The Magnetic Field from Cylindrical Arc Coils and Magnets: A Compendium with New Analytic Solutions for Radial Magnetisation and Azimuthal Current

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Supplemental material. An online repository can be found at: https://github.com/AUMAG/mag-cyl-field. The online repository may contain future updates. This document contains all analytic/integral formulations and explicit analytic solutions for the PM's (Section 1) and Coils (Section 2). Analytic solutions are compared to the original numeric integral to 8 decimal places. Comparison with FEA is given in the published article. For completeness, additional comparisons are given in Sections 3 and 4 for the Green's function *integrals and elliptic integral derivatives. Section 5 contains analytic force equations that were compared to semi-analytical field methods.*

- If you are reading this in PDF format, note the code is truncated. There is a Mathematica notebook ‘.nb’ file that contains full functionality and a package ‘.wl’ file that contains the nomenclature. If you do not have a paid/trial version of Mathematica, this notebook can be opened in a free player (e.g Wolfram Player https://www.wolfram.com/player/) in order to view/copy the equations. There are also built-in or addon functions that can convert the equations to MATLAB, LATEX, Python, etc...
- The code below is not written to optimise computational speed, but to be readable, portable, and correct.
- For simplicity, the wrapper functions provided do not handle list inputs for ρ, φ, z.
- Similarly, algorithms for computing a partial sum of β,δ (η,ζ,ι) are omitted and functions take a direct input for the number of terms. The number of terms are identical between all series at one field point; therefore, we only evaluate the volume coil (Section 2.3) for P=150 terms to reduce computation time. Mathematica is holding these terms symbolically before converting to a numeric value, where there ends up being a large number of ‘0.’ terms truncated to $MachinePrecision. This is not an issue with an algorithm and adaptive convergence parameters.
- Mathematica uses complex numbers for the evaluation of ArcTan, ArcTanh,... and as such these return real numbers of the form a+0.i when converted to a numerical value using '//N'. As such we additionally use '//Chop' to remove numerical 0.i terms.
- We also use ‘//Chop’ for the comparison of the analytic and numeric result, as dependent on the numeric procedure, we only expect accuracy to a certain precision.
- Function handles are called from ‘ResultTable[]’ to return either the cylindrical or Cartesian (on axis) field components at a particular point.
- 0.3 Geometry, 0.4 Constants, and 0.5 Field Points can be freely changed. Default values conform with those of the article cited in 0.1.

0.1 Citations for this work

M. Forbes, W.S.P Robertson, A.C. Zander, J.J.H. Paulides, "The Magnetic Field from Cylindrical Arc Coils and Magnets: A Compendium with New Analytic Solutions for Radial Magnetisation and Azimuthal Current"

```
@Article {Forbes2024,
    author = {Forbes, M. and Robertson, W.S.P. and Zander, A.C. and Paulides, J.J.H},
```

journal = {Advanced Physics Research},
title = {The Magnetic Field from Cylindrical Arc Coils and Magnets: A Compendium with New Analytic Solutions for Radial Magnetisation and Azimuthal Current},
doi = {10.1002/apxr.202300136},
publisher = {Wiley},
}

0.2 Variables and Functions

In[1]:=
```
SetDirectory[NotebookDirectory[]];
<< Nomenclature`
<< Carlson` (*Used for some backwards compatibility with a spec
IntOptions ={WorkingPrecision→$MachinePrecision, AccuracyGoal→
```

0.3 Geometry

In[5]:=

$\rho' = \left\{\frac{3}{1000}, \frac{8}{1000}\right\};\ \varphi' = \left\{-\frac{\pi}{6}, \frac{3\pi}{5}\right\};\ z' = \left\{\frac{1}{1000}, \frac{5}{1000}\right\};$ (*Defini

0.4 Constants

In[6]:=

$u0 = 4\pi*10^{-7};$
$M = 955*10^{3};$ (*A/m*)
$I\varphi = 20;$ (*A*)
$K\varphi = 4*10^{4};$ (*A/m*)
$J\varphi = 1*10^{6};$ (*A/m^2*)
$\varphi☆ = \frac{\pi}{6};$ (*Diametric magnetisation direction*)
$P = 150;$ (*Number of terms in partial sum*)

0.4 Field Points

Seven field points to test all equations in article.

In[13]:=
```
ρ1 = 9/1000; φ1 = 5π/24; z1 = 31/10000; (*Standard*)
ρ2 = 7/1000; φ2 = φ1; z2 = z1; (*Inside magnet/coil*)
ρ3 = 0; φ3 = 0; z3 = z′[[2]]; (*On magnet/coil axis*)
ρ4 = ρ1; φ4 = φ′[[1]]; z4 = z′[[2]]; (*Singular plane 1*)
ρ5 = ρ′[[2]]; φ5 = -φ1; z5 = z′[[2]]; (*Singular plane 2*)
ρ6 = ρ′[[2]]; φ6 = φ′[[1]]; z6 = 6/1000; (*Singular plane 3*)
ρ7 = 2/1000; φ7 = φ1; z7 = z1; (*Coil high field area*)
```

0.5 Results Format

In[20]:=
```
MagCylField[Md_,Mρ_,Mφ_,Mz_,φ☆_,P_,ρ′_,ρ_,φ′_,φ_,z′_,z_] :=
    Module[{},
        If[Md ≠ 0,(*Diametric*)
            ResultTable[Md,φ☆,P,ρ′,ρ,φ′,φ,z′,z,"M⊥",BdAna,BdN
        ];
        If[Mρ ≠ 0,(*Radial*)
            ResultTable[Mρ,φ☆,P,ρ′,ρ,φ′,φ,z′,z,"Mρ",BρAna,BρN
        ];
        If[Mφ ≠ 0,(*Azimuthal*)
            ResultTable[Mφ,φ☆,P,ρ′,ρ,φ′,φ,z′,z,"Mφ",BφAna,BφN
        ];
        If[Mz ≠ 0,(*Azimuthal*)
            ResultTable[Mz,φ☆,P,ρ′,ρ,φ′,φ,z′,z,"Mz",BzAna,BzN
        ];
        Null
    ]

CoilFilamentField[I_,ρ′_,ρ_,φ′_,φ_,z′_,z_]:= ResultTable[I,0,(
CoilDiscField[K_,P_,ρ′_,ρ_,φ′_,φ_,z′_,z_] := ResultTable[K,0,
CoilShellField[K_,ρ′_,ρ_,φ′_,φ_,z′_,z_]   := ResultTable[K,0,(
CoilCylField [J_,P_,ρ′_,ρ_,φ′_,φ_,z′_,z_] := ResultTable[J,0,

ResultTable[M_,φ☆_,P_,ρ′_,ρ_,φ′_,φ_,z′_,z_,mag_,anaFn_,numFn_
```

```
    Module[{Bana,Bnum,Bcom,heading,tab},
        If[ρ==0,
            Bana = N[anaFnAxis[M,φ☆,ρ',φ',z',z],$MachinePreci
            Bnum =   numFnAxis[M,φ☆,ρ',φ',z',z];
            Bcom = Chop[Bana-Bnum,10^-8];
            heading = {"Bx","By","Bz"} (*Axis, Cartesian*),
            Bana = N[anaFn[M,φ☆,P,ρ',ρ,φ',φ,z',z],$MachinePre
            Bnum =   numFn[M,φ☆,ρ',ρ,φ',φ,z',z];
            Bcom = Chop[Bana-Bnum,10^-8];
            heading = {"Bρ","Bφ","Bz"} (*Cylindrical*);
        ];
        tab = TableForm[{Bana,Bnum,Bcom}, TableHeadings -> {{"
        CellPrint@ExpressionCell[RawBoxes[ToBoxes[tab] /. (x :
    ]

ResultTableForce[M_,Mp_,ρ'_,ρ_,φ'_,φ_,z'_,z_,V_,P_,U_,O_,anaF
    Module[{Fana,Fnum,Fcom,heading,tab},
            Fana = N[anaFn[M,Mp,ρ',ρ,φ',φ,z',z,V,P,U,O],$Mach
            Fnum =   numFn[M,Mp,ρ',ρ,φ',φ,z',z];
            Fcom = Chop[Fana-Fnum,10^-8];
            heading = {"Fx","Fy","Fz"} ;
        tab = TableForm[{Fana,Fnum,Fcom}, TableHeadings -> {{"
        CellPrint@ExpressionCell[tab, "Output"]
    ]
```

0.6 Example field solutions

Evaluate the same magnet geometry, at the same field point, with equal magnetisation in the diametric, radial, azimuthal, and axial directions. Creates a table for each magnetisation direction and compares to 8 decimal places the analytic solution and numeric integral.

```
In[•]:= MagCylField[M,M,M,M,φ☆,P,ρ′,ρ1,φ′,φ1,z′,z1] (*Note: This expr
```

M⊥	Bρ	Bφ
Analytic	0.3588672233516815	0.01541822529046387
Numeric	0.3588672228056431	0.01541822528724981
Comparison 8dp	0	0

Mρ	Bρ	Bφ
Analytic	0.2448856704190372	-0.000306026032949424З
Numeric	0.2448856701063103	-0.00030602603308330
Comparison 8dp	0	0

Mφ	Bρ	Bφ
Analytic	-0.000986646361453491З	-0.027353091874110
Numeric	-0.00098664636141892	-0.027353091873834
Comparison 8dp	0	0

Mz	Bρ	Bφ
Analytic	0.01242509986271859	-0.0000139329893382662
Numeric	0.01242509986169668	-0.0000139329893381925
Comparison 8dp	0	0

0.7 Timed solutions

Altered tables to those in Section 0.5, including RepeatedTiming[] and an additional table to show computational efficiency.

In[ ]:=
```
MagCylFieldTimed[Md_,Mρ_,Mφ_,Mz_,φ☆_,P_,ρ'_,ρ_,φ'_,φ_,z'_,z_
    Module[{},
        If[Md ≠ 0,(*Diametric*)
            ResultTableTimed[Md,φ☆,P,ρ',ρ,φ',φ,z',z,"M⊥",BdAı
        ];
        If[Mρ ≠ 0,(*Radial*)
            ResultTableTimed[Mρ,φ☆,P,ρ',ρ,φ',φ,z',z,"Mρ",BρAı
        ];
        If[Mφ ≠ 0,(*Azimuthal*)
            ResultTableTimed[Mφ,φ☆,P,ρ',ρ,φ',φ,z',z,"Mφ",BφAı
        ];
        If[Mz ≠ 0,(*Azimuthal*)
            ResultTableTimed[Mz,φ☆,P,ρ',ρ,φ',φ,z',z,"Mz",BzAn
        ];
        Null
    ]
ResultTableTimed[M_,φ☆_,P_,ρ'_,ρ_,φ'_,φ_,z'_,z_,mag_,anaFn_,ı
    Module[{Bana,Bnum,Bcom,heading,tab,ta,ba,tn,bn,time},
        If[ρ==0,
            {ta,ba} = RepeatedTiming[
                Bana = N[anaFnAxis[M,φ☆,ρ',φ',z',z],$MachineP
            {tn,bn} = RepeatedTiming[
                Bnum =   numFnAxis[M,φ☆,ρ',φ',z',z];];
            Bcom = Chop[Bana-Bnum,10^-8];
            heading = {"Bx","By","Bz"} (*Axis, Cartesian*),
            {ta,ba} = RepeatedTiming[
                Bana = N[anaFn[M,φ☆,P,ρ',ρ,φ',φ,z',z],$Machir
            {tn,bn} = RepeatedTiming[
                Bnum =   numFn[M,φ☆,ρ',ρ,φ',φ,z',z];];
            Bcom = Chop[Bana-Bnum,10^-8];
            heading = {"Bρ","Bφ","Bz"} (*Cylindrical*);
        ];
        time = {ta,tn};
        tab = TableForm[{Bana,Bnum,Bcom}, TableHeadings -> {{"
        CellPrint@ExpressionCell[RawBoxes[ToBoxes[tab] /. (x :

    ]
```

```
In[•]:= MagCylFieldTimed[M,M,M,M,φ☆,P,ρ′,ρ1,φ′,φ1,z′,z1] (*Note: This
```

M ⊥	Bρ	Bφ
Analytic	0.3588672233516815	0.01541822529046387
Numeric	0.3588672228056431	0.01541822528724981
Comparison 8dp	0	0

Mρ	Bρ	Bφ
Analytic	0.2448856704190372	-0.0003060260329494243
Numeric	0.2448856701063103	-0.000306026033083330
Comparison 8dp	0	0

Mφ	Bρ	Bφ
Analytic	-0.0009866463614534913	-0.027353091874116
Numeric	-0.00098664636141892	-0.027353091873834
Comparison 8dp	0	0

Mz	Bρ	Bφ
Analytic	0.01242509986271859	-0.000013932989338266
Numeric	0.01242509986169668	-0.000013932989338192
Comparison 8dp	0	0

1.0 Permanent Magnets

1.1 Diametric Magnetisation

1.1.0 Equations

Analytic and Numeric function handles. Returns B={Bρ,Bφ,Bz} or B={Bx,By,Bz} (on axis).

In[∘]:=

```
BdAna[M_,φ☆_,P_,ρp_,ρ_,φp_,φ_,zp_,z_] := M u0/4π (∑_{m=1}^{2} ∑_{n=1}^{2} ∑_{q=1}^{2} (-1)^(m+n+

BdAnaAxis[M_,φ☆_,ρp_,φp_,zp_,z_] := M u0/4π (∑_{m=1}^{2} ∑_{n=1}^{2} ∑_{q=1}^{2} (-1)^(m+n+q) BdS

BdNum[M_,φ☆_,ρp_,ρ_,φp_,φ_,zp_,z_] := M u0/4π (∑_{m=1}^{2} (-1)^m NIntegrat
                                                  + ∑_{q=1}^{2} (-1)^q NIntegrat
                                                  + Md[φ☆,ρp,ρ,φp,φ,

BdNumAxis[M_,φ☆_,ρp_,φp_,zp_,z_] := M u0/4π (∑_{m=1}^{2} (-1)^m NIntegrate[
                                                  + ∑_{q=1}^{2} (-1)^q NInt
                                                  + MdAxis[φ☆,ρ
```

Magnetisation vector for field inside magnet. Returns M={Mρ,Mφ,Mz}.

