Number Theory

Coding Club

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Introduction

- **Number Theory?** Study of integers, especially positive integers
- **Format?** You'll see the following:
  - Mathematical definitions with examples
  - Some interesting algorithms
  - Applications to cryptography
- **Why Math?** Some math background will enable you to make more sophisticated software.
- **These Slides?** See our GitHub repository.
- **If you don't know coding:** Check appendix in main document on GitHub, or find a guide online. (We won't do much in this presentation.)

Any questions?

Outline

Divisibility

Definition (Divisibility)

A nonzero integer a is a divisor of an integer b if $b = ak$ for some integer k.

- When a divides b, we write "$a \mid b$".
- When a does not divide b, we write "$a \nmid b$".

Example

- $5 \mid 15$ because $15 = 5 \cdot 3$, and 3 is an integer.
- $6 \nmid 15$ because $15 = 6 \cdot 2.5$, and 2.5 is not an integer.
- For all n, $n \mid 0$. (Why?)

Division Algorithm

Theorem (The Division Algorithm)

For integers a and m with $m > 0$, there exist unique integers q and r such that

$$a = mq + r,$$

where $0 \leqslant r < m$. We may write $a \bmod m$ to refer to this unique r.

Example

- If $a = 17$ and $m = 5$, $17 = 5 \cdot 3 + 2$. Note that $0 \leqslant 2 < 5$.
- If $a = -17$ and $m = 5$, $-17 = 5 \cdot -4 + 3$. Also, $-17 \bmod 5 = 3$.

Division Algorithm (Cont.)

In the C programming language, % gives the remainder.

```
int a = 17, m = 5;
int r = a % m;
printf("%d",r);
```

This outputs 2.

Note! This isn't the same as $a \bmod m$. See this example:

```
int a = -17, m = 5;
int r = a % m;
printf("%d",r);
```

This prints -2, instead of $3 = -17 \bmod 5$.

Caesar Cipher

- **Encryption:** Transforming a plain text message into cipher text to hide its content.
- **Decryption:** Reverting the cipher text to plain text.
- **Key:** Determines "parameters" for the encryption and decryption.
  - Usually agreed upon by sender and receiver.
- **Caesar Cipher:** "Shift" alphabet by the key number.

Caesar Cipher (Cont.)

Example

Alphabet shifted by key $k = 3$.

$$\begin{array}{cccccccccc} A & B & C & D & E & \cdots & W & X & Y & Z \\ D & E & F & G & H & \cdots & Z & A & B & C \end{array}$$

Message:

I HAVE INVENTED A NEW SALAD, TELL THE GREEKS.

Replace each letter with its correspondent:

L KDYH LQYHQWHG D QHZ VDODG, WHOO WKH JUHHNV.

GCD

Definition (Greatest Common Divisor)

Let a, b, c be integers. If $c \mid a$ and $c \mid b$, then c is a *common divisor* of a and b. The largest such c is the greatest common divisor of a and b, and is denoted $\gcd(a, b)$.

Theorem (Bézout's Identity)

Let a, b, d be integers with $d = \gcd(a, b)$. For each multiple of d, there exists a pair of integers x, y such that $ax + by$ is equal to this multiple.

The Euclidean Algorithm

Algorithm (Euclidean Algorithm)

Given two integers m and n, find $\gcd(m, n)$.

1. [Find remainder.] Divide m by n and let r be the remainder.
2. [Is it zero?] If r is 0, the algorithm terminates; n is the answer.
3. [Reduce.] Set m to n, then n to r, and go back to Step 1.

Example: Euclidean Algorithm

Let $m = 119, n = 544$.

1. Set $r = 119 \bmod 544 = 119$.
2. $119 \neq 0$, so set m to 544, then n to 119.
3. Set $r = 544 \bmod 119 = 68$ (since $544 = 4 \times 119 + 68$).
4. $68 \neq 0$, so set m to 119, then n to 68.
5. Set $r = 119 \bmod 68 = 51$.
6. $51 \neq 0$, so set m to 68, then n to 51.
7. Set $r = 68 \bmod 51 = 17$.
8. $17 \neq 0$, so set m to 51, then n to 17.
9. Set $r = 51 \bmod 17 = 0$.
10. $r = 0$, so terminate. $\gcd(119, 544) = 17$.

Prime Numbers

Definition (Prime Number)

A prime number p is a positive integer that has no divisors apart from 1 and p.

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, \ldots$$

The Sieve of Eratosthenes

Algorithm (Sieve of Eratosthenes)

Generate a list of all prime numbers less than or equal to a positive integer n.

1. [Initialize.] Create a list of consecutive integers from 2 to n. Let $p = 2$.
2. [Remove composites.] Remove all multiples of p from the list, except p itself.
3. [Iterate.] If there is an integer greater than p in the list, set p to be the smallest such integer, and go to Step 2. Otherwise, terminate; all numbers in the list are prime.

Example: Sieve of Eratosthenes

Initialize

	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Example: Sieve of Eratosthenes (Cont.)

