

Explaining “Theorems for Free” and parametricity

A tutorial, with code examples in Scala

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Parametricity: a theory about certain code refactorings

Expected properties of code that manipulates collections:

- First extract user information, then convert stream to list; or first convert to list, then extract user information:
`db.getRows.toList.map(getUserInfo)` gives the same result as
`db.getRows.map(getUserInfo).toList`
- First extract user information, then exclude invalid rows; or first exclude invalid rows, then extract user information:
`db.getRows.map(getUserInfo).filter(isValid)` gives the same result as
`db.getRows.filter(getUserInfo andThen isValid).map(getUserInfo)`
- These refactorings are guaranteed to be correct
 - ▶ because `_.toList` is a **natural transformation** `Stream[A] => List[A]`
 - ▶ ... and `_.filter` is also a natural transformation in disguise
- Natural transformations “work the same way for all types”
 - ▶ ... and satisfy the “naturality laws”

Refactoring rules written as equations

- The refactoring involving `toList`:

```
def toList[A]: Stream[A] => List[A]
```

For any function $f: A \Rightarrow B$:

```
_.toList.map(f) == _.map(f).toList
```

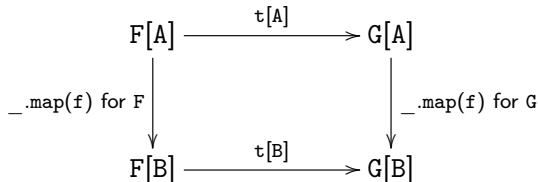
- The refactoring involving `filter`:

```
def filter[A]: Stream[A] => (A => Boolean) => Stream[A]
```

For any function $f: A \Rightarrow B$ and predicate $p: B \Rightarrow \text{Boolean}$:

```
_.filter(f andThen p).map(f) == _.map(f).filter(p)
```

- Generally: applying $t[A]: F[A] \Rightarrow G[A]$ *before* `_.map(f)` equals applying $t[B]: F[B] \Rightarrow G[B]$ *after* `_.map(f)`, for any function $f: A \Rightarrow B$



- This is called a **naturality law**

Naturality laws: further examples

Naturality law for a function $t[A]: F[A] \Rightarrow G[A]$ is an equation that permutes the order of t and of $_.map(f)$, for an arbitrary function $f: A \Rightarrow B$

- We expect it to hold if the code works the same way for all types
- Intuition: t rearranges data in a collection, “not looking” at values

Further examples:

- The headOption method: `headOption: List[A] => Option[A]`

$$list.map(f).headOption == list.headOption.map(f)$$

- Reverse a list: `reverse[A]: List[A] => List[A]`

$$list.map(f).reverse == list.reverse.map(f)$$

- The pure method: `pure[A]: A => L[A]` or `Id[A] => L[A]`

$$pure(x).map(f) == pure(f(x))$$

- Get length: `length[A]: List[A] => Int` or `List[A] => Const[Int, A]`

$$length(list.map(f)) == length(list)$$

Naturality laws in typeclasses

Another use of naturality laws is in implementing typeclasses

- Typeclasses require type constructors with methods `map`, `filter`, `fold`, `flatMap`, `pure`, and others

To be useful for programming, the methods must satisfy certain laws

- `map`: identity, composition
- `filter`: identity, composition, partial function, naturality
- `fold` (traverse): identity, composition, naturality
- `flatMap`: identity, associativity, naturality
- `pure`: naturality

We need to check the laws when implementing new typeclass instances

If naturality holds for `flatMap`, `filter`, and `pure`, then:

- The methods `flatMap` and `flatten: F[F[A]] => F[A]` are equivalent
- The methods `filter` and `deflate: F[Option[A]] => F[A]` are equivalent
- The methods `pure: A => F[A]` and `unit: F[Unit]` are equivalent
 - ▶ Can simplify the definitions of some typeclasses

Naturality laws and “theorems for free”

- The name “theorems for free” comes from a **1989 paper** by P. Wadler
 - ▶ *From the type of a polymorphic function we can derive a theorem it satisfies. Every function of the same type satisfies the same theorem.*
- The “theorems for free” are laws that come in two flavors:
 - ▶ Naturality laws (most often seen in practice)
 - ▶ Dinaturality laws (rarely seen) – a generalized form of naturality laws
- The **parametricity theorem** says:
 - ▶ Any **fully parametric** code $t: P[A] \Rightarrow Q[A]$ satisfies a dinaturality law
 - ▶ There is a recipe for writing that law
 - ▶ One independent law is obtained per type parameter
 - ▶ In most cases, it is actually a naturality law
- Usually, typeclass instances are written in fully parametric code
 - ▶ Then it is not necessary to verify the naturality laws
 - ▶ Can simplify some typeclass definitions
 - ▶ Naturality laws save us time and simplify code

What did “theorems for free” ever do for us programmers?

