

# INTRODUCTION TO ABSTRACT ALGEBRA

MATH 350, YALE UNIVERSITY, FALL 2018

These are lecture notes for MATH 350a, “Introduction to Abstract Algebra,” taught by Marketa Havlickova at Yale University during the fall of 2018. These notes are not official, and have not been proofread by the instructor for the course. These notes live in my lecture notes respository at

<https://github.com/jopetty/lecture-notes/tree/master/MATH-350>.

If you find any errors, please open a bug report describing the error, and label it with the course identifier, or open a pull request so I can correct it.

## Contents

Syllabus	1
References	1
1 August 31, 2018	2
1.1 <i>Symmetries of a regular <math>n</math>-gon</i>	3
1.2 <i>Symmetric group on <math>n</math> elements</i>	4
2 September 4, 2018	5
2.1 <i>Fields <math>n</math> stuff</i>	6
3 September 7, 2018	7
3.1 <i>Homomorphisms</i>	7
4 September 10, 2018	9
4.1 <i>Representation Theory</i>	9
4.2 <i>Isomorphisms and Equality</i>	10

5	September 12, 2018	11
	5.1 <i>Kernels</i> . . . . .	11
6	September 14, 2018	14
	6.1 <i>Cyclic Groups</i> . . . . .	14
7	September 17, 2018	17
	7.1 <i>Finite Cyclic Groups</i> . . . . .	17
	7.2 <i>Subgroups</i> . . . . .	18
8	September 19, 2018	19
9	September 21, 2018	22
10	September 24, 2018	23
	10.1 <i>Quotient Groups</i> . . . . .	23
11	September 26, 2018	25
	11.1 <i>Mapping from <math>G</math> to <math>G/H</math></i> . . . . .	25
	11.2 <i>Testing Normality</i> . . . . .	26
12	Friday, 28 September 2018	27
	12.1 <i>Product Subgroups</i> . . . . .	27
	12.2 <i>Isomorphism Theorems</i> . . . . .	28

## Syllabus

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<b>Instructor</b>	Marketa Havlickova, <a href="mailto:miki.havlickova@yale.edu">miki.havlickova@yale.edu</a>
<b>Lecture</b>	MWF 10:30–11:20 AM, LOM 205
<b>Recitation</b>	TBA
<b>Textbook</b>	Dummit and Foote. <i>Abstract Algebra</i> . 3rd ed. John Wiley & Sons, 2004
<b>Midterms</b>	Wednesday, October 10, 2018 Wednesday, November 14, 2018
<b>Final</b>	Monday, December 17, 2018, 2:00–5:30 PM

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Abstract Algebra is the study of mathematical structures carrying notions of “multiplication” and/or “addition”. Though the rules governing these structures seem familiar from our middle and high school training in algebra, they can manifest themselves in a beautiful variety of different ways. The notion of a group, a structure carrying only multiplication, has its classical origins in the study of roots of polynomial equations and in the study of symmetries of geometrical objects. Today, group theory plays a role in almost all aspects of higher mathematics and has important applications in chemistry, computer science, materials science, physics, and in the modern theory of communications security. The main topics covered will be (finite) group theory, homomorphisms and isomorphism theorems, subgroups and quotient groups, group actions, the Sylow theorems, ring theory, ideals and quotient rings, Euclidean domains, principal ideal domains, unique factorization domains, module theory, and vector space theory. Time permitting, we will investigate other topics. This will be a heavily proof-based course with homework requiring a significant investment of time and thought. The course is essential for all students interested in studying higher mathematics, and it would be helpful for those considering majors such as computer science and theoretical physics.

Your final grade for the course will be determined by

$$\max \left\{ \begin{array}{l} 25\% \text{ homework} + 20\% \text{ exam 1} + 20\% \text{ exam 2} + 35\% \text{ final} \\ 25\% \text{ homework} + 10\% \text{ exam 1} + 20\% \text{ exam 2} + 45\% \text{ final} \\ 25\% \text{ homework} + 20\% \text{ exam 1} + 10\% \text{ exam 2} + 45\% \text{ final} \end{array} \right\}.$$

## References

[DF04] Dummit and Foote. *Abstract Algebra*. 3rd ed. John Wiley & Sons, 2004.

# 1 August 31, 2018

As always, Miki began class at precisely 10:25 AM. She wrote a review of last lecture on the board, and then posed the following question as a warm up. She also talked about how the DUS department is arguing over whether money should be spent on T-shirts or chocolate (Miki thinks chocolate).

**Problem 1** (Warm Up). Are these groups?

- (a)  $(\mathbb{Z}/n\mathbb{Z}, \times)$ ;
- (b)  $(\mathbb{Z}/n\mathbb{Z} \setminus \{0\}, \times)$

*Solution.* The solutions to the warm-up

*Solution to (a).* No, since 0 has no inverse. □

*Solution to (b).* No, this only works when  $n$  is prime. For any factors  $a, b$  of  $n$ ,  $a \times b = 0$ , which isn't in the group. We say that  $(\mathbb{Z}/p\mathbb{Z}, \times)$  is a group for all prime  $p$ . □



**Theorem 1** (Fermat's little theorem). For prime  $p$  and composite  $a = np$ , then  $a^{p-1} \equiv 1 \pmod{p}$ .

**Lemma 1.** If  $\bar{a} \in \mathbb{Z}/p\mathbb{Z} \setminus \{0\}$ , then  $\bar{a}$  has an inverse in  $(\mathbb{Z}/p\mathbb{Z} \setminus \{0\}, \times)^*$ .

**Definition 1** (Units). A unit is something which has an inverse. The units of a group are denoted by putting an asterisk after the group, eg  $(\mathbb{Z}/p\mathbb{Z} \setminus \{0\}, \times)^*$ .

Units

**Example 1.** For integers modulo 4,  $(\mathbb{Z}/4\mathbb{Z}, \times)^* = \{\bar{1}, \bar{3}\}$ .

**Problem 2** (On Homework). What are the conditions for determining the units of a group? We know it must have an inverse, but that's hard to check. Instead, we know that  $a$  is a unit if and only if  $\gcd(a, n) = 1$ . Prove this.

## 1.1 Symmetries of a regular $n$ -gon

Miki is angry with the book because she doesn't like how it treats symmetries, I think because she wants  $D_{2n}$  to be called  $D_n$ .

Miki drew a triangle on the board, and began talking about the different operations we can perform on that triangle to preserve symmetries. She introduced  $s$  to mean a reflection, and  $r$  to mean a rotation. For a triangle, there are three distinct reflections,

$$s = \{s_1, s_2, s_3\},$$

where  $s_i$  is the reflection across the line  $OA_i$ . We can also rotate the triangle in two directions.

We know that these are all the symmetries, since we can count the permutations of the triangle. We've exhausted then, so we know that there can't be any more elements of the triangle-symmetry group  $D_6$ . In fact, because of the permutation fact, we know that  $|D_{2n}| = 2n$ . Some other observations about  $D_{2n}$ :

- $s^2 = e \implies s = s^{-1}$ ;
- rotating twice clockwise is the same as rotating counterclockwise, so these aren't unique elements;
- $r^n = e$
- $rs = s_2$ , so  $s_n$  is just a combination of  $r$  and  $s$  — then we can generate the entire group with just  $r$  and  $s$ .

These things lead us to discover a new object.

