interval $[0, 1]$ is closed but $[0, 1)$ is not. To show that something isn't a limit point, use the minimum distance between this point and the interval. This must be positive since otherwise it would be in the interval. Then let your r be smaller than this, and the open ball with this radius centered at this point will not intersect the original interval. Generalize to higher dimensions as needed.

Corollary 1.2. *Given any metric space X, we know that $\emptyset$ is closed. Furthermore, $\bar{B}(y, r) = B[y, r] = \{y \mid d(x, y) \leq r\}$ is closed for any r.*

Proposition 1.5. *Let $A \subset X$. We know that A is open if and only if $A^\complement$ is closed.*

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Recall from last lecture the definitions of open and closed sets.

Proposition 2.1. *The union of finitely many closed sets is closed. The intersection of arbitrarily many closed sets is closed.*

Note that $(\bigcup S_i)^{\complement} = \bigcap S_i^{\complement}$ and $(\bigcap S_i)^{\complement} = \bigcup S_i^{\complement}$.

Proof. By the above property, we konw that the union of finitely many closed sets is the complement of the intersection of finitely many open sets, which is open. We also know that the intersection of closed sets is the complement of the union of their complements, which are open, so the second part of the proposition is true as well. ∎

Example 2.1 (Counterexamples to illustrate the conditions)**.**

1. In $\mathbb{R}$, we know that $\bigcup \left[\frac{1}{n}, 1 - \frac{1}{n}\right]$ for $n \geq 2 = (0, 1)$. This demonstrates the necessity of "finiteness" in our proposition.

Definition (Closure)**.** For $A \subset X$ the closure of A is the set of all limit points of A, usually written as $\bar{A}$. *Closure*

Proposition 2.2. *For any $A \subset X$, we know that $\bar{A}$ is closed and in fact is the smallest closed set containing A.*

Proof that $\bar{A}$ is closed. We prove $\bar{A}$ is closed by showing that $X \setminus \bar{A}$ is open. For all $x \in X \setminus \bar{A}$ we know that x is not a limit point of A since otherwise it would be an element of $\bar{A}$. Then there exists some $r > 0$ such that $B(x, r) \subset X \setminus A$. Since $B(x, r)$ is open we know in fact that $B(x, r) \subset X \setminus \bar{A}$. Then $X \setminus \bar{A}$ is open, so we know that $\bar{A}$ is closed. ∎

Proof that $\bar{A}$ is the smallest closed set containing A. Let B be a closed subset containing A. We want to show that $\bar{A} \subset B$. If $x \in \bar{A}$ then x is a limit point of A and so x is a limit point of B as well since B is closed and contains A. Since B is closed, B must contain all limit points, so $x \in B$ as well. Then $\bar{A} \subset B$. ∎

Corollary 2.1. *A set $A \subset X$ is closed if and only if $A = \bar{A}$ and it is open if and only if $A = A^\circ$.*

Proposition 2.3. *For $x \in X$ and $r > 0$, consider the set $B[x, r] = \{y \in X \mid d(x, y) \leq r\}$ is closed. This is called the closed ball of radius r centered at x.*

Proof. We show that the complement of $B[x, r]$ is open. Let $z \in B[x, r]^{\complement}$. Then $d(x, z) > r$. Let $r_0 = d(x, z) - r$. Then $B(z, r_0)$ does not intersect $B[x, r]$. If this were false, then there would exist a y such that $d(y, z) < r_0$ and $d(x, y) \leq r$ which would mean that $d(x, z) < r_0 + r = d(x, z)$ which is a contradiction. ∎

2.1 Compact Sets

Definition (Covering)**.** Let $B \subset X$ be nonempty. A collection $\mathcal{U} = \{U_\alpha \mid \alpha \in I\}$ of subsets of X is called a cover(ing) if $B \subset \bigcup U_\alpha$. If every U_α is open then this $\mathcal{U}$ is an *open cover* of B. A subcollection $\mathcal{V} = \{U_\alpha \mid \alpha \in J\}$ and $J \subset I$ is called a subcover of $\mathcal{U}$ if $B \subset \bigcup U_\alpha$ where $\alpha \in J$. *Covering*

Definition (Compact set)**.** A nonempty set B is compact if every open cover of B admits a finite subcover, so if $B \subset \mathcal{U}$ where $\mathcal{U}$ is open then there exists some finite collection $\mathcal{V}$ which still contains B. *Compact set*

Example 2.2.