In[∘]:=

```
Md[φ☆_,ρp_,ρ_,φp_,φ_,zp_,z_] := If[InsideVolume[ρp,ρ,φp,φ,zp

MdAxis[φ☆_,ρp_,zp_,z_] := If[InsideVolumeAxis[ρp,zp,z],4π{Cos
```

Special cases of the geometry for the analytic function handle. Replaces BdAna[] (Cylindrical) or BdAnaAxis[] (Cartesian, on axis).

```
In[•]:= BdAna[M_,φ☆_,P_,ρp_,ρ_,{0,2π},φ_,zp_,z_] := M u0/4π (∑_{m=1}^{2} ∑_{n=1}^{2} (-1)^{m+r

BdAnaAxis[M_,φ☆_,ρp_,{0,2π},zp_,z_] := M u0/4π (∑_{m=1}^{2} ∑_{n=1}^{2} (-1)^{m+n} BdSu

BdAna[M_,φ☆_,P_,{0,ρp_},ρ_,φp_,φ_,zp_,z_] := M u0/4π (∑_{n=1}^{2} ∑_{q=1}^{2} (-1)^{n

BdAna[M_,φ☆_,P_,{0,ρp_},ρ_,{0,2π},φ_,zp_,z_] := M u0/4π (∑_{n=1}^{2} (-1)
```

Integrands to be solved for BdNum[] (Cylindrical) and BdNumAxis[] (Cartesian, on axis).

```
In[•]:= BdIntegrand1[φ☆_,ρp_,ρ_,φp_,φ_,zp_,z_] := {BdIρ1[φ☆,ρp,ρ,φp,
BdIntegrand2[φ☆_,ρp_,ρ_,φp_,φ_,zp_,z_] := {BdIρ2[φ☆,ρp,ρ,φp,

BdIρ1[φ☆_,ρp_,ρ_,φp_,φ_,zp_,z_] := ρp Cos[φ☆-φp] (ρ-ρp Cos[
BdIρ2[φ☆_,ρp_,ρ_,φp_,φ_,zp_,z_] := Sin[φ☆-φp] (ρ-ρp Cos[Φ[φ
BdIφ1[φ☆_,ρp_,ρ_,φp_,φ_,zp_,z_] := ρp^2 Cos[φ☆-φp] Sin[Φ[φ,φ
BdIφ2[φ☆_,ρp_,ρ_,φp_,φ_,zp_,z_] := ρp Sin[φ☆-φp] Sin[Φ[φ,φp
BdIz1[φ☆_,ρp_,ρ_,φp_,φ_,zp_,z_] := ρp Cos[φ☆-φp] Z[z,zp] G
BdIz2[φ☆_,ρp_,ρ_,φp_,φ_,zp_,z_] := Sin[φ☆-φp] Z[z,zp] G[ρ,ρ

BdIntegrandAxis1[φ☆_,ρp_,φp_,zp_,z_] := {BdIAxisx1[φ☆,ρp,φp,
BdIntegrandAxis2[φ☆_,ρp_,φp_,zp_,z_] := {BdIAxisx2[φ☆,ρp,φp,

BdIAxisx1[φ☆_,ρp_,φp_,zp_,z_] := -ρp^2 Cos[φ☆-φp] Cos[φp] (Z[z
BdIAxisx2[φ☆_,ρp_,φp_,zp_,z_] := -ρp Sin[φ☆-φp] Cos[φp] (Z[z
BdIAxisy1[φ☆_,ρp_,φp_,zp_,z_] := -ρp^2 Cos[φ☆-φp] Sin[φp] (Z[z
BdIAxisy2[φ☆_,ρp_,φp_,zp_,z_] := -ρp Sin[φ☆-φp] Sin[φp] (Z[z,
BdIAxisz1[φ☆_,ρp_,φp_,zp_,z_] := ρp Z[z,zp] Cos[φ☆-φp] (Z[z,z
BdIAxisz2[φ☆_,ρp_,φp_,zp_,z_] := Z[z,zp] Sin[φ☆-φp] (Z[z,zp]^2
```

Summands for BdAna[] and BdAnaAxis[].

```
In[•]:= BdSummand[φ☆_,ρp_,ρ_,φp_,φ_,zp_,z_] := {BdSρ1[φ☆,ρp,ρ,φp,φ,
BdSummand1[φ☆_,ρp_,ρ_,φp_,φ_,zp_,z_] := {BdSρ1[φ☆,ρp,ρ,φp,φ
BdSummand2[φ☆_,ρp_,ρ_,φp_,φ_,zp_,z_] := {BdSρ2[φ☆,ρp,ρ,φp,φ
```

$$\text{BdS}\rho1[\varphi☆\_,\rho p\_,\rho\_,\varphi p\_,\varphi\_,zp\_,z\_] := \frac{1}{\rho}\frac{\rho p\ Z[z,zp]}{R[\rho,\rho p,z,zp]}\text{Cos}[\varphi☆-\varphi]$$

$$\text{BdS}\rho2[\varphi☆\_,\rho p\_,\rho\_,\varphi p\_,\varphi\_,zp\_,z\_] := \text{Which}\Big[(\varphi p == \varphi+\pi)\ ||\ (\varphi p == \varphi-$$

$$\text{BdS}\varphi1[\varphi☆\_,\rho p\_,\rho\_,\varphi p\_,\varphi\_,zp\_,z\_] := \frac{2}{\rho}\ \rho p\ Z[z,zp]\frac{\text{Sin}[\varphi☆-\varphi]}{R[\rho,\rho p,z,zp]}$$

$$\text{BdS}\varphi2[\varphi☆\_,\rho p\_,\rho\_,\varphi p\_,\varphi\_,zp\_,z\_] := \text{Which}\Big[(\varphi p == \varphi+\pi)\ ||\ (\varphi p == \varphi-$$

$$\text{BdSz1}[\varphi☆\_,\rho p\_,\rho\_,\varphi p\_,\varphi\_,zp\_,z\_] := \frac{2\ \rho p\ \text{Cos}[\varphi☆-\varphi]}{R[\rho,\rho p,z,zp]}\Big(\text{EllipticF}$$

$$\text{BdSz2}[\varphi☆\_,\rho p\_,\rho\_,\varphi p\_,\varphi\_,zp\_,z\_] := \text{Log}\big[\rho p-\rho\ \text{Cos}[\Phi[\varphi,\varphi p]]+\text{G}[$$

$$\text{BdSummandAxis}[\varphi☆\_,\rho p\_,\varphi p\_,zp\_,z\_] := \{\text{BdSAxisx}[\varphi☆,\rho p,\varphi p,zp,z$$

$$\text{BdSAxisx}[\varphi☆\_,\rho p\_,\varphi p\_,zp\_,z\_] := \frac{Z[z,zp]\ \ (2\varphi p\ \text{Cos}[\varphi☆]-\text{Sin}[\varphi}{4\ \sqrt{Z[z,zp]^2+\rho p^2}}$$

$$\text{BdSAxisy}[\varphi☆\_,\rho p\_,\varphi p\_,zp\_,z\_] := \frac{Z[z,zp]\ \ (-\text{Cos}[\varphi☆-2\ \varphi p]+2\ \varphi p}{4\ \sqrt{Z[z,zp]^2+\rho p^2}}$$

$$\text{BdSAxisz}[\varphi☆\_,\rho p\_,\varphi p\_,zp\_,z\_] := \Bigg(-\frac{\rho p}{\sqrt{Z[z,zp]^2+\rho p^2}}+\text{ArcTanh}\Big[\frac{}{\sqrt{}}$$

$$\text{BdSummandAS}[\varphi☆\_,\rho p\_,\rho\_,\varphi\_,zp\_,z\_] := \{\text{BdSAS}\rho[\varphi☆,\rho p,\rho,\varphi,zp,z]$$

$$\text{BdSAS}\rho[\varphi☆\_,\rho p\_,\rho\_,\varphi\_,zp\_,z\_] := 2\text{Cos}[\varphi☆-\varphi]\frac{\rho p\ Z[z,zp]}{\rho\ R[\rho,\rho p,z,zp]}\Bigg($$

$$\text{BdSAS}\varphi[\varphi☆\_,\rho p\_,\rho\_,\varphi\_,zp\_,z\_] := 4\ \text{Sin}[\varphi☆-\varphi]\frac{\rho p\ Z[z,zp]}{\rho\ R[\rho,\rho p,z,zp]}$$

$$\text{BdSASz}[\varphi☆\_,\rho p\_,\rho\_,\varphi\_,zp\_,z\_] := \frac{4\ \rho p\ \text{Cos}[\varphi☆-\varphi]}{R[\rho,\rho p,z,zp]}\Big(2\text{EllipticD}\big[k$$

Singularities in the summands of BdAna[] or BdAnaAxis[].

In[∘]:=
```
(*Along the axis*)
```

```
BdSAxisx[φ☆_,ρp_,φp_,zp_,zp_]:=0
BdSAxisy[φ☆_,ρp_,φp_,zp_,zp_]:=0
BdSAxisz[φ☆_,ρp_,φp_,zp_,zp_]:=(Log[ρp]-1) Sin[φ☆-φp]
(*Along the axis & axisymmetric*)
BdSAxisx[φ☆_,ρp_,{0,2π},zp_,z_] := π Z[z,zp] Cos[φ☆] / Sqrt[Z[z,zp]^2+ρp^2]
BdSAxisy[φ☆_,ρp_,{0,2π},zp_,z_] := π Z[z,zp] Sin[φ☆] / Sqrt[Z[z,zp]^2+ρp^2]
BdSAxisz[φ☆_,ρp_,{0,2π},zp_,z_] := 0
BdSAxisx[φ☆_,ρp_,{0,2π},zp_,zp_] := 0
BdSAxisy[φ☆_,ρp_,{0,2π},zp_,zp_] := 0
BdSAxisz[φ☆_,ρp_,{0,2π},zp_,zp_] := 0
(*On the shell plane*)
EllipticPiT[1,ϕ_,k_] := EllipticFT[ϕ,k]-1/(1-k) (EllipticET[ϕ,
(*On the shell plane & axisymmetric*)
BdSASρ[φ☆_,ρp_,ρp_,φ_,zp_,z_] := 2 (Z[z,zp] Cos[φ☆-φ]) / R[ρp,ρp,z,zp] (Elli
BdSASφ[φ☆_,ρp_,ρp_,φ_,zp_,z_] := 4 Z[z,zp] Sin[φ☆-φ] / R[ρp,ρp,z,zp] Ellip
(*On the section plane*)
BdSρ2[φ☆_,ρp_,ρ_,φp_,φp_,zp_,z_] := -ArcTanh[R̄[ρ,ρp,z,zp] / Z[z,zp]] S
BdSφ2[φ☆_,ρp_,ρ_,φp_,φp_,zp_,z_] := 0
(*On the disc plane*)
BdSρ1[φ☆_,ρp_,ρ_,φp_,φ_,zp_,zp_] := 0
BdSρ2[φ☆_,ρp_,ρ_,φp_,φ_,zp_,zp_] := 0
BdSφ1[φ☆_,ρp_,ρ_,φp_,φ_,zp_,zp_] := 0
BdSφ2[φ☆_,ρp_,ρ_,φp_,φ_,zp_,zp_] := 0
(*On the axial line*)
BdSρ1[φ☆_,ρp_,ρp_,φp_,φp_,zp_,z_] := Cos[φ☆-φp] Z[z,zp] / R[ρp,ρp,z,z
BdSρ2[φ☆_,ρp_,ρp_,φp_,φp_,zp_,z_] := -Log[Abs[Z[z,zp]]]Sign[
BdSφ1[φ☆_,ρp_,ρp_,φp_,φp_,zp_,z_] := 2 Sin[φ☆-φp] Z[z,z / R[ρp,ρp,z
BdSφ2[φ☆_,ρp_,ρp_,φp_,φp_,zp_,z_] := 0
(*On the azimuthal line*)
BdSρ1[φ☆_,ρp_,ρp_,φp_,φ_,zp_,zp_] := 0
```

```
BdSρ2[φ☆_,ρp_,ρp_,φp_,φ_,zp_,zp_] := 0
BdSφ1[φ☆_,ρp_,ρp_,φp_,φ_,zp_,zp_] := 0
BdSφ2[φ☆_,ρp_,ρp_,φp_,φ_,zp_,zp_] := 0
BdSz1[φ☆_,ρp_,ρp_,φp_,φ_,zp_,zp_] := Cos[φ☆-φ] (ArcTanh[Sin
(*On the radial line*)
BdSρ1[φ☆_,ρp_,ρ_,φp_,φp_,zp_,zp_] := 0
BdSρ2[φ☆_,ρp_,ρ_,φp_,φp_,zp_,zp_] := 0
BdSφ1[φ☆_,ρp_,ρ_,φp_,φp_,zp_,zp_] := 0
BdSφ2[φ☆_,ρp_,ρ_,φp_,φp_,zp_,zp_] := 0
BdSz2[φ☆_,ρp_,ρ_,φp_,φp_,zp_,zp_] := -Sign[ϱ̅[ρ,ρp]]Log[Abs[ϱ̅
```

1.1.1 Standard - Outside Magnet

In[•]:=
```
MagCylField[M,0,0,0,φ☆,0,ρ′,ρ1,φ′,φ1,z′,z1]
```

M ⊥	Bρ	Bφ
Analytic	0.3588672233516815	0.01541822529046387
Numeric	0.3588672228056431	0.01541822528724981
Comparison 8dp	0	0

1.1.2 Special Case a. - Inside Magnet

In[•]:=
```
MagCylField[M,0,0,0,φ☆,0,ρ′,ρ2,φ′,φ2,z′,z2]
```

M ⊥	Bρ	Bφ
Analytic	0.6966643256463426	−0.1368270731830296
Numeric	0.6966643258557762	−0.13682707315087393
Comparison 8dp	0	0

1.1.3 Special Case b. - On Magnet Axis

In[•]:=
```
MagCylField[M,0,0,0,φ☆,0,ρ′,ρ3,φ′,φ3,z′,z3]
```

M ⊥	Bx	By
Analytic	0.06907925427658275	0.05798271989375209
Numeric	0.0690792542700186	0.0579827199700861
Comparison 8dp	0	0

1.1.4 Special Case c. - Axisymmetric

In[∘]:=
```
MagCylField[M,0,0,0,φ☆,0,ρ′,ρ1,{0,2π},φ1,z′,z1]
```

M ⊥	Bρ	Bφ
Analytic	0.3898898941670998	0.01766718672333797
Numeric	0.3898898941265560	0.01766718672565229
Comparison 8dp	0	0

1.1.5 Special Case d. - Solid

In[∘]:=
```
MagCylField[M,0,0,0,φ☆,0,{0,ρ′[[2]]},ρ1,φ′,φ1,z′,z1]
```

M ⊥	Bρ	Bφ
Analytic	0.3768042165296648	0.01650501972700407
Numeric	0.3768042159820810	0.01650501972378489
Comparison 8dp	0	0

1.1.6 Special Case e. - Axisymmetric & Solid

In[∘]:=
```
MagCylField[M,0,0,0,φ☆,0,{0,ρ′[[2]]},ρ1,{0,2π},φ1,z′,z1]
```

M ⊥	Bρ	Bφ
Analytic	0.4201134535501512	0.01963054065204408
Numeric	0.4201134535186059	0.01963054065431834
Comparison 8dp	0	0

1.1.7 Singularities b,c,f. - Singular plane 1

In[∘]:=
```
MagCylField[M,0,0,0,φ☆,0,ρ′,ρ4,φ′,φ4,z′,z4]
```

M ⊥	Bρ	Bφ
Analytic	−0.05283942296707968	−0.10630808491530856
Numeric	−0.0528394229808127	−0.1063080843063326
Comparison 8dp	0	0

1.1.8 Singularities a,c,e. - Singular plane 2

In[•]:=
```
MagCylField[M,0,0,0,φ☆,0,ρ',ρ5,φ',φ5,z',z5]
```

M ⊥	Bρ	Bφ
Analytic	−0.12706206084430823	−0.01442552150406100
Numeric	−0.1270620610434591	−0.0144255205837600
Comparison 8dp	0	0

1.1.9 Singularities a,b,d. - Singular plane 3

In[•]:=
```
MagCylField[M,0,0,0,φ☆,0,ρ',ρ6,φ',φ6,z',z6]
```

M ⊥	Bρ	Bφ
Analytic	−0.08575035760538238	−0.08687164959114616
Numeric	−0.0857503575964678	−0.0868716487826758
Comparison 8dp	0	0

1.1.10 (not in article) - On Magnet Axis & Axisymmetric

NIntegrate struggles with Bz.

