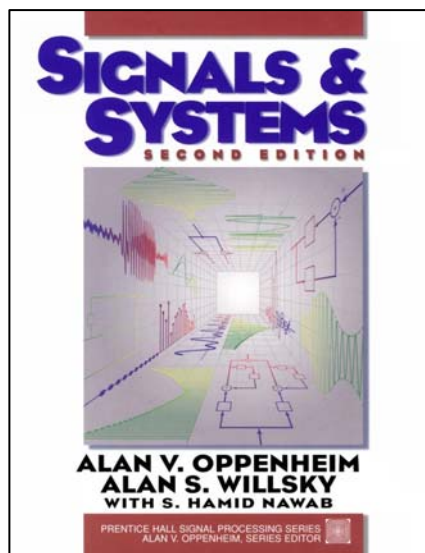


Spring 2010

# 信號與系統 Signals and Systems

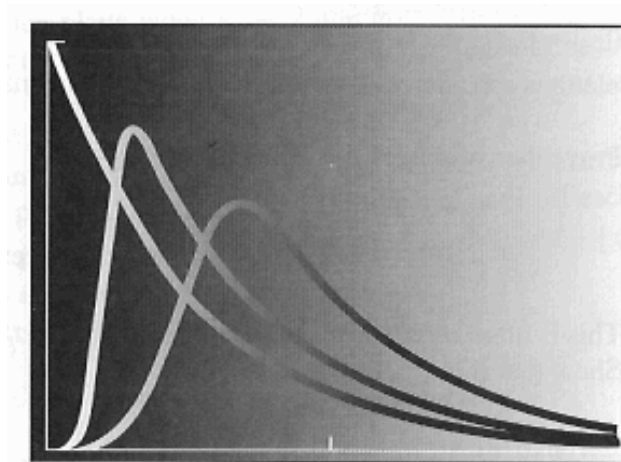
## Chapter SS-2 Linear Time-Invariant Systems



Feng-Li Lian

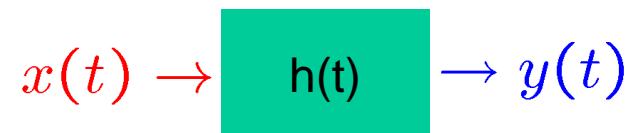
NTU-EE

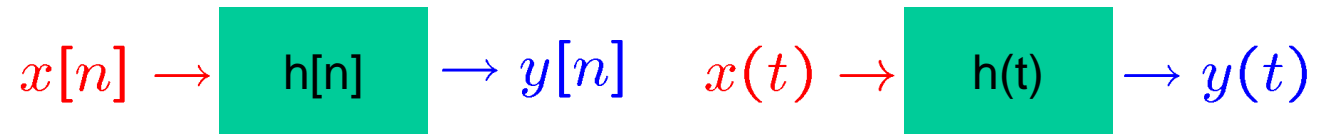
Feb10 – Jun10



Figures and images used in these lecture notes are adopted from  
“Signals & Systems” by Alan V. Oppenheim and Alan S. Willsky, 1997

- Discrete-Time Linear Time-Invariant Systems
  - The convolution sum
- Continuous-Time Linear Time-Invariant Systems
  - The convolution integral
- Properties of Linear Time-Invariant Systems
- Causal Linear Time-Invariant Systems  
Described by Differential & Difference Equations
- Singularity Functions





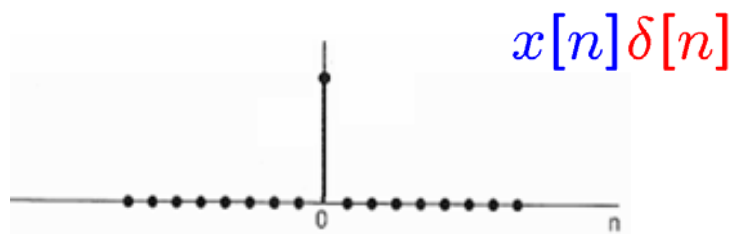
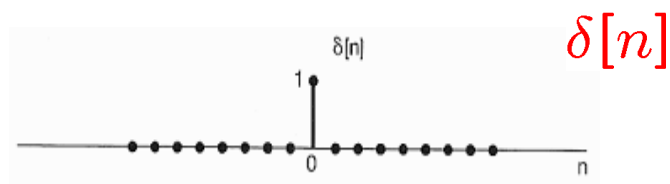
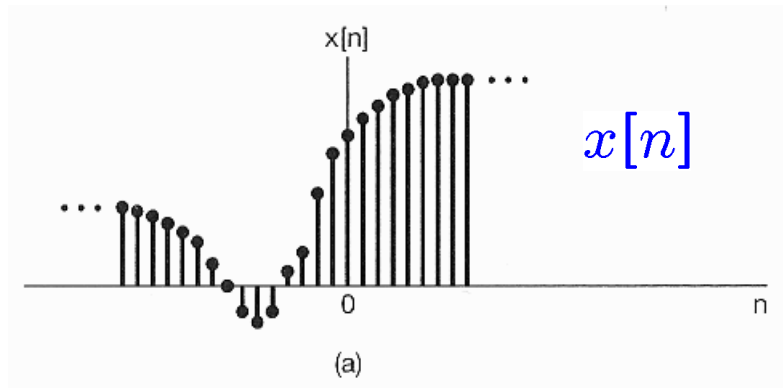
Signals

Systems

## ■ Sample by Unit Impulse

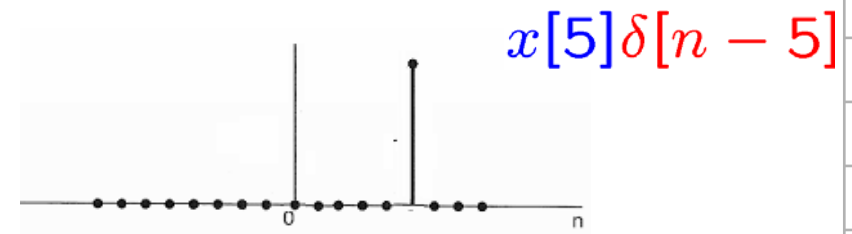
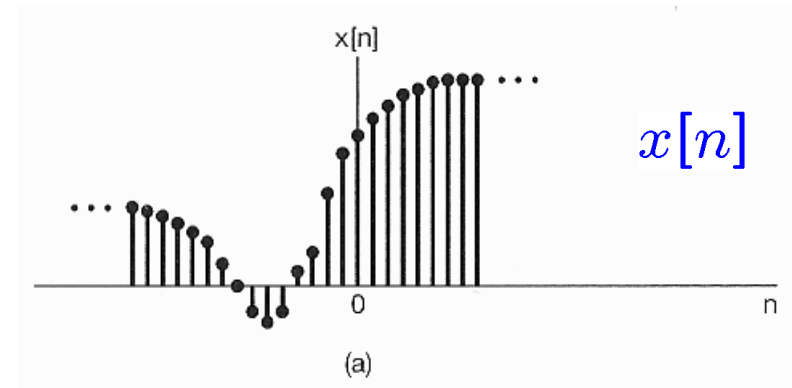
### ■ For $x[n]$

$$x[n]\delta[n] = x[0]\delta[n]$$

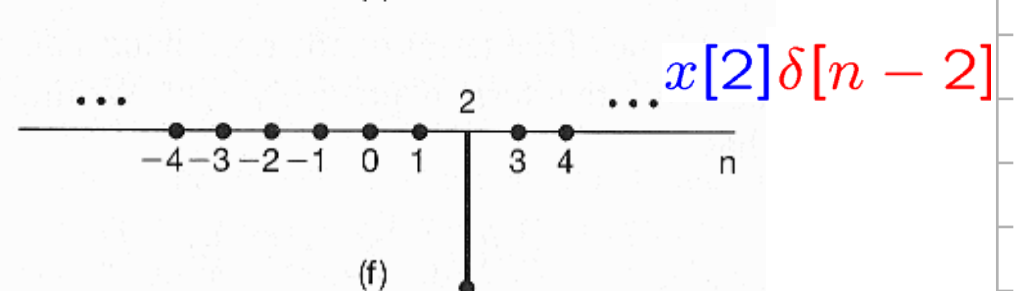
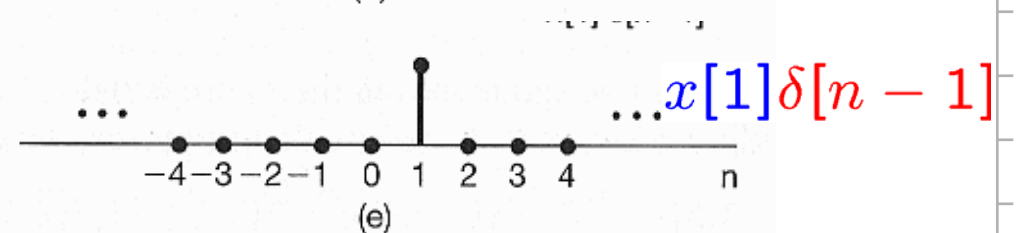
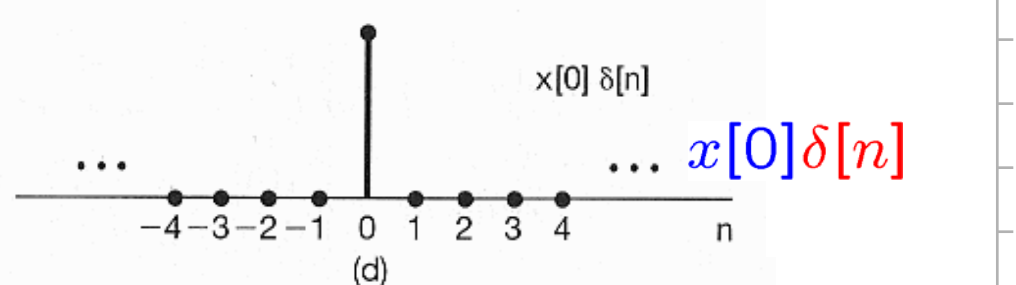
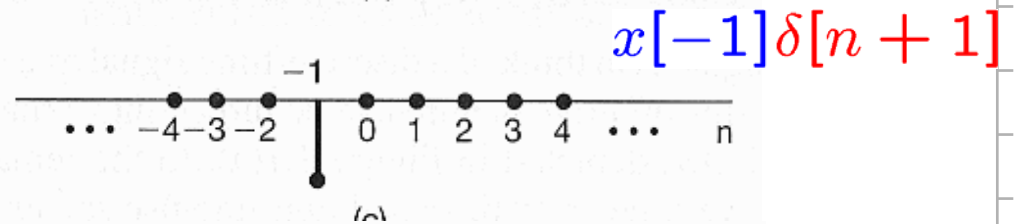
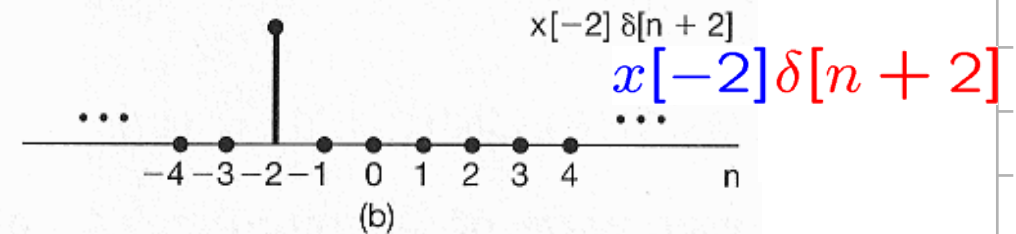
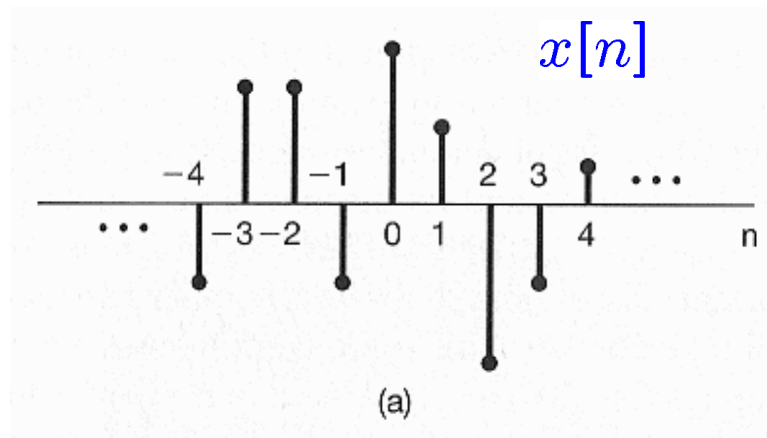


### ■ More generally,

$$x[n]\delta[n - n_0] = x[n_0]\delta[n - n_0]$$



## Representation of DT Signals by Impulses



## ■ Representation of DT Signals by Impulses:

- More generally,

$$\begin{aligned}x[n] &= \cdots + x[-3]\delta[n+3] + x[-2]\delta[n+2] \\&\quad + x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] \\&\quad + x[2]\delta[n-2] + x[3]\delta[n-3] + \cdots \\&= \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k]\end{aligned}$$

- The sifting property of the DT unit impulse
- $x[n]$  = a superposition of scaled versions of shifted unit impulses  $\delta[n-k]$

## ■ DT Unit Impulse Response & Convolution Sum:

input  $\longrightarrow$  Linear System  $\longrightarrow$  output

$\delta[n]$   $\longrightarrow$  Linear System  $\longrightarrow h_0[n]$

$\delta[n - 1]$   $\longrightarrow$  Linear System  $\longrightarrow h_1[n]$

$\delta[n - 2]$   $\longrightarrow$  Linear System  $\longrightarrow h_2[n]$

⋮

$\delta[n - k]$   $\longrightarrow$  Linear System  $\longrightarrow h_k[n]$

## ■ DT Unit Impulse Response & Convolution Sum:

$$x[n] \longrightarrow \text{Linear System} \longrightarrow y[n]$$

$$x[0] \cdot \delta[n] \longrightarrow \text{Linear System} \longrightarrow h_0[n] \cdot x[0]$$

$$x[1] \cdot \delta[n-1] \longrightarrow \text{Linear System} \longrightarrow h_1[n] \cdot x[1]$$

$$x[2] \cdot \delta[n-2] \longrightarrow \text{Linear System} \longrightarrow h_2[n] \cdot x[2]$$

⋮

$$x[k] \cdot \delta[n-k] \longrightarrow \text{Linear System} \longrightarrow h_k[n] \cdot x[k]$$

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k] \delta[n-k] \implies y[n] = \sum_{k=-\infty}^{+\infty} x[k] h_k[n]$$

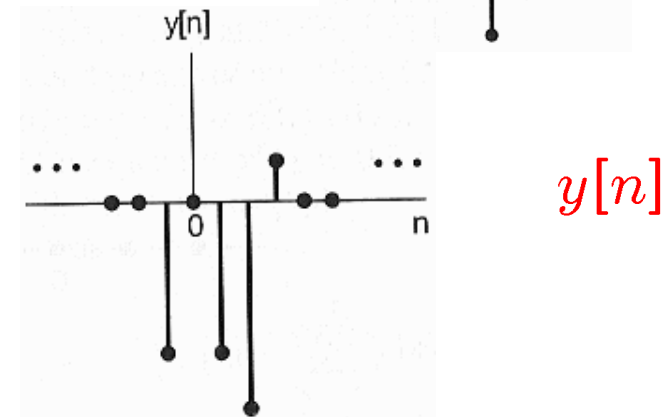
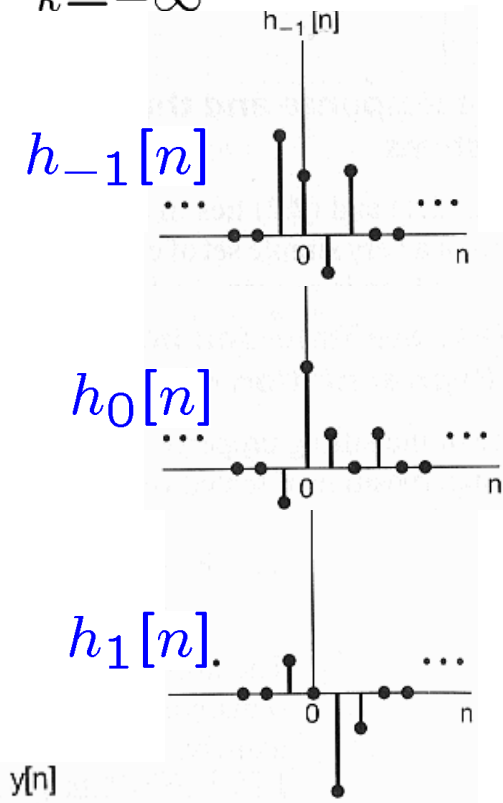
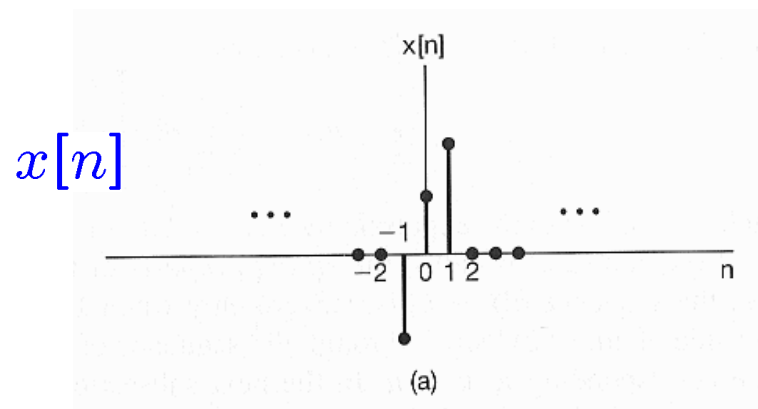
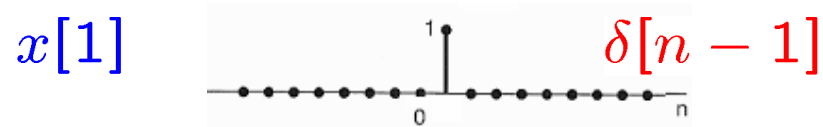
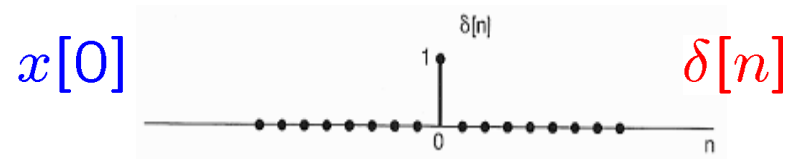


