A Generalised Maxwell Stress Tensor for Semi-Analytic Force and Torque Between Permanent Magnets, Coils, and Soft Iron

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Supplemental material (https://doi.org/10.1016/j.apm.2025.116050). A related online repository can be found at: https://github.com/AUMAG/mag-gmst-force. This document uses the Generalised Maxwell Stress Tensor (GMST) and MST to calculate 3D force and torque on a region using elemental magnetic field solutions. Provided is the package <u>MagCylField</u> (Section 0.3) that can be used to calculate the magnetic field from cylindrical arc magnets and coils. A single case study is (currently) shown where the force and torque is calculated between diametrically magnetised arcs using a cuboid closed-surface mesh (Section 1). The results are compared to an FEA model herein and in the Appendix of the published article.

- If you are reading this in PDF format, note the code is truncated. There is a Mathematica notebook '.nb' file that contains full functionality and a package '.wl' file that contains the nomenclature and magnetic field equations. If you do not have a paid/trial version of Mathematica, this notebook can be opened in a free player (e.g Wolfram Player https://www.wolfram.com/player/) in order to view/copy the tensors/equations. There are also built-in or addon functions that can convert the equations to MATLAB, L^AT_EX, Python, etc...
- The code below is not written to optimise computational speed. This serves as a visualisation with example usage.
- The field points (and sometimes source points) need to be rounded to \$MachineEpsilon to ensure the memoization for special cases correctly executes. This otherwise leads to undefined or infinite results.
- The magnetic field functions will return values in Cylindrical coordinates unless ρ=0.
- It is useful to operate in both Cylindrical and Cartesian coordinates for the calculations and visualisations (and to check results). There is no error handling if a mesh field point is along the axis as the point in Cylindrical coordinates is undefined. It is suggested to avoid ρ=0 (x=0 && y=0), even though the field functions will accept this and return a correct value (in Cartesian coordinates).
- Mathematica uses complex numbers for the evaluation of ArcTan, ArcTanh,... and as such these return real numbers of the form a+0.i when converted to a numerical value using '//N' (this is useful for not keeping all values symbolic). We use '//Chop' to remove numerical 0.i terms.

0.1 Citations for this work

M. Forbes, W.S.P Robertson, A.C. Zander, J. Vidler, J.J.H. Paulides, "A Generalised Maxwell Stress Tensor for Semi-Analytic Force and Torque Between Permanent Magnets, Coils, and Soft Iron"

```
@Article {Forbes2025,
     author = {Forbes, M. and Robertson, W.S.P. and Zander, A.C. and Vidler, J. and Paulides, J.J.H},
     journal = {Applied Mathematical Modelling},
     title = {A Generalised Maxwell Stress Tensor for Semi-Analytic Force and Torque Between Permanent Magnets, Coils, and Soft Iron},
     doi = {10.1016/j.apm.2025.116050},
     publisher = {Elsevier},
     pages = {116050},
}
```

M. Forbes, W.S.P Robertson, A.C. Zander, J.J.H. Paulides, "The Magnetic Field from Cylindrical Arc Coils and Magnets: A Compendium with New Analytic Solutions for Radial Magnetisation and Azimuthal Current"

```
@Article {Forbes2024,
     author = {Forbes, M. and Robertson, W.S.P. and Zander, A.C. and Paulides, J.J.H},
     journal = {Advanced Physics Research},
     title = {The Magnetic Field from Cylindrical Arc Coils and Magnets: A Compendium with New Analytic Solutions for Radial Magnetisation and Azimuthal Current},
     doi = {10.1002/apxr.202300136},
     publisher = {Wiley},
     number = {7},
     pages = {2300136},
     volume = {3}
}
```

0.2 Tensors

FGMST and TGMST are the Generalised Maxwell Stress Tensors (GMST) for force and torque. FMST and TMST are the Maxwell Stress Tensors (MST) for force and torque.

```
In[ ]:= FMST[Bx_,By_,Bz_] := u0⁻¹{{1/2 (Bx²-By²-Bz²),Bx By,Bx Bz},{Bx By,1/2 (-Bx²+By²-Bz²),By Bz},{Bx Bz,By Bz,1/2 (-Bx²-By²+Bz²)}};

       TMST[x_,y_,z_,Bx_,By_,Bz_] := u0⁻¹{{-z Bx By+y Bx Bz,-z By²+y By Bz+1/2 z (Bx²+By²+Bz²),-z By Bz+y Bz²-1/2 y (Bx²+By²+Bz²)},{z Bx²-x Bx Bz

       FGMST[BEx_,BEy_,BEz_,BIx_,BIy_,BIz_] := u0⁻¹{{BEx BIx-BEy BIy-BEz BIz,BEy BIx+BEx BIy,BEz BIx+BEx BIz},{BEy BIx+BEx BIy,-BEx BIx+BEy B

       TGMST[x_,y_,z_,BEx_,BEy_,BEz_,BIx_,BIy_,BIz_] := u0⁻¹{{-z(BEx BIy+BEy BIx)+y(BEx BIz+BEz BIx),y(BEy BIz+BEz BIy)+z(BEx BIx-BEy BIy+BE

       u0 FGMST[Bₓ"E",Bᵧ"E",B_z"E",Bₓ"I",Bᵧ"I",B_z"I"]//MatrixForm
       u0 TGMST[x,y,z,Bₓ"E",Bᵧ"E",B_z"E",Bₓ"I",Bᵧ"I",B_z"I"]//MatrixForm
```

