The Magnetic Field from Cylindrical Arc Coils and Magnets: A Compendium with New Analytic Solutions for Radial Magnetisation and Azimuthal Current

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Supplemental material. An online repository can be found at: https://github.com/AUMAG/mag-cyl-field (once accepted for publication). The online repository may contain future updates. This document contains all analytic/integral formulations and explicit analytic solutions for the PM's (Section 1) and Coils (Section 2). Analytic solutions are compared to the original numeric integral to 8 decimal places. Comparison with FEA is given in the published article. For completeness, additional comparisons are given in Sections 3 and 4 for the Green's function integrals and elliptic integral derivatives. Section 5 contains the analytic force equations that were compared to semi-analytical field methods.

- If you are reading this in PDF format, note the code is truncated. There is a Mathematica notebook '.nb' file that contains full functionality and a package '.wl' file that contains the nomenclature. If you do not have a paid/trial version of Mathematica, this notebook can be opened in a free player (e.g Wolfram Player https://www.wolfram.com/player/) in order to view/copy the equations. There are also built-in or addon functions that can convert the equations to MATLAB, LATEX, Python, etc...
- The code below is not written to optimise computational speed, but to be readable, portable, and correct.
- For simplicity, the wrapper functions provided do not handle list inputs for ρ, φ, z.
- Similarly, algorithms for computing a partial sum of β,δ (η,ζ,ι) are omitted and functions take a direct input for the number of terms. The number of terms are identical between all series at one field point; therefore, we only evaluate the volume coil (Section 2.3) for P=150 terms to reduce computation time. Mathematica is holding these terms symbolically before converting to a numeric value, where there ends up being a large number of '0.' terms truncated to $MachinePrecision. This is not an issue with an algorithm and adaptive convergence parameters.
- Mathematica uses complex numbers for the evaluation of ArcTan, ArcTanh,... and as such these return real numbers of the form a+0.i when converted to a numerical value using '//N'. As such we additionally use '//Chop' to remove numerical 0.i terms.
- We also use '//Chop' for the comparison of the analytic and numeric result, as dependent on the numeric procedure, we only expect accuracy to a certain precision.
- Function handles are called from 'ResultTable[]' to return either the cylindrical or Cartesian (on axis) field components at a particular point.
- 0.3 Geometry, 0.4 Constants, and 0.5 Field Points can be freely changed. Default values conform with those of the article cited in 0.1.

0.1 Citations for this work

M. Forbes, W.S.P Robertson, A.C. Zander, J.J.H. Paulides, "The Magnetic Field from Cylindrical Arc Coils and Magnets: A Compendium with New Analytic Solutions for Radial Magnetisation and Azimuthal Current"

```
@Article {Forbes2024,
    author = {Forbes, M. and Robertson, W.S.P. and Zander, A.C. and Paulides, J.J.H},
    journal = {Advanced Physics Research},
    title = {The Magnetic Field from Cylindrical Arc Coils and Magnets: A Compendium with New Analytic Solutions for Radial Magnetisation and Azimuthal Current},
    doi = {10.1002/apxr.202300136},
    publisher = {Wiley},
 }
```

0.2 Variables and Functions

In[ ]:=

```
SetDirectory[NotebookDirectory[]];
<< Nomenclature`
<< Carlson` (*Used for some backwards compatibility with a special case of EllipticF[ϕ,1], and not required in 12.3 or later: http
IntOptions ={WorkingPrecision→$MachinePrecision, AccuracyGoal→53Log[10,2], PrecisionGoal→24Log[10,2]};
```

0.3 Geometry

In[ ]:=

$\rho' = \left\{\frac{3}{1000}, \frac{8}{1000}\right\}$; $\varphi' = \left\{-\frac{\pi}{6}, \frac{3\pi}{5}\right\}$; $z' = \left\{\frac{1}{1000}, \frac{5}{1000}\right\}$; (*Definite integral limits*)

0.4 Constants

In[∘]:=
```
u0 = 4π*10^-7;
M = 955*10^3; (*A/m*)
Iφ = 20; (*A*)
Kφ = 4*10^4; (*A/m*)
Jφ = 1*10^6; (*A/m^2*)
φ☆ = π/6; (*Diametric magnetisation direction*)
P = 150;(*Number of terms in partial sum*)
```

0.4 Field Points

Seven field points to test all equations in article.

In[∘]:=
```
ρ1 = 9/1000; φ1 = 5π/24; z1 = 31/10000; (*Standard*)
ρ2 = 7/1000; φ2 = φ1; z2 = z1; (*Inside magnet/coil*)
ρ3 = 0; φ3 = 0; z3 = z'[[2]]; (*On magnet/coil axis*)
ρ4 = ρ1; φ4 = φ'[[1]]; z4 = z'[[2]]; (*Singular plane 1*)
ρ5 = ρ'[[2]]; φ5 = -φ1; z5 = z'[[2]]; (*Singular plane 2*)
ρ6 = ρ'[[2]]; φ6 = φ'[[1]]; z6 = 6/1000; (*Singular plane 3*)
ρ7 = 2/1000; φ7 = φ1; z7 = z1; (*Coil high field area*)
```

0.5 Results Format

In[∘]:=
```
MagCylField[Md_,Mρ_,Mφ_,Mz_,φ☆_,P_,ρ'_,ρ_,φ'_,φ_,z'_,z_] :=
    Module[{},
        If[Md ≠ 0,(*Diametric*)
            ResultTable[Md,φ☆,P,ρ',ρ,φ',φ,z',z,"M⊥",BdAna,BdNum,BdAnaAxis,BdNumAxis];
```

```
        ];
        If[Mρ ≠ 0,(*Radial*)
            ResultTable[Mρ,φ☆,P,ρ',ρ,φ',φ,z',z,"Mρ",BρAna,BρNum,BρAnaAxis,BρNumAxis];
        ];
        If[Mφ ≠ 0,(*Azimuthal*)
            ResultTable[Mφ,φ☆,P,ρ',ρ,φ',φ,z',z,"Mφ",BφAna,BφNum,BφAnaAxis,BφNumAxis];
        ];
        If[Mz ≠ 0,(*Azimuthal*)
            ResultTable[Mz,φ☆,P,ρ',ρ,φ',φ,z',z,"Mz",BzAna,BzNum,BzAnaAxis,BzNumAxis];
        ];
        Null
    ]

CoilFilamentField[I_,ρ'_,ρ_,φ'_,φ_,z'_,z_]:= ResultTable[I,0,0,ρ'[[2]],ρ,φ',φ,z'[[1]],z,"Iφ",BiAna,BiNum,BiAnaAxis,BiNumAxis];
CoilDiscField[K_,P_,ρ'_,ρ_,φ'_,φ_,z'_,z_] := ResultTable[K,0,P,ρ',ρ,φ',φ,z'[[1]],z,"Kφ",BkAna,BkNum,BkAnaAxis,BkNumAxis];
CoilShellField[K_,ρ'_,ρ_,φ'_,φ_,z'_,z_]   := ResultTable[K,0,0,ρ'[[2]],ρ,φ',φ,z',z,"Kφ",BsAna,BsNum,BsAnaAxis,BsNumAxis];
CoilCylField [J_,P_,ρ'_,ρ_,φ'_,φ_,z'_,z_] := ResultTable[J,0,P,ρ',ρ,φ',φ,z',z,"Jφ",BcAna,BcNum,BcAnaAxis,BcNumAxis];

ResultTable[M_,φ☆_,P_,ρ'_,ρ_,φ'_,φ_,z'_,z_,mag_,anaFn_,numFn_,anaFnAxis_,numFnAxis_] :=
    Module[{Bana,Bnum,Bcom,heading,tab},
        If[ρ==0,
            Bana = N[anaFnAxis[M,φ☆,ρ',φ',z',z],$MachinePrecision] //Chop;
            Bnum =   numFnAxis[M,φ☆,ρ',φ',z',z];
            Bcom = Chop[Bana-Bnum,10^-8];
            heading = {"Bx","By","Bz"} (*Axis, Cartesian*),
            Bana = N[anaFn[M,φ☆,P,ρ',ρ,φ',φ,z',z],$MachinePrecision] //Chop;
            Bnum =   numFn[M,φ☆,ρ',ρ,φ',φ,z',z];
            Bcom = Chop[Bana-Bnum,10^-8];
            heading = {"Bρ","Bφ","Bz"} (*Cylindrical*);
        ];
        tab = TableForm[{Bana,Bnum,Bcom}, TableHeadings -> {{"Analytic", "Numeric", "Comparison 8dp"}, heading}];
        CellPrint@ExpressionCell[RawBoxes[ToBoxes[tab] /. (x : GridBox[___]) :>  MapAt[mag &, x, {1, 1, 1}]], "Output"]
    ]
```

```
ResultTableForce[M_,Mp_,ρ'_,ρ_,φ'_,φ_,z'_,z_,V_,P_,U_,O_,anaFn_,numFn_] :=
    Module[{Fana,Fnum,Fcom,heading,tab},
            Fana = N[anaFn[M,Mp,ρ',ρ,φ',φ,z',z,V,P,U,O],$MachinePrecision] //Chop;
            Fnum =   numFn[M,Mp,ρ',ρ,φ',φ,z',z];
            Fcom = Chop[Fana-Fnum,10^-8];
            heading = {"Fx","Fy","Fz"} ;
        tab = TableForm[{Fana,Fnum,Fcom}, TableHeadings -> {{"Analytic", "Numeric", "Comparison 8dp"}, heading}];
        CellPrint@ExpressionCell[tab, "Output"]
    ]
```

0.6 Example field solutions

Evaluate the same magnet geometry, at the same field point, with equal magnetisation in the diametric, radial, azimuthal, and axial directions. Creates a table for each magnetisation direction and compares to 8 decimal places the analytic solution and numeric integral.

In[ ]:=
```
MagCylField[M,M,M,M,φ☆,P,ρ',ρ1,φ',φ1,z',z1] (*Note: This expression will not evaluate without the functions in Section 1 initiali
```

M⊥	Bρ	Bφ	Bz
Analytic	0.3588672233516815	0.01541822529046387	0.01232062003099335
Numeric	0.3588672228056431	0.01541822528724981	0.01232062003001142
Comparison 8dp	0	0	0

Mρ	Bρ	Bφ	Bz
Analytic	0.2448856704190372	-0.0003060260329494243	0.00965560666092249
Numeric	0.2448856701063103	-0.00030602603308330	0.0096555968222767
Comparison 8dp	0	0	0

Mφ	Bρ	Bφ	Bz
Analytic	-0.0009866463614534913	-0.02735309187411041	-0.00002238181997612098
Numeric	-0.00098664636141892	-0.02735309187383460	-0.0000223818199760637
Comparison 8dp	0	0	0

Mz	Bρ	Bφ	Bz
Analytic	0.01242509986271859	-0.00001393298933826625	-0.2358090801524554
Numeric	0.01242509986169668	-0.0000139329893381925	-0.2358090801086101
Comparison 8dp	0	0	0

0.7 Timed solutions

Altered tables to those in Section 0.5, including RepeatedTiming[] and an additional table to show computational efficiency.

In[∘]:=
```
MagCylFieldTimed[Md_,Mρ_,Mφ_,Mz_,φ☆_,P_,ρ'_,ρ_,φ'_,φ_,z'_,z_] :=
    Module[{},
        If[Md ≠ 0,(*Diametric*)
            ResultTableTimed[Md,φ☆,P,ρ',ρ,φ',φ,z',z,"M⊥",BdAna,BdNum,BdAnaAxis,BdNumAxis];
        ];
        If[Mρ ≠ 0,(*Radial*)
            ResultTableTimed[Mρ,φ☆,P,ρ',ρ,φ',φ,z',z,"Mρ",BρAna,BρNum,BρAnaAxis,BρNumAxis];
        ];
        If[Mφ ≠ 0,(*Azimuthal*)
            ResultTableTimed[Mφ,φ☆,P,ρ',ρ,φ',φ,z',z,"Mφ",BφAna,BφNum,BφAnaAxis,BφNumAxis];
        ];
        If[Mz ≠ 0,(*Azimuthal*)
            ResultTableTimed[Mz,φ☆,P,ρ',ρ,φ',φ,z',z,"Mz",BzAna,BzNum,BzAnaAxis,BzNumAxis];
        ];
        Null
    ]
ResultTableTimed[M_,φ☆_,P_,ρ'_,ρ_,φ'_,φ_,z'_,z_,mag_,anaFn_,numFn_,anaFnAxis_,numFnAxis_] :=
    Module[{Bana,Bnum,Bcom,heading,tab,ta,ba,tn,bn,time},
        If[ρ==0,
            {ta,ba} = RepeatedTiming[
                Bana = N[anaFnAxis[M,φ☆,ρ',φ',z',z],$MachinePrecision] //Chop;];
            {tn,bn} = RepeatedTiming[
                Bnum =   numFnAxis[M,φ☆,ρ',φ',z',z];];
            Bcom = Chop[Bana-Bnum,10^-8];
            heading = {"Bx","By","Bz"} (*Axis, Cartesian*),
            {ta,ba} = RepeatedTiming[
                Bana = N[anaFn[M,φ☆,P,ρ',ρ,φ',φ,z',z],$MachinePrecision] //Chop;];
            {tn,bn} = RepeatedTiming[
                Bnum =   numFn[M,φ☆,ρ',ρ,φ',φ,z',z];];
            Bcom = Chop[Bana-Bnum,10^-8];
```

```
            heading = {"Bρ","Bφ","Bz"} (*Cylindrical*);
        ];
        time = {ta,tn};
        tab = TableForm[{Bana,Bnum,Bcom}, TableHeadings -> {{"Analytic", "Numeric", "Comparison 8dp"}, heading}];
        CellPrint@ExpressionCell[RawBoxes[ToBoxes[tab] /. (x : GridBox[___]) :> MapAt[mag &, x, {1, 1, 1}]]×TableForm[{time},Tal

    ]
```

In[ ]:=
```
MagCylFieldTimed[M,M,M,M,φ☆,P,ρ′,ρ1,φ′,φ1,z′,z1] (*Note: This expression will not evaluate without the functions in Section 1 ini
```

M⊥	Bρ	Bφ	Bz
Analytic	0.3588672233516815	0.01541822529046387	0.01232062003099335
Numeric	0.3588672228056431	0.01541822528724981	0.01232062003001142
Comparison 8dp	0	0	0

	Analytic	Numeric
Time	0.018	3.6

Mρ	Bρ	Bφ	Bz
Analytic	0.2448856704190372	−0.0003060260329494243	0.00965560666092249
Numeric	0.2448856701063103	−0.00030602603308330	0.0096555968222767
Comparison 8dp	0	0	0

	Analytic	Numeric
Time	1.03	4.78

Mφ	Bρ	Bφ	Bz
Analytic	−0.0009866463614534913	−0.02735309187411041	−0.00002238181997612098
Numeric	−0.00098664636141892	−0.02735309187383460	−0.0000223818199760637
Comparison 8dp	0	0	0

	Analytic	Numeric
Time	0.0050	0.0959

Mz	Bρ	Bφ	Bz
Analytic	0.01242509986271859	−0.00001393298933826625	−0.2358090801524554
Numeric	0.01242509986169668	−0.0000139329893381925	−0.2358090801086101
Comparison 8dp	0	0	0

	Analytic	Numeric
Time	0.0076	16.3

1.0 Permanent Magnets

1.1 Diametric Magnetisation

1.1.0 Equations

Analytic and Numeric function handles. Returns B={Bρ,Bφ,Bz} or B={Bx,By,Bz} (on axis).

In[ ]:=

```
BdAna[M_,φ☆_,P_,ρp_,ρ_,φp_,φ_,zp_,z_] := (M u0)/(4π) (Sum[Sum[Sum[(-1)^(m+n+q) BdSummand[φ☆,ρp[[m]],ρ,φp[[q]],φ,zp[[n]],z], {q,1,2}], {n,1,2}], {m,1,2}] + Md[φ☆,ρp,ρ,φp,φ,z

BdAnaAxis[M_,φ☆_,ρp_,φp_,zp_,z_] := (M u0)/(4π) (Sum[Sum[Sum[(-1)^(m+n+q) BdSummandAxis[φ☆,ρp[[m]],φp[[q]],zp[[n]],z], {q,1,2}], {n,1,2}], {m,1,2}] + MdAxis[φ☆,ρp,zp,z])

BdNum[M_,φ☆_,ρp_,ρ_,φp_,φ_,zp_,z_] := (M u0)/(4π) (Sum[(-1)^m NIntegrate[BdIntegrand1[φ☆,ρp[[m]],ρ,dφp,φ,dzp,z],{dzp,zp[[1]],zp[[2]]},{d
    + Sum[(-1)^q NIntegrate[BdIntegrand2[φ☆,dρp,ρ,φp[[q]],φ,dzp,z],{dzp,zp[[1]],zp[[2]]},{d
    + Md[φ☆,ρp,ρ,φp,φ,zp,z])

BdNumAxis[M_,φ☆_,ρp_,φp_,zp_,z_] := (M u0)/(4π) (Sum[(-1)^m NIntegrate[BdIntegrandAxis1[φ☆,ρp[[m]],dφp,dzp,z],{dzp,zp[[1]],zp[[2]]},{dφp,
    + Sum[(-1)^q NIntegrate[BdIntegrandAxis2[φ☆,dρp,φp[[q]],dzp,z],{dzp,zp[[1]],zp[[2]
    + MdAxis[φ☆,ρp,zp,z])
```

Magnetisation vector for field inside magnet. Returns M={Mρ,Mφ,Mz}.

In[ ]:=

```
Md[φ☆_,ρp_,ρ_,φp_,φ_,zp_,z_] := If[InsideVolume[ρp,ρ,φp,φ,zp,z],4π{Cos[φ☆-φ],Sin[φ☆-φ],0},{0,0,0}]

MdAxis[φ☆_,ρp_,zp_,z_] := If[InsideVolumeAxis[ρp,zp,z],4π{Cos[φ☆],Sin[φ☆],0},{0,0,0}]
```

Special cases of the geometry for the analytic function handle. Replaces BdAna[] (Cylindrical) or BdAnaAxis[] (Cartesian, on axis).

In[•]:=
```
BdAna[M_,φ☆_,P_,ρp_,ρ_,{0,2π},φ_,zp_,z_] := (M u0)/(4π) (Sum[Sum[(-1)^(m+n) BdSummandAS[φ☆,ρp[[m]],ρ,φ,zp[[n]],z], {m,1,2}], {n,1,2}] + Md[φ☆,ρp,ρ,{0,2π},π,zp,z
BdAnaAxis[M_,φ☆_,ρp_,{0,2π},zp_,z_] := (M u0)/(4π) (Sum[Sum[(-1)^(m+n) BdSummandAxis[φ☆,ρp[[m]],{0,2π},zp[[n]],z], {m,1,2}], {n,1,2}] + MdAxis[φ☆,ρp,zp,z])
BdAna[M_,φ☆_,P_,{0,ρp_},ρ_,φp_,φ_,zp_,z_] := (M u0)/(4π) (Sum[Sum[(-1)^(n+q) (BdSummand1[φ☆,ρp,ρ,φp[[q]],φ,zp[[n]],z] + Sum[(-1)^m BdSummand2[φ☆, {m,1,2}] ..., {n,1,2}], {q,1,2}]
BdAna[M_,φ☆_,P_,{0,ρp_},ρ_,{0,2π},φ_,zp_,z_] := (M u0)/(4π) (Sum[(-1)^n BdSummandAS[φ☆,ρp,ρ,φ,zp[[n]],z], {n,1,2}] + Md[φ☆,{0,ρp},ρ,{0,2π},π,zp,z]
```

Integrands to be solved for BdNum[] (Cylindrical) and BdNumAxis[] (Cartesian, on axis).