# DT LTI Systems: Convolution Sum

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NTUEE-SS2-LTI-9

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k] \delta[n - k]$$

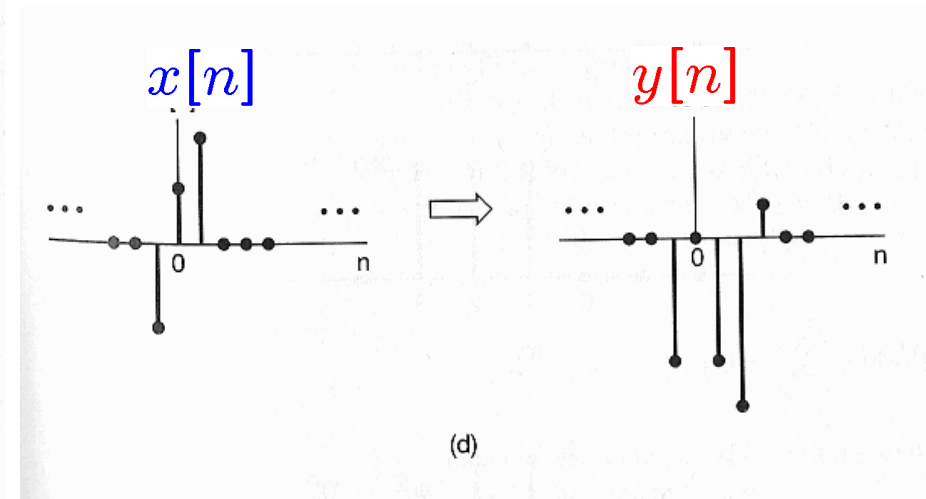
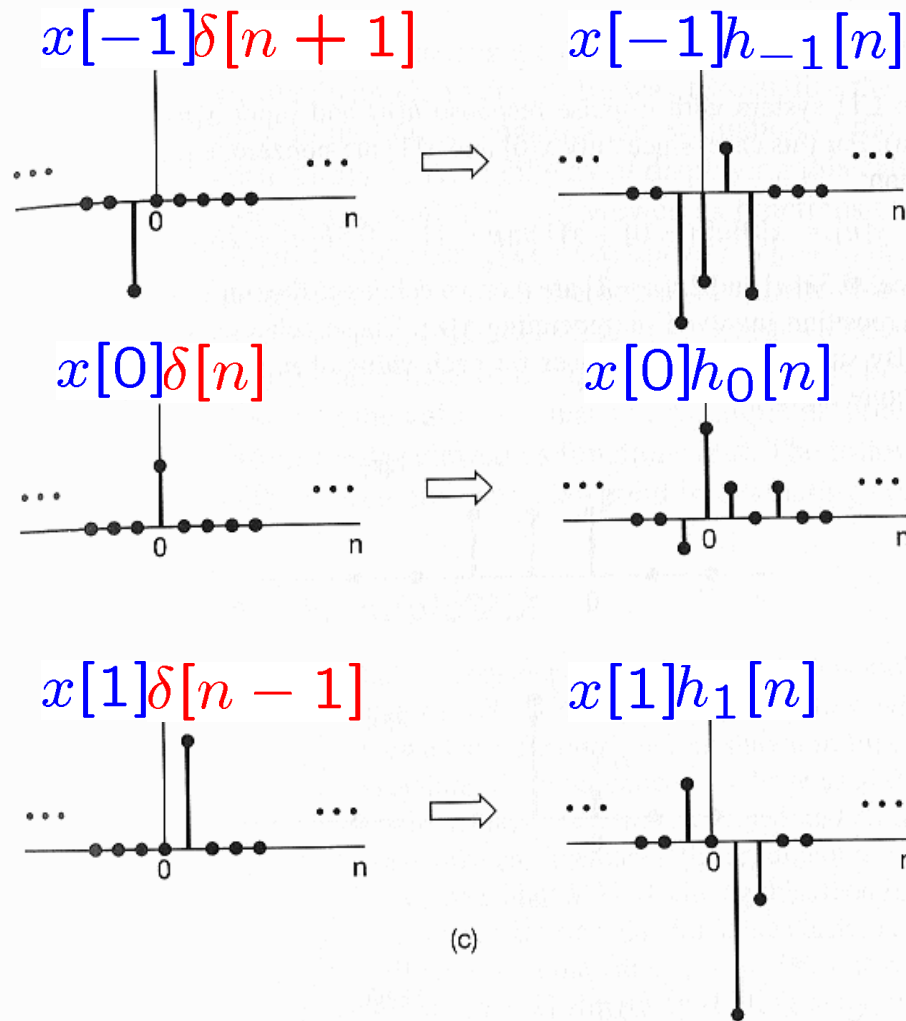
$$\Rightarrow y[n] = \sum_{k=-\infty}^{+\infty} x[k] h_k[n]$$



# DT LTI Systems: Convolution Sum

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NTUEE-SS2-LTI-10

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k] \delta[n - k] \implies y[n] = \sum_{k=-\infty}^{+\infty} x[k] h_k[n]$$



$x[n] \longrightarrow \text{Linear System} \longrightarrow y[n]$

- If the linear system (L) is also time-invariant (TI)

- Then,

$$h_k[n] = h_0[n - k] = h[n - k]$$

- Hence, for an LTI system,

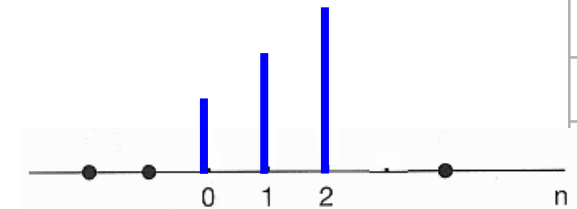
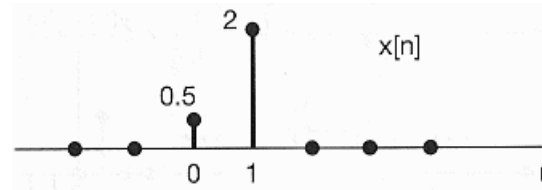
$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n - k] = \sum_{k=-\infty}^{+\infty} x[n - k]h[k]$$

- Known as the convolution of  $x[n]$  &  $h[n]$
  - Referred as the convolution sum or superposition sum

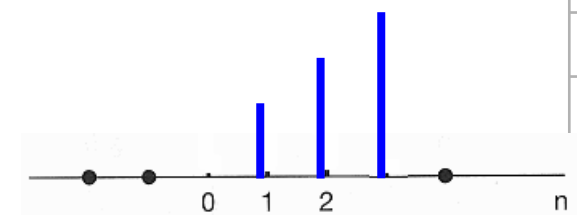
- Symbolically,  $y[n] = x[n] * h[n] = h[n] * x[n]$

## ■ Example 2.1:

$$x[n] \longrightarrow h[n] \longrightarrow y[n]$$

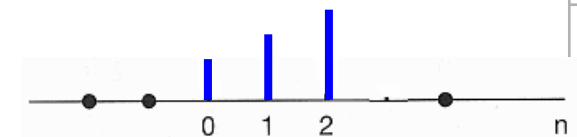


$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

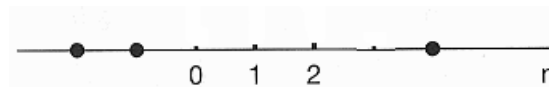
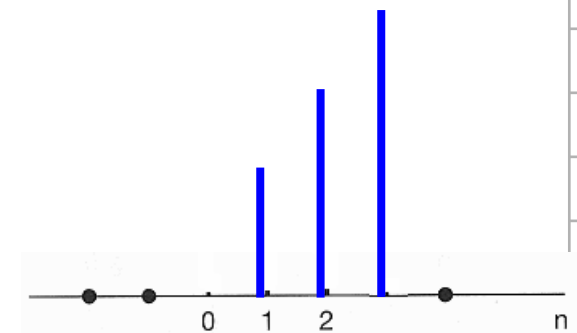


$$= \cdots + x[-1]h[n+1] + x[0]h[n] + x[1]h[n-1] + x[2]h[n-2] + \cdots$$

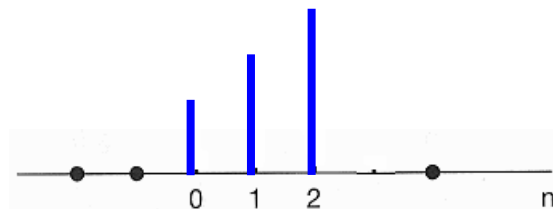
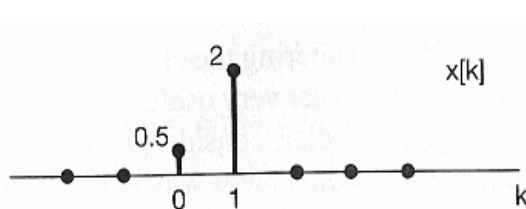
$$y[n] = x[0]h[n-0] + x[1]h[n-1]$$



$$= 0.5h[n] + 2h[n-1]$$

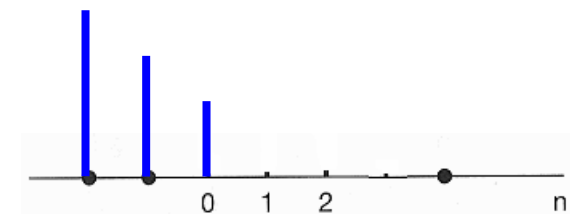


■ Example 2.2:  $y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$   $x[n] \longrightarrow h[n] \longrightarrow y[n]$



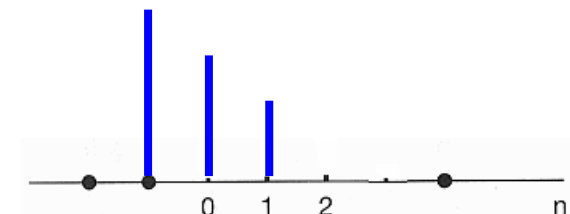
$$y[0] = \sum_{k=-\infty}^{+\infty} x[k]h[0-k]$$

$$= \cdots + x[-1]h[1] + x[0]h[0] + x[1]h[-1] + x[2]h[-2] + \cdots = 0.5$$

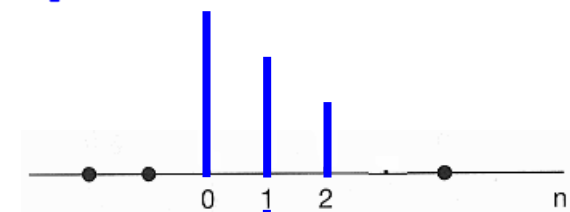


$$y[1] = \sum_{k=-\infty}^{+\infty} x[k]h[1-k]$$

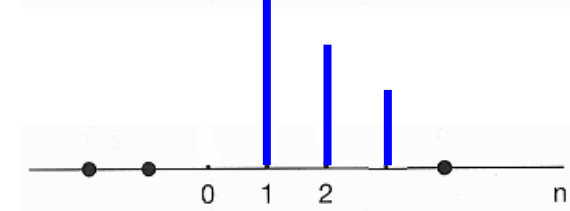
$$= \cdots + x[-1]h[2] + x[0]h[1] + x[1]h[0] + x[2]h[-1] + \cdots = 4$$



$$y[2] = \sum_{k=-\infty}^{+\infty} x[k]h[2-k] = 5.5$$

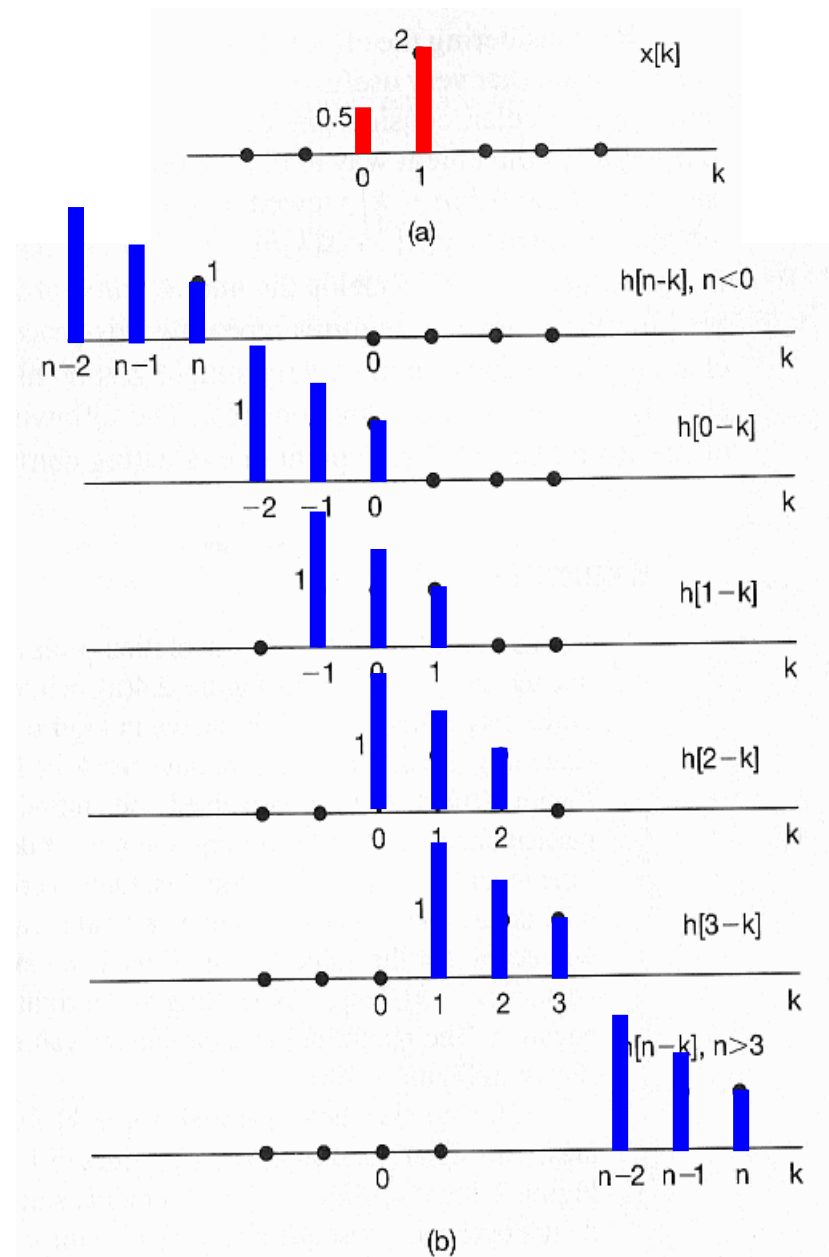
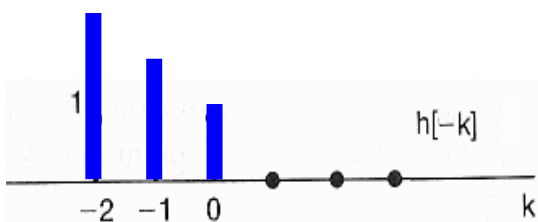
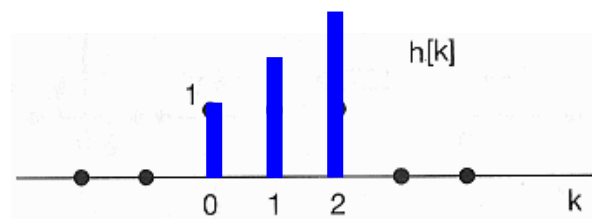


$$y[3] = \sum_{k=-\infty}^{+\infty} x[k]h[3-k] = 4.0$$

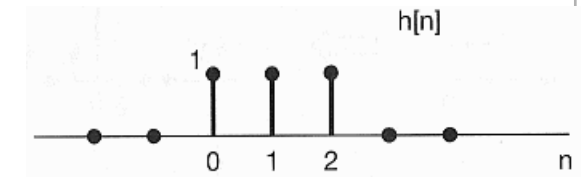
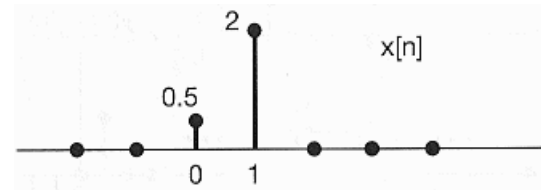