$p = 2$

	2	3		5		7		9	
11		13		15		17		19	
21		23		25		27		29	
31		33		35		37		39	
41		43		45		47		49	
51		53		55		57		59	
61		63		65		67		69	
71		73		75		77		79	
81		83		85		87		89	
91		93		95		97		99	

Example: Sieve of Eratosthenes (Cont.)

$p = 3$

	2	3		5		7			
11		13				17		19	
		23		25				29	
31				35		37			
41		43				47		49	
		53		55				59	
61				65		67			
71		73				77		79	
		83		85				89	
91				95		97			

Example: Sieve of Eratosthenes (Cont.)

$$p = 5$$

	2	3		5		7			
11		13				17		19	
		23						29	
31						37			
41		43				47		49	
		53						59	
61						67			
71		73				77		79	
		83						89	
91						97			

Example: Sieve of Eratosthenes (Cont.)

$p = 7$

	2	3		5		7			
11		13				17		19	
		23						29	
31						37			
41		43				47			
		53						59	
61						67			
71		73						79	
		83						89	
						97			

Optimization: We can stop if $p > \sqrt{n}$.

Theorems About Primes

Theorem (Euclid's Lemma)

If a prime number p divides the product ab of two integers a and b, then p must divide at least one of a or b.

Theorem (Fundamental Theorem of Arithmetic)

Every integer greater than 1 can be represented uniquely as a product of prime powers.

Relatively Prime Numbers

Definition (Relatively Prime Numbers)

Let a and b be integers. If $\gcd(a, b) = 1$, then a and b are said to be relatively prime.

Definition (Euler's Totient Function)

Let n be an integer. $\phi(n)$ counts how many of the positive integers up to n are relatively prime to n.

Proposition

- Whenever n is prime, $\phi(n) = n - 1$.
- For any two relatively prime numbers m and n, $\phi(mn) = \phi(m)\phi(n)$.

Recap

- We defined divisibility and went over the division algorithm.
- Caesar Cipher.
- Greatest Common Divisor, Bézout's Identity, and the Euclidean Algorithm.
- Prime numbers and the Sieve of Eratosthenes.

Modular Arithmetic

Definition (Congruence Modulo m)

For integers a, b, m, if $m \mid (a - b)$, then we say that a is congruent to b modulo m, and write $a \equiv b \pmod{m}$.

Example

- $9 \equiv 21 \pmod{6}$ because $6 \mid (21 - 9)$.
  According to the Division Algorithm, $21 = 6 \cdot 3 + 3$ and $9 = 6 \cdot 1 + 3$.
  Remainders are the same!
- $-17 \equiv 4 \pmod{7}$ because $7 \mid (4 - (-17))$.

Properties

Proposition (Modular Arithmetic)

Suppose that $a \equiv b \pmod{m}$. Then, the following is true for all integers k.

- $a + k \equiv b + k \pmod{m}$.
- If $c \equiv d \pmod{m}$, then $ac \equiv bd \pmod{m}$.
- $a^k \equiv b^k \pmod{m}$.

Definition (Modular Multiplicative Inverse)

Given relatively prime integers a, m, there exists an integer a^{-1} such that $a^{-1}a \equiv 1 \pmod{m}$. We call a^{-1} the modular multiplicative inverse of a.

Affine Cipher

- Generalization of the Caesar Cipher.
- First multiply modulo 26, then shift (add modulo 26).

Algorithm (Affine Cipher Encryption)

1. [Choose key.] Choose an integer $0 < a < 26$ relatively prime to 26, and any integer $0 \leqslant b < 26$.
2. [Encrypt.] For each letter, take its numerical value x. Find the integer $0 \leqslant y < 26$ such that $y \equiv ax + b \pmod{26}$. Replace by the letter corresponding to y.

Affine Cipher Decryption

Assume $b = 0$. We know $y \equiv ax \pmod{26}$.
Since $\gcd(a, 26) = 1$, there must be a^{-1} such that

$$a^{-1}ax \equiv x \equiv a^{-1}y \pmod{26}.$$

Pairs:

a	1	3	5	7	9	11	15	17	19	21	23	25
a^{-1}	1	9	21	15	3	19	7	23	11	5	17	25

What if $b \neq 0$, so that $ax + b \equiv y \pmod{26}$? Then $ax \equiv y - b \pmod{26}$, so

$$x \equiv a^{-1}y - a^{-1}b \pmod{26}.$$

Example: Affine Cipher Encryption

0	1	2	3	4	5	6	7	8	9	10	11	12
A	B	C	D	E	F	G	H	I	J	K	L	M
13	14	15	16	17	18	19	20	21	22	23	24	25
N	O	P	Q	R	S	T	U	V	W	X	Y	Z

AFFINE NOT LINEAR

Suppose $a = 3, b = 7$. Encryption function $E(x) = ax + b \bmod 26$, where x is the character being encrypted. So $E(A) = 3 \cdot 0 + 7 = 7 = H$.
Continuing:

$E(F) = W$	$E(N) = U$	$E(O) = X$	$E(L) = O$
$E(I) = F$	$E(E) = T$	$E(T) = M$	$E(R) = G$

Cipher text:

HWWFUT UXM OFUTHG

Residues

- Suppose you have a set of moduli $m_1, m_2, \ldots, m_k$, and an integer x.
- "Residues" $u_1 = x \bmod m_1$, $u_2 = x \bmod m_2$, ...
- Modular representation of x in this system is

$$(u_1, u_2, \ldots, u_k).$$

Example

Three moduli $m_1 = 8, m_2 = 21, m_3 = 5$. Let's choose $x = 127$. Then $u_1 = 7, u_2 = 1, u_3 = 2$. So x can be represented as $(7, 1, 2)$.