- “Theorems for free” guarantee naturality laws for fully parametric code
- To use “theorems for free” in practice, programmers need to:
 - ▶ Recognize and write fully parametric code
 - ▶ Be able to write the refactoring that follows from naturality laws
 - ▶ Be able to implement `_.map` for any covariant functor (as well as `_.contramap` for any contravariant functor)
 - ▶ Recognize that the refactoring is guaranteed by parametricity
 - ▶ Recognize simplifications in typeclasses

Fully parametric code: example

Fully parametric code: “works in the same way for all types”

- Example of a fully parametric type:

```
case class Data[A, B](x: Either[A, B] => B, y: (A, B) => A)
```

- Example of a fully parametric function:

```
def headOpt[A]: List[A] => Option[A] = {  
  case Nil => None  
  case head :: tail => Some(head)  
}
```

- The code does not use explicit types
- Naturality laws express the programmer’s intuition about the properties of fully parametric code

Example of code that is *not* fully parametric:

- A bad implementation of `headOpt` that has special code for `Int` type

```
def headOptBad[A]: List[A] => Option[A] = {  
  case Nil                => None  
  case (head: Int) :: tail => Some((head + 100).asInstanceOf[A])  
  case head :: tail       => Some(head)  
}
```

- The code uses explicit run-time type detection
 - ▶ But the code is still purely functional and referentially transparent
 - ▶ “Full parametricity” is a stronger restriction on code

The function `headOptBad` fails the naturality law:

```
scala> headOptBad( List(1, 2, 3).map(x => s"value = $x") )  
res0: Option[String] = Some(value = 1)
```

```
scala> headOptBad(List(1, 2, 3)) .map(x => s"value = $x")  
res1: Option[String] = Some(value = 101)
```

Full parametricity: The price of “free theorems”

“Free theorems” only apply to **fully parametric** code:

- All argument types are combinations of type parameters
- All type parameters are treated as unknown, arbitrary types
- No hard-coded values of specific types (`123: Int` or `"abc": String`)
- No side effects (printing, `var x`, mutating values, writing files, networking, starting or stopping new threads, GUI events, etc.)
- No `null`, no throwing of exceptions, no run-time type comparison
- No run-time code loading, no external libraries with unknown code

“Fully parametric” is a stronger restriction than “purely functional” (referentially transparent)

Purely functional code is fully parametric if restricted to using only `Unit` type or type parameters

- No hard-coded values of specific types, and no run-time type detection

Fully parametric programs are written using the 9 code constructions:

```
def fmap[A, B](f: A => B): List[(A, A)] => List[(B, B)] = { // 3
  case Nil => Nil
// 8 1 1,7
  case head :: tail => (f (head._1), f (head._2)) :: fmap(f)(tail)
// 8 6 2 4 6 5 2 4 6 7 9
} // This code uses each of the nine allowed constructions.
```

- 1 Use `Unit` value (or equivalent type), e.g. `()`, `Nil`, `None`
- 2 Use bound variable (a given argument of the function)
- 3 Create a function: `{ x => expr(x) }`
- 4 Use a function: `f(x)`
- 5 Create a tuple: `(a, b)`
- 6 Use a tuple: `p._1`
- 7 Create a disjunctive value: `Left[A, B](x)`
- 8 Use a disjunctive value: `{ case ... => ... }` (pattern-matching)
- 9 Use a recursive call: e.g., `fmap(f)(tail)` within the code of `fmap`

Approaches to using and proving the parametricity theorem

Using the parametricity theorem à la Wadler is difficult

- The “theorems for free” (Reynolds; Wadler) approach needs to replace functions (one-to-one or many-to-one) by “relations” (many-to-many)
 - ▶ Derive a law with relation variables, then replace them by functions
 - ▶ Need to learn how to work with relation values and relation types
 - ▶ Need to guess how to replace relations by functions in the end
- Alternative approach: analysis of dinatural transformations derives the naturality laws directly (Bainbridge et al.; Backhouse; de Lataillade)
 - ▶ See also a 2019 paper by Voigtländer
 - ▶ No need to use relations
 - ▶ For any type signature, can quickly write the naturality law

Dinatural transformations and profunctors

Some methods do *not* have the type signature of the form $F[A] \Rightarrow G[A]$ where both F and G are functors (or both are contrafunctors)

- $\text{find}[A]: (A \Rightarrow \text{Boolean}) \Rightarrow \text{List}[A] \Rightarrow \text{Option}[A]$
- $\text{fold}[A, B]: \text{List}[A] \Rightarrow B \Rightarrow (A \Rightarrow B \Rightarrow B) \Rightarrow B$ with respect to B
 - ▶ The type parameter is in contravariant and covariant positions at once
 - ▶ This gives us neither a functor nor a contrafunctor
 - ★ But we can identify the variance for each occurrence of type parameter
- Solution: use a **profunctor** $P[X, Y]$ (contravariant in X , covariant in Y) but set equal type parameters: $P[A, A]$
- Within any fully parametric type signature, each occurrence of a type parameter is either covariant or contravariant