## Example 2.2:

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$



■ Example 2.1:  $x[n] \longrightarrow h[n] \longrightarrow y[n]$

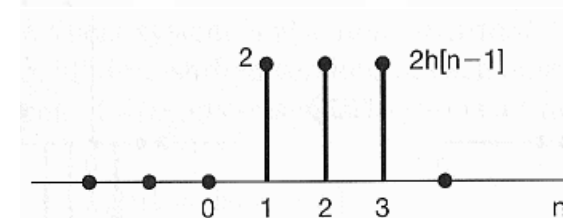
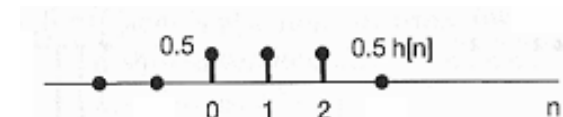


$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

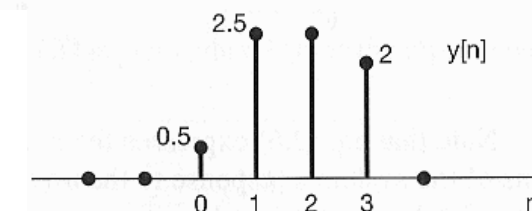
$$= \cdots + x[-1]h[n+1] + x[0]h[n] + x[1]h[n-1] + x[2]h[n-2] + \cdots$$

$$y[n] = x[0]h[n-0] + x[1]h[n-1]$$

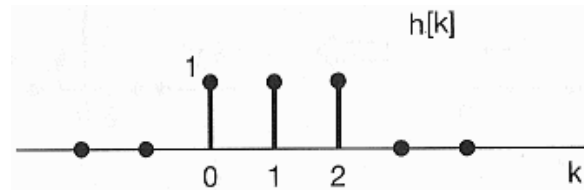
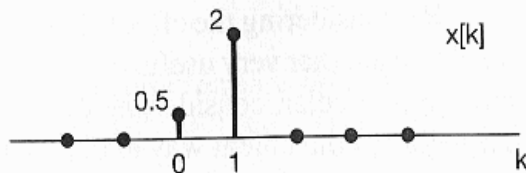
$$= 0.5h[n] + 2h[n-1]$$



(b)

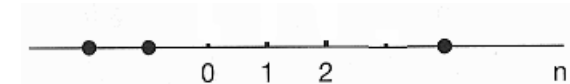


■ Example 2.2:  $y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$   $x[n] \longrightarrow h[n] \longrightarrow y[n]$



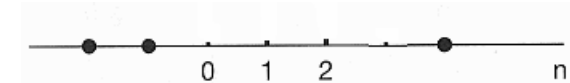
$$y[0] = \sum_{k=-\infty}^{+\infty} x[k]h[0-k]$$

$$= \cdots + x[-1]h[1] + x[0]h[0] + x[1]h[-1] + x[2]h[-2] + \cdots = 0.5$$



$$y[1] = \sum_{k=-\infty}^{+\infty} x[k]h[1-k] = 2.5$$

$$= \cdots + x[-1]h[2] + x[0]h[1] + x[1]h[0] + x[2]h[-1] + \cdots = 2.5$$



$$y[2] = \sum_{k=-\infty}^{+\infty} x[k]h[2-k] = 2.5$$

$$y[n] = 0 \text{ for } n < 0$$

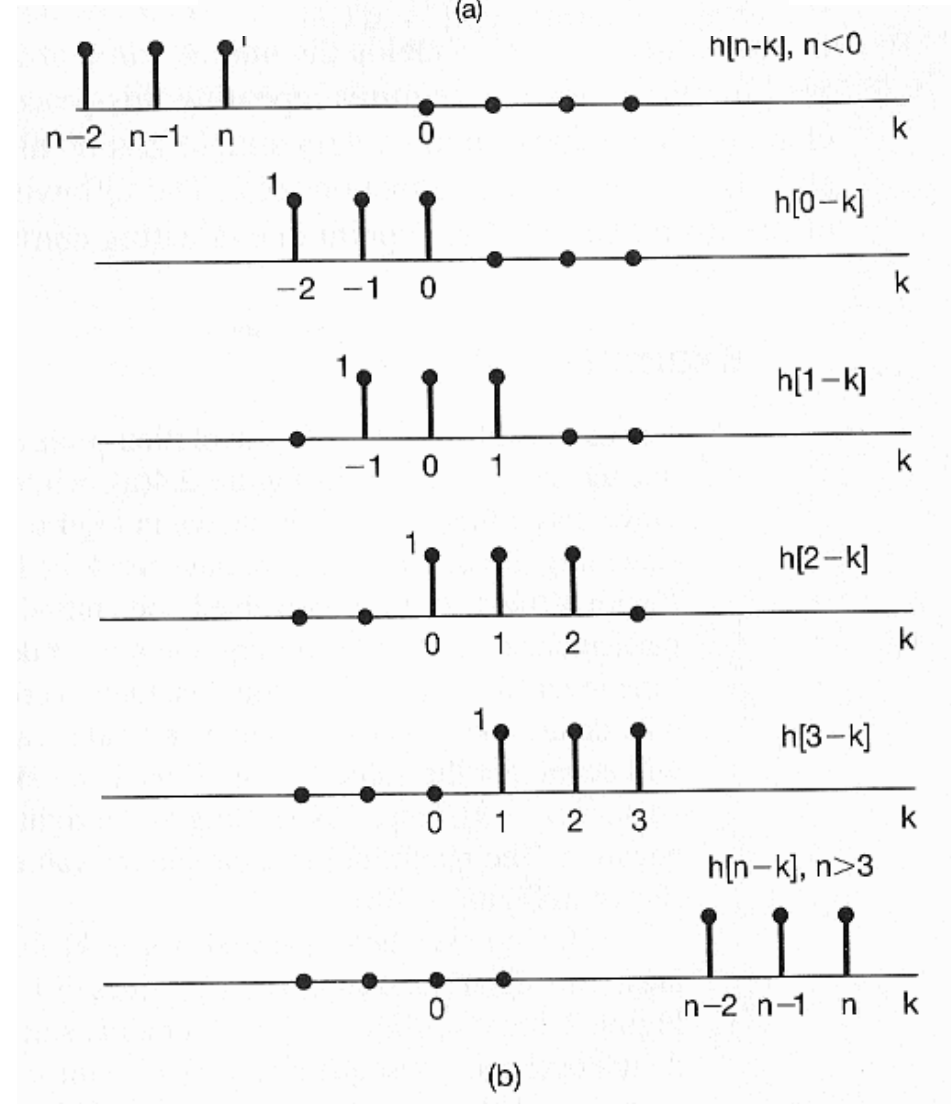
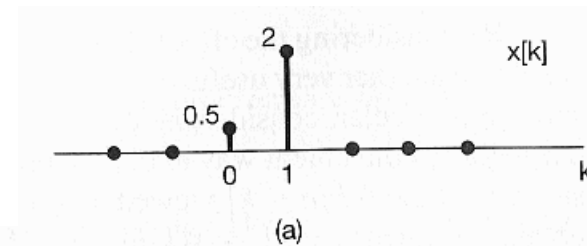
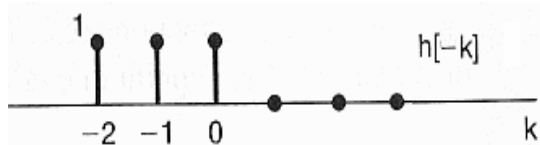
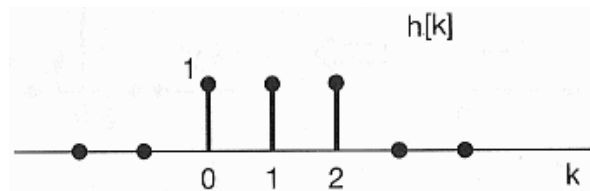
$$y[3] = \sum_{k=-\infty}^{+\infty} x[k]h[3-k] = 2.0$$

$$y[n] = 0 \text{ for } n > 3$$



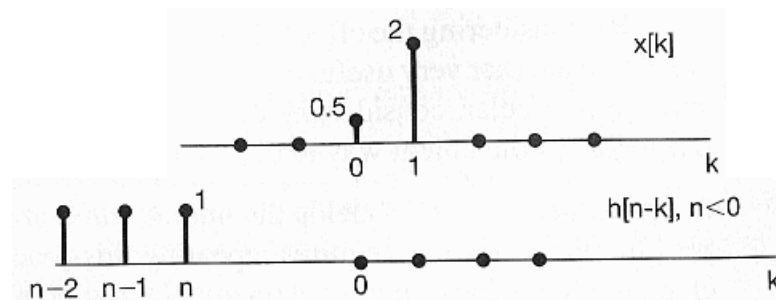
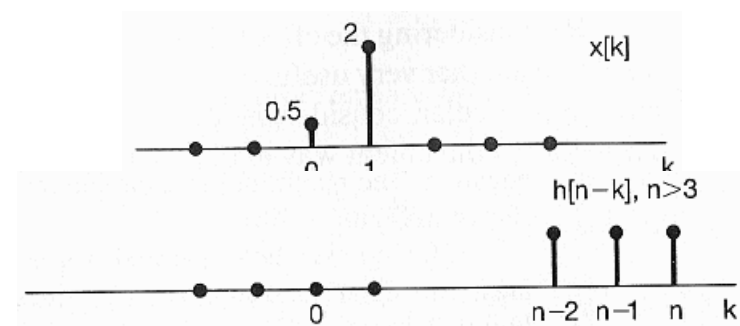
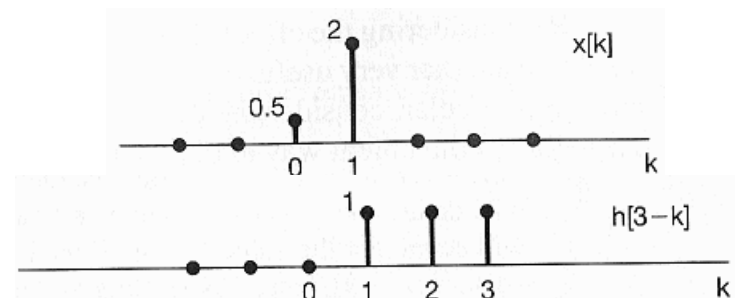
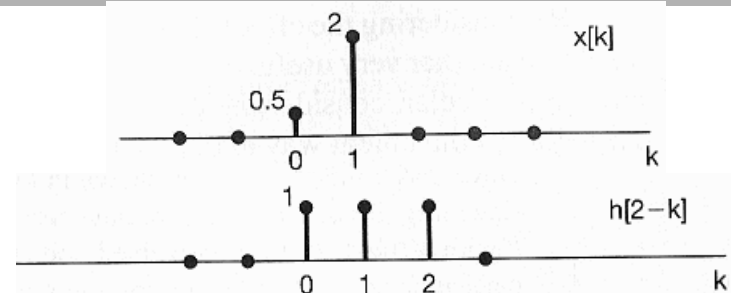
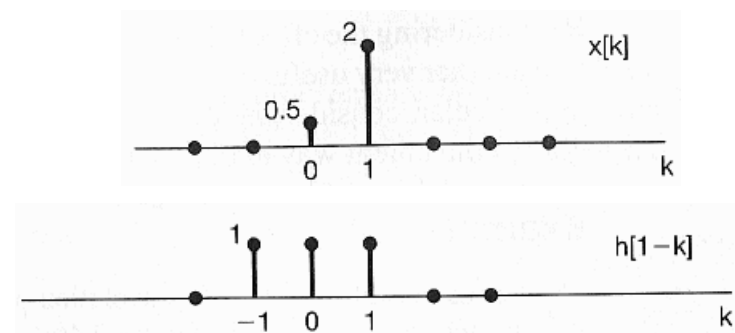
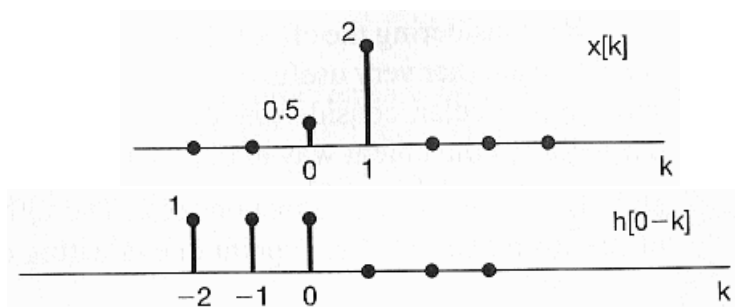
## ■ Example 2.2:

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

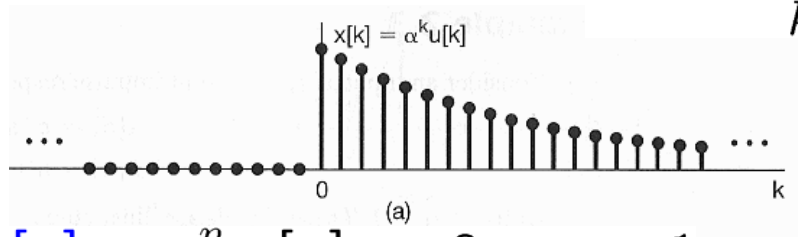


## ■ Example 2.2:

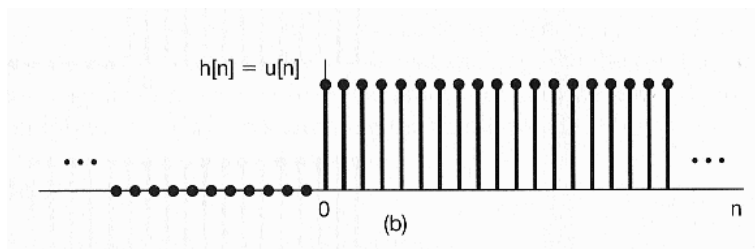
$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$



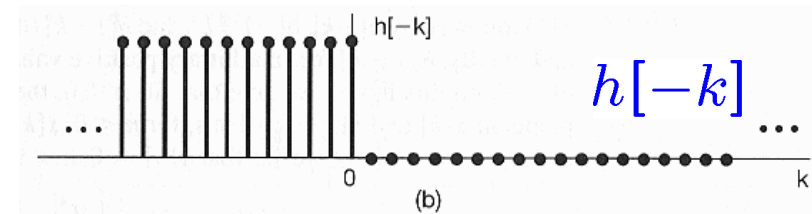
■ Example 2.3:  $y[n] = \sum_{k=-\infty}^{+\infty} x[k] h[n-k]$



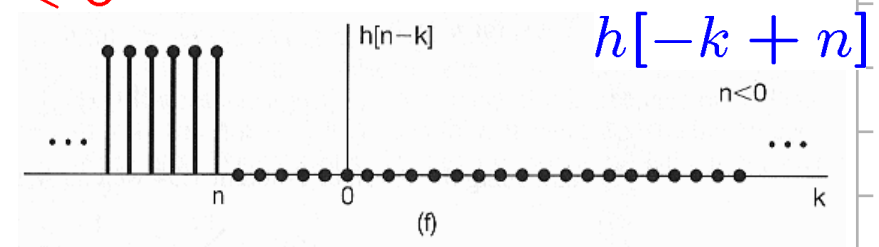
$$x[n] = \alpha^n u[n], \quad 0 < \alpha < 1$$