**Definition 2** (Generators). For a group  $G$ , the generators of  $G$  is a set  $S = \{a, b, \dots : a, b, \dots \in G\}$  where  $G$  is equal to all possible combinations of elements of  $S$ . For  $D_{2n}$ , we could say that  $D_{2n}$  is generated by  $r$  and  $s$ . Usually there isn't a way to guess the generators of a group easily.

*Generators*

**Definition 3** (Relations). A relation is a way of writing equivalent elements of groups. For example, in  $D_{2n}$ ,

*Relations*

$$r^n \equiv 1, \quad s^2 \equiv 1, \quad sr \equiv r^2s.$$

Relations allow us to define how we can commute elements of the group.

**Definition 4** (Presentation). A presentation of a group are the generators combined with the relations necessary to create the group. The largest group which is generated from the generators and which satisfies the relations, and has no other relations, is our group. A presentation

*Presentation*

is written as  $\langle a, b \mid \text{relations between } a \text{ and } b \rangle$ , where  $a$  and  $b$  are the generators of the group.

Now Miki told us that the group of the symmetries of a regular  $n$ -gon is the dihedral group of order  $2n$ , written either as  $\{D_{2n} \text{ or } D_n\}$ , depending on if you are a representation theorist or not.

**Problem 3 (HW).** Why is the order of  $D_{2n}$  always  $2n$ ?

## 1.2 Symmetric group on $n$ elements

Miki defined the symmetric group on  $n$  elements  $S_n$ , which is just the permutations of  $n$  elements. Notice that  $D_{2n}$  is a subgroup of  $S_n$ . We know that the order of  $S_n = n!$  and the order of  $D_{2n} = 2n$ .

[Insert diagrams of different ways to denote permutations, like the cycle notation]

## 2 September 4, 2018

**Definition 5.** For a set  $\Omega$ , the symmetric group on  $\Omega$  is  $S_\Omega = \{\text{bijective maps } \Omega \rightarrow \Omega\}$ . For  $n \in \mathbb{N}$ , we say that  $S_n = S_{\{1, \dots, n\}}$ . This is usually called the symmetric group on  $n$  letters.

Let's consider this example for  $S_4$  (warning, there's some cyclic decomposition for  $g_1, g_2$ ?)

**Example 2.** Consider the following maps  $g_1, g_2 \in S_4$ ,

$g_1$	$g_2$
$1 \rightarrow 2$	$1 \rightarrow 3$
$2 \rightarrow 1$	$2 \rightarrow 1$
$3 \rightarrow 4$	$3 \rightarrow 2$
$4 \rightarrow 2$	$4 \rightarrow 4$

We can also write these as  $g_1 = (12)(34)$  and  $g_2 = (132)(4)$ . In this notation, how do we multiply things? E.g., what is  $g_2 g_1$ ? Well, we can write this naively as  $(132)(4)(12)(34)$ , but we don't want to repeat any numbers. Let's see what happens to 1:

$$(132)(4)(12)(34) \cdot 1 = (132)(4)(12) \cdot 1 = (132) \cdot 2 = 1.$$

For 2, we get

$$(132)(4)(12)(34) \cdot 2 = 3.$$

For 3, this comes  $g_2 g_1 \cdot 3 = 4$ , and for 4 we have  $g_2 g_1 \cdot 4 = 2$ . Then  $g_2 g_1 = (1)(234)$ . Unfortunately, doing this sort of element-wise reduction is the fastest way to multiply anything.

**Problem 4.** Someone asked the question "does order matter?" E.g., is it true that  $(12)(34) = (34)(12)$  always?

*Solution.* No. They are the same. Also,  $(abc) = (bca)$ ; as long as the sign of the permutation of the cycle elements is +1, it won't matter how you order the elements of a cycle. ■

**Problem 5.** Does order matter when there is a number repeated (when the cycles are not disjoint)? E.g. does  $g_1g_2 = g_2g_1$ ?

*Solution.* Yeah, order does matter. Consider that  $(12)(13) \neq (13)(12)$ . This means that, in general,  $S_n$  is not abelian. ■

**Problem 6.** Consider  $S_5$ , where  $g = (123)(45)$  and  $h = (12345)$ . Find  $g^2, g^{-1}, h^{-1}$ . Fun fact, it's easy.

These facts lead us to an interesting and useful conclusion.

**Proposition 1.** For any  $g \in S_n$ , we can write  $g$  as a product of disjoint cycles.

This gives us an interesting observation for  $S_n$ .

**Proposition 2.** Let  $g \in S_n$  be written as the product of disjoint cycles. Then the order of  $g$  is the least common multiple of the orders of the disjoint cycles.

## 2.1 Fields $n$ stuff

**Definition 6.** A field  $k$  is a triple  $(F, +, \times)$  where  $(F, +)$  and  $(F \setminus \{0\}, \times)$  are groups where  $F^\times = F \setminus \{0\}$  and where multiplication distributes over addition. Some canonical examples are  $\mathbb{Q}$ ,  $\mathbb{R}$ ,  $\mathbb{C}$ ,  $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$  for prime  $p$ .

A brief note on finite fields: for a finite field  $\mathbb{F}$ , we know that  $|\mathbb{F}| = p^n$  for some prime  $p$  and some  $n \geq 1$ .

Now that we have fields, we can get matrices for free. Consider the canonical matrix group  $\text{GL}_n(k)$  of invertible matrices with entries in  $k$ .

**Example 3.** Consider  $\text{GL}_2(\mathbb{F}_2)$  where  $\mathbb{F}_2 = \{\bar{1}, \bar{2}\}$  (note that this is just  $\mathbb{Z}/2\mathbb{Z}$ ). What is the order of  $\text{GL}_2(\mathbb{F}_2)$ ?

*Proof.* There are six. Any element cannot have three or four zeros in it, nor two zeros in the same row or column. Then just count the total possibilities. ■



### 3 September 7, 2018

Some facts about finite fields.

1. For all prime  $p$ , there exists a field  $\mathbb{F}_p$  where  $|\mathbb{F}_p| = p$ ;
2. For all prime  $p$  and  $n > 0$ , there exists a field  $\mathbb{F}$  where  $|\mathbb{F}| = p^n$ ;
3. Every finite field has order  $p^n$  for some prime  $p$  and some  $n > 0$ .

From last time, we know for all prime  $p$  and all  $n > 0$ , there exists a field with  $p^n$  elements. However, the naïve choice for this field isn't always right.

**Example 4.** Consider  $\mathbb{F}_4$ ; what could it be?

*Solution.* It can't be  $\mathbb{Z}/4\mathbb{Z}$ , since inverses aren't unique as 4 isn't prime, and so  $(\mathbb{Z}/4\mathbb{Z} \setminus \{0\}, \times)$  isn't a group. However, it could be the direct product of  $\mathbb{Z}/2\mathbb{Z}$  with itself; i.e.,  $\mathbb{F}^4 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ . Whatever it is, we know it has elements  $\{\bar{0}, \bar{1}, x, x+1\}$  which satisfies  $x^2 + x + 1 = 0$ ,  $\bar{1} + \bar{1} = 0$ , and  $x + x = 0$ . Then  $x^2 = -x - 1 = x + 1$ . ■

#### 3.1 Homomorphisms

**Definition 7** (Homomorphism). A group homomorphism is a map  $\varphi : (G, *) \rightarrow (H, \times)$  which preserves the operations between the groups, so  $\varphi(a * b) = \varphi(a) \times \varphi(b)$ . Usually, this is just abbreviated into  $\varphi(ab) = \varphi(a)\varphi(b)$ .

