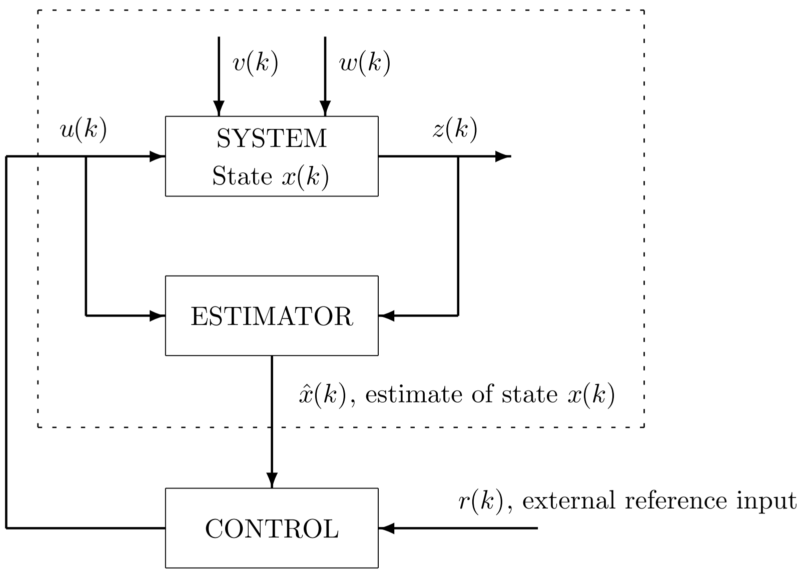
# 10. Observer-Based Control and the Separation Principle

An important application of state estimation is feedback control: an estimate of the state (typically the mean estimate) is used as input to a state-feedback controller. This scheme is called observer[1]-based control, and it is a common way of designing an output-feedback controller (i.e. a controller that does not have access to perfect state measurements).

In principle, any type of state estimator discussed in this class can be combined with a state-feedback controller. We illustrate observer-based control for linear time-invariant (LTI) systems and also point to potential issues (such as stability) for other types of systems. For an LTI system, we consider the combination of an LTI observer (such as a steady-state Kalman Filter) with a static-gain state-feedback controller and analyze the resulting closed-loop system. For this control structure, it is possible to provide straightforward stability and even optimality guarantees, which are more difficult or even impossible to obtain for general systems and observer/controller structures.

## 10.1 A Common Strategy for Control

Many modern methods for control design assume knowledge of the state $x(k)$; that is, these methods can be used to design *state-feedback* controllers. If, however, perfect state measurements are not available (i.e. if $z(k) \neq x(k)$), one can use an estimate $\hat{x}(k)$ of the state instead as the input to the state-feedback controller, as shown in the diagram below.




In this class, you learned how to design state estimators/observers under the assumption that $u(k)$ is given

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[1] *Observer* and *estimator* are used synonymously here. In literature, there is no clear distinction between the two. The notion of *observer* tends to be used for deterministic settings (no noise) where the state vector is not directly measured, while *estimator* is typically used for stochastic problems.

(the dashed block above). In other classes (for example, *Dynamic Programming and Optimal Control*), you may have learned how to design controllers (i.e. to find the input $u(k)$) under the assumption that the system state is known. The above scheme is the combination of the two designs.

Important question: *Does it make sense to separate the two steps of state estimator and state-feedback controller design?*

- Often yes, sometimes in a provable way.
- We discuss the case of LTI systems in detail, and we prove that the feedback control system resulting from the combination of a stable LTI observer with a stable static-gain controller is stable. Even more, if both designs are optimal, the combination of the two is optimal as well (in the sense of minimizing a quadratic cost).
- In practice, the separation of the two design problems allows you to manage the design complexity by focusing on the design of a state estimator without caring about the controller, and vice versa.
- There are counterexamples, however, where a stable state estimator and a stable state-feedback controller together yield an unstable closed-loop system (see examples in the problem set).