$$h[n] = u[n]$$

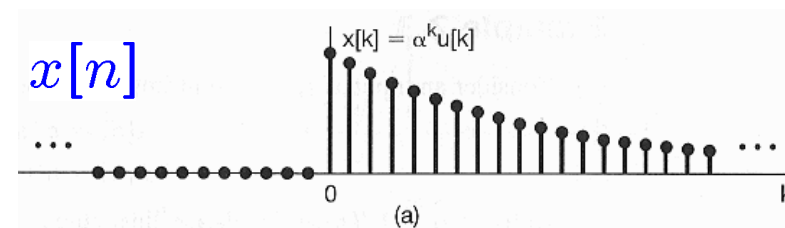


$$n < 0$$

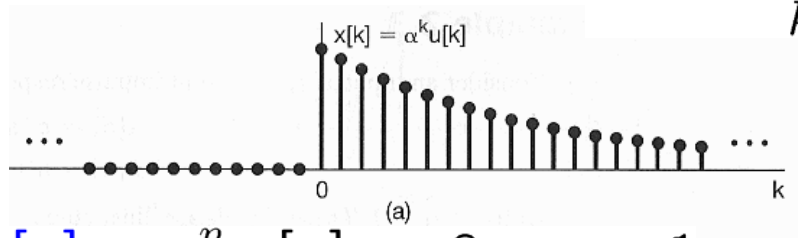


for  $n < 0$ ,  $x[k] h[n-k] = 0$

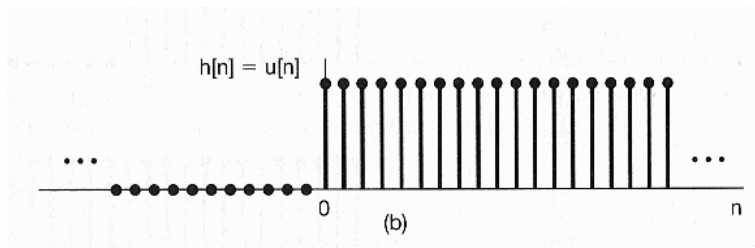
$$\Rightarrow y[n] = 0$$



■ Example 2.3:  $y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$

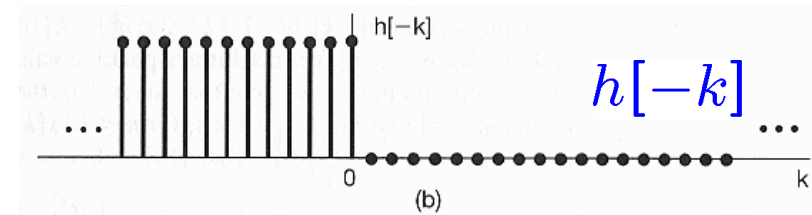


$$x[n] = \alpha^n u[n], \quad 0 < \alpha < 1$$

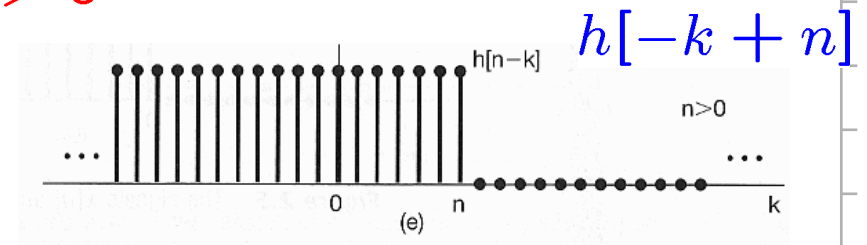


$$h[n] = u[n]$$

$$n > 0$$



$$h[-k]$$

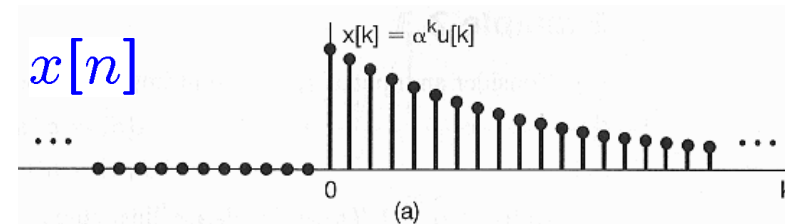


$$h[-k + n]$$

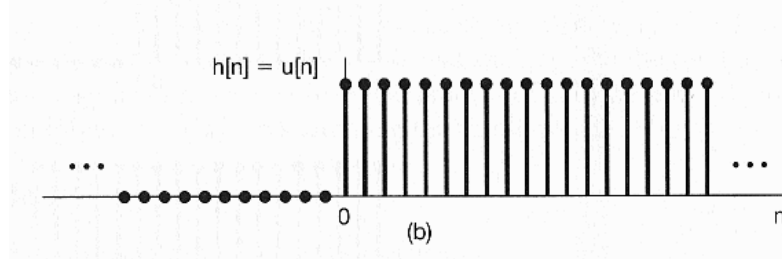
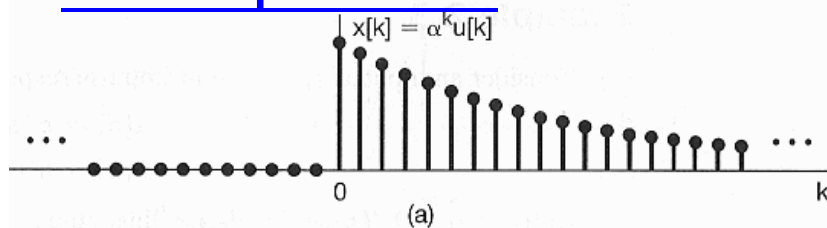
for  $n \geq 0$ ,

$$x[k] h[n-k] = \begin{cases} \alpha^k, & 0 \leq k \leq n \\ 0, & \text{otherwise} \end{cases}$$

$$\Rightarrow y[n] = \sum_{k=0}^n \alpha^k = \frac{1 - \alpha^{n+1}}{1 - \alpha}$$



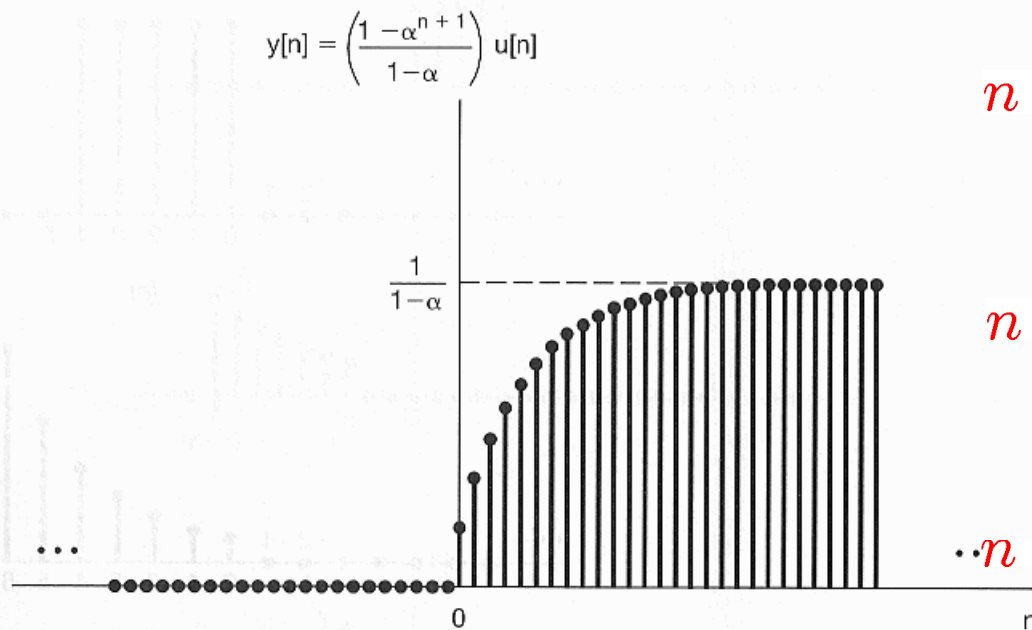
## ■ Example 2.3:



$$y[n] = \sum_{k=-\infty}^{+\infty} x[k] h[n-k]$$

for all  $n$ ,  $y[n] = \left( \frac{1 - \alpha^{n+1}}{1 - \alpha} \right) u[n]$

$$\alpha = \frac{7}{8}$$

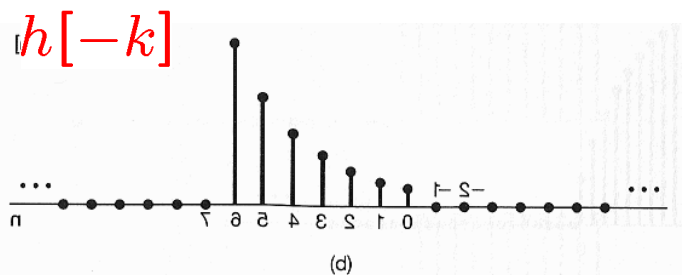
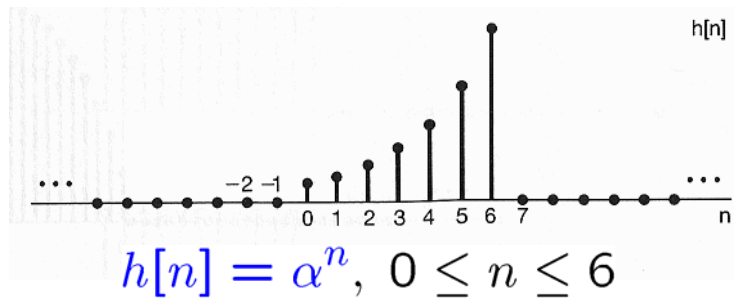
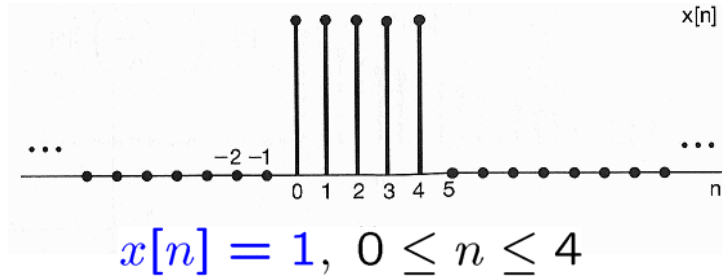


$n = 0$       $y[0] = \frac{1 - \frac{7}{8}}{1 - \frac{7}{8}} = 1$

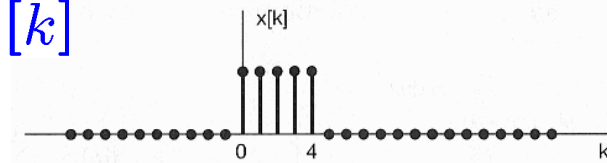
$n = 1$       $y[1] = \frac{1 - (\frac{7}{8})^2}{1 - \frac{7}{8}} = \frac{15}{8}$

$n \rightarrow \infty$       $y[n] = \frac{1 - 0}{1 - \frac{7}{8}} = 8$

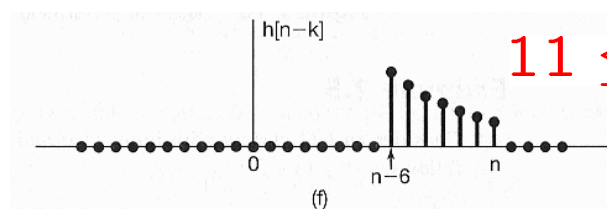
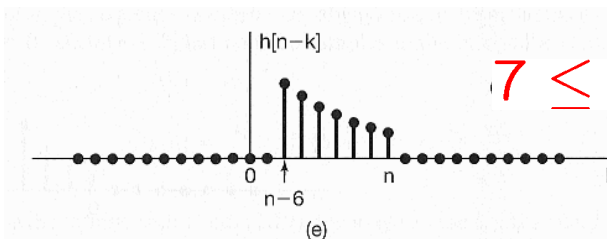
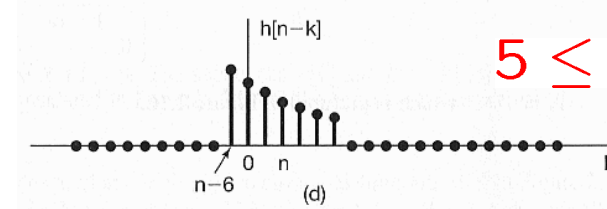
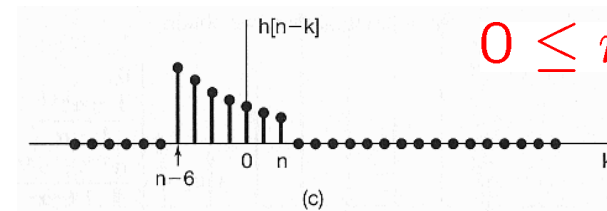
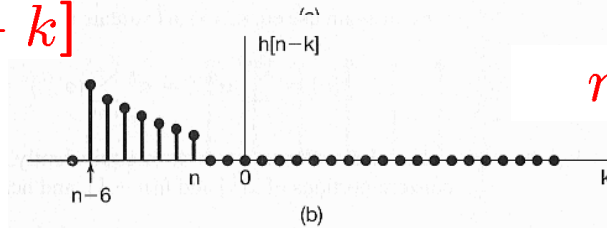
## Example 2.4:

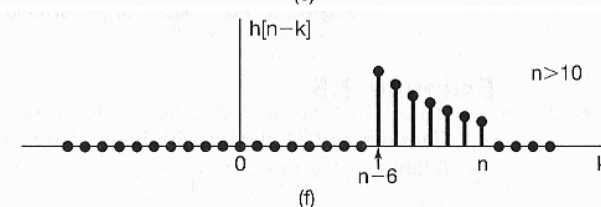
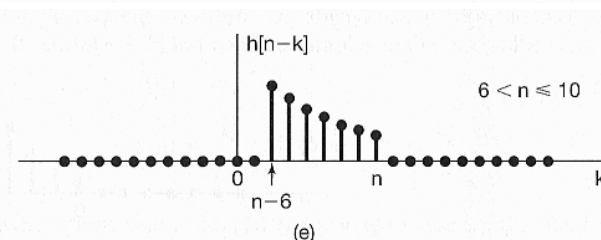
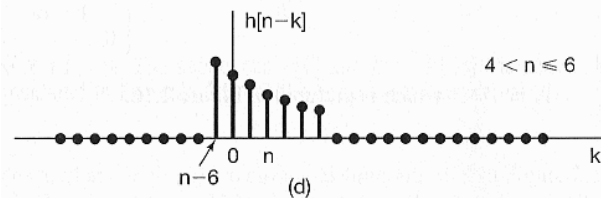
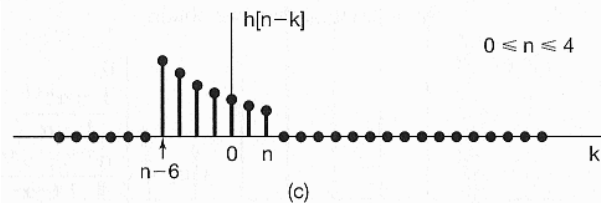
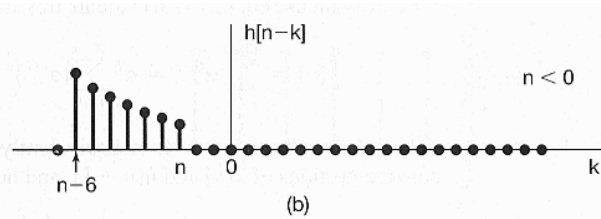
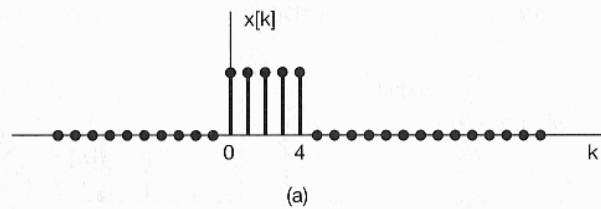


$x[k]$



$h[n-k]$





for  $n < 0$ ,  $x[k] h[n-k] = 0 \Rightarrow y[n] = 0$

for  $0 \leq n \leq 4$ ,  $x[k] h[n-k] = \begin{cases} \alpha^{n-k}, & 0 \leq k \leq n \\ 0, & \text{otherwise} \end{cases}$

$$\Rightarrow y[n] = \sum_{k=0}^n \alpha^{n-k} = \frac{1 - \alpha^{n+1}}{1 - \alpha}$$

for  $4 < n \leq 6$ ,  $x[k] h[n-k] = \begin{cases} \alpha^{n-k}, & 0 \leq k \leq 4 \\ 0, & \text{otherwise} \end{cases}$

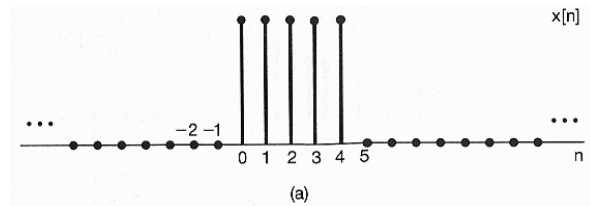
$$\Rightarrow y[n] = \sum_{k=0}^4 \alpha^{n-k} = \frac{\alpha^{n-4} - \alpha^{n+1}}{1 - \alpha}$$

for  $6 < n \leq 10$ ,  $x[k] h[n-k] = \begin{cases} \alpha^{n-k}, & (n-6) \leq k \leq 4 \\ 0, & \text{otherwise} \end{cases}$

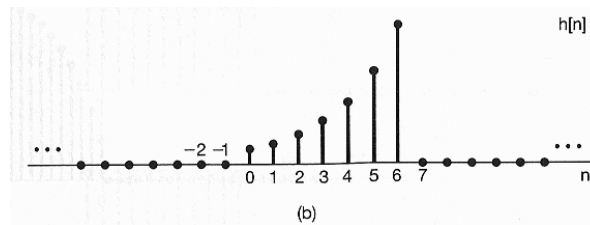
$$\Rightarrow y[n] = \sum_{k=n-6}^4 \alpha^{n-k} = \frac{\alpha^{n-4} - \alpha^7}{1 - \alpha}$$

for  $n > 10$ ,  $y[n] = 0$

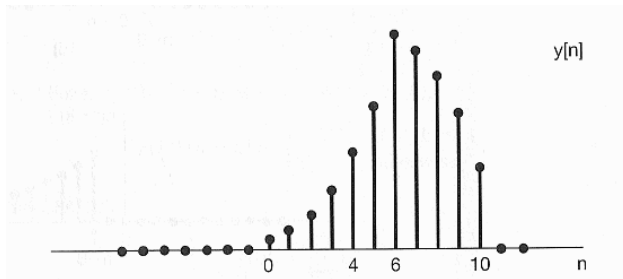
$$x[n] \longrightarrow h[n] \longrightarrow y[n]$$



$$x[n] = 1, 0 \leq n \leq 4$$



$$h[n] = \alpha^n, 0 \leq n \leq 6$$

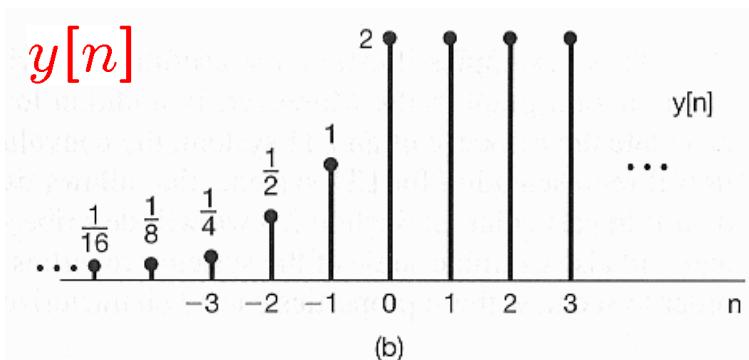
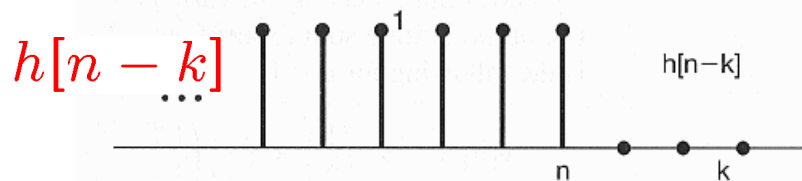
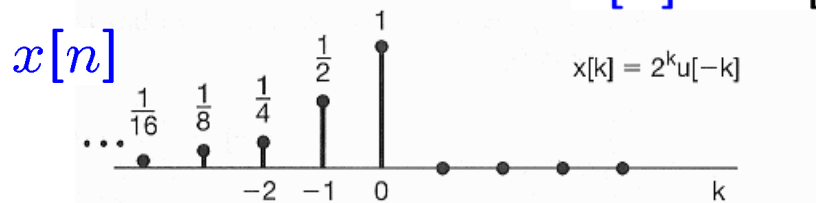
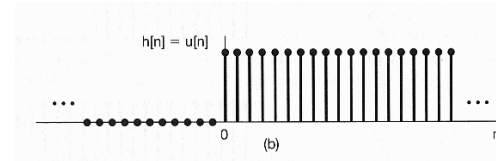


$$y[n] = \begin{cases} 0, & n < 0 \\ \frac{1 - \alpha^{n+1}}{1 - \alpha}, & 0 \leq n \leq 4 \\ \frac{\alpha^{n-4} - \alpha^{n+1}}{1 - \alpha}, & 4 < n \leq 6 \\ \frac{\alpha^{n-4} - \alpha^7}{1 - \alpha}, & 6 < n \leq 10 \\ 0, & 10 < n \end{cases}$$



