Forces between magnets and multipole arrays of magnets: A Matlab implementation

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Abstract

This is the user guide and documented implementation of a set of Matlab functions for calculating the forces (and stiffnesses) between cuboid permanent magnets and between multipole arrays of the same.

This document is still evolving. The documentation for the source code, especially, is rather unclear/non-existent at present. The user guide, however, should contain the bulk of the information needed to use this code.

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1 User guide

(See Section 2 for installation instructions.)

1.1 Forces between magnets

The function `magnetforces` is used to calculate both forces and stiffnesses between magnets. The syntax is as follows:

```
forces = magnetforces(magnet_fixed, magnet_float, displ);
   ... = magnetforces( ... , 'force');
   ... = magnetforces( ... , 'stiffness');
   ... = magnetforces( ... , 'torque');
   ... = magnetforces( ... , 'x');
   ... = magnetforces( ... , 'y');
   ... = magnetforces( ... , 'z');
```

`magnetforces` takes three mandatory inputs to specify the position and magnetisation of the first and second magnets and the displacement between them. Optional arguments appended indicate whether to calculate force and/or torque and/or stiffness and whether to calculate components in x- and/or y- and/or z- components respectively. The force[1] is calculated as that imposed on the second magnet; for this reason, I often call the first magnet the 'fixed' magnet and the second 'floating'.

Outputs You must match up the output arguments according to the requested calculations. For example, when only calculating torque, the syntax is

```
T = magnetforces(magnet_fixed, magnet_float, displ,'torque');
```

Similarly, when calculating all three of force/stiffness/torque, write

```
[F S T] = magnetforces(magnet_fixed, magnet_float, displ,...
              'force','stiffness','torque');
```

The ordering of 'force', 'stiffness', 'torque' affects the order of the output arguments. As shown in the original example, if no calculation type is requested then the forces only are calculated.

Cuboid magnets The first two inputs are structures containing the following fields:

`magnet.dim` A (3×1) vector of the side-lengths of the magnet.
`magnet.grade` The 'grade' of the magnet as a string such as `'N42'`.

[1] From now I will omit most mention of calculating torques and stiffnesses; assume whenever I say 'force' I mean 'force *and/or* stiffness *and/or* torque'

`magnet.magdir` A vector representing the direction of the magnetisation. This may be either a (3×1) vector in cartesian coordinates or a (2×1) vector in spherical coordidates.

Instead of specifying a magnet grade, you may explicitly input the remanence magnetisation of the magnet direction with

`magnet.magn` The remanence magnetisation of the magnet in Tesla.

Note that when not specified, the `magn` value B_r is calculated from the magnet grade N using $B_r = 2\sqrt{N/100}$.

In cartesian coordinates, the `magdir` vector is interpreted as a unit vector; it is only used to calculate the direction of the magnetisation. In other words, writing `[1;0;0]` is the same as `[2;0;0]`, and so on. In spherical coordinates (θ, ϕ), θ is the vertical projection of the angle around the x–y plane ($\theta = 0$ coincident with the x-axis), and ϕ is the angle from the x–y plane towards the z-axis. In other words, the following unit vectors are equivalent:

$$(1,0,0)_{\text{cartesian}} \equiv (0,0)_{\text{spherical}}$$
$$(0,1,0)_{\text{cartesian}} \equiv (90,0)_{\text{spherical}}$$
$$(0,0,1)_{\text{cartesian}} \equiv (0,90)_{\text{spherical}}$$

N.B. θ and ϕ must be input in degrees, not radians. This seemingly odd decision was made in order to calculate quantities such as $\cos(\pi/2) = 0$ exactly rather than to machine precision.[2]

If you are calculating the torque on the second magnet, then it is assumed that the centre of rotation is at the centroid of the second magnet. If this is not the case, the centre of rotation of the second magnet can be specified with

`magnet_float.lever` A (3×1) vector of the centre of rotation (or $(3 \times D)$ if necessary; see D below).

Cylindrical magnets/coils If the dimension of the magnet (`magnet.dim`) only has two elements, or the `magnet.type` is '`cylinder`', the forces are calculated between two cylindrical magnets.

Only the force between coaxial cylinders can be calculated at present; this is still an area of active investigation.

`magnet.dim` A (2×1) vector containing, respectively, the magnet radius and length.

`magnet.dir` Alignment direction of the cylindrical magnets; '`x`' or '`y`' or '`z`' (default). E.g., for an alignment direction of '`z`', the faces of the cylinder will be oriented in the x–y plane.

A 'thin' magnetic coil can be modelled in the same way as a magnet, above; instead of specifying a magnetisation, however, use the following:

[2] Try for example comparing the logical comparisons `cosd(90)==0` versus `cos(pi)==0`.

`coil.turns` A scalar representing the number of axial turns of the coil.
`coil.current` Scalar coil current flowing CCW-from-top.

A 'thick' magnetic coil contains multiple windings in the radial direction and requires further specification. The complete list of variables to describe a thick coil, which requires `magnet.type` to be '`coil`' are

`coil.dim` A (3×1) vector containing, respectively, the inner coil radius, the outer coil radius, and the coil length.
`coil.turns` A (2×1) containing, resp., the number of radial turns and the number of axial turns of the coil.
`coil.current` Scalar coil current flowing CCW-from-top.

Again, only coaxial displacements and forces can be investigated at this stage.

Displacement inputs The third mandatory input is `displ`, which is a matrix of displacement vectors between the two magnets. `displ` should be a $(3 \times D)$ matrix, where D is the number of displacements over which to calculate the forces. The size of `displ` dictates the size of the output force matrix; `forces` (etc.) will be also of size $(3 \times D)$.

Example Using `magnetforces` is rather simple. A magnet is set up as a simple structure like

```
magnet_fixed = struct(...
  'dim'   , [0.02 0.012 0.006], ...
  'magn'  , 0.38, ...
  'magdir', [0 0 1] ...
);
```

with something similar for `magnet_float`. The displacement matrix is then built up as a list of (3×1) displacement vectors, such as

```
displ = [0; 0; 1]*linspace(0.01,0.03);
```

And that's about it. For a complete example, see '`examples/magnetforces_example.m`'.

1.2 Forces between multipole arrays of magnets

Because multipole arrays of magnets are more complex structures than single magnets, calculating the forces between them requires more setup as well. The syntax for calculating forces between multipole arrays follows the same style as for single magnets:

```
     forces = multipoleforces(array_fixed, array_float, displ);
stiffnesses = multipoleforces( ... , 'stiffness');
      [f s] = multipoleforces( ... , 'force', 'stiffness');
```

```
... = multipoleforces( ... , 'x');
... = multipoleforces( ... , 'y');
... = multipoleforces( ... , 'z');
```

Because multipole arrays can be defined in various ways, there are several overlapping methods for specifying the structures defining an array. Please escuse a certain amount of dryness in the information to follow; more inspiration for better documentation will come with feedback from those reading this document!

Linear Halbach arrays A minimal set of variables to define a linear multipole array are:

`array.type` Use '`linear`' to specify an array of this type.
`array.align` One of '`x`', '`y`', or '`z`' to specify an alignment axis along which successive magnets are placed.
`array.face` One of '`+x`', '`+y`', '`+z`', '`-x`', '`-y`', or '`-z`' to specify which direction the 'strong' side of the array faces.
`array.msize` A (3×1) vector defining the size of each magnet in the array.
`array.Nmag` The number of magnets composing the array.
`array.magn` The magnetisation magnitude of each magnet.
`array.magdir_rotate` The amount of rotation, in degrees, between successive magnets.

Notes:

- The array must `face` in a direction orthogonal to its alignment.
- '`up`' and '`down`' are defined as synonyms for facing '`+z`' and '`-z`', respectively, and '`linear`' for array type '`linear-x`'.
- Singleton input to `msize` assumes a cube-shaped magnet.

The variables above are the minimum set required to specify a multipole array. In addition, the following array variables may be used instead of or as well as to specify the information in a different way:

`array.magdir_first` This is the angle of magnetisation in degrees around the direction of magnetisation rotation for the first magnet. It defaults to $\pm 90°$ depending on the facing direction of the array.
`array.length` The total length of the magnet array in the alignment direction of the array. If this variable is used then `width` and `height` (see below) must be as well.
`array.width` The dimension of the array orthogonal to the alignment and facing directions.
`array.height` The height of the array in the facing direction.
`array.wavelength` The wavelength of magnetisation. Must be an integer number of magnet lengths.

`array.Nwaves` The number of wavelengths of magnetisation in the array, which is probably always going to be an integer.

`array.Nmag_per_wave` The number of magnets per wavelength of magnetisation (e.g., `Nmag_per_wave` of four is equivalent to `magdir_rotate` of 90°).

`array.gap` Air-gap between successive magnet faces in the array. Defaults to zero.

Notes:

- `array.mlength`+`array.width`+`array.height` may be used as a synonymic replacement for `array.msize`.
- When using `Nwaves`, an additional magnet is placed on the end for symmetry.
- Setting `gap` does not affect `length` *or* `mlength`! That is, when `gap` is used, `length` refers to the total length of magnetic material placed end-to-end, not the total length of the array including the gaps.

Planar Halbach arrays Most of the information above follows for planar arrays, which can be thought of as a superposition of two orthogonal linear arrays.

`array.type` Use '`planar`' to specify an array of this type.

`array.align` One of '`xy`' (default), '`yz`', or '`xz`' for a plane with which to align the array.

`array.width` This is now the 'length' in the second spanning direction of the planar array. E.g., for the array '`planar-xy`', 'length' refers to the x-direction and 'width' refers to the y-direction. (And 'height' is z.)

`array.mwidth` Ditto for the width of each magnet in the array.

All other variables for linear Halbach arrays hold analogously for planar Halbach arrays; if desired, two-element input can be given to specify different properties in different directions.

Planar quasi-Halbach arrays This magnetisation pattern is simpler than the planar Halbach array described above.

`array.type` Use '`quasi-halbach`' to specify an array of this type.

`array.Nwaves` There are always four magnets per wavelength for the quasi-Halbach array. Two elements to specify the number of wavelengths in each direction, or just one if the same in both.

`array.Nmag` Instead of `Nwaves`, in case you want a non-integer number of wavelengths (but that would be weird).

Patchwork planar array

`array.type` Use '`patchwork`' to specify an array of this type.

`array.Nmag` There isn't really a 'wavelength of magnetisation' for this one; or rather, there is but it's trivial. So just define the number of magnets per side, instead. (Two-element for different sizes of one-element for an equal number of magnets in both directions.)

Arbitrary arrays Until now we have assumed that magnet arrays are composed of magnets with identical sizes and regularly-varying magnetisation directions. Some facilities are provided to generate more general/arbitrary–shaped arrays.

`array.type` Should be '`generic`' but may be omitted.

`array.mcount` The number of magnets in each direction, say (X, Y, Z).

`array.msize_array` An $(X, Y, Z, 3)$-length matrix defining the magnet sizes for each magnet of the array.

`array.magdir_fn` An anonymous function that takes three input variables (i, j, k) to calculate the magnetisation for the (i, j, k)-th magnet in the (x, y, z)-directions respectively.

`array.magn` At present this still must be singleton-valued. This will be amended at some stage to allow `magn_array` input to be analogous with `msize` and `msize_array`.

This approach for generating magnet arrays has been little-tested. Please inform me of associated problems if found.

2 Meta-information

Obtaining The latest version of this package may be obtained from the GitHub repository http://github.com/wspr/magcode with the following command:

```
git clone git://github.com/wspr/magcode.git
```

Installing It may be installed in Matlab simply by adding the '`matlab/`' subdirectory to the Matlab path; e.g., adding the following to your `startup.m` file: (if that's where you cloned the repository)

```
addpath ~/magcode/matlab
```

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[3] http://www.apache.org/licenses/LICENSE-2.0

Contributing and feedback Please report problems and suggestions at the GitHub issue tracker.[4]

The Matlab source code is written using Norman Ramsey's NOWEB literate programming tool.[5] After it is installed, use `make code` to extract the Matlab files `magnetforces.m` and `multipoleforces.m`, as well as extracting the test suite (such as it is, for now). Running `make doc` will compiling the documentation you are currently reading.

About this file This is a 'literate programming' approach to writing Matlab code using NOWEB[6]. To be honest I don't know if it's any better than simply using the Matlab programming language directly. The big advantage for me is that you have access to the entire LaTeX document environment, which gives you access to vastly better tools for cross-referencing, maths typesetting, structured formatting, bibliography generation, and so on.

The downside is obviously that you miss out on Matlab's IDE with its integrated M-Lint program, debugger, profiler, and so on. Depending on one's work habits, this may be more or less of limiting factor to using literate programming in this way.

[4] http://github.com/wspr/magcode/issues
[5] http://www.cs.tufts.edu/~nr/noweb/
[6] http://www.cs.tufts.edu/~nr/noweb/

3 Documented code in a literate programming style

3.1 Calculating forces between magnets

This is the source of some code to calculate the forces and/or stiffnesses between two cuboid-shaped magnets with arbitary displacements and magnetisation direction. (A cuboid is like a three dimensional rectangle; its faces are all orthogonal but may have different side lengths.)
The main function is |magnetforces|, which takes three mandatory arguments: |magnet_fixed|, |magnet_float|, and |displ|. These will be described in more detail below.

Optional string arguments may be any combination of `'force'`, and/or `'stiffness'` to indicate which calculations should be output. If no calculation is specified, `'force'` is the default.

Displacement between cylindrical magnets may only be in the axial direction; displacement between cuboid magnets in all three directions.

Rather than specifying a magnetisation for a cylindrical magnet, it may be considered a thin current-carrying coil with a certain number of turns.

