7. The Kalman Filter as State Observer

The Kalman Filter can be used to estimate states that are not directly accessible through a sensor measurement; that is, states that do not explicitly appear in the measurement equation for $z(k)$. In last lecture's example, for instance, position and velocity estimates are obtained from a position measurement only. The KF can infer information about states that are not directly measured by exploiting possible couplings of the states through the system dynamics. In this sense, the KF is sometimes referred to as a *state observer*.

In this lecture, we discuss *observability* and *detectability* as conditions that guarantee that all states of a linear, time-invariant system can reliably be estimated by a KF (in the sense that the error variances of all states remain bounded and do not tend to infinity as time progresses). In addition, the discussion of asymptotic properties of the KF will allow us to derive the *Steady-State Kalman Filter*, which is a time-invariant implementation of the KF.

7.1 Model

For this lecture, we restrict the model of Sec. 6.1 to time-invariant systems and stationary distributions. That is, $A(k) = A$, $B(k) = B$, $H(k) = H$, $Q(k) = Q$, and $R(k) = R$ are constant.

$$\begin{aligned} x(k) &= Ax(k-1) + Bu(k-1) + v(k-1) \qquad && x(0) \sim \mathcal{N}(x_0, P_0), v(k-1) \sim \mathcal{N}(0, Q) \\ z(k) &= Hx(k) + w(k) && w(k) \sim \mathcal{N}(0, R) \end{aligned}$$

where $x(k) \in \mathbb{R}^n$, $z(k) \in \mathbb{R}^m$.

7.2 Observability

We first introduce the concept of observability for a deterministic system, i.e. without noise ($v(k-1) = 0$, $w(k) = 0$). Can we reconstruct $x(0)$ from measurements $z(0)$, $z(1)$, $z(2)$, etc.?[1]

We have

$$\begin{aligned} z(0) &= Hx(0) \\ z(1) &= Hx(1) = HAx(0) + HBu(0) \\ z(2) &= Hx(2) = HA^2x(0) + HBu(1) + HABu(0) \\ &\vdots \\ z(n-1) &= Hx(n-1) = HA^{n-1}x(0) + HBu(n-2) + \cdots + HA^{n-2}Bu(0), \end{aligned}$$

which we rewrite as:

$$\underbrace{\begin{bmatrix} H \\ HA \\ \vdots \\ HA^{n-1} \end{bmatrix}}_{=:\mathcal{O}} x(0) = \begin{bmatrix} z(0) \\ z(1) \\ \vdots \\ z(n-1) \end{bmatrix} - \begin{bmatrix} 0 & 0 & \cdots & 0 \\ HB & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ HA^{n-2}B & HA^{n-3}B & \cdots & 0 \end{bmatrix} \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(n-1) \end{bmatrix}.$$

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[1]Note that once we know $x(0)$, we can reconstruct $x(k)$ for all k (for the deterministic case).

The right-hand side (RHS) is known. Hence, we can uniquely solve for $x(0)$ if and only if $\text{rank}(\mathcal{O}) = n$ (full column rank):

$$x(0) = (\mathcal{O}^T\mathcal{O})^{-1}\mathcal{O}^T \cdot \text{RHS}.$$

If $\text{rank}(\mathcal{O}) = n$, we say that the pair (A, H) is *observable*.

Observability conditions[2]

The pair (A, H) is observable.

$\Leftrightarrow$ For a deterministic LTI system ($x(k) = Ax(k\text{-}1) + Bu(k\text{-}1)$, $z(k) = Hx(k)$), knowledge of $z(0\!:\!n-1)$ and $u(0\!:\!n-1)$ suffices to determine $x(0)$.

$\Leftrightarrow$ $\text{rank}(\mathcal{O}) = n$.

$\Leftrightarrow$ $\begin{bmatrix} A - \lambda I \\ H \end{bmatrix}$ is full column rank for all $\lambda \in \mathbb{C}$ (PBH-Test).

- For the PBH-Test (PBH = Popov-Belevitch-Hautus), one only needs to check those λ that are eigenvalues of A. For all other values of λ, $A - \lambda I$ has full rank.
- Observability is the dual of reachability: (A, H) is observable $\Leftrightarrow$ (A^T, H^T) is reachable.
  - Reachable[3] means that a control sequence can be found such that an arbitrary final state can be reached from 0 in finite time.
  - (A, B) is reachable $\Leftrightarrow$ $\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$ has full row rank.

7.3 Asymptotic Properties of the Kalman Filter

For constant A, B, H, Q, and R, the KF is still time-varying:

$$\begin{aligned} P_p(k) &= AP_m(k\text{-}1)A^T + Q \\ K(k) &= P_p(k)H^T(HP_p(k)H^T + R)^{-1} \\ P_m(k) &= (I - K(k)H)P_p(k). \end{aligned}$$

What happens to the estimation error $e(k) = x(k) - \hat{x}_m(k)$ as $k \to \infty$? We already know that the filter is unbiased, i.e. $\text{E}[e(k)] = 0$ for all k if $\hat{x}_m(0) = x_0$. But what about the variance?

