سوال ۱)

$$y[n] = \sum_{K=-\infty}^{+\infty} h[K]x[n-K]$$

1. $$y[n] = \sum_{K=0}^{\infty} \left(\frac{1}{2}\right)^K e^{j\pi(n-K)} = e^{j\pi n} \cdot \sum_{K=0}^{\infty} \left(\frac{e^{-j\pi}}{2}\right)^K = \frac{e^{j\pi n}}{1-\frac{1}{2}e^{-j\pi}} = \frac{2}{3}(-1)^n$$

2. $$y[n] = \sum_{K=0}^{\infty} \left(\frac{1}{2}\right)^K e^{j\pi(n-K)/4}$$

$$= e^{j\pi n/4} \sum_{K=0}^{\infty} \left(\frac{e^{-j\pi/4}}{2}\right)^K = \frac{e^{j\pi n/4}}{1-\frac{1}{2}e^{-j(\pi/4)}}$$

قانون اویلر :
$$\boxed{\cos\theta = \frac{e^{j\theta}+e^{-j\theta}}{2}}$$

3. $$y[n] = \sum_{K=0}^{\infty} \left(\frac{1}{2}\right)^K \left[\frac{1}{2}e^{j\pi/8}\, e^{j\pi(n-K)/4} + \frac{1}{2}e^{-j\pi/8}\, e^{-j\pi(n-K)/4}\right]$$

$$= \frac{1}{2}e^{j\pi/8}\, e^{j\pi n/4} \sum_{K=0}^{\infty}\left(\frac{e^{-j\pi/4}}{2}\right)^K + \frac{1}{2}e^{-j(\pi/8)}\, e^{-j\pi n/4} \sum_{K=0}^{\infty}\left(\frac{e^{j\pi/4}}{2}\right)^K$$

سوال ۲)

1. $$x[n] = \sin\left(\frac{2\pi n}{3}\right)\cos\left(\frac{\pi n}{2}\right) = \frac{1}{۲}\left(\sin\frac{۷\pi n}{۴} + \sin\frac{\pi n}{۴}\right) = \frac{1}{۲}\left(\frac{1}{۲j}\left(e^{j\frac{۷\pi n}{۴}} - e^{-j\frac{۷\pi n}{۴}}\right) + \frac{1}{۲j}\left(e^{j\frac{\pi n}{۴}} - e^{-j\frac{\pi n}{۴}}\right)\right)$$

$$\sin a \cdot \cos b = \frac{1}{۲}\left(\sin(a+b) + \sin(a-b)\right) \qquad \omega_0 = \frac{\pi}{۴},\ N = ۱۲$$

$$\rightarrow a_{-۷} = -\frac{1}{۴j},\quad a_{۷} = \frac{1}{۴j},\quad a_{-1} = \frac{1}{۴j},\quad a_1 = \frac{1}{۴j}$$

2. $$N_0 = ۴,\quad a_K = \frac{1}{N_0}\sum_{K=\langle N\rangle} x[n]e^{-jK\omega_0 n} = \frac{1}{۴}\sum_{N=-۲}^{۳} x[n]e^{-jK\omega_0 n}$$
$$\omega_0 = \frac{۲\pi}{۴} = \frac{\pi}{۲}$$

$$= \frac{1}{۴}\left(-e^{-jK\omega_0(-۲)} + ۲e^{-jK\omega_0(-1)} + e^{-jK\omega_0(0)} + ۲e^{-jK\omega_0(1)} - e^{-jK\omega_0(۲)} + 0 \times \text{—}\right)$$

$$= \frac{1}{۴}\left(1 - ۲\cos\frac{۲K\pi}{۳}\right) + ۴\cos\left(\frac{K\pi}{۳}\right)$$

سوال ۳) ۱. صحیح است چون $\tilde{x}[n]$ با دوره تناوب ۱۰ متناوب است.

۲. غلط است چون $\tilde{x}[n]$ زوج نیست . ۳. $a_k e^{jk2\pi/5}$ حقیقی است ، زیرا سری فوریه‌ی $\tilde{x}[n+2]$ را مشخص می‌کند که مشخصا یک دنباله‌ی حقیقی و زوج است.

۴. $a_0 = 0$ صحیح است چون مجموع مقادیر $\tilde{x}[n]$ در یک تناوب برابر با صفر است.