$Out[ ]//MatrixForm=$

$$\begin{pmatrix} B_x^E B_x^I - B_y^E B_y^I - B_z^E B_z^I & B_x^I B_y^E + B_x^E B_y^I & B_x^I B_z^E + B_x^E B_z^I \\ B_x^I B_y^E + B_x^E B_y^I & -B_x^E B_x^I + B_y^E B_y^I - B_z^E B_z^I & B_y^I B_z^E + B_y^E B_z^I \\ B_x^I B_z^E + B_x^E B_z^I & B_y^I B_z^E + B_y^E B_z^I & -B_x^E B_x^I - B_y^E B_y^I + B_z^E B_z^I \end{pmatrix}$$

$Out[ ]//MatrixForm=$

$$\begin{pmatrix} -z\left(B_x^I B_y^E + B_x^E B_y^I\right) + y\left(B_x^I B_z^E + B_x^E B_z^I\right) & y\left(B_y^I B_z^E + B_y^E B_z^I\right) + z\left(B_x^E B_x^I - B_y^E B_y^I + B_z^E B_z^I\right) & -z\left(B_y^I B_z^E + B_y^E B_z^I\right) - y\left(B_x^E B_x^I + B_y^E B_y^I - B_z^E B_z^I\right) \\ -x\left(B_x^I B_z^E + B_x^E B_z^I\right) + z\left(B_x^E B_x^I - B_y^E B_y^I - B_z^E B_z^I\right) & z\left(B_x^I B_y^E + B_x^E B_y^I\right) - x\left(B_y^I B_z^E + B_y^E B_z^I\right) & z\left(B_x^I B_z^E + B_x^E B_z^I\right) + x\left(B_x^E B_x^I + B_y^E B_y^I - B_z^E B_z^I\right) \\ x\left(B_x^I B_y^E + B_x^E B_y^I\right) + y\left(-B_x^E B_x^I + B_y^E B_y^I + B_z^E B_z^I\right) & -y\left(B_x^I B_y^E + B_x^E B_y^I\right) - x\left(B_x^E B_x^I - B_y^E B_y^I + B_z^E B_z^I\right) & -y\left(B_x^I B_z^E + B_x^E B_z^I\right) + x\left(B_y^I B_z^E + B_y^E B_z^I\right) \end{pmatrix}$$

0.3 Import Cylindrical Magnetic Field Equations

The MagCylField package was created for this article using the work from our previous article https://doi.org/10.1002/apxr.202300136, with the supplemental material https://github.com/AUMAG/-mag-cyl-field.

This notebook file and package assume the user has Mathematica 12.3 or later. If this is not the case, you will need to include the (commented out) Carlson package in both files https://github.-com/tpfto/Carlson/tree/v1.1 oi.
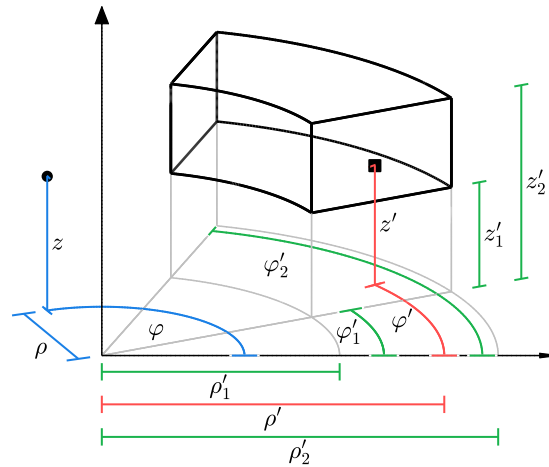

```
In[ ]:=  SetDirectory[NotebookDirectory[]];
         << MagCylField`  (* Needs["Carlson`"] *)
         (* << Carlson`   Used for some backwards compatibility with a special case of EllipticF[ϕ,1], and not required in 12.3 or later: https:/
```