Consider $P_p(k)$. Combining the above equations yields:

$$P_p(k\text{+}1) = AP_p(k)A^T + Q - AP_p(k)H^T\big(HP_p(k)H^T + R\big)^{-1}HP_p(k)A^T. \tag{7.1}$$

Does $P_p(k)$ converge to a finite matrix P_∞ as $k \to \infty$? Or $\|P_p(k)\| \to \infty$? Note that if $P_p(k)$ converges, then so do $K(k)$ and $P_m(k)$.

- It can be shown that observability is sufficient for convergence: if (A, H) is observable, then $\lim_{k\to\infty} P_p(k) = P_\infty$.
- That is, if we run the filter long enough, we get a stationary error distribution: $\lim_{k\to\infty} e(k) \sim \mathcal{N}(0, P_{m,\infty})$ with finite variance $\lim_{k\to\infty} P_m(k) = P_{m,\infty}$. Hence, the uncertainty in the state estimates remains bounded for all states and reaches a steady-state value.
- Intuition: all states are observable through the measurement $z(k)$; that is, the KF can obtain some information about all states. Since the KF is optimal, it will use this information, and the uncertainty in estimating the states (captured by the variance) will not grow unbounded.

[2] According to *Anderson, Moore, Optimal Filtering, Dover Publications, 2005*.

[3] In some textbooks, this is called *controllable*. In other books (such as *Anderson, Moore, Optimal Filtering, Dover Publications, 2005*, which we use here), however, controllable is defined to mean that one can steer an LTI system from any initial state to 0 in finite time; whereas reachable means that one can reach an arbitrary state from 0 in finite time. In this sense, reachable and controllable are not equivalent for discrete-time systems. Consider, for example, $A = 0$ and $B = 0$, which is controllable to 0, but not reachable.

Examples

Consider the system ($n = 2$, $m = 1$)

$$x(k) = Ax(k-1) + v(k-1) \qquad x(0) \sim \mathcal{N}(0, 10 \cdot I), v(k-1) \sim \mathcal{N}(0, I)$$
$$z(k) = Hx(k) + w(k) \qquad w(k) \sim \mathcal{N}(0, 1).$$

(Matlab files to simulate the following examples are provided on the class website.)

Case 1:

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 0.5 \end{bmatrix}, \quad H = \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad \Rightarrow \quad \mathcal{O} = \begin{bmatrix} H \\ HA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

Since the observability matrix has full rank, it follows that $\lim_{k\to\infty} P_p(k) = P_\infty$.

Case 2:

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 0.5 \end{bmatrix}, \quad H = \begin{bmatrix} 0 & 1 \end{bmatrix} \qquad \Rightarrow \quad \mathcal{O} = \begin{bmatrix} H \\ HA \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0.5 \end{bmatrix}$$

The observability matrix does not have full rank and $P_p(k)$ diverges. The state $x_1(k)$ is not observable and the corresponding error variance grows unbounded: $\text{Var}[e_1(k)] = P_p^{11}(k) \to \infty$ as $k \to \infty$.

Case 3:

$$A = \begin{bmatrix} 0.5 & 1 \\ 0 & 2 \end{bmatrix}, \quad H = \begin{bmatrix} 0 & 1 \end{bmatrix} \qquad \Rightarrow \quad \mathcal{O} = \begin{bmatrix} H \\ HA \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$$

Even though the observability matrix does not have full rank, $\lim_{k\to\infty} P_p(k) = P_\infty$.

The last example shows that observability is stronger than needed. Indeed, there is a weaker condition that guarantees the convergence of the KF variance: detectability.

7.4 Detectability

In words, a system is detectable if all its unstable modes are observable.

Detectability conditions

The pair (A, H) is detectable.

$$\Leftrightarrow \quad \begin{bmatrix} A - \lambda I \\ H \end{bmatrix} \text{ is full column rank for all } \lambda \in \mathbb{C} \text{ with } |\lambda| \geq 1 \text{ (PBH-Test).}$$

Furthermore, if (A, H) is not observable, then there exists a state transformation T such that

$$TAT^{-1} = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}, \quad HT^{-1} = \begin{bmatrix} H_1 & 0 \end{bmatrix}, \quad \text{and } (A_{11}, H_1) \text{ observable.}$$

Then: (A, H) is detectable $\Leftrightarrow$ A_{22} is stable (all eigenvalues have magnitude less than 1).

- The main idea behind detectability:

  Assume that there exists a λ with $|\lambda| \geq 1$ such that $\left[\begin{smallmatrix} A-\lambda I \\ H \end{smallmatrix}\right]$ is not full column rank. Then there exists a vector v, $v \neq 0$, such that

  $$(A - \lambda I)v = 0, \quad Hv = 0 \quad \Leftrightarrow \quad Av = \lambda v, \quad Hv = 0.$$

  That is, the vector v is a natural mode of the system that does not decay and that is not seen in the output of the system.