Inputs:	|magnet_fixed|	structure describing first magnet
	|magnet_float|	structure describing the second magnet
	|displ|	displacement between the magnets
	[*what to calculate*]	any of 'force', 'torque', 'stiffness'
Outputs:	|forces|	forces on the second magnet
	|stiffnesses|	stiffnesses on the second magnet
Magnet properties:	|type|	'cuboid' (default) or 'cylinder'
	|dim|	size of each magnet
	|magn|	magnetisation magnitude
	|magdir|	magnetisation direction
	|lever|	displacement from centre of rotation (second magnet only when calculating torques)
	|dir|	alignment direction (cylindrical magnets only)
	|current|	(cylinder only) current in the coil
	|turns|	(cylinder only) number of coil turns

10 ⟨ *magnetforces.m* 10⟩≡

```
function [varargout] = magnetforces(magnet_fixed, magnet_float, displ, varargin)
```

⟨ *Matlab help text (forces)* 48b⟩

⟨ *Parse calculation args* 17⟩
⟨ *Organise input displacements* 16⟩
⟨ *Initialise main variables* 12⟩

⟨ *Precompute rotations* 46a⟩

⟨ *Calculate for each displacement* 19⟩
⟨ *Return all results* 18⟩

⟨ *Function for calculating magnetisation from grade* 44⟩
⟨ *Function for resolving magnetisations* 45⟩
⟨ *Function for single force calculation* 20a⟩
⟨ *Function for single torque calculation* 21a⟩
⟨ *Function for single stiffness calculation* 20b⟩
⟨ *Functions for calculating forces and stiffnesses* 26⟩

```
end
```

Root chunk (not used in this document).

3.1.1 Variables and data structures

First of all, address the data structures required for the input and output. Because displacement of a single magnet has three components, plus sizes of the faces another three, plus magnetisation strength and direction (two) makes nine in total, we use one of Matlab's structures to pass the information into the function. Otherwise we'd have an overwhelming number of input arguments.

The input variables |magnet.dim| should be the entire side lengths of the magnets; these dimensions are halved when performing all of the calculations. (Because that's just how the maths is.)

We use spherical coordinates to represent magnetisation angle, where |phi| is the angle from the horizontal plane ($-\pi/2 \leq \phi \leq \pi/2$) and |theta| is the angle around the horizontal plane ($0 \leq \theta \leq 2\pi$). This follows Matlab's definition; other conventions are commonly used as well. Remember:

$$(1,0,0)_{\text{cartesian}} \equiv (0,0,1)_{\text{spherical}}$$
$$(0,1,0)_{\text{cartesian}} \equiv (\pi/2,0,1)_{\text{spherical}}$$
$$(0,0,1)_{\text{cartesian}} \equiv (0,\pi/2,1)_{\text{spherical}}$$

Cartesian components can also be used as input as well, in which case they are made into a unit vector before multiplying it by the magnetisation magnitude. Either way (between spherical or cartesian input), |J1| and |J2| are made into the magnetisation vectors in cartesian coordindates.

12 ⟨ *Initialise main variables* 12⟩≡

```
if ~isfield(magnet_fixed,'type')
  if length(magnet_fixed.dim) == 2
    magnet_fixed.type = 'cylinder';
  else
    magnet_fixed.type = 'cuboid';
  end
end

if ~isfield(magnet_float,'type')
  if length(magnet_float.dim) == 2
    magnet_float.type = 'cylinder';
  else
    magnet_float.type = 'cuboid';
  end
end

if isfield(magnet_fixed,'grade')
  if isfield(magnet_fixed,'magn')
    error('Cannot specify both ''magn'' and ''grade''.')
  else
    magnet_fixed.magn = grade2magn(magnet_fixed.grade);
  end
```

```
end

if isfield(magnet_float,'grade')
  if isfield(magnet_float,'magn')
    error('Cannot specify both ''magn'' and ''grade''.')
  else
    magnet_float.magn = grade2magn(magnet_float.grade);
  end
end

coil_bool = false;

if strcmp(magnet_fixed.type, 'coil')

  if ~strcmp(magnet_float.type, 'cylinder')
    error('Coil/magnet forces can only be calculated for cylindrical magnets.')
  end

  coil_bool = true;
  coil = magnet_fixed;
  magnet = magnet_float;
  magtype = 'cylinder';
  coil_sign = +1;

end

if strcmp(magnet_float.type, 'coil')

  if ~strcmp(magnet_fixed.type, 'cylinder')
    error('Coil/magnet forces can only be calculated for cylindrical magnets.')
  end

  coil_bool = true;
  coil = magnet_float;
  magnet = magnet_fixed;
  magtype = 'cylinder';
  coil_sign = -1;

end

if coil_bool

  error('to do')

else
```

```
  if ~strcmp(magnet_fixed.type, magnet_float.type)
    error('Magnets must be of same type')
  end
  magtype = magnet_fixed.type;


if strcmp(magtype,'cuboid')

  size1 = reshape(magnet_fixed.dim/2,[3 1]);
  size2 = reshape(magnet_float.dim/2,[3 1]);

  J1 = resolve_magnetisations(magnet_fixed.magn,magnet_fixed.magdir);
  J2 = resolve_magnetisations(magnet_float.magn,magnet_float.magdir);

  if calc_torque_bool
    if ~isfield(magnet_float,'lever')
      magnet_float.lever = [0; 0; 0];
    else
      ss = size(magnet_float.lever);
      if (ss(1)~=3) && (ss(2)==3)
        magnet_float.lever = magnet_float.lever'; % attempt [3 M] shape
      end
    end
  end

elseif strcmp(magtype,'cylinder')

  size1 = magnet_fixed.dim(:);
  size2 = magnet_float.dim(:);

  if ~isfield(magnet_fixed,'dir')
    magnet_fixed.dir = [0 0 1];
  end
  if ~isfield(magnet_float,'dir')
    magnet_float.dir = [0 0 1];
  end
  if abs(magnet_fixed.dir) ~= abs(magnet_float.dir)
    error('Cylindrical magnets must be oriented in the same direction')
  end

  if ~isfield(magnet_fixed,'magdir')
    magnet_fixed.magdir = [0 0 1];
  end
  if abs(magnet_fixed.dir) ~= abs(magnet_fixed.magdir)
    error('Cylindrical magnets must be magnetised in the same direction as their orientati
  end
```

```
  if ~isfield(magnet_float,'magdir')
    magnet_float.magdir = [0 0 1];
  end
  if abs(magnet_float.dir) ~= abs(magnet_float.magdir)
    error('Cylindrical magnets must be magnetised in the same direction as their orientati
  end

  cyldir = find(magnet_float.magdir ~= 0);
  cylnotdir = find(magnet_float.magdir == 0);
  if length(cyldir) ~= 1
    error('Cylindrical magnets must be aligned in one of the x, y or z directions')
  end

  magnet_float.magdir = magnet_float.magdir(:);
  magnet_fixed.magdir = magnet_fixed.magdir(:);
  magnet_float.dir = magnet_float.dir(:);
  magnet_fixed.dir = magnet_fixed.dir(:);

  if ~isfield(magnet_fixed,'magn')
    magnet_fixed.magn = 4*pi*1e-7*magnet_fixed.turns*magnet_fixed.current/magnet_fixed.dim
  end
  if ~isfield(magnet_float,'magn')
    magnet_float.magn = 4*pi*1e-7*magnet_float.turns*magnet_float.current/magnet_float.dim
  end

  J1 = magnet_fixed.magn*magnet_fixed.magdir;
  J2 = magnet_float.magn*magnet_float.magdir;

end

end
```

This definition is continued in chunk 43a.
This code is used in chunk 10.

Gotta check the displacement input for both functions. After sorting that out, we can initialise the output variables now we know how big they need to me.

16 ⟨ *Organise input displacements* 16⟩≡

```
if size(displ,1) == 3
  % all good
elseif size(displ,2) == 3
  displ = transpose(displ);
else
  error(['Displacements matrix should be of size (3, D) ',...
         'where D is the number of displacements.'])
end

Ndispl = size(displ,2);

if calc_force_bool
  forces_out = nan([3 Ndispl]);
end

if calc_stiffness_bool
  stiffnesses_out = nan([3 Ndispl]);
end

if calc_torque_bool
  torques_out = nan([3 Ndispl]);
end
```

This code is used in chunks 10 and 69a.

3.1.2 Wrangling user input and output

We now have a choice of calculations to take based on the user input. This chunk and the next are used in both `magnetforces.m` and `multipoleforces.m`.

17 ⟨ *Parse calculation args* 17⟩≡

```
debug_disp = @(str) disp([]);
calc_force_bool = false;
calc_stiffness_bool = false;
calc_torque_bool = false;

% Undefined calculation flags for the three directions:
calc_xyz = [-1; -1; -1];

for ii = 1:length(varargin)
  switch varargin{ii}
    case 'debug',      debug_disp = @(str) disp(str);
    case 'force',      calc_force_bool     = true;
    case 'stiffness',  calc_stiffness_bool = true;
    case 'torque',     calc_torque_bool    = true;
    case 'x',  calc_xyz(1) = 1;
    case 'y',  calc_xyz(2) = 1;
    case 'z',  calc_xyz(3) = 1;
    otherwise
      error(['Unknown calculation option ''',varargin{ii},'""'])
  end
end

% If none of |'x'|, |'y'|, |'z'| are specified, calculate all.
if all( calc_xyz == -1 )
  calc_xyz = [1; 1; 1];
end

calc_xyz( calc_xyz == -1 ) = 0;

if ~calc_force_bool && ~calc_stiffness_bool && ~calc_torque_bool
  varargin{end+1} = 'force';
  calc_force_bool = true;
end
```

This code is used in chunks 10 and 69a.

After all of the calculations have occured, they're placed back into |varargout|. (This happens at the very end, obviously.) Outputs are ordered in the same order as the inputs are specified.

18 ⟨ *Return all results* 18⟩≡

```
varargout = {};

for ii = 1:length(varargin)
  switch varargin{ii}
    case 'force'
      varargout{end+1} = forces_out;

    case 'stiffness'
      varargout{end+1} = stiffnesses_out;

    case 'torque'
      varargout{end+1} = torques_out;
  end
end
```

This code is used in chunks 10 and 69a.

The actual mechanics. The idea is that a multitude of displacements can be passed to the function and we iterate to generate a matrix of vector outputs.

19 ⟨ *Calculate for each displacement* 19⟩≡

```
if coil_bool

  forces_out = coil_sign*coil.dir*...
    forces_magcyl_shell_calc(mag.dim, coil.dim, squeeze(displ(cyldir,:)), J1(cyldir), coil

else

if strcmp(magtype,'cuboid')

if calc_force_bool
  for ii = 1:Ndispl
    forces_out(:,ii)  =  single_magnet_force(displ(:,ii));
  end
end

if calc_stiffness_bool
  for ii = 1:Ndispl
    stiffnesses_out(:,ii)  =  single_magnet_stiffness(displ(:,ii));
  end
end

if calc_torque_bool
  torques_out  =  single_magnet_torque(displ,magnet_float.lever);
end

elseif strcmp(magtype,'cylinder')

if strcmp(magtype,'cylinder')
  if any(displ(cylnotdir,:)~=0)
    error(['Displacements for cylindrical magnets may only be axial. ',...
           'I.e., only in the direction of their alignment.'])
  end
end

if calc_force_bool
  forces_out = magnet_fixed.dir*...
    forces_cyl_calc(size1, size2, squeeze(displ(cyldir,:)), J1(cyldir), J2(cyldir));
end

if calc_stiffness_bool
  error('Stiffness cannot be calculated for cylindrical magnets yet.')
```

```
end

if calc_torque_bool
  error('Torques cannot be calculated for cylindrical magnets yet.')
end

end

end
```

This code is used in chunk 10.

And this is what does the calculations.

20a ⟨ *Function for single force calculation* 20a⟩≡

```
function force_out = single_magnet_force(displ)

force_components = nan([9 3]);

⟨ Precompute displacement rotations 46b⟩
⟨ Print diagnostics 48a⟩
⟨ Calculate |x| force 22a⟩
⟨ Calculate |y| force 22b⟩
⟨ Calculate |z| force 21b⟩

force_out = sum(force_components);
end
```

This code is used in chunk 10.

And this is what does the calculations for stiffness.

20b ⟨ *Function for single stiffness calculation* 20b⟩≡

```
function stiffness_out = single_magnet_stiffness(displ)

stiffness_components = nan([9 3]);

⟨ Precompute displacement rotations 46b⟩
⟨ Print diagnostics 48a⟩
⟨ Calculate stiffnesses 23⟩

stiffness_out = sum(stiffness_components);
end
```

This code is used in chunk 10.

21a ⟨ *Function for single torque calculation* 21a⟩≡

```
function torques_out = single_magnet_torque(displ,lever)

torque_components = nan([size(displ) 9]);

⟨ Precompute torque displacement rotations 47a⟩
⟨ Print diagnostics 48a⟩
⟨ Calculate torques 25⟩

torques_out = sum(torque_components,3);
end
```

This code is used in chunk 10.

The easy one first, where our magnetisation components align with the direction expected by the force functions.

21b ⟨ *Calculate |z| force* 21b⟩≡

```
debug_disp('z-z force:')
force_components(9,:) = forces_calc_z_z( size1,size2,displ,J1,J2 );

debug_disp('z-y force:')
force_components(8,:) = forces_calc_z_y( size1,size2,displ,J1,J2 );

debug_disp('z-x force:')
force_components(7,:) = forces_calc_z_x( size1,size2,displ,J1,J2 );
```

This code is used in chunk 20a.

The other forces (i.e., |x| and |y| components) require a rotation to get the magnetisations correctly aligned. In the case of the magnet sizes, the lengths are just flipped rather than rotated (in rotation, sign is important). After the forces are calculated, rotate them back to the original coordinate system.

22a ⟨ *Calculate |x| force* 22a⟩≡

```
calc_xyz = swap_x_z(calc_xyz);

debug_disp('Forces x-x:')
force_components(1,:) = ...
  rotate_z_to_x( forces_calc_z_z(size1_x,size2_x,d_x,J1_x,J2_x) );

debug_disp('Forces x-y:')
force_components(2,:) = ...
  rotate_z_to_x( forces_calc_z_y(size1_x,size2_x,d_x,J1_x,J2_x) );

debug_disp('Forces x-z:')
force_components(3,:) = ...
  rotate_z_to_x( forces_calc_z_x(size1_x,size2_x,d_x,J1_x,J2_x) );

calc_xyz = swap_x_z(calc_xyz);
```

This code is used in chunk 20a.

Same again, this time making |y| the 'up' direction.

22b ⟨ *Calculate |y| force* 22b⟩≡

```
calc_xyz = swap_y_z(calc_xyz);

debug_disp('Forces y-x:')
force_components(4,:) = ...
  rotate_z_to_y( forces_calc_z_x(size1_y,size2_y,d_y,J1_y,J2_y) );

debug_disp('Forces y-y:')
force_components(5,:) = ...
  rotate_z_to_y( forces_calc_z_z(size1_y,size2_y,d_y,J1_y,J2_y) );

debug_disp('Forces y-z:')
force_components(6,:) = ...
  rotate_z_to_y( forces_calc_z_y(size1_y,size2_y,d_y,J1_y,J2_y) );

calc_xyz = swap_y_z(calc_xyz);
```

This code is used in chunk 20a.