## 10.2 LTI Observer

We consider a linear time-invariant (LTI) system:

$$\begin{aligned} x(k) &= A\,x(k{-}1) + Bu(k{-}1) + v(k{-}1) \\ z(k) &= Hx(k) + w(k), \end{aligned}$$

where $v(k{-}1)$ and $w(k)$ are zero-mean CRVs representing noise.

As usual, we are interested in constructing an estimate $\hat{x}(k)$ of the state $x(k)$. In the absence of noise, we require that $\hat{x}(k)$ converges to $x(k)$ as $k \to \infty$. For the case with noise, this corresponds to the mean of $\hat{x}(k)$ converging to the mean of $x(k)$.

We consider an LTI observer of the following form (also called a *Luenberger observer*):

$$\begin{aligned} \hat{x}(k) &= A\,\hat{x}(k{-}1) + Bu(k{-}1) + K\,(z(k) - \hat{z}(k)) \\ \hat{z}(k) &= H\,(A\,\hat{x}(k{-}1) + Bu(k{-}1))\,, \end{aligned}$$

where $K$ is a correction matrix that is to be designed, and $A\hat{x}(k{-}1) + Bu(k{-}1)$ is what we predict the state should be.

- The LTI observer has the same structure as the steady-state Kalman Filter (KF) derived in previous lectures:

  $$\hat{x}(k) = \big(I - K_\infty H\big)A\,\hat{x}(k{-}1) + \big(I - K_\infty H\big)B\,u(k{-}1) + K_\infty\,z(k).$$

  In fact, the steady-state KF is one way to design the observer gain matrix $K$.

- We have already analyzed the dynamics of the observer error $e(k) = x(k) - \hat{x}(k)$ in the context of the steady-state KF. In the absence of noise ($v(k{-}1) = 0,\ w(k) = 0$), we get

  $$\begin{aligned} e(k) &= A\,x(k{-}1) + Bu(k{-}1) - A\,\hat{x}(k{-}1) - Bu(k{-}1) - K\,(z(k) - \hat{z}(k)) \\ &= A\,e(k{-}1) - K\,(HA\,x(k{-}1) + HBu(k{-}1) - HA\,\hat{x}(k{-}1) - HBu(k{-}1)) \\ &= (I - KH)A\,e(k{-}1). \end{aligned}$$

  That is, $e(k) \to 0$ as $k \to \infty$ for all initial errors $e(0)$ *if and only if* $(I - KH)A$ is stable (i.e. all eigenvalues have magnitude less than one).

- Theorem from Linear Systems Theory:

  There exists such a $K$ if and only if $(A, HA)$ is detectable.

- Furthermore, one can show that: $(A, HA)$ is detectable if and only if $(A, H)$ is detectable. (**PSET 6: P1**)

**Summary**

If $(A, H)$ is detectable, we can construct a matrix $K$ such that $(I - KH)A$ is stable.

- Pole placement design: you can use the `place()` command in Matlab to find a $K$ that places the eigenvalues of the error dynamics corresponding to the observable modes at desired locations (unobservable modes remain unchanged). (**PSET 6: P2**)
- Steady-state KF design: yields the *optimal* $K$ (in the sense of minimizing the steady-state mean squared error), given the noise statistics of $v(k)$ and $w(k)$.

**Alternative formulation**

An alternative formulation, which also appears often in literature[2], uses the previous measurement $z(k{-}1)$ in its update equation instead of $z(k)$:

$$\begin{aligned} \hat{x}(k) &= A\,\hat{x}(k{-}1) + Bu(k{-}1) + K\left(z(k{-}1) - \hat{z}(k{-}1)\right) \\ \hat{z}(k{-}1) &= H\hat{x}(k{-}1). \end{aligned}$$

- The error dynamics for this observer are given by

  $$e(k) = (A - KH)\,e(k{-}1),$$

  and one can find a $K$ that yields a stable $A - KH$ if and only if $(A, H)$ is detectable.
- This observer essentially involves a delay since the estimate $\hat{x}(k)$ depends on the measurement $z(k{-}1)$ rather than $z(k)$.