سوال ۴)

1. $$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-jn\omega} = \sum_{n=-\infty}^{\infty}\left(\frac{1}{4}\right)^n u[n]e^{-j\omega n} = \sum_{n=0}^{\infty}\left(\frac{1}{4}e^{-j\omega}\right)^n$$

$$= \frac{1}{1-\frac{1}{4}e^{-j\omega}}$$

$$\boxed{\sum_{n=0}^{\infty} a^n = \frac{1}{1-a} \quad |a|<1}$$

2. $$x[n] = (a^n \sin\omega_0 n)u[n]$$

می‌دانیم $$(\sin\omega_0 n)x[n] \rightarrow \frac{1}{2j}\left[X(\omega-\omega_0) - X(\omega+\omega_0)\right]$$

$$\underbrace{a^n u[n]}_{x_1[n]} \rightarrow \frac{1}{1-ae^{-j\omega}} = X_1(e^{j\omega})$$

$$X(e^{j\omega}) = \frac{1}{2j}\left[\frac{1}{1-ae^{-j(\omega-\omega_0)}} - \frac{1}{1-ae^{-j(\omega+\omega_0)}}\right]$$

3. $x[n] = u[n] - u[n-4] \rightarrow X(e^{j\omega}) = \frac{1}{1-e^{-j\omega}} - \frac{e^{-j4\omega}}{1-e^{-j\omega}}$

4. $x[n] = (\frac{1}{4})^n u[n+2] = (\frac{1}{4})^{n+2} (\frac{1}{4})^{-2} u[n+2] = 16 (\frac{1}{4})^{n+2} u[n+2]$

$$16(\tfrac{1}{4})^n u[n] \rightarrow \frac{16}{1-\frac{1}{4}e^{-j\omega}} \Rightarrow 16(\tfrac{1}{4})^{n+2} u[n+2] \rightarrow \frac{16e^{j2\omega}}{1-\frac{1}{4}e^{-j\omega}}$$

سوال ۵) الف)

$$y[n] - \tfrac{1}{2}y[n] = x[n] \rightarrow Y(e^{j\omega})(1-\tfrac{1}{2}e^{-j\omega}) = X(e^{j\omega})$$

$$\rightarrow H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1}{1-\frac{1}{2}e^{-j\omega}}$$

ب)

1. $x[n] = \delta[n] \rightarrow X(e^{j\omega}) = 1 \rightarrow Y(e^{j\omega}) = H(e^{j\omega}) \cdot X(e^{j\omega}) = H(e^{j\omega})$
$\rightarrow y[n] = (\frac{1}{2})^n u[n]$

2. $x[n] = \delta[n-n_0] \rightarrow X(e^{j\omega}) = e^{-j\omega n_0} \rightarrow Y(e^{j\omega}) = \frac{e^{-j\omega n_0}}{1-\frac{1}{2}e^{-j\omega}} \rightarrow y[n] = (\frac{1}{2})^{n-n_0} u[n-n_0]$

3. $x[n] = (\frac{3}{4})^n u[n] \rightarrow X(e^{j\omega}) = \frac{1}{1-\frac{3}{4}e^{-j\omega}} \rightarrow Y(e^{j\omega}) = \frac{-2}{1-\frac{1}{2}e^{-j\omega}} + \frac{3}{1-\frac{3}{4}e^{-j\omega}}$

$$\rightarrow y[n] = -2(\tfrac{1}{2})^n u[n] + 3(\tfrac{3}{4})^n u[n]$$

سوال ۶) الف)

$$h_1[n] = (\tfrac{1}{2})^n u[n] \rightarrow H_1(e^{j\omega}) = \frac{1}{1-\frac{1}{2}e^{-j\omega}} \Rightarrow H(e^{j\omega}) = \frac{1}{2}\left[\frac{1}{1-\frac{1}{2}e^{-j(\omega-\frac{\pi}{2})}} + \frac{1}{1-\frac{1}{2}e^{-j(\omega+\frac{\pi}{2})}}\right]$$

$$x[n] \rightarrow X(e^{j\omega}) \Rightarrow x[n]\cos\omega_0 n \rightarrow \tfrac{1}{2}\left(X(e^{j(\omega-\omega_0)}) + X(e^{j(\omega+\omega_0)})\right)$$

ب)

$$x[n] = \cos n\tfrac{\pi}{2} = \tfrac{1}{2}e^{jn\frac{\pi}{2}} + \tfrac{1}{2}e^{-jn\frac{\pi}{2}} \qquad x[n] = e^{j\omega_k n},\ h[n] \rightarrow y[n] = e^{j\omega_k n} H(\omega_k)$$

$$H(\tfrac{\pi}{2}) = \tfrac{1}{2}\left(\frac{1}{1-\frac{1}{2}} + \frac{1}{1+\frac{1}{2}}\right) = \tfrac{1}{2}(2+\tfrac{2}{3}) = \tfrac{4}{3}$$

$$H(-\tfrac{\pi}{2}) = \tfrac{1}{2}\left(\frac{1}{1+\frac{1}{2}} + \frac{1}{1-\frac{1}{2}}\right) = \tfrac{4}{3}$$

$$\Rightarrow y[n] = \tfrac{2}{3}e^{jn\frac{\pi}{2}} + \tfrac{2}{3}e^{-jn\frac{\pi}{2}} = \tfrac{4}{3}\cos\tfrac{\pi}{2}n$$

سوال ۷)

1. $x[n] \rightarrow X(e^{j\omega}) \qquad x_m[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j(\omega-\omega_0)}) e^{jn\omega} d\omega =$