In[•]:=
```
BdIntegrand1[φ☆_,ρp_,ρ_,φp_,φ_,zp_,z_] := {BdIρ1[φ☆,ρp,ρ,φp,φ,zp,z], BdIφ1[φ☆,ρp,ρ,φp,φ,zp,z], BdIz1[φ☆,ρp,ρ,φp,φ,zp,z]}
BdIntegrand2[φ☆_,ρp_,ρ_,φp_,φ_,zp_,z_] := {BdIρ2[φ☆,ρp,ρ,φp,φ,zp,z], BdIφ2[φ☆,ρp,ρ,φp,φ,zp,z], BdIz2[φ☆,ρp,ρ,φp,φ,zp,z]}

BdIρ1[φ☆_,ρp_,ρ_,φp_,φ_,zp_,z_] := ρp Cos[φ☆-φp] (ρ-ρp Cos[Φ[φ,φp]]) G[ρ,ρp,φ,φp,z,zp]^3
BdIρ2[φ☆_,ρp_,ρ_,φp_,φ_,zp_,z_] := Sin[φ☆-φp] (ρ-ρp Cos[Φ[φ,φp]]) G[ρ,ρp,φ,φp,z,zp]^3
BdIφ1[φ☆_,ρp_,ρ_,φp_,φ_,zp_,z_] := ρp^2 Cos[φ☆-φp] Sin[Φ[φ,φp]] G[ρ,ρp,φ,φp,z,zp]^3
BdIφ2[φ☆_,ρp_,ρ_,φp_,φ_,zp_,z_] := ρp Sin[φ☆-φp] Sin[Φ[φ,φp]] G[ρ,ρp,φ,φp,z,zp]^3
BdIz1[φ☆_,ρp_,ρ_,φp_,φ_,zp_,z_] := ρp Cos[φ☆-φp] Z[z,zp]  G[ρ,ρp,φ,φp,z,zp]^3
BdIz2[φ☆_,ρp_,ρ_,φp_,φ_,zp_,z_] := Sin[φ☆-φp] Z[z,zp] G[ρ,ρp,φ,φp,z,zp]^3

BdIntegrandAxis1[φ☆_,ρp_,φp_,zp_,z_] := {BdIAxisx1[φ☆,ρp,φp,zp,z], BdIAxisy1[φ☆,ρp,φp,zp,z], BdIAxisz1[φ☆,ρp,φp,zp,z]}
BdIntegrandAxis2[φ☆_,ρp_,φp_,zp_,z_] := {BdIAxisx2[φ☆,ρp,φp,zp,z], BdIAxisy2[φ☆,ρp,φp,zp,z], BdIAxisz2[φ☆,ρp,φp,zp,z]}

BdIAxisx1[φ☆_,ρp_,φp_,zp_,z_] := -ρp^2 Cos[φ☆-φp] Cos[φp] (Z[z,zp]^2+ρp^2)^(-3/2)
BdIAxisx2[φ☆_,ρp_,φp_,zp_,z_] := -ρp  Sin[φ☆-φp] Cos[φp] (Z[z,zp]^2+ρp^2)^(-3/2)
BdIAxisy1[φ☆_,ρp_,φp_,zp_,z_] := -ρp^2 Cos[φ☆-φp] Sin[φp] (Z[z,zp]^2+ρp^2)^(-3/2)
BdIAxisy2[φ☆_,ρp_,φp_,zp_,z_] := -ρp Sin[φ☆-φp] Sin[φp] (Z[z,zp]^2+ρp^2)^(-3/2)
BdIAxisz1[φ☆_,ρp_,φp_,zp_,z_] := ρp Z[z,zp] Cos[φ☆-φp] (Z[z,zp]^2+ρp^2)^(-3/2)
BdIAxisz2[φ☆_,ρp_,φp_,zp_,z_] := Z[z,zp] Sin[φ☆-φp] (Z[z,zp]^2+ρp^2)^(-3/2)
```

Summands for BdAna[] and BdAnaAxis[].

In[•]:=
```
BdSummand[φ☆_,ρp_,ρ_,φp_,φ_,zp_,z_] := {BdSρ1[φ☆,ρp,ρ,φp,φ,zp,z] + BdSρ2[φ☆,ρp,ρ,φp,φ,zp,z], BdSφ1[φ☆,ρp,ρ,φp,φ,zp,z] + BdSφ2
BdSummand1[φ☆_,ρp_,ρ_,φp_,φ_,zp_,z_] := {BdSρ1[φ☆,ρp,ρ,φp,φ,zp,z], BdSφ1[φ☆,ρp,ρ,φp,φ,zp,z], BdSz1[φ☆,ρp,ρ,φp,φ,zp,z]}
```

```
BdSummand2[φ☆_,ρp_,ρ_,φp_,φ_,zp_,z_] := {BdSρ2[φ☆,ρp,ρ,φp,φ,zp,z], BdSφ2[φ☆,ρp,ρ,φp,φ,zp,z], BdSz2[φ☆,ρp,ρ,φp,φ,zp,z]}

BdSρ1[φ☆_,ρp_,ρ_,φp_,φ_,zp_,z_] := 1/ρ (ρp Z[z,zp])/R[ρ,ρp,z,zp] Cos[φ☆-φ] (2EllipticDT[ϕ[φ,φp],k[ρ,ρp,z,zp]^2]-(1+(2 ϱ̅[ρ,ρp])/(κ[ρ, ρp]^2 ϱ[ρ,ρp]))EllipticF
BdSρ2[φ☆_,ρp_,ρ_,φp_,φ_,zp_,z_] := Which[(φp==φ+π) || (φp==φ-π),ArcTanh[R[ρ,ρp,z,zp]/Z[z,zp]] Sin[φ☆-φp],True,Sin[φ☆-φp] (ArcTan[ϒ[ρ,ρ
BdSφ1[φ☆_,ρp_,ρ_,φp_,φ_,zp_,z_] := 2/ρ ρp Z[z,zp] Sin[φ☆-φ]/R[ρ,ρp,z,zp] (ϱ̅[ρ,ρp]^2/(4 ρ ρp) (EllipticPiT[κ[ρ,ρp]^2,ϕ[φ,φp],k[ρ,ρp,z,zp]^2]-EllipticFT[ϕ
BdSφ2[φ☆_,ρp_,ρ_,φp_,φ_,zp_,z_] := Which[(φp==φ+π)|| (φp==φ-π),0,True,Sin[φ☆-φp] (ArcTan[ϒ[ρ,ρp,φ,φp,z,zp]]Cos[Φ[φ,φp]] + Sin[Φ
BdSz1[φ☆_,ρp_,ρ_,φp_,φ_,zp_,z_] := (2 ρp Cos[φ☆-φ])/R[ρ,ρp,z,zp] (EllipticFT[ϕ[φ,φp],k[ρ,ρp,z,zp]^2]-2EllipticDT[ϕ[φ,φp],k[ρ,ρp,z,zp]^2])+ Sin[
BdSz2[φ☆_,ρp_,ρ_,φp_,φ_,zp_,z_] := Log[ρp-ρ Cos[Φ[φ,φp]]+G[ρ,ρp,φ,φp,z,zp]^-1] Sin[φ☆-φp]

BdSummandAxis[φ☆_,ρp_,φp_,zp_,z_] := {BdSAxisx[φ☆,ρp,φp,zp,z], BdSAxisy[φ☆,ρp,φp,zp,z], BdSAxisz[φ☆,ρp,φp,zp,z]}

BdSAxisx[φ☆_,ρp_,φp_,zp_,z_] := (Z[z,zp] (2φp Cos[φ☆]-Sin[φ☆-2φp]))/(4 Sqrt[Z[z,zp]^2+ρp^2])-ArcTanh[Sqrt[Z[z,zp]^2+ρp^2]/Z[z,zp]]Cos[φp]Sin[φ☆-φp]
BdSAxisy[φ☆_,ρp_,φp_,zp_,z_] := (Z[z,zp] (-Cos[φ☆-2 φp]+2 φp Sin[φ☆]))/(4 Sqrt[Z[z,zp]^2+ρp^2])-ArcTanh[Sqrt[Z[z,zp]^2+ρp^2]/Z[z,zp]]Sin[φ☆-φp]Sin[φp]
BdSAxisz[φ☆_,ρp_,φp_,zp_,z_] := (-ρp/Sqrt[Z[z,zp]^2+ρp^2]+ArcTanh[ρp/Sqrt[Z[z,zp]^2+ρp^2]]) Sin[φ☆-φp]


BdSummandAS[φ☆_,ρp_,ρ_,φ_,zp_,z_] := {BdSASρ[φ☆,ρp,ρ,φ,zp,z], BdSASφ[φ☆,ρp,ρ,φ,zp,z], BdSASz[φ☆,ρp,ρ,φ,zp,z]}

BdSASρ[φ☆_,ρp_,ρ_,φ_,zp_,z_] := 2Cos[φ☆-φ] (ρp Z[z,zp])/(ρ R[ρ,ρp,z,zp]) ((1+(2 ϱ̅[ρ,ρp])/(κ[ρ, ρp]^2 ϱ[ρ,ρp]))EllipticK[k[ρ,ρp,z,zp]^2]-2EllipticD[k[ρ,ρp,z,z
BdSASφ[φ☆_,ρp_,ρ_,φ_,zp_,z_] := 4 Sin[φ☆-φ] (ρp Z[z,zp])/(ρ R[ρ,ρp,z,zp]) (EllipticD[k[ρ,ρp,z,zp]^2]+ϱ̅[ρ,ρp]^2/(4 ρ ρp) (EllipticK[k[ρ,ρp,z,zp]^2]-Ellipti
```

```
BdSASz[φ☆_,ρp_,ρ_,φ_,zp_,z_] := (4 ρp Cos[φ☆-φ])/R[ρ,ρp,z,zp] (2EllipticD[k[ρ,ρp,z,zp]^2]-EllipticK[k[ρ,ρp,z,zp]^2])
```

Singularities in the summands of BdAna[] or BdAnaAxis[].

```
In[•]:= (*Along the axis*)
BdSAxisx[φ☆_,ρp_,φp_,zp_,zp_]:=0
BdSAxisy[φ☆_,ρp_,φp_,zp_,zp_]:=0
BdSAxisz[φ☆_,ρp_,φp_,zp_,zp_]:=(Log[ρp]-1) Sin[φ☆-φp]
(*Along the axis & axisymmetric*)
BdSAxisx[φ☆_,ρp_,{0,2π},zp_,z_] := (π Z[z,zp] Cos[φ☆])/Sqrt[Z[z,zp]^2+ρp^2]
BdSAxisy[φ☆_,ρp_,{0,2π},zp_,z_] := (π Z[z,zp] Sin[φ☆])/Sqrt[Z[z,zp]^2+ρp^2]
BdSAxisz[φ☆_,ρp_,{0,2π},zp_,z_] := 0
BdSAxisx[φ☆_,ρp_,{0,2π},zp_,zp_] := 0
BdSAxisy[φ☆_,ρp_,{0,2π},zp_,zp_] := 0
BdSAxisz[φ☆_,ρp_,{0,2π},zp_,zp_] := 0
(*On the shell plane*)
EllipticPiT[1,ϕ_,k_] := EllipticFT[ϕ,k]-1/(1-k) (EllipticET[ϕ,k]-Sqrt[1-k Sin[ϕ]^2] Tan[ϕ]);
(*On the shell plane & axisymmetric*)
BdSASρ[φ☆_,ρp_,ρp_,φ_,zp_,z_] := 2 (Z[z,zp] Cos[φ☆-φ])/R[ρp,ρp,z,zp] (EllipticK[k[ρp,ρp,z,zp]^2]-2 EllipticD[k[ρp,ρp,z,zp]^2])
BdSASφ[φ☆_,ρp_,ρp_,φ_,zp_,z_] := 4 Z[z,zp] Sin[φ☆-φ]/R[ρp,ρp,z,zp] EllipticD[k[ρp,ρp,z,zp]^2]
(*On the section plane*)
BdSρ2[φ☆_,ρp_,ρ_,φp_,φp_,zp_,z_] := -ArcTanh[OverBar[R][ρ,ρp,z,zp]/Z[z,zp]] Sin[φ☆-φp]
BdSφ2[φ☆_,ρp_,ρ_,φp_,φp_,zp_,z_] := 0
(*On the disc plane*)
BdSρ1[φ☆_,ρp_,ρ_,φp_,φ_,zp_,zp_] := 0
BdSρ2[φ☆_,ρp_,ρ_,φp_,φ_,zp_,zp_] := 0
```

```
BdSφ1[φ☆_,ρp_,ρ_,φp_,φ_,zp_,zp_] := 0
BdSφ2[φ☆_,ρp_,ρ_,φp_,φ_,zp_,zp_] := 0
(*On the axial line*)
BdSρ1[φ☆_,ρp_,ρp_,φp_,φp_,zp_,z_] := Cos[φ☆-φp] Z[z,zp]/R[ρp,ρp,z,zp] (EllipticK[k[ρp,ρp,z,zp]^2]-2EllipticD[k[ρp,ρp,z,zp]^2]) - 1/(2 ρp^2) Sin[
BdSρ2[φ☆_,ρp_,ρp_,φp_,φp_,zp_,z_] := -Log[Abs[Z[z,zp]]]Sign[Z[z,zp]]Sin[φ☆-φp]
BdSφ1[φ☆_,ρp_,ρp_,φp_,φp_,zp_,z_] := 2 Sin[φ☆-φp] Z[z,zp]/R[ρp,ρp,z,zp] EllipticD[k[ρp,ρp,z,zp]^2]-Sign[Z[z,zp]]Cos[φ☆-φp] (Z[z,zp]^2/(2 ρp^2) + 
BdSφ2[φ☆_,ρp_,ρp_,φp_,φp_,zp_,z_] := 0
(*On the azimuthal line*)
BdSρ1[φ☆_,ρp_,ρp_,φp_,φ_,zp_,zp_] := 0
BdSρ2[φ☆_,ρp_,ρp_,φp_,φ_,zp_,zp_] := 0
BdSφ1[φ☆_,ρp_,ρp_,φp_,φ_,zp_,zp_] := 0
BdSφ2[φ☆_,ρp_,ρp_,φp_,φ_,zp_,zp_] := 0
BdSz1[φ☆_,ρp_,ρp_,φp_,φ_,zp_,zp_] := Cos[φ☆-φ] (ArcTanh[Sin[ϕ[φ,φp]]]-2 Sin[ϕ[φ,φp]]) Sign[Φ[φ,φp]]-(2-√2 √(1-Cos[Φ[φ,φp]])
(*On the radial line*)
BdSρ1[φ☆_,ρp_,ρ_,φp_,φp_,zp_,zp_] := 0
BdSρ2[φ☆_,ρp_,ρ_,φp_,φp_,zp_,zp_] := 0
BdSφ1[φ☆_,ρp_,ρ_,φp_,φp_,zp_,zp_] := 0
BdSφ2[φ☆_,ρp_,ρ_,φp_,φp_,zp_,zp_] := 0
BdSz2[φ☆_,ρp_,ρ_,φp_,φp_,zp_,zp_] := -Sign[ϱ[ρ,ρp]]Log[Abs[ϱ[ρ,ρp]]] Sin[φ☆-φp]
```

1.1.1 Standard - Outside Magnet

In[]:=

```
MagCylField[M,0,0,0,φ☆,0,ρ',ρ1,φ',φ1,z',z1]
```

M⊥	Bρ	Bφ	Bz
Analytic	0.3588672233516815	0.01541822529046387	0.01232062003099335
Numeric	0.3588672228056431	0.01541822528724981	0.01232062003001142
Comparison 8dp	0	0	0

1.1.2 Special Case a. - Inside Magnet

```
In[ ]:= MagCylField[M,0,0,0,φ☆,0,ρ',ρ2,φ',φ2,z',z2]
```

M ⊥	Bρ	Bφ	Bz
Analytic	0.6966643256463426	−0.1368270731830296	0.01255680063730398
Numeric	0.6966643258557762	−0.13682707315087393	0.01255680064801926
Comparison 8dp	0	0	0

1.1.3 Special Case b. - On Magnet Axis

```
In[ ]:= MagCylField[M,0,0,0,φ☆,0,ρ',ρ3,φ',φ3,z',z3]
```

M ⊥	Bx	By	Bz
Analytic	0.06907925427658275	0.05798271989375209	−0.0924213908277805
Numeric	0.0690792542700186	0.0579827199700861	−0.0924213908376251
Comparison 8dp	0	0	0

1.1.4 Special Case c. - Axisymmetric

```
In[ ]:= MagCylField[M,0,0,0,φ☆,0,ρ',ρ1,{0,2π},φ1,z',z1]
```

M ⊥	Bρ	Bφ	Bz
Analytic	0.3898898941670998	0.01766718672333797	0.01281246823194358
Numeric	0.3898898941265560	0.01766718672565229	0.01281246823236362
Comparison 8dp	0	0	0

1.1.5 Special Case d. - Solid

```
In[ ]:= MagCylField[M,0,0,0,φ☆,0,{0,ρ'[[2]]},ρ1,φ',φ1,z',z1]
```

M ⊥	Bρ	Bφ	Bz
Analytic	0.3768042165296648	0.01650501972700407	0.01265950791630013
Numeric	0.3768042159820810	0.01650501972378489	0.01265950791531691
Comparison 8dp	0	0	0

1.1.6 Special Case e. - Axisymmetric & Solid

In[∘]:=
```
MagCylField[M,0,0,0,φ☆,0,{0,ρ'[[2]]},ρ1,{0,2π},φ1,z',z1]
```

M ⊥	Bρ	Bφ	Bz
Analytic	0.4201134535501512	0.01963054065204408	0.01333389852194592
Numeric	0.4201134535186059	0.01963054065431834	0.01333389852237476
Comparison 8dp	0	0	0

1.1.7 Singularities b,c,f. - Singular plane 1

In[∘]:=
```
MagCylField[M,0,0,0,φ☆,0,ρ',ρ4,φ',φ4,z',z4]
```

M ⊥	Bρ	Bφ	Bz
Analytic	-0.05283942296707968	-0.10630808491530856	0.005014311029636042
Numeric	-0.0528394229808127	-0.1063080843063326	0.0050143107684108
Comparison 8dp	0	0	0

1.1.8 Singularities a,c,e. - Singular plane 2

In[∘]:=
```
MagCylField[M,0,0,0,φ☆,0,ρ',ρ5,φ',φ5,z',z5]
```

M ⊥	Bρ	Bφ	Bz
Analytic	-0.12706206084430823	-0.01442552150406100	-0.06078349497863217
Numeric	-0.1270620610434591	-0.0144255205837600	-0.0607834949649726
Comparison 8dp	0	0	0

1.1.9 Singularities a,b,d. - Singular plane 3

In[∘]:=
```
MagCylField[M,0,0,0,φ☆,0,ρ',ρ6,φ',φ6,z',z6]
```

M ⊥	Bρ	Bφ	Bz
Analytic	-0.08575035760538238	-0.08687164959114616	-0.03159198910244035
Numeric	-0.0857503575964678	-0.0868716487826758	-0.0315919890768606
Comparison 8dp	0	0	0

1.1.10 (not in article) - On Magnet Axis & Axisymmetric

NIntegrate struggles with Bz.

In[∘]:=
```
MagCylField[M,0,0,0,φ☆,0,ρ',ρ3,{0,2π},φ3,z',z3]// Quiet
```

M ⊥	Bx	By	Bz
Analytic	0.0916633480745974	0.05292185868569116	0
Numeric	0.0916633480754929	0.0529218586946682	$0.\times10^{-18}$
Comparison 8dp	0	0	0

1.1.11 (not in article) - Axisymmetric & Singular plane 3

In[∘]:=
```
MagCylField[M,0,0,0,φ☆,0,ρ',ρ6,{0,2π},φ6,z',z6]// Quiet
```

M ⊥	Bρ	Bφ	Bz
Analytic	−0.006033024943461395	−0.08602519603606986	0.1305260496557415
Numeric	−0.00603302493568282	−0.0860251960250795	0.1305260495723831
Comparison 8dp	0	0	0

1.2 Radial Magnetisation

1.2.0 Equations

Analytic and Numeric function handles. Returns B={Bρ,Bφ,Bz} or B={Bx,By,Bz} (on axis).

In[∘]:=

$$\text{BρAna}[\text{M\_},\varphi☆\text{\_},\text{P\_},\rho\text{p\_},\rho\text{\_},\varphi\text{p\_},\varphi\text{\_},\text{zp\_},\text{z\_}] := \frac{\text{M u0}}{4\pi}\sum_{m=1}^{2}\sum_{n=1}^{2}\sum_{q=1}^{2}(-1)^{m+n+q}\ \text{BρSummand}[\text{P},\rho\text{p}[[m]],\rho,\varphi\text{p}[[q]],\varphi,\text{zp}[[n]],\text{z}]$$

$$\text{BρAnaAxis}[\text{M\_},\varphi☆\text{\_},\rho\text{p\_},\varphi\text{p\_},\text{zp\_},\text{z\_}] := \frac{\text{M u0}}{4\pi}\sum_{m=1}^{2}\sum_{n=1}^{2}\sum_{q=1}^{2}(-1)^{m+n+q}\ \text{BρSummandAxis}[\rho\text{p}[[m]],\varphi\text{p}[[q]],\text{zp}[[n]],\text{z}]$$

$$\text{BρNum}[\text{M\_},\varphi☆\text{\_},\rho\text{p\_},\rho\text{\_},\varphi\text{p\_},\varphi\text{\_},\text{zp\_},\text{z\_}] := \frac{\text{M u0}}{4\pi}\Bigg(\sum_{n=1}^{2}(-1)^{n}\ \text{NIntegrate}\big[\text{BρIntegrand1}[\text{d}\rho\text{p},\rho,\text{d}\varphi\text{p},\varphi,\text{zp}[[n]],\text{z}],\{\text{d}\rho\text{p},\rho\text{p}[[1]],\rho\text{p}[[2]]\},\{\text{d}\varphi\text{p},\ldots$$

$$+\sum_{q=1}^{2}(-1)^{q}\ \text{NIntegrate}\big[\text{BρIntegrand2}[\text{d}\rho\text{p},\rho,\varphi\text{p}[[q]],\varphi,\text{dzp},\text{z}],\{\text{dzp},\text{zp}[[1]],\text{zp}[[2]]\},\ldots$$

$$\text{BρNumAxis}[\text{M\_},\varphi☆\text{\_},\rho\text{p\_},\varphi\text{p\_},\text{zp\_},\text{z\_}] := \frac{\text{M u0}}{4\pi}\Bigg(\sum_{n=1}^{2}(-1)^{n}\ \text{NIntegrate}\big[\text{BρIntegrandAxis1}[\text{d}\rho\text{p},\text{d}\varphi\text{p},\text{zp}[[n]],\text{z}],\{\text{d}\rho\text{p},\rho\text{p}[[1]],\rho\text{p}[[2]]\},\{\text{d}\varphi\text{p},\varphi\text{p}[\ldots$$

$$+\sum_{q=1}^{2}(-1)^{q}\ \text{NIntegrate}\big[\text{BρIntegrandAxis2}[\text{d}\rho\text{p},\varphi\text{p}[[q]],\text{dzp},\text{z}],\{\text{dzp},\text{zp}[[1]],\text{zp}[[2]]\},\ldots$$

Special cases of the geometry for the analytic function handle. Replaces BρAna[] (Cylindrical) or BρAnaAxis[] (Cartesian, on axis).