■ Example 2.5:  $x[n] = 2^n u[-n]$   $x[n] \longrightarrow h[n] \longrightarrow y[n]$

$$h[n] = u[n]$$



$$\text{for } n \geq 0, \quad y[n] = \sum_{k=-\infty}^0 x[k] h[n-k] = \sum_{k=-\infty}^0 2^k$$

$$= \sum_{r=0}^{\infty} \left(\frac{1}{2}\right)^r = \frac{1}{1 - (1/2)} = 2$$

$$\text{for } n < 0, \quad y[n] = \sum_{k=-\infty}^n x[k] h[n-k] = \sum_{k=-\infty}^n 2^k$$

$$= \sum_{l=-n}^{\infty} \left(\frac{1}{2}\right)^l = \sum_{m=0}^{\infty} \left(\frac{1}{2}\right)^{m-n}$$

$$= \left(\frac{1}{2}\right)^{-n} \sum_{m=0}^{\infty} \left(\frac{1}{2}\right)^m = 2^n \cdot 2 = 2^{n+1}$$

- Discrete-Time Linear Time-Invariant Systems

- The convolution sum  $y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k] \quad y[n] = x[n] * h[n]$

- Continuous-Time Linear Time-Invariant Systems

- The convolution integral

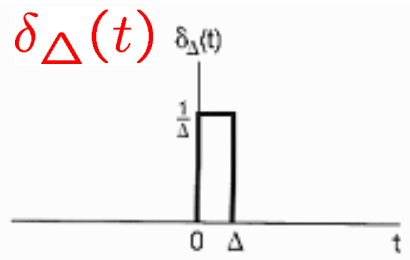
- Properties of Linear Time-Invariant Systems

- Causal Linear Time-Invariant Systems

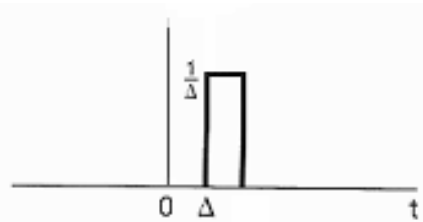
Described by Differential & Difference Equations

- Singularity Functions

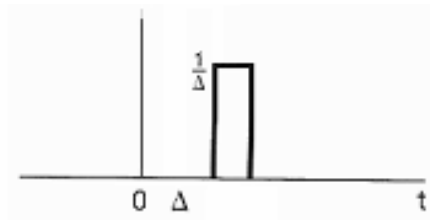
## Representation of CT Signals by Impulses:



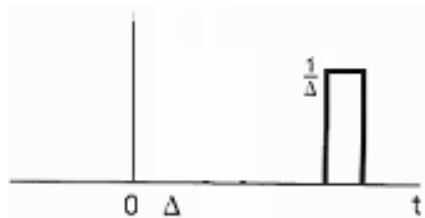
$$\delta_{\Delta}(t) = \begin{cases} \frac{1}{\Delta}, & 0 \leq t < \Delta \\ 0, & \text{otherwise} \end{cases}$$



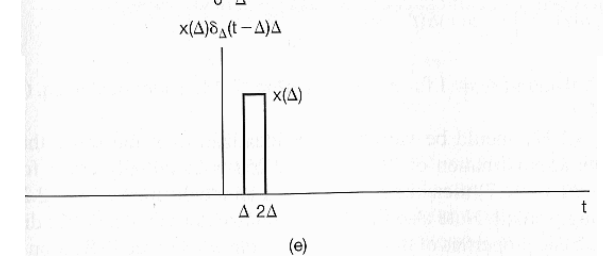
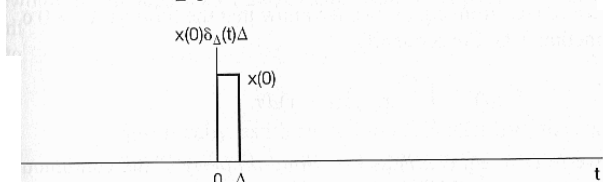
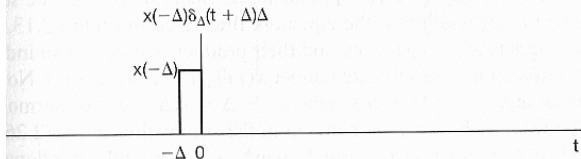
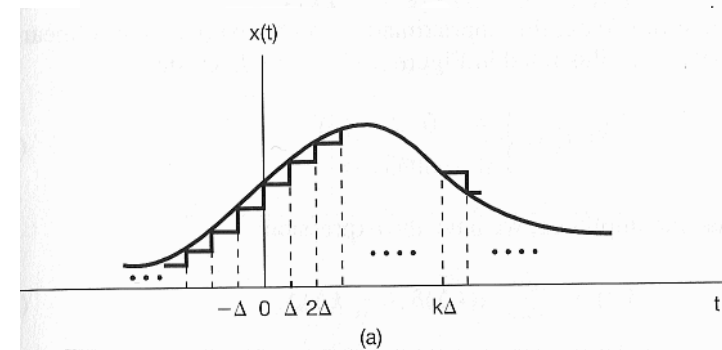
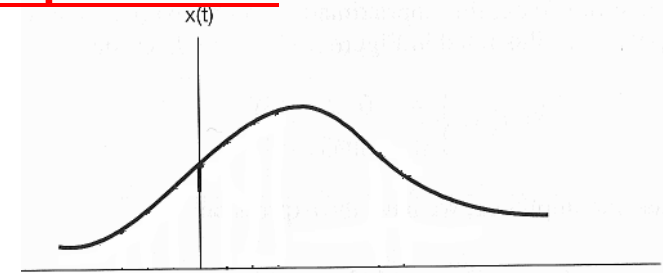
$$\delta_{\Delta}(t - \Delta)$$



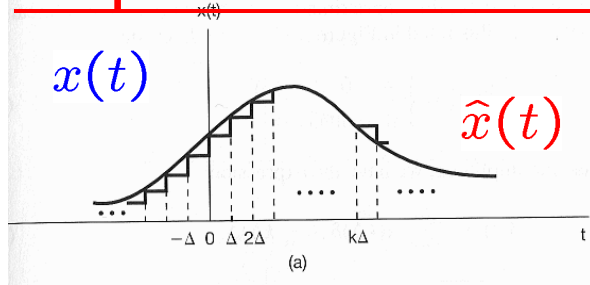
$$\delta_{\Delta}(t - 2\Delta)$$



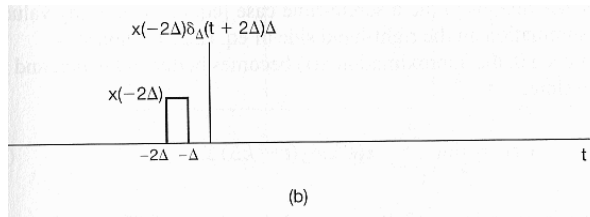
$$\delta_{\Delta}(t - k\Delta)$$



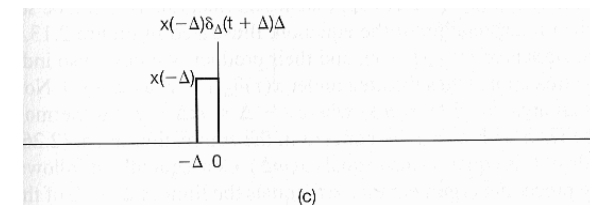
## Representation of CT Signals by Impulses:



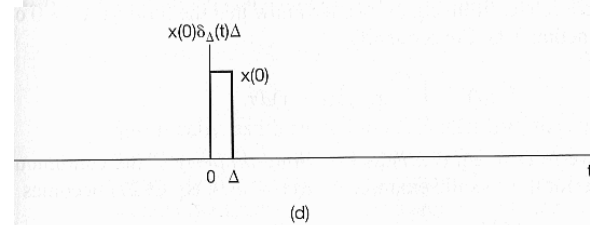
$$\hat{x}(t) = \sum_{k=-\infty}^{+\infty} x(k\Delta) \delta_{\Delta}(t - k\Delta) \Delta$$



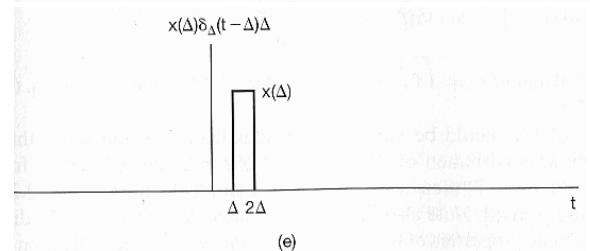
$$x(t) = \lim_{\Delta \rightarrow 0} \sum_{k=-\infty}^{+\infty} x(k\Delta) \delta_{\Delta}(t - k\Delta) \Delta$$



$$x(t) = \int_{-\infty}^{+\infty} x(\tau) \delta(t - \tau) d\tau$$

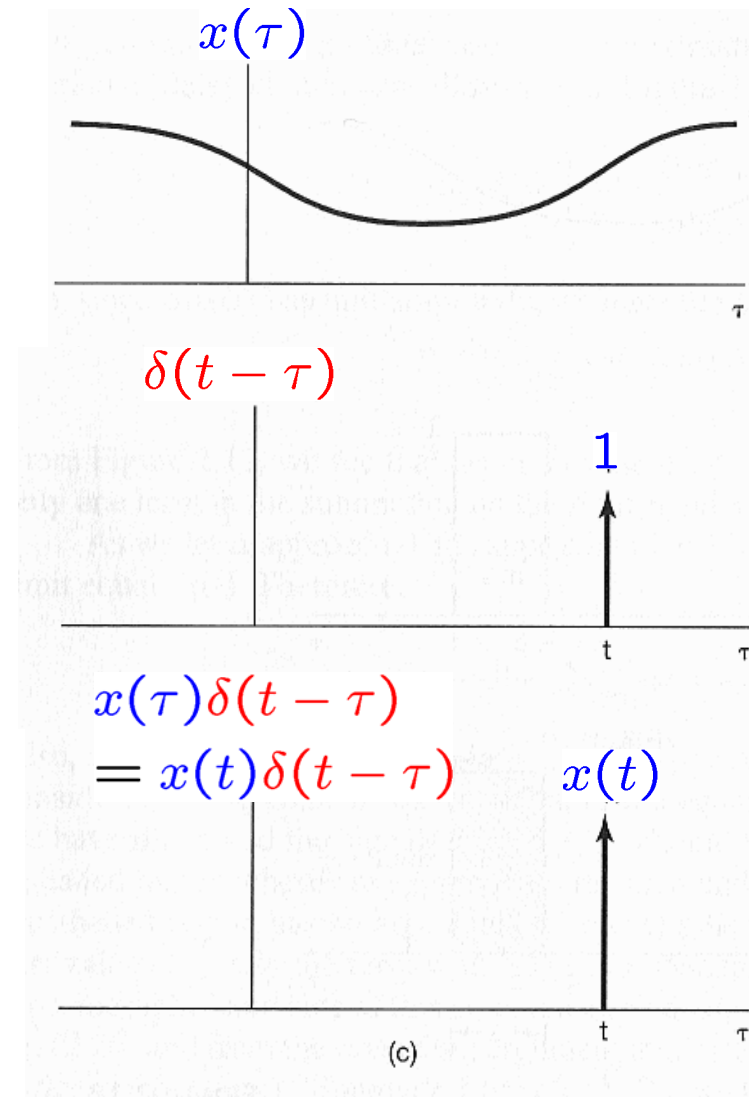
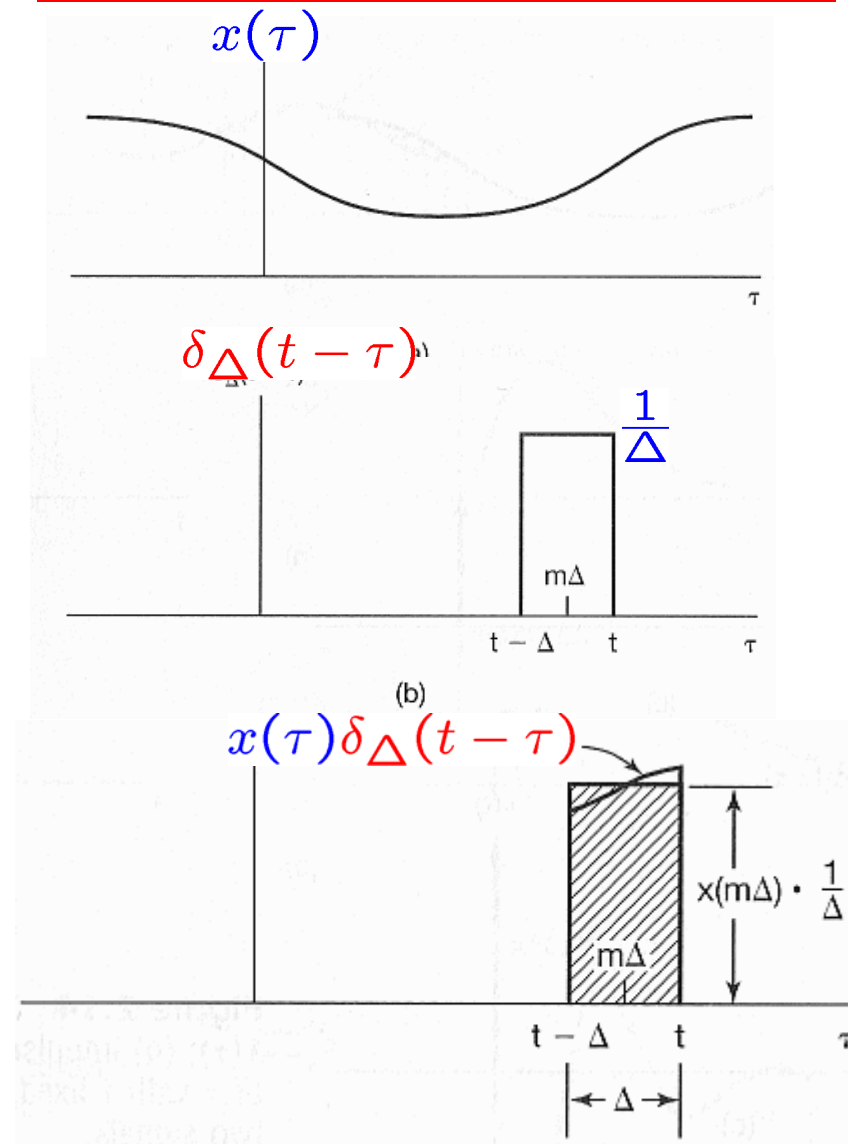


the **sifting property** of CT impulse



$x(t)$  = an integral of weighted,  
shifted impulses

## ■ Graphical interpretation:



## ■ CT Impulse Response & Convolution Integral:

input  $\longrightarrow$  Linear System  $\longrightarrow$  output

$\delta_{\Delta}(t) \longrightarrow$  Linear System  $\longrightarrow \hat{h}_{0\Delta}(t)$

$\delta_{\Delta}(t - 1\Delta) \longrightarrow$  Linear System  $\longrightarrow \hat{h}_{1\Delta}(t)$

$\delta_{\Delta}(t - 2\Delta) \longrightarrow$  Linear System  $\longrightarrow \hat{h}_{2\Delta}(t)$

$\vdots$

$\delta_{\Delta}(t - k\Delta) \longrightarrow$  Linear System  $\longrightarrow \hat{h}_{k\Delta}(t)$

## ■ CT Impulse Response & Convolution Integral:

input  $\longrightarrow$  Linear System  $\longrightarrow$  output

$$x(0\Delta) \quad \delta_{\Delta}(t) \longrightarrow \text{Linear System} \longrightarrow \hat{h}_{0\Delta}(t) \quad x(0\Delta)$$

$$x(1\Delta) \quad \delta_{\Delta}(t - 1\Delta) \longrightarrow \text{Linear System} \longrightarrow \hat{h}_{1\Delta}(t) \quad x(1\Delta)$$