Same as all the above. Except not really. Because stiffness isn't the same sort of vector quantity (if at all, really) as force, we simply 'flip' the directions around between different coordinate systems rather than rotate them.

23 ⟨ *Calculate stiffnesses* 23⟩≡

```
debug_disp('z-x stiffness:')
stiffness_components(7,:) = ...
  stiffnesses_calc_z_x( size1,size2,displ,J1,J2 );

debug_disp('z-y stiffness:')
stiffness_components(8,:) = ...
  stiffnesses_calc_z_y( size1,size2,displ,J1,J2 );

debug_disp('z-z stiffness:')
stiffness_components(9,:) = ...
  stiffnesses_calc_z_z( size1,size2,displ,J1,J2 );

calc_xyz = swap_x_z(calc_xyz);

debug_disp('x-x stiffness:')
stiffness_components(1,:) = ...
  swap_x_z( stiffnesses_calc_z_z( size1_x,size2_x,d_x,J1_x,J2_x ) );

debug_disp('x-y stiffness:')
stiffness_components(2,:) = ...
  swap_x_z( stiffnesses_calc_z_y( size1_x,size2_x,d_x,J1_x,J2_x ) );

debug_disp('x-z stiffness:')
stiffness_components(3,:) = ...
  swap_x_z( stiffnesses_calc_z_x( size1_x,size2_x,d_x,J1_x,J2_x ) );

calc_xyz = swap_x_z(calc_xyz);

calc_xyz = swap_y_z(calc_xyz);

debug_disp('y-x stiffness:')
stiffness_components(4,:) = ...
  swap_y_z( stiffnesses_calc_z_x( size1_y,size2_y,d_y,J1_y,J2_y ) );

debug_disp('y-y stiffness:')
stiffness_components(5,:) = ...
  swap_y_z( stiffnesses_calc_z_z( size1_y,size2_y,d_y,J1_y,J2_y ) );

debug_disp('y-z stiffness:')
stiffness_components(6,:) = ...
```

```
    swap_y_z( stiffnesses_calc_z_y( size1_y,size2_y,d_y,J1_y,J2_y ) );

  calc_xyz = swap_y_z(calc_xyz);
```

This code is used in chunk 20b.

25 ⟨ *Calculate torques* 25⟩≡

```
debug_disp('Torque: z-z:')
torque_components(:,:,9) = torques_calc_z_z( size1,size2,displ,lever,J1,J2 );

debug_disp('Torque z-y:')
torque_components(:,:,8) = torques_calc_z_y( size1,size2,displ,lever,J1,J2 );

debug_disp('Torque z-x:')
torque_components(:,:,7) = torques_calc_z_x( size1,size2,displ,lever,J1,J2 );

calc_xyz = swap_x_z(calc_xyz);

debug_disp('Torques x-x:')
torque_components(:,:,1) = ...
  rotate_z_to_x( torques_calc_z_z(size1_x,size2_x,d_x,l_x,J1_x,J2_x) );

debug_disp('Torques x-y:')
torque_components(:,:,2) = ...
  rotate_z_to_x( torques_calc_z_y(size1_x,size2_x,d_x,l_x,J1_x,J2_x) );

debug_disp('Torques x-z:')
torque_components(:,:,3) = ...
  rotate_z_to_x( torques_calc_z_x(size1_x,size2_x,d_x,l_x,J1_x,J2_x) );

calc_xyz = swap_x_z(calc_xyz);

calc_xyz = swap_y_z(calc_xyz);

debug_disp('Torques y-x:')
torque_components(:,:,4) = ...
  rotate_z_to_y( torques_calc_z_x(size1_y,size2_y,d_y,l_y,J1_y,J2_y) );

debug_disp('Torques y-y:')
torque_components(:,:,5) = ...
  rotate_z_to_y( torques_calc_z_z(size1_y,size2_y,d_y,l_y,J1_y,J2_y) );

debug_disp('Torques y-z:')
torque_components(:,:,6) = ...
  rotate_z_to_y( torques_calc_z_y(size1_y,size2_y,d_y,l_y,J1_y,J2_y) );

calc_xyz = swap_y_z(calc_xyz);
```

This code is used in chunk 21a.

3.1.3 Functions for calculating forces, torques, and stiffnesses

The calculations for forces between differently-oriented cuboid magnets are all directly from the literature. The stiffnesses have been derived by differentiating the force expressions, but that's the easy part.

26 ⟨ *Functions for calculating forces and stiffnesses* 26⟩≡

⟨ *Parallel magnets force calculation* 27⟩
⟨ *Orthogonal magnets force calculation* 29⟩

⟨ *Parallel magnets stiffness calculation* 40⟩
⟨ *Orthogonal magnets stiffness calculation* 41⟩

⟨ *Parallel magnets torque calculation* 37⟩

⟨ *Cylindrical magnets force calculation* 33⟩
⟨ *Magnet/coil coaxial force function (shell)* 36⟩

⟨ *Helper functions* 47b⟩

This code is used in chunk 10.

The expressions here follow directly from Akoun and Yonnet [1].

Inputs:	|size1|=|(a,b,c)|	the half dimensions of the fixed magnet
	|size2|=|(A,B,C)|	the half dimensions of the floating magnet
	|displ|=|(dx,dy,dz)|	distance between magnet centres
	|(J,J2)|	magnetisations of the magnet in the z-direction
Outputs:	|forces_xyz|=|(Fx,Fy,Fz)|	Forces of the second magnet

27 ⟨ *Parallel magnets force calculation* 27⟩≡

```
function calc_out = forces_calc_z_z(size1,size2,offset,J1,J2)

J1 = J1(3);
J2 = J2(3);

⟨ Initialise subfunction variables 42b⟩

if calc_xyz(1)
  component_x = ...
    + multiply_x_log_y( 0.5*(v.^2-w.^2), r-u ) ...
    + multiply_x_log_y( u.*v, r-v ) ...
    + v.*w.*atan1(u.*v,r.*w) ...
    + 0.5*r.*u;
end

if calc_xyz(2)
  component_y = ...
    + multiply_x_log_y( 0.5*(u.^2-w.^2), r-v ) ...
    + multiply_x_log_y( u.*v, r-u ) ...
    + u.*w.*atan1(u.*v,r.*w) ...
    + 0.5*r.*v;
end

if calc_xyz(3)
  component_z = ...
    - multiply_x_log_y( u.*w, r-u ) ...
    - multiply_x_log_y( v.*w, r-v ) ...
    + u.*v.*atan1(u.*v,r.*w) ...
    - r.*w;
end

⟨ Finish up 43b⟩
```

This code is used in chunk 26.

Orthogonal magnets forces given by Yonnet and Allag [3]. Note those equations seem to be written to calculate the force on the first magnet due to the second, so we negate all the values to get the force on the latter instead.

29 ⟨ *Orthogonal magnets force calculation* 29⟩≡

```
function calc_out = forces_calc_z_y(size1,size2,offset,J1,J2)

J1 = J1(3);
J2 = J2(2);

⟨ Initialise subfunction variables 42b⟩

allag_correction = -1;

if calc_xyz(1)
  component_x = ...
    - multiply_x_log_y ( v .* w , r-u ) ...
    + multiply_x_log_y ( v .* u , r+w ) ...
    + multiply_x_log_y ( u .* w , r+v ) ...
    - 0.5 * u.^2 .* atan1( v .* w , u .* r ) ...
    - 0.5 * v.^2 .* atan1( u .* w , v .* r ) ...
    - 0.5 * w.^2 .* atan1( u .* v , w .* r );
  component_x = allag_correction*component_x;
end

if calc_xyz(2)
  component_y = ...
    0.5 * multiply_x_log_y( u.^2 - v.^2 , r+w ) ...
    - multiply_x_log_y( u .* w , r-u ) ...
    - u .* v .* atan1( u .* w , v .* r ) ...
    - 0.5 * w .* r;
  component_y = allag_correction*component_y;
end

if calc_xyz(3)
  component_z = ...
    0.5 * multiply_x_log_y( u.^2 - w.^2 , r+v ) ...
    - multiply_x_log_y( u .* v , r-u ) ...
    - u .* w .* atan1( u .* v , w .* r ) ...
    - 0.5 * v .* r;
  component_z = allag_correction*component_z;
end

⟨ Finish up 43b⟩
```

This definition is continued in chunk 32.
This code is used in chunk 26.

This is the same calculation with Janssen's equations instead. By default this code never runs, but if you like it can be enabled to prove that the equations are consistent.

31 ⟨ *Test against Janssen results* 31⟩≡

```
S=u;
T=v;
U=w;
R=r;

component_x_ii = ...
    ( 0.5*atan1(U,S)+0.5*atan1(T.*U,S.*R) ).*S.^2 ...
    + T.*S - 3/2*U.*S - multiply_x_log_y( S.*T , U+R )-T.^2 .* atan1(S,T) ...
    + U.* ( U.* ( ...
              0.5*atan1(S,U)+0.5*atan1(S.*T,U.*R) ...
            ) ...
          - multiply_x_log_y( T , S+R )+multiply_x_log_y(S,R-T) ...
          ) ...
        + 0.5*T.^2 .* atan1(S.*U,T.*R)...
;

component_y_ii = ...
        0.5*U.*(R-2*S)+...
        multiply_x_log_y( 0.5*(T.^2-S.^2) , U+R )+...
        S.*T.*( atan1(U,T)+atan1(S.*U,T.*R) )+...
        multiply_x_log_y( S.*U , R-S )...
;

component_z_ii = ...
        0.5*T.*(R-2*S)+...
        multiply_x_log_y( 0.5*(U.^2-S.^2), T+R )+...
        S.*U.*( atan1(T,U)+atan1(S.*T,U.*R) )+...
        multiply_x_log_y( S.*T , R-S )...
;

xx = index_sum.*component_x;
xx_ii = index_sum.*component_x_ii;
assert( abs(sum(xx(:)) - sum(xx_ii(:))) < 1e-8 )

yy = index_sum.*component_y;
yy_ii = index_sum.*component_y_ii;
assert( abs(sum(yy(:)) - sum(yy_ii(:))) < 1e-8 )

zz = index_sum.*component_z;
zz_ii = index_sum.*component_z_ii;
```

```
assert( abs(sum(zz(:)) - sum(zz_ii(:))) < 1e-8 )

component_x = component_x_ii;
component_y = component_y_ii;
component_z = component_z_ii;
```

Root chunk (not used in this document).

The improvement in processing time between typing in the actual equals compared to just transforming the z–y case isn't worth the tedium of actually doing it.

32 ⟨ *Orthogonal magnets force calculation* 29⟩+≡

```
function calc_out = forces_calc_z_x(size1,size2,offset,J1,J2)

calc_xyz = swap_x_y(calc_xyz);

forces_xyz = forces_calc_z_y(...
  swap_x_y(size1), swap_x_y(size2), rotate_x_to_y(offset),...
  J1, rotate_x_to_y(J2) );

calc_xyz = swap_x_y(calc_xyz);
calc_out = rotate_y_to_x( forces_xyz );

end
```

Cylindrical magnets are incomplete.

33 ⟨ *Cylindrical magnets force calculation* 33⟩≡

```
function calc_out = forces_cyl_calc(size1,size2,h_gap,J1,J2)

% inputs

r1 = size1(1);
r2 = size2(1);

% implicit

z = nan(4,length(h_gap));
z(1,:) = -size1(2)/2;
z(2,:) =  size1(2)/2;
z(3,:) = h_gap - size2(2)/2;
z(4,:) = h_gap + size2(2)/2;

C_d = zeros(size(h_gap));

for ii = [1 2]

  for jj = [3 4]

    a1 = z(ii,:) - z(jj,:);
    a2 = 1 + ( (r1-r2)./a1 ).^2;
    a3 = sqrt( (r1+r2).^2 + a1.^2 );
    a4 = 4*r1.*r2./( (r1+r2).^2 + a1.^2 );

    [K, E, PI] = ellipkepi( a4./(1-a2) , a4 );

    a2_ind = ( a2 == 1 | isnan(a2) );
    if all(a2_ind)% singularity at a2=1 (i.e., equal radii)
      PI_term(a2_ind) = 0;
    elseif all(~a2_ind)
      PI_term = (1-a1.^2./a3.^2).*PI;
    else % this branch just for completeness
      PI_term = zeros(size(a2));
      PI_term(~a2_ind) = (1-a1.^2/a3.^2).*PI;
    end

    f_z = a1.*a2.*a3.*( K - E./a2 - PI_term );

    f_z(abs(a1)<eps)=0; % singularity at a1=0 (i.e., coincident faces)
```

```
    C_d = C_d + (-1)^(ii+jj).*f_z;

  end

end

calc_out = J1*J2/(8*pi*1e-7)*C_d;

end


function [k,e,PI] = ellipkepi(a,m)

% Complete elliptic integrals calculated with the arithmetric-geometric mean
% algorithms contained here: http://dlmf.nist.gov/19.8
%
% Valid for a <= 1 and m <= 1

a0 = 1;
g0 = sqrt(1-m);
s0 = m;
nn = 0;

p0 = sqrt(1-a);
Q0 = 1;
Q1 = 1;
QQ = Q0;

w1 = ones(size(m));

while max(Q1(:)) > eps

  % for Elliptic I
  a1 = (a0+g0)/2;
  g1 = sqrt(a0.*g0);

  % for Elliptic II
  nn = nn + 1;
  c1 = (a0-g0)/2;
  w1 = 2^nn*c1.^2;
  s0 = s0 + w1;

  % for Elliptic III
  rr = p0.^2+a0.*g0;
  p1 = rr./(2.*p0);
  Q1 = 0.5*Q0.*(p0.^2-a0.*g0)./rr;
```

```
  QQ = QQ+Q1;

  a0 = a1;
  g0 = g1;
  Q0 = Q1;
  p0 = p1;

end

k = pi./(2*a1);
e = k.*(1-s0/2);
PI = pi./(4.*a1).*(2+a./(1-a).*QQ);

im = find(m == 1);
if ~isempty(im)
  k(im) = inf;
  e(im) = ones(length(im),1);
  PI(im) = inf;
end

end
```

This code is used in chunk 26.