In[ ]:=

```
BρAna[M_,φ☆_,P_,ρp_,ρ_,{0,2π},φ_,zp_,z_] := (M u0)/(4π) Sum[Sum[(-1)^(m+n) BρSummandAS[P,ρp[[m]],ρ,φ,zp[[n]],z], {n,1,2}], {m,1,2}]
BρAnaAxis[M_,φ☆_,ρp_,{0,2π},zp_,z_] := (M u0)/(4π) Sum[Sum[(-1)^(m+n) BρSummandAxis[ρp[[m]],{0,2π},zp[[n]],z], {n,1,2}], {m,1,2}]
```

Integrands to be solved for BρNum[] (Cylindrical) and BρNumAxis[] (Cartesian, on axis).

In[ ]:=

```
BρIntegrand1[ρp_,ρ_,φp_,φ_,zp_,z_] := {BρIρ1[ρp,ρ,φp,φ,zp,z], BρIφ1[ρp,ρ,φp,φ,zp,z], BρIz[ρp,ρ,φp,φ,zp,z]}
BρIntegrand2[ρp_,ρ_,φp_,φ_,zp_,z_] := {BρIρ2[ρp,ρ,φp,φ,zp,z], BρIφ2[ρp,ρ,φp,φ,zp,z], 0}

BρIρ1[ρp_,ρ_,φp_,φ_,zp_,z_] := -Z[z,zp]  ρp Cos[Φ[φ,φp]] G[ρ,ρp,φ,φp,z,zp]^3
BρIρ2[ρp_,ρ_,φp_,φ_,zp_,z_] := -ρp Sin[Φ[φ,φp]] G[ρ,ρp,φ,φp,z,zp]^3
BρIφ1[ρp_,ρ_,φp_,φ_,zp_,z_] := Z[z,zp] ρp Sin[Φ[φ,φp]] G[ρ,ρp,φ,φp,z,zp]^3
BρIφ2[ρp_,ρ_,φp_,φ_,zp_,z_] := (ρ-ρp Cos[Φ[φ,φp]]) G[ρ,ρp,φ,φp,z,zp]^3
BρIz [ρp_,ρ_,φp_,φ_,zp_,z_] := ρp (-ρp+ρ Cos[Φ[φ,φp]]) G[ρ,ρp,φ,φp,z,zp]^3

BρIntegrandAxis1[ρp_,φp_,zp_,z_] := {BρIAxisx1[ρp,φp,zp,z], BρIAxisy1[ρp,φp,zp,z], BρIAxisz[ρp,φp,zp,z]}
BρIntegrandAxis2[ρp_,φp_,zp_,z_] := {BρIAxisx2[ρp,φp,zp,z], BρIAxisy2[ρp,φp,zp,z],0}

BρIAxisx1[ρp_,φp_,zp_,z_] := -Z[z,zp] ρp Cos[φp] (Z[z,zp]^2+ρp^2)^(-3/2)
BρIAxisx2[ρp_,φp_,zp_,z_] := ρp Sin[φp] (Z[z,zp]^2+ρp^2)^(-3/2)
BρIAxisy1[ρp_,φp_,zp_,z_] := -Z[z,zp] ρp Sin[φp] (Z[z,zp]^2+ρp^2)^(-3/2)
BρIAxisy2[ρp_,φp_,zp_,z_] := -ρp Cos[φp] (Z[z,zp]^2+ρp^2)^(-3/2)
BρIAxisz [ρp_,φp_,zp_,z_] := -ρp^2 (Z[z,zp]^2+ρp^2)^(-3/2)
```

Summands for BρAna[] and BρAnaAxis[].

In[∘]:=

```
BρSummand[P_,ρp_,ρ_,φp_,φ_,zp_,z_] := {BρSρ1[ρp,ρ,φp,φ,zp,z] + BρSρ2[ρp,ρ,φp,φ,zp,z], BρSφ1[ρp,ρ,φp,φ,zp,z] + BρSφ2[ρp,ρ,φp,φ,

BρSρ1[ρp_,ρ_,φp_,φ_,zp_,z_] := -(2/ρ) (ρp Z[z,zp])/(R[ρ,ρp,z,zp]) (EllipticFT[ϕ[φ,φp],k[ρ,ρp,z,zp]^2] - (ρ L[ρ,z,zp])/(Z[z,zp]^2 ρp) ((S[ρ,ρp,z,zp]^2)/(a[ρ,z,zp]^2) EllipticPiT[ā[ρ,z,

BρSρ2[ρp_,ρ_,φp_,φ_,zp_,z_] := Which[(φp==φ+π) || (φp==φ-π),0,True,-Cos[Φ[φ,φp]]ArcTan[ϒ[ρ,ρp,φ,φp,z,zp]]-Sin[Φ[φ,φp]]ArcTanh[G

BρSφ1[ρp_,ρ_,φp_,φ_,zp_,z_] := (Z[z,zp])/ρ Log[ρp-ρ Cos[Φ[φ,φp]]+G[ρ,ρp,φ,φp,z,zp]^-1]

BρSφ2[ρp_,ρ_,φp_,φ_,zp_,z_] := Which[(φp==φ+π) || (φp==φ-π),ArcTanh[(R[ρ,ρp,z,zp])/(Z[z,zp])],True,+ArcTan[ϒ[ρ,ρp,φ,φp,z,zp]] Sin[Φ[φ,φp]]-

BρSz[P_,ρp_,ρ_,φp_,φ_,zp_,z_] := -((2 ρp)/(R[ρ,ρp,z,zp]) EllipticFT[ϕ[φ,φp],k[ρ,ρp,z,zp]^2]+φp α1[ρ,ρp,z,zp]+ φp β1[ρ,ρp,z,zp,P] - δ1ξηι[

BρSummandAxis[ρp_,φp_,zp_,z_] := {BρSAxisx[ρp,φp,zp,z], BρSAxisy[ρp,φp,zp,z], BρSAxisz[ρp,φp,zp,z]}

BρSAxisx[ρp_,φp_,zp_,z_] :=  Sin[φp] ((Z[z,zp])/(Sqrt[Z[z,zp] ^2+ρp^2])+ArcTanh[(Sqrt[Z[z,zp]^2+ρp^2])/(Z[z,zp])])

BρSAxisy[ρp_,φp_,zp_,z_] := -Cos[φp] ((Z[z,zp])/(Sqrt[Z[z,zp]^2+ρp^2])+ArcTanh[(Sqrt[Z[z,zp]^2+ρp^2])/(Z[z,zp])])

BρSAxisz[ρp_,φp_,zp_,z_] := φp ((ρp)/(Sqrt[Z[z,zp]^2+ρp^2])- ArcTanh[(ρp)/(Sqrt[Z[z,zp]^2+ρp^2])])

BρSummandAS[P_,ρp_,ρ_,φ_,zp_,z_] := {BρSASρ[ρp,ρ,φ,zp,z], BρSASφ[ρp,ρ,φ,zp,z], BρSASz[P,ρp,ρ,φ,zp,z]}

BρSASρ[ρp_,ρ_,φ_,zp_,z_] := (4/ρ) (ρp Z[z,zp])/(R[ρ,ρp,z,zp]) (EllipticK[k[ρ,ρp,z,zp]^2]-(ρ L[ρ,z,zp])/(Z[z,zp]^2 ρp) ((S[ρ,ρp,z,zp]^2)/(a[ρ,z,zp]^2) EllipticPi[ā[ρ,z,zp]^2,k[ρ,ρp,z,
BρSASφ[ρp_,ρ_,φ_,zp_,z_] := 0
BρSASz[P_,ρp_,ρ_,φ_,zp_,z_] := ((4ρp)/(R[ρ,ρp,z,zp]) EllipticK[k[ρ,ρp,z,zp]^2]-2π α1[ρ,ρp,z,zp]-2π β1[ρ,ρp,z,zp,P])
```

Singularities in the summands of BρAna[] or BρAnaAxis[].

In[∘]:=

```
(*Along the axis*)
BρSAxisx[ρp_,φp_,zp_,zp_] := 0
```

```
BρSAxisy[ρp_,φp_,zp_,zp_] := 0
BρSAxisz[ρp_,φp_,zp_,zp_] := -φp Log[ρp]
(*Along the axis & axisymmetric*)
BρSAxisx[ρp_,{0,2π},zp_,z_] := 0
BρSAxisy[ρp_,{0,2π},zp_,z_] := 0
BρSAxisz[ρp_,{0,2π},zp_,z_] := BρSAxisz[ρp,2π,zp,z]
BρSAxisx[ρp_,{0,2π},zp_,zp_] := 0
BρSAxisy[ρp_,{0,2π},zp_,zp_] := 0
BρSAxisz[ρp_,{0,2π},zp_,zp_] := BρSAxisz[ρp,2π,zp,zp]
(*Solid*)
BρSρ1[0,ρ_,φp_,φ_,zp_,z_] := -L[ρ,z,zp]/ρ ArcTan[ρ Sin[Φ[φ,φp]]/Z[z,zp]]
BρSρ1[0,ρ_,φp_,φ_,zp_,zp_] := 0
(*Solid & axisymmetric*)
BρSASρ[0,ρ_,φ_,zp_,z_] := 0
(*On the shell plane*)
EllipticPiT[1,ϕ_,k_] := EllipticFT[ϕ,k]-1/(1-k) (EllipticET[ϕ,k]-Sqrt[1-k Sin[ϕ]^2] Tan[ϕ]);
(*On the shell plane & axisymmetric*)
BρSASρ[ρp_,ρ_,φ_,zp_,zp_] := 0
(*On the section plane*)
BρSρ2[ρp_,ρ_,φp_,φp_,zp_,z_] := 0
BρSφ2[ρp_,ρ_,φp_,φp_,zp_,z_] := -ArcTanh[OverBar[R][ρ,ρp,z,zp]/Z[z,zp]]
(*On the disc plane*)
BρSρ1[ρp_,ρ_,φp_,φ_,zp_,zp_] := 0
BρSρ2[ρp_,ρ_,φp_,φ_,zp_,zp_] := 0
BρSφ1[ρp_,ρ_,φp_,φ_,zp_,zp_] := 0
BρSφ2[ρp_,ρ_,φp_,φ_,zp_,zp_] := 0
(*On the axial line*)
BρSρ2[ρp_,ρp_,φp_,φp_,zp_,z_] := 0
BρSφ2[ρp_,ρp_,φp_,φp_,zp_,z_] := -Log[Abs[Z[z,zp]]] Sign[Z[z,zp]]
(*On the azimuthal line*)
BρSρ1[ρp_,ρp_,φp_,φ_,zp_,zp_] := 0
BρSρ2[ρp_,ρp_,φp_,φ_,zp_,zp_] := 0
BρSφ1[ρp_,ρp_,φp_,φ_,zp_,zp_] := 0
BρSφ2[ρp_,ρp_,φp_,φ_,zp_,zp_] := 0
```

```
EllipticFT[ϕ_,1]:=Sin[ϕ]CarlsonRC[1,Cos[ϕ]^2]
(*On the radial line*)
BρSρ1[ρp_,ρ_,φp_,φp_,zp_,zp_] := 0
BρSρ2[ρp_,ρ_,φp_,φp_,zp_,zp_] := 0
BρSφ1[ρp_,ρ_,φp_,φp_,zp_,zp_] := 0
BρSφ2[ρp_,ρ_,φp_,φp_,zp_,zp_] := 0
```

1.2.1 Standard - Outside Magnet

In[ ]:=
```
MagCylField[0,M,0,0,0,150,ρ',ρ1,φ',φ1,z',z1]
```

Mρ	Bρ	Bφ	Bz
Analytic	0.2448856704190372	-0.0003060260329494243	0.009655606660922 49
Numeric	0.2448856701063103	-0.00030602603308330	0.0096555968222767
Comparison 8dp	0	0	0

1.2.2 Special Case a. - Inside Magnet

In[ ]:=
```
MagCylField[0,M,0,0,0,200,ρ',ρ2,φ',φ2,z',z2]
```

Mρ	Bρ	Bφ	Bz
Analytic	0.5489154592264350	-0.00006130244558783496	0.005130179778262334
Numeric	0.5489154591261374	-0.00006130244558542	0.00513017934144725
Comparison 8dp	0	0	0

1.2.3 Special Case b. - On Magnet Axis

In[ ]:=
```
MagCylField[0,M,0,0,0,0,ρ',ρ3,φ',φ3,z',z3]
```

Mρ	Bx	By	Bz
Analytic	0.1344445158028215	0.10887102224775870	-0.1207055650454725
Numeric	0.1344445157685432	0.1088710222296142	-0.1207055650001518
Comparison 8dp	0	0	0

1.2.4 Special Case c. - Axisymmetric

NIntegrate struggles with Bφ.

```
In[•]:= MagCylField[0,M,0,0,0,150,ρ',ρ1,{0,2π},φ1,z',z1] //Quiet
```

Mρ	Bρ	Bφ	Bz
Analytic	0.2112280841430892	0	0.00930958672344238
Numeric	0.2112280841851579	$0.\times10^{-18}$	0.0093095770720707
Comparison 8dp	0	0	0

1.2.5 Special Case d. - Solid

```
In[•]:= MagCylField[0,M,0,0,0,150,{0,ρ'[[2]]},ρ1,φ',φ1,z',z1]
```

Mρ	Bρ	Bφ	Bz
Analytic	0.2572853367868567	−0.0004488894971618596	0.00990814170475206
Numeric	0.2572853369280047	−0.00044888949695013	0.0099081319164752
Comparison 8dp	0	0	0

1.2.6 Special Case e. - Axisymmetric & Solid

NIntegrate struggles with Bφ.

```
In[•]:= MagCylField[0,M,0,0,0,150,{0,ρ'[[2]]},ρ1,{0,2π},φ1,z',z1]//Quiet
```

Mρ	Bρ	Bφ	Bz
Analytic	0.2161116638799649	0	0.00946993030238639
Numeric	0.2161116637943994	$0.\times10^{-17}$	0.0094699206463423
Comparison 8dp	0	0	0

1.2.7 Singularities b,c,f. - Singular plane 1

```
In[•]:= MagCylField[0,M,0,0,0,600,ρ',ρ4,φ',φ4,z',z4]
```

Mρ	Bρ	Bφ	Bz
Analytic	0.04575438476808203	−0.10541986120998000	0.08474434821709217
Numeric	0.04575438476471416	−0.1054198612281754	0.0847443409988348
Comparison 8dp	0	0	0

1.2.8 Singularities a,c,e. - Singular plane 2

In[ ]:=
```
MagCylField[0,M,0,0,0,500,ρ',ρ5,φ',φ5,z',z5]
```

Mρ	Bρ	Bφ	Bz
Analytic	-0.06364161691744196	-0.09932523777628700	0.03837016740099248
Numeric	-0.0636416167461836	-0.0993252378337986	0.03837016147362449
Comparison 8dp	0	0	0

1.2.9 Singularities a,b,d. - Singular plane 3

In[ ]:=
```
MagCylField[0,M,0,0,0,650,ρ',ρ6,φ',φ6,z',z6]
```

Mρ	Bρ	Bφ	Bz
Analytic	-0.06633792348273685	-0.07011471596517585	0.07855349604063655
Numeric	-0.0663379233926231	-0.0701147160567020	0.07855349152769346
Comparison 8dp	0	0	0

1.2.10 (not in article) - On Magnet Axis & Axisymmetric

In[ ]:=
```
MagCylField[0,M,0,0,0,0,ρ',ρ3,{0,2π},φ3,z',z3]
```

Mρ	Bx	By	Bz
Analytic	0	0	-0.3148840827273196
Numeric	$1.169847043839314\times10^{-21}$	$0.\times10^{-17}$	-0.3148840826090916
Comparison 8dp	0	0	0

1.2.11 (not in article) - Axisymmetric & Singular plane 1

In[ ]:=
```
MagCylField[0,M,0,0,0,650,ρ',ρ4,{0,2π},φ4,z',z4]//Quiet
```

Mρ	Bρ	Bφ	Bz
Analytic	0.08202777975183896	0	0.1675255032847301
Numeric	0.0820277797271617	$0.\times10^{-18}$	0.1675254970192023
Comparison 8dp	0	0	0

1.3 Azimuthal Magnetisation

1.3.0 Equations

Analytic and Numeric function handles. Returns B={Bρ,Bφ,Bz} or B={Bx,By,Bz} (on axis).

In[ ]:=
```
BφAna[M_,φ☆_,P_,ρp_,ρ_,φp_,φ_,zp_,z_] := M u0/(4π) (Σ_{m=1}^{2} Σ_{n=1}^{2} Σ_{q=1}^{2} (-1)^(m+n+q) BφSummand[ρp[[m]],ρ,φp[[q]],φ,zp[[n]],z] + Mφ[ρp,ρ,φp,φ,zp,z])
BφAnaAxis[M_,φ☆_,ρp_,φp_,zp_,z_] := M u0/(4π) (Σ_{m=1}^{2} Σ_{n=1}^{2} Σ_{q=1}^{2} (-1)^(m+n+q) BφSummandAxis[ρp[[m]],φp[[q]],zp[[n]],z])
BφNum[M_,φ☆_,ρp_,ρ_,φp_,φ_,zp_,z_] := M u0/(4π) (Σ_{q=1}^{2} (-1)^q NIntegrate[BφIntegrand[dρp,ρ,φp[[q]],φ,dzp,z],{dzp,zp[[1]],zp[[2]]},{dρp,ρp
    + Mφ[ρp,ρ,φp,φ,zp,z])
BφNumAxis[M_,φ☆_,ρp_,φp_,zp_,z_] := M u0/(4π) Σ_{q=1}^{2} (-1)^q NIntegrate[BφIntegrandAxis[dρp,φp[[q]],dzp,z],{dzp,zp[[1]],zp[[2]]},{dρp,ρp[[1
```

Magnetisation vector for field inside magnet. Returns M={Mρ,Mφ,Mz}.

In[ ]:=
```
Mφ[ρp_,ρ_,φp_,φ_,zp_,z_] := If[InsideVolume[ρp,ρ,φp,φ,zp,z],4π{0,1,0},{0,0,0}]
```

Special cases of the geometry for the analytic function handle. Replaces BφAna[] (Cylindrical) or BφAnaAxis[] (Cartesian, on axis).

In[ ]:=
```
BφAna[M_,φ☆_,P_,ρp_,ρ_,{0,2π},φ_,zp_,z_] := M u0/(4π) (Mφ[ρp,ρ,{0,2π},π,zp,z])
BφAnaAxis[M_,φ☆_,ρp_,{0,2π},zp_,z_] := {0,0,0}
```

Integrands to be solved for BφNum[] (Cylindrical) and BφNumAxis[] (Cartesian, on axis).

In[ ]:=
```
BφIntegrand[ρp_,ρ_,φp_,φ_,zp_,z_] := {BφIρ[ρp,ρ,φp,φ,zp,z], BφIφ[ρp,ρ,φp,φ,zp,z], BφIz[ρp,ρ,φp,φ,zp,z]}

BφIρ[ρp_,ρ_,φp_,φ_,zp_,z_] := (ρ-ρp Cos[Φ[φ,φp]]) G[ρ,ρp,φ,φp,z,zp]^3
BφIφ[ρp_,ρ_,φp_,φ_,zp_,z_] := ρp Sin[Φ[φ,φp]] G[ρ,ρp,φ,φp,z,zp]^3
BφIz[ρp_,ρ_,φp_,φ_,zp_,z_] := Z[z,zp] G[ρ,ρp,φ,φp,z,zp]^3

BφIntegrandAxis[ρp_,φp_,zp_,z_] := {BφIAxisx[ρp,φp,zp,z], BφIAxisy[ρp,φp,zp,z], BφIAxisz[ρp,φp,zp,z]}

BφIAxisx[ρp_,φp_,zp_,z_] := -ρp Cos[φp] (Z[z,zp]^2+ρp^2)^(-3/2)
BφIAxisy[ρp_,φp_,zp_,z_] := -ρp Sin[φp] (Z[z,zp]^2+ρp^2)^(-3/2)
BφIAxisz[ρp_,φp_,zp_,z_] := Z[z,zp] (Z[z,zp]^2+ρp^2)^(-3/2)
```

Summands for BφAna[] and BφAnaAxis[].