Brief README.

```
In[●]:=   (*  CylB𝑑: Diametric magnetisation
          CylBρ: Radial magnetisation
          CylBφ: Azimuthal magnetisation,
          CylB𝑧: Axial magnetisation,
          CylB𝐼f: Azmimuthal current filament,
          CylB𝒦𝑑: Azmimuthal current disc,
          CylB𝒦𝑠: Azmimuthal current shell,
          CylB𝒥𝑣: Azmimuthal current volume  *)
       CylB𝑑::usage
```

Out[●]= Diametric magnetisation
       CylB𝑑[M,φ☆,{ρ₁',ρ₂'},ρ,{φ₁',φ₂'},φ,{z₁',z₂'},z] with: M magnetisation
        magnitude (A/m), φ☆ magnetision direction (rad), r' source limits (m,rad,m), r field point (m,rad,m).
       Returns the magnetic field in cylindrical coordinates for ρ≠0 and Cartesian coordinates for ρ=0.
       Points on the magnet surface are (potentially) undefined.
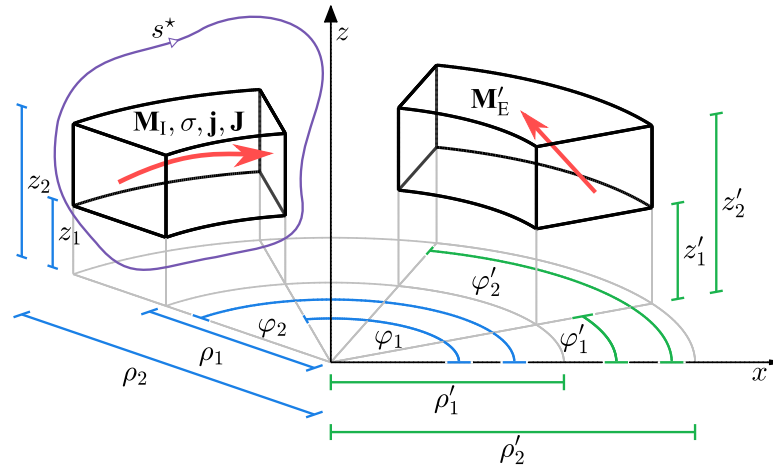
0.4 Constants
0.5 Plot functions
0.6 Mesh functions
0.7 Transforms
0.8 Figure and result table

1.0 Case study 1: Diametrically magnetised PMs

1.1 Geometry

PM details.

$In[\bullet]:=$
```
ME = 955*10³;  (*A/m*)
MI = 800*10³;  (*A/m*)
```

$$\rho' = \left\{ \frac{5}{1000}, \frac{10}{1000} \right\}; \quad z' = \left\{ \frac{0}{1000}, \frac{5}{1000} \right\}; \quad (*\text{ME definite integral limits}*)$$

$$\rho = \left\{ \frac{5}{1000}, \frac{10}{1000} \right\}; \quad \varphi = \left\{ \frac{\pi}{32}, \frac{\pi}{4} \right\}; \quad z = \left\{ \frac{6}{1000}, \frac{11}{1000} \right\}; \quad (*\text{MI definite integral limits}*)$$

$$\text{meshTol} = \left\{ \frac{2}{1000}, \frac{1}{2000} \right\}; \quad (* \text{ Scaling for mesh spacing \{figure, calculation\} } *)$$

```
noMags=3;  (* Number of magnets in lower ring -- FEA result only valid for 3 *)
```

$$\varphi \star = \frac{2\pi}{\text{noMags}+1} \left(\# - \frac{1}{2} \right) \& /@ \text{Range[1,noMags+1];} (* \text{ Diametric magnetisation directions } *)$$

1.2 Primary result

Mesh details.

In[ ]:= $\quad$ x☆ = $\left\{\dfrac{3}{1000},\dfrac{21}{2000}\right\}$; y☆= $\left\{\dfrac{0}{1000},\dfrac{15}{2000}\right\}$; z☆ = $\left\{\dfrac{11}{2000},\dfrac{23}{2000}\right\}$; $\quad$ (*GMST definite integral limits*)

Results table and visualisation plot.

Accuracy and computational cost increases with increasing meshTol.