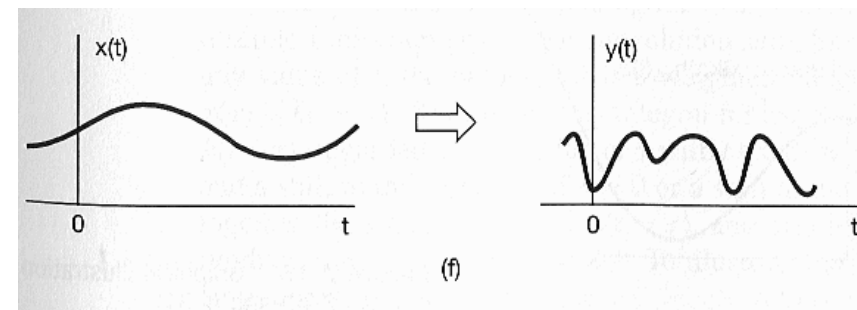
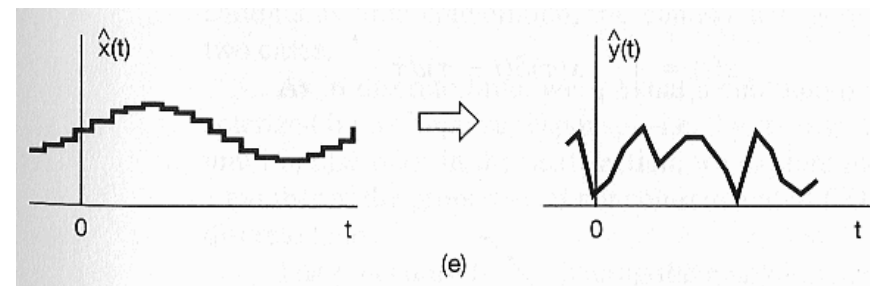
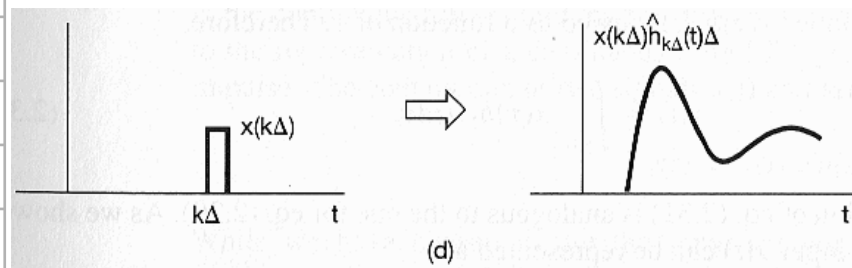
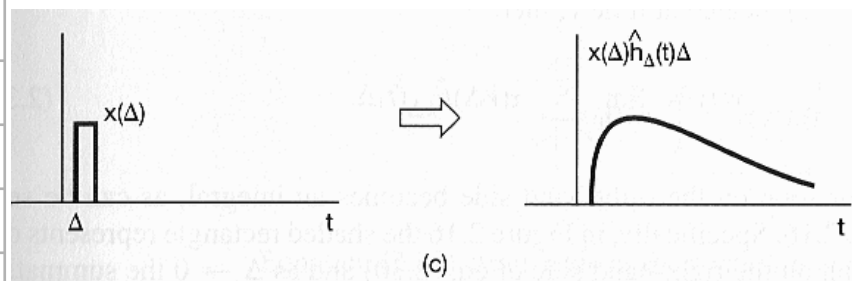
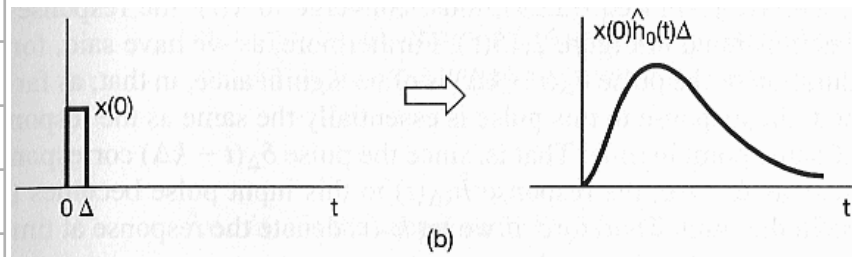
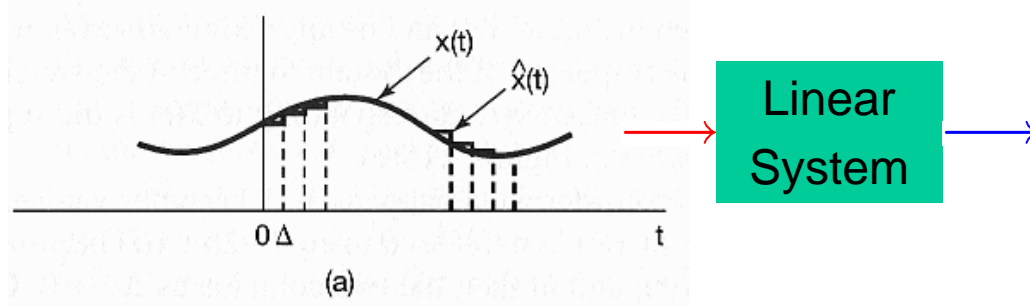
$$x(2\Delta) \quad \delta_{\Delta}(t - 2\Delta) \longrightarrow \text{Linear System} \longrightarrow \hat{h}_{2\Delta}(t) \quad x(2\Delta)$$

⋮

$$x(k\Delta) \quad \delta_{\Delta}(t - k\Delta) \longrightarrow \text{Linear System} \longrightarrow \hat{h}_{k\Delta}(t) \quad x(k\Delta)$$

$$\hat{x}(t) = \sum_{k=-\infty}^{+\infty} x(k\Delta) \delta_{\Delta}(t - k\Delta) \Delta \implies \hat{y}(t) = \sum_{k=-\infty}^{+\infty} x(k\Delta) \hat{h}_{k\Delta}(t) \Delta$$

# CT LTI Systems: Convolution Integral





## ■ CT Unit Impulse Response & Convolution Integral:

$$\hat{y}(t) = \sum_{k=-\infty}^{+\infty} x(k\Delta) \hat{h}_{k\Delta}(t) \Delta$$

$$y(t) = \lim_{\Delta \rightarrow 0} \sum_{k=-\infty}^{+\infty} x(k\Delta) \hat{h}_{k\Delta}(t) \Delta$$

$$y(t) = \int_{-\infty}^{+\infty} x(\tau) h_{\tau}(t) d\tau$$

$$\delta(t - \tau) \longrightarrow \text{Linear System} \longrightarrow h_{\tau}(t)$$

$$x(t) \longrightarrow \text{Linear System} \longrightarrow y(t)$$

$$x(t) = \int_{-\infty}^{+\infty} x(\tau) \delta(t - \tau) d\tau \quad \Longrightarrow \quad y(t) = \int_{-\infty}^{+\infty} x(\tau) h_{\tau}(t) d\tau$$

- If the **linear system (L)** is also **time-invariant (TI)**

- Then,

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$$h_{\tau}(t) = h_0(t - \tau) = h(t - \tau)$$

- Hence, for an LTI system,

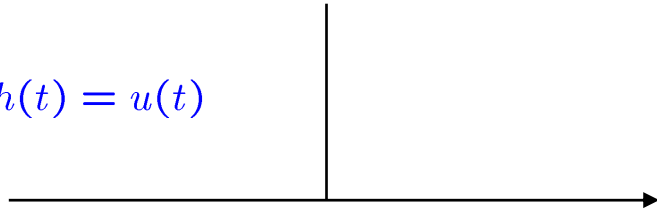
$$y(t) = \int_{-\infty}^{+\infty} x(\tau) h(t - \tau) d\tau = \int_{-\infty}^{+\infty} h(\tau) x(t - \tau) d\tau$$

- Known as the **convolution** of **x(t)** & **h(t)**
  - Referred as the **convolution integral** or the **superposition integral**

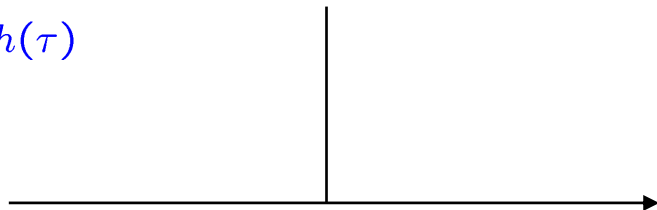
- Symbolically,  $y(t) = x(t) * h(t) = h(t) * x(t)$

■ Example 2.6:  $y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$

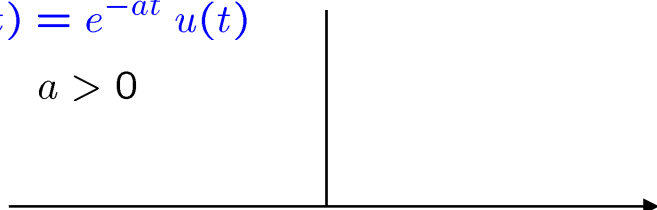
$h(t) = u(t)$



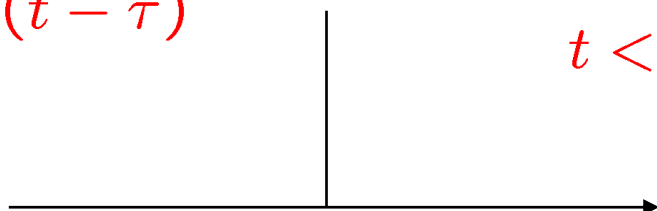
$h(\tau)$



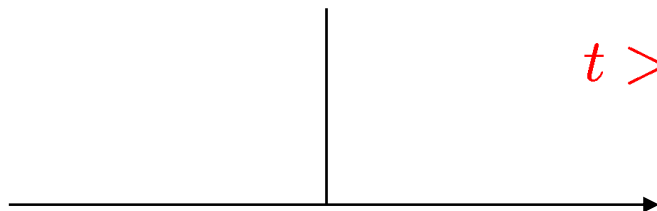
$x(t) = e^{-at} u(t)$   
 $a > 0$



$h(t-\tau)$



$t < 0$



$t > 0$

for  $t < 0$ ,  $x(\tau) h(t-\tau) = 0$

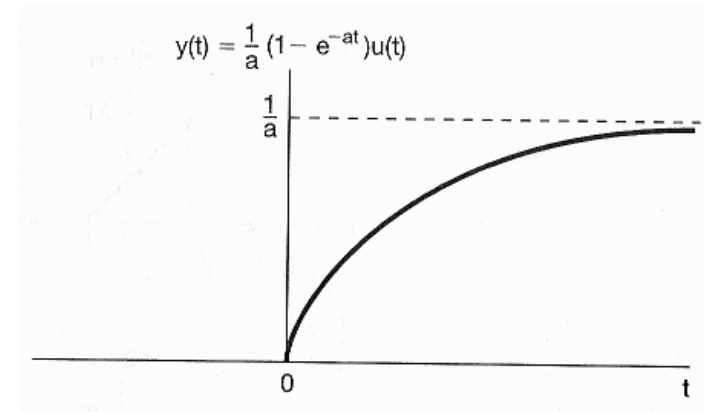
$\Rightarrow y(t) = \int_{-\infty}^t 0 d\tau = 0$

for  $t \geq 0$ ,  $x(\tau) h(t-\tau) = \begin{cases} e^{-a\tau}, & 0 \leq \tau \leq t \\ 0, & \text{otherwise} \end{cases}$

$\Rightarrow y(t) = \int_0^t e^{-a\tau} d\tau$

$= -\frac{1}{a} e^{-a\tau} \Big|_0^t$

$= \frac{1}{a} (1 - e^{-at})$



■ Example 2.7:  $y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$

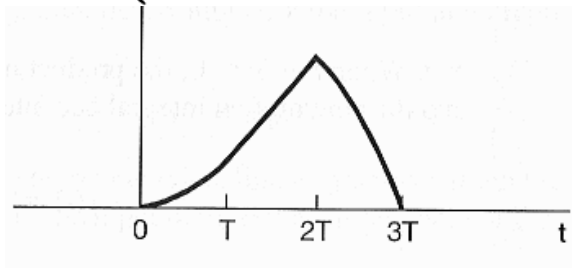
$$x(t) = \begin{cases} 1, & 0 < t < T \\ 0, & \text{otherwise} \end{cases}$$



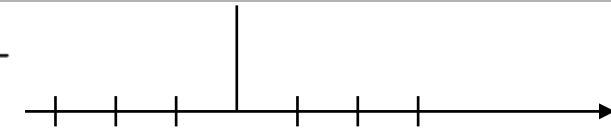
$$h(t) = \begin{cases} t, & 0 < t < 2T \\ 0, & \text{otherwise} \end{cases}$$



$$y(t) = \begin{cases} 0, & t < 0 \\ \frac{1}{2}t^2, & 0 < t < T \\ Tt - \frac{1}{2}T^2, & T < t < 2T \\ -\frac{1}{2}t^2 + Tt + \frac{3}{2}T^2, & 2T < t < 3T \\ 0, & 3T < t \end{cases}$$

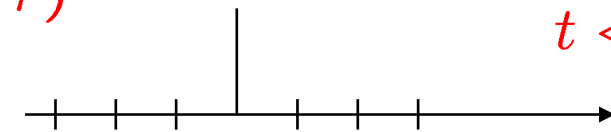


$x(\tau)$

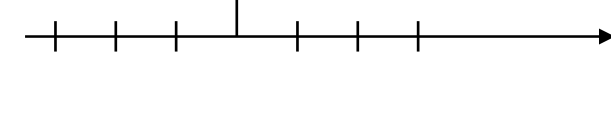


$h(t-\tau)$

$t < 0$

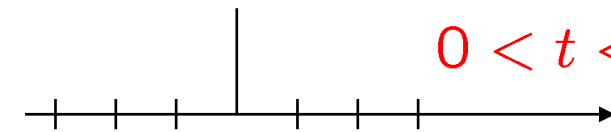


$h(\tau)$

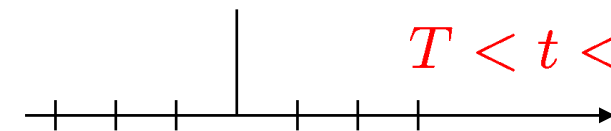


$h(-\tau)$

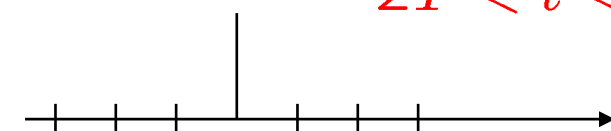
$0 < t < T$



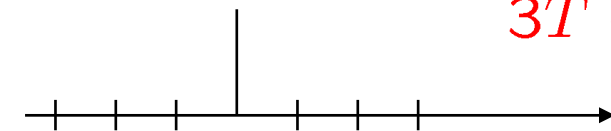
$T < t < 2T$



$2T < t < 3T$



$3T < t$

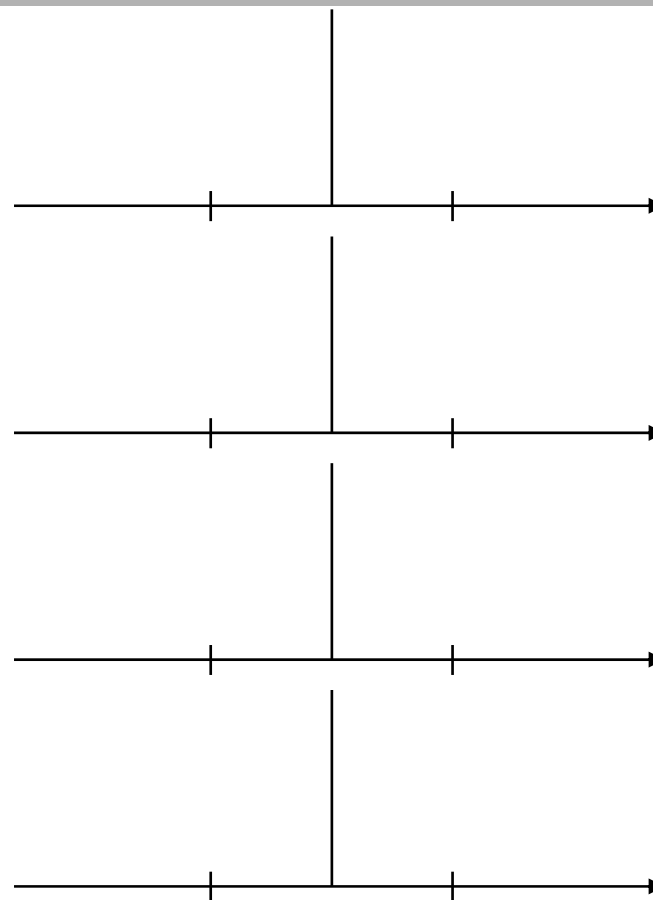


■ Example 2.8:  $x(t) = e^{2t}u(-t)$

$$h(t) = u(t - 3)$$

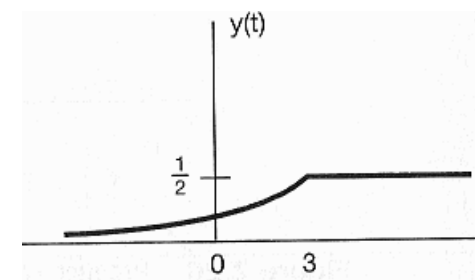
$$h(-\tau)$$

$$h(t - \tau)$$



$$\text{for } t - 3 \leq 0, \quad y(t) = \int_{-\infty}^{t-3} e^{2\tau} d\tau = \frac{1}{2} e^{2(t-3)}$$

$$\text{for } t - 3 \geq 0, \quad y(t) = \int_{-\infty}^0 e^{2\tau} d\tau = \frac{1}{2}$$