In[ ]:=
```
BφSummand[ρp_,ρ_,φp_,φ_,zp_,z_] := {BφSρ[ρp,ρ,φp,φ,zp,z] , BφSφ[ρp,ρ,φp,φ,zp,z], BφSz[ρp,ρ,φp,φ,zp,z]}

BφSρ[ρp_,ρ_,φp_,φ_,zp_,z_] := Which[(φp==φ+π) || (φp==φ-π),ArcTanh[R[ρ,ρp,z,zp]/Z[z,zp]],True,ArcTan[ϒ[ρ,ρp,φ,φp,z,zp]]Sin[Φ[φ,φp]]-Cos

BφSφ[ρp_,ρ_,φp_,φ_,zp_,z_] := Which[(φp==φ+π)|| (φp==φ-π),0,True,ArcTan[ϒ[ρ,ρp,φ,φp,z,zp]]Cos[Φ[φ,φp]] + Sin[Φ[φ,φp]]ArcTanh[G

BφSz[ρp_,ρ_,φp_,φ_,zp_,z_] := Log[ρp-ρ Cos[Φ[φ,φp]]+G[ρ,ρp,φ,φp,z,zp]^-1]


BφSummandAxis[ρp_,φp_,zp_,z_] := {BφSAxisx[ρp,φp,zp,z], BφSAxisy[ρp,φp,zp,z], BφSAxisz[ρp,φp,zp,z]}


BφSAxisx[ρp_,φp_,zp_,z_] := -ArcTanh[Sqrt[Z[z,zp]^2+ρp^2]/Z[z,zp]]Cos[φp]

BφSAxisy[ρp_,φp_,zp_,z_] := -ArcTanh[Sqrt[Z[z,zp]^2+ρp^2]/Z[z,zp]]Sin[φp]

BφSAxisz[ρp_,φp_,zp_,z_] := ArcTanh[ρp/Sqrt[Z[z,zp]^2+ρp^2]]
```

Singularities in the summands of BφAna[] or BφAnaAxis[].

In[ ]:=
```
(*Along the axis*)
BφSAxisx[ρp_,φp_,zp_,zp_] := 0
BφSAxisy[ρp_,φp_,zp_,zp_] := 0
BφSAxisz[ρp_,φp_,zp_,zp_] := Log[ρp]
(*On the section plane*)
BφSρ[ρp_,ρ_,φp_,φp_,zp_,z_] := -ArcTanh[R̄[ρ,ρp,z,zp]/Z[z,zp]]
BφSφ[ρp_,ρ_,φp_,φp_,zp_,z_] := 0
(*On the disc plane*)
BφSρ[ρp_,ρ_,φp_,φ_,zp_,zp_] := 0
BφSφ[ρp_,ρ_,φp_,φ_,zp_,zp_] := 0
(*On the axial line*)
BφSρ[ρp_,ρp_,φp_,φp_,zp_,z_] := -Log[Abs[Z[z,zp]]]Sign[Z[z,zp]]
BφSφ[ρp_,ρp_,φp_,φp_,zp_,z_] := 0
(*On the azimuthal line*)
BφSρ[ρp_,ρp_,φp_,φ_,zp_,zp_] := 0
BφSφ[ρp_,ρp_,φp_,φ_,zp_,zp_] := 0
(*On the radial line*)
BφSρ[ρp_,ρ_,φp_,φp_,zp_,zp_] := 0
BφSφ[ρp_,ρ_,φp_,φp_,zp_,zp_] := 0
BφSz[ρp_,ρ_,φp_,φp_,zp_,zp_] := -Sign[ϱ̄[ρ,ρp]]Log[Abs[ϱ̄[ρ,ρp]]]
```

1.3.1 Standard - Outside Magnet

In[ ]:=
```
MagCylField[0,0,M,0,0,0,ρ',ρ1,φ',φ1,z',z1]
```

Mφ	Bρ	Bφ	Bz
Analytic	-0.0009866463614534913	-0.02735309187411041	-0.00002238181997612098
Numeric	-0.00098664636141892	-0.02735309187383460	-0.0000223818199760637
Comparison 8dp	0	0	0

1.3.2 Special Case a. - Inside Magnet

In[ ]:=
```
MagCylField[0,0,M,0,0,0,ρ',ρ2,φ',φ2,z',z2]
```

Mφ	Bρ	Bφ	Bz
Analytic	−0.0012124488065541426	1.1527249218186003	−0.00004258378179621846
Numeric	−0.00121244880652350	1.15272492181848386	−0.000042583781796078
Comparison 8dp	0	0	0

1.3.3 Special Case b. - On Magnet Axis

In[ ]:=
```
MagCylField[0,0,M,0,0,0,ρ',ρ3,φ',φ3,z',z3]
```

Mφ	Bx	By	Bz
Analytic	0.06928254939210094	−0.08555682324180471	0
Numeric	0.06928254938118138	−0.08555682322832017	$0. \times 10^{-18}$
Comparison 8dp	0	0	0

1.3.4 Special Case c. - Axisymmetric

In[ ]:=
```
MagCylField[0,0,M,0,0,0,ρ',ρ1,{0,2π},φ1,z',z1]
```

Mφ	Bρ	Bφ	Bz
Analytic	0	0	0
Numeric	$0. \times 10^{-18}$	$0. \times 10^{-18}$	$0. \times 10^{-19}$
Comparison 8dp	0	0	0

1.3.5 Special Case d. - Solid

In[ ]:=
```
MagCylField[0,0,M,0,0,0,{0,ρ'[[2]]},ρ1,φ',φ1,z',z1]
```

Mφ	Bρ	Bφ	Bz
Analytic	−0.0012549369978166019	−0.03240929395082253	−0.00002681206264645387
Numeric	−0.00125493699761732	−0.03240929395149236	−0.0000268120626457422
Comparison 8dp	0	0	0

1.3.6 Special Case e. - Axisymmetric & Solid

In[ ]:=
```
MagCylField[0,0,M,0,0,0,{0,ρ'[[2]]},ρ1,{0,2π},φ1,z',z1]
```

Mφ	Bρ	Bφ	Bz
Analytic	0	0	0
Numeric	$0.\times10^{-17}$	$0.\times10^{-18}$	$0.\times10^{-19}$
Comparison 8dp	0	0	0

1.3.7 Singularities b,c,f. - Singular plane 1

In[ ]:=
```
MagCylField[0,0,M,0,0,0,ρ',ρ4,φ',φ4,z',z4]
```

Mφ	Bρ	Bφ	Bz
Analytic	−0.1305570199440887	−0.002611825775922538	−0.07915012386811429
Numeric	−0.1305570199604018	−0.002611825775599110	−0.07915012419882970
Comparison 8dp	0	0	0

1.3.8 Singularities a,c,e. - Singular plane 2

In[ ]:=
```
MagCylField[0,0,M,0,0,0,ρ',ρ5,φ',φ5,z',z5]
```

Mφ	Bρ	Bφ	Bz
Analytic	−0.1306098964594674	0.10456601136784070	−0.1218312629689864
Numeric	−0.1306098965151097	0.1045660111848111	−0.1218312629672508
Comparison 8dp	0	0	0

1.3.9 Singularities a,b,d. - Singular plane 3

In[ ]:=
```
MagCylField[0,0,M,0,0,0,ρ',ρ6,φ',φ6,z',z6]
```

Mφ	Bρ	Bφ	Bz
Analytic	−0.07776878671122339	−0.003099650187275807	−0.1340002458099955
Numeric	−0.0777687868094261	−0.003099650149955470	−0.1340002457557562
Comparison 8dp	0	0	0

1.3.10 (not in article) - On Magnet Axis & Axisymmetric

In[ ]:=
```
MagCylField[0,0,M,0,0,0,ρ',ρ3,{0,2π},φ3,z',z3]
```

Mφ	Bx	By	Bz
Analytic	0	0	0
Numeric	$0.\times10^{-17}$	0	$0.\times10^{-18}$
Comparison 8dp	0	0	0

1.4 Axial Magnetisation

1.4.0 Equations

Analytic and Numeric function handles. Returns B={Bρ,Bφ,Bz} or B={Bx,By,Bz} (on axis).

In[ ]:=
```
BzAna[M_,φ☆_,P_,ρp_,ρ_,φp_,φ_,zp_,z_] := (M u0)/(4π) Σ_{m=1}^{2} Σ_{n=1}^{2} Σ_{q=1}^{2} (-1)^(m+n+q) BzSummand[ρp[[m]],ρ,φp[[q]],φ,zp[[n]],z]
BzAnaAxis[M_,φ☆_,ρp_,φp_,zp_,z_] := (M u0)/(4π) Σ_{m=1}^{2} Σ_{n=1}^{2} Σ_{q=1}^{2} (-1)^(m+n+q) BzSummandAxis[ρp[[m]],φp[[q]],zp[[n]],z]
BzNum[M_,φ☆_,ρp_,ρ_,φp_,φ_,zp_,z_] := (M u0)/(4π) ( Σ_{m=1}^{2} (-1)^m NIntegrate[BzIntegrand1[ρp[[m]],ρ,dφp,φ,dzp,z],{dzp,zp[[1]],zp[[2]]},{dφp,φ
      + Σ_{q=1}^{2} (-1)^q NIntegrate[BzIntegrand2[dρp,ρ,φp[[q]],φ,dzp,z],{dzp,zp[[1]],zp[[2]]},
BzNumAxis[M_,φ☆_,ρp_,φp_,zp_,z_] := (M u0)/(4π) ( Σ_{m=1}^{2} (-1)^m NIntegrate[BzIntegrandAxis1[ρp[[m]],dφp,dzp,z],{dzp,zp[[1]],zp[[2]]},{dφp,φp[
      + Σ_{q=1}^{2} (-1)^q NIntegrate[BzIntegrandAxis2[dρp,φp[[q]],dzp,z],{dzp,zp[[1]],zp[[2]]},
```

Special cases of the geometry for the analytic function handle. Replaces BzAna[] (Cylindrical) or BzAnaAxis[] (Cartesian, on axis).

In[∘]:=

$$\text{BzAna}[M\_,\varphi☆\_,P\_,\rho p\_,\rho\_,\{0,2\pi\},\varphi\_,zp\_,z\_] := \frac{M\ u0}{4\pi}\sum_{m=1}^{2}\sum_{n=1}^{2}(-1)^{m+n}\ \text{BzSummandAS}[\rho p[[m]],\rho,\varphi,zp[[n]],z]$$

$$\text{BzAnaAxis}[M\_,\varphi☆\_,\rho p\_,\{0,2\pi\},zp\_,z\_] := \frac{M\ u0}{4\pi}\sum_{m=1}^{2}\sum_{n=1}^{2}(-1)^{m+n}\ \text{BzSummandAxis}[\rho p[[m]],\{0,2\pi\},zp[[n]],z]$$

$$\text{BzAna}[M\_,\varphi☆\_,P\_,\{0,\rho p\_\},\rho\_,\varphi p\_,\varphi\_,zp\_,z\_] := \frac{M\ u0}{4\pi}\sum_{n=1}^{2}\sum_{q=1}^{2}(-1)^{n+q}\left(\text{BzSummand1}[\rho p,\rho,\varphi p[[q]],\varphi,zp[[n]],z] + \sum_{m=1}^{2}(-1)^{m}\text{BzSummand2}[(m-1)\rho p\right.$$

$$\text{BzAna}[M\_,\varphi☆\_,P\_,\{0,\rho p\_\},\rho\_,\{0,2\pi\},\varphi\_,zp\_,z\_] := \frac{M\ u0}{4\pi}\sum_{n=1}^{2}(-1)^{n}\ \text{BzSummandAS}[\rho p,\rho,\varphi,zp[[n]],z]$$

Integrands to be solved for BzNum[] (Cylindrical) and BzNumAxis[] (Cartesian, on axis).

In[∘]:=

```
BzIntegrand1[ρp_,ρ_,φp_,φ_,zp_,z_] := {BzIρ1[ρp,ρ,φp,φ,zp,z], BzIφ1[ρp,ρ,φp,φ,zp,z], BzIz1[ρp,ρ,φp,φ,zp,z]}
BzIntegrand2[ρp_,ρ_,φp_,φ_,zp_,z_] := {BzIρ2[ρp,ρ,φp,φ,zp,z], BzIφ2[ρp,ρ,φp,φ,zp,z], BzIz2[ρp,ρ,φp,φ,zp,z]}

BzIρ1[ρp_,ρ_,φp_,φ_,zp_,z_] := ρp Cos[Φ[φ,φp]] Z[z,zp] G[ρ,ρp,φ,φp,z,zp]^3
BzIρ2[ρp_,ρ_,φp_,φ_,zp_,z_] := Sin[Φ[φ,φp]] Z[z,zp] G[ρ,ρp,φ,φp,z,zp]^3
BzIφ1[ρp_,ρ_,φp_,φ_,zp_,z_] := -ρp Sin[Φ[φ,φp]] Z[z,zp] G[ρ,ρp,φ,φp,z,zp]^3
BzIφ2[ρp_,ρ_,φp_,φ_,zp_,z_] := Z[z,zp]Cos[Φ[φ,φp]] G[ρ,ρp,φ,φp,z,zp]^3
BzIz1[ρp_,ρ_,φp_,φ_,zp_,z_] := ρp (ρp - ρ Cos[Φ[φ,φp]])  G[ρ,ρp,φ,φp,z,zp]^3
BzIz2[ρp_,ρ_,φp_,φ_,zp_,z_] := -ρ Sin[Φ[φ,φp]] G[ρ,ρp,φ,φp,z,zp]^3

BzIntegrandAxis1[ρp_,φp_,zp_,z_] := {BzIAxisx1[ρp,φp,zp,z], BzIAxisy1[ρp,φp,zp,z], BzIAxisz[ρp,φp,zp,z]}
BzIntegrandAxis2[ρp_,φp_,zp_,z_] := {BzIAxisx2[ρp,φp,zp,z], BzIAxisy2[ρp,φp,zp,z],0}

BzIAxisx1[ρp_,φp_,zp_,z_] := ρp Z[z,zp] Cos[φp] (Z[z,zp]^2+ρp^2)^(-3/2)
BzIAxisx2[ρp_,φp_,zp_,z_] := -Z[z,zp] Sin[φp] (Z[z,zp]^2+ρp^2)^(-3/2)
BzIAxisy1[ρp_,φp_,zp_,z_] := ρp Z[z,zp] Sin[φp] (Z[z,zp]^2+ρp^2)^(-3/2)
BzIAxisy2[ρp_,φp_,zp_,z_] := Z[z,zp] Cos[φp] (Z[z,zp]^2+ρp^2)^(-3/2)
BzIAxisz[ρp_,φp_,zp_,z_] := ρp^2 (Z[z,zp]^2+ρp^2)^(-3/2)
```

Summands for BzAna[] and BzAnaAxis[].

In[ ]:=

```
BzSummand[ρp_,ρ_,φp_,φ_,zp_,z_] := {BzSρ1[ρp,ρ,φp,φ,zp,z] + BzSρ2[ρp,ρ,φp,φ,zp,z], BzSφ1[ρp,ρ,φp,φ,zp,z] + BzSφ2[ρp,ρ,φp,φ,zp,z
BzSummand1[ρp_,ρ_,φp_,φ_,zp_,z_] := {BzSρ1[ρp,ρ,φp,φ,zp,z], BzSφ1[ρp,ρ,φp,φ,zp,z], BzSz1[ρp,ρ,φp,φ,zp,z]}
BzSummand2[ρp_,ρ_,φp_,φ_,zp_,z_] := {BzSρ2[ρp,ρ,φp,φ,zp,z], BzSφ2[ρp,ρ,φp,φ,zp,z], BzSz2[ρp,ρ,φp,φ,zp,z]}
```

$$\text{BzS}\rho1[\rho p\_,\rho\_,\varphi p\_,\varphi\_,zp\_,z\_] := \frac{2\ \rho p}{R[\rho,\rho p,zp,z]}\left(\text{EllipticFT}\left[\phi[\varphi,\varphi p],k[\rho,\rho p,z,zp]^2\right]-2\text{EllipticDT}\left[\phi[\varphi,\varphi p],k[\rho,\rho p,z,zp]^2\right]\right)$$

$$\text{BzS}\rho2[\rho p\_,\rho\_,\varphi p\_,\varphi\_,zp\_,z\_] := \text{Log}\left[\rho p-\rho\ \text{Cos}[\Phi[\varphi,\varphi p]]+G[\rho,\rho p,\varphi,\varphi p,z,zp]^{-1}\right]\text{Sin}[\Phi[\varphi,\varphi p]]$$

$$\text{BzS}\varphi1[\rho p\_,\rho\_,\varphi p\_,\varphi\_,zp\_,z\_] := \frac{1}{\rho}G[\rho,\rho p,\varphi,\varphi p,z,zp]^{-1}$$

$$\text{BzS}\varphi2[\rho p\_,\rho\_,\varphi p\_,\varphi\_,zp\_,z\_] := \text{Log}\left[\rho p-\rho\ \text{Cos}[\Phi[\varphi,\varphi p]]+G[\rho,\rho p,\varphi,\varphi p,z,zp]^{-1}\right]\text{Cos}[\Phi[\varphi,\varphi p]]$$

$$\text{BzSz1}[\rho p\_,\rho\_,\varphi p\_,\varphi\_,zp\_,z\_] := \frac{Z[z,zp]}{R[\rho,\rho p,zp,z]}\left(\text{EllipticFT}\left[\phi[\varphi,\varphi p],k[\rho,\rho p,z,zp]^2\right]+\frac{\rho p^2-\rho^2}{\varrho[\rho,\rho p]^2}\text{EllipticPiT}\left[\varkappa[\rho,\rho p]^2,\phi[\varphi,\varphi p],k[\rho,\rho p,z,\right.\right.$$

$$\text{BzSz2}[\rho p\_,\rho\_,\varphi p\_,\varphi\_,zp\_,z\_] := \text{Which}[(\varphi p==\varphi+\pi)\,||\,(\varphi p==\varphi-\pi),0,\text{True},-\text{ArcTan}[\Upsilon[\rho,\rho p,\varphi,\varphi p,z,zp]]]$$

$$\text{BzSummandAxis}[\rho p\_,\varphi p\_,zp\_,z\_] := \{\text{BzSAxisx}[\rho p,\varphi p,zp,z],\ \text{BzSAxisy}[\rho p,\varphi p,zp,z],\ \text{BzSAxisz}[\rho p,\varphi p,zp,z]\}$$

$$\text{BzSAxisx}[\rho p\_,\varphi p\_,zp\_,z\_] := \text{Sin}[\varphi p]\left(\frac{\rho p}{\sqrt{Z[z,zp]^2+\rho p^2}}-\text{ArcTanh}\left[\frac{\rho p}{\sqrt{Z[z,zp]^2+\rho p^2}}\right]\right)$$

$$\text{BzSAxisy}[\rho p\_,\varphi p\_,zp\_,z\_] := -\text{Cos}[\varphi p]\left(\frac{\rho p}{\sqrt{Z[z,zp]^2+\rho p^2}}-\text{ArcTanh}\left[\frac{\rho p}{\sqrt{Z[z,zp]^2+\rho p^2}}\right]\right)$$

$$\text{BzSAxisz}[\rho p\_,\varphi p\_,zp\_,z\_] := -\frac{Z[z,zp]\ \varphi p}{\sqrt{Z[z,zp]^2+\rho p^2}}$$

$$\text{BzSummandAS}[\rho p\_,\rho\_,\varphi\_,zp\_,z\_] := \{\text{BzSAS}\rho[\rho p,\rho,\varphi,zp,z],\ \text{BzSAS}\varphi[\rho p,\rho,\varphi,zp,z],\ \text{BzSASz}[\rho p,\rho,\varphi,zp,z]\}$$

$$\text{BzSAS}\rho[\rho p\_,\rho\_,\varphi\_,zp\_,z\_] := \frac{-4\ \rho p}{R[\rho,\rho p,zp,z]}\left(\text{EllipticK}\left[k[\rho,\rho p,z,zp]^2\right]-2\text{EllipticD}\left[k[\rho,\rho p,z,zp]^2\right]\right)$$

$$\text{BzSAS}\varphi[\rho p\_,\rho\_,\varphi\_,zp\_,z\_] := 0$$

$$\text{BzSASz}[\rho p\_,\rho\_,\varphi\_,zp\_,z\_] := \frac{-2\ Z[z,zp]}{R[\rho,\rho p,zp,z]}\left(\text{EllipticK}\left[k[\rho,\rho p,z,zp]^2\right]+\frac{\rho p^2-\rho^2}{\varrho[\rho,\rho p]^2}\text{EllipticPi}\left[\varkappa[\rho,\rho p]^2,k[\rho,\rho p,z,zp]^2\right]\right)$$

Singularities in the summands of BzAna[] or BzAnaAxis[].