*Homomorphism*

**Lemma 2.** Let  $\varphi$  be a homomorphism. Then  $\varphi(1_G) = 1_H$ , and  $\varphi(a^{-1}) = \varphi(a)^{-1}$ .

*Proof.* We know that  $1 \cdot 1 = 1$ . Then  $\varphi(1 \cdot 1) = \varphi(1)\varphi(1) = \varphi(1)$ . Then multiply by  $\varphi(1)^{-1}$ , and we have that  $\varphi(1) = 1$ . Consider then that  $1 = aa^{-1}$ , so  $\varphi(1) = \varphi(a)\varphi(a^{-1}) = 1$  (by the previous result). Then  $1 = \varphi(a)\varphi(a^{-1})$ , so  $\varphi(a^{-1}) = \varphi(a)^{-1}$  since inverses are unique. ■

**Example 5** (Examples of Homomorphisms).

1. The identity map  $g \mapsto g$ ;
2. The determinant  $\det : \text{GL}_n(\mathbb{R}) \rightarrow (R^\times, \times)$ ;
3. The map  $(\mathbb{Z}, +) \rightarrow (\mathbb{Z}_n, +)$  where  $a \mapsto \bar{a}$ ;
4. Let  $g \in G$ . Then we have a map  $(\mathbb{Z}, +) \rightarrow G$  where  $n \mapsto g^n$ .

**Definition 8** (Isomorphism). An isomorphism is a bijective homomorphism. Note that the inverse of an isomorphism is also a group isomorphism.

Isomorphism

What does it mean for two things to be isomorphic? Well, it means that anything you care about can be preserved under a sufficiently good map, so two isomorphic groups aren't the same, but they're "the same." As an example of why they aren't actually the same, consider that  $(\mathbb{Z}_2, +)$  and  $(\mathbb{Z}_3^\times, \times)$  are isomorphic. These groups don't have the same elements or the same operations, but they are isomorphic to one another.

**Lemma 3.** Let  $\phi : G \rightarrow H$  be a homomorphism where  $g_i \mapsto h_i$ . Then any relation on  $\{g_i\}$  is satisfied by  $\{h_i\}$ . For example, if  $G$  is abelian then  $H$  is abelian as well.

**Corollary 1.** If  $G = \langle g_1, \dots, g_n \mid \text{relations} \rangle$ , and if  $h_1, \dots, h_n \in H$  satisfy the same relations, then there exists a homomorphism  $\phi : G \rightarrow H$  where  $g_i \mapsto h_i$ . However, any map which does preserve these relations need not be surjective nor injective. This means that presentations aren't enough to determine group isometry. Worse, minimal generating sets may not even have the same size for distinct generators. For example  $\{1\}$  and  $\{2, 3\}$  both generate  $\mathbb{Z}_6$ .

**Corollary 2.** Homomorphisms don't actually preserve order, since if  $g^n = 1$  then  $\phi(g^n) = 1$ , but the order of  $\phi(g^n)$  might just be a divisor of  $n$ , not  $n$  itself.

**Definition 9** (Subgroup). A subgroup  $H$  of  $G$  is a group where the set of  $H$  is a subset of  $G$  and  $H$  inherits its operation from  $G$ . Formally,  $H$  is a subgroup of  $G$  if the following are satisfied:

Subgroup

- $e \in G$ ;
- $a \in H \implies a^{-1} \in H$ ;
- $a, b \in H \implies ab \in H$ .

## 4 September 10, 2018

A useful fact about orders a generated subgroups.

**Lemma 4.** *Let  $x \in G$ , and let  $\langle x \rangle \subset G$ . Then  $|x| = |\langle x \rangle|$ .*

Today, we're gonna connect the notion of a homomorphism and the notion of a group. Let  $\phi : H \rightarrow G$  be a homomorphism. Then the image  $\phi(H)$  is a subgroup of  $G$ . Why is this true? Well, trivially,  $\phi(H)$  is a subset of  $G$ . Since  $\phi$  is a homomorphism, we know that  $\phi(1) = 1$  and so  $1 \in \phi(H)$ . Since  $\phi$  is multiplicative,  $\phi(a), \phi(b) \in \phi(H) \implies \phi(a)\phi(b) \in \phi(H)$ . And since  $\phi(a)^{-1} = \phi(a^{-1})$ ,  $\phi(H)$  contains inverses for all  $\phi(a) \in \phi(H)$ . However, there's not much that we can say about  $\phi(H)$  in relation to  $G$ , other than the fact that it must not be larger than  $G$ . However, if  $\phi : H \hookrightarrow G$  is injective, then  $H$  and  $\phi(H)$  are isomorphic, so there's a copy of  $H$  inside of  $G$ .

### 4.1 Representation Theory

Miki says that she's not supposed to talk about representation theory in this class but she can't resist mentioning it here when we discuss group actions.

**Definition 10** (Group Action). Let  $G$  be a group. A group action is a map  $\phi : G \times A \rightarrow A$ , where  $A$  is a set on which  $G$  is acting, which obeys the following axioms.

*Group Action*

1. The identity in  $G$  becomes the identity map, so  $\phi(1_G, a) = a$  for all  $a \in A$ ;
2. The action  $\phi$  is associative, so  $\phi(g, \phi(h, a)) = \phi(gh, a)$ .

The simplest example of a group action is the *trivial action*, which is simply the map  $\phi(g, a) = a$  for any  $a \in A$  and any  $g \in G$ . Another example is *translation*, where we map each  $a$  to  $a + n$  from some  $n$ . *Reflection* is where we map  $a$  to  $-a$ .

**Example 6.** Some food for thought: the group operation is also an action on that group.

Some facts about group actions.

**Lemma 5.** *For all  $g \in G$ , we get a map  $\sigma_g : A \rightarrow A$  where  $a \mapsto g \cdot a$ ; then  $\sigma_g$  is bijective since it's just a permutation of  $a$ ; then we have a map  $\pi : G \rightarrow S_A : g \mapsto \sigma_g$ . This map  $\pi$  is a group homomorphism. However, we don't know that  $\pi$  is necessarily injective.*

*Proof.* We know that  $\sigma_g$  is bijective since it has an inverse in  $\sigma_{g^{-1}}$ . Since it's bijective, we know that  $\sigma_g$  is a permutation, and so is an element of  $S_A$ . Then consider that  $\pi(gh)(a) = (gh)(a) = g \cdot (h \cdot a) = \pi(g)\pi(h)(a)$ , so  $\pi$  is multiplicative. ■

**Example 7.** Let  $A = G$ , so our action is left multiplication  $* : G \times G \rightarrow G : (g, a) \mapsto ga$ . Since multiplication already fulfills the requirements for group actions, we know this forms a valid action. For this action, look at the map  $\pi : G \rightarrow S_G$ . We know  $\pi(g) = \sigma_g$  is bijective. Then  $\pi$  is injective. This fact gives us that every finite group is isomorphic to a subgroup of  $S_n$  for some  $n$ , since  $G \cong \pi(G) \subset S_n$  for  $n = |G|$ .