1. $\mathbb{Z} \subset \mathbb{R}$ is not compact since $\bigcup B(n, 1/2)$ for all $n \in \mathbb{Z}$ is an open cover of $\mathbb{Z}$ which does not admit any finite subcover.
2. $\mathbb{R}$ is not compact. Consider $B(n, 1)$ for all $z \in \mathbb{Z}$. This is an open cover of $\mathbb{R}$ which does not admit any finite subcover.
3. $(0, 1]$ is not compact. Consider the cover created by $\bigcup (1/n, \infty) = (0, \infty)$. This is an open cover of $(0, 1]$ but no finite subset of this will contain $(0, 1]$ so it admits no finite subcover.

Definition (Bounded)**.** A set $A \subset X$ is bounded if there exists some $r > 0$ such that $A \subset B(x, r)$ for some $x \in X$. *Bounded*

Proposition 2.4. *Any compact subset of a metric space is closed and bounded.*

Proof. Let B be a compact subset of X. First we will show that B is closed, or equivalently that $B^\complement$ is open. Let $x \in B^\complement$. Let $U_n = \{y \in X \mid d(x, y) > \frac{1}{n}\}$. This is the set $B[x, \frac{1}{n}]^\complement$, which is open since it's the complement of a closed ball. Consider that $\bigcup U_n = X \setminus \{x\}$. In particular, this union contains B so $B \subset \bigcup U_n$. Since B is compact, there must be some finite subcover of $\bigcup U_n$ so $B \subset U_k$ for some k. Then $B(x, \frac{1}{n}) \cap B = \emptyset$ so $B(x, \frac{1}{n}) \subset B^\complement$. Since x was arbitrary, we know that $B^\complement$ is open.

Next we will show that B is bounded. Let $x \in B$ and consider $B \subset \bigcup B(x, n)$. Since B is compact there is some k such that $B \subset B(x, k)$, and so B is bounded. ■

Corollary 2.2. *In a metric space* (X, d) *the set* X *is always closed.*

Example 2.3 (Note on the converse)**.** The converse of this is not true. In an arbitrary metric space, not all closed and bounded sets are compact; consider $\mathbb{Z}$ equipped with the discrete metric. In this metric, $\mathbb{Z} \subset B(0, 2)$ and it is closed since it contains all of its limit points. However, it is not compact since $\mathbb{Z} \subset \bigcup B(n, \frac{1}{2})$ but this cover admits no finite subcover.

Theorem 2.1 (Heine-Borel)**.** *Any closed and bounded subset of* $\mathbb{R}^n$ *is compact.*

Lemma 2.1. *Any closed subset of a compact set is compact.*

Proof. Let B be a compact subset of X and C is a closed subset of B. Let $\mathcal{U}$ (defined in the normal way) be an open cover of C, so $C \subset \mathcal{U}$. Then $B \subset \mathcal{U} \cup (X \setminus C) = X$. Since $X \setminus C$ is open we know that $\mathcal{U} \cup (X \setminus C)$ is an open cover of B. Since B is compact, we know that $\mathcal{U} \cup (X \setminus C)$ admits some finite cover $U_\alpha \cup \cdots \cup U_\zeta \cup (X \setminus C)$. Then it must be true that $C \subset U_\alpha \cup \cdots \cup U_\zeta$, so there is a finte open cover of C so it is compact. ■