In[ ]:=
```
(*Along the axis*)
BzSAxisx[ρp_,φp_,zp_,zp_] := Sin[φp](1-Log[ρp] )
BzSAxisy[ρp_,φp_,zp_,zp_] := Cos[φp](Log[ρp]-1)
BzSAxisz[ρp_,φp_,zp_,zp_] := 0
(*Along the axis & axisymmetric*)
BzSAxisx[ρp_,{0,2π},zp_,z_] := 0
BzSAxisy[ρp_,{0,2π},zp_,z_] := 0
BzSAxisz[ρp_,{0,2π},zp_,z_] := BzSAxisz[ρp,2π,zp,z]
BzSAxisx[ρp_,{0,2π},zp_,zp_] := 0
BzSAxisy[ρp_,{0,2π},zp_,zp_] := 0
BzSAxisz[ρp_,{0,2π},zp_,zp_] := 0
(*On the shell plane*)
BzSz1[ρp_,ρp_,φp_,φ_,zp_,z_] := Z[z,zp]/Sqrt[Z[z,zp]^2+4 ρp^2] EllipticFT[ϕ[φ,φp], 4 ρp^2/(Z[z,zp]^2+4 ρp^2)]
(*On the shell plane & axisymmetric *)
BzSASz[ρp_,ρp_,φ_,zp_,z_] := -2 Z[z,zp]/R[ρp,ρp,zp,z] EllipticK[k[ρp,ρp,z,zp]^2]
(*On the section plane*)
BzSz2[ρp_,ρ_,φp_,φp_,zp_,z_] := 0
(*On the axial line*)
BzSz2[ρp_,ρp_,φp_,φp_,zp_,z_] := 0
(*On the azimuthal line*)
EllipticFT[ϕ_,1]:=Sin[ϕ]CarlsonRC[1,Cos[ϕ]^2]
BzSz1[ρp_,ρp_,φp_,φ_,zp_,zp_] := 0
(*On the radial line*)
BzSρ2[ρp_,ρ_,φp_,φp_,zp_,zp_] := 0
BzSφ2[ρp_,ρ_,φp_,φp_,zp_,zp_] := -Sign[ϱ[ρ,ρp]]Log[Abs[ϱ[ρ,ρp]]]
```

1.4.1 Standard - Outside Magnet

In[ ]:=
```
MagCylField[0,0,0,M,0,0,ρ',ρ1,φ',φ1,z',z1]
```

Mz	Bρ	Bφ	Bz
Analytic	0.01242509986271859	−0.00001393298933826625	−0.2358090801524554
Numeric	0.01242509986169668	−0.0000139329893381925	−0.2358090801086101
Comparison 8dp	0	0	0

1.4.2 Special Case a. - Inside Magnet

In[ ]:=
```
MagCylField[0,0,0,M,0,0,ρ',ρ2,φ',φ2,z',z2]
```

Mz	Bρ	Bφ	Bz
Analytic	0.01266084849525560	−0.00003269425069322492	0.6633431753061298
Numeric	0.01266084849536185	−0.0000326942506934173	0.6633431748479679
Comparison 8dp	0	0	0

1.4.3 Special Case b. - On Magnet Axis

In[ ]:=
```
MagCylField[0,0,0,M,0,0,ρ',ρ3,φ',φ3,z',z3]
```

Mz	Bx	By	Bz
Analytic	−0.07272021717981136	−0.05888767076268449	−0.08114684998472645
Numeric	−0.0727202171871756	−0.05888767076910802	−0.0811468499945612
Comparison 8dp	0	0	0

1.4.4 Special Case c. - Axisymmetric

NIntegrate struggles with Bφ.

In[ ]:=
```
MagCylField[0,0,0,M,0,0,ρ',ρ1,{0,2π},φ1,z',z1]//Quiet
```

Mz	Bρ	Bφ	Bz
Analytic	0.01292302651525148	0	−0.2579006638843391
Numeric	0.01292302651543708	$4.\times10^{-18}$	−0.2579006638181462
Comparison 8dp	0	0	0

1.4.5 Special Case d. - Solid

```
In[•]:= MagCylField[0,0,0,M,0,0,{0,ρ'[[2]]},ρ1,φ',φ1,z',z1]
```

Mz	Bρ	Bφ	Bz
Analytic	0.01276677951668406	−0.00001493928657643710	−0.2448778172082850
Numeric	0.01276677951565720	−0.0000149392865761592	−0.2448778171647654
Comparison 8dp	0	0	0

1.4.6 Special Case e. - Axisymmetric & Solid

NIntegrate struggles with Bφ.

```
In[•]:= MagCylField[0,0,0,M,0,0,{0,ρ'[[2]]},ρ1,{0,2π},φ1,z',z1]//Quiet
```

Mz	Bρ	Bφ	Bz
Analytic	0.01344895620667155	0	−0.2733431824759750
Numeric	0.01344895620688026	$4.\times10^{-18}$	−0.2733431824120061
Comparison 8dp	0	0	0

1.4.7 Singularities b,c,f. - Singular plane 1

```
In[•]:= MagCylField[0,0,0,M,0,0,ρ',ρ4,φ',φ4,z',z4]
```

Mz	Bρ	Bφ	Bz
Analytic	0.1131007576718941	−0.05950872524189693	−0.05720141207922482
Numeric	0.1131007576909566	−0.0595087255655965	−0.05720141208092251
Comparison 8dp	0	0	0

1.4.8 Singularities a,c,e. - Singular plane 2

```
In[•]:= MagCylField[0,0,0,M,0,0,ρ',ρ5,φ',φ5,z',z5]
```

Mz	Bρ	Bφ	Bz
Analytic	0.08547085115221370	−0.1011947666062023	−0.05944892351147916
Numeric	0.0854708511544623	−0.1011947665760598	−0.0594489236528541
Comparison 8dp	0	0	0

1.4.9 Singularities a,b,d. - Singular plane 3

In[ ]:=
```
MagCylField[0,0,0,M,0,0,ρ′,ρ6,φ′,φ6,z′,z6]
```

Mz	Bρ	Bφ	Bz
Analytic	0.1286619619883900	−0.1107623052135217	0.05821848296015986
Numeric	0.1286619619891644	−0.1107623051291430	0.05821848293333856
Comparison 8dp	0	0	0

1.4.10 (not in article) - On Magnet Axis & Axisymmetric

In[ ]:=
```
MagCylField[0,0,0,M,0,0,ρ′,ρ3,{0,2π},φ3,z′,z3]
```

Mz	Bx	By	Bz
Analytic	0	0	−0.2116874347427647
Numeric	$-1.157683990029317\times10^{-21}$	$0.\times10^{-18}$	−0.2116874347684205
Comparison 8dp	0	0	0

1.4.11 (not in article) - Axisymmetric & Singular plane 3

In[ ]:=
```
MagCylField[0,0,0,M,0,0,ρ′,ρ6,{0,2π},φ6,z′,z6]//Quiet
```

Mz	Bρ	Bφ	Bz
Analytic	0.2610520993114831	0	0.1113993900639533
Numeric	0.2610520991272603	$0.\times10^{-18}$	0.1113993901132542
Comparison 8dp	0	0	0

2.0 Coils with Azimuthal Current Density

Integrands to be solved for B{i,k,s,c}Num[] (Cylindrical) and B{i,k,s,c}NumAxis[] (Cartesian, on axis). Common between filament, disc, shell, volume.

In[•]:=
```
BcIntegrand[ρp_,ρ_,φp_,φ_,zp_,z_] := {BcIρ[ρp,ρ,φp,φ,zp,z], BcIφ[ρp,ρ,φp,φ,zp,z], BcIz[ρp,ρ,φp,φ,zp,z]}

BcIρ[ρp_,ρ_,φp_,φ_,zp_,z_] := ρp Cos[Φ[φ,φp]] Z[z,zp] G[ρ,ρp,φ,φp,z,zp]^3
BcIφ[ρp_,ρ_,φp_,φ_,zp_,z_] := -ρp Sin[Φ[φ,φp]] Z[z,zp] G[ρ,ρp,φ,φp,z,zp]^3
BcIz[ρp_,ρ_,φp_,φ_,zp_,z_] := ρp (ρp - ρ Cos[Φ[φ,φp]]) G[ρ,ρp,φ,φp,z,zp]^3

BcIntegrandAxis[ρp_,φp_,zp_,z_] := {BcIAxisx[ρp,φp,zp,z], BcIAxisy[ρp,φp,zp,z], BcIAxisz[ρp,φp,zp,z]}

BcIAxisx[ρp_,φp_,zp_,z_] := ρp Z[z,zp] Cos[φp] (Z[z,zp]^2+ρp^2)^(-3/2)
BcIAxisy[ρp_,φp_,zp_,z_] := ρp Z[z,zp] Sin[φp] (Z[z,zp]^2+ρp^2)^(-3/2)
BcIAxisz[ρp_,φp_,zp_,z_] := ρp^2 (Z[z,zp]^2+ρp^2)^(-3/2)
```

2.1 Filament

2.1.0 Equations

Analytic and Numeric function handles. Returns B={Bρ,Bφ,Bz} or B={Bx,By,Bz} (on axis).

In[•]:=
```
BiAna[I_,φ☆_,P_,ρp_,ρ_,φp_,φ_,zp_,z_] := (I u0)/(4π) Sum[(-1)^q BiSummand[ρp,ρ,φp[[q]],φ,zp,z], {q,1,2}]
BiAnaAxis[I_,φ☆_,ρp_,φp_,zp_,z_] := (I u0)/(4π) Sum[(-1)^q BiSummandAxis[ρp,φp[[q]],zp,z], {q,1,2}]
BiNum[I_,φ☆_,ρp_,ρ_,φp_,φ_,zp_,z_] := (I u0)/(4π) NIntegrate[BcIntegrand[ρp,ρ,dφp,φ,zp,z],{dφp,φp[[1]],φp[[2]]},Evaluate[IntOptions]]
BiNumAxis[I_,φ☆_,ρp_,φp_,zp_,z_] := (I u0)/(4π) NIntegrate[BcIntegrandAxis[ρp,dφp,zp,z],{dφp,φp[[1]],φp[[2]]},Evaluate[IntOptions]]
```

Special cases of the geometry for the analytic function handle. Replaces BiAna[] (Cylindrical) or BiAnaAxis[] (Cartesian, on axis).

In[•]:=
```
BiAna[I_,φ☆_,P_,ρp_,ρ_,{0,2π},φ_,zp_,z_] := (I u0)/(4π) BiSummandAS[ρp,ρ,φ,zp,z]
BiAnaAxis[I_,φ☆_,ρp_,{0,2π},zp_,z_] := (I u0)/(4π) BiSummandAxis[ρp,{0,2π},zp,z]
```

Summands for BiAna[] and BiAnaAxis[].

In[ ]:=

```
BiSummand[ρp_,ρ_,φp_,φ_,zp_,z_] := {BiSρ[ρp,ρ,φp,φ,zp,z], BiSφ[ρp,ρ,φp,φ,zp,z], BiSz[ρp,ρ,φp,φ,zp,z]}
```

$$\text{BiS}\rho[\rho p\_,\rho\_,\varphi p\_,\varphi\_,zp\_,z\_] := \frac{Z[z,zp]}{\rho}\left(\frac{1}{R[\rho,\rho p,z,zp]}\left(\text{EllipticFT}\left[\phi[\varphi,\varphi p],k[\rho,\rho p,z,zp]^2\right]-\left(1+\frac{1}{2}\overline{k}[\rho,\rho p,z,zp]^2\right)\text{EllipticET}\left[\phi[\varphi,\varphi p],k[\rho\right.\right.\right.$$

$$\text{BiS}\varphi[\rho p\_,\rho\_,\varphi p\_,\varphi\_,zp\_,z\_] := -\frac{Z[z,zp]}{\rho}G[\rho,\rho p,\varphi,\varphi p,z,zp]$$

$$\text{BiSz}[\rho p\_,\rho\_,\varphi p\_,\varphi\_,zp\_,z\_] := -\frac{1}{R[\rho,\ \rho p,z,zp]}\left(\text{EllipticFT}\left[\phi[\varphi,\varphi p],k[\rho,\ \rho p,z,zp]^2\right]-\frac{\overline{T}[\rho,\rho p,z,zp]^2}{\overline{R}[\rho,\rho p,z,zp]^2}\text{EllipticET}\left[\phi[\varphi,\varphi p],k[\rho,\ \rho p,z,zp]^2\right.\right.$$

```
BiSummandAxis[ρp_,φp_,zp_,z_] := {BiSAxisx[ρp,φp,zp,z], BiSAxisy[ρp,φp,zp,z], BiSAxisz[ρp,φp,zp,z]}
```

$$\text{BiSAxisx}[\rho p\_,\varphi p\_,zp\_,z\_] := \frac{Z[z,zp]\ \rho p\ \text{Sin}[\varphi p]}{\left(Z[z,zp]^2+\rho p^2\right)^{3/2}}$$

$$\text{BiSAxisy}[\rho p\_,\varphi p\_,zp\_,z\_] := -\frac{Z[z,zp]\ \rho p\ \text{Cos}[\varphi p]}{\left(Z[z,zp]^2+\rho p^2\right)^{3/2}}$$

$$\text{BiSAxisz}[\rho p\_,\varphi p\_,zp\_,z\_] := \frac{\rho p^2\ \varphi p}{\left(Z[z,zp]^2+\rho p^2\right)^{3/2}}$$

```
BiSummandAS[ρp_,ρ_,φ_,zp_,z_] := {BiSASρ[ρp,ρ,φ,zp,z], BiSASφ[ρp,ρ,φ,zp,z], BiSASz[ρp,ρ,φ,zp,z]}
```

$$\text{BiSAS}\rho[\rho p\_,\rho\_,\varphi\_,zp\_,z\_] := \frac{2\ Z[z,zp]}{\rho\ R[\rho p,\rho,z,zp]}\left(\frac{T[\rho,\rho p,z,zp]^2}{\overline{R}[\rho,\rho p,z,zp]^2}\ \text{EllipticE}\left[k[\rho p,\rho,z,zp]^2\right]-\text{EllipticK}\left[k[\rho p,\rho,z,zp]^2\right]\right)$$

```
BiSASφ[ρp_,ρ_,φ_,zp_,z_] := 0
```

$$\text{BiSASz}[\rho p\_,\rho\_,\varphi\_,zp\_,z\_] := \frac{2}{R[\rho,\rho p,z,zp]}\left(\text{EllipticK}\left[k[\rho,\rho p,z,zp]^2\right]-\frac{\overline{T}[\rho,\rho p,z,zp]^2}{\overline{R}[\rho,\rho p,z,zp]^2}\ \text{EllipticE}\left[k[\rho,\rho p,z,zp]^2\right]\ \right)$$

Singularities in the summands of BiAna[] or BiAnaAxis[].

In[∘]:=
```
(*Along the axis & axisymmetric*)
BiSAxisx[ρp_,{0,2π},zp_,z_] := 0
BiSAxisy[ρp_,{0,2π},zp_,z_] := 0
BiSAxisz[ρp_,{0,2π},zp_,z_] := BiSAxisz[ρp,2π,zp,z]
(*On the azimuthal line*)
BiSρ[ρp_,ρp_,φp_,φ_,zp_,zp_] := 0
BiSφ[ρp_,ρp_,φp_,φ_,zp_,zp_] := 0
BiSz[ρp_,ρp_,φp_,φ_,zp_,zp_] := -Sign[Φ[φ,φp]] ArcTanh[Sin[ϕ[φ,φp]]] / (2 ρp)
```

2.1.1 Standard - Outside Coil Radii

In[∘]:=
```
CoilFilamentField[Iφ,ρ',ρ1,φ',φ1,z',z1]
```

Iφ	Bρ	Bφ	Bz
Analytic	0.001372714157642409	$1.698760127815627\times10^{-6}$	-0.0003967696850031862
Numeric	0.001372714157642415	$1.698760127815627\times10^{-6}$	-0.0003967696850031852
Comparison 8dp	0	0	0

2.1.2 Special Case a. - Inside Coil Radii

In[∘]:=
```
CoilFilamentField[Iφ,ρ',ρ2,φ',φ2,z',z2]
```

Iφ	Bρ	Bφ	Bz
Analytic	0.001534719063283769	$2.418938548515046\times10^{-6}$	0.001069419149543071
Numeric	0.001534719063283777	$2.418938548515046\times10^{-6}$	0.001069419149543071
Comparison 8dp	0	0	0

2.1.3 Special Case b. - On Coil Axis

In[∘]:=
```
CoilFilamentField[Iφ,ρ',ρ3,φ',φ3,z',z3]
```

Iφ	Bx	By	Bz
Analytic	0.0001297864403851998	0.0001050989871491504	0.0004308553035038312
Numeric	0.0001297864403851998	0.0001050989871491504	0.0004308553035038312
Comparison 8dp	0	0	0

2.1.4 Special Case c. - Axisymmetric

In[ ]:=
```
CoilFilamentField[Iφ,ρ',ρ1,{0,2π},φ1,z',z1]
```

Iφ	Bρ	Bφ	Bz
Analytic	0.001360094620733514	0	−0.0001562732832471367
Numeric	0.001360094620733503	$5.564487148325225 \times 10^{-25}$	−0.0001562732832471367
Comparison 8dp	0	0	0

2.1.5 Singularities b,c,f. - Singular plane 1

In[ ]:=
```
CoilFilamentField[Iφ,ρ',ρ4,φ',φ4,z',z4]
```

Iφ	Bρ	Bφ	Bz
Analytic	0.0003770114716661066	0.0001612909935983802	0.00005207000999609287
Numeric	0.0003770114716661066	0.0001612909935983802	0.00005207000999609286
Comparison 8dp	0	0	0

2.1.6 Singularities a,c,e. - Singular plane 2

In[ ]:=
```
CoilFilamentField[Iφ,ρ',ρ5,φ',φ5,z',z5]
```

Iφ	Bρ	Bφ	Bz
Analytic	0.0002938289080965867	0.0001785499735766973	0.0001780958179363176
Numeric	0.0002938289080965867	0.0001785499735766973	0.0001780958179363176
Comparison 8dp	0	0	0

2.1.7 Singularities a,b,d. - Singular plane 3

In[ ]:=
```
CoilFilamentField[Iφ,ρ',ρ6,φ',φ6,z',z6]
```

Iφ	Bρ	Bφ	Bz
Analytic	0.0003156861113994550	0.0001706445231121001	0.0001429507104140370
Numeric	0.0003156861113994550	0.0001706445231121001	0.0001429507104140370
Comparison 8dp	0	0	0

2.1.8 - On Coil Axis & Axisymmetric

In[∘]:=
```
CoilFilamentField[Iφ,ρ',ρ3,{0,2π},φ3,z',z3]
```

Iφ	Bx	By	Bz
Analytic	0	0	0.001123970356966516
Numeric	$5.363823723160991\times10^{-43}$	$-2.422790896503524\times10^{-44}$	0.001123970356966516
Comparison 8dp	0	0	0

2.2 Disc

2.2.0 Equations

Analytic and Numeric function handles. Returns B={Bρ,Bφ,Bz} or B={Bx,By,Bz} (on axis).

In[∘]:=

$$\text{BkAna}[K\_,\varphi☆\_,P\_,\rho p\_,\rho\_,\varphi p\_,\varphi\_,zp\_,z\_] := \frac{K\ u0}{4\pi}\sum_{m=1}^{2}\sum_{q=1}^{2}(-1)^{m+q}\ \text{BkSummand}[P,\rho p[[m]],\rho,\varphi p[[q]],\varphi,zp,z]$$

$$\text{BkAnaAxis}[K\_,\varphi☆\_,\rho p\_,\varphi p\_,zp\_,z\_] := \frac{K\ u0}{4\pi}\sum_{m=1}^{2}\sum_{q=1}^{2}(-1)^{m+q}\ \text{BkSummandAxis}[\rho p[[m]],\varphi p[[q]],zp,z]$$

$$\text{BkNum}[K\_,\varphi☆\_,\rho p\_,\rho\_,\varphi p\_,\varphi\_,zp\_,z\_] := \frac{K\ u0}{4\pi}\ \text{NIntegrate}[\text{BcIntegrand}[d\rho p,\rho,d\varphi p,\varphi,zp,z],\{d\rho p,\rho p[[1]],\rho p[[2]]\},\{d\varphi p,\varphi p[[1]],\varphi p[[2]]$$

$$\text{BkNumAxis}[K\_,\varphi☆\_,\rho p\_,\varphi p\_,zp\_,z\_] := \frac{K\ u0}{4\pi}\ \text{NIntegrate}[\text{BcIntegrandAxis}[d\rho p,d\varphi p,zp,z],\{d\rho p,\rho p[[1]],\rho p[[2]]\},\{d\varphi p,\varphi p[[1]],\varphi p[[2]]\},$$

Special cases of the geometry for the analytic function handle. Replaces BkAna[] (Cylindrical) or BkAnaAxis[] (Cartesian, on axis).