In[ ]:= $\quad$ **DiametricExample["Standard",noMags,meshTol,x☆,y☆,z☆,φ☆,ME,ρ',z',MI,ρ,φ,z]**

	Elements	F_x, N	F_y, N	F_z, N	T_x, mN·m	T_y, mN·m	T_z, mN·m
MST	1008	0.372	0.124	1.017	−0.713	−1.021	1.308
GMST	1008	0.373	0.123	1.014	−0.699	−0.995	1.29
FEA	N/A	0.371	0.123	1.013	−0.698	−1.002	1.287
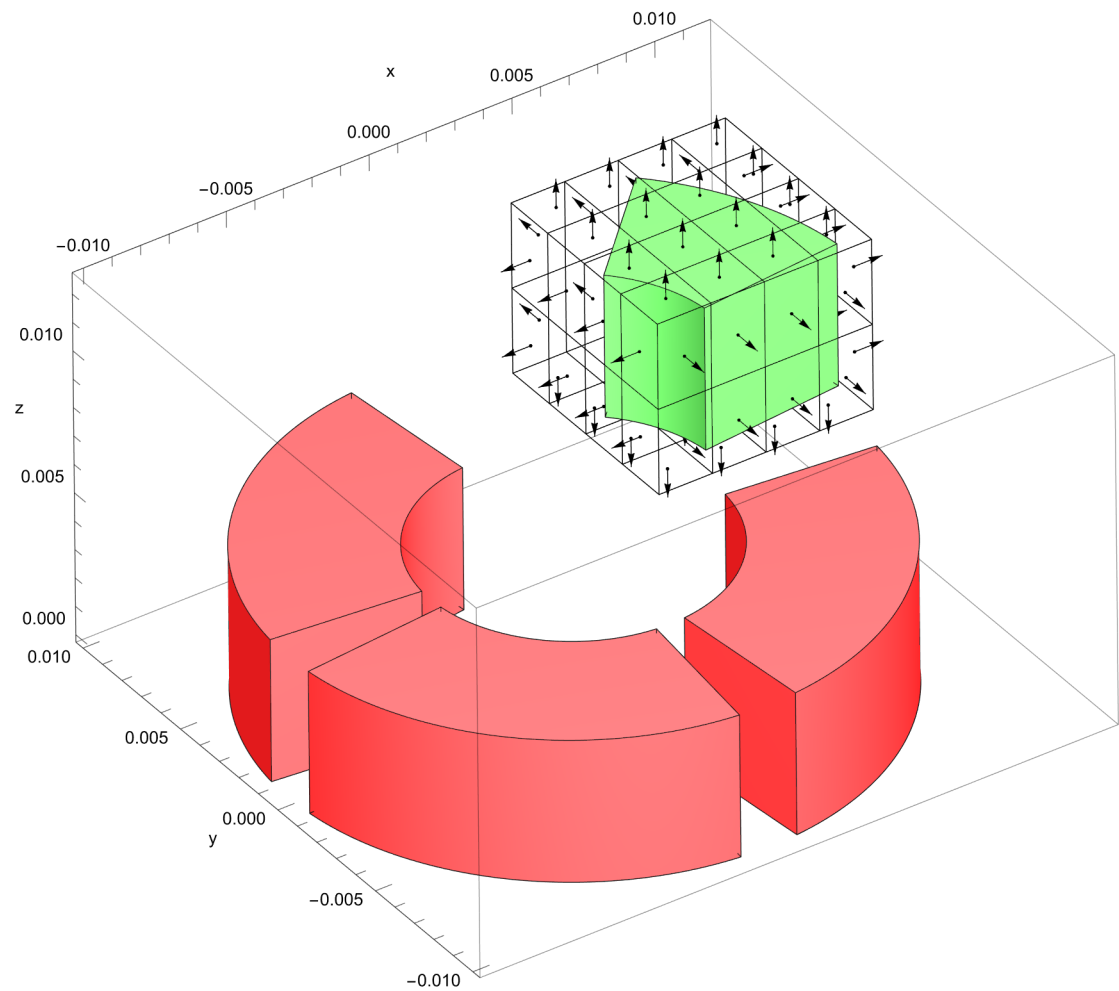
1.3 Inverse result of §1.2

Mesh details.

$In[\circ]:=$ `x☆2 =` $\left\{-\dfrac{21}{2000}, \dfrac{21}{2000}\right\}$`; y☆2=` $\left\{-\dfrac{21}{2000}, \dfrac{21}{2000}\right\}$`; z☆2 =` $\left\{-\dfrac{1}{2000}, \dfrac{11}{2000}\right\}$`; (*GMST definite integral limits*)`

BE and BI reversed (or primed and unprimed reversed)

$In[\circ]:=$ `DiametricExample["Inverse",noMags,meshTol,x☆2,y☆2,z☆2,φ☆,ME,ρ',z',MI,ρ,φ,z]`

	Elements	F_x, N	F_y, N	F_z, N	T_x, mN·m	T_y, mN·m	T_z, mN·m
MST	5376	−0.37	−0.121	−1.012	0.695	0.998	−1.284
GMST	5376	−0.371	−0.122	−1.013	0.696	1.002	−1.286
FEA	N/A	−0.371	−0.123	−1.013	0.698	1.002	−1.287