COMP3811: Computer Graphics
Tutorial 1
Trigonometry Recap

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Objective

In this module you will use a considerable amount of trigonometry. It may have been a while since you've seen trigonometry, and in secondary school some results will not have been proven rigorously. In this tutorial, we will review the basic trig functions and prove the sine rule, the cosine rule, the Pythagorean Theorem. We will derive an important relationship between the scalar product of two vectors and the angle between them, and introduce the cross product of two vectors.

It is not the intention to insult anyone's intelligence and attendance for this tutorial will not be monitored, but you will need to make sure you are completely familiar with these results: in particular I will expect that you can prove each result.

Definitions

Just for reference, here are the definitions of the three most important trig functions: Given a right-

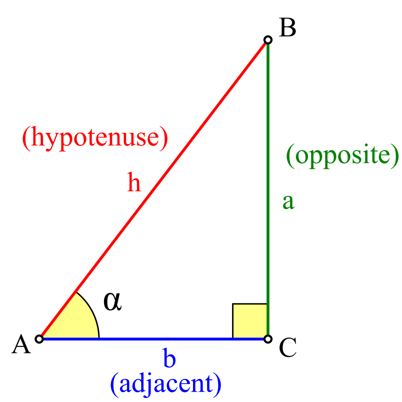

Figure 1: Definition of triangle sides.

angled triangle (Fig. 1) the sine of angle α is defined as the ratio of the opposite over the hypotenuse, and the cosine as the ratio of the adjacent over the hypotenuse:

$$\sin\alpha = \frac{a}{h} \qquad (1)$$
$$\cos\alpha = \frac{b}{h}$$

The Pythagorean Theorem

- Show that for *any* triangle surfaces area equals half times base times height

Consider Fig. 2.

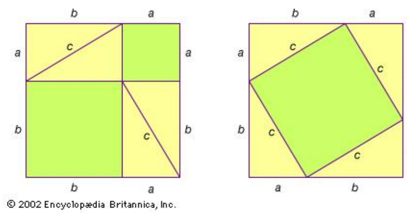

Figure 2: Set up for a geometrical proof of the Pythagorean Theorem.

- Find an algebraic expression for the area of the large square.
- Find an algebraic expression for the area of one of the triangles.
- Use both expressions to find an algebraic expression for the area of the small square.
- Observe that you have now a proof of the Pythagorean Theorem. Explain this carefully.

Cosine and Sine Rule

The cosine rule can be considered to be a generalization of the Pythagorean Theorem. Using the

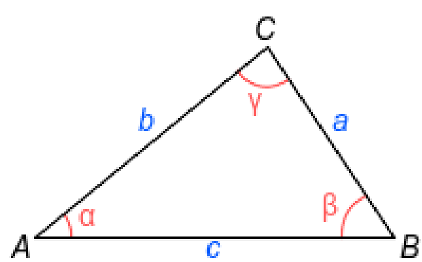

Figure 3: Triangle with notations

notations from Fig. 3 the rule is:

$$a^2 = b^2 + c^2 - 2bc\cos\alpha$$

- Use a symmetry argument to find two other expressions of this rule.
- Prove the cosine rule. Hint: split the triangle into two right-angled ones; use the Pythagorean Theorem.

While you're at it, the sine rule is:

$$\sin\alpha/a = \sin\beta/b = \sin\gamma/c$$

- Prove the sine rule.

The Scalar Product

We will speak about vectors in great an gory detail later, but I will assume you are familiar with the basics: a vector $\vec{a}$ in two dimensions represents a displacement and can be represented by two numbers:

$$\vec{a} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

This can be loosely interpreted as "*move one to the right and three upwards*". Vectors can be used to represent sides from triangles in this way: starting from the origin two vectors $\vec{a}$ and $\vec{b}$ each can represent the side of a triangle. The length of the side represented by $\vec{a}$ is then given by:

$$|\,\vec{a}\,| = \sqrt{a_x^2 + a_y^2}$$

where

$$\vec{a} = \begin{pmatrix} a_x \\ a_y \end{pmatrix}$$

So a_x is the so-called x-component off $\vec{a}$.

The scalar (or dot) product is now defined as:

$$\vec{a}\cdot\vec{b} \equiv a_x b_x + a_y b_y$$

- Prove the following relationship using the cosine rule:

$$\vec{a}\cdot\vec{b} = |\,\vec{a}\,||\,\vec{b}\,|\cos\theta,$$

where θ is the angle between vectors $\vec{a}$ and $\vec{b}$.