In[∘]:=

$$\text{BkAna}[K\_,\varphi☆\_,P\_,\rho p\_,\rho\_,\{0,2\pi\},\varphi\_,zp\_,z\_] := \frac{K\ u0}{4\pi}\sum_{m=1}^{2}(-1)^{m}\ \text{BkSummandAS}[P,\rho p[[m]],\rho,\varphi,zp,z]$$

$$\text{BkAnaAxis}[K\_,\varphi☆\_,\rho p\_,\{0,2\pi\},zp\_,z\_] := \frac{K\ u0}{4\pi}\sum_{m=1}^{2}(-1)^{m}\ \text{BkSummandAxis}[\rho p[[m]],\{0,2\pi\},zp,z]$$

Summands for BkAna[] and BkAnaAxis[].

In[ ]:=

```
BkSummand[P_,ρp_,ρ_,φp_,φ_,zp_,z_] := {BkSρ[ρp,ρ,φp,φ,zp,z], BkSφ[ρp,ρ,φp,φ,zp,z], BkSz[P,ρp,ρ,φp,φ,zp,z]}

BkSρ[ρp_,ρ_,φp_,φ_,zp_,z_] := 2/ρ (ρp Z[z,zp])/R[ρ,ρp,z,zp] (EllipticFT[ϕ[φ,φp],k[ρ,ρp,z,zp]^2]-(ρ L[ρ,z,zp])/(Z[z,zp]^2 ρp) (S[ρ,ρp,z,zp]^2/a[ρ,z,zp]^2 EllipticPiT[ā[ρ,z,zp
BkSφ[ρp_,ρ_,φp_,φ_,zp_,z_] := -Z[z,zp]/ρ Log[ρp-ρ Cos[Φ[φ,φp]]+G[ρ,ρp,φ,φp,z,zp]^-1]
BkSz[P_,ρp_,ρ_,φp_,φ_,zp_,z_] := (2 ρp)/R[ρ,ρp,z,zp] EllipticFT[ϕ[φ,φp],k[ρ,ρp,z,zp]^2]+φp α1[ρ,ρp,z,zp]+ φp β1[ρ,ρp,z,zp,P] - δ1ξηι[ρ,

BkSummandAxis[ρp_,φp_,zp_,z_] := {BkSAxisx[ρp,φp,zp,z], BkSAxisy[ρp,φp,zp,z], BkSAxisz[ρp,φp,zp,z]}

BkSAxisx[ρp_,φp_,zp_,z_] := -Sin[φp] Z[z,zp]/Sqrt[Z[z,zp] ^2+ρp^2]
BkSAxisy[ρp_,φp_,zp_,z_] := Cos[φp] Z[z,zp]/Sqrt[Z[z,zp]^2+ρp^2]
BkSAxisz[ρp_,φp_,zp_,z_] := -φp (ρp/Sqrt[Z[z,zp]^2+ρp^2]- ArcTanh[ρp/Sqrt[Z[z,zp]^2+ρp^2]])

BkSummandAS[P_,ρp_,ρ_,φ_,zp_,z_] := {BkSASρ[ρp,ρ,φ,zp,z], BkSASφ[ρp,ρ,φ,zp,z], BkSASz[P,ρp,ρ,φ,zp,z]}

BkSASρ[ρp_,ρ_,φ_,zp_,z_] := -4/ρ (ρp Z[z,zp])/R[ρ,ρp,z,zp] (EllipticK[k[ρ,ρp,z,zp]^2]-(ρ L[ρ,z,zp])/(Z[z,zp]^2 ρp) (S[ρ,ρp,z,zp]^2/a[ρ,z,zp]^2 EllipticPi[ā[ρ,z,zp]^2,k[ρ,ρp,z
BkSASφ[ρp_,ρ_,φ_,zp_,z_] := 0
BkSASz[P_,ρp_,ρ_,φ_,zp_,z_] := -(4ρp)/R[ρ,ρp,z,zp] EllipticK[k[ρ,ρp,z,zp]^2]+2π α1[ρ,ρp,z,zp]+2π β1[ρ,ρp,z,zp,P]
```

Singularities in the summands of BkAna[] or BkAnaAxis[].

In[ ]:=
```
(*Along the axis*)
BkSAxisx[ρp_,φp_,zp_,zp_] := 0
BkSAxisy[ρp_,φp_,zp_,zp_] := 0
BkSAxisz[ρp_,φp_,zp_,zp_] := φp Log[ρp]
(*Along the axis & axisymmetric*)
BkSAxisx[ρp_,{0,2π},zp_,z_] := 0
BkSAxisy[ρp_,{0,2π},zp_,z_] := 0
BkSAxisz[ρp_,{0,2π},zp_,z_] := BkSAxisz[ρp,2π,zp,z]
(*Solid*)
BkSρ[0,ρ_,φp_,φ_,zp_,z_] := L[ρ,z,zp]/ρ ArcTan[ρ Sin[Φ[φ,φp]]/Z[z,zp]]
BkSρ[0,ρ_,φp_,φ_,zp_,zp_] := 0
(*Solid & axisymmetric*)
BkSASρ[0,ρ_,φ_,zp_,z_] := 0
(*On the disc plane*)
BkSρ[ρp_,ρ_,φp_,φ_,zp_,zp_] := 0
BkSφ[ρp_,ρ_,φp_,φ_,zp_,zp_] := 0
(*On the disc plane & axisymmetric*)
BkSASρ[ρp_,ρ_,φ_,zp_,zp_] := 0
(*On the azimuthal line*)
EllipticFT[ϕ_,1]:=Sin[ϕ]CarlsonRC[1,Cos[ϕ]^2]
BkSρ[ρp_,ρp_,φp_,φ_,zp_,zp_] := 0
BkSφ[ρp_,ρp_,φp_,φ_,zp_,zp_] := 0
(*On the radial line*)
BkSρ[ρp_,ρ_,φp_,φp_,zp_,zp_] := 0
BkSφ[ρp_,ρ_,φp_,φp_,zp_,zp_] := 0
```

2.2.1 Standard - Outside Coil Radii

In[ ]:=
```
CoilDiscField[Kφ,100,ρ',ρ1,φ',φ1,z',z1]
```

Kφ	Bρ	Bφ	Bz
Analytic	0.004550646471690198	0.00001486261867018283	−0.004089840391961120
Numeric	0.004550646471947809	0.00001486261867065866	−0.004089833754297460
Comparison 8dp	0	0	0

2.2.2 Special Case a. - Inside Coil Radii

In[∘]:=
```
CoilDiscField[Kφ,150,ρ',ρ2,φ',φ2,z',z2]
```

Kφ	Bρ	Bφ	Bz
Analytic	0.01013923265250101	0.00002505900107230787	−0.001280424692817354
Numeric	0.01013923264982671	0.00002505900107113981	−0.001280424610557872
Comparison 8dp	0	0	0

2.2.3 Special Case b. - On Coil Axis

In[∘]:=
```
CoilDiscField[Kφ,0,ρ',ρ3,φ',φ3,z',z3]
```

Kφ	Bx	By	Bz
Analytic	0.002047652044440496	0.001658155931127027	0.004393791852333189
Numeric	0.002047652043569552	0.001658155930824412	0.004393791854502072
Comparison 8dp	0	0	0

2.2.4 Special Case c. - Axisymmetric

NIntegrate struggles with Bφ.

In[∘]:=
```
CoilDiscField[Kφ,100,ρ',ρ1,{0,2π},φ1,z',z1]//Quiet
```

Kφ	Bρ	Bφ	Bz
Analytic	0.004412345974681498	0	−0.002214796040884351
Numeric	0.004412345973088094	$-1.410421035147059\times10^{-19}$	−0.002214789554461825
Comparison 8dp	0	0	0

2.2.5 Special Case d. - Solid

In[∘]:=
```
CoilDiscField[Kφ,100,{0,ρ'[[2]]},ρ1,φ',φ1,z',z1]
```

Kφ	Bρ	Bφ	Bz
Analytic	0.004710462151164272	0.00001760013686144884	−0.004574725506618468
Numeric	0.004710462152513212	0.00001760013686158787	−0.004574718867992251
Comparison 8dp	0	0	0

2.2.6 Special Case e. - Axisymmetric & Solid

NIntegrate struggles with Bφ.

In[∘]:=
```
CoilDiscField[Kφ,100,{0,ρ'[[2]]},ρ1,{0,2π},φ1,z',z1]//Quiet
```

Kφ	Bρ	Bφ	Bz
Analytic	0.004518329230250307	0	-0.002341160773793059
Numeric	0.004518329227550135	$-2.233301434601139\times10^{-19}$	-0.002341154285474315
Comparison 8dp	0	0	0

2.2.7 Singularities b,c,f. - Singular plane 1

In[∘]:=
```
CoilDiscField[Kφ,50,ρ',ρ4,φ',φ4,z',z4]
```

Kφ	Bρ	Bφ	Bz
Analytic	0.001807018177682073	0.001052865287292513	-0.0003772521706449185
Numeric	0.001807018177554557	0.001052865287213675	-0.0003772521127762577
Comparison 8dp	0	0	0

2.2.8 Singularities a,c,e. - Singular plane 2

In[∘]:=
```
CoilDiscField[Kφ,50,ρ',ρ5,φ',φ5,z',z5]
```

Kφ	Bρ	Bφ	Bz
Analytic	0.001714110762320366	0.001310352196154153	0.0003146652144706637
Numeric	0.001714110761827329	0.001310352196075857	0.0003146652222908744
Comparison 8dp	0	0	0

2.2.9 Singularities a,b,d. - Singular plane 3

In[∘]:=
```
CoilDiscField[Kφ,50,ρ',ρ6,φ',φ6,z',z6]
```

Kφ	Bρ	Bφ	Bz
Analytic	0.001871178367543343	0.001277701647764240	0.0001772489313626284
Numeric	0.001871178367547036	0.001277701647679863	0.0001772489314075168
Comparison 8dp	0	0	0

2.2.10 - On Coil Axis & Axisymmetric

In[∘]:=
```
CoilDiscField[Kφ,0,ρ′,ρ3,{0,2π},φ3,z′,z3]
```

Kφ	Bx	By	Bz
Analytic	0	0	0.01146206570173875
Numeric	$4.899882906133252\times10^{-23}$	$-9.27052056626888\times10^{-24}$	0.01146206570739671
Comparison 8dp	0	0	0

2.2.11 (not in article) - Axisymmetric & Singular plane 3

In[∘]:=
```
CoilDiscField[Kφ,100,ρ′,ρ6,{0,2π},φ6,z′,z6] //Quiet
```

Kφ	Bρ	Bφ	Bz
Analytic	0.003488669562401596	0	0.001067140995447426
Numeric	0.003488669563036198	$4.824185584218220\times10^{-13}$	0.001067140994383038
Comparison 8dp	0	0	0

2.3 Shell

2.3.0 Equations

Analytic and Numeric function handles. Returns B={Bρ,Bφ,Bz} or B={Bx,By,Bz} (on axis).

In[∘]:=
```
BsAna[K_,φ☆_,P_,ρp_,ρ_,φp_,φ_,zp_,z_] := (K u0)/(4π) Sum[Sum[(-1)^(n+q) BsSummand[ρp,ρ,φp[[q]],φ,zp[[n]],z], {q,1,2}], {n,1,2}]
BsAnaAxis[K_,φ☆_,ρp_,φp_,zp_,z_] := (K u0)/(4π) Sum[Sum[(-1)^(n+q) BsSummandAxis[ρp,φp[[q]],zp[[n]],z], {q,1,2}], {n,1,2}]
BsNum[K_,φ☆_,ρp_,ρ_,φp_,φ_,zp_,z_] := (K u0)/(4π) NIntegrate[BcIntegrand[ρp,ρ,dφp,φ,dzp,z],{dzp,zp[[1]],zp[[2]]},{dφp,φp[[1]],φp[[2]]
BsNumAxis[K_,φ☆_,ρp_,φp_,zp_,z_] := (K u0)/(4π) NIntegrate[BcIntegrandAxis[ρp,dφp,dzp,z],{dzp,zp[[1]],zp[[2]]},{dφp,φp[[1]],φp[[2]]},
```

Special cases of the geometry for the analytic function handle. Replaces BsAna[] (Cylindrical) or BsAnaAxis[] (Cartesian, on axis).

In[•]:=
```
BsAna[K_,φ☆_,P_,ρp_,ρ_,{0,2π},φ_,zp_,z_] := (K u0)/(4π) Sum[(-1)^n BsSummandAS[ρp,ρ,φ,zp[[n]],z], {n,1,2}]
BsAnaAxis[K_,φ☆_,ρp_,{0,2π},zp_,z_] := (K u0)/(4π) Sum[(-1)^n BsSummandAxis[ρp,{0,2π},zp[[n]],z], {n,1,2}]
```

Summands for BsAna[] and BsAnaAxis[].

In[•]:=
```
BsSummand[ρp_,ρ_,φp_,φ_,zp_,z_] := {BsSρ[ρp,ρ,φp,φ,zp,z], BsSφ[ρp,ρ,φp,φ,zp,z], BsSz[ρp,ρ,φp,φ,zp,z]}

BsSρ[ρp_,ρ_,φp_,φ_,zp_,z_] := (2 ρp)/(R[ρ,ρp,zp,z]) (EllipticFT[ϕ[φ,φp],k[ρ,ρp,z,zp]^2]-2EllipticDT[ϕ[φ,φp],k[ρ,ρp,z,zp]^2])
BsSφ[ρp_,ρ_,φp_,φ_,zp_,z_] := 1/ρ G[ρ,ρp,φ,φp,z,zp]^-1
BsSz[ρp_,ρ_,φp_,φ_,zp_,z_] := (Z[z,zp])/(R[ρ,ρp,zp,z]) (EllipticFT[ϕ[φ,φp],k[ρ,ρp,z,zp]^2]+ (ρp^2-ρ^2)/(ϱ[ρ,ρp]^2) EllipticPiT[ϰ[ρ,ρp]^2,ϕ[φ,φp],k[ρ,ρp,z,z

BsSummandAxis[ρp_,φp_,zp_,z_] := {BsSAxisx[ρp,φp,zp,z], BsSAxisy[ρp,φp,zp,z], BsSAxisz[ρp,φp,zp,z]}

BsSAxisx[ρp_,φp_,zp_,z_] :=  Sin[φp] ρp/Sqrt[Z[z,zp]^2+ρp^2]
BsSAxisy[ρp_,φp_,zp_,z_] := -Cos[φp] ρp/Sqrt[Z[z,zp]^2+ρp^2]
BsSAxisz[ρp_,φp_,zp_,z_] := - (Z[z,zp] φp)/Sqrt[Z[z,zp]^2+ρp^2]

BsSummandAS[ρp_,ρ_,φ_,zp_,z_] := {BsSASρ[ρp,ρ,φ,zp,z], BsSASφ[ρp,ρ,φ,zp,z], BsSASz[ρp,ρ,φ,zp,z]}

BsSASρ[ρp_,ρ_,φ_,zp_,z_] := (4 ρp)/(R[ρ,ρp,zp,z]) (2EllipticD[k[ρ,ρp,z,zp]^2]-EllipticK[k[ρ,ρp,z,zp]^2])
BsSASφ[ρp_,ρ_,φ_,zp_,z_] := 0
BsSASz[ρp_,ρ_,φ_,zp_,z_] := (-2 Z[z,zp])/(R[ρ,ρp,zp,z]) (EllipticK[k[ρ,ρp,z,zp]^2]+ (ρp^2-ρ^2)/(ϱ[ρ,ρp]^2) EllipticPi[ϰ[ρ,ρp]^2,k[ρ,ρp,z,zp]^2])
```

Singularities in the summands of BsAna[] or BsAnaAxis[].

In[ ]:=
```
(*Along the axis*)
BsSAxisx[ρp_,φp_,zp_,zp_] := Sin[φp]
BsSAxisy[ρp_,φp_,zp_,zp_] := -Cos[φp]
BsSAxisz[ρp_,φp_,zp_,zp_] := 0
(*Along the axis & axisymmetric*)
BsSAxisx[ρp_,{0,2π},zp_,z_] := 0
BsSAxisy[ρp_,{0,2π},zp_,z_] := 0
BsSAxisz[ρp_,{0,2π},zp_,z_] := BsSAxisz[ρp,2π,zp,z]
BsSAxisx[ρp_,{0,2π},zp_,zp_] := 0
BsSAxisy[ρp_,{0,2π},zp_,zp_] := 0
BsSAxisz[ρp_,{0,2π},zp_,zp_] := 0
(*On the shell plane*)
BsSz[ρp_,ρp_,φp_,φ_,zp_,z_] := Z[z,zp]/Sqrt[Z[z,zp]^2+4 ρp^2] EllipticFT[ϕ[φ,φp], 4 ρp^2/(Z[z,zp]^2+4 ρp^2)]
(*On the shell plane & axisymmetric*)
BsSASz[ρp_,ρp_,φ_,zp_,z_] := -2 Z[z,zp]/R[ρp,ρp,zp,z] EllipticK[k[ρp,ρp,z,zp]^2]
(*On the azimuthal line*)
EllipticFT[ϕ_,1]:=Sin[ϕ]CarlsonRC[1,Cos[ϕ]^2]
```

2.3.1 Standard - Outside Coil Radii

In[ ]:=
```
CoilShellField[Kφ,ρ',ρ1,φ',φ1,z',z1]
```

Kφ	Bρ	Bφ	Bz
Analytic	0.0005681218241650236	$6.511476224250626\times10^{-7}$	-0.01342080647548769
Numeric	0.0005681218241220527	$6.511476224270352\times10^{-7}$	-0.01342080647369517
Comparison 8dp	0	0	0

2.3.2 Special Case a. - Inside Coil Radii

In[ ]:=
```
CoilShellField[Kφ,ρ',ρ2,φ',φ2,z',z2]
```

Kφ	Bρ	Bφ	Bz
Analytic	0.0006356826497196410	$9.28724216507752\times10^{-7}$	0.02217975675423922
Numeric	0.0006356826497240293	$9.28724216509253\times10^{-7}$	0.02217975673515034
Comparison 8dp	0	0	0

2.3.3 Special Case b. - On Coil Axis

In[ ]:=
```
CoilShellField[Kφ,ρ',ρ3,φ',φ3,z',z3]
```

Kφ	Bx	By	Bz
Analytic	0.0006127684497726228	0.0004962101066715267	0.004308553035038312
Numeric	0.0006127684497739614	0.0004962101066695900	0.004308553035038092
Comparison 8dp	0	0	0

2.3.4 Special Case c. - Axisymmetric

NIntegrate struggles with Bφ.

In[ ]:=
```
CoilShellField[Kφ,ρ',ρ1,{0,2π},φ1,z',z1]//Quiet
```

Kφ	Bρ	Bφ	Bz
Analytic	0.0005633070662480231	0	-0.01144892910894136
Numeric	0.0005633070662567649	$1.600183994864209\times10^{-19}$	-0.01144892910626204
Comparison 8dp	0	0	0

2.3.5 Singularities b,c,f. - Singular plane 1

In[ ]:=
```
CoilShellField[Kφ,ρ',ρ4,φ',φ4,z',z4]
```

Kφ	Bρ	Bφ	Bz
Analytic	0.005009379856705172	0.001167520097808665	-0.003071447681757532
Numeric	0.005009379857527716	0.001167520098129617	-0.003071447681849082
Comparison 8dp	0	0	0

2.3.6 Singularities a,c,e. - Singular plane 2

In[ ]:=
```
CoilShellField[Kφ,ρ',ρ5,φ',φ5,z',z5]
```

Kφ	Bρ	Bφ	Bz
Analytic	0.003220598895996764	0.001286646043698470	0.002190520177620032
Numeric	0.003220598896130437	0.001286646044934483	0.002190520172020854
Comparison 8dp	0	0	0

2.3.7 Singularities a,b,d. - Singular plane 3

In[ ]:=
```
CoilShellField[Kφ,ρ',ρ6,φ',φ6,z',z6]
```

Kφ	Bρ	Bφ	Bz
Analytic	0.005908193994816346	0.001609408420570709	0.001749837323422153
Numeric	0.005908193994780027	0.001609408421723598	0.001749837323221755
Comparison 8dp	0	0	0

2.3.8 - On Coil Axis & Axisymmetric

In[ ]:=
```
CoilShellField[Kφ,ρ',ρ3,{0,2π},φ3,z',z3]
```

Kφ	Bx	By	Bz
Analytic	0	0	0.01123970356966516
Numeric	$1.533582990691132\times10^{-23}$	$-1.548082264520142\times10^{-24}$	0.01123970356966459
Comparison 8dp	0	0	0

2.3.9 (not in article) - Axisymmetric & Singular plane 3

In[ ]:=
```
CoilShellField[Kφ,ρ',ρ6,{0,2π},φ6,z',z6] //Quiet
```

Kφ	Bρ	Bφ	Bz
Analytic	0.01169236924966334	0	0.004205497057936947
Numeric	0.01169236924174034	$1.874673356262654\times10^{-19}$	0.004205497060167882
Comparison 8dp	0	0	0

2.3 Volume

2.2.0 Equations

Analytic and Numeric function handles. Returns B={Bρ,Bφ,Bz} or B={Bx,By,Bz} (on axis).