Chinese Remainder Theorem

In above example, between 1 and $m_1m_2m_3 = 840$ inclusive, 127 is the only number with representation $(7, 1, 2)$!

Theorem (Chinese Remainder Theorem)

Let $m_1, m_2, \ldots, m_k$ be positive integers that are relatively prime in pairs. Let $m = m_1m_2\cdots m_k$, and let $a, u_1, u_2, \ldots, u_k$ be integers. Then there is exactly one x such that

$$a \leqslant x < a + m, \quad \text{and} \quad x \equiv u_i \pmod{m_i} \quad \text{for } 1 \leqslant i \leqslant k.$$

a allows for an offset. We took $a = 1$ above, but could choose any value.

RSA

- **Asymmetric encryption** (two keys)
  - **Public key** shared with anyone, used for encryption
  - **Private key** known only to receiver, used for decryption
- RSA's security relies on difficulty of factorizing large primes.

RSA

Algorithm (RSA Encryption)

1. [Choose key.] Choose two primes p and q, and an integer e such that $(p-1)(q-1)$ and e are relatively prime.
2. [Encrypt.] For each letter, take its numerical value x, and replace it with the letter corresponding to $y = (x^e \bmod pq)$.

Decryption: Find integer d for which $ed \equiv 1 \pmod{(p-1)(q-1)}$. Then take $x = y^d \bmod pq$. Yes, that's it.

But why does this work?

Example: RSA

65	66	67	68	69	70	71	72	73	74	75	76	77
A	B	C	D	E	F	G	H	I	J	K	L	M
78	79	80	81	82	83	84	85	86	87	88	89	90
N	O	P	Q	R	S	T	U	V	W	X	Y	Z

KEEP ON KEEPING ON

Suppose $p = 17, q = 19$. Then $pq = 323$, and $(p-1)(q-1) = 288$. Let's say $e = 5$, then $d = 173$. Encryption function is $E(x) = x^5 \bmod 323$.

$E(K) = 75^5 \bmod 323 = 113$ $\quad E(\text{␣}) = 32^5 \bmod 323 = 223$ $\quad E(I) = 73^5 \bmod 323 = 99$

$E(E) = 69^5 \bmod 323 = 103$ $\quad E(O) = 79^5 \bmod 323 = 129$ $\quad E(G) = 71^5 \bmod 323 = 124$

$E(P) = 16^5 \bmod 323 = 207$ $\quad E(N) = 78^5 \bmod 323 = 108$

Cipher "text":

$$113, 103, 103, 207, 223, 129, 108, 223, 113,$$
$$103, 103, 207, 99, 108, 124, 223, 129, 108$$

Euler's Theorem

We need this theorem first.

Theorem (Euler's Theorem)

For integers a and n, if they are relatively prime, then

$$a^{\phi(n)} \equiv 1 \pmod{n}, \quad \text{or equivalently } a^{\phi(n)+1} \equiv a \pmod{n}.$$

So $x^{\phi(pq)} \equiv 1 \pmod{pq}$, which implies that $x^{k\phi(pq)+1} \equiv x \pmod{pq}$.

Correctness of RSA Decryption

Given $ed \equiv 1 \pmod{(p-1)(q-1)}$.

Proof.

We know that $x^{p-1} \equiv 1 \pmod{p}$ and $x^{q-1} \equiv 1 \pmod{q}$.
So $x^{k(p-1)(q-1)+1} \equiv x \pmod{p}$ and $x^{k(p-1)(q-1)+1} \equiv x \pmod{q}$.
Since $ed \equiv 1 \pmod{\phi(pq)}$, there is k such that $ed = k\phi(pq) + 1$. That is, $ed = k(p-1)(q-1) + 1$.
Substitute:

$$x^{ed} \equiv x \pmod{p} \quad \text{and} \quad x^{ed} \equiv x \pmod{q}.$$

So $x^{ed} \equiv x \pmod{pq}$. ∎

That's All!

Most of this was based on the following:

- The Art of Computer Programming (Knuth) — Chapter 4, sections 4.3.2 and 4.5.4
- Concrete Mathematics (Graham, Knuth, Patashnik) — Chapter 4
- Number Theory (Andrews) — Chapters 1 through 4
- Proofs: A Long-Form Mathematics Textbook (Cummings) — Chapter 2
- Handbook of Applied Cryptography (Menezes, Oorschot, Vanstone) — Section 8.2