Magnet/coil forces are more complete in the Mathematica implementation.

36 ⟨ *Magnet/coil coaxial force function (shell)* 36⟩≡

```
function Fz = forces_magcyl_shell_calc(magsize,coilsize,displ,Jmag,Nrz,I)

Jcoil = 4*pi*1e-7*Nrz(2)*I/coil.dim(3);

shell_forces = nan([length(displ) Nrz(1)]);

for rr = 1:Nrz(1)

  this_radius = coilsize(1)+(rr-1)/(Nrz(1)-1)*(coilsize(2)-coilsize(1));
  shell_size = [this_radius, coilsize(3)];

  shell_forces(:,rr) = forces_cyl_calc(magsize,shell_size,displ,Jmag,Jcoil)

end

Fz = sum(shell_forces,2);

end
```

This code is used in chunk 26.

The expressions here follow directly from Janssen et al. [2]. The code below was largely written by Allan Liu; thanks! We have checked it against Janssen's own Matlab code and the two give identical output.

Inputs:	|size1|=|(a1,b1,c1)|	the half dimensions of the fixed magnet
	|size2|=|(a2,b2,c2)|	the half dimensions of the floating magnet
	|displ|=|(a,b,c)|	distance between magnet centres
	|lever|=|(d,e,f)|	distance between floating magnet and its centre of rotation
	|(J,J2)|	magnetisations of the magnet in the z-direction
Outputs:	|forces_xyz|=|(Fx,Fy,Fz)|	Forces of the second magnet

37 ⟨ *Parallel magnets torque calculation* 37⟩≡

```
function calc_out = torques_calc_z_z(size1,size2,offset,lever,J1,J2)

br1 = J1(3);
br2 = J2(3);

if br1==0 || br2==0
  debug_disp('Zero magnetisation')
  calc_out = 0*offset;
  return
end

a1 = size1(1);
b1 = size1(2);
c1 = size1(3);

a2 = size2(1);
b2 = size2(2);
c2 = size2(3);

a = offset(1,:);
b = offset(2,:);
c = offset(3,:);

d = a+lever(1,:);
e = b+lever(2,:);
f = c+lever(3,:);

Tx=zeros([1 size(offset,2)]);
Ty=Tx;
Tz=Tx;

for ii=[0,1]
  for jj=[0,1]
```

```
for kk=[0,1]
  for ll=[0,1]
    for mm=[0,1]
      for nn=[0,1]

        Cu=(-1)^ii.*a1-d;
        Cv=(-1)^kk.*b1-e;
        Cw=(-1)^mm.*c1-f;

        u=a-(-1)^ii.*a1+(-1)^jj.*a2;
        v=b-(-1)^kk.*b1+(-1)^ll.*b2;
        w=c-(-1)^mm.*c1+(-1)^nn.*c2;

        s=sqrt(u.^2+v.^2+w.^2);

        Ex=(1/8).*(...
          -2.*Cw.*(-4.*v.*u+s.^2+2.*v.*s)-...
          w.*(-8.*v.*u+s.^2+8.*Cv.*s+6.*v.*s)+...
          2.*(2.*Cw+w).*(u.^2+w.^2).*log(v+s)+...
          4.*(...
            2.*Cv.*u.*w.*acoth(u./s) + ...
            w.*(v.^2+2.*Cv.*v-w.*(2.*Cw+w)).*acoth(v./s) - ...
            u.*(...
              2*w.*(Cw+w).*atan(v./w) + ...
              2*v.*(Cw+w).*log(s-u) + ...
              (w.^2+2.*Cw.*w-v.*(2.*Cv+v)).*atan( u.*v./(w.*s) ) ...
            )...
          )...
        );

        Ey=(1/8)*...
          ((2.*Cw+w).*u.^2-8.*u.*v.*(Cw+w)+8.*u.*v.*(Cw+w).*log(s-v)...
          +4.*Cw.*u.*s+6.*w.*s.*u+(2.*Cw+w).*(v.^2+w.^2)+...
          4.*w.*(w.^2+2.*Cw.*w-u.*(2.*Cu+u)).*acoth(u./s)+...
          4.*v.*(-2.*Cu.*w.*acoth(v./s)+2.*w.*(Cw+w).*atan(u./w)...
          +(w.^2+2.*Cw.*w-u.*(2.*Cu+u)).*atan(u.*v./(w.*s)))...
          -2.*(2.*Cw+w).*(v.^2+w.^2).*log(u+s)+8.*Cu.*w.*s);

        Ez=(1/36).*(-u.^3-18.*v.*u.^2-6.*u.*(w.^2+6.*Cu...
          .*v-3.*v.*(2.*Cv+v)+3.*Cv.*s)+v.*(v.^2+6.*(w.^2+...
          3.*Cu.*s))+6.*w.*(w.^2-3.*v.*(2.*Cv+v)).*atan(u./w)...
          -6.*w.*(w.^2-3.*u.*(2.*Cu+u)).*atan(v./w)-9.*...
          (2.*(v.^2+2.*Cv.*v-u.*(2.*Cu+u)).*w.*atan(u.*v./(w.*s))...
          -2.*u.*(2.*Cu+u).*v.*log(s-u)-(2.*Cv+v).*(v.^2-w.^2)...
          .*log(u+s)+2.*u.*v.*(2.*Cv+v).*log(s-v)+(2.*Cu+...
          u).*(u.^2-w.^2).*log(v+s)));
```

```
            Tx=Tx+(-1)^(ii+jj+kk+ll+mm+nn)*Ex;
            Ty=Ty+(-1)^(ii+jj+kk+ll+mm+nn)*Ey;
            Tz=Tz+(-1)^(ii+jj+kk+ll+mm+nn)*Ez;

          end
        end
      end
    end
  end
end

calc_out = real([Tx; Ty; Tz].*br1*br2/(16*pi^2*1e-7));

end

function calc_out = torques_calc_z_y(size1,size2,offset,lever,J1,J2)

if J1(3)~=0 && J2(2)~=0
  error('Torques cannot be calculated for orthogonal magnets yet.')
end

calc_out = 0*offset;

end

function calc_out = torques_calc_z_x(size1,size2,offset,lever,J1,J2)

if J1(3)~=0 && J2(1)~=0
  error('Torques cannot be calculated for orthogonal magnets yet.')
end

calc_out = 0*offset;

end
```

This code is used in chunk 26.

Stiffness calculations are simply differentiated (in Mathematica) from the forces.

40 ⟨ *Parallel magnets stiffness calculation* 40⟩≡

```
function calc_out = stiffnesses_calc_z_z(size1,size2,offset,J1,J2)

J1 = J1(3);
J2 = J2(3);

⟨ Initialise subfunction variables 42b⟩

if calc_xyz(1) || calc_xyz(3)
  component_x = - r - (u.^2 .*v)./(u.^2+w.^2) - v.*log(r-v) ;
end

if calc_xyz(2) || calc_xyz(3)
  component_y = - r - (v.^2 .*u)./(v.^2+w.^2) - u.*log(r-u) ;
end

if calc_xyz(3)
  component_z = - component_x - component_y;
end

⟨ Finish up 43b⟩
```

This code is used in chunk 26.

Orthogonal magnets stiffnesses derived from Yonnet and Allag [3]. First the z–y magnetisation.

41 ⟨ *Orthogonal magnets stiffness calculation* 41⟩≡

```
function calc_out = stiffnesses_calc_z_y(size1,size2,offset,J1,J2)

J1 = J1(3);
J2 = J2(2);

⟨ Initialise subfunction variables 42b⟩

if calc_xyz(1) || calc_xyz(3)
  component_x =  ((u.^2 .*v)./(u.^2 + v.^2)) + (u.^2 .*w)./(u.^2 + w.^2) ...
       - u.*atan1(v.*w,r.*u) + multiply_x_log_y( w , r + v ) + ...
       + multiply_x_log_y( v , r + w );
end

if calc_xyz(2) || calc_xyz(3)
  component_y = - v/2 + (u.^2 .*v)./(u.^2 + v.^2) - (u.*v.*w)./(v.^2 + w.^2) ...
       -  u.*atan1(u.*w,r.*v) - multiply_x_log_y( v , r + w );
end

if calc_xyz(3)
  component_z = - component_x - component_y;
end

⟨ Finish up 43b⟩
```

This definition is continued in chunk 42a.
This code is used in chunk 26.

Now the z–x magnetisation, which is z–y rotated.

42a ⟨ *Orthogonal magnets stiffness calculation* 41⟩+≡

```
function calc_out = stiffnesses_calc_z_x(size1,size2,offset,J1,J2)

calc_xyz = swap_x_y(calc_xyz);

stiffnesses_xyz = stiffnesses_calc_z_y(...
  swap_x_y(size1), swap_x_y(size2), rotate_x_to_y(offset),...
  J1, rotate_x_to_y(J2) );

calc_xyz = swap_x_y(calc_xyz);
calc_out = swap_x_y(stiffnesses_xyz);

end
```

Some shared setup code. First |return| early if either of the magnetisations are zero — that's the trivial solution. Assume that the magnetisation has already been rounded down to zero if necessary; i.e., that we don't need to check for |J1| or |J2| are less than |1e-12| or whatever.

42b ⟨ *Initialise subfunction variables* 42b⟩≡

```
if (J1==0 || J2==0)
  debug_disp('Zero magnetisation.')
  calc_out  =  [0; 0; 0];
  return;
end

u = offset(1) + size2(1)*(-1).^index_j - size1(1)*(-1).^index_i;
v = offset(2) + size2(2)*(-1).^index_l - size1(2)*(-1).^index_k;
w = offset(3) + size2(3)*(-1).^index_q - size1(3)*(-1).^index_p;
r = sqrt(u.^2+v.^2+w.^2);
```

This code is used in chunks 27, 29, 40, and 41.

Here are some variables used above that only need to be computed once. The idea here is to vectorise instead of using |for| loops because it allows more convenient manipulation of the data later on.

43a ⟨ *Initialise main variables* 12⟩+≡

```
magconst = 1/(4*pi*(4*pi*1e-7));

[index_i, index_j, index_k, index_l, index_p, index_q] = ndgrid([0 1]);

index_sum = (-1).^(index_i+index_j+index_k+index_l+index_p+index_q);
```

And some shared finishing code.

43b ⟨ *Finish up* 43b⟩≡

```
if calc_xyz(1)
  component_x = index_sum.*component_x;
else
  component_x = 0;
end

if calc_xyz(2)
  component_y = index_sum.*component_y;
else
  component_y = 0;
end

if calc_xyz(3)
  component_z = index_sum.*component_z;
else
  component_z = 0;
end

calc_out = J1*J2*magconst .* ...
  [ sum(component_x(:)) ;
    sum(component_y(:)) ;
    sum(component_z(:)) ] ;

debug_disp(calc_out')

end
```

This code is used in chunks 27, 29, 40, and 41.

3.1.4 Setup code

Magnet 'strength' can be specified using either `magn` or `grade`. In the latter case, this should be a string such as `'N42'`, from which the |magn| is automatically calculated using the equation

$$B_r = 2\sqrt{\mu_0 [BH]_{\max}}$$

where $[BH]_{\max}$ is the numeric value given in the grade in MG Oe. I.e., an N42 magnet has $[BH]_{\max} = 42\,\mathrm{MG\,Oe}$. Since $1\,\mathrm{MG\,Oe} = 100/(4\pi)\,\mathrm{kJ/m^3}$, the calculation simplifies to

$$B_r = 2\sqrt{N/100}$$

where N is the numeric grade in MG Oe. Easy.

44 ⟨ *Function for calculating magnetisation from grade* 44⟩≡

```
function magn = grade2magn(grade)

if isnumeric(grade)
  magn = 2*sqrt(grade/100);
else
  if strcmp(grade(1),'N')
    magn = 2*sqrt(str2num(grade(2:end))/100);
  else
    magn = 2*sqrt(str2num(grade)/100);
  end
end

end
```

This code is used in chunk 10.

Magnetisation directions are specified in either cartesian or spherical coordinates. Since this is shared code, it's sent to the end to belong in a nested function.

We don't use Matlab's |sph2cart| here, because it doesn't calculate zero accurately (because it uses radians and |cos(pi/2)| can only be evaluated to machine precision of pi rather than symbolically).

45 ⟨ *Function for resolving magnetisations* 45⟩≡

```
function J = resolve_magnetisations(magn,magdir)

if length(magdir)==2
  J_r = magn;
  J_t = magdir(1);
  J_p = magdir(2);
  J   = [ J_r * cosd(J_p) * cosd(J_t)  ; ...
          J_r * cosd(J_p) * sind(J_t)  ; ...
          J_r * sind(J_p) ];
else
  if all(magdir == zeros(size(magdir)) )
    J = [0; 0; 0];
  else
    J = magn*magdir/norm(magdir);
    J = reshape(J,[3 1]);
  end
end

end
```

This code is used in chunk 10.

Forces due to magnetisations in x and y are calculated by rotating the original expressions. The rotated magnet sizes and magnetisation vectors are calculated here once only.

The rotation matrices are precalculated to avoid performing the matrix multiplications each time.

46a ⟨ *Precompute rotations* 46a⟩≡

```
if strcmp(magtype,'cuboid')

swap_x_y = @(vec) vec([2 1 3],:);
swap_x_z = @(vec) vec([3 2 1],:);
swap_y_z = @(vec) vec([1 3 2],:);

rotate_z_to_x = @(vec) [  vec(3,:);  vec(2,:); -vec(1,:) ] ; % Ry( 90)
rotate_x_to_z = @(vec) [ -vec(3,:);  vec(2,:);  vec(1,:) ] ; % Ry(-90)

rotate_y_to_z = @(vec) [  vec(1,:); -vec(3,:);  vec(2,:) ] ; % Rx( 90)
rotate_z_to_y = @(vec) [  vec(1,:);  vec(3,:); -vec(2,:) ] ; % Rx(-90)

rotate_x_to_y = @(vec) [ -vec(2,:);  vec(1,:);  vec(3,:) ] ; % Rz( 90)
rotate_y_to_x = @(vec) [  vec(2,:); -vec(1,:);  vec(3,:) ] ; % Rz(-90)

size1_x = swap_x_z(size1);
size2_x = swap_x_z(size2);
J1_x    = rotate_x_to_z(J1);
J2_x    = rotate_x_to_z(J2);

size1_y = swap_y_z(size1);
size2_y = swap_y_z(size2);
J1_y    = rotate_y_to_z(J1);
J2_y    = rotate_y_to_z(J2);

end
```

This code is used in chunk 10.

