Observe that, although this way of writing one vector as a combination of the others

$$\vec{s}_0 = c_1\vec{s}_1 + c_2\vec{s}_2 + \cdots + c_n\vec{s}_n$$

visually sets off $\vec{s}_0$, algebraically there is nothing special about that vector in that equation. For any $\vec{s}_i$ with a coefficient c_i that is non-0 we can rewrite to isolate $\vec{s}_i$.

$$\vec{s}_i = (1/c_i)\vec{s}_0 + \cdots + (-c_{i-1}/c_i)\vec{s}_{i-1} + (-c_{i+1}/c_i)\vec{s}_{i+1} + \cdots + (-c_n/c_i)\vec{s}_n$$

When we don't want to single out any vector we will instead say that $\vec{s}_0, \vec{s}_1, \ldots, \vec{s}_n$ are in a *linear relationship* and put all of the vectors on the same side. The next result rephrases the linear independence definition in this style. It is how we usually compute whether a finite set is dependent or independent.

1.5 Lemma A subset S of a vector space is linearly independent if and only if among its elements the only linear relationship $c_1\vec{s}_1 + \cdots + c_n\vec{s}_n = \vec{0}$ (with $\vec{s}_i \neq \vec{s}_j$ for all $i \neq j$) is the trivial one $c_1 = 0, \ldots, c_n = 0$.

Proof If S is linearly independent then no vector $\vec{s}_i$ is a linear combination of other vectors from S so there is no linear relationship where some of the $\vec{s}$'s have nonzero coefficients.

If S is not linearly independent then some $\vec{s}_i$ is a linear combination $\vec{s}_i = c_1\vec{s}_1 + \cdots + c_{i-1}\vec{s}_{i-1} + c_{i+1}\vec{s}_{i+1} + \cdots + c_n\vec{s}_n$ of other vectors from S. Subtracting $\vec{s}_i$ from both sides gives a relationship involving a nonzero coefficient, the -1 in front of $\vec{s}_i$. QED

1.6 Example In the vector space of two-wide row vectors, the two-element set $\{(40\ \ 15), (-50\ \ 25)\}$ is linearly independent. To check this, take

$$c_1 \cdot (40\ \ 15) + c_2 \cdot (-50\ \ 25) = (0\ \ 0)$$

and solve the resulting system.

$$\begin{array}{rl} 40c_1 - 50c_2 = 0 \\ 15c_1 + 25c_2 = 0 \end{array} \xrightarrow{-(15/40)\rho_1+\rho_2} \begin{array}{rl} 40c_1 - \quad 50c_2 = 0 \\ (175/4)c_2 = 0 \end{array}$$

Both c_1 and c_2 are zero. So the only linear relationship between the two given row vectors is the trivial relationship.

In the same vector space, the set $\{(40\ \ 15), (20\ \ 7.5)\}$ is linearly dependent since we can satisfy $c_1 \cdot (40\ \ 15) + c_2 \cdot (20\ \ 7.5) = (0\ \ 0)$ with $c_1 = 1$ and $c_2 = -2$.

1.7 Example The set $\{1 + x, 1 - x\}$ is linearly independent in $\mathcal{P}_2$, the space of quadratic polynomials with real coefficients, because

$$0 + 0x + 0x^2 = c_1(1 + x) + c_2(1 - x) = (c_1 + c_2) + (c_1 - c_2)x + 0x^2$$

are $\vec{v}_1, \vec{v}_2 \in V$ such that $\vec{w}_1 = f(\vec{v}_1)$ and $\vec{w}_2 = f(\vec{v}_2)$. Then

$$\begin{aligned} f^{-1}(c_1 \cdot \vec{w}_1 + c_2 \cdot \vec{w}_2) &= f^{-1}\big(\, c_1 \cdot f(\vec{v}_1) + c_2 \cdot f(\vec{v}_2)\,\big) \\ &= f^{-1}(\, f\big(c_1\vec{v}_1 + c_2\vec{v}_2)\,\big) = c_1\vec{v}_1 + c_2\vec{v}_2 = c_1 \cdot f^{-1}(\vec{w}_1) + c_2 \cdot f^{-1}(\vec{w}_2) \end{aligned}$$

since $f^{-1}(\vec{w}_1) = \vec{v}_1$ and $f^{-1}(\vec{w}_2) = \vec{v}_2$. With that, by Lemma 1.11's second statement, this map preserves structure. QED

2.2 Theorem Isomorphism is an equivalence relation between vector spaces.

Proof We must prove that the relation is symmetric, reflexive, and transitive.