In[•]:=
```
MagCylField[M,0,0,0,φ☆,0,ρ',ρ3,{0,2π},φ3,z',z3]// Quiet
```

M ⊥	Bx	By
Analytic	0.0916633480745974	0.05292185868569116
Numeric	0.0916633480754929	0.0529218586946682
Comparison 8dp	0	0

1.1.11 (not in article) - Axisymmetric & Singular plane 3

In[•]:=
```
MagCylField[M,0,0,0,φ☆,0,ρ',ρ6,{0,2π},φ6,z',z6]// Quiet
```

M ⊥	Bρ	Bφ
Analytic	−0.006033024943461395	−0.0860251960360698
Numeric	−0.00603302493568282	−0.0860251960250795
Comparison 8dp	0	0

1.2 Radial Magnetisation

1.2.0 Equations

Analytic and Numeric function handles. Returns B={Bρ,Bφ,Bz} or B={Bx,By,Bz} (on axis).

```
In[•]:= BρAna[M_,φ☆_,P_,ρp_,ρ_,φp_,φ_,zp_,z_] := M u0/(4π) Sum[Sum[Sum[(-1)^(m+n+q)
        BρAnaAxis[M_,φ☆_,ρp_,φp_,zp_,z_] := M u0/(4π) Sum[Sum[Sum[(-1)^(m+n+q) BρSu
        BρNum[M_,φ☆_,ρp_,ρ_,φp_,φ_,zp_,z_] := M u0/(4π) (Sum[(-1)^n NIntegrat
                                              + Sum[(-1)^q NInt
        BρNumAxis[M_,φ☆_,ρp_,φp_,zp_,z_] := M u0/(4π) (Sum[(-1)^n NIntegrate[
                                              + Sum[(-1)^q NInt
```

(sums: m=1 to 2, n=1 to 2, q=1 to 2)

Special cases of the geometry for the analytic function handle. Replaces BρAna[] (Cylindrical) or BρAnaAxis[] (Cartesian, on axis).

```
In[•]:= BρAna[M_,φ☆_,P_,ρp_,ρ_,{0,2π},φ_,zp_,z_] := M u0/(4π) Sum[Sum[(-1)^(m+n)
        BρAnaAxis[M_,φ☆_,ρp_,{0,2π},zp_,z_] := M u0/(4π) Sum[Sum[(-1)^(m+n) BρSum
```

(sums: m=1 to 2, n=1 to 2)

Integrands to be solved for BρNum[] (Cylindrical) and BρNumAxis[] (Cartesian, on axis).

In[•]:=
```
BρIntegrand1[ρp_,ρ_,φp_,φ_,zp_,z_] := {BρIρ1[ρp,ρ,φp,φ,zp,z],
BρIntegrand2[ρp_,ρ_,φp_,φ_,zp_,z_] := {BρIρ2[ρp,ρ,φp,φ,zp,z],

BρIρ1[ρp_,ρ_,φp_,φ_,zp_,z_] := -Z[z,zp]  ρp Cos[Φ[φ,φp]] G[ρ
BρIρ2[ρp_,ρ_,φp_,φ_,zp_,z_] := -ρp Sin[Φ[φ,φp]] G[ρ,ρp,φ,φp,
BρIφ1[ρp_,ρ_,φp_,φ_,zp_,z_] := Z[z,zp] ρp Sin[Φ[φ,φp]] G[ρ,ρ
BρIφ2[ρp_,ρ_,φp_,φ_,zp_,z_] := (ρ-ρp Cos[Φ[φ,φp]]) G[ρ,ρp,φ,
BρIz [ρp_,ρ_,φp_,φ_,zp_,z_] := ρp (-ρp+ρ Cos[Φ[φ,φp]]) G[ρ,ρ

BρIntegrandAxis1[ρp_,φp_,zp_,z_] := {BρIAxisx1[ρp,φp,zp,z], Bρ
BρIntegrandAxis2[ρp_,φp_,zp_,z_] := {BρIAxisx2[ρp,φp,zp,z], Bρ

BρIAxisx1[ρp_,φp_,zp_,z_] := -Z[z,zp] ρp Cos[φp] (Z[z,zp]^2+ρp^2
BρIAxisx2[ρp_,φp_,zp_,z_] := ρp Sin[φp] (Z[z,zp]^2+ρp^2)^(-3/2)
BρIAxisy1[ρp_,φp_,zp_,z_] := -Z[z,zp] ρp Sin[φp] (Z[z,zp]^2+ρp^2
BρIAxisy2[ρp_,φp_,zp_,z_] := -ρp Cos[φp] (Z[z,zp]^2+ρp^2)^(-3/2)
BρIAxisz [ρp_,φp_,zp_,z_] := -ρp^2 (Z[z,zp]^2+ρp^2)^(-3/2)
```

Summands for BρAna[] and BρAnaAxis[].

In[•]:=
```
BρSummand[P_,ρp_,ρ_,φp_,φ_,zp_,z_] := {BρSρ1[ρp,ρ,φp,φ,zp,z]

BρSρ1[ρp_,ρ_,φp_,φ_,zp_,z_] := -2/ρ (ρp Z[z,zp])/(R[ρ,ρp,z,zp]) (EllipticFT[ϕ

BρSρ2[ρp_,ρ_,φp_,φ_,zp_,z_] := Which[(φp==φ+π) || (φp==φ-π),0,

BρSφ1[ρp_,ρ_,φp_,φ_,zp_,z_] := Z[z,zp]/ρ Log[ρp-ρ Cos[Φ[φ,φp]]

BρSφ2[ρp_,ρ_,φp_,φ_,zp_,z_] := Which[(φp==φ+π) || (φp==φ-π),Ar

BρSz[P_,ρp_,ρ_,φp_,φ_,zp_,z_] := -((2 ρp)/(R[ρ,ρp,z,zp]) EllipticFT[ϕ[

BρSummandAxis[ρp_,φp_,zp_,z_] := {BρSAxisx[ρp,φp,zp,z], BρSAx

BρSAxisx[ρp_,φp_,zp_,z_] :=  Sin[φp] (Z[z,zp]/Sqrt[Z[z,zp] ^2+ρp^2]+ArcTanh

BρSAxisy[ρp_,φp_,zp_,z_] := -Cos[φp] (Z[z,zp]/Sqrt[Z[z,zp]^2+ρp^2]+ArcTanh[

BρSAxisz[ρp_,φp_,zp_,z_] := φp (ρp/Sqrt[Z[z,zp]^2+ρp^2]- ArcTanh[ ———/Sqrt[Z[z

BρSummandAS[P_,ρp_,ρ_,φ_,zp_,z_] := {BρSASρ[ρp,ρ,φ,zp,z], BρS

BρSASρ[ρp_,ρ_,φ_,zp_,z_] := 4/ρ (ρp Z[z,zp])/(R[ρ,ρp,z,zp]) (EllipticK[k[ρ,ρp

BρSASφ[ρp_,ρ_,φ_,zp_,z_] := 0

BρSASz[P_,ρp_,ρ_,φ_,zp_,z_] := ((4ρp)/(R[ρ,ρp,z,zp]) EllipticK[k[ρ,ρp
```

Singularities in the summands of BρAna[] or BρAnaAxis[].

In[•]:=
```
(*Along the axis*)
```

```
BρSAxisx[ρp_,φp_,zp_,zp_] := 0
BρSAxisy[ρp_,φp_,zp_,zp_] := 0
BρSAxisz[ρp_,φp_,zp_,zp_] := -φp Log[ρp]
(*Along the axis & axisymmetric*)
BρSAxisx[ρp_,{0,2π},zp_,z_] := 0
BρSAxisy[ρp_,{0,2π},zp_,z_] := 0
BρSAxisz[ρp_,{0,2π},zp_,z_] := BρSAxisz[ρp,2π,zp,z]
BρSAxisx[ρp_,{0,2π},zp_,zp_] := 0
BρSAxisy[ρp_,{0,2π},zp_,zp_] := 0
BρSAxisz[ρp_,{0,2π},zp_,zp_] := BρSAxisz[ρp,2π,zp,zp]
(*Solid*)
BρSρ1[0,ρ_,φp_,φ_,zp_,z_] := -(L[ρ,z,zp]/ρ) ArcTan[(ρ Sin[Φ[φ,φp]
                                                      Z[z,zp]
BρSρ1[0,ρ_,φp_,φ_,zp_,zp_] := 0
(*Solid & axisymmetric*)
BρSASρ[0,ρ_,φ_,zp_,z_] := 0
(*On the shell plane*)
EllipticPiT[1,ϕ_,k_] := EllipticFT[ϕ,k]-1/(1-k) (EllipticET[ϕ,
(*On the shell plane & axisymmetric*)
BρSASρ[ρp_,ρ_,φ_,zp_,zp_] := 0
(*On the section plane*)
BρSρ2[ρp_,ρ_,φp_,φp_,zp_,z_] := 0
BρSφ2[ρp_,ρ_,φp_,φp_,zp_,z_] := -ArcTanh[R̄[ρ,ρp,z,zp]/Z[z,zp]]
(*On the disc plane*)
BρSρ1[ρp_,ρ_,φp_,φ_,zp_,zp_] := 0
BρSρ2[ρp_,ρ_,φp_,φ_,zp_,zp_] := 0
BρSφ1[ρp_,ρ_,φp_,φ_,zp_,zp_] := 0
BρSφ2[ρp_,ρ_,φp_,φ_,zp_,zp_] := 0
(*On the axial line*)
BρSρ2[ρp_,ρp_,φp_,φp_,zp_,z_] := 0
BρSφ2[ρp_,ρp_,φp_,φp_,zp_,z_] := -Log[Abs[Z[z,zp]]] Sign[Z[z,
(*On the azimuthal line*)
BρSρ1[ρp_,ρp_,φp_,φ_,zp_,zp_] := 0
BρSρ2[ρp_,ρp_,φp_,φ_,zp_,zp_] := 0
BρSφ1[ρp_,ρp_,φp_,φ_,zp_,zp_] := 0
BρSφ2[ρp_,ρp_,φp_,φ_,zp_,zp_] := 0
EllipticFT[ϕ_,1]:=Sin[ϕ]CarlsonRC[1,Cos[ϕ]^2]
(*On the radial line*)
```

```
BρSρ1[ρp_,ρ_,φp_,φp_,zp_,zp_] := 0
BρSρ2[ρp_,ρ_,φp_,φp_,zp_,zp_] := 0
BρSφ1[ρp_,ρ_,φp_,φp_,zp_,zp_] := 0
BρSφ2[ρp_,ρ_,φp_,φp_,zp_,zp_] := 0
```

1.2.1 Standard - Outside Magnet

In[∘]:=
```
MagCylField[0,M,0,0,0,150,ρ′,ρ1,φ′,φ1,z′,z1]
```

Mρ	Bρ	Bφ
Analytic	0.2448856704190372	−0.0003060260329494243
Numeric	0.2448856701063103	−0.00030602603308330
Comparison 8dp	0	0

1.2.2 Special Case a. - Inside Magnet

In[∘]:=
```
MagCylField[0,M,0,0,0,200,ρ′,ρ2,φ′,φ2,z′,z2]
```

Mρ	Bρ	Bφ
Analytic	0.5489154592264350	−0.0000613024455878349
Numeric	0.5489154591261374	−0.00006130244558542
Comparison 8dp	0	0

1.2.3 Special Case b. - On Magnet Axis

In[∘]:=
```
MagCylField[0,M,0,0,0,0,ρ′,ρ3,φ′,φ3,z′,z3]
```

Mρ	Bx	By
Analytic	0.1344445158028215	0.10887102224775870
Numeric	0.1344445157685432	0.1088710222296142
Comparison 8dp	0	0

1.2.4 Special Case c. - Axisymmetric

NIntegrate struggles with Bφ.

In[•]:=
```
MagCylField[0,M,0,0,0,150,ρ',ρ1,{0,2π},φ1,z',z1] //Quiet
```

Mρ	Bρ	Bφ	Bz
Analytic	0.2112280841430892	0	0.00930958€
Numeric	0.2112280841851579	$0.\times10^{-18}$	0.009309577
Comparison 8dp	0	0	0

1.2.5 Special Case d. - Solid

In[•]:=
```
MagCylField[0,M,0,0,0,150,{0,ρ'[[2]]},ρ1,φ',φ1,z',z1]
```

Mρ	Bρ	Bφ
Analytic	0.2572853367868567	-0.0004488894971618596
Numeric	0.2572853369280047	-0.00044888949695013
Comparison 8dp	0	0

1.2.6 Special Case e. - Axisymmetric & Solid

NIntegrate struggles with Bφ.

In[•]:=
```
MagCylField[0,M,0,0,0,150,{0,ρ'[[2]]},ρ1,{0,2π},φ1,z',z1]//Qu
```

Mρ	Bρ	Bφ	Bz
Analytic	0.2161116638799649	0	0.00946993€
Numeric	0.2161116637943994	$0.\times10^{-17}$	0.00946992€
Comparison 8dp	0	0	0

1.2.7 Singularities b,c,f. - Singular plane 1

In[•]:=
```
MagCylField[0,M,0,0,0,600,ρ',ρ4,φ',φ4,z',z4]
```

Mρ	Bρ	Bφ
Analytic	0.04575438476808203	-0.1054198612099800
Numeric	0.04575438476471416	-0.1054198612281754
Comparison 8dp	0	0

1.2.8 Singularities a,c,e. - Singular plane 2

In[ ]:=
```
MagCylField[0,M,0,0,0,500,ρ',ρ5,φ',φ5,z',z5]
```

Mρ	Bρ	Bφ
Analytic	-0.06364161691744196	-0.09932523777628700
Numeric	-0.0636416167461836	-0.0993252378337986
Comparison 8dp	0	0

1.2.9 Singularities a,b,d. - Singular plane 3

In[ ]:=
```
MagCylField[0,M,0,0,0,650,ρ',ρ6,φ',φ6,z',z6]
```

Mρ	Bρ	Bφ
Analytic	-0.06633792348273685	-0.07011471596517585
Numeric	-0.0663379233926231	-0.0701147160567020
Comparison 8dp	0	0

1.2.10 (not in article) - On Magnet Axis & Axisymmetric

In[ ]:=
```
MagCylField[0,M,0,0,0,0,ρ',ρ3,{0,2π},φ3,z',z3]
```

Mρ	Bx	By	Bz
Analytic	0	0	-0.31
Numeric	$1.169847043839314\times10^{-21}$	$0.\times10^{-17}$	-0.31
Comparison 8dp	0	0	0

1.2.11 (not in article) - Axisymmetric & Singular plane 1

In[ ]:=
```
MagCylField[0,M,0,0,0,650,ρ',ρ4,{0,2π},φ4,z',z4]//Quiet
```

Mρ	Bρ	Bφ	Bz
Analytic	0.08202777975183896	0	0.16752556
Numeric	0.0820277797271617	$0.\times10^{-18}$	0.16752549
Comparison 8dp	0	0	0

1.3 Azimuthal Magnetisation

1.3.0 Equations

Analytic and Numeric function handles. Returns B={Bρ,Bφ,Bz} or B={Bx,By,Bz} (on axis).

```
In[•]:=  BφAna[M_,φ☆_,P_,ρp_,ρ_,φp_,φ_,zp_,z_] := M u0/(4π) (∑_{m=1}^{2} ∑_{n=1}^{2} ∑_{q=1}^{2} (-1)^(m+n-
         BφAnaAxis[M_,φ☆_,ρp_,φp_,zp_,z_] := M u0/(4π) (∑_{m=1}^{2} ∑_{n=1}^{2} ∑_{q=1}^{2} (-1)^(m+n+q) BφS
         BφNum[M_,φ☆_,ρp_,ρ_,φp_,φ_,zp_,z_] := M u0/(4π) (∑_{q=1}^{2} (-1)^q NIntegrat
                                                                  + Mφ[ρp,ρ,φp,
         BφNumAxis[M_,φ☆_,ρp_,φp_,zp_,z_] := M u0/(4π) ∑_{q=1}^{2} (-1)^q NIntegrate[B
```

Magnetisation vector for field inside magnet. Returns M={Mρ,Mφ,Mz}.

```
In[•]:=  Mφ[ρp_,ρ_,φp_,φ_,zp_,z_] := If[InsideVolume[ρp,ρ,φp,φ,zp,z],4
```

Special cases of the geometry for the analytic function handle. Replaces BφAna[] (Cylindrical) or BφAnaAxis[] (Cartesian, on axis).

```
In[•]:=  BφAna[M_,φ☆_,P_,ρp_,ρ_,{0,2π},φ_,zp_,z_] := M u0/(4π) (Mφ[ρp,ρ,{0
         BφAnaAxis[M_,φ☆_,ρp_,{0,2π},zp_,z_] := {0,0,0}
```

Integrands to be solved for BφNum[] (Cylindrical) and BφNumAxis[] (Cartesian, on axis).