## 10.3 Static State-Feedback Control

We now consider the design of a controller without paying attention to the state estimation problem by simply assuming that we have perfect state information $(z(k) = x(k))$. Furthermore, we assume that we have no process noise and consider the deterministic system

$$\begin{aligned} x(k) &= Ax(k{-}1) + Bu(k{-}1) \\ z(k) &= x(k), \end{aligned}$$

for which we seek to design a linear, static feedback law

$$u(k) = Fz(k) = Fx(k)$$

by choosing the static gain matrix $F$.

- The closed-loop dynamics are:

  $$x(k) = (A + BF)\,x(k{-}1).$$

  Hence, the system is stable if and only if $A + BF$ is stable.
- From Linear Systems Theory: One can find such an $F$ if and only if $(A, B)$ is *stabilizable*.
- Pole placement design: place (controllable) poles at desired closed-loop pole locations.
- *Linear Quadratic Regulator* (LQR) design: find $F$ that minimizes the quadratic cost

  $$J_{\text{LQR}} = \sum_{k=0}^{\infty} x^T(k)\bar{Q}\,x(k) + u^T(k)\bar{R}u\,(k),$$

  where $\bar{Q} = \bar{Q}^T \geq 0$ and $\bar{R} = \bar{R}^T \geq 0$ are weighting matrices.

[2] This form is discussed, for example, in the class *Digital Control Systems*.

Aside[3]: Interestingly, this $F$ can be found using the same tools that we used to obtain the steady-state KF gain $K_\infty$, namely by solving the DARE. In particular, under the assumptions that $(A, B)$ is stabilizable and $(A, G)$ is detectable with $\bar{Q} = GG^T$, the optimal stabilizing controller is given by

$$F = -(B^T PB + \bar{R})^{-1} B^T PA,$$

where $P = P^T \geq 0$ is the unique positive semidefinite solution to the DARE

$$P = A^T PA + \bar{Q} - A^T PB\big(B^T PB + \bar{R}\big)^{-1} B^T PA.$$

Notice the substitutions $A \to A^T$, $H \to B^T$, $Q \to \bar{Q}$, and $R \to \bar{R}$ when comparing this equation to the DARE we obtained for the steady-state KF. Because of this relationship, the steady-state KF design and the LQR design are often referred to as *dual* problems.

Now, what happens if we do not have perfect state measurements ($z(k) \neq x(k)$), but we use $\hat{x}(k)$ instead of $x(k)$ for feedback? Will the system still be stable? This question is answered by the separation principle.

## 10.4 Separation Principle

We consider the deterministic system (the case with noise is analyzed in (**PSET 6: P3**))

$$\begin{aligned} x(k) &= Ax(k\text{–}1) + Bu(k\text{–}1) \\ z(k) &= Hx(k) \end{aligned}$$

with observer and controller given by

$$\begin{aligned} \hat{x}(k) &= A\hat{x}(k\text{–}1) + Bu(k\text{–}1) + K\left(z(k) - \hat{z}(k)\right) \\ \hat{z}(k) &= H\left(A\hat{x}(k\text{–}1) + Bu(k\text{–}1)\right) \\ u(k) &= F\hat{x}(k). \end{aligned}$$

We assume that $(I - KH)A$ and $(A + BF)$ are stable; that is, both the estimator dynamics and the state-feedback dynamics are stable. Question: is the overall system stable?

- We derive the closed-loop dynamics.