To check reflexivity, that any space is isomorphic to itself, consider the identity map. It is clearly one-to-one and onto. This shows that it preserves linear combinations.

$$\mathrm{id}(c_1 \cdot \vec{v}_1 + c_2 \cdot \vec{v}_2) = c_1\vec{v}_1 + c_2\vec{v}_2 = c_1 \cdot \mathrm{id}(\vec{v}_1) + c_2 \cdot \mathrm{id}(\vec{v}_2)$$

Symmetry, that if V is isomorphic to W then also W is isomorphic to V, holds by Lemma 2.1 since each isomorphism map from V to W is paired with an isomorphism from W to V.

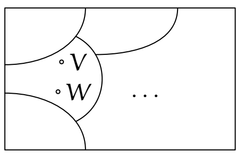
To finish we must check transitivity, that if V is isomorphic to W and W is isomorphic to U then V is isomorphic to U. Let $f\colon V \to W$ and $g\colon W \to U$ be isomorphisms. Consider their composition $g \circ f\colon V \to U$. Because the composition of correspondences is a correspondence, we need only check that the composition preserves linear combinations.

$$\begin{aligned} g \circ f\,\big(c_1 \cdot \vec{v}_1 + c_2 \cdot \vec{v}_2\big) &= g\big(\, f(\, c_1 \cdot \vec{v}_1 + c_2 \cdot \vec{v}_2\,)\,\big) \\ &= g\big(\, c_1 \cdot f(\vec{v}_1) + c_2 \cdot f(\vec{v}_2)\,\big) \\ &= c_1 \cdot g\big(f(\vec{v}_1)\big) + c_2 \cdot g\big(f(\vec{v}_2)\big) \\ &= c_1 \cdot (g \circ f)\,(\vec{v}_1) + c_2 \cdot (g \circ f)\,(\vec{v}_2) \end{aligned}$$

Thus the composition is an isomorphism. QED

Since it is an equivalence, isomorphism partitions the universe of vector spaces into classes: each space is in one and only one isomorphism class.

All finite dimensional vector spaces:



$V \cong W$

The next result characterizes these classes by dimension. That is, we can describe each class simply by giving the number that is the dimension of all of the spaces in that class.

3.19 What is the projection of $\vec{v}$ into M along N if $\vec{v} \in M$?

3.20 Show that if $M \subseteq \mathbb{R}^n$ is a subspace with orthonormal basis $\langle \vec{\kappa}_1, \ldots, \vec{\kappa}_n \rangle$ then the orthogonal projection of $\vec{v}$ into M is this.

$$(\vec{v} \bullet \vec{\kappa}_1) \cdot \vec{\kappa}_1 + \cdots + (\vec{v} \bullet \vec{\kappa}_n) \cdot \vec{\kappa}_n$$

✓ 3.21 Prove that the map $p: V \to V$ is the projection into M along N if and only if the map $\mathrm{id} - p$ is the projection into N along M. (Recall the definition of the difference of two maps: $(\mathrm{id} - p)(\vec{v}) = \mathrm{id}(\vec{v}) - p(\vec{v}) = \vec{v} - p(\vec{v})$.)

3.22 Show that if a vector is perpendicular to every vector in a set then it is perpendicular to every vector in the span of that set.

3.23 True or false: the intersection of a subspace and its orthogonal complement is trivial.

3.24 Show that the dimensions of orthogonal complements add to the dimension of the entire space.

3.25 Suppose that $\vec{v}_1, \vec{v}_2 \in \mathbb{R}^n$ are such that for all complements $M, N \subseteq \mathbb{R}^n$, the projections of $\vec{v}_1$ and $\vec{v}_2$ into M along N are equal. Must $\vec{v}_1$ equal $\vec{v}_2$? (If so, what if we relax the condition to: all orthogonal projections of the two are equal?)

✓ 3.26 Let M, N be subspaces of $\mathbb{R}^n$. The perp operator acts on subspaces; we can ask how it interacts with other such operations.

(a) Show that two perps cancel: $(M^\perp)^\perp = M$.

(b) Prove that $M \subseteq N$ implies that $N^\perp \subseteq M^\perp$.

(c) Show that $(M + N)^\perp = M^\perp \cap N^\perp$.

✓ 3.27 The material in this subsection allows us to express a geometric relationship that we have not yet seen between the range space and the null space of a linear map.