In[ ]:=

$$\text{BcAna}[J\_,\varphi☆\_,P\_,\rho p\_,\rho\_,\varphi p\_,\varphi\_,zp\_,z\_] := \frac{J\ u0}{4\pi}\sum_{m=1}^{2}\sum_{n=1}^{2}\sum_{q=1}^{2}(-1)^{m+n+q}\ \text{BcSummand}[P,\rho p[[m]],\rho,\varphi p[[q]],\varphi,zp[[n]],z]$$

$$\text{BcAnaAxis}[J\_,\varphi☆\_,\rho p\_,\varphi p\_,zp\_,z\_] := \frac{J\ u0}{4\pi}\sum_{m=1}^{2}\sum_{n=1}^{2}\sum_{q=1}^{2}(-1)^{m+n+q}\ \text{BcSummandAxis}[\rho p[[m]],\varphi p[[q]],zp[[n]],z]$$

$$\text{BcNum}[J\_,\varphi☆\_,\rho p\_,\rho\_,\varphi p\_,\varphi\_,zp\_,z\_] := \frac{J\ u0}{4\pi}\ \text{NIntegrate}[\text{BcIntegrand}[d\rho p,\rho,d\varphi p,\varphi,dzp,z],\{d\rho p,\rho p[[1]],\rho p[[2]]\},\{d\varphi p,\varphi p[[1]],\varphi p[[2$$

$$\text{BcNumAxis}[J\_,\varphi☆\_,\rho p\_,\varphi p\_,zp\_,z\_] := \frac{J\ u0}{4\pi}\ \text{NIntegrate}[\text{BcIntegrandAxis}[d\rho p,d\varphi p,dzp,z],\{d\rho p,\rho p[[1]],\rho p[[2]]\},\{d\varphi p,\varphi p[[1]],\varphi p[[2]]\}$$

Special cases of the geometry for the analytic function handle. Replaces BcAna[] (Cylindrical) or BcAnaAxis[] (Cartesian, on axis).

In[ ]:=

$$\text{BcAna}[J\_,\varphi☆\_,P\_,\rho p\_,\rho\_,\{0,2\pi\},\varphi\_,zp\_,z\_] := \frac{J\ u0}{4\pi}\sum_{m=1}^{2}\sum_{n=1}^{2}(-1)^{m+n}\ \text{BcSummandAS}[P,\rho p[[m]],\rho,\varphi,zp[[n]],z]$$

$$\text{BcAnaAxis}[J\_,\varphi☆\_,\rho p\_,\{0,2\pi\},zp\_,z\_] := \frac{J\ u0}{4\pi}\sum_{m=1}^{2}\sum_{n=1}^{2}(-1)^{m+n}\ \text{BcSummandAxis}[\rho p[[m]],\{0,2\pi\},zp[[n]],z]$$

Summands for BcAna[] and BcAnaAxis[].

In[ ]:=

```
BcSummand[P_,ρp_,ρ_,φp_,φ_,zp_,z_] := {BcSρ[P,ρp,ρ,φp,φ,zp,z], BcSφ[ρp,ρ,φp,φ,zp,z], BcSz[P,ρp,ρ,φp,φ,zp,z]}

BcSρ[P_,ρp_,ρ_,φp_,φ_,zp_,z_] := 1/2 (R[ρ,ρp,z,zp] 4/3 (EllipticFT[ϕ[φ,φp],k[ρ,ρp,z,zp]^2] + (-2+k[ρ,ρp,z,zp]^2) EllipticDT[ϕ[φ,φp],k[ρ,ρ
BcSφ[ρp_,ρ_,φp_,φ_,zp_,z_] := 1/(2ρ) ( (ρp-ρ Cos[Φ[φ,φp]]) G[ρ,ρp,φ,φp,z,zp]^-1 + (Z[z,zp]^2+ρ^2 Sin[Φ[φ,φp]]^2) Log[ρp-ρ Cos[Φ[φ,φp]]+G
BcSz[P_,ρp_,ρ_,φp_,φ_,zp_,z_] := φp α3[ρ,ρp,z,zp] + 1/2 Sign[Z[z,zp]] ρp (2φp α2[ρ,ρp,z,zp] + φp β2[ρ,ρp,z,zp,P] - φp Sqrt[π]/8 β3[ρ,ρp,z,zp,P]

BcSummandAxis[ρp_,φp_,zp_,z_] := {BcSAxisx[ρp,φp,zp,z], BcSAxisy[ρp,φp,zp,z], BcSAxisz[ρp,φp,zp,z]}

BcSAxisx[ρp_,φp_,zp_,z_] := Sin[φp] Sqrt[Z[z,zp]^2+ρp^2]
BcSAxisy[ρp_,φp_,zp_,z_] := -Cos[φp] Sqrt[Z[z,zp]^2+ρp^2]
BcSAxisz[ρp_,φp_,zp_,z_] := -φp Z[z,zp] ArcTanh[ρp / Sqrt[Z[z,zp]^2+ρp^2]]

BcSummandAS[P_,ρp_,ρ_,φ_,zp_,z_] := {BcSASρ[P,ρp,ρ,φ,zp,z], BcSASφ[ρp,ρ,φ,zp,z], BcSASz[P,ρp,ρ,φ,zp,z]}

BcSASρ[P_,ρp_,ρ_,φ_,zp_,z_] := π ρ (α1[ρ,ρp,z,zp]+β1[ρ,ρp,z,zp,P]+ ϑ[ρ,ρp,z,zp,P])- R[ρ,ρp,z,zp] 4/3 (EllipticK[k[ρ,ρp,z,zp]^2] + (-
BcSASφ[ρp_,ρ_,φ_,zp_,z_] := 0
BcSASz[P_,ρp_,ρ_,φ_,zp_,z_] := 2π(α3[ρ,ρp,z,zp] + 1/2 ρp Sign[Z[z,zp]] (2 α2[ρ,ρp,z,zp]+β2[ρ,ρp,z,zp,P] - Sqrt[π]/8 β3[ρ,ρp,z,zp,P]))
```

Singularities in the summands of BcAna[] or BcAnaAxis[].

In[ ]:=
```
(*Along the axis*)
BcSAxisz[ρp_,φp_,zp_,zp_] := 0
(*Along the axis & axisymmetric*)
BcSAxisx[ρp_,{0,2π},zp_,z_] := 0
BcSAxisy[ρp_,{0,2π},zp_,z_] := 0
BcSAxisz[ρp_,{0,2π},zp_,z_] := BcSAxisz[ρp,2π,zp,z]
(*On the azimuthal line*)
EllipticFT[ϕ_,1]:=Sin[ϕ]CarlsonRC[1,Cos[ϕ]^2]
(*On the radial line*)
BcSφ[ρp_,ρ_,φp_,φp_,zp_,zp_] := (*-Abs[ϱ[ρ,ρp]] ϱ[ρ,ρp]/(2 ρ)*)-(Abs[ϱ[ρ,ρp]] ρp (ρp-2 ρ))/(2 ρ ϱ[ρ,ρp])
```

2.4.1 Standard - Outside Coil

NIntegrate struggles with Bφ.

In[ ]:=
```
CoilCylField[Jφ,P,ρ',ρ7,φ',φ7,z',z7]//Quiet
```

Jφ	Bρ	Bφ	Bz
Analytic	0.00002947429112029849	$3.524997169679487\times10^{-7}$	0.001513201804832494
Numeric	0.00002947429112124344	$3.524997225520252\times10^{-7}$	0.001571624392843605
Comparison 8dp	0	0	-0.000058422588011112

2.4.2 Special Case b. - Axisymmetric

NIntegrate struggles with Bφ.

In[ ]:=
```
CoilCylField[Jφ,P,ρ',ρ7,{0,2π},φ7,z',z7]//Quiet
```

Jφ	Bρ	Bφ	Bz
Analytic	0.00002069362960441219	0	0.002431624306568482
Numeric	0.00002464071192591216	$1.021109903288754\times10^{-15}$	0.002490047237424913
Comparison 8dp	$-3.94708232149997\times10^{-6}$	0	-0.000058422930856431

2.4.3 Special Case a. - On Coil Axis

In[ ]:=
```
CoilCylField[Jφ,0,ρ′,ρ3,φ′,φ3,z′,z3]
```

Jφ	Bx	By	Bz
Analytic	0.0001531921124431557	0.0001240525266678817	0.0007230367439804593
Numeric	0.0001531921124201073	0.0001240525266482019	0.0007230367444749626
Comparison 8dp	0	0	0

2.4.4 Special Case a,b - On Coil Axis & Axisymmetric

In[ ]:=
```
CoilCylField[Jφ,0,ρ′,ρ3,{0,2π},φ3,z′,z3]
```

Jφ	Bx	By	Bz
Analytic	0	0	0.001886182810383807
Numeric	$3.712049951100202\times10^{-24}$	$-4.910735032908079\times10^{-25}$	0.001886182811673816
Comparison 8dp	0	0	0

2.4.5 Singularities b,c,f. - Singular plane 1

In[ ]:=
```
CoilCylField[Jφ,P,ρ′,ρ4,φ′,φ4,z′,z4]
```

Jφ	Bρ	Bφ	Bz
Analytic	0.0001841132305583547	0.00007498203826667723	-0.0001835655901368976
Numeric	0.0001860126163046093	0.00007498203826589994	-0.0001846274379018721
Comparison 8dp	$-1.8993857462547\times10^{-6}$	0	$1.0618477649745\times10^{-6}$

2.4.6 Singularities a,c,e. - Singular plane 2

In[ ]:=
```
CoilCylField[Jφ,P,ρ′,ρ5,φ′,φ5,z′,z5]
```

Jφ	Bρ	Bφ	Bz
Analytic	0.0001886679203860053	0.0001000973144813222	-0.00003550475181454477
Numeric	0.0001834555924935078	0.0001000973144525731	-0.00003565053740774642
Comparison 8dp	$5.2123278924976\times10^{-6}$	0	$1.4578559320165\times10^{-7}$

2.4.7 Singularities a,b,d. - Singular plane 3

In[ ]:=
```
CoilCylField[Jφ,P,ρ',ρ6,φ',φ6,z',z6]
```

Jφ	Bρ	Bφ	Bz
Analytic	0.0003035211228690777	0.0001273292407013310	-0.00006923686016875202
Numeric	0.0003070881388353115	0.0001273292406406753	-0.00006944687051424545
Comparison 8dp	$-3.5670159662338\times10^{-6}$	0	$2.100103454934\times10^{-7}$

3.0 Green's Function Integrals

This section is simply a comparison of the integral transforms, discussed in part 3 of the article.

3.0.0 Integrals and Analytic Solutions

In[ ]:=
```
Compare[P_,ρ'_,ρ_,φ'_,φ_,z'_,z_,numFn_,anaFn1_,anaFn2_] :=
  Module[{Bana1,Bana2,Bnum,heading},
    Bnum =   numFn[ρ',ρ,φ',φ,z',z];
    Bana1 = N[anaFn1[P,ρ',ρ,φ',φ,z',z],$MachinePrecision];
    Bana2=N[anaFn2[P,ρ',ρ,φ',φ,z',z],$MachinePrecision];
    If[anaFn2==0,Bana2=None];
    TableForm[{Bnum,Bana1,Bana2}, TableHeadings -> {{"Numeric","Analytic Form 1", "Analytic Form 2"}}]
  ]
(*Azimuthal integral*)
Gdφp[ρp_,ρ_,φp_,φ_,zp_,z_] := NIntegrate[G[ρ,ρp,φ,dφp,z,zp],{dφp,φp[[1]],φp[[2]]},Evaluate[IntOptions]]
```

$$Gd\varphi p1[P\_,\rho p\_,\rho\_,\varphi p\_,\varphi\_,zp\_,z\_] := -\frac{2}{R[\rho,\rho p,z,zp]}\sum_{q=1}^{2}(-1)^q\ \mathrm{EllipticF}\left[\phi[\varphi,\varphi p[[q]]],k[\rho,\rho p,z,zp]^2\right]$$

$$Gd\varphi p2[P\_,\rho p\_,\rho\_,\varphi p\_,\varphi\_,zp\_,z\_] := -\frac{2}{R[\rho,\rho p,z,zp]}\sum_{q=1}^{2}(-1)^q\ \mathrm{EllipticFT}\left[\phi[\varphi,\varphi p[[q]]],k[\rho,\rho p,z,zp]^2\right]$$

```
(*Radial integral*)
Gdρp[ρp_,ρ_,φp_,φ_,zp_,z_] := NIntegrate[G[ρ,dρp,φ,φp,z,zp],{dρp,ρp[[1]],ρp[[2]]},Evaluate[IntOptions]]
```

$$Gd\rho p1[P\_,\rho p\_,\rho\_,\varphi p\_,\varphi\_,zp\_,z\_] := \sum_{m=1}^{2}(-1)^m\ (\alpha 1[\rho,\rho p[[m]],z,zp] + \beta 1[\rho,\rho p[[m]],z,zp,P] + \gamma 1[\rho,\rho p[[m]],\varphi,\varphi p,z,zp,P])$$

```
(*Axial integral*)
Gdzp[ρp_,ρ_,φp_,φ_,zp_,z_] := NIntegrate[G[ρ,ρp,φ,φp,z,dzp],{dzp,zp[[1]],zp[[2]]},Evaluate[IntOptions]]
```

```
Gdzp1[P_,ρp_,ρ_,φp_,φ_,zp_,z_] := -∑_{n=1}^{2}(-1)^n Sign[Z[z,zp[[n]]]] (α2[ρ,ρp,z,zp[[n]]] + 1/2 β2[ρ,ρp,z,zp[[n]],P] + γ2[ρ,ρp,φ,φp,z,zp[
(*Radial surface integral*)
GdρpdφP[ρp_,ρ_,φp_,φ_,zp_,z_] := NIntegrate[G[ρ,dρp,φ,dφp,z,zp],{dρp,ρp[[1]],ρp[[2]]},{dφp,φp[[1]],φp[[2]]},Evaluate[IntOption
Gdρpdφp1[P_,ρp_,ρ_,φp_,φ_,zp_,z_] := ∑_{m=1}^{2}∑_{q=1}^{2}(-1)^{m+q} (φp[[q]]×α1[ρ,ρp[[m]],z,zp] + φp[[q]]×β1[ρ,ρp[[m]],z,zp,P] - δ1[ρ,ρp[[m]],φ,φ
Gdρpdφp2[P_,ρp_,ρ_,φp_,φ_,zp_,z_] := ∑_{m=1}^{2}∑_{q=1}^{2}(-1)^{m+q} (φp[[q]]×α1[ρ,ρp[[m]],z,zp] + φp[[q]]×β1[ρ,ρp[[m]],z,zp,P] - δ1ξηι[ρ,ρp[[m]]
(*Axial surface integral*)
Gdzpdφp[ρp_,ρ_,φp_,φ_,zp_,z_] := NIntegrate[G[ρ,ρp,φ,dφp,z,dzp],{dzp,zp[[1]],zp[[2]]},{dφp,φp[[1]],φp[[2]]},Evaluate[IntOptions
Gdzpdφp1[P_,ρp_,ρ_,φp_,φ_,zp_,z_] :=-1/2 ∑_{n=1}^{2}∑_{q=1}^{2}(-1)^{n+q} Sign[Z[z,zp[[n]]]](2φp[[q]]×α2[ρ,ρp,z,zp[[n]]] + φp[[q]]×β2[ρ,ρp,z,zp[[n]]
Gdzpdφp2[P_,ρp_,ρ_,φp_,φ_,zp_,z_] :=-1/2 ∑_{n=1}^{2}∑_{q=1}^{2}(-1)^{n+q} Sign[Z[z,zp[[n]]]](2φp[[q]]×α2[ρ,ρp,z,zp[[n]]] + φp[[q]]×β2[ρ,ρp,z,zp[[n]]
(*Volume integral*)
Gdρpdφpdzp[ρp_,ρ_,φp_,φ_,zp_,z_] := NIntegrate[G[ρ,dρp,φ,dφp,z,dzp],{dρp,ρp[[1]],ρp[[2]]},{dzp,zp[[1]],zp[[2]]},{dφp,φp[[1]],φ
Gdρpdφpdzp1[P_,ρp_,ρ_,φp_,φ_,zp_,z_] := ∑_{m=1}^{2}∑_{n=1}^{2}∑_{q=1}^{2}(-1)^{m+n+q} (φp[[q]]×α3[ρ,ρp[[m]],z,zp[[n]]] - 1/2 Sign[Z[z,zp[[n]]]] ρp[[m]] (1/8 √π φp
Gdρpdφpdzp2[P_,ρp_,ρ_,φp_,φ_,zp_,z_] := ∑_{m=1}^{2}∑_{n=1}^{2}∑_{q=1}^{2}(-1)^{m+n+q} (φp[[q]]×α3[ρ,ρp[[m]],z,zp[[n]]] - 1/2 Sign[Z[z,zp[[n]]]] ρp[[m]] (1/8 √π φp
```

3.0.1 Azimuthal Integral

In[∘]:=

```
Compare[0,ρ'[[2]],ρ1,φ',φ1,z'[[1]],z1,Gdφp,Gdφp1,Gdφp2]
```

Out[∘]//TableForm=

Numeric	520.8982064725782
Analytic Form 1	520.8982064725861
Analytic Form 2	520.8982064725861

3.0.2 Radial Integral

In[ ]:=
```
Compare[P,ρ',ρ1,φ'[[1]],φ1,z'[[1]],z1,Gdρp,Gdρp1,0]
```

Out[ ]//TableForm=

Numeric	0.5607377420756752
Analytic Form 1	0.5607377421188464
Analytic Form 2	None

3.0.3 Axial Integral

In[ ]:=
```
Compare[200,ρ'[[2]],ρ1,φ'[[1]],φ1,z',z1,Gdzp,Gdzp1,0]
```

Out[ ]//TableForm=

Numeric	0.4187967839218372
Analytic Form 1	0.4187978710349870
Analytic Form 2	None

3.0.4 Radial Surface Integral

In[ ]:=
```
Compare[P,ρ',ρ1,φ',φ1,z'[[1]],z1,Gdρpdφp,Gdρpdφp1,Gdρpdφp2]
```

Out[ ]//TableForm=

Numeric	2.133688224453506
Analytic Form 1	2.133677478713198
Analytic Form 2	2.133688203712503

3.0.5 Axial Surface Integral

In[ ]:=
```
Compare[250,ρ'[[2]],ρ1,φ',φ1,z',z1,Gdzpdφp,Gdzpdφp1,Gdzpdφp2]
```

Out[ ]//TableForm=

Numeric	2.536820273844129
Analytic Form 1	2.530666999720498
Analytic Form 2	2.536092152698103

3.0.6 Volume Integral

In[ ]:=
```
Compare[100,ρ',ρ1,φ',φ1,z',z1,GdρpdφpdzP,GdρpdφpdzP1,GdρpdφpdzP2]
```

Out[ ]//TableForm=

Numeric	0.00927987090388250
Analytic Form 1	0.00926414493132398
Analytic Form 2	0.00927635460527792

4.0 Magnetic field derivates

This section has the example derivatives given in part 6.2 of the article. Integrals with respect to t are not true for all φ' (ϕ).