In[•]:=
```
BφIntegrand[ρp_,ρ_,φp_,φ_,zp_,z_] := {BφIρ[ρp,ρ,φp,φ,zp,z], B

BφIρ[ρp_,ρ_,φp_,φ_,zp_,z_] := (ρ-ρp Cos[Φ[φ,φp]]) G[ρ,ρp,φ,
BφIφ[ρp_,ρ_,φp_,φ_,zp_,z_] := ρp Sin[Φ[φ,φp]] G[ρ,ρp,φ,φp,z,
BφIz[ρp_,ρ_,φp_,φ_,zp_,z_] := Z[z,zp] G[ρ,ρp,φ,φp,z,zp]^3

BφIntegrandAxis[ρp_,φp_,zp_,z_] := {BφIAxisx[ρp,φp,zp,z], BφI

BφIAxisx[ρp_,φp_,zp_,z_] := -ρp Cos[φp] (Z[z,zp]^2+ρp^2)^(-3/2)
BφIAxisy[ρp_,φp_,zp_,z_] := -ρp Sin[φp] (Z[z,zp]^2+ρp^2)^(-3/2)
BφIAxisz[ρp_,φp_,zp_,z_] := Z[z,zp] (Z[z,zp]^2+ρp^2)^(-3/2)
```

Summands for BφAna[] and BφAnaAxis[].

In[•]:=
```
BφSummand[ρp_,ρ_,φp_,φ_,zp_,z_] := {BφSρ[ρp,ρ,φp,φ,zp,z] , Bφ

BφSρ[ρp_,ρ_,φp_,φ_,zp_,z_] := Which[(φp==φ+π) || (φp==φ-π),Ar

BφSφ[ρp_,ρ_,φp_,φ_,zp_,z_] := Which[(φp==φ+π)|| (φp==φ-π),0,T

BφSz[ρp_,ρ_,φp_,φ_,zp_,z_] := Log[ρp-ρ Cos[Φ[φ,φp]]+G[ρ,ρp,φ


BφSummandAxis[ρp_,φp_,zp_,z_] := {BφSAxisx[ρp,φp,zp,z], BφSAx

BφSAxisx[ρp_,φp_,zp_,z_] := -ArcTanh[Sqrt[Z[z,zp]^2+ρp^2]/Z[z,zp]] Cos[φp]

BφSAxisy[ρp_,φp_,zp_,z_] := -ArcTanh[Sqrt[Z[z,zp]^2+ρp^2]/Z[z,zp]] Sin[φp]

BφSAxisz[ρp_,φp_,zp_,z_] := ArcTanh[ρp/Sqrt[Z[z,zp]^2+ρp^2]]
```

Singularities in the summands of BφAna[] or BφAnaAxis[].

In[ ]:=
```
(*Along the axis*)
BφSAxisx[ρp_,φp_,zp_,zp_] := 0
BφSAxisy[ρp_,φp_,zp_,zp_] := 0
BφSAxisz[ρp_,φp_,zp_,zp_] := Log[ρp]
(*On the section plane*)
BφSρ[ρp_,ρ_,φp_,φp_,zp_,z_] := -ArcTanh[R̄[ρ,ρp,z,zp]/Z[z,zp]]
BφSφ[ρp_,ρ_,φp_,φp_,zp_,z_] := 0
(*On the disc plane*)
BφSρ[ρp_,ρ_,φp_,φ_,zp_,zp_] := 0
BφSφ[ρp_,ρ_,φp_,φ_,zp_,zp_] := 0
(*On the axial line*)
BφSρ[ρp_,ρp_,φp_,φp_,zp_,z_] := -Log[Abs[Z[z,zp]]]Sign[Z[z,zp
BφSφ[ρp_,ρp_,φp_,φp_,zp_,z_] := 0
(*On the azimuthal line*)
BφSρ[ρp_,ρp_,φp_,φ_,zp_,zp_] := 0
BφSφ[ρp_,ρp_,φp_,φ_,zp_,zp_] := 0
(*On the radial line*)
BφSρ[ρp_,ρ_,φp_,φp_,zp_,zp_] := 0
BφSφ[ρp_,ρ_,φp_,φp_,zp_,zp_] := 0
BφSz[ρp_,ρ_,φp_,φp_,zp_,zp_] := -Sign[ϱ̄[ρ,ρp]]Log[Abs[ϱ̄[ρ,ρp
```

1.3.1 Standard - Outside Magnet

In[ ]:=
```
MagCylField[0,0,M,0,0,0,ρ',ρ1,φ',φ1,z',z1]
```

Mφ	Bρ	Bφ
Analytic	-0.0009866463614534913	-0.027353091874110
Numeric	-0.00098664636141892	-0.027353091873834
Comparison 8dp	0	0

1.3.2 Special Case a. - Inside Magnet

In[•]:=
```
MagCylField[0,0,M,0,0,0,ρ',ρ2,φ',φ2,z',z2]
```

Mφ	Bρ	Bφ
Analytic	-0.0012124488065541426	1.1527249218186003
Numeric	-0.00121244880652350	1.1527249218184838
Comparison 8dp	0	0

1.3.3 Special Case b. - On Magnet Axis

In[•]:=
```
MagCylField[0,0,M,0,0,0,ρ',ρ3,φ',φ3,z',z3]
```

Mφ	Bx	By
Analytic	0.06928254939210094	-0.08555682324180471
Numeric	0.06928254938118138	-0.08555682322832017
Comparison 8dp	0	0

1.3.4 Special Case c. - Axisymmetric

In[•]:=
```
MagCylField[0,0,M,0,0,0,ρ',ρ1,{0,2π},φ1,z',z1]
```

Mφ	Bρ	Bφ	Bz
Analytic	0	0	0
Numeric	$0. \times 10^{-18}$	$0. \times 10^{-18}$	$0. \times 10^{-19}$
Comparison 8dp	0	0	0

1.3.5 Special Case d. - Solid

In[•]:=
```
MagCylField[0,0,M,0,0,0,{0,ρ'[[2]]},ρ1,φ',φ1,z',z1]
```

Mφ	Bρ	Bφ
Analytic	-0.0012549369978166019	-0.032409293950822
Numeric	-0.00125493699761732	-0.032409293951492
Comparison 8dp	0	0

1.3.6 Special Case e. - Axisymmetric & Solid

In[∘]:=
```
MagCylField[0,0,M,0,0,0,{0,ρ'[[2]]},ρ1,{0,2π},φ1,z',z1]
```

Mφ	Bρ	Bφ	Bz
Analytic	0	0	0
Numeric	$0.\times10^{-17}$	$0.\times10^{-18}$	$0.\times10^{-19}$
Comparison 8dp	0	0	0

1.3.7 Singularities b,c,f. - Singular plane 1

In[∘]:=
```
MagCylField[0,0,M,0,0,0,ρ',ρ4,φ',φ4,z',z4]
```

Mφ	Bρ	Bφ
Analytic	-0.1305570199440887	-0.002611825775922538
Numeric	-0.1305570199604018	-0.002611825775599110
Comparison 8dp	0	0

1.3.8 Singularities a,c,e. - Singular plane 2

In[∘]:=
```
MagCylField[0,0,M,0,0,0,ρ',ρ5,φ',φ5,z',z5]
```

Mφ	Bρ	Bφ
Analytic	-0.1306098964594674	0.10456601136784070
Numeric	-0.1306098965151097	0.1045660111848111
Comparison 8dp	0	0

1.3.9 Singularities a,b,d. - Singular plane 3

In[∘]:=
```
MagCylField[0,0,M,0,0,0,ρ',ρ6,φ',φ6,z',z6]
```

Mφ	Bρ	Bφ
Analytic	-0.07776878671122339	-0.00309965018727580
Numeric	-0.0777687868094261	-0.00309965014995547
Comparison 8dp	0	0

1.3.10 (not in article) - On Magnet Axis & Axisymmetric

```
In[•]:= MagCylField[0,0,M,0,0,0,ρ′,ρ3,{0,2π},φ3,z′,z3]
```

Mφ	Bx	By	Bz
Analytic	0	0	0
Numeric	$0.\times 10^{-17}$	0	$0.\times 10^{-18}$
Comparison 8dp	0	0	0

1.4 Axial Magnetisation

1.4.0 Equations

Analytic and Numeric function handles. Returns B={Bρ,Bφ,Bz} or B={Bx,By,Bz} (on axis).

In[•]:= BzAna[M_,φ☆_,P_,ρp_,ρ_,φp_,φ_,zp_,z_] := $\frac{M\ u0}{4\pi}\sum_{m=1}^{2}\sum_{n=1}^{2}\sum_{q=1}^{2}(-1)^{m+n+q}$

BzAnaAxis[M_,φ☆_,ρp_,φp_,zp_,z_] := $\frac{M\ u0}{4\pi}\sum_{m=1}^{2}\sum_{n=1}^{2}\sum_{q=1}^{2}(-1)^{m+n+q}$ BzSu

BzNum[M_,φ☆_,ρp_,ρ_,φp_,φ_,zp_,z_] := $\frac{M\ u0}{4\pi}\left(\sum_{m=1}^{2}(-1)^{m}\right.$ NIntegrat

$+\sum_{q=1}^{2}(-1)^{q}$ NInt

BzNumAxis[M_,φ☆_,ρp_,φp_,zp_,z_] := $\frac{M\ u0}{4\pi}\left(\sum_{m=1}^{2}(-1)^{m}\right.$ NIntegrate[

$+\sum_{q=1}^{2}(-1)^{q}$ NInt

Special cases of the geometry for the analytic function handle. Replaces BzAna[] (Cylindrical) or BzAnaAxis[] (Cartesian, on axis).

In[∘]:=
```
BzAna[M_,φ☆_,P_,ρp_,ρ_,{0,2π},φ_,zp_,z_] := (M u0)/(4π) Sum[Sum[(-1)^(m+n)
BzAnaAxis[M_,φ☆_,ρp_,{0,2π},zp_,z_] := (M u0)/(4π) Sum[Sum[(-1)^(m+n) BzSum
BzAna[M_,φ☆_,P_,{0,ρp_},ρ_,φp_,φ_,zp_,z_] := (M u0)/(4π) Sum[Sum[(-1)^(n+
BzAna[M_,φ☆_,P_,{0,ρp_},ρ_,{0,2π},φ_,zp_,z_] := (M u0)/(4π) Sum[(-1)^n
```
(sums shown: m=1 to 2, n=1 to 2; n=1 to 2, q=1 to 2; n=1 to 2)

Integrands to be solved for BzNum[] (Cylindrical) and BzNumAxis[] (Cartesian, on axis).

In[∘]:=
```
BzIntegrand1[ρp_,ρ_,φp_,φ_,zp_,z_] := {BzIρ1[ρp,ρ,φp,φ,zp,z],
BzIntegrand2[ρp_,ρ_,φp_,φ_,zp_,z_] := {BzIρ2[ρp,ρ,φp,φ,zp,z],

BzIρ1[ρp_,ρ_,φp_,φ_,zp_,z_] := ρp Cos[Φ[φ,φp]] Z[z,zp] G[ρ,ρ
BzIρ2[ρp_,ρ_,φp_,φ_,zp_,z_] := Sin[Φ[φ,φp]] Z[z,zp] G[ρ,ρp,φ
BzIφ1[ρp_,ρ_,φp_,φ_,zp_,z_] := -ρp Sin[Φ[φ,φp]] Z[z,zp] G[ρ,
BzIφ2[ρp_,ρ_,φp_,φ_,zp_,z_] := Z[z,zp]Cos[Φ[φ,φp]] G[ρ,ρp,φ,
BzIz1[ρp_,ρ_,φp_,φ_,zp_,z_] := ρp (ρp - ρ Cos[Φ[φ,φp]]) G[ρ
BzIz2[ρp_,ρ_,φp_,φ_,zp_,z_] := -ρ Sin[Φ[φ,φp]] G[ρ,ρp,φ,φp,z

BzIntegrandAxis1[ρp_,φp_,zp_,z_] := {BzIAxisx1[ρp,φp,zp,z], Bz
BzIntegrandAxis2[ρp_,φp_,zp_,z_] := {BzIAxisx2[ρp,φp,zp,z], Bz

BzIAxisx1[ρp_,φp_,zp_,z_] := ρp Z[z,zp] Cos[φp] (Z[z,zp]^2+ρp^2)
BzIAxisx2[ρp_,φp_,zp_,z_] := -Z[z,zp] Sin[φp] (Z[z,zp]^2+ρp^2)^(-3/
BzIAxisy1[ρp_,φp_,zp_,z_] := ρp Z[z,zp] Sin[φp] (Z[z,zp]^2+ρp^2)
BzIAxisy2[ρp_,φp_,zp_,z_] := Z[z,zp] Cos[φp] (Z[z,zp]^2+ρp^2)^(-3/2
BzIAxisz[ρp_,φp_,zp_,z_] := ρp^2 (Z[z,zp]^2+ρp^2)^(-3/2)
```

Summands for BzAna[] and BzAnaAxis[].

In[•]:=
```
BzSummand[ρp_,ρ_,φp_,φ_,zp_,z_] := {BzSρ1[ρp,ρ,φp,φ,zp,z] + B
BzSummand1[ρp_,ρ_,φp_,φ_,zp_,z_] := {BzSρ1[ρp,ρ,φp,φ,zp,z], B
BzSummand2[ρp_,ρ_,φp_,φ_,zp_,z_] := {BzSρ2[ρp,ρ,φp,φ,zp,z], B
```

$$\text{BzS}\rho 1[\rho p\_,\rho\_,\varphi p\_,\varphi\_,zp\_,z\_] := \frac{2\ \rho p}{R[\rho,\rho p,zp,z]}\Big(\text{EllipticFT}\big[\phi[\varphi,\varphi$$

$$\text{BzS}\rho 2[\rho p\_,\rho\_,\varphi p\_,\varphi\_,zp\_,z\_] := \text{Log}\big[\rho p-\rho\ \text{Cos}[\Phi[\varphi,\varphi p]]+G[\rho,\rho p,$$

$$\text{BzS}\varphi 1[\rho p\_,\rho\_,\varphi p\_,\varphi\_,zp\_,z\_] := \frac{1}{\rho}G[\rho,\rho p,\varphi,\varphi p,z,zp]^{-1}$$

$$\text{BzS}\varphi 2[\rho p\_,\rho\_,\varphi p\_,\varphi\_,zp\_,z\_] := \text{Log}\big[\rho p-\rho\ \text{Cos}[\Phi[\varphi,\varphi p]]+G[\rho,\rho p,$$

$$\text{BzSz1}[\rho p\_,\rho\_,\varphi p\_,\varphi\_,zp\_,z\_] := \frac{Z[z,zp]}{R[\rho,\rho p,zp,z]}\Big(\text{EllipticFT}\big[\phi[\varphi,$$

$$\text{BzSz2}[\rho p\_,\rho\_,\varphi p\_,\varphi\_,zp\_,z\_] := \text{Which}[(\varphi p==\varphi+\pi)\ ||\ (\varphi p==\varphi-\pi),0,$$

```
BzSummandAxis[ρp_,φp_,zp_,z_] := {BzSAxisx[ρp,φp,zp,z], BzSAxi
```

$$\text{BzSAxisx}[\rho p\_,\varphi p\_,zp\_,z\_] := \text{Sin}[\varphi p]\left(\frac{\rho p}{\sqrt{Z[z,zp]^2+\rho p^2}}-\text{ArcTanh}\Big[\right.$$

$$\text{BzSAxisy}[\rho p\_,\varphi p\_,zp\_,z\_] := -\text{Cos}[\varphi p]\left(\frac{\rho p}{\sqrt{Z[z,zp]^2+\rho p^2}}-\text{ArcTanh}\Big[\right.$$

$$\text{BzSAxisz}[\rho p\_,\varphi p\_,zp\_,z\_] := -\frac{Z[z,zp]\ \varphi p}{\sqrt{Z[z,zp]^2+\rho p^2}}$$

```
BzSummandAS[ρp_,ρ_,φ_,zp_,z_] := {BzSASρ[ρp,ρ,φ,zp,z], BzSASφ
```

$$\text{BzSAS}\rho[\rho p\_,\rho\_,\varphi\_,zp\_,z\_] := \frac{-4\ \rho p}{R[\rho,\rho p,zp,z]}\Big(\text{EllipticK}\big[k[\rho,\rho p,$$

$$\text{BzSAS}\varphi[\rho p\_,\rho\_,\varphi\_,zp\_,z\_] := 0$$

$$\text{BzSASz}[\rho p\_,\rho\_,\varphi\_,zp\_,z\_] := \frac{-2\ Z[z,zp]}{R[\rho,\rho p,zp,z]}\Big(\text{EllipticK}\big[k[\rho,\rho p,z$$

Singularities in the summands of BzAna[] or BzAnaAxis[].