- Detectability is weaker than observability: (A, H) observable $\Rightarrow$ (A, H) detectable.
- Duality: (A, H) is detectable $\Leftrightarrow$ (A^T, H^T) is stabilizable.
- It can be shown that if (A, H) detectable, then $\lim_{k\to\infty} P_p(k) = P_\infty$.

7.5 The Steady-State Kalman Filter

Motivation: if the KF variance converges, then so does the KF gain: $\lim_{k\to\infty} K(k) = K_\infty$. Using the constant gain K_∞ instead of the time-varying gain $K(k)$ simplifies the implementation of the filter (there is no need to compute or store $K(k)$). This filter is called the *Steady-State KF*.

How to compute K_∞?

Assume $P_p(k)$ converges to P_∞. Then, (7.1) reads

$$P_\infty = AP_\infty A^T + Q - AP_\infty H^T \big(HP_\infty H^T + R\big)^{-1} HP_\infty A^T.$$

- This is an algebraic equation in P_∞, called the *Discrete Algebraic Riccati Equation* (DARE).
- Efficient methods exist for solving it (under certain assumptions on the problem parameters, see below); Matlab implementation: `dare(A',H',Q,R)`.
- The steady-state KF gain then is: $K_\infty = P_\infty H^T (HP_\infty H^T + R)^{-1}$.

Steady-state estimator

The steady-state KF equations with $\hat{x}(k) := \hat{x}_m(k)$:

$$\begin{aligned}\hat{x}(k) &= \big(I - K_\infty H\big)A\,\hat{x}(k{-}1) + \big(I - K_\infty H\big)B\,u(k{-}1) + K_\infty\, z(k)\\ &= \hat{A}\,\hat{x}(k{-}1) + \hat{B}\,u(k) + K_\infty\, z(k),\end{aligned}$$

a linear time-invariant system.

Estimation error:

$$e(k) = x(k) - \hat{x}(k) = \underbrace{(I - K_\infty H)A}_{\substack{\text{stability}\\ \text{important!}}}\, e(k{-}1) + (I - K_\infty H)\, v(k) - K\, w(k).$$

- Want $(I - K_\infty H)A$ to be stable for the error not to diverge.
- Mean: $\mathrm{E}[e(k)] = (I - K_\infty H)A\,\mathrm{E}[e(k{-}1)]$.

  What if $\mathrm{E}[e(0)] = x_0 - \hat{x}_m(0) \neq 0$ (we may not know $x_0 = \mathrm{E}[x(0)]$)? We have: $\mathrm{E}[e(k)] \to 0$ as $k \to \infty$ for any initial $\mathrm{E}[e(0)]$ if and only if $(I - K_\infty H)A$ is stable.

What can go wrong?

- $P_p(k)$ does not converge as $k \to \infty$.
- $P_p(k)$ converges, but to different solutions for different initial $P_p(1)$. This is not desirable – which one should we use to compute K_∞?
- $(I - K_\infty H)A$ is unstable.

All these are addressed by the following theorem.

Theorem[4]

Assume $R > 0$ and $Q \geq 0$, and let G be any matrix such that $Q = GG^T$ (can always be done for symmetric matrices). Then:

(A, H) is detectable and (A, G) is stabilizable.

$\Leftrightarrow$ The DARE has a unique positive semidefinite solution $P_\infty \geq 0$.

[4] Adapted from *Simon, Optimal State Estimation, Wiley, 2006* and *Anderson, Moore, Optimal Filtering, Dover Publications, 2005.* See these and references therein for proofs.

Furthermore, the resulting $(I - K_\infty H)A$ is stable, and

$$\lim_{k\to\infty} P_p(k) = P_\infty \qquad \text{for any initial } P_p(1) \geq 0 \text{ (and, hence, any } P_m(0) = P_0 \geq 0).$$

Interpretation of the two conditions:

- (A, H) is detectable: can observe all unstable modes.
- (A, G) is stabilizable: noise excites unstable parts (guaranteed if $Q > 0$).
- Examples where one of these is not satisfied are discussed in the recitation.

7.6 Remarks

- Why did we not discuss observability and detectability when deriving the KF?

  The KF is the optimal state estimator (for a linear system and Gaussian distributions) irrespective of whether the system is observable, detectable, or neither. The KF does the best it can, even if the measurements do not provide sufficient information for reliably estimating all states.
- Observability and detectability are properties of the system, and not of the estimation algorithm. Hence, they cannot be altered by using a different state estimation algorithm, but only by modifying the system (for example, by placing an additional sensor).
- Observability and detectability can also be defined for time-varying or nonlinear systems. However, conditions for checking them are usually not as straightforward as for the linear time-invariant case.