*Proof.* Suppose  $\sigma_g = \sigma_h$ . Then  $\sigma_g(a) = \sigma_h(a)$  for all  $a \in A = G$ . In particular, let  $a = 1$ . The  $g = g \cdot 1 = h \cdot 1 = h$ , so  $g = h$ . Then  $\pi(g) = \pi(h)$  if and only if  $g = h$ . ■

## 4.2 Isomorphisms and Equality

Why do we bother saying that groups are isomorphic instead of just saying that groups are “equal.” Consider  $D_8$  acting on a square  $\square$ . There is a subgroup  $H_1 = \langle r^2 \rangle = \{1, r^2\}$ . We know that  $H_1 \cong \mathbb{Z}/2\mathbb{Z}$ . There is also the subgroup  $H_2 = \langle s \rangle = \{1, s\}$ . We know that  $H_2$  is also isomorphic to  $\mathbb{Z}/2\mathbb{Z}$ . However, it's pretty clear that  $H_1 \neq H_2$  even though  $H_1 \cong H_2$ . Then isomorphic groups can be distinguished by their group actions.

## 5 September 12, 2018

“Oh, I erased my smiley face. How sad.” (she did not sound sad)

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Miki

Today we’ll officially state something we covered last time.

**Theorem 2** (Caley’s Theorem). *Every finite group  $G$  is isomorphic to a subgroup of  $S_n$  for some  $n$ .*

*Proof.* Let  $n = |G|$ . ■

### 5.1 Kernels

Let’s discuss formally the idea of a kernel of a homomorphism and a kernel of a group action.

**Definition 11** (Kernel). Let  $\phi : G \rightarrow H$  be a homomorphism. Then the kernel of  $\phi$ , written  $\ker \phi$ , is the set of all elements in  $G$  which are mapped to the identity in  $H$ ; i.e.,  $\ker \phi = \{g \mid \phi(g) = 1_H\}$ .

*Kernel*

**Definition 12.** Suppose  $G$  acts on  $A$  by  $\pi$ . Then the kernel of the action is the set of all elements of  $G$  which act trivially on  $A$ ; i.e.,  $\ker \pi = \{g \mid ga = a \text{ for all } a \in A\}$ .

**Example 8.** Consider the action  $\phi : \text{GL}_2(\mathbb{R}) \rightarrow (\mathbb{R}^\times, \times) : A \mapsto \det A$ . Then the kernel of  $\phi$  are all matrices with determinant 1, called  $\text{SL}_2(\mathbb{R})$ .

**Definition 13** (Stabalizer). Let  $\pi : G \times A \rightarrow A$  be a group action, and fix  $a \in A$ . The *stabalizer* is  $G_a = \{g \in G \mid ga = a\}$ . By this definition, the kernel is contained within any stabalizer, and in fact is equal to the intersection of all stabalizers.

*Stabalizer*

**Example 9.** Let  $G = \text{GL}_2(\mathbb{R})$  and let  $A = \mathbb{R}^2$  defined with the usual action (vector-matrix multiplication). What is the kernel of this action? Then let  $c = (0, 1)^\top \in \mathbb{R}^2$ . What is the stabalizer of  $c$ ?

**Corollary 3.** *The kernel of an action is a subgroup of  $G$ , and  $G_a$  is a subgroup of  $G$  for any fixed  $a \in A$ .*

**Definition 14** (Orbit). Fix  $a \in A$ . The orbit of  $a$  is the image of  $a$  under the group action; i.e.,  $O_a = \{ga \mid g \in G\}$ . Intuitively, it's everywhere  $a$  can go under a specific group action. Notice that the orbits partition  $A$ , and so are equivalence classes in  $A$ .

Orbit

**Example 10.** Let  $G = \text{GL}_2(\mathbb{R})$  and let  $A = \mathbb{R}^2$  defined with the usual action (vector-matrix multiplication). What is the orbit of  $(1, 0)^\top$ ?

**Definition 15** (Faithful). An action is faithful if the kernel is the identity. This means that the base element of the action must be the identity. This tells us that  $G$  is injective into  $S_A$ .

Faithful

**Example 11.** Consider  $D_8$  acting on a square (technically the set  $A = \{1, 2, 3, 4\}$ ). The orbit  $O_1$  is all possible vertices, since you can rotate any vertex to any position. The stabilizer is  $\{1, s\}$ .

**Lemma 6.** As it turns out, for a fixed  $a \in A$ , we see that  $|O_a||G_a| = |G|$ . We'll prove this later. (Orbit-Stabilizer Theorem I think?)

**Definition 16** (Conjugation). Consider the action  $\pi : G \times G \rightarrow G : (g, a) \mapsto gag^{-1}$ . This action is known as *conjugation*.

Conjugation

**Definition 17** (Centralizer). Let  $S \subset G$ . The *centralizer* of  $S$  in  $G$ , written  $C_G(S) = \{g \in G \mid gsg^{-1} = s \text{ for all } s \in S\}$ . This is the set of things that fix  $S$  in  $G$  pointwise under conjugation. By definition, this is the set of elements in  $G$  which commute with all elements in  $S$ . In the case that  $S = \{s\}$  we see that  $C_G(S) = G_S$ .

Centralizer

**Definition 18** (Normalizer). Let  $S \subset G$ . The *normalizer* of  $S$  in  $G$  is  $N_G(S) = \{g \in G \mid gSg^{-1} = S\}$ . Essentially, this is just a centralizer on a set, except that it may permute the elements of  $S$ . Then  $C_G(S) \subset N_G(S)$ .

Normalizer

**Example 12.** Suppose that  $G$  is abelian. For any  $S \subset G$ , we see that  $C_G(S) = N_G(S) = G$ .

**Example 13.** Let  $G = S_3$ , and let  $S = G$ . What is the normalizer of  $S$ ? (It's the whole thing since  $G$  is closed under its operation.) What is the centralizer of  $S$ ? (It's the identity.)

## 6 September 14, 2018

“Why do we get struck by lightning when we reach a contradiction? I don’t know, it’s usually a good thing.” ♣

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Miki

**Definition 19** (Center). The center of a group  $G$  is  $Z(G) = \{g \in G \mid gs = sg \text{ for all } s \in G\}$ ; i.e.,  $Z(G) = C_G(G)$ , so it’s the centralizer of the whole group.

Center

Why do we care so much about conjugation? We give all these special names to the sets of conjugation, like the Normalizer, Stabilizer, and Centralizer. We also know that conjugation preserves the order of an element, so  $|a| = |gag^{-1}|$ .

**Problem 7.** What is the center of  $D_8$ ? We know the identity must be in the center. What about  $r^2$ ? We know it commutes with  $s$ , and  $sr^2 = r^{-2}s = r^2s$ , so it commutes with  $s$  as well; Since  $r$  and  $s$  generate the group, we know that it be in the center as well. So  $Z(D_8) = \{1, r^2\}$ .

### 6.1 Cyclic Groups

**Proposition 3.** Let  $G$  be a group, and let  $x \in G$ . For  $m, n \in \mathbb{Z}$ , if  $x^n = x^m = 1$  then  $x^d = 1$  where  $d = \gcd(m, n)$ .