4.0.0 Derivatives and Analytic Solutions

```
Compare2[ρ'_,ρ_,φ'_,φ_,z'_,z_,numFn1_,numFn2_,anaFn_] :=
    Module[{Bana,Bnum1,Bnum2,heading},
        Bnum1 = N[numFn1[ρ',ρ,φ',φ,z',z],$MachinePrecision];
        Bnum2 = N[numFn2[ρ',ρ,φ',φ,z',z],$MachinePrecision];
        Bana =  N[anaFn[ρ',ρ,φ',φ,z',z],$MachinePrecision];
        If[numFn2==0,Bnum2=None];
        TableForm[{Bnum1,Bnum2,Bana}, TableHeadings -> {{"Numeric Form 1", "Numeric Form 2", "Analytic Form"}}]
    ]
(*Elliptic integral of the first kind*)
dρeFn[ρp_,ρ_,φp_,φ_,zp_,z_] := Sum[(-1)^q (D[EllipticF[ϕ[φ,φp[[q]]],k[dρ,ρp,z,zp]^2],dρ])/. dρ→ ρ, {q,1,2}]

dφeFn[ρp_,ρ_,φp_,φ_,zp_,z_] := Sum[(-1)^q (D[EllipticF[ϕ[dφ,φp[[q]]],k[ρ,ρp,z,zp]^2],dφ])/. dφ→ φ, {q,1,2}]

dzeFn[ρp_,ρ_,φp_,φ_,zp_,z_] := Sum[(-1)^q (D[EllipticF[ϕ[φ,φp[[q]]],k[ρ,ρp,dz,zp]^2],dz])/. dz→ z, {q,1,2}]
```

$$\text{d}\rho\text{eFt}[\rho p\_,\rho\_,\varphi p\_,\varphi\_,zp\_,z\_] := \sum_{q=1}^{2}(-1)^q\ \text{NIntegrate}\Big[\frac{2\ \rho p\left(Z[z,zp]^2-\rho^2+\rho p^2\right)}{R[\rho,\rho p,z,zp]^4}\ \frac{t^2}{\left(1-k[\rho,\rho p,z,zp]^2\ t^2\ \right)}\ \frac{1}{\sqrt{\left(1-t^2\right)\ \left(1-k[\rho,\rho p,z,zp]^2\ t^2\right)}},\{t,$$

$$\text{dzeFt}[\rho p\_,\rho\_,\varphi p\_,\varphi\_,zp\_,z\_] := \sum_{q=1}^{2}(-1)^q \text{ NIntegrate}\Big[-\frac{Z[z,zp]k[\rho,\rho p,z,zp]^2}{R[\rho,\rho p,z,zp]^2}\frac{t^2}{(1-k[\rho,\rho p,z,zp]^2\ t^2\ )}\frac{1}{\sqrt{(1-t^2)\ (1-k[\rho,\rho p,z,zp]^2\ t^2)}},\{t,$$

$$\text{d}\rho\text{eFa}[\rho p\_,\rho\_,\varphi p\_,\varphi\_,zp\_,z\_] := \sum_{q=1}^{2}(-1)^q\frac{Z[z,zp]^2-\rho^2+\rho p^2}{2\ \rho\ R[\rho,\rho p,z,zp]^2}\Big(\text{EllipticPiT}\big[k[\rho,\rho p,z,zp]^2,\phi[\varphi,\varphi p[[q]]],k[\rho,\rho p,z,zp]^2\big]-\text{EllipticFT}\big[\phi[\varphi,\varphi$$

$$\text{d}\varphi\text{eFa}[\rho p\_,\rho\_,\varphi p\_,\varphi\_,zp\_,z\_] := \sum_{q=1}^{2}(-1)^q\ \frac{1}{2\ \sqrt{(1-k[\rho,\rho p,z,zp]^2\ \text{Sin}[\phi[\varphi,\varphi p[[q]]]]^2)}}$$

$$\text{dzeFa}[\rho p\_,\rho\_,\varphi p\_,\varphi\_,zp\_,z\_] := \sum_{q=1}^{2}(-1)^q\frac{Z[z,zp]}{R[\rho,\rho p,z,zp]^2}\Big(\text{EllipticF}\big[\phi[\varphi,\varphi p[[q]]],k[\rho,\rho p,z,zp]^2\big]-\text{EllipticPi}\big[k[\rho,\rho p,z,zp]^2,\phi[\varphi,\varphi p[[q]]]$$

(*Elliptic integral of the second kind*)

$$\text{d}\rho\text{eEn}[\rho p\_,\rho\_,\varphi p\_,\varphi\_,zp\_,z\_] := \sum_{q=1}^{2}(-1)^q\ \Big(\text{D}\big[\text{EllipticE}\big[\phi[\varphi,\varphi p[[q]]],k[\text{d}\rho,\rho p,z,zp]^2\big],\text{d}\rho\big]\Big)\text{/. d}\rho\rightarrow\ \rho$$

$$\text{d}\rho\text{eEt}[\rho p\_,\rho\_,\varphi p\_,\varphi\_,zp\_,z\_] := \sum_{q=1}^{2}(-1)^q\ \text{NIntegrate}\Big[-\frac{2\ \rho p\big(Z[z,zp]^2-\rho^2+\rho p^2\big)}{R[\rho,\rho p,z,zp]^4}\frac{t^2}{\sqrt{(1-t^2)\ (1-k[\rho,\rho p,z,zp]^2\ t^2)}},\{t,0,\text{Sin}[\phi[\varphi,\varphi p[[q]]]]\}.$$

$$\text{d}\rho\text{eEa}[\rho p\_,\rho\_,\varphi p\_,\varphi\_,zp\_,z\_] := \sum_{q=1}^{2}(-1)^q\frac{-2\ \rho p\big(Z[z,zp]^2-\rho^2+\rho p^2\big)}{R[\rho,\rho p,z,zp]^4}\text{EllipticDT}\big[\phi[\varphi,\varphi p[[q]]],k[\rho,\rho p,z,zp]^2\big]$$

(*Elliptic integral of the third kind*)

$$\text{d}\rho\text{ePn}[\rho p\_,\rho\_,\varphi p\_,\varphi\_,zp\_,z\_] := \sum_{q=1}^{2}(-1)^q\ \Big(\text{D}\big[\text{EllipticPi}\big[\varkappa[\text{d}\rho,\rho p]^2,\phi[\varphi,\varphi p[[q]]],k[\text{d}\rho,\rho p,z,zp]^2\big],\text{d}\rho\big]\Big)\text{/. d}\rho\rightarrow\ \rho$$

$$\text{d}\rho\text{ePt}[\rho p\_,\rho\_,\varphi p\_,\varphi\_,zp\_,z\_] := \sum_{q=1}^{2}(-1)^q\ \text{NIntegrate}\Big[\Big(\frac{k[\rho,\rho p,z,zp]^2\big(Z[z,zp]^2-\rho^2+\rho p^2\big)}{2\ \ \rho\ R[\rho,\rho p,z,zp]^2\ \ (1-k[\rho,\rho p,z,zp]^2t^2)}-\frac{\varkappa[\rho,\rho p]^2\ \overline{\varrho}[\rho,\rho p]}{\rho\ \varrho[\rho,\rho p]\ (1-\varkappa[\rho,\rho p]^2t^2)}\Big)\frac{}{(1-\varkappa[\rho,\rho p}$$

$$\text{d}\rho\text{ePa}[\rho p\_,\rho\_,\varphi p\_,\varphi\_,zp\_,z\_] := \sum_{q=1}^{2}(-1)^q\Big(\Big(\frac{1}{2\ \rho}-\frac{1}{\overline{\varrho}[\rho,\rho p]}-\frac{\varrho[\rho,\rho p]}{R[\rho,\rho p,z,zp]^2}\Big)\text{EllipticPi}\big[k[\rho,\rho p,z,zp]^2,\phi[\varphi,\varphi p[[q]]],k[\rho,\rho p,z,zp]^2\ \ \big]+\Big(-\frac{}{\overline{\varrho}[}$$

(*Regularised beta function*)

$$\text{d}\rho\text{rBn}[\rho p\_,\rho\_,a\_,b\_,zp\_,z\_] :=\ \ \Big(\text{D}\big[\text{BetaRegularized}\big[zs[\text{d}\rho,\rho p,z,zp]^2,a,b\big],\text{d}\rho\big]\Big)\text{/. d}\rho\rightarrow\ \rho$$

$$\text{d}\rho\text{rBa}[P\_,\rho p\_,\rho\_,a\_,b\_,zp\_,z\_] := \frac{-2\rho}{\text{Beta}[a,b]T[\rho,\rho p,z,zp]^2}zs[\rho,\rho p,z,zp]^{2a}\big(1-zs[\rho,\rho p,z,zp]^2\big)^{b-1}$$

4.0.1 $\frac{\partial}{\partial\rho}$ Elliptic integral of the first kind

In[ ]:=
```
Compare2[ρ'[[2]],ρ1,{π/4,4π/3},φ1,z'[[1]],z1,dρeFn,dρeFt,dρeFa]
```

Out[ ]//TableForm=

Numeric Form 1	69.52930154354390
Numeric Form 2	69.52930154354402
Analytic Form	69.52930154354390

4.0.2 $\frac{\partial}{\partial \varphi}$ Elliptic integral of the first kind

In[ ]:=
```
Compare2[ρ'[[2]],ρ1,{π/4,4π/3},φ1,z'[[1]],z1,dφeFn,0,dφeFa]
```

Out[ ]//TableForm=

Numeric Form 1	-2.813617119399465
Numeric Form 2	None
Analytic Form	-2.813617119399465

4.0.3 $\frac{\partial}{\partial z}$ Elliptic integral of the first kind

In[ ]:=
```
Compare2[ρ'[[2]],ρ1,{π/4,4π/3},φ1,z'[[1]],z1,dzeFn,dzeFt,dzeFa]
```

Out[ ]//TableForm=

Numeric Form 1	208.7535820767244
Numeric Form 2	208.7535820767247
Analytic Form	208.7535820767244

4.0.4 $\frac{\partial}{\partial \rho}$ Elliptic integral of the second kind

In[ ]:=
```
Compare2[ρ'[[2]],ρ1,{π/4,4π/3},φ1,z'[[1]],z1,dρeEn,dρeEt,dρeEa]
```

Out[ ]//TableForm=

Numeric Form 1	-4.561890931702129
Numeric Form 2	-4.561890931702118
Analytic Form	-4.561890931702129

4.0.5 $\frac{\partial}{\partial\rho}$ Elliptic integral of the third kind

In[]:=
```
Compare2[ρ'[[2]],ρ1,{π/4,4π/3},φ1,z'[[1]],z1,dρePn,dρePt,dρePa]
```

Out[]//TableForm=

Numeric Form 1	25 513.98100004374
Numeric Form 2	25 513.98100004376
Analytic Form	25 513.98100004374

4.0.6 $\frac{\partial}{\partial\rho}$ Regularised beta function

In[]:=
```
Compare[0,ρ'[[2]],ρ1,4,6,z'[[1]],z1,dρrBn,dρrBa,0]//N
```

Out[]//TableForm=

Numeric	-124.787
Analytic Form 1	-124.787
Analytic Form 2	None

5.0 Forces between Axially Magnetised Permanent Magnets

5.0.0 Equations

Analytic and Numeric function handles. Returns F={Fx,Fy,Fz}.

In[∘]:=

```
FzAna[M_,Mp_,ρp_,ρ_,φp_,φ_,zp_,z_,V_,P_,U_,O_] := (M Mp u0)/(4π) Σ_{m=1}^{2} Σ_{n=1}^{2} Σ_{q=1}^{2} Σ_{mp=1}^{2} Σ_{np=1}^{2} Σ_{qp=1}^{2} (-1)^(m+n+q+mp+np+qp) FzSummand[ρp[[mp]],ρ[[m]],φp[[qp]],φ[[

FzNum[M_,Mp_,ρp_,ρ_,φp_,φ_,zp_,z_] := (M Mp u0)/(4π) ( Σ_{np=1}^{2} (-1)^np NIntegrate[FzIntegrand1[dρp,dρ,dφp,dφ,zp[[np]],dz],{dρ,ρ[[1]],ρ[[2]]},{
    Σ_{m=1}^{2} Σ_{np=1}^{2} (-1)^(m+np) NIntegrate[FzIntegrand2[dρp,ρ[[m]],dφp,dφ,zp[[np]],dz],{dφ,φ[[1]],φ
    Σ_{n=1}^{2} Σ_{qp=1}^{2} (-1)^(n+qp) NIntegrate[FzIntegrand3[dρp,dρ,φp[[qp]],dφ,dzp,z[[n]]],{dρ,ρ[[1]],ρ
    + Σ_{m=1}^{2} Σ_{mp=1}^{2} (-1)^(m+mp) NIntegrate[FzIntegrand4[ρp[[mp]],ρ[[m]],dφp,dφ,dzp,dz],{dφ,φ[[1]],
    + Σ_{m=1}^{2} Σ_{qp=1}^{2} (-1)^(m+qp) NIntegrate[FzIntegrand5[dρp,ρ[[m]],φp[[qp]],dφ,dzp,dz],{dφ,φ[[1]],
    + Σ_{q=1}^{2} Σ_{mp=1}^{2} (-1)^(q+mp) NIntegrate[FzIntegrand6[ρp[[mp]],dρ,dφp,φ[[q]],dzp,dz],{dρ,ρ[[1]],
    + Σ_{q=1}^{2} Σ_{qp=1}^{2} (-1)^(q+qp) NIntegrate[FzIntegrand7[dρp,dρ,φp[[qp]],φ[[q]],dzp,dz],{dρ,ρ[[1]],
```

Special cases of the geometry for the analytic function handle. Replaces FzAna[] or FzNum[].

```
FzAna[M_,Mp_,ρp_,ρ_,{0,2π},{0,2π},zp_,z_,V_,P_,U_,O_] := 2Mp M u0 Σ_{m=1}^{2} Σ_{n=1}^{2} Σ_{mp=1}^{2} Σ_{np=1}^{2} (-1)^(m+n+mp+np) {0,0,FzSummandAS[ρp[[mp]],ρ[[m]],zp[[np

FzAna[M_,Mp_,{0,ρp_},{0,ρ_},{0,2π},{0,2π},zp_,z_,V_,P_,U_,O_] := 2 Mp M u0 Σ_{n=1}^{2} Σ_{np=1}^{2} (-1)^(n+np) FzSummandAS[ρp,ρ,zp[[np]],z[[n]]]

FzNum[M_,Mp_,ρp_,ρ_,{0,2π},{0,2π},zp_,z_] := (M Mp u0)/2 Σ_{m=1}^{2} Σ_{mp=1}^{2} (-1)^(m+mp) {0,0,NIntegrate[FzIntegrandAS[ρp[[m]],ρ[[mp]],dφp,0,dzp,dz],{
```

Integrands to be solved for FzNum[].

```
FzIntegrandAS[ρp_,ρ_,φp_,φ_,zp_,z_] := -ρp ρ Z[z,zp] Cos[φ-φp]G[ρ,ρp,φ,φp,z,zp]^3

FzIntegrand1[ρp_,ρ_,φp_,φ_,zp_,z_] := {-ρp Z[z,zp]Cos[φ]G[ρ,ρp,φ,φp,z,zp]^3,-ρp (z-zp) Sin[φ]G[ρ,ρp,φ,φp,z,zp]^3,0}
FzIntegrand2[ρp_,ρ_,φp_,φ_,zp_,z_] := {ρp ρ Z[z,zp]Cos[φ]G[ρ,ρp,φ,φp,z,zp]^3,ρp ρ(z-zp)Sin[φ]G[ρ,ρp,φ,φp,z,zp]^3,0}
FzIntegrand3[ρp_,ρ_,φp_,φ_,zp_,z_] := {-ρp Z[z,zp]Sin[φ]G[ρ,ρp,φ,φp,z,zp]^3,ρp(z-zp)Cos[φ]G[ρ,ρp,φ,φp,z,zp]^3,0}
FzIntegrand4[ρp_,ρ_,φp_,φ_,zp_,z_] := {0,0,-Z[z,zp] ρ ρp Cos[φ-φp]G[ρ,ρp,φ,φp,z,zp]^3}
FzIntegrand5[ρp_,ρ_,φp_,φ_,zp_,z_] := {0,0,-ρ Z[z,zp] Sin[φ-φp]G[ρ,ρp,φ,φp,z,zp]^3}
FzIntegrand6[ρp_,ρ_,φp_,φ_,zp_,z_] := {0,0,Z[z,zp] ρp Sin[φ-φp]G[ρ,ρp,φ,φp,z,zp]^3}
FzIntegrand7[ρp_,ρ_,φp_,φ_,zp_,z_] := {0,0,-Z[z,zp] Cos[φ-φp]G[ρ,ρp,φ,φp,z,zp]^3}
```

Summands and ancillary functions for FzAna[].

```
FzSummand[ρp_,ρ_,φp_,φ_,zp_,z_,V_,P_,U_,O_] := {FzSx[ρp,ρ,φp,φ,zp,z,V,P,U], FzSy[ρp,ρ,φp,φ,zp,z,V,P,U], FzSz[ρp,ρ,φp,φ,zp,z,V,P,

FzSummandAS[ρp_,ρ_,zp_,z_] := Z[z,zp]/R[ρ,ρp,z,zp] (ϱ[ρ,ρp]^2/4 (EllipticPi[ϰ[ρ,ρp]^2,k[ρ,ρp,z,zp]^2]-EllipticK[k[ρ,ρp,z,zp]^2])-ρ ρp EllipticD

FzSx[ρp_,ρ_,φp_,φ_,zp_,z_,V_,P_,U_]:= Sin[φ] Sum[(-φp ς[0,p,ρp,ρ,zp,z]- 1/(4Sin[φ]) (Cos[2 φ-φp]+2 φ Sin[φp]) ς[1,p,ρp,ρ,zp,z]-Sum[ν Cos, {ν,2,V}], {p,0,P}]
FzSy[ρp_,ρ_,φp_,φ_,zp_,z_,V_,P_,U_]:= Cos[φ] Sum[(φp ς[0,p,ρp,ρ,zp,z]- 1/(4Cos[φ]) (-2 φ Cos[φp]+Sin[2 φ-φp]) ς[1,p,ρp,ρ,zp,z]- Sum[ν C, {ν,2,V}], {p,0,P}]
FzSz[ρp_,ρ_,φp_,φ_,zp_,z_,V_,P_,U_,O_]:= Sum[(-Cos[φ-φp] (1+2 p) ς[0,p,0,0,0,ρp,ρ,zp,z]- 1/8 (4 φ φp-2 φp^2+Cos[2 (φ-φp)]) (1+2 p+1) ς, {p,0,P}]

ς[ν_,p_,ρp_,ρ_,zp_,z_]:= If[ν==0,1,2] ρ^2/(2 Sqrt[π]) (((z-zp)^2+ρ^2)/ρ^2)^(1/2-p-ν/2) Gamma[1/2+2 p+ν]/(p! (p+ν)!) Beta[ρp^2/((z-zp)^2+ρ^2+ρp^2), 1/2 (2+2 p+ν), 1/2 (-1+2 p+ν)]
ς[ν_,p_,u_,ρp_,ρ_,zp_,z_]:= If[ν==0,1,2] ρp^2/(4 Sqrt[π]) (1+(z-zp)^2/ρp^2)^(-p-ν/2-u) Gamma[1+p+ν/2] Gamma[1/2+2 p+u+ν]/(Gamma[2+p+u+ν/2] p! (p+ν)!) Beta[ρ^2/((z-zp)^2+ρ^2+ρp^2), 1/2 (1+2 p+ν), 1/2
ς[ν_,p_,u_,o_,l_,ρp_,ρ_,zp_,z_]:= If[ν==0,1,2] (1+2 p+ν) (ρ ρp Sign[(z-zp)])/(8 Sqrt[π]) ((ρ ρp)/(ρ^2+ρp^2))^(2 p+ν) (((ρp^2)/(ρ^2+ρp^2))^u+l ((ρ^2)/(ρ^2+ρp^2))^u) ((ρ^2)/(ρ^2+ρp^2))^o Gamm/Gamma[
```

Singularities in the summands of FzAna[].

```
In[•]:= (*On the shell plane & axisymmetric*)
FzSummandAS[ρp_,ρp_,zp_,z_] :=- Z[z,zp]ρp^2/R[ρp,ρp,z,zp] EllipticD[k[ρp,ρp,z,zp]^2]
```

5.0.1 An axisymmetric force

Hollow rings.

In[ ]:=
```
ResultTableForce[800*10^3,-955*10^3,{5*10^-3,10*10^-3},{5*10^-3,8*10^-3},{0,2π},{0,2π},{0,4*10^-3},{5*10^-3,10*10^-3},0,0,0,0,FzAna,FzNum]
```

	Fx	Fy	Fz
Analytic	0	0	17.43010334628681
Numeric	0	0	17.43010333790363
Comparison 8dp	0	0	0

Solid rings (m-summations removed).

```
ResultTableForce[800*10^3,-955*10^3,{0,10*10^-3},{0,8*10^-3},{0,2π},{0,2π},{0,4*10^-3},{5*10^-3,10*10^-3},0,0,0,0,FzAna,FzNum]
```

	Fx	Fy	Fz
Analytic	0	0	17.43010334628681
Numeric	0	0	17.43010333790363
Comparison 8dp	0	0	0

5.0.2 A non-axisymmetric force

The partial sum is computed without an algorithm, and a low number of terms have been chosen.

In[ ]:=
```
ResultTableForce[800*10^3,-955*10^3,{5*10^-3,10*10^-3},{5*10^-3,8*10^-3},{-π/6,π/6},{-π/12,5π/12},{0,4*10^-3},{10*10^-3,15*10^-3},5,30,15,5,FzAna,F
```

	Fx	Fy	Fz
Analytic	−0.1737413964903685	0.1709873453068254	0.4523015329859225
Numeric	−0.173396819398153	0.171120681310334	0.453936214106906
Comparison 8dp	−0.000344577092216	−0.000133336003508	−0.001634681120983