In[•]:=
```
(*Along the axis*)
BzSAxisx[ρp_,φp_,zp_,zp_] := Sin[φp](1-Log[ρp] )
BzSAxisy[ρp_,φp_,zp_,zp_] := Cos[φp](Log[ρp]-1)
BzSAxisz[ρp_,φp_,zp_,zp_] := 0
(*Along the axis & axisymmetric*)
BzSAxisx[ρp_,{0,2π},zp_,z_] := 0
BzSAxisy[ρp_,{0,2π},zp_,z_] := 0
BzSAxisz[ρp_,{0,2π},zp_,z_] := BzSAxisz[ρp,2π,zp,z]
BzSAxisx[ρp_,{0,2π},zp_,zp_] := 0
BzSAxisy[ρp_,{0,2π},zp_,zp_] := 0
BzSAxisz[ρp_,{0,2π},zp_,zp_] := 0
(*On the shell plane*)
BzSz1[ρp_,ρp_,φp_,φ_,zp_,z_] := Z[z,zp]/Sqrt[Z[z,zp]^2+4 ρp^2] EllipticFT[
(*On the shell plane & axisymmetric *)
BzSASz[ρp_,ρp_,φ_,zp_,z_] := -2 Z[z,zp]/R[ρp,ρp,zp,z] EllipticK[k[ρp,ρp,
(*On the section plane*)
BzSz2[ρp_,ρ_,φp_,φp_,zp_,z_] := 0
(*On the axial line*)
BzSz2[ρp_,ρp_,φp_,φp_,zp_,z_] := 0
(*On the azimuthal line*)
EllipticFT[ϕ_,1]:=Sin[ϕ]CarlsonRC[1,Cos[ϕ]^2]
BzSz1[ρp_,ρp_,φp_,φ_,zp_,zp_] := 0
(*On the radial line*)
BzSρ2[ρp_,ρ_,φp_,φp_,zp_,zp_] := 0
BzSφ2[ρp_,ρ_,φp_,φp_,zp_,zp_] := -Sign[ϱ[ρ,ρp]]Log[Abs[ϱ[ρ,ρ
```

1.4.1 Standard - Outside Magnet

In[•]:=
```
MagCylField[0,0,0,M,0,0,ρ',ρ1,φ',φ1,z',z1]
```

Mz	Bρ	Bφ
Analytic	0.01242509986271859	-0.000013932989338266
Numeric	0.01242509986169668	-0.000013932989338192
Comparison 8dp	0	0

1.4.2 Special Case a. - Inside Magnet

In[∘]:=
```
MagCylField[0,0,0,M,0,0,ρ′,ρ2,φ′,φ2,z′,z2]
```

Mz	Bρ	Bφ
Analytic	0.01266084849525560	-0.000032694250693224
Numeric	0.01266084849536185	-0.000032694250693417
Comparison 8dp	0	0

1.4.3 Special Case b. - On Magnet Axis

In[∘]:=
```
MagCylField[0,0,0,M,0,0,ρ′,ρ3,φ′,φ3,z′,z3]
```

Mz	Bx	By
Analytic	-0.07272021717981136	-0.05888767076268449
Numeric	-0.0727202171871756	-0.05888767076910802
Comparison 8dp	0	0

1.4.4 Special Case c. - Axisymmetric

NIntegrate struggles with Bφ.

In[∘]:=
```
MagCylField[0,0,0,M,0,0,ρ′,ρ1,{0,2π},φ1,z′,z1]//Quiet
```

Mz	Bρ	Bφ	Bz
Analytic	0.01292302651525148	0	-0.257900
Numeric	0.01292302651543708	$4.\times 10^{-18}$	-0.257900
Comparison 8dp	0	0	0

1.4.5 Special Case d. - Solid

In[∘]:=
```
MagCylField[0,0,0,M,0,0,{0,ρ′[[2]]},ρ1,φ′,φ1,z′,z1]
```

Mz	Bρ	Bφ
Analytic	0.01276677951668406	-0.000014939286576437
Numeric	0.01276677951565720	-0.000014939286576159
Comparison 8dp	0	0

1.4.6 Special Case e. - Axisymmetric & Solid

NIntegrate struggles with Bφ.

In[∘]:=

```
MagCylField[0,0,0,M,0,0,{0,ρ'[[2]]},ρ1,{0,2π},φ1,z',z1]//Quie
```

Mz	Bρ	Bφ	Bz
Analytic	0.01344895620667155	0	-0.273343
Numeric	0.01344895620688026	$4. \times 10^{-18}$	-0.273343
Comparison 8dp	0	0	0

1.4.7 Singularities b,c,f. - Singular plane 1

In[∘]:=

```
MagCylField[0,0,0,M,0,0,ρ',ρ4,φ',φ4,z',z4]
```

Mz	Bρ	Bφ
Analytic	0.1131007576718941	-0.05950872524189693
Numeric	0.1131007576909566	-0.0595087255655965
Comparison 8dp	0	0

1.4.8 Singularities a,c,e. - Singular plane 2

In[∘]:=

```
MagCylField[0,0,0,M,0,0,ρ',ρ5,φ',φ5,z',z5]
```

Mz	Bρ	Bφ
Analytic	0.08547085115221370	-0.1011947666062023
Numeric	0.0854708511544623	-0.1011947665760598
Comparison 8dp	0	0

1.4.9 Singularities a,b,d. - Singular plane 3

In[∘]:=

```
MagCylField[0,0,0,M,0,0,ρ',ρ6,φ',φ6,z',z6]
```

Mz	Bρ	Bφ
Analytic	0.1286619619883900	-0.1107623052135217
Numeric	0.1286619619891644	-0.1107623051291430
Comparison 8dp	0	0

1.4.10 (not in article) - On Magnet Axis & Axisymmetric

In[∘]:=
```
MagCylField[0,0,0,M,0,0,ρ',ρ3,{0,2π},φ3,z',z3]
```

Mz	Bx	By	Bz
Analytic	0	0	-0.2
Numeric	$-1.157683990029317\times10^{-21}$	$0.\times10^{-18}$	-0.2
Comparison 8dp	0	0	0

1.4.11 (not in article) - Axisymmetric & Singular plane 3

In[∘]:=
```
MagCylField[0,0,0,M,0,0,ρ',ρ6,{0,2π},φ6,z',z6]//Quiet
```

Mz	Bρ	Bφ	Bz
Analytic	0.2610520993114831	0	0.11139939(
Numeric	0.2610520991272603	$0.\times10^{-18}$	0.11139939(
Comparison 8dp	0	0	0

2.0 Coils with Azimuthal Current Density

Integrands to be solved for B{i,k,s,c}Num[] (Cylindrical) and B{i,k,s,c}NumAxis[] (Cartesian, on axis). Common between filament, disc, shell, volume.

In[∘]:=
```
BcIntegrand[ρp_,ρ_,φp_,φ_,zp_,z_] := {BcIρ[ρp,ρ,φp,φ,zp,z], E

BcIρ[ρp_,ρ_,φp_,φ_,zp_,z_] := ρp Cos[Φ[φ,φp]] Z[z,zp] G[ρ,ρp
BcIφ[ρp_,ρ_,φp_,φ_,zp_,z_] := -ρp Sin[Φ[φ,φp]] Z[z,zp] G[ρ,ρ
BcIz[ρp_,ρ_,φp_,φ_,zp_,z_] := ρp (ρp - ρ Cos[Φ[φ,φp]]) G[ρ,ρ

BcIntegrandAxis[ρp_,φp_,zp_,z_] := {BcIAxisx[ρp,φp,zp,z], BcIA

BcIAxisx[ρp_,φp_,zp_,z_] := ρp Z[z,zp] Cos[φp] (Z[z,zp]^2+ρp^2)^-3
BcIAxisy[ρp_,φp_,zp_,z_] := ρp Z[z,zp] Sin[φp] (Z[z,zp]^2+ρp^2)^-3
BcIAxisz[ρp_,φp_,zp_,z_] := ρp^2 (Z[z,zp]^2+ρp^2)^-3/2
```

2.1 Filament

2.1.0 Equations

Analytic and Numeric function handles. Returns B={Bρ,Bφ,Bz} or B={Bx,By,Bz} (on axis).

```
In[•]:= BiAna[I_,φ☆_,P_,ρp_,ρ_,φp_,φ_,zp_,z_] := (I u0)/(4π) Sum[(-1)^q BiSumma
        BiAnaAxis[I_,φ☆_,ρp_,φp_,zp_,z_] := (I u0)/(4π) Sum[(-1)^q BiSummandAxi
        BiNum[I_,φ☆_,ρp_,ρ_,φp_,φ_,zp_,z_] := (I u0)/(4π) NIntegrate[BcInte
        BiNumAxis[I_,φ☆_,ρp_,φp_,zp_,z_] := (I u0)/(4π) NIntegrate[BcIntegra
```

(sums run over q=1 to 2)

Special cases of the geometry for the analytic function handle. Replaces BiAna[] (Cylindrical) or BiAnaAxis[] (Cartesian, on axis).

```
In[•]:= BiAna[I_,φ☆_,P_,ρp_,ρ_,{0,2π},φ_,zp_,z_] := (I u0)/(4π) BiSummandAS
        BiAnaAxis[I_,φ☆_,ρp_,{0,2π},zp_,z_] := (I u0)/(4π) BiSummandAxis[ρp
```

Summands for BiAna[] and BiAnaAxis[].

In[•]:=

```
BiSummand[ρp_,ρ_,φp_,φ_,zp_,z_] := {BiSρ[ρp,ρ,φp,φ,zp,z], BiS

BiSρ[ρp_,ρ_,φp_,φ_,zp_,z_] := Z[z,zp]/ρ (1/R[ρ,ρp,z,zp] (Elliptic
BiSφ[ρp_,ρ_,φp_,φ_,zp_,z_] := -Z[z,zp]/ρ G[ρ,ρp,φ,φp,z,zp]
BiSz[ρp_,ρ_,φp_,φ_,zp_,z_] := -1/R[ρ, ρp,z,zp] (EllipticFT[ϕ[φ,

BiSummandAxis[ρp_,φp_,zp_,z_] := {BiSAxisx[ρp,φp,zp,z], BiSAxi

BiSAxisx[ρp_,φp_,zp_,z_] := Z[z,zp] ρp Sin[φp]/(Z[z,zp]^2+ρp^2)^(3/2)
BiSAxisy[ρp_,φp_,zp_,z_] := -Z[z,zp] ρp Cos[φp]/(Z[z,zp]^2+ρp^2)^(3/2)
BiSAxisz[ρp_,φp_,zp_,z_] := ρp^2 φp/(Z[z,zp]^2+ρp^2)^(3/2)

BiSummandAS[ρp_,ρ_,φ_,zp_,z_] := {BiSASρ[ρp,ρ,φ,zp,z], BiSASφ

BiSASρ[ρp_,ρ_,φ_,zp_,z_] := 2 Z[z,zp]/(ρ R[ρp,ρ,z,zp]) (T[ρ,ρp,z,zp]^2/R̄[ρ,ρp,z,zp]^2 El
BiSASφ[ρp_,ρ_,φ_,zp_,z_] := 0
BiSASz[ρp_,ρ_,φ_,zp_,z_] := 2/R[ρ,ρp,z,zp] (EllipticK[k[ρ,ρp,z,
```

Singularities in the summands of BiAna[] or BiAnaAxis[].

In[∘]:=
```
(*Along the axis & axisymmetric*)
BiSAxisx[ρp_,{0,2π},zp_,z_] := 0
BiSAxisy[ρp_,{0,2π},zp_,z_] := 0
BiSAxisz[ρp_,{0,2π},zp_,z_] := BiSAxisz[ρp,2π,zp,z]
(*On the azimuthal line*)
BiSρ[ρp_,ρp_,φp_,φ_,zp_,zp_] := 0
BiSφ[ρp_,ρp_,φp_,φ_,zp_,zp_] := 0
BiSz[ρp_,ρp_,φp_,φ_,zp_,zp_] := -Sign[Φ[φ,φp]] ArcTanh[Sin[ϕ[
                                               ─────────────
                                                    2 ρp
```

2.1.1 Standard - Outside Coil Radii

In[∘]:=
```
CoilFilamentField[Iφ,ρ',ρ1,φ',φ1,z',z1]
```

Iφ	Bρ	Bφ
Analytic	0.001372714157642409	1.698760127815627 × 10
Numeric	0.001372714157642415	1.698760127815627 × 10
Comparison 8dp	0	0

2.1.2 Special Case a. - Inside Coil Radii

In[∘]:=
```
CoilFilamentField[Iφ,ρ',ρ2,φ',φ2,z',z2]
```

Iφ	Bρ	Bφ
Analytic	0.001534719063283769	2.418938548515046 × 10
Numeric	0.001534719063283777	2.418938548515046 × 10
Comparison 8dp	0	0

2.1.3 Special Case b. - On Coil Axis

In[∘]:=
```
CoilFilamentField[Iφ,ρ',ρ3,φ',φ3,z',z3]
```

Iφ	Bx	By
Analytic	0.0001297864403851998	0.000105098987149156
Numeric	0.0001297864403851998	0.000105098987149156
Comparison 8dp	0	0

2.1.4 Special Case c. - Axisymmetric

In[ ]:=
```
CoilFilamentField[Iφ,ρ′,ρ1,{0,2π},φ1,z′,z1]
```

Iφ	Bρ	Bφ
Analytic	0.001360094620733514	0
Numeric	0.001360094620733503	5.564487148325225 × 10
Comparison 8dp	0	0

2.1.5 Singularities b,c,f. - Singular plane 1

In[ ]:=
```
CoilFilamentField[Iφ,ρ′,ρ4,φ′,φ4,z′,z4]
```

Iφ	Bρ	Bφ
Analytic	0.0003770114716661066	0.000161290993598386
Numeric	0.0003770114716661066	0.000161290993598386
Comparison 8dp	0	0

2.1.6 Singularities a,c,e. - Singular plane 2

In[ ]:=
```
CoilFilamentField[Iφ,ρ′,ρ5,φ′,φ5,z′,z5]
```

Iφ	Bρ	Bφ
Analytic	0.0002938289080965867	0.000178549973576697
Numeric	0.0002938289080965867	0.000178549973576697
Comparison 8dp	0	0

2.1.7 Singularities a,b,d. - Singular plane 3

In[ ]:=
```
CoilFilamentField[Iφ,ρ′,ρ6,φ′,φ6,z′,z6]
```

Iφ	Bρ	Bφ
Analytic	0.0003156861113994550	0.000170644523112106
Numeric	0.0003156861113994550	0.000170644523112106
Comparison 8dp	0	0

2.1.8 - On Coil Axis & Axisymmetric

In[∘]:=
```
CoilFilamentField[Iφ,ρ',ρ3,{0,2π},φ3,z',z3]
```

Iφ	Bx	By
Analytic	0	0
Numeric	$5.363823723160991\times10^{-43}$	-2.42279089650352
Comparison 8dp	0	0

2.2 Disc

2.2.0 Equations

Analytic and Numeric function handles. Returns B={Bρ,Bφ,Bz} or B={Bx,By,Bz} (on axis).