*Proof.* Use the Euclidean Algorithm. We know there are integers  $a, b \in \mathbb{Z}$  where  $d = am + bn$ , so  $x^d = x^{am+bn} = (x^a)^m (x^b)^n = 1^a 1^b = 1$ . ■

**Corollary 4.** If  $x^m = 1$  then  $|x|$  divides  $m$  if  $m$  is finite.

*Proof.* If  $m = 0$ , we are done since everything divides zero. Assume that  $1 \leq m < \infty$ . Let  $n = |x| \leq m < \infty$  be finite. Let  $d = \gcd(m, n)$ , so  $x^d = 1$ . We know that  $d$  divides  $n$ , and since  $n$  is the smallest power of  $x$  to be the identity, we know that  $d = n$ . *A priori*, we know that  $d$  divides  $m$  so  $d$  must divide  $m$  as well. ■

**Proposition 4.** Let  $x \in G$ , and let  $a \in \mathbb{Z} \setminus \{0\}$ .

1. If  $|x| = \infty$ , then  $|x^a| = \infty$ ;
2. If  $|x| = n < \infty$ , then  $|x^a| = n / \gcd(a, n)$ .

*Proof.* The proof of (1) is omitted, and left as an exercise to the student. For (2), let’s focus on the special case that  $a$  divides  $n$ . If  $x^n = 1$  then  $(x^a)^{n/a} = x^n = 1$ .



Then  $|x^a|$  is at most  $n/a$ . Suppose by way of contraction that the order  $d$  is strictly less than  $n/a$ . Then  $x^{ad} = 1 \implies 1 \leq ad < n$ , but  $|x| = n$ . This is a contradiction, so the order of  $x^a$  must be exactly  $n/a$ . In the case that  $a$  does not divide  $n$ , play around with this to get the more general conclusion (the logic is the same). ■

**Definition 20** (Cyclic Group). A group  $G$  is cyclic if there exists an  $x \in G$  such that  $G = \langle x \rangle$ . As a note, it's not always easy to tell since there could be other presentations of a group which are not single elements. Always remember that presentations are not unique.

*Cyclic Group*

**Problem 8.** Let  $G = \langle a, b \mid a^2 = b^3 = 1, ab = ba \rangle$ . Show that  $G$  is cyclic.

**Corollary 5.** All cyclic groups must be abelian, since any  $g \in G$  is generated by some  $x^a$ , and  $x$  always commutes with itself.

**Example 14** (Infinite Cyclic Groups). Throughout, let  $G = \langle x \rangle$ , and assume that  $|x| = \infty$ .

**Proposition 5.** The order of  $G$  is  $\infty$ . Then  $x^m \neq x^n$  for all distinct  $m, n \in \mathbb{Z}$ .

*Proof.* Let  $m < n$ . Suppose by way of contradiction that  $x^m = x^n$ . Then  $x^{n-m} = 1$ , which cannot happen since  $n - m > 0$  and  $|x| = \infty$ . Then  $|G| = \infty$ . ■

**Proposition 6.** Such  $G$  must be isomorphic to  $(\mathbb{Z}, +)$ .

*Proof.* Define a map  $\phi : \mathbb{Z} \rightarrow G : n \mapsto x^n$ . This map is well defined. It also respects multiplication since  $m + n \mapsto x^m x^n$ . It is injective by Proposition 1, and it is surjective by Proposition 1 since  $G$  is generated completely by  $x$ . Then  $\phi$  is an isomorphism. ■

**Proposition 7.** Such a group  $G$  is generated by  $x^n$  if and only if  $n = \pm 1$ .

*Proof.* Left as an exercise to the student. ■

**Proposition 8.** Every subgroup of  $G$  is cyclic of the form  $H = \langle x^n \rangle$  for some  $n \in \mathbb{Z}$ .

*Proof.* Suppose that  $x^n = 1$ . Then  $H$  is obviously cyclic. On the other hand, if  $H \neq \langle 1 \rangle$ . Let  $n = \min\{k > 0 \mid x^k \in H\}$ . This can't be empty, so there is an  $n$ . Then  $\langle x^n \rangle \subset H$ . Take some other element  $x^m \in H$ , and let

$d = \gcd(m, n) = am + bn$ . Then  $x^d = (x^m)^a(x^n)^b \in H$  but  $1 \leq d \leq n$ , so  $d = n$ . Then  $n$  divides  $m$ , and so  $x^m \in \langle x^n \rangle$ . The  $\langle x^n \rangle = H$ , and so  $H$  is cyclic. ■

**Corollary 6.** *Every non-trivial subgroup of  $\mathbb{Z}$  is isomorphic to  $\mathbb{Z}$ .*

**Corollary 7.** *For some cyclic  $G$ , we know that  $\langle x^n \rangle = \langle x^{-n} \rangle \subset G$ . Then all non-trivial subgroups correspond to  $\mathbb{Z}_{>0}$ .*

## 7 September 17, 2018

“The only thing I learned for years was how to count hedgehogs in a field.” (In the midst of a wonderful and inspirational talk about being a mathematician.)

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Miki

### 7.1 Finite Cyclic Groups

Today, we'll cover finite cyclic groups. This will be very similar to the previous lecture on infinite cyclic groups. As a reminder, here are the propositions for infinite cyclic groups:

**Proposition 9** (Infinite Cyclic Groups). *Let  $G$  be an infinite cyclic group.*

1. *The order of  $G$  is infinite, with  $G = \{\dots, x^{-1}, 1, x, x^2, \dots\}$  all distinct.*
2. *The group  $G$  is isomorphic to  $\mathbb{Z}$ .*
3. *The group  $G$  is generated by  $x^n$  if and only if  $n = \pm 1$ .*
4. *Every subgroup  $G$  is cyclic.*

Now for the finite case.

**Proposition 10** (Finite Cyclic Groups). *Let  $G = \langle x \rangle$  with  $|G| = n < \infty$ .*

- P1. *The group  $G$  is exactly  $\{1, x, \dots, x^{n-1}\}$ .*
- P2. *The group  $G$  is isomorphic to  $\mathbb{Z}/n\mathbb{Z}$ .*
- P3. *The group  $G$  is generated by  $x^k$  if and only if  $\gcd(k, n) = 1$ .*
- P4. *Every subgroup of  $G$  is also cyclic. That is, for all  $k > 0$  where  $k$  divides  $n$  we get a subgroup  $H$  of order  $k$  generated by  $x^{n/k}$ .*

*Proof of P1.* We know that  $1, \dots, x^{n-1}$  are all in  $G$ . Suppose that  $x^a = x^b$  for some distinct  $a, b$ . Then  $x^{b-a} = 1$  for  $0 < b - a < n$ , which is a contradiction since  $|x| = n$ . In fact, this set enumerates  $G$ . Suppose that  $x^k \in G$  for some  $k \in \mathbb{Z}$ . We use the division algorithm to write that  $k = an + r$  for some  $a, r \in \mathbb{Z}$ . Then  $x^k = x^{an+r} = (x^n)^a x^r = x^r$ , so  $x^k$  is in  $G$ . ■

*Proof of P2.* Let  $\phi : \mathbb{Z}/n\mathbb{Z} \rightarrow G : \bar{k} \mapsto x^k$  where  $k$  is any representative of  $\bar{k} \in \mathbb{Z}$ . To show that  $\phi$  is well-defined, consider another representative  $\ell$  of  $\bar{k} \in \mathbb{Z}$ . Then  $\ell = k + an$ , so  $x^\ell = x^{k+an} = x^k (x^n)^a = x^k$ . To show that  $\phi$  is a homomorphism,

consider that  $\phi(\bar{m} + \bar{n}) = x^{m+n} = x^m x^n$ , so  $\phi$  is multiplicative. Finally, we know that  $\phi$  is surjective and injective by P1. This tells us that, up to an isomorphism, there are only really two cyclic groups;  $\mathbb{Z}$  if the group is of infinite order, or  $\mathbb{Z}/n\mathbb{Z}$  if it is finite. ■

*Proof of P3.* This is more of a sketch. Recall that  $\langle |x^k| \rangle = |x^k|$ , and this is  $n$  if and only if  $\gcd(k, n) = 1$ . In general,  $|x^k| = n/\gcd(k, n)$ . ■

*Proof of P4.* Exactly the same as the infinite case. ■

Now that we've covered cyclic groups, it's helpful to introduce some notation to represent them.