$\omega' = \omega - \omega_0$ تغییر متغیر $\qquad x_m[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega'}) e^{j(\omega_0+\omega')} d\omega = x[n]e^{j\omega_0 n}$

2. $\frac{1}{2\pi}\int_{2\pi} Re\{X(e^{j\omega})\} e^{j\omega n} d\omega = \frac{1}{2\pi}\int_{2\pi} \frac{1}{2}(X(e^{j\omega}) + X^*(e^{j\omega}))e^{j\omega n} d\omega$

$$= \frac{1}{2\pi}\left(\int_{2\pi} X(e^{j\omega}) e^{-j\omega n} d\omega\right)^* + \tfrac{1}{2}x[n] = \tfrac{1}{2}(x^*[-n] + x[n])$$

سوال ۷)

3. $\frac{1}{2\pi}\int_{2\pi} Im\{X(e^{jw})\} e^{jwn} dw = \frac{1}{2\pi}\int_{2\pi}\left[\frac{X(e^{jw}) - X^*(e^{jw})}{2j}\right] e^{jwn} dw$

$= \frac{1}{2j} x[n] - \frac{1}{2j}\left(\frac{1}{2\pi}\int_{2\pi} X(e^{jw}) e^{-jwn} dw\right)^* = \frac{1}{2j}\left(x[n] - x^*[-n]\right)$

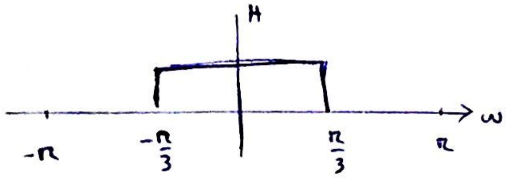
4. $|X(e^{jw})|^2 = X(e^{jw}) . X^*(e^{jw})$ $\quad \frac{1}{2\pi}\int_{2\pi} X^*(e^{jw}) e^{jwn} dw = \left(\frac{1}{2\pi}\int_{2\pi} X(e^{jw}) e^{-jwn} dw\right)^*$

$= x^*[-n]$

$\Rightarrow x_m[n] = x[n] * x^*[-n]$

سوال ۸) الف)

می دانیم $\frac{\sin nW}{n\pi} \longrightarrow X(e^{jw}) = \begin{cases} 1 & |w| \leq W \\ 0 & o.w \end{cases}$ $\Rightarrow \frac{\sin n\frac{\pi}{3}}{n\pi} \longrightarrow H(e^{jw}) = \begin{cases} 1 & |w| \leq \frac{\pi}{3} \\ 0 & o.w \end{cases}$

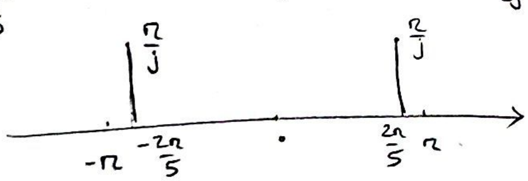


ب)

1. $x[n] = \sin\frac{2\pi}{5}n = \frac{1}{2j}\left(e^{j\frac{2\pi}{5}n} - e^{-j\frac{2\pi}{5}n}\right)$ $\quad X(e^{jw}) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(w - kw_0)$

$\frac{2\pi}{\frac{2\pi}{5}} = N_0 \longrightarrow w_0 = \frac{2\pi}{5}$ $\Rightarrow a_1 = \frac{1}{2j}, \ a_{-1} = \frac{-1}{2j}$

$= \frac{\pi}{j}\delta(w - \frac{2\pi}{5}) + \frac{\pi}{j}\delta(w + \frac{2\pi}{5}) \quad -\pi < w < \pi$

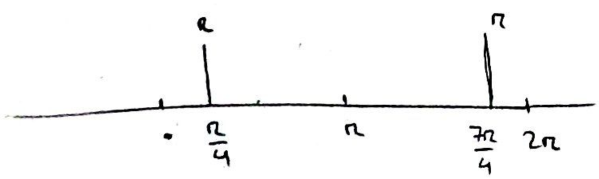


$\frac{5\pi}{15} = \frac{\pi}{3} < \frac{2\pi}{5} = \frac{6\pi}{15}$ $\quad y[n] = 0$

2. $(-1)^n \cos\frac{3\pi}{4}n = x[n]$ $\quad \cos\frac{3\pi}{4}n \longrightarrow \pi\delta(w - \frac{3\pi}{4}) + \pi\delta(w + \frac{3\pi}{4})$

$(-1)^n \cos\frac{3\pi}{4}n \longrightarrow \pi\delta(w - \frac{3\pi}{4} - \pi) + \pi\delta(w + \frac{3\pi}{4} - \pi)$

$= \pi\delta(w - \frac{7\pi}{4}) + \pi\delta(w - \frac{\pi}{4})$



$\frac{\pi}{4} < \frac{\pi}{3} \rightarrow y[n] = x[n] = \cos\frac{n\pi}{4}$