The dot product can be extended to an arbitrary number of dimensions:

$$\vec{a}\cdot\vec{b} \equiv \sum_{J=1}^{N} a_j b_j$$

- The relationship between dot product and cosine also holds in three and more dimensions. Argue that this suggests that the dot product is invariant under rotations (this is in fact true).

The dot product is very remarkable: it relates two bags of numbers: (the components of the two vectors; numbers that can be stored, read from file etc.) via a simple algebraic formula to a geometrical concept: the projection from one vector onto another (Fig. 4).

- Find an algebraic expression that relates the components of two vectors that are perpendicular.

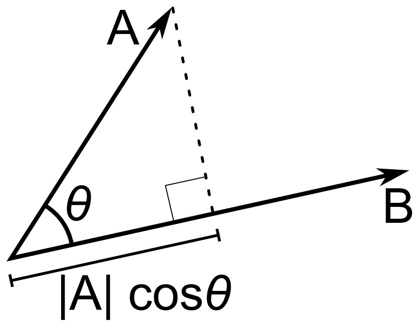

Figure 4: Projection of one vector onto another via the scalar product

The Cross Product

The cross product can be defined in two dimensions, but it makes more sense in three dimensions or higher: let:

$$\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

and

$$\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

The cross product is another vector defined by:

$$\vec{a} \times \vec{b} \equiv \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}$$

It also has a remarkable geometrical interpretation. Consider two vectors $\vec{a}$ and $\vec{b}$ that lie in the $x - y$ plane. So

$$\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ 0 \end{pmatrix}$$

and

$$\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ 0 \end{pmatrix}$$

so that the cross product is :

$$\vec{a} \times \vec{b} \equiv \begin{pmatrix} 0 \\ 0 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}$$

- Show that the surface of the parallelogram spanned by $\vec{a}$ and $\vec{b}$ is $| a || b | \sin\theta$.
- Show that $a_1 b_2 - a_2 b_1 = | a || b | \sin\theta$. *This is a slightly harder question; if you get stuck, move on*

Again we find a relationship between the geometrical properties of two vectors and an algebraic expression of its components! Moreover, one that generalises again (in a way we will explore later) to higher dimensions.

Another interesting property:

- Calculate $\vec{a} \cdot (\vec{a} \times \vec{b})$.
- Are the brackets necessary?
- What do you conclude from the answer?

Three dimensional coordinate systems have an orientation: they can be left- or right-handed. We will work mostly in Right-handed coordinate systems, but there are transformations that change the handedness of a 3D structure.

- State a transformation that changes handedness

There is a visual way for establishing whether a coordinate system is left- or right-handed:

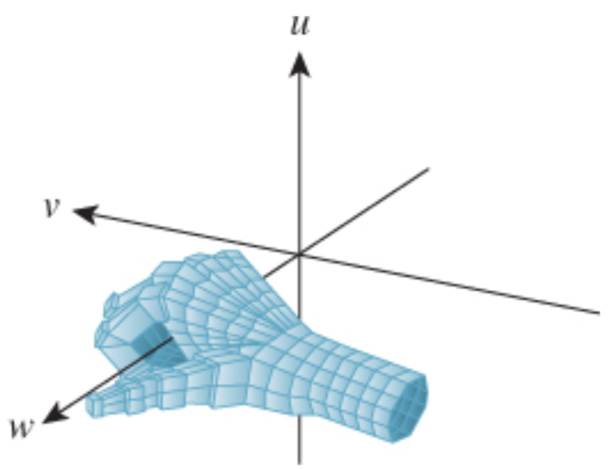

Figure 7.8: The uvw directions form a right-handed coordinate system.

Figure 5: Taken from Hughes *et al.* (Fig. 7.8)

Place your little finger (pinky) in the direction of the positive u-axis, and curl it in the direction of the positive v-axis, closing your fist. If your thumb points in the direction of the positive w-axis, your coordinate system is right-handed, otherwise it is left-handed.

Finally the cross product can be used to establish handedness: Consider

$$\vec{e_x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

and

$$\vec{e_y} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

- Calculate $\vec{e_x} \times \vec{e_y}$.
- Calculate $\vec{e_y} \times \vec{e_x}$.
- Explain the outcome of the last question in terms of the outcome of the penultimate one.
- What does the cross product express about handedness?