**Notation** ( $\mathbb{Z}_n, C_n$ ). We write the multiplicative cyclic group of order  $n$  as  $\mathbb{Z}_n$ . The additive cyclic group of order  $n$ , which we've been writing as  $\mathbb{Z}/n\mathbb{Z}$ , is commonly written as  $C_n$ .

$\mathbb{Z}_n, C_n$

## 7.2 Subgroups

[DF04] uses the notation  $S \subset G$  to mean that  $S$  is a subset of  $G$ , and  $H \leq G$  to mean that  $H$  is a subgroup of  $G$ .

**Definition 21** (Subgroup). Let  $S \subset G$  be nonempty. Let  $H = \{a_1^{\varepsilon_1} \cdots a_k^{\varepsilon_k}\}$  where  $a_i \in S$  and  $\varepsilon_i = \pm 1$  for  $k \in \mathbb{Z}_{\geq 0}$ . This sequence of  $a_i^{\varepsilon_i}$  is called a *word*. Note that  $a_i$  need not be distinct. Then  $H$  is a subgroup of  $G$ .

*Subgroup*

*Proof.* First, we know that  $H \subset G$ . We know that  $1 \in H$ . Since any concatenation of words is also a word, we know that  $H$  is closed under multiplication. Finally, since  $((ab)^n)^{-1} = b^{-n}a^{-n}$  we know that the inverses of a word are words themselves, and so  $H$  is closed under inversion. ■

## 8 September 19, 2018

“Funny things happen with groups,  
which is why they’re fun!”

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Miki

Recall from last time how we defined a subgroup  $H$  of  $G$  in terms of words where the powers of each element was  $\pm 1$ . If  $G$  is abelian we can combine elements of like bases to get powers which can be any integral value. If we assume that  $|a_i| = d_i$  is finite for all  $a_i \in H$ , then we know that  $|H| \leq d_1 \cdots d_k$ . This gives us a limit on the order of a subgroup; if  $G$  is abelian then the order of a subgroup is bounded above by the product of the orders of the generating elements. On the other hand, if  $G$  is not abelian then this does not always hold. Consider  $G = \langle a, b \mid a^2 = b^2 = 1 \rangle$ . If  $G$  isn’t commutative, then  $(ab)^n \neq a^n b^n$  for all  $n$  and so we can just create infinitely many words by appending  $ab$  to one another and so the order is infinite.

**Lemma 7.** *Let  $G = \{a_1^{n_1} \cdots a_k^{n_k}\}$  be abelian, and let each  $a_i$  have finite order  $d_i$ . Then  $|G| \leq d_1 \cdots d_i$ .*

**Proposition 11.** *Let  $G$  be a group and let  $\mathcal{L}$  be a collection of subgroups of  $G$ . Then*

$$K = \bigcap_{L \in \mathcal{L}} L$$

*is a subgroup of  $G$ .*

**Definition 22** (Subgroup). Let  $S \subset G$  and let  $\mathcal{L} = \{L \leq G \mid s \in L\}$ .

*Subgroup*

Then the subgroup generated by  $S$  is

$$K = \bigcap_{L \in \mathcal{L}} L.$$

What do we know from this definition? Well,  $S \subset K$  and  $K \leq G$ . We want to say that  $K \in \mathcal{L}$  is the minimal element, so  $K = L_i$  for some  $i$ .

**Definition 23** (Minimal Element). Let  $\mathcal{M}$  be a collection of subsets of  $G$ . A minimal element is an element  $M$  of  $\mathcal{M}$  such that if  $M' \in \mathcal{M}$  and  $M' \subset M$  then  $M = M'$ . It’s like “the smallest element” except there could be multiple minimal elements.

*Minimal Element*

We want to show that  $K$  is the minimal element of  $\mathcal{L}$ .

*Proof  $K$  is minimal.* Let  $L \in \mathcal{L}$ . Then  $K \subset L$ . Then either  $K = L$  or  $L$  is not minimal. ■

*Proof  $K$  is the minimal element.* Suppose there is another minimal  $M$  in  $\mathcal{L}$ . By definition  $K \subset M$  so by minimality  $M = K$ . ■

**Proposition 12.** *Our two definitions for subgroup (generated by words  $H$  and minimal element of collection  $K$ ) containing elements of  $S \subset G$  are equivalent.*

*Proof.*  $H \leq G$  and  $S \subset H$  by the construction of 1-letter words. Then  $H \in \mathcal{L}$  so  $K \subset H$ . On the other hand,  $S \subset K$  and  $K$  is a group. Then  $K$  contains all inverses and products of elements in  $S$ , so it contains all words and therefore contains  $H$ . Then  $H \subset K$ . Putting this together, we have that  $K = H$ . ■

**Definition 24** (Lattice). Given a group  $G$ , a lattice is a diagram showing all subgroups of  $G$  which shows containment between the subgroups.

*Lattice*

**Figure 1:** Lattice Diagram of  $C_2$

Recall from last time that for  $C_n$  the subgroups are paired with the divisors  $k$  of  $n$ ; then  $\langle k \rangle$  generates subgroup of order  $n/k$ .

**Figure 2:** Lattice Diagram of  $C_4$

**Figure 3:** Lattice Diagram of  $C_8$

**Figure 4:** Lattice Diagram of  $C_6$

**Figure 5:** Lattice Diagram of  $S_3$

## 9 September 21, 2018

“Oh I erased my smiley face again.  
How sad.” (She did not sound sad.)

---

Miki

**Problem 9** (Warm up). Draw the lattice diagram for  $C_{12}$ .

**Figure 6:** Lattice for  $C_{12}$

*Finish this*



## 10 September 24, 2018

“This is where the fun begins.”  
(slightly paraphrased)

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Miki

### 10.1 Quotient Groups

For the rest of this section, keep in mind the example of  $\mathbb{Z}/n\mathbb{Z}$ . This is kind of like the prototypical example for quotient groups.

**Definition 25** (Coset). Let  $H \leq G$ . The left coset of  $H$  in  $G$  is a set of the form  $aH = \{ah \mid h \in H\} \subset G$  for a fixed  $a \in G$ . The right coset of  $H$  in  $G$  is a set of the form  $Hb = \{hb \mid h \in H\} \subset G$  for a fixed  $b \in G$ .

*Coset*

We said previously that left multiplication permutes the elements of  $H$  (this was called the left regular action), and in particular we know that  $|aH| = |H|$ . We can see this trivially by simply multiplying each  $ah$  by  $a^{-1}$ . Note that this coset is usually *not* a subgroup; if  $a^{-1} \notin H$  then  $e \notin aH$ .