In[∘]:=
```
BkAna[K_,φ☆_,P_,ρp_,ρ_,φp_,φ_,zp_,z_] := (K u0)/(4π) Σ_{m=1}^{2} Σ_{q=1}^{2} (-1)^(m+q) BkS

BkAnaAxis[K_,φ☆_,ρp_,φp_,zp_,z_] := (K u0)/(4π) Σ_{m=1}^{2} Σ_{q=1}^{2} (-1)^(m+q) BkSumman

BkNum[K_,φ☆_,ρp_,ρ_,φp_,φ_,zp_,z_] := (K u0)/(4π) NIntegrate[BcInte

BkNumAxis[K_,φ☆_,ρp_,φp_,zp_,z_] := (K u0)/(4π) NIntegrate[BcIntegr
```

Special cases of the geometry for the analytic function handle. Replaces BkAna[] (Cylindrical) or BkAnaAxis[] (Cartesian, on axis).

In[∘]:=
```
BkAna[K_,φ☆_,P_,ρp_,ρ_,{0,2π},φ_,zp_,z_] := (K u0)/(4π) Σ_{m=1}^{2} (-1)^m BkS

BkAnaAxis[K_,φ☆_,ρp_,{0,2π},zp_,z_] := (K u0)/(4π) Σ_{m=1}^{2} (-1)^m BkSumman
```

Summands for BkAna[] and BkAnaAxis[].

In[•]:=

```
BkSummand[P_,ρp_,ρ_,φp_,φ_,zp_,z_] := {BkSρ[ρp,ρ,φp,φ,zp,z],

BkSρ[ρp_,ρ_,φp_,φ_,zp_,z_] := 2/ρ ρp Z[z,zp]/R[ρ,ρp,z,zp] (EllipticFT[ϕ[φ
BkSφ[ρp_,ρ_,φp_,φ_,zp_,z_] := -Z[z,zp]/ρ Log[ρp-ρ Cos[Φ[φ,φp]]
BkSz[P_,ρp_,ρ_,φp_,φ_,zp_,z_] := 2 ρp/R[ρ,ρp,z,zp] EllipticFT[ϕ[φ,

BkSummandAxis[ρp_,φp_,zp_,z_] := {BkSAxisx[ρp,φp,zp,z], BkSAxi

BkSAxisx[ρp_,φp_,zp_,z_] := -Sin[φp] Z[z,zp]/Sqrt[Z[z,zp]^2+ρp^2]

BkSAxisy[ρp_,φp_,zp_,z_] := Cos[φp] Z[z,zp]/Sqrt[Z[z,zp]^2+ρp^2]

BkSAxisz[ρp_,φp_,zp_,z_] := -φp (ρp/Sqrt[Z[z,zp]^2+ρp^2] - ArcTanh[ /Sqrt[Z[

BkSummandAS[P_,ρp_,ρ_,φ_,zp_,z_] := {BkSASρ[ρp,ρ,φ,zp,z], BkS

BkSASρ[ρp_,ρ_,φ_,zp_,z_] := -4/ρ ρp Z[z,zp]/R[ρ,ρp,z,zp] (EllipticK[k[ρ,ρ
BkSASφ[ρp_,ρ_,φ_,zp_,z_] := 0
BkSASz[P_,ρp_,ρ_,φ_,zp_,z_] := -4ρp/R[ρ,ρp,z,zp] EllipticK[k[ρ,ρp
```

Singularities in the summands of BkAna[] or BkAnaAxis[].

In[•]:=
```
(*Along the axis*)
BkSAxisx[ρp_,φp_,zp_,zp_] := 0
BkSAxisy[ρp_,φp_,zp_,zp_] := 0
BkSAxisz[ρp_,φp_,zp_,zp_] := φp Log[ρp]
(*Along the axis & axisymmetric*)
BkSAxisx[ρp_,{0,2π},zp_,z_] := 0
BkSAxisy[ρp_,{0,2π},zp_,z_] := 0
BkSAxisz[ρp_,{0,2π},zp_,z_] := BkSAxisz[ρp,2π,zp,z]
(*Solid*)
BkSρ[0,ρ_,φp_,φ_,zp_,z_] := L[ρ,z,zp]/ρ ArcTan[ρ Sin[Φ[φ,φp]]/Z[z,zp]]
BkSρ[0,ρ_,φp_,φ_,zp_,zp_] := 0
(*Solid & axisymmetric*)
BkSASρ[0,ρ_,φ_,zp_,z_] := 0
(*On the disc plane*)
BkSρ[ρp_,ρ_,φp_,φ_,zp_,zp_] := 0
BkSφ[ρp_,ρ_,φp_,φ_,zp_,zp_] := 0
(*On the disc plane & axisymmetric*)
BkSASρ[ρp_,ρ_,φ_,zp_,zp_] := 0
(*On the azimuthal line*)
EllipticFT[ϕ_,1]:=Sin[ϕ]CarlsonRC[1,Cos[ϕ]^2]
BkSρ[ρp_,ρp_,φp_,φ_,zp_,zp_] := 0
BkSφ[ρp_,ρp_,φp_,φ_,zp_,zp_] := 0
(*On the radial line*)
BkSρ[ρp_,ρ_,φp_,φp_,zp_,zp_] := 0
BkSφ[ρp_,ρ_,φp_,φp_,zp_,zp_] := 0
```

2.2.1 Standard - Outside Coil Radii

In[•]:=
```
CoilDiscField[Kφ,100,ρ',ρ1,φ',φ1,z',z1]
```

Kφ	Bρ	Bφ
Analytic	0.004550646471690198	0.000014862618670182ε
Numeric	0.004550646471947809	0.000014862618670658ε
Comparison 8dp	0	0

2.2.2 Special Case a. - Inside Coil Radii

In[∘]:=
```
CoilDiscField[Kφ,150,ρ′,ρ2,φ′,φ2,z′,z2]
```

Kφ	Bρ	Bφ
Analytic	0.01013923265250101	0.000025059001072307871
Numeric	0.01013923264982671	0.000025059001071139811
Comparison 8dp	0	0

2.2.3 Special Case b. - On Coil Axis

In[∘]:=
```
CoilDiscField[Kφ,0,ρ′,ρ3,φ′,φ3,z′,z3]
```

Kφ	Bx	By
Analytic	0.002047652044440496	0.001658155931127027
Numeric	0.002047652043569552	0.001658155930824412
Comparison 8dp	0	0

2.2.4 Special Case c. - Axisymmetric

NIntegrate struggles with Bφ.

In[∘]:=
```
CoilDiscField[Kφ,100,ρ′,ρ1,{0,2π},φ1,z′,z1]//Quiet
```

Kφ	Bρ	Bφ
Analytic	0.004412345974681498	0
Numeric	0.004412345973088094	$-1.410421035147059\times 1$
Comparison 8dp	0	0

2.2.5 Special Case d. - Solid

In[∘]:=
```
CoilDiscField[Kφ,100,{0,ρ′[[2]]},ρ1,φ′,φ1,z′,z1]
```

Kφ	Bρ	Bφ
Analytic	0.004710462151164272	0.00001760013686144881
Numeric	0.004710462152513212	0.00001760013686158781
Comparison 8dp	0	0

2.2.6 Special Case e. - Axisymmetric & Solid

NIntegrate struggles with Bφ.

In[∘]:=
```
CoilDiscField[Kφ,100,{0,ρ'[[2]]},ρ1,{0,2π},φ1,z',z1]//Quiet
```

Kφ	Bρ	Bφ
Analytic	0.004518329230250307	0
Numeric	0.004518329227550135	-2.233301434601139×1
Comparison 8dp	0	0

2.2.7 Singularities b,c,f. - Singular plane 1

In[∘]:=
```
CoilDiscField[Kφ,50,ρ',ρ4,φ',φ4,z',z4]
```

Kφ	Bρ	Bφ
Analytic	0.001807018177682073	0.001052865287292513
Numeric	0.001807018177554557	0.001052865287213675
Comparison 8dp	0	0

2.2.8 Singularities a,c,e. - Singular plane 2

In[∘]:=
```
CoilDiscField[Kφ,50,ρ',ρ5,φ',φ5,z',z5]
```

Kφ	Bρ	Bφ
Analytic	0.001714110762320366	0.001310352196154153
Numeric	0.001714110761827329	0.001310352196075857
Comparison 8dp	0	0

2.2.9 Singularities a,b,d. - Singular plane 3

In[∘]:=
```
CoilDiscField[Kφ,50,ρ',ρ6,φ',φ6,z',z6]
```

Kφ	Bρ	Bφ
Analytic	0.001871178367543343	0.001277701647764240
Numeric	0.001871178367547036	0.001277701647679863
Comparison 8dp	0	0

2.2.10 - On Coil Axis & Axisymmetric

In[•]:=
```
CoilDiscField[Kφ,0,ρ',ρ3,{0,2π},φ3,z',z3]
```

Kφ	Bx	By
Analytic	0	0
Numeric	$4.899882906133252\times10^{-23}$	-9.27052056626888
Comparison 8dp	0	0

2.2.11 (not in article) - Axisymmetric & Singular plane 3

In[•]:=
```
CoilDiscField[Kφ,100,ρ',ρ6,{0,2π},φ6,z',z6] //Quiet
```

Kφ	Bρ	Bφ
Analytic	0.003488669562401596	0
Numeric	0.003488669563036198	4.824185584218220 × 10
Comparison 8dp	0	0

2.3 Shell

2.3.0 Equations

Analytic and Numeric function handles. Returns B={Bρ,Bφ,Bz} or B={Bx,By,Bz} (on axis).

In[•]:=

$$\text{BsAna}[K\_,\varphi☆\_,P\_,\rho p\_,\rho\_,\varphi p\_,\varphi\_,zp\_,z\_] := \frac{K\ u0}{4\pi}\sum_{n=1}^{2}\sum_{q=1}^{2}(-1)^{n+q}\ \text{BsS}$$

$$\text{BsAnaAxis}[K\_,\varphi☆\_,\rho p\_,\varphi p\_,zp\_,z\_] := \frac{K\ u0}{4\pi}\sum_{n=1}^{2}\sum_{q=1}^{2}(-1)^{n+q}\ \text{BsSumman}$$

$$\text{BsNum}[K\_,\varphi☆\_,\rho p\_,\rho\_,\varphi p\_,\varphi\_,zp\_,z\_] := \frac{K\ u0}{4\pi}\ \text{NIntegrate}[\text{BcInte}$$

$$\text{BsNumAxis}[K\_,\varphi☆\_,\rho p\_,\varphi p\_,zp\_,z\_] := \frac{K\ u0}{4\pi}\ \text{NIntegrate}[\text{BcIntegr}$$

Special cases of the geometry for the analytic function handle. Replaces BsAna[] (Cylindrical) or BsAnaAxis[] (Cartesian, on axis).

```
In[•]:= BsAna[K_,φ☆_,P_,ρp_,ρ_,{0,2π},φ_,zp_,z_] := (K u0)/(4π) ∑_{n=1}^{2} (-1)^n BsS

        BsAnaAxis[K_,φ☆_,ρp_,{0,2π},zp_,z_] := (K u0)/(4π) ∑_{n=1}^{2} (-1)^n BsSumman
```

Summands for BsAna[] and BsAnaAxis[].

```
In[•]:= BsSummand[ρp_,ρ_,φp_,φ_,zp_,z_] := {BsSρ[ρp,ρ,φp,φ,zp,z], BsS

        BsSρ[ρp_,ρ_,φp_,φ_,zp_,z_] := (2 ρp)/(R[ρ,ρp,zp,z]) (EllipticFT[ϕ[φ,φp
        BsSφ[ρp_,ρ_,φp_,φ_,zp_,z_] := 1/ρ G[ρ,ρp,φ,φp,z,zp]^-1
        BsSz[ρp_,ρ_,φp_,φ_,zp_,z_] := (Z[z,zp])/(R[ρ,ρp,zp,z]) (EllipticFT[ϕ[φ,φ

        BsSummandAxis[ρp_,φp_,zp_,z_] := {BsSAxisx[ρp,φp,zp,z], BsSAxi

        BsSAxisx[ρp_,φp_,zp_,z_] :=  Sin[φp] ρp/Sqrt[Z[z,zp]^2+ρp^2]

        BsSAxisy[ρp_,φp_,zp_,z_] := -Cos[φp] ρp/Sqrt[Z[z,zp]^2+ρp^2]

        BsSAxisz[ρp_,φp_,zp_,z_] := - (Z[z,zp] φp)/Sqrt[Z[z,zp]^2+ρp^2]

        BsSummandAS[ρp_,ρ_,φ_,zp_,z_] := {BsSASρ[ρp,ρ,φ,zp,z], BsSASφ

        BsSASρ[ρp_,ρ_,φ_,zp_,z_] := (4 ρp)/(R[ρ,ρp,zp,z]) (2EllipticD[k[ρ,ρp
        BsSASφ[ρp_,ρ_,φ_,zp_,z_] := 0
        BsSASz[ρp_,ρ_,φ_,zp_,z_] := (-2 Z[z,zp])/(R[ρ,ρp,zp,z]) (EllipticK[k[ρ,ρp,z
```

Singularities in the summands of BsAna[] or BsAnaAxis[].

In[•]:=
```
(*Along the axis*)
BsSAxisx[ρp_,φp_,zp_,zp_] := Sin[φp]
BsSAxisy[ρp_,φp_,zp_,zp_] := -Cos[φp]
BsSAxisz[ρp_,φp_,zp_,zp_] := 0
(*Along the axis & axisymmetric*)
BsSAxisx[ρp_,{0,2π},zp_,z_] := 0
BsSAxisy[ρp_,{0,2π},zp_,z_] := 0
BsSAxisz[ρp_,{0,2π},zp_,z_] := BsSAxisz[ρp,2π,zp,z]
BsSAxisx[ρp_,{0,2π},zp_,zp_] := 0
BsSAxisy[ρp_,{0,2π},zp_,zp_] := 0
BsSAxisz[ρp_,{0,2π},zp_,zp_] := 0
(*On the shell plane*)
BsSz[ρp_,ρp_,φp_,φ_,zp_,z_] := Z[z,zp]/Sqrt[Z[z,zp]^2+4 ρp^2] EllipticFT[ϕ
(*On the shell plane & axisymmetric*)
BsSASz[ρp_,ρp_,φ_,zp_,z_] := -2 Z[z,zp]/R[ρp,ρp,zp,z] EllipticK[k[ρp,ρp,
(*On the azimuthal line*)
EllipticFT[ϕ_,1]:=Sin[ϕ]CarlsonRC[1,Cos[ϕ]^2]
```

2.3.1 Standard - Outside Coil Radii

In[•]:=
```
CoilShellField[Kφ,ρ′,ρ1,φ′,φ1,z′,z1]
```

Kφ	Bρ	Bφ
Analytic	0.0005681218241650236	6.511476224250626 × 1
Numeric	0.0005681218241220527	6.511476224270352 × 1
Comparison 8dp	0	0

2.3.2 Special Case a. - Inside Coil Radii

In[•]:=
```
CoilShellField[Kφ,ρ′,ρ2,φ′,φ2,z′,z2]
```

Kφ	Bρ	Bφ
Analytic	0.0006356826497196410	9.28724216507752 × 10
Numeric	0.0006356826497240293	9.28724216509253 × 10
Comparison 8dp	0	0

2.3.3 Special Case b. - On Coil Axis

In[∘]:=
```
CoilShellField[Kφ,ρ',ρ3,φ',φ3,z',z3]
```

Kφ	Bx	By
Analytic	0.0006127684497726228	0.00049621010667152(
Numeric	0.0006127684497739614	0.00049621010666959(
Comparison 8dp	0	0

2.3.4 Special Case c. - Axisymmetric

NIntegrate struggles with Bφ.