(a) Represent $f: \mathbb{R}^3 \to \mathbb{R}$ given by

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \mapsto 1v_1 + 2v_2 + 3v_3$$

with respect to the standard bases and show that

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

is a member of the perp of the null space. Prove that $\mathscr{N}(f)^\perp$ is equal to the span of this vector.

(b) Generalize that to apply to any $f: \mathbb{R}^n \to \mathbb{R}$.

(c) Represent $f: \mathbb{R}^3 \to \mathbb{R}^2$

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \mapsto \begin{pmatrix} 1v_1 + 2v_2 + 3v_3 \\ 4v_1 + 5v_2 + 6v_3 \end{pmatrix}$$

with respect to the standard bases and show that

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$$

the entries from the original matrix come one per row, and also one per column.

$$\begin{aligned}\begin{vmatrix}2&1&-1\\4&3&\boxed{0}\\2&1&5\end{vmatrix}&=\begin{vmatrix}2&0&0\\0&3&0\\0&0&5\end{vmatrix}+\begin{vmatrix}2&0&0\\0&0&\boxed{0}\\0&1&0\end{vmatrix}\\&\quad+\begin{vmatrix}0&1&0\\4&0&0\\0&0&5\end{vmatrix}+\begin{vmatrix}0&1&0\\0&0&\boxed{0}\\2&0&0\end{vmatrix}\\&\quad+\begin{vmatrix}0&0&-1\\4&0&0\\0&1&0\end{vmatrix}+\begin{vmatrix}0&0&-1\\0&3&0\\2&0&0\end{vmatrix}\end{aligned}$$

In that expansion we can bring out the scalars.

$$\begin{aligned}&=(2)(3)(5)\begin{vmatrix}1&0&0\\0&1&0\\0&0&1\end{vmatrix}+(2)(\boxed{0})(1)\begin{vmatrix}1&0&0\\0&0&1\\0&1&0\end{vmatrix}\\&\quad+(1)(4)(5)\begin{vmatrix}0&1&0\\1&0&0\\0&0&1\end{vmatrix}+(1)(\boxed{0})(2)\begin{vmatrix}0&1&0\\0&0&1\\1&0&0\end{vmatrix}\\&\quad+(-1)(4)(1)\begin{vmatrix}0&0&1\\1&0&0\\0&1&0\end{vmatrix}+(-1)(3)(2)\begin{vmatrix}0&0&1\\0&1&0\\1&0&0\end{vmatrix}\end{aligned}$$

To finish, evaluate those six determinants by row-swapping them to the identity matrix, keeping track of the sign changes.

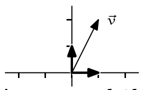
$$\begin{aligned}&=30\cdot(+1)+0\cdot(-1)\\&\quad+20\cdot(-1)+0\cdot(+1)\\&\quad-4\cdot(+1)-6\cdot(-1)=12\end{aligned}$$

That example captures this subsection's new calculation scheme. Multilinearity expands a determinant into many separate determinants, each with one entry from the original matrix per row. Most of these have one row that is a multiple of another so we omit them. We are left with the determinants that have one entry per row and column from the original matrix. Factoring out the scalars further reduces the determinants that we must compute to the one-entry-per-row-and-column matrices where all entries are 1's.

Recall Definition Three.IV.3.14, that a *permutation matrix* is square, with entries 0's except for a single 1 in each row and column. We now introduce a notation for permutation matrices.

1.28 We say that matrices H and G are *similar* if there is a nonsingular matrix P such that $H = P^{-1}GP$ (we will study this relation in Chapter Five). Show that similar matrices have the same determinant.

1.29 We usually represent vectors in $\mathbb{R}^2$ with respect to the standard basis so vectors in the first quadrant have both coordinates positive.



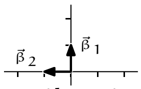
$$\mathrm{Rep}_{\mathcal{E}_2}(\vec{v}) = \begin{pmatrix} +3 \\ +2 \end{pmatrix}$$

Moving counterclockwise around the origin, we cycle thru four regions:

$$\cdots \longrightarrow \begin{pmatrix} + \\ + \end{pmatrix} \longrightarrow \begin{pmatrix} - \\ + \end{pmatrix} \longrightarrow \begin{pmatrix} - \\ - \end{pmatrix} \longrightarrow \begin{pmatrix} + \\ - \end{pmatrix} \longrightarrow \cdots.$$

Using this basis

$$B = \langle \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix} \rangle$$



gives the same counterclockwise cycle. We say these two bases have the same *orientation*.

(a) Why do they give the same cycle?

(b) What other configurations of unit vectors on the axes give the same cycle?