**Example 15.** Let  $G = \mathbb{Z}$ , and let  $H = 2\mathbb{Z}$ . Consider the cosets  $0 + H$  and  $1 + H$  (these are just the even integers and the odd integers). In particular,  $0 \notin 1 + H$  and so  $1 + H \not\leq G$ .

*The Right and Left cosets here are equal, which is always true of  $G$  is abelian.*

Notice that in the above example, the cosets are disjoint and partition the group into equivalence classes. In general this is a true statement.

**Lemma 8.** *The cosets of  $H$  partition  $G$  into equivalence classes, with the relation  $a \sim b$  if and only if  $a = bh$  for some  $h \in H$ . In particular,  $a \sim b$  if and only if  $aH = bH$ , and so the cosets defined by those elements are identical.*

**Corollary 8.** *The order of the cosets divides the order of  $G$ . In particular,  $|G| = |H| \cdot [G : H]$  where  $[G : H]$  is the index of  $H$  in  $G$  and is the number of (left OR right) cosets of  $H$  in  $G$ .*

In the example with  $\mathbb{Z}$  and  $2\mathbb{Z}$ , let's try to make these cosets behave like groups. Consider that  $(0 + H) + (1 + H) = (1 + H)$  (which just says that an even plus an odd equals an odd). We also have a homomorphism  $\pi : \mathbb{Z} \rightarrow 2\mathbb{Z} : n \mapsto \bar{n} = n + H$ . This maps integers to cosets. Note that  $\pi$  respects the operations in each group! This is kind of what defines “adding even and odd integers” in the languages of sets.

**Notation.** Let  $0 + H = \pi^{-1}(\bar{0}) = \{n \in \mathbb{Z} \mid \pi(n) = \bar{0}\}$ ; this is the preimage of  $\pi$  or the fiber of  $\pi$  above 0. Yes this is overloaded notation, and no  $\pi$  does not have an inverse (it's pretty clearly *not* injective.)

Note that it doesn't really matter which elements we send into  $\pi$  as long as they are both of the same coset, so  $\pi(a) = \pi(b)$  if and only if  $a \sim b$ . Additionally, note that since  $\pi$  is a homomorphism we can say that  $\pi(\bar{1} + \bar{2}) = \pi(\bar{1}) + \pi(\bar{2})$ .

Now, in making these cosets into groups we want them to inherit their operation from the parent group (so we can't just make up multiplications to suit our needs).

**Definition 26.** Let  $A, B \subset G$ . Then  $AB = \{ab \mid a \in A, b \in B\} \subset G$ . In particular, note that  $HH = H$  and  $(1H \cdot 1H = 1H)$ .

**Example 16** (Things Go Wrong). Let  $G = S_3$  and let  $H = \langle(23)\rangle = \{1, (23)\}$ . The cosets of  $H$  are  $1H = (23)H$ ,  $(12)H = (123)H = \{(12), (123)\}$ , and  $(13)H = (132)H = \{(13), (132)\}$ . Now consider  $1H \cdot (12)H = \{(12), (123), (132), (13)\}$ . In particular, note that this isn't a coset (it has too many elements!). We would have wanted that  $1H \cdot (12)H = (12)H$  but this doesn't happen. Then there is not quotient group  $G/H$ .

What just happened? Why can't we create a group out of the cosets of  $S_3$ ? We wanted that  $aH \cdot bH = abH$  but this didn't happen; essentially, we want  $b$  and  $H$  to commute, so we want that the left and right cosets to be equal to one another.

**Example 17.** Let  $G = S_3$  and let  $H = \langle(123)\rangle = A_3$ . This is the alternating group on three letters. As always,  $1H = H1 = H$ . Note that  $(12)H$  contains *the only other elements of  $G$*  which aren't in  $1H$ , and so  $(12)H = H(12) = G \setminus 1H = G \setminus H1$ . This happens when  $[G : H] = 2$  even though  $G$  is not abelian. Then  $S_3 \setminus A_3 = G \setminus H$  and  $\bar{a}\bar{b} = \overline{ab}$  so multiplication is well defined.

## 11 September 26, 2018

Let  $N$  be a group...I'll call it  $N$   
suggestively

---

Miki

Recall from last class that we found an example of a non-abelian group and a subgroup for which the left and right cosets of the group were the same; in this case, it was  $G = S_3$  and  $H = A_3$ .

**Definition 27** (Quotient Group). Let  $H \leq G$ . The *quotient group*  $G/H$  is a group whose elements are the left cosets of  $H$ . The set for this group is known as the quotient set, and the operation for the group is inherited from  $G$  such that  $gH \star kH = gkH$ . Note that  $(gH, \star)$  does not always form a group, so it isn't guaranteed that  $G/H$  exists for any  $G, H$ .

*Quotient Group*

### 11.1 Mapping from $G$ to $G/H$

Given a group  $G$  and a quotient group  $G/H$  we can find a very natural mapping  $\pi : G \rightarrow G/H$  where  $g \mapsto gH$ . This map sends elements to their coset, and  $\pi(a) = \pi(b)$  if and only if  $aH = bH$ ; then the fibers of  $\pi$  are the left cosets of  $H$ , and  $\ker \pi = H$ . This is why we call it the quotient group — it's like we're dividing out by  $H$ . Note that this homomorphism is always going to be surjective since there's no member of  $G$  which isn't in some coset of  $H$  as they partition  $G$ .

**Definition 28** (Normal Subgroup). Let  $N \leq G$ . Then  $N$  is normal if and only if the left and right cosets are the same, so  $gN = Ng$ . If  $N$  is normal then  $G/N$  forms a quotient group. Note that this does not mean that  $gn = ng$  so  $g$  and  $n$  do not commute necessarily, but the cosets are preserved. This is equivalent to saying that  $N_N(G) = G$  but  $C_N(G)$  is not necessarily  $G$ .

*Normal Subgroup*

**Notation** ( $\trianglelefteq$ ). We write  $N \trianglelefteq G$  to mean that  $N$  is a normal subgroup of  $G$ .

$\trianglelefteq$

**Theorem 3.** *The quotient group  $G/N$  exists if and only if  $N \trianglelefteq G$ .*

*Proof that  $N \trianglelefteq G$  is sufficient.* Observe that  $(aN)(bN) = abN$  if  $N$  is normal. Then group multiplication is well defined. Observe also that  $(aN)^{-1} = a^{-1}N$ , so the group is closed under inversion, and by definition our multiplication is associative. Then  $G/N$  forms a group if  $N$  is normal in  $G$ . ■

*Proof that  $N \trianglelefteq G$  is necessary.* Suppose  $H \leq G$  is not normal. Then there is some  $g \in G$  for which  $gH \neq Hg$ . Then we know that  $1HgH \neq gH$ , and our group operation  $\star$  cannot hold. ■

Not that  $|G/N| = |G|/|N| = [G : N]$  if  $G$  is finite, which we already knew but it's worth remembering it.

## 11.2 Testing Normality

**Proposition 13.** *The following are equivalent:*

- $N \trianglelefteq G$ ;
- $gNg^{-1} \subset N$  for all  $g \in G$  (note this implies they are equal since conjugation is injective);
- $N$  is the kernel of some homomorphism  $\pi : G \rightarrow H$  for some  $H \leq G$ .