And the rotated displacement vectors are calculated once per loop:

46b ⟨ *Precompute displacement rotations* 46b⟩≡

```
d_x  = rotate_x_to_z(displ);
d_y  = rotate_y_to_z(displ);
```

This code is used in chunk 20.

For torque need lever arm as well:

47a ⟨ *Precompute torque displacement rotations* 47a⟩≡

```
d_x  = rotate_x_to_z(displ);
d_y  = rotate_y_to_z(displ);

l_x = rotate_x_to_z(lever);
l_y = rotate_y_to_z(lever);
```

This code is used in chunk 21a.

The equations contain two singularities. Specifically, the equations contain terms of the form $x\log(y)$, which becomes |NaN| when both x and y are zero since $\log(0)$ is negative infinity.

This function computes $x\log(y)$, special-casing the singularity to output zero, instead. (This is indeed the value of the limit.)

47b ⟨ *Helper functions* 47b⟩≡

```
function out = multiply_x_log_y(x,y)
  out = x.*log(y);
  out(~isfinite(out))=0;
end
```

This definition is continued in chunk 47c.
This code is used in chunk 26.

Also, we're using |atan| instead of |atan2| (otherwise the wrong results are calculated — I guess I don't totally understand that), which becomes a problem when trying to compute |atan(0/0)| since |0/0| is |NaN|.

This function computes |atan| but takes two arguments.

47c ⟨ *Helper functions* 47b⟩+≡

```
function out = atan1(x,y)
  out = zeros(size(x));
  ind = x~=0 & y~=0;
  out(ind) = atan(x(ind)./y(ind));
end
```

Let's print some information to the terminal to aid debugging. This is especially important (for me) when looking at the rotated coordinate systems.

48a ⟨ *Print diagnostics* 48a⟩≡

```
debug_disp('  ')
debug_disp('CALCULATING THINGS')
debug_disp('==================')
debug_disp('Displacement:')
debug_disp(displ')
debug_disp('Magnetisations:')
debug_disp(J1')
debug_disp(J2')
```

This code is used in chunks 20 and 21a.

When users type `help magnetforces` this is what they see.

48b ⟨ *Matlab help text (forces)* 48b⟩≡

```
%% MAGNETFORCES  Calculate forces between two cuboid magnets
%
% Finish this off later. Please read the PDF documentation instead for now.
%
```

This code is used in chunk 10.

3.2 Test files for magnet forces

The chunks that follow are designed to be saved into individual files and executed automatically to check for (a) correctness and (b) regression problems as the code evolves.

How do I know if the code produces the correct forces? Well, for many cases I can compare with published values in the literature. Beyond that, I'll be setting up some tests that I can logically infer should produce the same results (such as mirror-image displacements) and test that.

There are many Matlab unit test frameworks but I'll be using a fairly low-tech method. In time this test suite should be (somehow) useable for all implementations of `magnetocode`, not just Matlab. But I haven't thought about doing anything like that, yet.

Because I'm lazy, just run the tests manually for now. This script must be run twice if it updates itself.

49 ⟨ *testall.m* 49⟩≡

```
clc

test001a
test001b
test001c
test001d
testcyl01
testcuboidtorque01
testgrades01
```

This definition is continued in chunk 90b.
Root chunk (not used in this document).

Force testing. This test checks that square magnets produce the same forces in the each direction when displaced in positive and negative |x|, |y|, and |z| directions, respectively. In other words, this tests the function |forces_calc_z_y| directly. Both positive and negative magnetisations are used.

50 ⟨ *test001a.m* 50⟩≡

```
clear all
disp('=================')
fprintf('TEST 001a: ')

magnet_fixed.dim = [0.04 0.04 0.04];
magnet_float.dim = magnet_fixed.dim;

magnet_fixed.magn = 1.3;
magnet_float.magn = 1.3;
offset = 0.1;

⟨ Test z–z magnetisations 51a⟩
⟨ Assert magnetisations tests 57a⟩

⟨ Test x–x magnetisations 51b⟩
⟨ Assert magnetisations tests 57a⟩

⟨ Test y–y magnetisations 52⟩
⟨ Assert magnetisations tests 57a⟩

fprintf('passed\n')
disp('=================')
```

Root chunk (not used in this document).

Testing vertical forces.

51a ⟨ *Test z–z magnetisations* 51a⟩≡

```
f = [];

for ii = [1, -1]
  magnet_fixed.magdir = [0 ii*90]; % $\pm z$
  for jj = [1, -1]
    magnet_float.magdir = [0 jj*90];
    for kk = [1, -1]
      displ = kk*[0 0 offset];
      f(:,end+1) = magnetforces(magnet_fixed,magnet_float,displ);
    end
  end
end

dirforces = chop( f(3,:), 8 );
otherforces = f([1 2],:);
```

This code is used in chunk 50.

Testing horizontal x forces.

51b ⟨ *Test x–x magnetisations* 51b⟩≡

```
f = [];

for ii = [1, -1]
  magnet_fixed.magdir = [90+ii*90 0]; % $\pm x$
  for jj = [1, -1]
    magnet_float.magdir = [90+jj*90 0];
    for kk = [1, -1]
      displ = kk*[offset 0 0];
      f(:,end+1) = magnetforces(magnet_fixed,magnet_float,displ);
    end
  end
end

dirforces = chop( f(1,:), 8 );
otherforces = f([2 3],:);
```

This code is used in chunk 50.

Testing horizontal y forces.

52 ⟨ *Test y–y magnetisations* 52⟩≡

```
f = [];

for ii = [1, -1]
  magnet_fixed.magdir = [ii*90 0]; % $\pm y$
  for jj = [1, -1]
    magnet_float.magdir = [jj*90 0];
    for kk = [1, -1]
      displ = kk*[0 offset 0];
      f(:,end+1) = magnetforces(magnet_fixed,magnet_float,displ);
    end
  end
end

dirforces = chop( f(2,:), 8 );
otherforces = f([1 3],:);
```

This code is used in chunk 50.

This test does the same thing but for orthogonally magnetised magnets.

53 ⟨ *test001b.m* 53⟩≡

```
clear all
disp('=================')
fprintf('TEST 001b: ')

magnet_fixed.dim = [0.04 0.04 0.04];
magnet_float.dim =  magnet_fixed.dim;

magnet_fixed.magn = 1.3;
magnet_float.magn = 1.3;

⟨ Test ZYZ 54a⟩
⟨ Assert magnetisations tests 57a⟩

⟨ Test ZXZ 54b⟩
⟨ Assert magnetisations tests 57a⟩

⟨ Test ZXX 56⟩
⟨ Assert magnetisations tests 57a⟩

⟨ Test ZYY 55⟩
⟨ Assert magnetisations tests 57a⟩

fprintf('passed\n')
disp('=================')
```

Root chunk (not used in this document).

z–y magnetisations, z displacement.

54a ⟨ *Test ZYZ* 54a⟩≡

```
fzyz = [];

for ii = [1, -1]
  for jj = [1, -1]
    for kk = [1, -1]

      magnet_fixed.magdir = ii*[0  90];  % $\pm z$
      magnet_float.magdir = jj*[90 0];  % $\pm y$
      displ = kk*[0 0 0.1];  % $\pm z$
      fzyz(:,end+1) = magnetforces(magnet_fixed,magnet_float,displ);

    end
  end
end

dirforces = chop( fzyz(2,:), 8 );
otherforces = fzyz([1 3],:);
```

This code is used in chunk 53.

z–x magnetisations, z displacement.

54b ⟨ *Test ZXZ* 54b⟩≡

```
fzxz = [];

for ii = [1, -1]
  for jj = [1, -1]
    for kk = [1, -1]

      magnet_fixed.magdir = ii*[0  90];  % $\pm z$
      magnet_float.magdir = [90+jj*90 0];  % $\pm x$
      displ = kk*[0.1 0 0];  % $\pm x$
      fzxz(:,end+1) = magnetforces(magnet_fixed,magnet_float,displ);

    end
  end
end

dirforces = chop( fzxz(3,:), 8 );
otherforces = fzxz([1 2],:);
```

This code is used in chunk 53.

z–y magnetisations, y displacement.

55 ⟨ *Test ZYY* 55⟩≡

```
fzyy = [];

for ii = [1, -1]
  for jj = [1, -1]
    for kk = [1, -1]

      magnet_fixed.magdir = ii*[0  90];  % $\pm z$
      magnet_float.magdir = jj*[90 0];  % $\pm y$
      displ = kk*[0 0.1 0];  % $\pm y$
      fzyy(:,end+1) = magnetforces(magnet_fixed,magnet_float,displ);

    end
  end
end

dirforces = chop( fzyy(3,:), 8 );
otherforces = fzyy([1 2],:);
```

This code is used in chunk 53.

z–x magnetisations, x displacement.

56 ⟨ *Test ZXX* 56⟩≡

```
fzxx = [];

for ii = [1, -1]
  for jj = [1, -1]
    for kk = [1, -1]

      magnet_fixed.magdir = ii*[0  90];  % $\pm z$
      magnet_float.magdir = [90+jj*90 0];  % $\pm x$
      displ = kk*[0 0 0.1];  % $\pm z$
      fzxx(:,end+1) = magnetforces(magnet_fixed,magnet_float,displ);

    end
  end
end

dirforces = chop( fzxx(1,:), 8 );
otherforces = fzxx([2 3],:);
```

This code is used in chunk 53.

The assertions, common between directions.

57a ⟨ *Assert magnetisations tests* 57a⟩≡

```
assert ( ...
             all( abs( otherforces(:) ) < 1e-11 ) , ...
             'Orthogonal forces should be zero' ...
           )
assert ( ...
             all( abs(dirforces) == abs(dirforces(1)) ) , ...
             'Force magnitudes should be equal' ...
           )
assert ( ...
             all( dirforces(1:4) == -dirforces(5:8) ) , ...
             'Forces should be opposite with reversed fixed magnet magnetisation' ...
           )
assert ( ...
             all( dirforces([1 3 5 7]) == -dirforces([2 4 6 8]) ) , ...
             'Forces should be opposite with reversed float magnet magnetisation' ...
           )
```

This code is used in chunks 50 and 53.

Now try combinations of displacements.

57b ⟨ *test001c.m* 57b⟩≡

```
clear all
disp('=================')
fprintf('TEST 001c: ')

magnet_fixed.dim = [0.04 0.04 0.04];
magnet_float.dim =  magnet_fixed.dim;

magnet_fixed.magn = 1.3;
magnet_float.magn = 1.3;

⟨ Test combinations ZZ 58⟩
⟨ Assert combinations tests 60⟩
⟨ Test combinations ZY 59⟩
⟨ Assert combinations tests 60⟩

fprintf('passed\n')
disp('=================')
```

Root chunk (not used in this document).

Tests.

58 ⟨ *Test combinations ZZ* 58⟩≡

```
f = [];

for ii = [-1 1]
  for jj = [-1 1]
    for xx = 0.12*[-1, 1]
      for yy = 0.12*[-1, 1]
        for zz = 0.12*[-1, 1]

              magnet_fixed.magdir = [0  ii*90];  % $z$
              magnet_float.magdir = [0  jj*90];  % $z$
              displ = [xx yy zz];
              f(:,end+1) = magnetforces(magnet_fixed,magnet_float,displ);

        end
      end
    end
  end
end

f = chop( f , 8 );

uniquedir = f(3,:);
otherdir  = f([1 2],:);
```

This code is used in chunk 57b.

Tests.

59 ⟨ *Test combinations ZY* 59⟩≡

```
f = [];

for ii = [-1 1]
  for jj = [-1 1]
    for xx = 0.12*[-1, 1]
      for yy = 0.12*[-1, 1]
        for zz = 0.12*[-1, 1]

              magnet_fixed.magdir = [0  ii*90];  % $\pm z$
              magnet_float.magdir = [jj*90  0];  % $\pm y$
              displ = [xx yy zz];
              f(:,end+1) = magnetforces(magnet_fixed,magnet_float,displ);

        end
      end
    end
  end
end

f = chop( f , 8 );

uniquedir = f(1,:);
otherdir  = f([2 3],:);
```

This code is used in chunk 57b.

Shared tests, again.

60 ⟨ *Assert combinations tests* 60⟩≡

```
test1 = abs(diff(abs(f(1,:))))<1e-10 ;
test2 = abs(diff(abs(f(2,:))))<1e-10 ;
test3 = abs(diff(abs(f(3,:))))<1e-10 ;
assert ( all(test1) && all(test2) && all(test3) , ...
         'All forces in a single direction should be equal' )

test = abs(diff(abs(otherdir))) < 1e-11;
assert ( all(test) , 'Orthogonal forces should be equal' )

test1 = f(:,1:8) == f(:,25:32);
test2 = f(:,9:16) == f(:,17:24);
assert ( all( test1(:) ) && all( test2(:)) , ...
             'Reverse magnetisation shouldn''t make a difference' )
```

This code is used in chunk 57b.

Now we want to try non-orthogonal magnetisation.

61a ⟨ *test001d.m* 61a⟩≡

```
clear all
disp('=================')
fprintf('TEST 001d: ')

magnet_fixed.dim = [0.04 0.04 0.04];
magnet_float.dim =  magnet_fixed.dim;

% Fixed parameters:
magnet_fixed.magn = 1.3;
magnet_float.magn = 1.3;
magnet_fixed.magdir = [0  90];  % $z$
displ = 0.12*[1 1 1];

⟨ Test XY superposition 61b⟩
⟨ Assert superposition 63⟩
⟨ Test XZ superposition 62a⟩
⟨ Assert superposition 63⟩
⟨ Test planar superposition 62b⟩
⟨ Assert superposition 63⟩

fprintf('passed\n')
disp('=================')
```

Root chunk (not used in this document).

Test with a magnetisation unit vector of $(1, 1, 0)$.

61b ⟨ *Test XY superposition* 61b⟩≡

```
magnet_float.magdir = [45  0];  % $\vec e_x+\vec e_y$
f1 = magnetforces(magnet_fixed,magnet_float,displ);

% Components:
magnet_float.magdir = [0  0];  % $\vec e_x$
fc1 = magnetforces(magnet_fixed,magnet_float,displ);

magnet_float.magdir = [90  0];  % $\vec e_y$
fc2 = magnetforces(magnet_fixed,magnet_float,displ);

f2 = (fc1+fc2)/sqrt(2);
```

This code is used in chunk 61a.

