Group Theory

Rubik's Cube and the Permutation Group

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Coding Club at the American University of Ras Al Khaimah

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Introduction

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Any questions?

Outline

Review of Sets

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Subset: $B \subseteq A$, all elements of B are in A.

Functions

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Definition (Operation)

$* : X \times X \to X$, assigns to each pair of elements x, y from X an element of X.
This is denoted $x * y$

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Notational shortcut: ab instead of $a * b$.

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It's a group.

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- Inverse of ab is $b^{-1}a^{-1}$

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- A subgroup H of a group G always contains the identity.
- The inverse of any $h \in H$ is the same as in G, and is also a part of H.

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What is $\langle 5 \rangle$?

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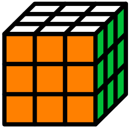
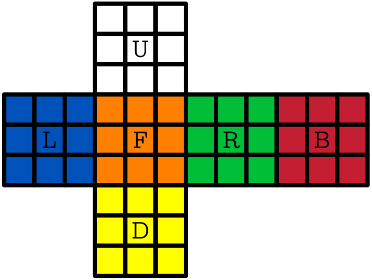
- All cosets of H, including H itself (He), are the same size.
- The union of all cosets of H produces the group set of G.
- (Lagrange's Theorem) The number of cosets, $(G : H)$, is equal to $\frac{|G|}{|H|}$.

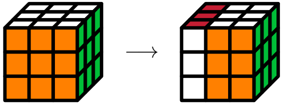
Here's a Rubik's Cube:

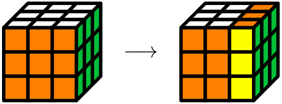


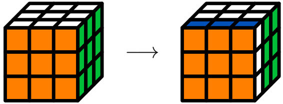


The Rubik's Cube Group

The group is generated by the following set of rotations:

L:


R:


F:


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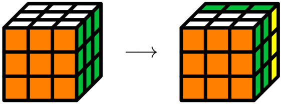
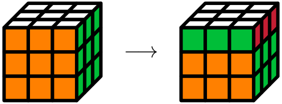


U:


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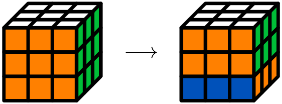

This group has $2^{27}3^{14}5^{3}7^{2}11 = 43,252,003,274,489,856,000$ elements!

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$$G_0 = \langle L, R, F, B, U, D \rangle \qquad |G_0| = 4.3 \times 10^{19}$$

$$G_1 = \langle L, R, F, B, U^2, D^2 \rangle \qquad |G_1| = 2.1 \times 10^{16}$$

$$G_2 = \langle L, R, F^2, B^2, U^2, D^2 \rangle \qquad |G_2| = 1.9 \times 10^{10}$$

$$G_3 = \langle L^2, R^2, F^2, B^2, U^2, D^2 \rangle \qquad |G_3| = 6.6 \times 10^5$$

$$G_4 = \langle 1 \rangle \qquad |G_4| = 1$$

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$$\begin{aligned}
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G_1 &= \langle L, R, F, B, U^2, D^2 \rangle & |G_1| &= 2.1 \times 10^{16} \\
G_2 &= \langle L, R, F^2, B^2, U^2, D^2 \rangle & |G_2| &= 1.9 \times 10^{10} \\
G_3 &= \langle L^2, R^2, F^2, B^2, U^2, D^2 \rangle & |G_3| &= 6.6 \times 10^{5} \\
G_4 &= \langle 1 \rangle & |G_4| &= 1
\end{aligned}$$

G_{i+1} is a subgroup of G_i, and thus has cosets in G_i.

Nested Subgroups of the Rubik's Cube Group

The Rubik's Cube group G may be split into these subgroups:

$$
\begin{aligned}
G_0 &= \langle L, R, F, B, U, D \rangle & |G_0| &= 4.3 \times 10^{19} \\
G_1 &= \langle L, R, F, B, U^2, D^2 \rangle & |G_1| &= 2.1 \times 10^{16} \\
G_2 &= \langle L, R, F^2, B^2, U^2, D^2 \rangle & |G_2| &= 1.9 \times 10^{10} \\
G_3 &= \langle L^2, R^2, F^2, B^2, U^2, D^2 \rangle & |G_3| &= 6.6 \times 10^{5} \\
G_4 &= \langle 1 \rangle & |G_4| &= 1
\end{aligned}
$$

G_{i+1} is a subgroup of G_i, and thus has cosets in G_i.

Definition (Coset Space)

The set of all right cosets of H in G is called its right coset space, and is denoted $H \setminus G$.

Thistlethwaite's Algorithm

$$G_4 \leqslant G_3 \leqslant G_2 \leqslant G_1 \leqslant G_0$$

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Strategy: Traverse cosets in $G_{i+1} \setminus G_i$ to reach G_{i+1}. Repeat until G_4 reached.

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- From G_0 to G_1, we traverse a coset space of cardinality 2048, using all legal moves. Goal is to *orient* all the edges.

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- $|G_2 \setminus G_1| = 1082565$. Goal is to orient all the corners.

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- $|G_4 \setminus G_3| = 663,552$. Goal is to re-orient the corners and position the edges, solving the cube.

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Move choices are done using look-up tables.

That's All, Folks!

Thank you :)

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(Sources will be sent later)

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Questions? Comments? Concerns?

qr codes or something idk