*Proof that 1  $\implies$  2.* Let  $g \in G$  and  $n \in N$ . We know that  $gN = Ng$ , so there exists  $n' \in N$  such that  $ng = n'g$ . Multiply on the right by  $g^{-1}$  and we see that  $gng^{-1} = n'$ , and so  $gng^{-1} \in N$  for all  $g, n$ . ■

*Proof that 2  $\implies$  1.* Literally just reverse the above procedure. ■

*Proof that 1  $\implies$  3.* Let  $H = G/N$ . Then we know that  $\ker \pi = N$  where  $\pi : G \rightarrow G/N : g \mapsto gN$ . Then, rather trivially, we know  $N$  is the kernel for some homomorphism if  $N \trianglelefteq G$ . ■

*Proof that 3  $\implies$  2.* We know that  $N = \ker \pi$  for some  $\pi : G \rightarrow H$ . Then take any  $g \in G$  and  $n \in N$ , and consider that  $\pi(gng^{-1}) = \pi(g)\pi(n)\pi(g^{-1}) = \pi(g)\pi(g^{-1})$  since  $n \in \ker \pi$ , and then we conclude that  $\pi(g)\pi(g^{-1}) = 1$  and so we know that  $gng^{-1} \in \ker \pi$  so  $gng^{-1} \in N$  for all  $n \in N$  and for all  $g \in G$ . ■

## 12 Friday, 28 September 2018

“I’ll leave the cosets for later, where later means 15 seconds from now.”

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Miki

“Continuous math is not allowed...don’t tell anyone I said that.”

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Miki

Recall Lagrange’s Theorem, where if  $G$  is a finite group and  $H \leq G$  then  $|H|$  divides  $|G|$ ; in fact,  $|G|/|H| = [G : H]$ .

**Corollary 9.** *Let  $G$  be a finite group and let  $x \in G$ . Then  $|x|$  divides  $|G|$  since  $x$  generates a cyclic subgroup of order  $|x|$ , so  $|x| = |\langle x \rangle|$  which must divide  $|G|$  by Lagrange.*

**Corollary 10.** *If  $|G| = p$  is prime, then  $|G| \cong Z_p$ .*

*Proof.* Since  $|G| \neq 1$  there exists  $x \in G$  which is not the identity. Then consider  $\langle x \rangle$ . The order of this cyclic group must divide  $p$ , and since  $p$  is prime it must equal  $p$ , and so  $G = \langle x \rangle$  which means it is isomorphic to  $Z_p$ . ■

If we have some  $n \in \mathbb{Z}_{>0}$  where  $n$  divides  $|G|$  for some  $G$ , it isn’t guaranteed that there exists some  $H \leq G$  where  $|H| = n$ , and/or there isn’t always an  $x \in G$  where  $|x| = n$ . For example, consider  $G = S_3$  and  $n = 6$ . However, if prime  $p$  divides  $|G|$  then there exists an  $x \in G$  where  $|x| = p$  — Miki says she will prove this later.

### 12.1 Product Subgroups

Let  $G$  be a group and let  $H, K \leq G$ . Let’s consider the product of  $HK$ , which we recall is defined as

$$HK = \{hk \mid h \in H, k \in K\}.$$

This may or may not be a subgroup. In general it is not.

**Example 18.** Let  $G = S_3$ , and let  $H = \langle (12) \rangle$  and let  $K = \langle (13) \rangle$ . Then  $HK = \{1, (12), (13), (132)\}$  which is not a subgroup of  $S_3$  since 4 does not divide 6.

What can we say about  $HK$  anyways?

**Proposition 14.** *The order of  $HK$  is at most  $|H||K|$ . In fact,*

$$|HK| = \frac{|K||K|}{|H \cap K|}.$$

*Proof.* We know that  $HK$  is the union of left cosets of  $K$  where

$$HK = \bigcup_{h \in H} hK.$$

Consider  $a, b \in H$ . We know that  $aK = bK$  if and only if  $a^{-1}b \in K$  which is true if and only if  $a^{-1}b \in K \cap H$ . This means that  $aK \cap H = bK \cap H$ . Then we've reduce the problem to counting the number of distinct cosets  $hK$  which is just the index, so it is  $|H|/|K \cap H|$ . Multiplying through by the size of  $K$ , we find that

$$|HK| = \frac{|H||K|}{|K \cap H|}. \quad \blacksquare$$

Now we can answer when  $HK$  is a subgroup; it happens if and only if  $HK = KH$ . Intuitively, this happens only when  $hkh'k' \in HK$  which can happen if and only if we can commute the  $h$  and  $k$  elements. It is sufficient to say that  $H$  is in the normalizer of  $K$  or vice-versa. Another sufficient condition is to say that  $K \trianglelefteq G$ , or the other way around. Note that neither of these conditions is necessary.

$$H \leq N_G H \implies hK = Kh \implies hk = k'h,$$

but we only need that  $hk = k'h'$ . That is, we only need that  $hK = Kh'$  which is a weaker condition than being in the normalizer.

## 12.2 Isomorphism Theorems

**Theorem 4** (First Isomorphism Theorem). *Given a surjective homomorphism  $\phi : G \rightarrow H$ , we know that  $H \cong G/\ker \phi$ .*

*Proof.* This was the definition of  $G/\ker \phi$ , since  $\ker \phi \trianglelefteq G$ . See the previous lecture notes for a more in-depth explanation.  $\blacksquare$

**Example 19.** Consider  $\text{GL}_2(\mathbb{F}_3)$  and let  $\phi = \det : G \rightarrow \mathbb{F}_3^\times$ . Then  $\ker \phi = \text{SL}_2(\mathbb{F}_3)$ , and  $\text{GL}_2(\mathbb{F}_3)/\text{SL}_2(\mathbb{F}_3) \cong \mathbb{F}_3^\times$ . Since  $\text{GL}_2(\mathbb{F}_3)$  has 48 and  $\mathbb{F}_3^\times$  has 2 elements then we know that  $\text{SL}_2(\mathbb{F}_3)$  is of order 24.

**Theorem 5** (Second Isomorphism Theorem). *Let  $G$  be a group with  $H, K \leq G$  and let  $H \leq N_G K$ . Then  $HK/L \cong H/H \cap K$ .*

*Proof.* We know several things.

- $HK \leq H$  since  $H \leq N_G K$ ;
- $K \leq HK$ , since we know that  $H \leq N_G K$  and  $K \leq N_G K$  so  $K \leq HK$ ;
- Now we can take the quotient  $HK/K$ , which is the left cosets of  $K$  in  $HK$ . We have shown that  $hK = h'K$  if and only if  $hH \cap K = h'H \cap K$ . Then define the map  $\pi : H \rightarrow HK/K$  defined by  $h \mapsto hK$ . This is a homomorphism since  $hKh'K = hh'K$  since that's how we defined multiplication. Then  $\ker \pi$  is all elements  $h$  of  $H$  which map to the identity coset which happens if and only if  $h \in K$ , so  $\ker \pi = \{h \in H \cap K\}$ . Then by the First Isomorphism Theorem,  $H/H \cap K \cong HK/K$ . ■

**Example 20.** Let  $G = S_3$ , let  $K = A_3$ , and let  $H = \langle (12) \rangle$ . We know that  $HK = S_3$  and  $H \cap K = \{e\}$ . Then we know that  $HK/K = S_3/A_3 \cong \langle (12) \rangle/1 \cong Z_2$ .