(c) Find the determinants of the matrices formed from those (ordered) bases.

(d) What other counterclockwise cycles are possible, and what are the associated determinants?

(e) What happens in $\mathbb{R}^1$?

(f) What happens in $\mathbb{R}^3$?

A fascinating general-audience discussion of orientations is in [Gardner].

1.30 *This question uses material from the optional Determinant Functions Exist subsection.* Prove Theorem 1.5 by using the permutation expansion formula for the determinant.

✓ 1.31 **(a)** Show that this gives the equation of a line in $\mathbb{R}^2$ thru (x_2, y_2) and (x_3, y_3).

$$\begin{vmatrix} x & x_2 & x_3 \\ y & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

(b) [Petersen] Prove that the area of a triangle with vertices (x_1, y_1), (x_2, y_2), and (x_3, y_3) is

$$\frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix}.$$

(c) [Math. Mag., Jan. 1973] Prove that the area of a triangle with vertices at (x_1, y_1), (x_2, y_2), and (x_3, y_3) whose coordinates are integers has an area of N or $N/2$ for some positive integer N.

We will prove this using projective geometry. (We've drawn Euclidean figures because that is the more familiar image. To consider them as projective figures we can imagine that, although the line segments shown are parts of great circles and so are curved, the model has such a large radius compared to the size of the figures that the sides appear in our sketch to be straight.)

For the proof we need a preliminary lemma [Coxeter]: if W, X, Y, Z are four points in the projective plane, no three of which are collinear, then there are homogeneous coordinate vectors $\vec{w}$, $\vec{x}$, $\vec{y}$, and $\vec{z}$ for the projective points, and a basis B for $\mathbb{R}^3$, satisfying this.

$$\mathrm{Rep}_B(\vec{w})=\begin{pmatrix}1\\0\\0\end{pmatrix}\quad \mathrm{Rep}_B(\vec{x})=\begin{pmatrix}0\\1\\0\end{pmatrix}\quad \mathrm{Rep}_B(\vec{y})=\begin{pmatrix}0\\0\\1\end{pmatrix}\quad \mathrm{Rep}_B(\vec{z})=\begin{pmatrix}1\\1\\1\end{pmatrix}$$

To prove the lemma, because W, X, and Y are not on the same projective line, any homogeneous coordinate vectors $\vec{w}_0$, $\vec{x}_0$, and $\vec{y}_0$ do not line on the same plane through the origin in $\mathbb{R}^3$ and so form a spanning set for $\mathbb{R}^3$. Thus any homogeneous coordinate vector for Z is a combination $\vec{z}_0=a\cdot\vec{w}_0+b\cdot\vec{x}_0+c\cdot\vec{y}_0$. Then let the basis be $B=\langle\vec{w},\vec{x},\vec{y}\rangle$ and take $\vec{w}=a\cdot\vec{w}_0$, $\vec{x}=b\cdot\vec{x}_0$, $\vec{y}=c\cdot\vec{y}_0$, and $\vec{z}=\vec{z}_0$.

To prove Desargue's Theorem use the lemma to fix homogeneous coordinate vectors and a basis.

$$\mathrm{Rep}_B(\vec{t}_1)=\begin{pmatrix}1\\0\\0\end{pmatrix}\quad \mathrm{Rep}_B(\vec{u}_1)=\begin{pmatrix}0\\1\\0\end{pmatrix}\quad \mathrm{Rep}_B(\vec{v}_1)=\begin{pmatrix}0\\0\\1\end{pmatrix}\quad \mathrm{Rep}_B(\vec{o})=\begin{pmatrix}1\\1\\1\end{pmatrix}$$

The projective point T_2 is incident on the projective line OT_1 so any homogeneous coordinate vector for T_2 lies in the plane through the origin in $\mathbb{R}^3$ that is spanned by homogeneous coordinate vectors of O and T_1:

$$\mathrm{Rep}_B(\vec{t}_2)=a\begin{pmatrix}1\\1\\1\end{pmatrix}+b\begin{pmatrix}1\\0\\0\end{pmatrix}$$

for some scalars a and b. Hence the homogeneous coordinate vectors of members T_2 of the line OT_1 are of the form on the left below. The forms for U_2 and V_2 are similar.

$$\mathrm{Rep}_B(\vec{t}_2)=\begin{pmatrix}t_2\\1\\1\end{pmatrix}\qquad \mathrm{Rep}_B(\vec{u}_2)=\begin{pmatrix}1\\u_2\\1\end{pmatrix}\qquad \mathrm{Rep}_B(\vec{v}_2)=\begin{pmatrix}1\\1\\v_2\end{pmatrix}$$

3.3 Example The only transformation on the trivial space $\{\vec{0}\}$ is $\vec{0} \mapsto \vec{0}$. This map has no eigenvalues because there are no non-$\vec{0}$ vectors $\vec{v}$ mapped to a scalar multiple $\lambda \cdot \vec{v}$ of themselves.