  As before, let $e(k) = x(k) - \hat{x}(k)$. When deriving the estimation error dynamics, we did not make *any* assumptions on $u(k\text{–}1)$, so the derivation certainly holds for $u(k\text{–}1) = F\hat{x}(k\text{–}1)$. Consequently,

  $$e(k) = (I - KH)Ae(k\text{–}1).$$

  For the state equation, we get

  $$\begin{aligned} x(k) &= Ax(k\text{–}1) + BF\hat{x}(k\text{–}1) \\ &= Ax(k\text{–}1) + BF\left(\hat{x}(k\text{–}1) + x(k\text{–}1) - x(k\text{–}1)\right) \\ &= (A + BF)x(k\text{–}1) - BFe(k\text{–}1), \end{aligned}$$

  which, together with the above, yields the closed-loop dynamics

  $$\begin{bmatrix} x(k) \\ e(k) \end{bmatrix} = \begin{bmatrix} A + BF & -BF \\ 0 & (I - KH)A \end{bmatrix} = \begin{bmatrix} x(k\text{–}1) \\ e(k\text{–}1) \end{bmatrix}.$$

- The eigenvalues of the closed-loop dynamics are given by the eigenvalues of $(I - KH)A$ and $(A + BF)$. Therefore, the overall system is stable. This is called the *separation principle*[4] for LTI systems.

[3] A detailed treatment of this is beyond the scope of this class. See, for example, *Anderson, Moore, Optimal Control – Linear Quadratic Methods, Dover Publications, 2007* or *Bertsekas, Dynamic Programming and Optimal Control, Athena Scientific, Vol. 1/2, 2005/2007.*

[4] We use *separation principle* to denote the fact that the combination of a stable state estimator with a stable state-feedback controller yields a *stable* closed-loop system. In the next section, we use *separation theorem* to refer to the *optimality* of this scheme. The meaning of these terms may, however, be different in some literature.

- Including noise in the analysis does not affect stability, there will simply be $v(k-1)$ and $w(k)$ terms driving the system. (**PSET 6: P3**)
- Under some mild assumptions, the above analysis generalizes to the time-varying case.
- In general, the separation principle does *not* hold for nonlinear systems. (**PSET 6: P5**)

## 10.5 Separation Theorem

It turns out that we can say something even stronger for LTI systems. Consider the case with noise:

$$
\begin{aligned}
x(k) &= Ax(k-1) + Bu(k-1) + v(k-1) \qquad & v(k-1) &\sim \mathcal{N}(0, Q) \\
z(k) &= Hx(k) + w(k) & w(k) &\sim \mathcal{N}(0, R).
\end{aligned}
$$

The control objective is to find the control policy that minimizes

$$J_{\text{LQG}} \;=\; \lim_{N\to\infty} \mathrm{E}\left[\frac{1}{N}\sum_{k=0}^{N-1}\left(x^T(k)\bar{Q}\,x(k) + u^T(k)\bar{R}\,u(k)\right)\right]$$

where $u(k)$ can depend on current and past measurements $z(1{:}k)$ (causal strategy).

Then the *optimal* strategy is[5]:

1. Design a steady-state KF, which *does not* depend on $\bar{Q}$ and $\bar{R}$. The filter provides an estimate $\hat{x}(k)$ of $x(k)$.
2. Design an optimal state-feedback strategy $u(k) = Fx(k)$ for the deterministic LQR problem

   $$x(k) = Ax(k-1) + Bu(k-1)$$

   that minimizes

   $$J_{\text{LQR}} \;=\; \sum_{k=0}^{\infty} x^T(k)\bar{Q}\,x(k) + u^T(k)\bar{R}\,u(k),$$

   which *does not* depend on the noise statistics $Q$ and $R$.
3. Put both together.

This control design is called *Linear Quadratic Gaussian* (LQG) control. The fact that the combination of the optimal state estimator (steady-state KF) and the optimal state-feedback controller (LQR) is the globally optimal control strategy is often referred to as the *separation theorem* for LTI systems and quadratic cost.

[5] The separation theorem is stated here without a proof. For a proof and further details refer to, for example, *Bertsekas, Dynamic Programming and Optimal Control, Athena Scientific, Vol. 1, 2005* or *Åström, Wittenmark, Computer-Controlled Systems, Dover Publications, 2011.*