Test with a magnetisation unit vector of $(1, 0, 1)$.

62a ⟨ *Test XZ superposition* 62a⟩≡

```
magnet_float.magdir = [0  45];  % $\vec e_y+\vec e_z$
f1 = magnetforces(magnet_fixed,magnet_float,displ);

% Components:
magnet_float.magdir = [0  0];  % $\vec e_x$
fc1 = magnetforces(magnet_fixed,magnet_float,displ);

magnet_float.magdir = [0  90];  % $\vec e_z$
fc2 = magnetforces(magnet_fixed,magnet_float,displ);

f2 = (fc1+fc2)/sqrt(2);
```

This code is used in chunk 61a.

Test with a magnetisation unit vector of $(1, 1, 1)$. This is about as much as I can be bothered testing for now. Things seem to be working.

62b ⟨ *Test planar superposition* 62b⟩≡

```
[t p r] = cart2sph(1/sqrt(3),1/sqrt(3),1/sqrt(3));
magnet_float.magdir = [t p]*180/pi;  % $\vec e_y+\vec e_z+\vec e_z$
f1 = magnetforces(magnet_fixed,magnet_float,displ);

% Components:
magnet_float.magdir = [0  0];  % $\vec e_x$
fc1 = magnetforces(magnet_fixed,magnet_float,displ);

magnet_float.magdir = [90  0];  % $\vec e_y$
fc2 = magnetforces(magnet_fixed,magnet_float,displ);

magnet_float.magdir = [0  90];  % $\vec e_z$
fc3 = magnetforces(magnet_fixed,magnet_float,displ);

f2 = (fc1+fc2+fc3)/sqrt(3);
```

This code is used in chunk 61a.

The assertion is the same each time.

63 ⟨ *Assert superposition* 63⟩≡

```
assert ( ...
              isequal ( chop( f1 , 4 ) , chop ( f2 , 4 ) ) , ...
              'Components should sum due to superposition' ...
            )
```

This code is used in chunk 61a.

Now check that components are calculated correctly.

64 ⟨ *test001e.m* 64⟩≡

```
clear all
disp('=================')
fprintf('TEST 001e: ')

magnet_fixed.dim = [0.03 0.04 0.05];
magnet_float.dim = [0.055 0.045 0.035];

magnet_fixed.magn = 1;
magnet_float.magn = 1;

magnet_fixed.magdir = [30  50];
magnet_fixed.magdir = [60  45];

displ = [0.1 0.09 0.11];

f_all = magnetforces(magnet_fixed,magnet_float,displ);
f_x = magnetforces(magnet_fixed,magnet_float,displ,'x');
f_y = magnetforces(magnet_fixed,magnet_float,displ,'y');
f_z = magnetforces(magnet_fixed,magnet_float,displ,'z');

assert( all(f_all==[f_x(1); f_y(2); f_z(3)]) , ...
  'Forces components calculated separately shouldn''t change.')

k_all = magnetforces(magnet_fixed,magnet_float,displ,'stiffness');
k_x = magnetforces(magnet_fixed,magnet_float,displ,'stiffness','x');
k_y = magnetforces(magnet_fixed,magnet_float,displ,'stiffness','y');
k_z = magnetforces(magnet_fixed,magnet_float,displ,'stiffness','z');

assert( all(k_all==[k_x(1); k_y(2); k_z(3)]) , ...
  'Stiffness components calculated separately shouldn''t change.')

fprintf('passed\n')
disp('=================')
```

Root chunk (not used in this document).

Cylindrical magnets testing.

65 ⟨ *testcyl01.m* 65⟩≡

```
clear all
disp('=================')
fprintf('TEST cylinder forces: ')

magnet_fixed.dim = [0.02 0.04];
magnet_float.dim = magnet_fixed.dim;

magnet_fixed.magn = 1.3;
magnet_float.magn = 1.3;

magnet_fixed.type = 'cylinder';
magnet_float.type = magnet_fixed.type;

magnet_fixed.dir  = [0 0 1];
magnet_float.dir  = [0 0 1]; % must be same

magnet_fixed.magdir  = [0 0  1];
magnet_float.magdir  = [0 0 -1]; % must be aligned

F = magnetforces(magnet_fixed,magnet_float,[0 0 0.05]);

assert( round(1000*F(3)) == 265537 , 'forces between cylindrical magnets' );

fprintf('passed\n')
disp('=================')
```

Root chunk (not used in this document).

Cuboid torques tests.

66 ⟨ *testcuboidtorque01.m* 66⟩≡

```
clear all
disp('=================')
fprintf('TEST cuboid torques: ')

magnet_fixed.dim = [0.04 0.04 0.02];
magnet_float.dim = magnet_fixed.dim;

magnet_fixed.magn = 1.3;
magnet_float.magn = 1.3;

magnet_fixed.magdir  = [0 0 1];
magnet_float.magdir  = [0 0 1]; % must be (anti-)parallel

T = magnetforces(magnet_fixed,magnet_float,[0.02 0 0.03],'torque');

assert( all( round(1e6*T) == [0 33877 0] ), 'incorrect reference torques between parallel


magnet_fixed.dim = [0.04 0.02 0.04];
magnet_float.dim = magnet_fixed.dim;

magnet_fixed.magn = 1.3;
magnet_float.magn = 1.3;

magnet_fixed.magdir  = [0 1 0];
magnet_float.magdir  = [0 1 0]; % must be (anti-)parallel

T2 = magnetforces(magnet_fixed,magnet_float,[0 0.03 0.02],'torque');

assert( all( round(1e6*T2) == [33877 0 0] ), 'incorrect reference torques between parallel


magnet_fixed.dim = [0.02 0.04 0.04];
magnet_float.dim = magnet_fixed.dim;

magnet_fixed.magn = 1.3;
magnet_float.magn = 1.3;

magnet_fixed.magdir  = [1 0 0];
magnet_float.magdir  = [1 0 0]; % must be (anti-)parallel

T3 = magnetforces(magnet_fixed,magnet_float,[0.03 0.02 0],'torque');
```

```
assert( all( round(1e6*T3) == [0 0 33877] ), 'incorrect reference torques between parallel

fprintf('passed\n')
disp('=================')
```

Root chunk (not used in this document).

Specifying grades.

68 ⟨ *testgrades01.m* 68⟩≡

```
clear all
disp('=================')
fprintf('TEST ''grade'' specification: ')

displ = [0.03 0.05 0.07];
magnet_fixed.dim = [0.01 0.02 0.03];
magnet_fixed.magn = 2*sqrt(42/100); % = 'N42'
magnet_fixed.magdir  = [1 0 0];
magnet_float = magnet_fixed;

magnet_fixed2.dim = [0.01 0.02 0.03];
magnet_fixed2.grade = 'N42';
magnet_fixed2.magdir  = [1 0 0];
magnet_float2 = magnet_fixed2;

magnet_fixed3.dim = [0.01 0.02 0.03];
magnet_fixed3.grade = '42';
magnet_fixed3.magdir  = [1 0 0];
magnet_float3 = magnet_fixed3;

magnet_fixed4.dim = [0.01 0.02 0.03];
magnet_fixed4.grade = '42';
magnet_fixed4.magdir  = [1 0 0];
magnet_float4 = magnet_fixed4;

F1 = magnetforces(magnet_fixed, magnet_float, displ);
F2 = magnetforces(magnet_fixed2,magnet_float2,displ);
F3 = magnetforces(magnet_fixed3,magnet_float3,displ);
F4 = magnetforces(magnet_fixed4,magnet_float4,displ);

assert( all( round(1e6*F1) == round(1e6*F2) ), 'grade spec should be consistent' )
assert( all( round(1e6*F1) == round(1e6*F3) ), 'grade spec should be consistent' )
assert( all( round(1e6*F1) == round(1e6*F4) ), 'grade spec should be consistent' )

fprintf('passed\n')
disp('=================')
```

Root chunk (not used in this document).

3.3 Forces between (multipole) magnet arrays

This function uses `magnetforces.m` to compute the forces between two multipole magnet arrays. As before, we can calculate either force and/or stiffness in all three directions.

The structure of the function itself should look fairly straightforward. Some of the code is repeated from |magnetforces| (an advantage of the literate programming approach) for parsing the inputs for which calculations to perform and return.

69a ⟨ *multipoleforces.m* 69a⟩≡

```
function [varargout] = multipoleforces(fixed_array, float_array, displ, varargin)

⟨ Matlab help text (multipole) 90a⟩

⟨ Parse calculation args 17⟩
⟨ Organise input displacements 16⟩
⟨ Initialise multipole variables 72⟩
⟨ Calculate array forces 71⟩
⟨ Return all results 18⟩

⟨ Multipole sub-functions 69b⟩

end
```

Root chunk (not used in this document).

And nested sub-functions.

69b ⟨ *Multipole sub-functions* 69b⟩≡

```
⟨ Create arrays from input variables 74⟩
⟨ Extrapolate variables from input 89⟩
```

This code is used in chunk 69a.

Table 1: Description of `multipoleforces` data structures.

Inputs:	|fixed_array|	structure describing first magnet array
	|float_array|	structure describing the second magnet array
	|displ|	displacement between first magnet of each array
	[*what to calculate*]	'force' and/or 'stiffness'
Outputs:	|forces|	forces on the second array
	|stiffnesses|	stiffnesses on the second array
Arrays:	|type|	See Table 2
	|align|	See Table 3
	|face|	See Table 4
	|mcount|	|[i j k]| magnets in each direction
	|msize|	size of each magnet
	|mgap|	gap between successive magnets
	|magn|	magnetisation magnitude
	|magdir_fn|	function to calculate the magnetisation direction

Table 2: Possibilities for the `type` of a multipole array.

`generic`	Magnetisation directions &c. are defined manually
`linear`	Linear Halbach array
`planar`	Planar Halbach array
`quasi-Halbach`	Quasi-Halbach planar array
`patchwork`	'Patchwork' planar array

Table 3: Axes or plane with which to align the array, set with `align`.

`x`, `y`, `z`	For linear arrays
`xy`, `yz`, `xz`	For planar arrays

Table 4: Facing direction for the strong side of the array, set with `face`.

`+x`, `-x`	Horizontal
`+y`, `-y`	Horizontal
`+z`, `-z`, `up`, `down`	Vertical

Although the input to these functions is described in the user guide, there's a quick summary in Tables 1 and 2.

3.3.1 Actual calculation of the forces

To calculate these forces, let's assume that we have two large arrays enumerating the positions and magnetisations of each individual magnet in each magnet array.

Required fields for each magnet array:

`total`	M total number of magnets in the array
`dim`	$(M \times 3)$ size of each magnet
`magloc`	$(M \times 3)$ location of each magnet from the local coodinate system of the array
`magn`	$(M \times 1)$ magnetisation magnitude of each magnet
`magdir`	$(M \times 2)$ magnetisation direction of each magnet in spherical coordinates
`size`	$(M \times 3)$ total actual dimensions of the array

Then it's just a matter of actually calculating each force and summing them together, as shown below. We'll discuss how to actually populate these data structures later.

71 ⟨ *Calculate array forces* 71⟩≡

```
for ii = 1:fixed_array.total

  fixed_magnet = struct(...
        'dim',    fixed_array.dim(ii,:), ...
        'magn',   fixed_array.magn(ii), ...
        'magdir', fixed_array.magdir(ii,:) ...
  );

  for jj = 1:float_array.total

    float_magnet = struct(...
      'dim',    float_array.dim(jj,:), ...
      'magn',   float_array.magn(jj), ...
      'magdir', float_array.magdir(jj,:) ...
    );

    mag_displ = displ_from_array_corners ...
                  - repmat(fixed_array.magloc(ii,:)',[1 Ndispl]) ...
                  + repmat(float_array.magloc(jj,:)',[1 Ndispl]) ;

    if calc_force_bool && ~calc_stiffness_bool
      array_forces(:,:,ii,jj) = ...
          magnetforces(fixed_magnet, float_magnet, mag_displ,varargin{:});
    elseif calc_stiffness_bool && ~calc_force_bool
      array_stiffnesses(:,:,ii,jj) = ...
```

```
            magnetforces(fixed_magnet, float_magnet, mag_displ,varargin{:});
        else
          [array_forces(:,:,ii,jj) array_stiffnesses(:,:,ii,jj)] = ...
            magnetforces(fixed_magnet, float_magnet, mag_displ,varargin{:});
        end

      end
    end

    if calc_force_bool
      forces_out = sum(sum(array_forces,4),3);
    end

    if calc_stiffness_bool
      stiffnesses_out = sum(sum(array_stiffnesses,4),3);
    end
```

This code is used in chunk 69a.

This is where it begins. This is basically just initialisation, but note the important |complete_array_from_input| function. This is what takes the high-level Halbach array (or whatever array) descriptions and translates them into a more direct (if tedious) form.

72 ⟨ *Initialise multipole variables* 72⟩≡

```
    part = @(x,y) x(y);

    fixed_array = complete_array_from_input(fixed_array);
    float_array = complete_array_from_input(float_array);

    if calc_force_bool
      array_forces = nan([3 Ndispl fixed_array.total float_array.total]);
    end

    if calc_stiffness_bool
      array_stiffnesses = nan([3 Ndispl fixed_array.total float_array.total]);
    end

    displ_from_array_corners = displ ...
      + repmat(fixed_array.size/2,[1 Ndispl]) ...
      - repmat(float_array.size/2,[1 Ndispl]);
```

This code is used in chunk 69a.

3.3.2 From user input to array generation

We separate the force calculation from transforming the inputs into an intermediate form used for that purpose. This will hopefully allow us a little more flexibility.

This is the magic abstraction behind |complete_array_from_input| that allows us to write readable input code describing multipole arrays in as little detail as possible.

As input variables for a linear multipole array, we want to use some combination of the following:

w	wavelength of magnetisation
l	length of the array without magnet gaps
N	number of wavelengths
d	magnet length
T	total number of magnets
M	number of magnets per wavelength
ϕ	rotation between successive magnets

These are related via the following equations of constraint:

$$w = Md \qquad l = Td \qquad N = T/M \qquad M = 360^\circ/\phi \tag{1}$$

Taking logarithms and writing in matrix form yields

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \log \begin{bmatrix} w \\ l \\ N \\ d \\ T \\ M \\ \phi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \log(360^\circ) \end{bmatrix} \tag{2}$$

We can use this matrix to compute whichever variables we need given enough inputs.