A **dinatural transformation** is a function $\tau[A]: P[A, A] \Rightarrow Q[A, A]$ where $P[X, Y]$ and $Q[X, Y]$ are some profunctors and τ satisfies the naturality law

- *All pure functions* have the type signature of a dinatural transformation

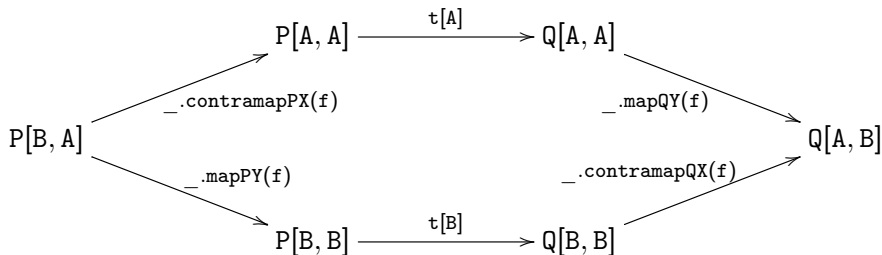
The general naturality law for dinatural transformations

Given $t[A]: P[A, A] \Rightarrow Q[A, A]$ where $P[X, Y]$ and $Q[X, Y]$ are profunctors
The naturality law requires that for any function $f: A \Rightarrow B$,

$p.\text{contramapPX}(f).\text{pipe}(t).\text{mapQY}(f) == p.\text{mapPY}(f).\text{pipe}(t).\text{contramapQX}(f)$

Both sides must give the same result when applied to arbitrary $p: P[B, A]$

- All naturality laws (also for `find`, `fold`) are derived in this way
- The code for `map` and `contramap` must be lawful and fully parametric



- This law reduces to natural transformation laws when P and Q are functors or contrafunctors

Example: deriving the naturality law for filter

Use a curried version of `_.filter` for convenience:

`def filt[A]: (A => Boolean) => F[A] => F[A]` for a filterable functor `F`

Rewrite as a dinatural transformation, `filt[A]: P[A, A] => Q[A, A]` with
`type P[X, Y] = X => Boolean` and `type Q[X, Y] = F[X] => F[Y]`

Write the code for `map` and `contramap` using the specific types of `P` and `Q`:

`p.contramapPX(f) == f andThen p`

`p.mapPY(f) == p`

`q.contramapQX(f) == _.map(f) andThen q`

`q.mapQY(f) = q andThen _.map(f)`

Now write the dinaturality law and simplify: `(f andThen p).pipe(filt)`
`andThen _.map(f)) == _.map(f) andThen p.pipe(filt)`

Rewriting in terms of `_.filter`, we obtain the naturality law of `filter`:
`_.filter(f andThen p).map(f) = _.map(f).filter(p)`

Other parametricity properties

- Bifunctor `map` calls commute if used with different type parameters:
For any value $b: B[X, Y]$, and any functions $f: X \Rightarrow P$, $g: Y \Rightarrow Q$,
the commutativity law holds: $b.\text{mapX}(f).\text{mapY}(g) == b.\text{mapY}(g).\text{mapX}(f)$

$$\begin{array}{ccc} B[X, Y] & \xrightarrow{\quad _.\text{mapX}(f) \quad} & B[P, Y] \\ \downarrow _.\text{mapY}(g) & & \downarrow _.\text{mapY}(g) \\ B[X, Q] & \xrightarrow{\quad _.\text{mapX}(f) \quad} & B[P, Q] \end{array}$$

- Example: `zio.IO[E, A]` has `_.mapError` and `_.map`
- A given functor's lawful and fully parametric method `map` is unique
 - ▶ Note: many typeclasses may admit several lawful, fully parametric, but inequivalent implementations of a typeclass instance for the same type constructor `F[A]`. For example, `Filterable`, `Monad`, `Applicative` instances are not always unique. But instances are unique for the functor and contrafunctor type classes.
- Analogous results for `contramap`, contrafunctors, and profunctors

- Fully parametric code enables powerful mathematical reasoning:
 - ▶ Naturality laws can be used for guaranteed correct refactoring
 - ▶ Naturality laws allow us to reduce the number of type parameters
 - ▶ In typeclass instances, all naturality laws hold, no need to check
 - ▶ Functor, contrafunctor, and profunctor typeclass instances are unique
 - ▶ Bifunctors and profunctors obey the commutativity law
- Full details and proofs are in the Appendix D of the upcoming book
 - ▶ Draft of the book: <https://github.com/winitzki/sofp>