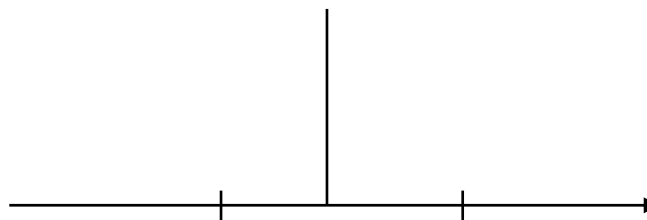
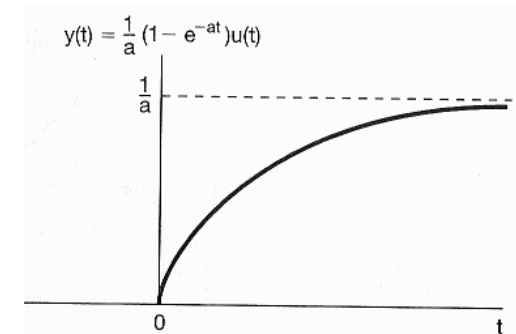
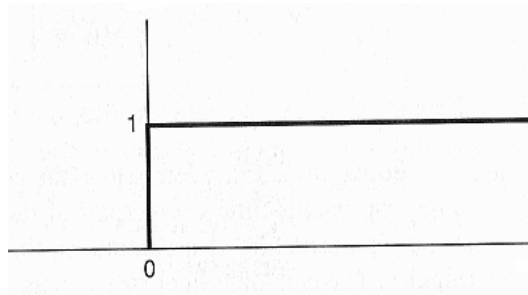
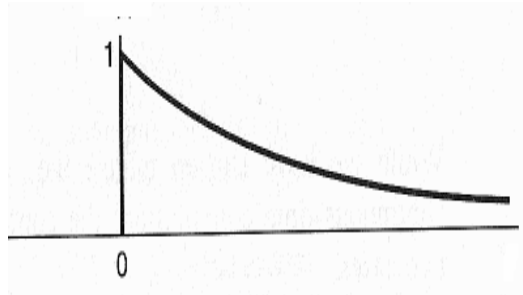


■ Signal and System..  $y(t) = x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau)h(t - \tau)d\tau$

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n - k] = \sum_{k=-\infty}^{+\infty} x[n - k]h[k] = x(t) * h(t)$$

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t - \tau)d\tau = \int_{-\infty}^{+\infty} h(\tau)x(t - \tau)d\tau = x[n] * h[n]$$

$$x(t) \rightarrow \boxed{h(t)} \rightarrow y(t)$$



- Discrete-Time Linear Time-Invariant Systems

- The **convolution sum**  $y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k] \quad y[n] = x[n] * h[n]$

- Continuous-Time Linear Time-Invariant Systems

- The **convolution integral**  $y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau \quad y(t) = x(t) * h(t)$

- Properties of Linear Time-Invariant Systems

- Causal Linear Time-Invariant Systems

Described by Differential & Difference Equations

- Singularity Functions

## ■ Convolution Sum & Integral of LTI Systems:

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k] = x[k] * h[n]$$

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau = x(t) * h(t)$$

$$x[n] \rightarrow \boxed{h[n]} \rightarrow y[n]$$

$$x(t) \rightarrow \boxed{h(t)} \rightarrow y(t)$$



## ■ Properties of LTI Systems

1. Commutative property
2. Distributive property
3. Associative property
4. With or without memory
5. Invertibility
6. Causality
7. Stability
8. Unit step response

$$y[n] = x[k] * h[n]$$

$$y(t) = x(t) * h(t)$$

$$a \times b = b \times a$$

$$a + b = b + a$$

$$a \times (b + c) = a \times b + a \times c$$

$$a \times (b \times c) = (a \times b) \times c$$

$$= \dots = a \times b \times c$$

$$x[n] \rightarrow \boxed{h[n]} \rightarrow y[n]$$

$$x(t) \rightarrow \boxed{h(t)} \rightarrow y(t)$$

$$\forall x[n] \rightarrow \forall y[n] \quad h[n] = ?$$

■ Commutative Property:  $n - k = r$

$$x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k] = \sum_{r=-\infty}^{+\infty} x[n-r]h[r]$$

$$= \sum_{r=-\infty}^{+\infty} h[r]x[n-r] = h[n] * x[n]$$

$$x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau \quad \begin{array}{l} t - \tau = \sigma \\ -d\tau = d\sigma \end{array}$$

$$\begin{aligned} &= \int_{+\infty}^{-\infty} x(t-\sigma)h(\sigma)(-d\sigma) = \int_{-\infty}^{+\infty} x(t-\sigma)h(\sigma)d\sigma \\ &= \int_{-\infty}^{+\infty} h(\sigma)x(t-\sigma)d\sigma = h(t) * x(t) \end{aligned}$$

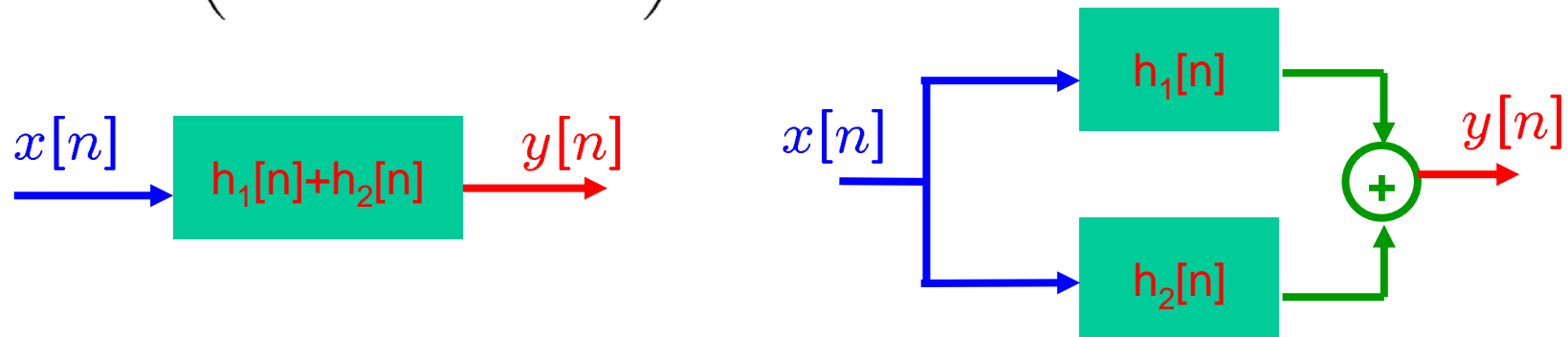
# ■ Distributive Property:

$$x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

$$x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

$$x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n]$$

$$x(t) * (h_1(t) + h_2(t)) = x(t) * h_1(t) + x(t) * h_2(t)$$



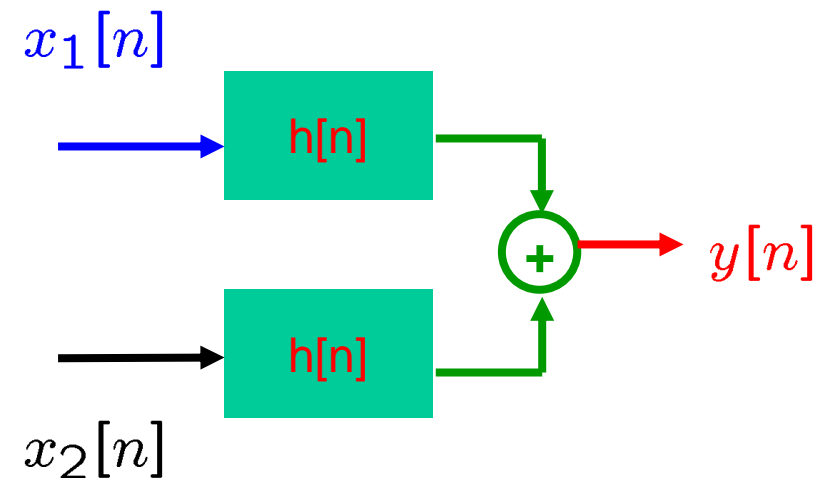
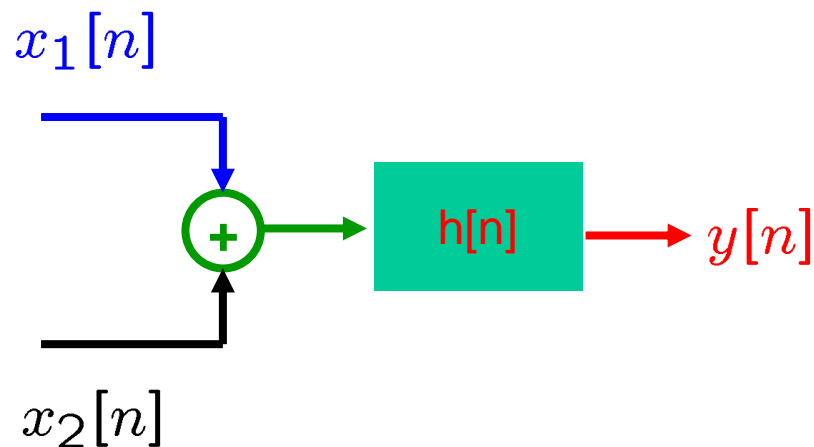
## ■ Distributive Property:

$$x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

$$x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

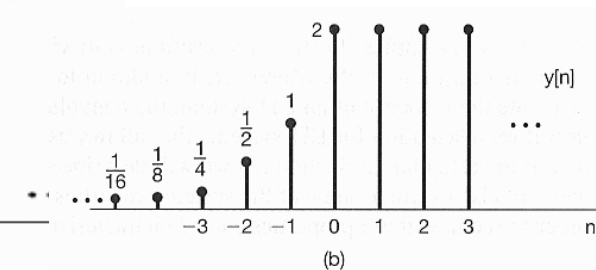
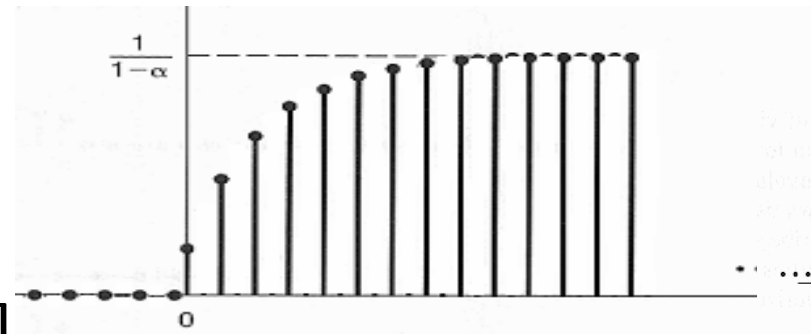
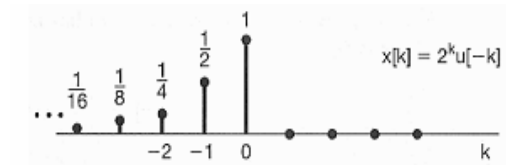
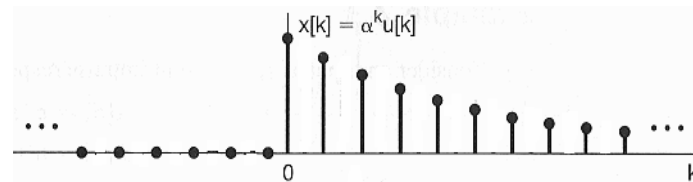
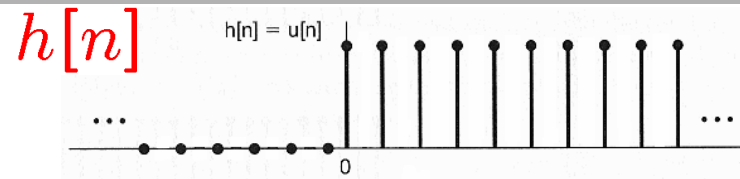
$$(x_1[n] + x_2[n]) * h[n] = x_1[n] * h[n] + x_2[n] * h[n]$$

$$(x_1(t) + x_2(t)) * h(t) = x_1(t) * h(t) + x_2(t) * h(t)$$



## Example 2.10

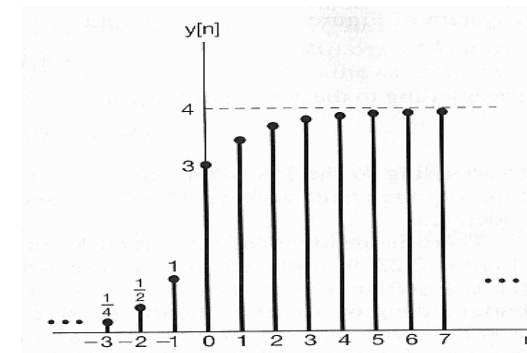
$$x[n] = \left(\frac{1}{2}\right)^n u[n] + 2^n u[-n]$$



$$y[n] = x[n] * h[n]$$

$$= \left( x_1[n] + x_2[n] \right) * h[n]$$

$$= x_1[n] * h[n] + x_2[n] * h[n]$$



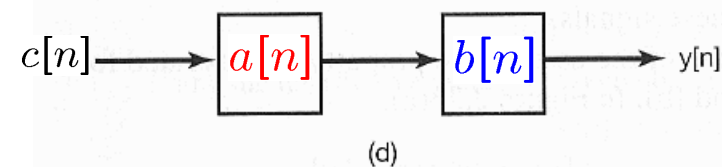
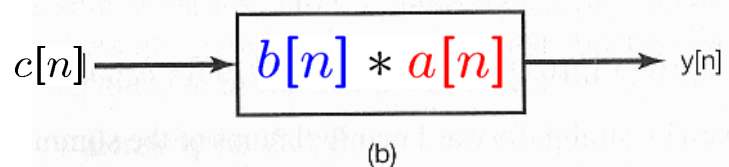
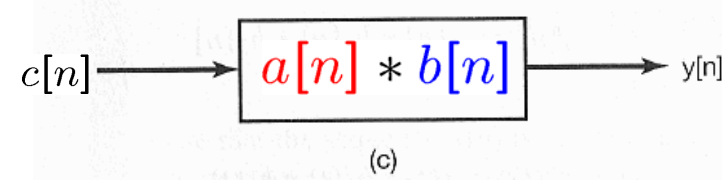
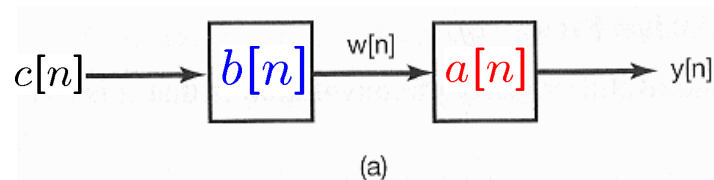
## ■ Associative Property:

$$a[n] * (b[n] * c[n]) = (a[n] * b[n]) * c[n]$$

$$x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

$$x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

$$a(t) * (b(t) * c(t)) = (a(t) * b(t)) * c(t)$$



## ■ Systems with or without memory

### ■ Memoryless systems

- Output depends only on the input at that same time

$$y[n] = (2x[n] - x[n]^2)^2$$

$$y(t) = Rx(t) \quad (\text{resistor})$$

### ■ Systems with memory

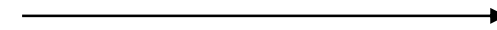
$$y[n] = \sum_{k=-\infty}^n x[k] \quad (\text{accumulator}) \quad y(t) = \frac{1}{C} \int_{-\infty}^t x(\tau) d\tau$$

$$y[n] = x[n-1] \quad (\text{delay})$$

## ■ Memoryless:

- A DT LTI system is memoryless if  $h[n] = 0$  for  $n \neq 0$

$$x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$



- The impulse response:  $h[n] = K\delta[n]$ ,  $K = h[0]$

- The convolution sum:  $y[n] = x[n] * h[n]$

$$= Kx[n]$$

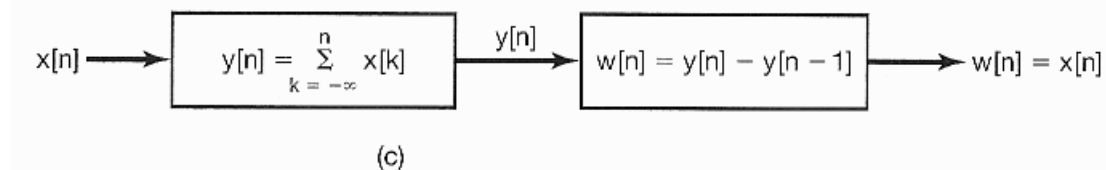
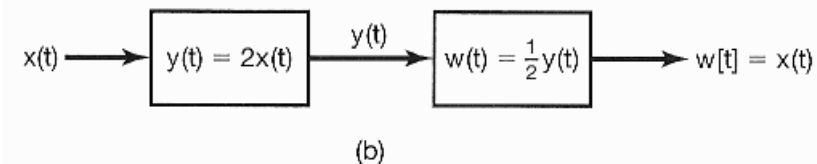
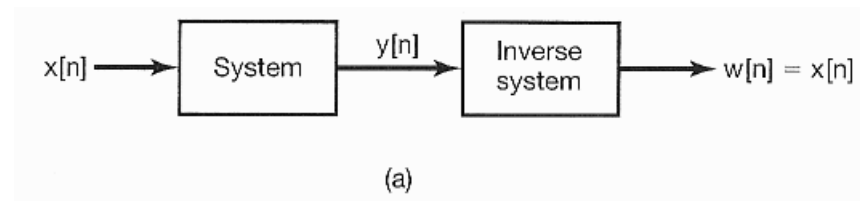
- Similarly, for CT LTI system:  $y(t) = x(t) * h(t) = Kx(t)$