3.4 Example Consider the homomorphism $t \colon \mathcal{P}_1 \to \mathcal{P}_1$ given by $c_0 + c_1 x \mapsto (c_0 + c_1) + (c_0 + c_1)x$. While the codomain $\mathcal{P}_1$ of t is two-dimensional, its range is one-dimensional $\mathscr{R}(t) = \{c + cx \mid c \in \mathbb{C}\}$. Application of t to a vector in that range will simply rescale the vector $c + cx \mapsto (2c) + (2c)x$. That is, t has an eigenvalue of 2 associated with eigenvectors of the form $c + cx$ where $c \neq 0$.

This map also has an eigenvalue of 0 associated with eigenvectors of the form $c - cx$ where $c \neq 0$.

The definition above is for maps. We can give a matrix version.

3.5 Definition A square matrix T has a scalar *eigenvalue* λ associated with the nonzero *eigenvector* $\vec{\zeta}$ if $T\vec{\zeta} = \lambda \cdot \vec{\zeta}$.

This extension of the definition for maps to a definition for matrices is natural but there is a point on which we must take care. The eigenvalues of a map are also the eigenvalues of matrices representing that map, and so similar matrices have the same eigenvalues. However, the eigenvectors can differ — similar matrices need not have the same eigenvectors. The next example explains.

3.6 Example These matrices are similar

$$T = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \qquad \hat{T} = \begin{pmatrix} 4 & -2 \\ 4 & -2 \end{pmatrix}$$

since $\hat{T} = PTP^{-1}$ for this P.

$$P = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \qquad P^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

The matrix T has two eigenvalues, $\lambda_1 = 2$ and $\lambda_2 = 0$. The first one is associated with this eigenvector.

$$T\vec{e}_1 = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} = 2\vec{e}_1$$

Suppose that T represents a transformation $t \colon \mathbb{C}^2 \to \mathbb{C}^2$ with respect to the standard basis. Then the action of this transformation t is simple.

$$\begin{pmatrix} x \\ y \end{pmatrix} \stackrel{t}{\longmapsto} \begin{pmatrix} 2x \\ 0 \end{pmatrix}$$

2.6 Corollary (Cauchy-Schwarz Inequality) For any $\vec{u}, \vec{v} \in \mathbb{R}^n$,

$$|\,\vec{u} \cdot \vec{v}\,| \leqslant |\,\vec{u}\,|\,|\vec{v}\,|$$

with equality if and only if one vector is a scalar multiple of the other.

Proof The Triangle Inequality's proof shows that $\vec{u} \cdot \vec{v} \leqslant |\vec{u}\,|\,|\vec{v}\,|$ so if $\vec{u} \cdot \vec{v}$ is positive or zero then we are done. If $\vec{u} \cdot \vec{v}$ is negative then this holds.

$$|\,\vec{u} \cdot \vec{v}\,| = -(\,\vec{u} \cdot \vec{v}\,) = (-\vec{u}\,) \cdot \vec{v} \leqslant |-\vec{u}\,|\,|\vec{v}\,| = |\vec{u}\,|\,|\vec{v}\,|$$

The equality condition is Exercise 19. QED

The Cauchy-Schwarz inequality assures us that the next definition makes sense because the fraction has absolute value less than or equal to one.

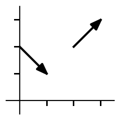
2.7 Definition The *angle* between two nonzero vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$ is

$$\theta = \arccos(\,\frac{\vec{u} \cdot \vec{v}}{|\vec{u}\,|\,|\vec{v}\,|}\,)$$

(if either is the zero vector then we take the angle to be a right angle).

2.8 Corollary Vectors from $\mathbb{R}^n$ are orthogonal, that is, perpendicular, if and only if their dot product is zero. They are parallel if and only if their dot product equals the product of their lengths.

2.9 Example These vectors are orthogonal.



$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0$$

We've drawn the arrows away from canonical position but nevertheless the vectors are orthogonal.

2.10 Example The $\mathbb{R}^3$ angle formula given at the start of this subsection is a special case of the definition. Between these two