However, we generally do not want an integer number of wavelengths of magnetisation in the magnet arrays; if $T = MN$ then we get small lateral forces that are undesirable for stability. We prefer instead to have $T = MN + 1$, but this cannot be represented by our linear (after taking logarithms) algebra above. Therefore, if the user requests a total number of wavelengths of magnetisation, we automatically add one end magnet to restore the symmetry of the forces.

More variables that can be set are:

ϕ_0	magnetisation direction of the first magnet
g	additional gap between adjacent magnet faces (optional)
e	array height (or magnet height)
f	array width (or magnet width)

For both technical reasons and reasons of convenience, the length of the array l does not take into account any specified magnet gap g. In other words, l is actually the length of the possibly discontiguous magnetic material; the length of the array will be $l + (N - 1)g$.

74 ⟨ *Create arrays from input variables* 74⟩≡

```
function array = complete_array_from_input(array)

if ~isfield(array,'type')
  array.type = 'generic';
end

⟨ Set alignment/facing directions 77⟩

switch array.type
  case 'linear'
⟨ Infer linear array variables 78⟩
  case 'linear-quasi'
⟨ Infer linear-quasi array variables 79⟩
  case 'planar'
⟨ Infer planar array variables 81⟩
  case 'quasi-halbach'
⟨ Infer quasi-Halbach array variables 82⟩
  case 'patchwork'
⟨ Infer patchwork array variables 83⟩
end

⟨ Array sizes 84⟩
⟨ Array magnetisation strengths 85⟩
⟨ Array magnetisation directions 86⟩

⟨ Fill in array structures 75⟩

end
```

This code is used in chunk 69b.

This is the part where those big data structures are filled up based on the user input data. I guess you could consider the process to consist of three stages. User input is the most abstract, from which the code above infers the other variables that have only been implied. Then the following code uses all that to construct a most basic description of the arrays, literally a listing of each magnet, its dimensions and position, and its magnetisation vector.

75 ⟨ *Fill in array structures* 75⟩≡

```
array.magloc = nan([array.total 3]);
array.magdir = array.magloc;
arrat.magloc_array = nan([array.mcount(1) array.mcount(2) array.mcount(3) 3]);

nn = 0;
for iii = 1:array.mcount(1)
  for jjj = 1:array.mcount(2)
    for kkk = 1:array.mcount(3)
      nn = nn + 1;
      array.magdir(nn,:) = array.magdir_fn(iii,jjj,kkk);
    end
  end
end

magsep_x = zeros(size(array.mcount(1)));
magsep_y = zeros(size(array.mcount(2)));
magsep_z = zeros(size(array.mcount(3)));

magsep_x(1) = array.msize_array(1,1,1,1)/2;
magsep_y(1) = array.msize_array(1,1,1,2)/2;
magsep_z(1) = array.msize_array(1,1,1,3)/2;

for iii = 2:array.mcount(1)
  magsep_x(iii) = array.msize_array(iii-1,1,1,1)/2 ...
                + array.msize_array(iii  ,1,1,1)/2 ;
end
for jjj = 2:array.mcount(2)
  magsep_y(jjj) = array.msize_array(1,jjj-1,1,2)/2 ...
                + array.msize_array(1,jjj  ,1,2)/2 ;
end
for kkk = 2:array.mcount(3)
  magsep_z(kkk) = array.msize_array(1,1,kkk-1,3)/2 ...
                + array.msize_array(1,1,kkk  ,3)/2 ;
end

magloc_x = cumsum(magsep_x);
magloc_y = cumsum(magsep_y);
```

```
magloc_z = cumsum(magsep_z);

for iii = 1:array.mcount(1)
  for jjj = 1:array.mcount(2)
    for kkk = 1:array.mcount(3)
      array.magloc_array(iii,jjj,kkk,:) = ...
        [magloc_x(iii); magloc_y(jjj); magloc_z(kkk)] ...
        + [iii-1; jjj-1; kkk-1].*array.mgap;
    end
  end
end
array.magloc = reshape(array.magloc_array,[array.total 3]);

array.size = squeeze( array.magloc_array(end,end,end,:) ...
           - array.magloc_array(1,1,1,:) ...
           + array.msize_array(1,1,1,:)/2 ...
           + array.msize_array(end,end,end,:)/2 );

debug_disp('Magnetisation directions')
debug_disp(array.magdir)

debug_disp('Magnet locations:')
debug_disp(array.magloc)
```

This code is used in chunk 74.

For all arrays that aren't |generic|, an alignment direction(s) and facing direction can be specified. By default, arrays face upwards and are aligned along x for linear arrays and on the x–y plane for planar.

77 ⟨ *Set alignment/facing directions* 77⟩≡

```
if ~isfield(array,'face')
  array.face = 'undefined';
end

linear_index = 0;
planar_index = [0 0];

switch array.type
  case 'generic'
  case 'linear',            linear_index = 1;
  case 'linear-quasi',        linear_index = 1;
  case 'planar',            planar_index = [1 2];
  case 'quasi-halbach',     planar_index = [1 2];
  case 'patchwork',         planar_index = [1 2];
  otherwise
    error(['Unknown array type ''',array.type,'''.'])
end

if ~isequal(array.type,'generic')
  if linear_index == 1
    if ~isfield(array,'align')
      array.align = 'x';
    end
    switch array.align
      case 'x', linear_index = 1;
      case 'y', linear_index = 2;
      case 'z', linear_index = 3;
    otherwise
      error('Alignment for linear array must be ''x'', ''y'', or ''z''.')
    end
  else
    if ~isfield(array,'align')
      array.align = 'xy';
    end
    switch array.align
      case 'xy', planar_index = [1 2];
      case 'yz', planar_index = [2 3];
      case 'xz', planar_index = [1 3];
    otherwise
      error('Alignment for planar array must be ''xy'', ''yz'', or ''xz''.')
```

```
    end
  end
end

switch array.face
  case {'+x','-x'},   facing_index = 1;
  case {'+y','-y'},   facing_index = 2;
  case {'up','down'}, facing_index = 3;
  case {'+z','-z'},   facing_index = 3;
  case 'undefined',   facing_index = 0;
end

if linear_index ~= 0
  if linear_index == facing_index
    error('Arrays cannot face into their alignment direction.')
  end
elseif ~isequal( planar_index, [0 0] )
  if any( planar_index == facing_index )
    error('Planar-type arrays can only face into their orthogonal direction')
  end
end
```

This code is used in chunk 74.

We need to finish off infering those variables that weren't specified but are implicit. This will be different for each type of multipole array, as you would have picked up on by now.

78 ⟨ *Infer linear array variables* 78⟩≡

```
array = extrapolate_variables(array);

array.mcount = ones(1,3);
array.mcount(linear_index) = array.Nmag;
```

This code is used in chunk 74.

The |linear-quasi| array is like the linear Halbach array but always has (except in the degenerate case) four magnets per wavelength. The magnet sizes are not equal.

79 ⟨ *Infer linear-quasi array variables* 79⟩≡

```
if isfield(array,'ratio') && isfield(array,'mlength')
  error('Cannot specify both ''ratio'' and ''mlength''.')
elseif ~isfield(array,'ratio') && ~isfield(array,'mlength')
  error('Must specify either ''ratio'' or ''mlength''.')
end


array.Nmag_per_wave = 4;
array.magdir_rotate = 90;

if isfield(array,'Nwaves')
  array.Nmag = array.Nmag_per_wave*array.Nwaves+1;
else
  error('''Nwaves'' must be specified.')
end

if isfield(array,'mlength')
  if numel(array.mlength) ~=2
    error('''mlength'' must have length two for linear-quasi arrays.')
  end
  array.ratio = array.mlength(2)/array.mlength(1);
else
  if isfield(array,'length')
    array.mlength(1) = 2*array.length/(array.Nmag*(1+array.ratio)+1-array.ratio);
    array.mlength(2) = array.mlength(1)*array.ratio;
  else
    error('''length'' must be specified.')
  end
end

array.mcount = ones(1,3);
array.mcount(linear_index) = array.Nmag;

array.msize = nan([array.mcount 3]);

[sindex_x sindex_y sindex_z] = ...
  meshgrid(1:array.mcount(1), 1:array.mcount(2), 1:array.mcount(3));

%% Because the array is linear, the |sindex| terms will be linear also.
```

```
all_indices = [1 1 1];
all_indices(linear_index) = 0;
all_indices(facing_index) = 0;
width_index = find(all_indices);

for ii = 1:array.Nmag
  array.msize(sindex_x(ii),sindex_y(ii),sindex_z(ii),linear_index) = ...
    array.mlength(mod(ii-1,2)+1);
  array.msize(sindex_x(ii),sindex_y(ii),sindex_z(ii),facing_index) = ...
    array.height;
  array.msize(sindex_x(ii),sindex_y(ii),sindex_z(ii),width_index) = ...
    array.width;
end
```

This code is used in chunk 74.

For now it's a bit more messy to do the planar array variables.

81 ⟨ *Infer planar array variables* 81⟩≡

```
if isfield(array,'length')
  if length(array.length) == 1
    if isfield(array,'width')
      array.length = [ array.length array.width ];
    else
      array.length = [ array.length array.length ];
    end
  end
end

if isfield(array,'mlength')
  if length(array.mlength) == 1
    if isfield(array.mwidth)
      array.mlength = [ array.mlength array.mwidth ];
    else
      array.mlength = [ array.mlength array.mlength ];
    end
  end
end

var_names = {'length','mlength','wavelength','Nwaves',...
             'Nmag','Nmag_per_wave','magdir_rotate'};

tmp_array1 = struct();
tmp_array2 = struct();
var_index = zeros(size(var_names));

for iii = 1:length(var_names)
  if isfield(array,var_names(iii))
    tmp_array1.(var_names{iii}) = array.(var_names{iii})(1);
    tmp_array2.(var_names{iii}) = array.(var_names{iii})(end);
  else
    var_index(iii) = 1;
  end
end

tmp_array1 = extrapolate_variables(tmp_array1);
tmp_array2 = extrapolate_variables(tmp_array2);

for iii = find(var_index)
  array.(var_names{iii}) = [tmp_array1.(var_names{iii}) tmp_array2.(var_names{iii})];
end
```

```
array.width  = array.length(2);
array.length = array.length(1);

array.mwidth  = array.mlength(2);
array.mlength = array.mlength(1);

array.mcount = ones(1,3);
array.mcount(planar_index) = array.Nmag;
```

This code is used in chunk 74.

The other two planar arrays are less complicated than the planar Halbach array above. Still lots of annoying variable-wrangling, though.

82 ⟨ *Infer quasi-Halbach array variables* 82⟩≡

```
if isfield(array,'mcount')
  if numel(array.mcount) ~=3
    error('''mcount'' must always have three elements.')
  end
elseif isfield(array,'Nwaves')
  if numel(array.Nwaves) > 2
    error('''Nwaves'' must have one or two elements only.')
  end
  array.mcount(facing_index) = 1;
  array.mcount(planar_index) = 4*array.Nwaves+1;
elseif isfield(array,'Nmag')
  if numel(array.Nmag) > 2
    error('''Nmag'' must have one or two elements only.')
  end
  array.mcount(facing_index) = 1;
  array.mcount(planar_index) = array.Nmag;
else
  error('Must specify the number of magnets (''mcount'' or ''Nmag'') or wavelengths (''Nwaves'')
end
```

This code is used in chunk 74.

Basically the same for the patchwork array but without worrying about wavelengths.

83 ⟨ *Infer patchwork array variables* 83⟩≡

```
if isfield(array,'mcount')
  if numel(array.mcount) ~=3
    error('''mcount'' must always have three elements.')
  end
elseif isfield(array,'Nmag')
  if numel(array.Nmag) > 2
    error('''Nmag'' must have one or two elements only.')
  end
  array.mcount(facing_index) = 1;
  array.mcount(planar_index) = array.Nmag;
else
  error('Must specify the number of magnets (''mcount'' or ''Nmag'')')
end
```

This code is used in chunk 74.

Sizes.

84 ⟨ *Array sizes* 84⟩≡

```
array.total = prod(array.mcount);

if ~isfield(array,'msize')
  array.msize = [NaN NaN NaN];
  if linear_index ~=0
    array.msize(linear_index) = array.mlength;
    array.msize(facing_index) = array.height;
    array.msize(isnan(array.msize)) = array.width;
  elseif ~isequal( planar_index, [0 0] )
    array.msize(planar_index) = [array.mlength array.mwidth];
    array.msize(facing_index) = array.height;
  else
    error('The array property ''msize'' is not defined and I have no way to infer it.')
  end
elseif numel(array.msize) == 1
  array.msize = repmat(array.msize,[3 1]);
end

if numel(array.msize) == 3
  array.msize_array = ...
      repmat(reshape(array.msize,[1 1 1 3]), array.mcount);
else
  if isequal([array.mcount 3],size(array.msize))
    array.msize_array = array.msize;
  else
    error('Magnet size ''msize'' must have three elements (or one element for a cube magnet)
  end
end
array.dim = reshape(array.msize_array, [array.total 3]);

if ~isfield(array,'mgap')
  array.mgap = [0; 0; 0];
elseif length(array.mgap) == 1
  array.mgap = repmat(array.mgap,[3 1]);
end
```

This code is used in chunk 74.

Magnetisation strength of each magnet.

85 ⟨ *Array magnetisation strengths* 85⟩≡

```
if ~isfield(array,'magn')
  if isfield(array,'grade')
    array.magn = grade2magn(array.grade);
  else
    array.magn = 1;
  end
end

if length(array.magn) == 1
  array.magn = repmat(array.magn,[array.total 1]);
else
  error('Magnetisation magnitude ''magn'' must be a single value.')
end
```

This code is used in chunk 74.

Magnetisation direction of each magnet.