In[∘]:=
```
CoilShellField[Kφ,ρ',ρ1,{0,2π},φ1,z',z1]//Quiet
```

Kφ	Bρ	Bφ
Analytic	0.0005633070662480231	0
Numeric	0.0005633070662567649	1.600183994864209 × 1
Comparison 8dp	0	0

2.3.5 Singularities b,c,f. - Singular plane 1

In[∘]:=
```
CoilShellField[Kφ,ρ',ρ4,φ',φ4,z',z4]
```

Kφ	Bρ	Bφ
Analytic	0.005009379856705172	0.001167520097808665
Numeric	0.005009379857527716	0.001167520098129617
Comparison 8dp	0	0

2.3.6 Singularities a,c,e. - Singular plane 2

In[∘]:=
```
CoilShellField[Kφ,ρ',ρ5,φ',φ5,z',z5]
```

Kφ	Bρ	Bφ
Analytic	0.003220598895996764	0.001286646043698470
Numeric	0.003220598896130437	0.001286646044934483
Comparison 8dp	0	0

2.3.7 Singularities a,b,d. - Singular plane 3

In[•]:=
```
CoilShellField[Kφ,ρ′,ρ6,φ′,φ6,z′,z6]
```

Kφ	Bρ	Bφ
Analytic	0.005908193994816346	0.001609408420570709
Numeric	0.005908193994780027	0.001609408421723598
Comparison 8dp	0	0

2.3.8 - On Coil Axis & Axisymmetric

In[•]:=
```
CoilShellField[Kφ,ρ′,ρ3,{0,2π},φ3,z′,z3]
```

Kφ	Bx	By
Analytic	0	0
Numeric	$1.533582990691132\times10^{-23}$	-1.54808226452014
Comparison 8dp	0	0

2.3.9 (not in article) - Axisymmetric & Singular plane 3

In[•]:=
```
CoilShellField[Kφ,ρ′,ρ6,{0,2π},φ6,z′,z6] //Quiet
```

Kφ	Bρ	Bφ
Analytic	0.01169236924966334	0
Numeric	0.01169236924174034	$1.874673356262654\times10^{-}$
Comparison 8dp	0	0

2.3 Volume

2.2.0 Equations

Analytic and Numeric function handles. Returns B={Bρ,Bφ,Bz} or B={Bx,By,Bz} (on axis).

```
In[•]:=  BcAna[J_,φ☆_,P_,ρp_,ρ_,φp_,φ_,zp_,z_] := (J u0)/(4π) Sum[Sum[Sum[(-1)^(m+n+q
         BcAnaAxis[J_,φ☆_,ρp_,φp_,zp_,z_] := (J u0)/(4π) Sum[Sum[Sum[(-1)^(m+n+q BcSu
         BcNum[J_,φ☆_,ρp_,ρ_,φp_,φ_,zp_,z_] := (J u0)/(4π) NIntegrate[BcInte
         BcNumAxis[J_,φ☆_,ρp_,φp_,zp_,z_] := (J u0)/(4π) NIntegrate[BcIntegra
```

$$\text{BcAna}[J\_,\varphi☆\_,P\_,\rho p\_,\rho\_,\varphi p\_,\varphi\_,zp\_,z\_] := \frac{J\ u0}{4\pi}\sum_{m=1}^{2}\sum_{n=1}^{2}\sum_{q=1}^{2}(-1)^{m+n+q}$$

$$\text{BcAnaAxis}[J\_,\varphi☆\_,\rho p\_,\varphi p\_,zp\_,z\_] := \frac{J\ u0}{4\pi}\sum_{m=1}^{2}\sum_{n=1}^{2}\sum_{q=1}^{2}(-1)^{m+n+q}\ \text{BcSu}$$

$$\text{BcNum}[J\_,\varphi☆\_,\rho p\_,\rho\_,\varphi p\_,\varphi\_,zp\_,z\_] := \frac{J\ u0}{4\pi}\ \text{NIntegrate}[\text{BcInte}$$

$$\text{BcNumAxis}[J\_,\varphi☆\_,\rho p\_,\varphi p\_,zp\_,z\_] := \frac{J\ u0}{4\pi}\ \text{NIntegrate}[\text{BcIntegra}$$

Special cases of the geometry for the analytic function handle. Replaces BcAna[] (Cylindrical) or BcAnaAxis[] (Cartesian, on axis).

In[•]:=

$$\text{BcAna}[J\_,\varphi☆\_,P\_,\rho p\_,\rho\_,\{0,2\pi\},\varphi\_,zp\_,z\_] := \frac{J\ u0}{4\pi}\sum_{m=1}^{2}\sum_{n=1}^{2}(-1)^{m+n}$$

$$\text{BcAnaAxis}[J\_,\varphi☆\_,\rho p\_,\{0,2\pi\},zp\_,z\_] := \frac{J\ u0}{4\pi}\sum_{m=1}^{2}\sum_{n=1}^{2}(-1)^{m+n}\ \text{BcSu}$$

Summands for BcAna[] and BcAnaAxis[].

In[•]:=
```
BcSummand[P_,ρp_,ρ_,φp_,φ_,zp_,z_] := {BcSρ[P,ρp,ρ,φp,φ,zp,z]

BcSρ[P_,ρp_,ρ_,φp_,φ_,zp_,z_] := 1/2 (R[ρ,ρp,z,zp] 4/3 (EllipticFT
BcSφ[ρp_,ρ_,φp_,φ_,zp_,z_] := 1/(2ρ) ( (ρp-ρ Cos[Φ[φ,φp]]) G[ρ,
BcSz[P_,ρp_,ρ_,φp_,φ_,zp_,z_] := φp α3[ρ,ρp,z,zp] + 1/2 Sign[Z[z,

BcSummandAxis[ρp_,φp_,zp_,z_] := {BcSAxisx[ρp,φp,zp,z], BcSAxi

BcSAxisx[ρp_,φp_,zp_,z_] := Sin[φp] Sqrt[Z[z,zp]^2+ρp^2]
BcSAxisy[ρp_,φp_,zp_,z_] := -Cos[φp] Sqrt[Z[z,zp]^2+ρp^2]
BcSAxisz[ρp_,φp_,zp_,z_] := -φp Z[z,zp] ArcTanh[ρp/Sqrt[Z[z,zp]^2+ρp

BcSummandAS[P_,ρp_,ρ_,φ_,zp_,z_] := {BcSASρ[P,ρp,ρ,φ,zp,z], B

BcSASρ[P_,ρp_,ρ_,φ_,zp_,z_] := π ρ (α1[ρ,ρp,z,zp]+β1[ρ,ρp,z,
BcSASφ[ρp_,ρ_,φ_,zp_,z_] := 0
BcSASz[P_,ρp_,ρ_,φ_,zp_,z_] := 2π (α3[ρ,ρp,z,zp] + 1/2 ρp Sign[Z[z
```

Singularities in the summands of BcAna[] or BcAnaAxis[].

In[∘]:=
```
(*Along the axis*)
BcSAxisz[ρp_,φp_,zp_,zp_] := 0
(*Along the axis & axisymmetric*)
BcSAxisx[ρp_,{0,2π},zp_,z_] := 0
BcSAxisy[ρp_,{0,2π},zp_,z_] := 0
BcSAxisz[ρp_,{0,2π},zp_,z_] := BcSAxisz[ρp,2π,zp,z]
(*On the azimuthal line*)
EllipticFT[ϕ_,1]:=Sin[ϕ]CarlsonRC[1,Cos[ϕ]^2]
(*On the radial line*)
BcSφ[ρp_,ρ_,φp_,φp_,zp_,zp_] := (*- Abs[ϱ[ρ,ρp]] ϱ[ρ,ρp] / (2 ρ) *) - Abs[ϱ[
```

2.4.1 Standard - Outside Coil

NIntegrate struggles with Bφ.

In[∘]:=
```
CoilCylField[Jφ,P,ρ′,ρ7,φ′,φ7,z′,z7]//Quiet
```

Jφ	Bρ	Bφ
Analytic	0.0000294742911202984 9	3.524997169679487 ×
Numeric	0.0000294742911212434 4	3.524997225520252 ×
Comparison 8dp	0	0

2.4.2 Special Case b. - Axisymmetric

NIntegrate struggles with Bφ.

In[∘]:=
```
CoilCylField[Jφ,P,ρ′,ρ7,{0,2π},φ7,z′,z7]//Quiet
```

Jφ	Bρ	Bφ
Analytic	0.00002069362960441219	0
Numeric	0.00002464071192591216	1.021109903288754 ›
Comparison 8dp	$-3.94708232149997\times10^{-6}$	0

2.4.3 Special Case a. - On Coil Axis

In[•]:=
```
CoilCylField[Jφ,0,ρ′,ρ3,φ′,φ3,z′,z3]
```

Jφ	Bx	By
Analytic	0.0001531921124431557	0.00012405252666788
Numeric	0.0001531921124201073	0.00012405252664820
Comparison 8dp	0	0

2.4.4 Special Case a,b - On Coil Axis & Axisymmetric

In[•]:=
```
CoilCylField[Jφ,0,ρ′,ρ3,{0,2π},φ3,z′,z3]
```

Jφ	Bx	By
Analytic	0	0
Numeric	$3.712049951100202\times10^{-24}$	-4.91073503290807
Comparison 8dp	0	0

2.4.5 Singularities b,c,f. - Singular plane 1

In[•]:=
```
CoilCylField[Jφ,P,ρ′,ρ4,φ′,φ4,z′,z4]
```

Jφ	Bρ	Bφ
Analytic	0.0001841132305583547	0.00007498203826667
Numeric	0.0001860126163046093	0.00007498203826589
Comparison 8dp	$-1.8993857462547\times10^{-6}$	0

2.4.6 Singularities a,c,e. - Singular plane 2

In[•]:=
```
CoilCylField[Jφ,P,ρ′,ρ5,φ′,φ5,z′,z5]
```

Jφ	Bρ	Bφ
Analytic	0.0001886679203860053	0.00010009731448132
Numeric	0.0001834555924935078	0.00010009731445257
Comparison 8dp	$5.2123278924976\times10^{-6}$	0

2.4.7 Singularities a,b,d. - Singular plane 3

In[ ]:=
```
CoilCylField[Jφ,P,ρ′,ρ6,φ′,φ6,z′,z6]
```

Jφ	Bρ	Bφ
Analytic	0.0003035211228690777	0.00012732924070133
Numeric	0.0003070881388353115	0.00012732924064067
Comparison 8dp	$-3.5670159662338\times10^{-6}$	0

3.0 Green’s Function Integrals

This section is simply a comparison of the integral transforms, discussed in part 3 of the article.

3.0.0 Integrals and Analytic Solutions

In[ ]:=
```
Compare[P_,ρ′_,ρ_,φ′_,φ_,z′_,z_,numFn_,anaFn1_,anaFn2_] :=
   Module[{Bana1,Bana2,Bnum,heading},
      Bnum =   numFn[ρ′,ρ,φ′,φ,z′,z];
      Bana1 = N[anaFn1[P,ρ′,ρ,φ′,φ,z′,z],$MachinePrecision]
      Bana2=N[anaFn2[P,ρ′,ρ,φ′,φ,z′,z],$MachinePrecision];
      If[anaFn2==0,Bana2=None];
      TableForm[{Bnum,Bana1,Bana2}, TableHeadings -> {{"Nume
   ]
(*Azimuthal integral*)
Gdφp[ρp_,ρ_,φp_,φ_,zp_,z_] := NIntegrate[G[ρ,ρp,φ,dφp,z,zp],{
```

$$\text{Gd}\varphi\text{p1}[P\_,\rho p\_,\rho\_,\varphi p\_,\varphi\_,zp\_,z\_] := -\frac{2}{R[\rho,\rho p,z,zp]}\sum_{q=1}^{2}(-1)^{q}\ \text{Ellip}$$

$$\text{Gd}\varphi\text{p2}[P\_,\rho p\_,\rho\_,\varphi p\_,\varphi\_,zp\_,z\_] := -\frac{2}{R[\rho,\rho p,z,zp]}\sum_{q=1}^{2}(-1)^{q}\ \text{Ellip}$$

```
(*Radial integral*)
Gdρp[ρp_,ρ_,φp_,φ_,zp_,z_] := NIntegrate[G[ρ,dρp,φ,φp,z,zp],{
```

$$\text{Gd}\rho\text{p1}[P\_,\rho p\_,\rho\_,\varphi p\_,\varphi\_,zp\_,z\_] := \sum_{m=1}^{2}(-1)^{m}\ (\alpha 1[\rho,\rho p[[m]],z,zp$$

```
(*Axial integral*)
Gdzp[ρp_,ρ_,φp_,φ_,zp_,z_] := NIntegrate[G[ρ,ρp,φ,φp,z,dzp],{
```

$$\text{Gdzp1}[P\_,\rho p\_,\rho\_,\varphi p\_,\varphi\_,zp\_,z\_] := -\sum_{n=1}^{2}(-1)^{n}\ \text{Sign}[Z[z,zp[[n]]]]$$

```
(*Radial surface integral*)
Gdρpdφp[ρp_,ρ_,φp_,φ_,zp_,z_] := NIntegrate[G[ρ,dρp,φ,dφp,z,z
Gdρpdφp1[P_,ρp_,ρ_,φp_,φ_,zp_,z_] := Σ_{m=1}^{2} Σ_{q=1}^{2} (-1)^(m+q) (φp[[q]]×α1
Gdρpdφp2[P_,ρp_,ρ_,φp_,φ_,zp_,z_] := Σ_{m=1}^{2} Σ_{q=1}^{2} (-1)^(m+q) (φp[[q]]×α1
(*Axial surface integral*)
Gdzpdφp[ρp_,ρ_,φp_,φ_,zp_,z_] := NIntegrate[G[ρ,ρp,φ,dφp,z,dz
Gdzpdφp1[P_,ρp_,ρ_,φp_,φ_,zp_,z_] :=-1/2 Σ_{n=1}^{2} Σ_{q=1}^{2} (-1)^(n+q) Sign[Z[z,
Gdzpdφp2[P_,ρp_,ρ_,φp_,φ_,zp_,z_] :=-1/2 Σ_{n=1}^{2} Σ_{q=1}^{2} (-1)^(n+q) Sign[Z[z,
(*Volume integral*)
Gdρpdφpdzp[ρp_,ρ_,φp_,φ_,zp_,z_] := NIntegrate[G[ρ,dρp,φ,dφp,
Gdρpdφpdzp1[P_,ρp_,ρ_,φp_,φ_,zp_,z_] := Σ_{m=1}^{2} Σ_{n=1}^{2} Σ_{q=1}^{2} (-1)^(m+n+q) (φp[[
Gdρpdφpdzp2[P_,ρp_,ρ_,φp_,φ_,zp_,z_] := Σ_{m=1}^{2} Σ_{n=1}^{2} Σ_{q=1}^{2} (-1)^(m+n+q) (φp[[
```

3.0.1 Azimuthal Integral

In[•]:=

```
Compare[0,ρ'[[2]],ρ1,φ',φ1,z'[[1]],z1,Gdφp,Gdφp1,Gdφp2]
```

Out[•]//TableForm=

Numeric	520.8982064725782
Analytic Form 1	520.8982064725861
Analytic Form 2	520.8982064725861

3.0.2 Radial Integral

In[•]:=

```
Compare[P,ρ',ρ1,φ'[[1]],φ1,z'[[1]],z1,Gdρp,Gdρp1,0]
```

Out[•]//TableForm=

Numeric	0.5607377420756752
Analytic Form 1	0.5607377421188464
Analytic Form 2	None

3.0.3 Axial Integral

In[ ]:=
```
Compare[200,ρ'[[2]],ρ1,φ'[[1]],φ1,z',z1,Gdzp,Gdzp1,0]
```

Out[ ]//TableForm=

Numeric	0.4187967839218372
Analytic Form 1	0.4187978710349870
Analytic Form 2	None

3.0.4 Radial Surface Integral

In[ ]:=
```
Compare[P,ρ',ρ1,φ',φ1,z'[[1]],z1,Gdρpdφp,Gdρpdφp1,Gdρpdφp2]
```

Out[ ]//TableForm=

Numeric	2.133688224453506
Analytic Form 1	2.133677478713198
Analytic Form 2	2.133688203712503

3.0.5 Axial Surface Integral

In[ ]:=
```
Compare[250,ρ'[[2]],ρ1,φ',φ1,z',z1,Gdzpdφp,Gdzpdφp1,Gdzpdφp2]
```

Out[ ]//TableForm=

Numeric	2.536820273844129
Analytic Form 1	2.530666999720498
Analytic Form 2	2.536092152698103

3.0.6 Volume Integral

In[ ]:=
```
Compare[100,ρ',ρ1,φ',φ1,z',z1,Gdρpdφpdzp,Gdρpdφpdzp1,Gdρpdφpdz
```

Out[ ]//TableForm=

Numeric	0.009279870903882 50
Analytic Form 1	0.00926414493132398
Analytic Form 2	0.00927635460527792

4.0 Magnetic field derivates

This section has the example derivatives given in part 6.2 of the article. Integrals with respect to t are not true for all φ' (ϕ).