86 ⟨ *Array magnetisation directions* 86⟩≡

```
if ~isfield(array,'magdir_fn')

  if ~isfield(array,'face')
    array.face = '+z';
  end

  switch array.face
    case {'up','+z','+y','+x'},   magdir_rotate_sign =  1;
    case {'down','-z','-y','-x'}, magdir_rotate_sign = -1;
  end

  if ~isfield(array,'magdir_first')
    array.magdir_first = magdir_rotate_sign*90;
  end

  magdir_fn_comp{1} = @(ii,jj,kk) 0;
  magdir_fn_comp{2} = @(ii,jj,kk) 0;
  magdir_fn_comp{3} = @(ii,jj,kk) 0;

  switch array.type
  case 'linear'
    magdir_theta = @(nn) ...
      array.magdir_first+magdir_rotate_sign*array.magdir_rotate*(nn-1);

    magdir_fn_comp{linear_index} = @(ii,jj,kk) ...
      cosd(magdir_theta(part([ii,jj,kk],linear_index)));

    magdir_fn_comp{facing_index} = @(ii,jj,kk) ...
      sind(magdir_theta(part([ii,jj,kk],linear_index)));

  case 'linear-quasi'

    magdir_theta = @(nn) ...
      array.magdir_first+magdir_rotate_sign*90*(nn-1);

    magdir_fn_comp{linear_index} = @(ii,jj,kk) ...
      cosd(magdir_theta(part([ii,jj,kk],linear_index)));

    magdir_fn_comp{facing_index} = @(ii,jj,kk) ...
      sind(magdir_theta(part([ii,jj,kk],linear_index)));

  case 'planar'
```

```
  magdir_theta = @(nn) ...
    array.magdir_first(1)+magdir_rotate_sign*array.magdir_rotate(1)*(nn-1);

  magdir_phi = @(nn) ...
    array.magdir_first(end)+magdir_rotate_sign*array.magdir_rotate(end)*(nn-1);

  magdir_fn_comp{planar_index(1)} = @(ii,jj,kk) ...
    cosd(magdir_theta(part([ii,jj,kk],planar_index(2))));

  magdir_fn_comp{planar_index(2)} = @(ii,jj,kk) ...
    cosd(magdir_phi(part([ii,jj,kk],planar_index(1))));

  magdir_fn_comp{facing_index} = @(ii,jj,kk) ...
    sind(magdir_theta(part([ii,jj,kk],planar_index(1)))) ...
    + sind(magdir_phi(part([ii,jj,kk],planar_index(2))));

case 'patchwork'

  magdir_fn_comp{planar_index(1)} = @(ii,jj,kk) 0;

  magdir_fn_comp{planar_index(2)} = @(ii,jj,kk) 0;

  magdir_fn_comp{facing_index} = @(ii,jj,kk) ...
    magdir_rotate_sign*(-1)^( ...
          part([ii,jj,kk],planar_index(1)) ...
          + part([ii,jj,kk],planar_index(2)) ...
          + 1 ...
        );

case 'quasi-halbach'

  magdir_fn_comp{planar_index(1)} = @(ii,jj,kk) ...
    sind(90*part([ii,jj,kk],planar_index(1))) ...
    * cosd(90*part([ii,jj,kk],planar_index(2)));

  magdir_fn_comp{planar_index(2)} = @(ii,jj,kk) ...
    cosd(90*part([ii,jj,kk],planar_index(1))) ...
    * sind(90*part([ii,jj,kk],planar_index(2)));

  magdir_fn_comp{facing_index} = @(ii,jj,kk) ...
    magdir_rotate_sign ...
    * sind(90*part([ii,jj,kk],planar_index(1))) ...
    * sind(90*part([ii,jj,kk],planar_index(2)));

otherwise
```

```
      error('Array property ''magdir_fn'' not defined and I have no way to infer it.')
    end

    array.magdir_fn = @(ii,jj,kk)   ...
      [ magdir_fn_comp{1}(ii,jj,kk) ...
        magdir_fn_comp{2}(ii,jj,kk) ...
        magdir_fn_comp{3}(ii,jj,kk) ];

  end
```

This code is used in chunk 74.

3.3.3 Sub-functions

89 ⟨ *Extrapolate variables from input* 89⟩≡

```
function array_out = extrapolate_variables(array)

var_names = {'wavelength','length','Nwaves','mlength',...
             'Nmag','Nmag_per_wave','magdir_rotate'};

if isfield(array,'Nwaves')
  mcount_extra  =  1;
else
  mcount_extra  =  0;
end

if isfield(array,'mlength')
  mlength_adjust = false;
else
  mlength_adjust = true;
end

variables = nan([7 1]);

for iii = 1:length(var_names);
  if isfield(array,var_names(iii))
    variables(iii) = array.(var_names{iii});
  end
end

var_matrix = ...
    [1,  0,  0, -1,  0, -1,  0;
     0,  1,  0, -1, -1,  0,  0;
     0,  0,  1,  0, -1,  1,  0;
     0,  0,  0,  0,  0,  1,  1];

var_results = [0 0 0 log(360)]';
variables = log(variables);

idx = ~isnan(variables);
var_known = var_matrix(:,idx)*variables(idx);
var_calc = var_matrix(:,~idx)\(var_results-var_known);
variables(~idx) = var_calc;
variables = exp(variables);

for iii = 1:length(var_names);
  array.(var_names{iii}) = variables(iii);
```

```
end

array.Nmag = round(array.Nmag) + mcount_extra;
array.Nmag_per_wave = round(array.Nmag_per_wave);

if mlength_adjust
  array.mlength  =  array.mlength * (array.Nmag-mcount_extra)/array.Nmag;
end

array_out = array;

end
```

This code is used in chunk 69b.

When users type `help multipoleforces` this is what they see.

90a ⟨ *Matlab help text (multipole)* 90a⟩≡

```
%% MULTIPOLEFORCES  Calculate forces between two multipole arrays of magnets
%
% Finish this off later. Please read the PDF documentation instead for now.
%
```

This code is used in chunk 69a.

3.4 Test files for multipole arrays

Not much here yet.

90b ⟨ *testall.m* 49⟩+≡

```
test002a
test002b
test002c
test002d

test003a
```

First test just to check the numbers aren't changing.

91 ⟨ *test002a.m* 91⟩≡

```
clear all
disp('=================')
fprintf('TEST 002a: ')

fixed_array = ...
  struct(...
        'type','linear', ...
        'align','x', ...
        'face','up', ...
        'length', 0.01, ...
        'width',  0.01, ...
        'height', 0.01, ...
        'Nmag_per_wave', 4, ...
        'Nwaves', 1, ...
        'magn', 1, ...
        'magdir_first', 90 ...
  );

float_array = fixed_array;
float_array.face = 'down';
float_array.magdir_first = -90;

displ = [0 0 0.02];

f_total = multipoleforces(fixed_array, float_array, displ);

assert( chop(f_total(3),5)==0.13909 , 'Regression shouldn''t fail');

fprintf('passed\n')
disp('=================')
```

Root chunk (not used in this document).

Test against single magnet.

92 ⟨ *test002b.m* 92⟩≡

```
clear all
disp('=================')
fprintf('TEST 002b: ')

fixed_array = ...
  struct(...
        'type','linear', ...
        'align','x', ...
        'face','up', ...
        'length', 0.01, ...
        'width',  0.01, ...
        'height', 0.01, ...
        'Nmag_per_wave', 1, ...
        'Nwaves', 1, ...
        'magn', 1, ...
        'magdir_first', 90 ...
  );

float_array = fixed_array;
float_array.face = 'down';
float_array.magdir_first = -90;

displ = [0 0 0.02];

f_total = multipoleforces(fixed_array, float_array, displ);

fixed_mag = struct('dim',[0.01 0.01 0.01],'magn',1,'magdir',[0  90]);
float_mag = struct('dim',[0.01 0.01 0.01],'magn',1,'magdir',[0 -90]);
f_mag = magnetforces(fixed_mag,float_mag,displ);

assert( chop(f_total(3),6) == chop(f_mag(3),6) );

fprintf('passed\n')
disp('=================')
```

Root chunk (not used in this document).

Test that linear arrays give consistent results regardless of orientation.

93 ⟨ *test002c.m* 93⟩≡

```
clear all
disp('=================')
fprintf('TEST 002c: ')

% Fixed parameters

fixed_array = ...
  struct(...
        'length', 0.10, ...
        'width',  0.01, ...
        'height', 0.01, ...
        'Nmag_per_wave', 4, ...
        'Nwaves', 1, ...
        'magn', 1, ...
        'magdir_first', 90 ...
  );

float_array = fixed_array;
float_array.magdir_first = -90;

f = nan([3 0]);

% The varying calculations

fixed_array.type = 'linear';
float_array.type = fixed_array.type;
fixed_array.align = 'x';
float_array.align = fixed_array.align;
fixed_array.face = 'up';
float_array.face = 'down';
displ = [0 0 0.02];
f(:,end+1) = multipoleforces(fixed_array, float_array, displ);

fixed_array.type = 'linear';
float_array.type = fixed_array.type;
fixed_array.align = 'x';
float_array.align = fixed_array.align;
fixed_array.face = '+y';
float_array.face = '-y';
displ = [0 0.02 0];
f(:,end+1) = multipoleforces(fixed_array, float_array, displ);
```

```
fixed_array.type = 'linear';
float_array.type = fixed_array.type;
fixed_array.align = 'y';
float_array.align = fixed_array.align;
fixed_array.face = 'up';
float_array.face = 'down';
displ = [0 0 0.02];
f(:,end+1) = multipoleforces(fixed_array, float_array, displ);

fixed_array.type = 'linear';
float_array.type = fixed_array.type;
fixed_array.align = 'y';
float_array.align = fixed_array.align;
fixed_array.face = '+x';
float_array.face = '-x';
displ = [0.02 0 0];
f(:,end+1) = multipoleforces(fixed_array, float_array, displ);


fixed_array.type = 'linear';
float_array.type = fixed_array.type;
fixed_array.align = 'z';
float_array.align = fixed_array.align;
fixed_array.face = '+x';
float_array.face = '-x';
displ = [0.02 0 0];
f(:,end+1) = multipoleforces(fixed_array, float_array, displ);

fixed_array.type = 'linear';
float_array.type = fixed_array.type;
fixed_array.align = 'z';
float_array.align = fixed_array.align;
fixed_array.face = '+y';
float_array.face = '-y';
displ = [0 0.02 0];
f(:,end+1) = multipoleforces(fixed_array, float_array, displ);

assert( all(chop(sum(f),4)==37.31) , ...
  'Arrays aligned in different directions should produce consistent results.');

fprintf('passed\n')
disp('=================')
```

Root chunk (not used in this document).

Test that planar arrays give consistent results regardless of orientation.

95 ⟨ *test002d.m* 95⟩≡

```
clear all
disp('=================')
fprintf('TEST 002d: ')

% Fixed parameters

fixed_array = ...
  struct(...
        'length', [0.10 0.10], ...
        'width',  0.10, ...
        'height', 0.01, ...
        'Nmag_per_wave', [4 4], ...
        'Nwaves', [1 1], ...
        'magn', 1, ...
        'magdir_first', [90 90] ...
  );

float_array = fixed_array;
float_array.magdir_first = [-90 -90];

f = nan([3 0]);

% The varying calculations

fixed_array.type = 'planar';
float_array.type = fixed_array.type;
fixed_array.align = 'xy';
float_array.align = fixed_array.align;
fixed_array.face = 'up';
float_array.face = 'down';
displ = [0 0 0.02];
f(:,end+1) = multipoleforces(fixed_array, float_array, displ);

fixed_array.type = 'planar';
float_array.type = fixed_array.type;
fixed_array.align = 'yz';
float_array.align = fixed_array.align;
fixed_array.face = '+x';
float_array.face = '-x';
displ = [0.02 0 0];
f(:,end+1) = multipoleforces(fixed_array, float_array, displ);
```

```
fixed_array.type = 'planar';
float_array.type = fixed_array.type;
fixed_array.align = 'xz';
float_array.align = fixed_array.align;
fixed_array.face = '+y';
float_array.face = '-y';
displ = [0 0.02 0];
f(:,end+1) = multipoleforces(fixed_array, float_array, displ);

ind = [3 4 8];

assert( all(round(f(ind) * 100)/100==589.05) ,  ...
  'Arrays aligned in different directions should produce consistent results.');

assert( all(f(~ind)<1e-10) ,  ...
  'These forces should all be (essentially) zero.');

fprintf('passed\n')
disp('=================')
```

Root chunk (not used in this document).

Check that the |linear-quasi| array gives same output as |linear| array for equivalent parameters.

97 ⟨ *test003a.m* 97⟩≡

```
clear all
disp('=================')
fprintf('TEST 003a: ')

displ = [0.02 0.02 0.02];

% Test against Halbach array with four magnets per wavelength

fixed_array = struct(...
        'type','linear', ...
        'align','x', ...
        'face','up', ...
        'length', 0.05, ...
        'width',  0.01, ...
        'height', 0.01, ...
        'Nmag_per_wave', 4, ...
        'Nwaves', 1 ...
  );

float_array = fixed_array;
float_array.face = 'down';

f1 = multipoleforces(fixed_array, float_array, displ);

fixed_array = struct(...
        'type','linear-quasi', ...
        'align','x', ...
        'face','up', ...
        'length', 0.05, ...
        'width',  0.01, ...
        'height', 0.01, ...
        'Nwaves', 1, ...
        'ratio', 1 ...
  );

float_array = fixed_array;
float_array.face = 'down';

f2 = multipoleforces(fixed_array, float_array, displ);

assert( all(chop(f1,6)==chop(f2,6)) , ...
```

```
  'linear (4mag) and linear-quasi should be equivalent');

% Test against Halbach array with two magnets per wavelength

fixed_array = struct(...
        'type','linear', ...
        'align','x', ...
        'face','up', ...
        'length', 0.03, ...
        'width',  0.01, ...
        'height', 0.01, ...
        'Nmag_per_wave', 2, ...
        'Nwaves', 1 ...
  );

float_array = fixed_array;
float_array.face = 'down';

f3 = multipoleforces(fixed_array, float_array, displ);

fixed_array = struct(...
        'type','linear-quasi', ...
        'align','x', ...
        'face','up', ...
        'length', 0.03, ...
        'width',  0.01, ...
        'height', 0.01, ...
        'Nwaves', 1, ...
        'ratio', 0 ...
  );

float_array = fixed_array;
float_array.face = 'down';

f4 = multipoleforces(fixed_array, float_array, displ);

assert( all(chop(f3,6)==chop(f4,6)) , ...
  'linear (2mag) and linear-quasi should be equivalent');

fprintf('passed\n')

disp('================')
```

Root chunk (not used in this document).

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