4.0.0 Derivatives and Analytic Solutions

```
Compare2[ρ'_,ρ_,φ'_,φ_,z'_,z_,numFn1_,numFn2_,anaFn_] :=
    Module[{Bana,Bnum1,Bnum2,heading},
        Bnum1 = N[numFn1[ρ',ρ,φ',φ,z',z],$MachinePrecision];
        Bnum2 = N[numFn2[ρ',ρ,φ',φ,z',z],$MachinePrecision];
        Bana =  N[anaFn[ρ',ρ,φ',φ,z',z],$MachinePrecision];
        If[numFn2==0,Bnum2=None];
        TableForm[{Bnum1,Bnum2,Bana}, TableHeadings -> {{"Nume
    ]
(*Elliptic integral of the first kind*)
```

$$\text{dρeFn}[\rho p\_,\rho\_,\varphi p\_,\varphi\_,zp\_,z\_] := \sum_{q=1}^{2}(-1)^{q}\ (\text{D}[\text{EllipticF}[\phi[\varphi,\varphi p[[$$

$$\text{dφeFn}[\rho p\_,\rho\_,\varphi p\_,\varphi\_,zp\_,z\_] := \sum_{q=1}^{2}(-1)^{q}\ (\text{D}[\text{EllipticF}[\phi[\text{d}\varphi,\varphi p[$$

$$\text{dzeFn}[\rho p\_,\rho\_,\varphi p\_,\varphi\_,zp\_,z\_] := \sum_{q=1}^{2}(-1)^{q}\ (\text{D}[\text{EllipticF}[\phi[\varphi,\varphi p[[$$

$$\text{dρeFt}[\rho p\_,\rho\_,\varphi p\_,\varphi\_,zp\_,z\_] := \sum_{q=1}^{2}(-1)^{q}\ \text{NIntegrate}\Big[\frac{2\ \rho p\,(\text{Z}[z,z}{\text{R}[\rho,\rho p,}$$

$$\text{dzeFt}[\rho p\_,\rho\_,\varphi p\_,\varphi\_,zp\_,z\_] := \sum_{q=1}^{2}(-1)^{q}\ \text{NIntegrate}\Big[-\frac{\text{Z}[z,zp]\,\text{k}[}{\text{R}[\rho,\rho p}$$

$$\text{dρeFa}[\rho p\_,\rho\_,\varphi p\_,\varphi\_,zp\_,z\_] := \sum_{q=1}^{2}(-1)^{q}\frac{\text{Z}[z,zp]^{2}-\rho^{2}+\rho p^{2}}{2\ \rho\ \text{R}[\rho,\rho p,z,zp]^{2}}(\text{Elli}$$

$$\text{dφeFa}[\rho p\_,\rho\_,\varphi p\_,\varphi\_,zp\_,z\_] := \sum_{q=1}^{2}(-1)^{q}\ \frac{1}{2\ \sqrt{(1-\text{k}[\rho,\rho p,z,zp]^{2}}}$$

$$\text{dzeFa}[\rho p\_,\rho\_,\varphi p\_,\varphi\_,zp\_,z\_] := \sum_{q=1}^{2}(-1)^{q}\frac{\text{Z}[z,zp]}{\text{R}[\rho,\rho p,z,zp]^{2}}(\text{Elliptic}$$

```
(*Elliptic integral of the second kind*)
```

$$\text{dρeEn}[\rho p\_,\rho\_,\varphi p\_,\varphi\_,zp\_,z\_] := \sum_{q=1}^{2}(-1)^{q}\ (\text{D}[\text{EllipticE}[\phi[\varphi,\varphi p[[$$

```
dρeEt[ρp_,ρ_,φp_,φ_,zp_,z_] := Sum[(-1)^q NIntegrate[- 2 ρp (Z[z,…  / R[ρ,ρp…, {q,1,2}]

dρeEa[ρp_,ρ_,φp_,φ_,zp_,z_] := Sum[(-1)^q (-2 ρp (Z[z,zp]^2-ρ^2+ρp^2)…) / R[ρ,ρp,z,zp]^4 …, {q,1,2}]

(*Elliptic integral of the third kind*)
dρePn[ρp_,ρ_,φp_,φ_,zp_,z_] := Sum[(-1)^q (D[EllipticPi[κ[dρ,ρp…, {q,1,2}]

dρePt[ρp_,ρ_,φp_,φ_,zp_,z_] := Sum[(-1)^q NIntegrate[((k[ρ,ρ… / (2 ρ R[ρ,…, {q,1,2}]

dρePa[ρp_,ρ_,φp_,φ_,zp_,z_] := Sum[(-1)^q (((1/(2 ρ) - 1/ϱ[ρ,ρp] - ϱ[ρ…/R[ρ,ρp…, {q,1,2}]

(*Regularised beta function*)
dρrBn[ρp_,ρ_,a_,b_,zp_,z_] := (D[BetaRegularized[zs[dρ,ρp,z,…

dρrBa[P_,ρp_,ρ_,a_,b_,zp_,z_] := -2ρ / (Beta[a,b] T[ρ,ρp,z,zp]^2) zs[ρ,…
```

4.0.1 $\frac{\partial}{\partial\rho}$ Elliptic integral of the first kind

In[•]:=

```
Compare2[ρ'[[2]],ρ1,{π/4,4 π/3},φ1,z'[[1]],z1,dρeFn,dρeFt,dρeFa]
```

Out[•]//TableForm=

Numeric Form 1	69.52930154354390
Numeric Form 2	69.52930154354402
Analytic Form	69.52930154354390

4.0.2 $\frac{\partial}{\partial\varphi}$ Elliptic integral of the first kind

In[•]:=

```
Compare2[ρ'[[2]],ρ1,{π/4,4 π/3},φ1,z'[[1]],z1,dφeFn,0,dφeFa]
```

Out[•]//TableForm=

Numeric Form 1	-2.813617119399465
Numeric Form 2	None
Analytic Form	-2.813617119399465

4.0.3 $\frac{\partial}{\partial z}$ Elliptic integral of the first kind

In[∘]:=
```
Compare2[ρ'[[2]],ρ1,{π/4,4π/3},φ1,z'[[1]],z1,dzeFn,dzeFt,dzeFa]
```

Out[∘]//TableForm=

Numeric Form 1	208.7535820767244
Numeric Form 2	208.7535820767247
Analytic Form	208.7535820767244

4.0.4 $\frac{\partial}{\partial \rho}$ Elliptic integral of the second kind

In[∘]:=
```
Compare2[ρ'[[2]],ρ1,{π/4,4π/3},φ1,z'[[1]],z1,dρeEn,dρeEt,dρeEa]
```

Out[∘]//TableForm=

Numeric Form 1	-4.561890931702129
Numeric Form 2	-4.561890931702118
Analytic Form	-4.561890931702129

4.0.5 $\frac{\partial}{\partial \rho}$ Elliptic integral of the third kind

In[∘]:=
```
Compare2[ρ'[[2]],ρ1,{π/4,4π/3},φ1,z'[[1]],z1,dρePn,dρePt,dρePa]
```

Out[∘]//TableForm=

Numeric Form 1	25 513.98100004374
Numeric Form 2	25 513.98100004376
Analytic Form	25 513.98100004374

4.0.6 $\frac{\partial}{\partial \rho}$ Regularised beta function

In[∘]:=
```
Compare[0,ρ'[[2]],ρ1,4,6,z'[[1]],z1,dρrBn,dρrBa,0]//N
```

Out[∘]//TableForm=

Numeric	-124.787
Analytic Form 1	-124.787
Analytic Form 2	None

5.0 Forces between Axially Magnetised Permanent Magnets

5.0.0 Equations

Analytic and Numeric function handles. Returns F={Fx,Fy,Fz}.

In[27]:=

```
FzAna[M_,Mp_,ρp_,ρ_,φp_,φ_,zp_,z_,V_,P_,U_,O_] := (M Mp u0)/(4π) ∑_{m=1}^{2}

FzNum[M_,Mp_,ρp_,ρ_,φp_,φ_,zp_,z_] := (M Mp u0)/(4π) ( ∑_{np=1}^{2} (-1)^np NInte
                                                        ∑_{m=1}^{2} ∑_{np=1}^{2} (-1)^(m+np) N
                                                        ∑_{n=1}^{2} ∑_{qp=1}^{2} (-1)^(n+qp) N
                                                       + ∑_{m=1}^{2} ∑_{mp=1}^{2} (-1)^(m+mp)
                                                       + ∑_{m=1}^{2} ∑_{qp=1}^{2} (-1)^(m+qp)
                                                       + ∑_{q=1}^{2} ∑_{mp=1}^{2} (-1)^(q+mp)
                                                       + ∑_{q=1}^{2} ∑_{qp=1}^{2} (-1)^(q+qp)
```

Special cases of the geometry for the analytic function handle. Replaces FzAna[] or FzNum[].

In[51]:=

```
FzAna[M_,Mp_,ρp_,ρ_,{0,2π},{0,2π},zp_,z_,V_,P_,U_,O_] := 2Mp

FzAna[M_,Mp_,{0,ρp_},{0,ρ_},{0,2π},{0,2π},zp_,z_,V_,P_,U_,O_]

FzNum[M_,Mp_,ρp_,ρ_,{0,2π},{0,2π},zp_,z_] := (M Mp u0)/2 ∑_{m=1}^{2} ∑_{mp=1}^{2} (-
```

Integrands to be solved for FzNum[].

In[32]:=
```
FzIntegrandAS[ρp_,ρ_,φp_,φ_,zp_,z_] := -ρp ρ Z[z,zp] Cos[φ-φp

FzIntegrand1[ρp_,ρ_,φp_,φ_,zp_,z_] := {-ρp Z[z,zp]Cos[φ]G[ρ,
FzIntegrand2[ρp_,ρ_,φp_,φ_,zp_,z_] := {ρp ρ Z[z,zp]Cos[φ]G[ρ
FzIntegrand3[ρp_,ρ_,φp_,φ_,zp_,z_] := {-ρp Z[z,zp]Sin[φ]G[ρ,
FzIntegrand4[ρp_,ρ_,φp_,φ_,zp_,z_] := {0,0,-Z[z,zp] ρ ρp Cos[
FzIntegrand5[ρp_,ρ_,φp_,φ_,zp_,z_] := {0,0,-ρ Z[z,zp] Sin[φ-φ
FzIntegrand6[ρp_,ρ_,φp_,φ_,zp_,z_] := {0,0,Z[z,zp] ρp Sin[φ-φ
FzIntegrand7[ρp_,ρ_,φp_,φ_,zp_,z_] := {0,0,-Z[z,zp] Cos[φ-φp]
```

Summands and ancillary functions for FzAna[].

In[40]:=
```
FzSummand[ρp_,ρ_,φp_,φ_,zp_,z_,V_,P_,U_,O_] := {FzSx[ρp,ρ,φp,
```

$$\texttt{FzSummandAS[}\rho p\texttt{\_,}\rho\texttt{\_,}zp\texttt{\_,}z\texttt{\_] := }\frac{\texttt{Z[}z,zp\texttt{]}}{\texttt{R[}\rho,\rho p,z,zp\texttt{]}}\left(\frac{\overline{\varrho}\texttt{[}\rho,\rho p\texttt{]}^2}{4}\left(\texttt{Elliptic}\right.\right.$$

$$\texttt{FzSx[}\rho p\texttt{\_,}\rho\texttt{\_,}\varphi p\texttt{\_,}\varphi\texttt{\_,}zp\texttt{\_,}z\texttt{\_,}V\texttt{\_,}P\texttt{\_,}U\texttt{\_]:= Sin[}\varphi\texttt{]}\sum_{p=0}^{P}\left(-\varphi p\ \varsigma\texttt{[0,}p\texttt{,}\rho p\texttt{,}\right.$$

$$\texttt{FzSy[}\rho p\texttt{\_,}\rho\texttt{\_,}\varphi p\texttt{\_,}\varphi\texttt{\_,}zp\texttt{\_,}z\texttt{\_,}V\texttt{\_,}P\texttt{\_,}U\texttt{\_]:= Cos[}\varphi\texttt{]}\sum_{p=0}^{P}\left(\varphi p\ \varsigma\texttt{[0,}p\texttt{,}\rho p\texttt{,}\rho\right.$$

$$\texttt{FzSz[}\rho p\texttt{\_,}\rho\texttt{\_,}\varphi p\texttt{\_,}\varphi\texttt{\_,}zp\texttt{\_,}z\texttt{\_,}V\texttt{\_,}P\texttt{\_,}U\texttt{\_,}O\texttt{\_]:= }\sum_{p=0}^{P}\left(\texttt{-Cos[}\varphi-\varphi p\texttt{] (1+2 }p\right.$$

$$\varsigma\texttt{[}\nu\texttt{\_,}p\texttt{\_,}\rho p\texttt{\_,}\rho\texttt{\_,}zp\texttt{\_,}z\texttt{\_]:= If[}\nu\texttt{==0,1,2]}\frac{\rho^2}{2\sqrt{\pi}}\left(\frac{\left((z-zp)^2+\rho^2\right)}{\rho^2}\right)^{\frac{1}{2}-p}$$

$$\varsigma\texttt{[}\nu\texttt{\_,}p\texttt{\_,}u\texttt{\_,}\rho p\texttt{\_,}\rho\texttt{\_,}zp\texttt{\_,}z\texttt{\_]:= If[}\nu\texttt{==0,1,2]}\frac{\rho p^2}{4\sqrt{\pi}}\left(1+\frac{(z-zp)^2}{\rho p^2}\right)^{-p-\frac{\nu}{2}}$$

$$\varsigma\texttt{[}\nu\texttt{\_,}p\texttt{\_,}u\texttt{\_,}o\texttt{\_,}l\texttt{\_,}\rho p\texttt{\_,}\rho\texttt{\_,}zp\texttt{\_,}z\texttt{\_]:= If[}\nu\texttt{==0,1,2] (1+2 }p+\nu\texttt{)}\ \frac{\rho\ \rho p}{}$$

Singularities in the summands of FzAna[].

In[48]:=
```
(*On the shell plane & axisymmetric*)
```
$$\texttt{FzSummandAS}[\rho p\_,\rho p\_,zp\_,z\_] := -\frac{Z[z,zp]\,\rho p^2}{R[\rho p,\rho p,z,zp]}\ \texttt{EllipticD}\big[k[\rho p,$$

5.0.1 An axisymmetric force

Hollow rings.

In[49]:=
$$\texttt{ResultTableForce}\big[800*10^3,-955*10^3,\{5*10^{-3},10*10^{-3}\},\{5*10^{-3},8*1$$

	Fx	Fy	Fz
Analytic	0	0	17.43010334628681
Numeric	0	0	17.43010333790363
Comparison 8dp	0	0	0

Solid rings (m-summations removed).

In[54]:=
$$\texttt{ResultTableForce}\big[800*10^3,-955*10^3,\{0,10*10^{-3}\},\{0,8*10^{-3}\},\{0,2\pi$$

	Fx	Fy	Fz
Analytic	0	0	21.90252744226638
Numeric	0	0	21.90252744338770
Comparison 8dp	0	0	0

5.0.2 A non-axisymmetric force

The partial sum is computed without an algorithm, and a low number of terms have been chosen.

In[∘]:=
$$\texttt{ResultTableForce}\big[800*10^3,-955*10^3,\{5*10^{-3},10*10^{-3}\},\{5*10^{-3},8*10$$

	Fx	Fy
Analytic	-0.1737413964903685	0.1709873453068254
Numeric	-0.173396819398153	0.17112068131034
Comparison 8dp	-0.000344577092216	-0.000133336003508