## ■ Invertibility & Inverse Systems

### ■ Invertible systems

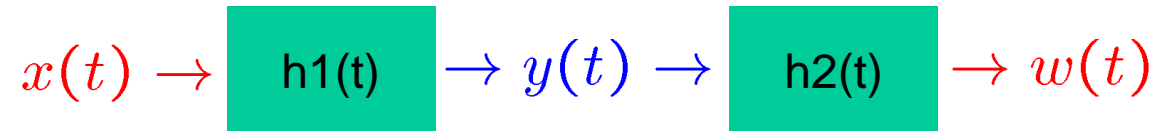
- Distinct inputs lead to distinct outputs



$y(t) = x(t)^2$  is not invertible

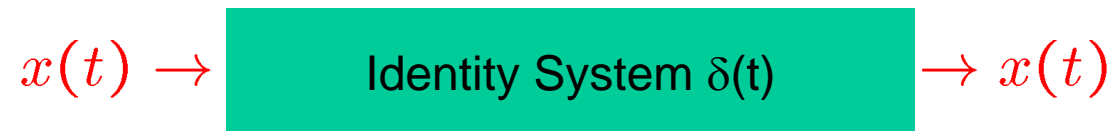
## ■ Invertibility:

$$x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau) h(t - \tau) d\tau$$



$$y(t) = x(t) * h_1(t) \quad w(t) = y(t) * h_2(t)$$

$$\Rightarrow w(t) = x(t) * h_1(t) * h_2(t)$$

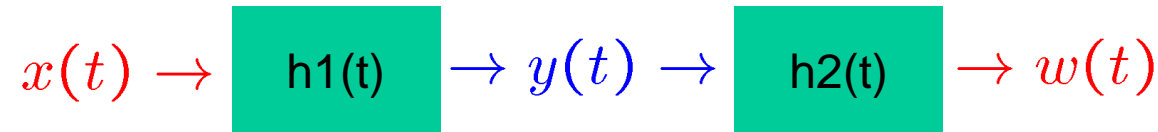


$$x(t) = x(t) * \delta(t)$$

$$\Rightarrow h_2(t) * h_1(t) = \delta(t)$$

■ Example 2.11: Pure time shift

$$x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau) h(t - \tau) d\tau$$



- $y(t) = x(t - t_0)$
- delay if  $t_0 > 0$
- advance if  $t_0 < 0$

$$\Rightarrow h_1(t) = \delta(t - t_0) \Rightarrow x(t) * \delta(t - t_0) = x(t - t_0)$$

$$\bullet w(t) = x(t) = y(t + t_0)$$

$$\Rightarrow h_2(t) = \delta(t + t_0) \Rightarrow y(t) * \delta(t + t_0) = y(t + t_0)$$

$$\Rightarrow h_1(t) * h_2(t) = \delta(t - t_0) * \delta(t + t_0) = \delta(t)$$

■ Example 2.12

$$x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k] h[n-k]$$



$$h_1[n] = u[n]$$

$$\Rightarrow y[n] = \sum_{k=-\infty}^{\infty} x[k] u[n-k] = \sum_{k=-\infty}^n x[k]$$

$\Rightarrow$  a **running-sum** operation

- Its inverse is a **first difference** operation:

$$w[n] = y[n] - y[n-1] \Rightarrow h_2[n] = \delta[n] - \delta[n-1]$$

$$\Rightarrow h_1[n] * h_2[n] = u[n] - u[n-1] = \delta[n]$$

## ■ Causality:

- The **output** of a **causal** system depends only on the **present** and **past** values of the **input** to the system
- Specifically,  $y[n]$  must **not** depend on  $x[k]$ , for  $k > n$

$$x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

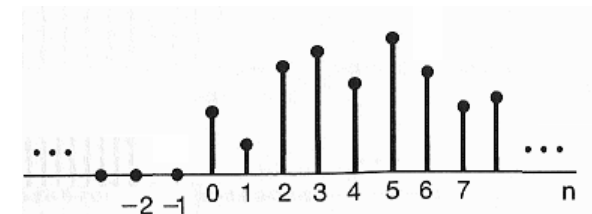
$$x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

$$h[n-k] = 0, \quad \text{for } k > n$$

$$h[n] = 0, \quad \text{for } n < 0$$

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

- It implies that the system is **initially rest**



- A **CT** LTI system is **causal** if

$$h(t) = 0, \quad \text{for } t < 0$$

## ■ Convolution Sum & Integral

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k] h[n-k]$$

$$= \sum_{k=-\infty}^n x[k] h[n-k]$$

$$= \sum_{k=0}^{\infty} h[k] x[n-k]$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau$$

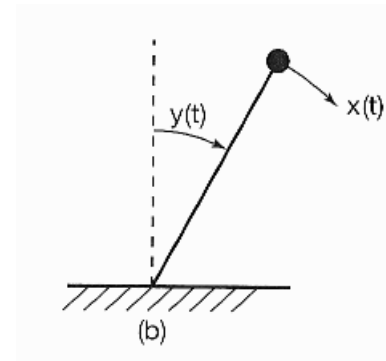
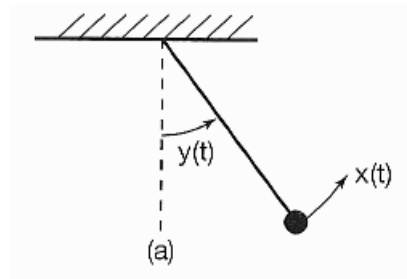
$$= \int_{-\infty}^t x(\tau) h(t-\tau) d\tau$$

$$= \int_0^{\infty} h(\tau) x(t-\tau) d\tau$$

## ■ Stability

### ■ Stable systems

- Small inputs lead to responses that **do not diverge**
- **Every** bounded input excites a **bounded** output
  - Bounded-input bounded-output stable (**BIBO stable**)
  - For all  $|x(t)| < a$ , then  $|y(t)| < b$ , for all  $t$



- **Balance** in a bank account?

$$y[n] = 1.01y[n-1] + x[n]$$

## ■ Stability:

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

- A system is **stable** if every **bounded input** produces a **bounded output**

$$x[n] \rightarrow \text{Stable LTI} \rightarrow y[n]$$

$$|x[n]| < B \quad \text{for all } n$$

$$|y[n]| = \sum_{k=-\infty}^{+\infty} h[k]x[n-k]$$

$$\Rightarrow y[n] \leq \sum_{k=-\infty}^{+\infty} h[k] x[n-k]$$

$$\Rightarrow y[n] \leq \left( \sum_{k=-\infty}^{+\infty} h[k] \right)$$

$$\text{if } \sum_{k=-\infty}^{+\infty} |h[k]| < \infty$$

absolutely summable

then,  $y[n]$  is bounded



## ■ Stability:

- For CT LTI stable system:

$$x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau) h(t - \tau) d\tau$$

$$x(t) \rightarrow \text{Stable LTI} \rightarrow y(t)$$

$$|x(t)| < B \quad \text{for all } t \quad |y(t)| = \left| \int_{-\infty}^{+\infty} h(\tau) x(t - \tau) d\tau \right|$$

$$\Rightarrow |y(t)| \leq \int_{-\infty}^{+\infty} |h(\tau)| |x(t - \tau)| d\tau$$

$$\Rightarrow |y(t)| \leq \left( \int_{-\infty}^{+\infty} |h(\tau)| d\tau \right)$$

if  $\int_{-\infty}^{+\infty} |h(\tau)| d\tau < \infty$  then,  $y(t)$  is bounded  
absolutely integrable

## ■ Example 2.13: Pure time shift

- $y[n] = x[n - n_0] \quad \& \quad h[n] = \delta[n - n_0]$

- $y(t) = x(t - t_0) \quad \& \quad h(t) = \delta(t - t_0)$

$$\Rightarrow \sum_{n=-\infty}^{+\infty} |h[n]| = \sum_{n=-\infty}^{+\infty} |\delta[n - n_0]| = 1 \quad \text{absolutely summable}$$

$$\Rightarrow \int_{-\infty}^{+\infty} |h(\tau)| d\tau = \int_{-\infty}^{+\infty} |\delta(\tau - t_0)| d\tau = 1 \quad \text{absolutely integrable}$$

$\Rightarrow$  A (CT or DT) pure time shift is stable

## ■ Example 2.13: Accumulator

$$\bullet y[n] = \sum_{k=-\infty}^n x[k] \quad \& \quad h[n] = u[n]$$

$$\bullet y(t) = \int_{-\infty}^t x(\tau) d\tau \quad \& \quad h(t) = u(t)$$

$$\Rightarrow \sum_{n=-\infty}^{+\infty} |h[n]| = \sum_{n=0}^{+\infty} |u[n]| = \infty \quad \text{NOT absolutely summable}$$

$$\Rightarrow \int_{-\infty}^{+\infty} |h(\tau)| d\tau = \int_0^{\infty} |u(\tau)| d\tau = \infty \quad \text{NOT absolutely integrable}$$

$\Rightarrow$  A accumulator or integrator is NOT stable

## ■ Unit Step Response:

$$h[n] = \delta[n] * h[n]$$

- For an LTI system, its impulse response is:

$$\delta[n] \rightarrow \boxed{\text{DT LTI}} \rightarrow h[n]$$

$$\delta(t) \rightarrow \boxed{\text{CT LTI}} \rightarrow h(t)$$

- Its unit step response is:

$$u[n] \rightarrow \boxed{\text{DT LTI}} \rightarrow s[n]$$

$$u(t) \rightarrow \boxed{\text{CT LTI}} \rightarrow s(t)$$

$$\Rightarrow s[n] = u[n] * h[n]$$

$$\Rightarrow s(t) = u(t) * h(t)$$

$$= \sum_{k=-\infty}^{+\infty} u[n-k]h[k]$$

$$= \int_{-\infty}^{+\infty} u(t-\tau)h(\tau)d\tau$$

$$= \sum_{k=-\infty}^n h[k]$$

$$= \int_{-\infty}^t h(\tau)d\tau$$

$$\Rightarrow h[n] = s[n] - s[n-1]$$

$$\Rightarrow h(t) = \frac{ds(t)}{dt}$$

## ■ Discrete-Time Linear Time-Invariant Systems

- The convolution sum  $y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k] \quad y[n] = x[n] * h[n]$

## ■ Continuous-Time Linear Time-Invariant Systems

- The convolution integral  $y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau \quad y(t) = x(t) * h(t)$

## ■ Properties of Linear Time-Invariant Systems

1. Commutative property

$$x(t) * h(t) = h(t) * x(t)$$

2. Distributive property

$$x(t) * (h_1(t) + h_2(t)) = x(t) * h_1(t) + x(t) * h_2(t)$$

3. Associative property

$$a(t) * (b(t) * c(t)) = (a(t) * b(t)) * c(t)$$

4. With or without memory

5. Invertibility

6. Causality

$$h[n] = 0 \text{ for } n \neq 0 \quad h(t) = 0, \text{ for } t < 0$$

7. Stability

8. Unit step response

$$h_2(t) * h_1(t) = \delta(t) \quad \text{if } \int_{-\infty}^{+\infty} |h(\tau)|d\tau < \infty$$

## ■ Causal Linear Time-Invariant Systems

Described by Differential & Difference Equations

## ■ Singularity Functions

$$h[n] = \left(\frac{1}{2}\right)^n u[n]$$

## ■ Singularity Functions

- CT unit impulse function is one of singularity functions

$$\delta(t) = u_0(t)$$

$$u(t) = u_{-1}(t)$$

$$\int_{-\infty}^{\infty} \delta(\tau) d\tau = u(t)$$

$$\frac{d}{dt} \delta(t) = u_1(t)$$

$$\int_{-\infty}^t u(\tau) d\tau = u_{-2}(t)$$

$$\frac{d^2}{dt^2} \delta(t) = u_2(t)$$

$$\int_{-\infty}^t \left( \int_{-\infty}^{\tau} u(\sigma) d\sigma \right) d\tau = u_{-3}(t)$$

$$\frac{d^k}{dt^k} \delta(t) = u_k(t)$$

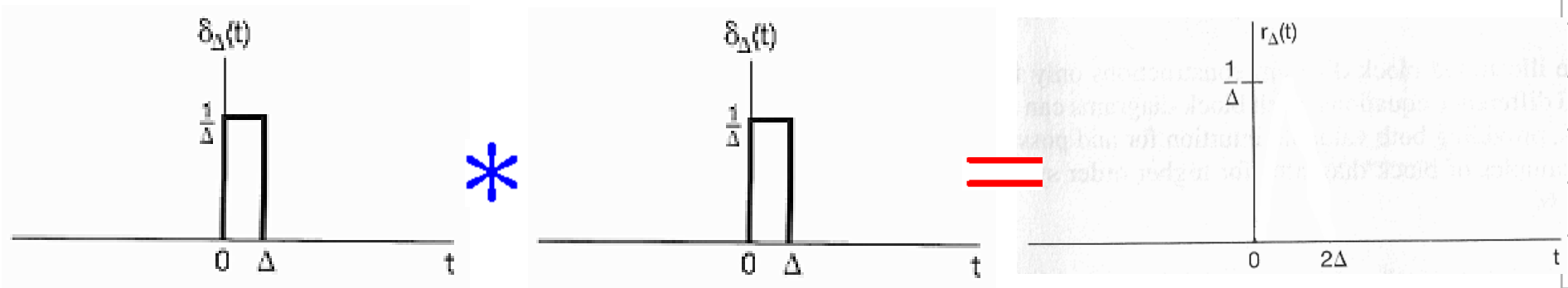
$$\int_{-\infty}^t \cdots \left( \int_{-\infty}^{\tau} u(\sigma) d\sigma \right) \cdots d\tau = u_{-k}(t)$$

## ■ Singularity Functions

$$x(t) = x(t) * \delta(t)$$

$$\delta(t) = \delta(t) * \delta(t)$$

$$r_{\Delta}(t) = \delta_{\Delta}(t) * \delta_{\Delta}(t)$$



$$\lim_{\Delta \rightarrow 0} \delta_{\Delta}(t) = \delta(t)$$

$$\Rightarrow \lim_{\Delta \rightarrow 0} r_{\Delta}(t) = \delta(t)$$

■ Example 2.16  $\frac{d}{dt}y(t) + 2y(t) = x(t)$

with initial-rest condition

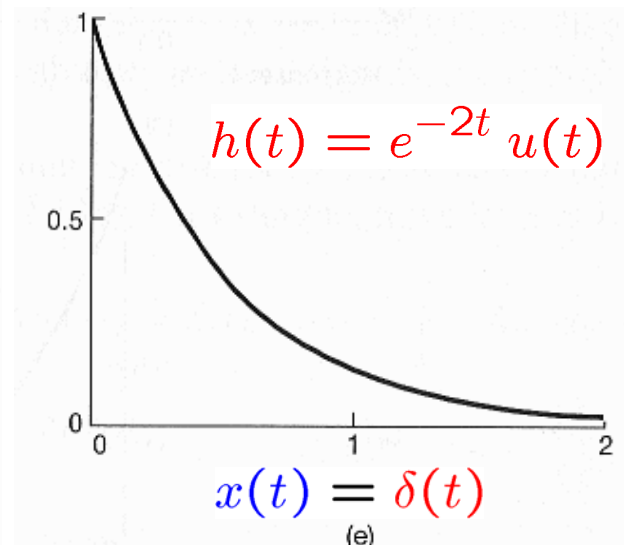
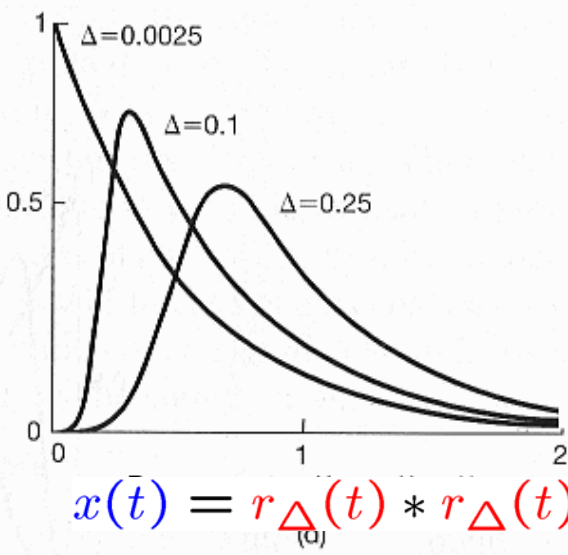
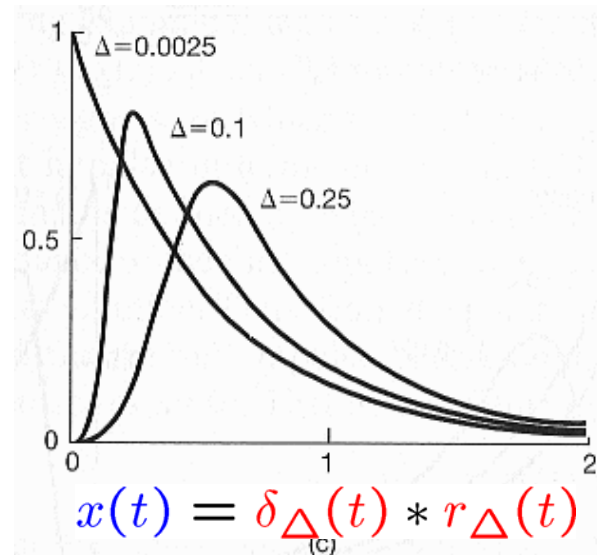
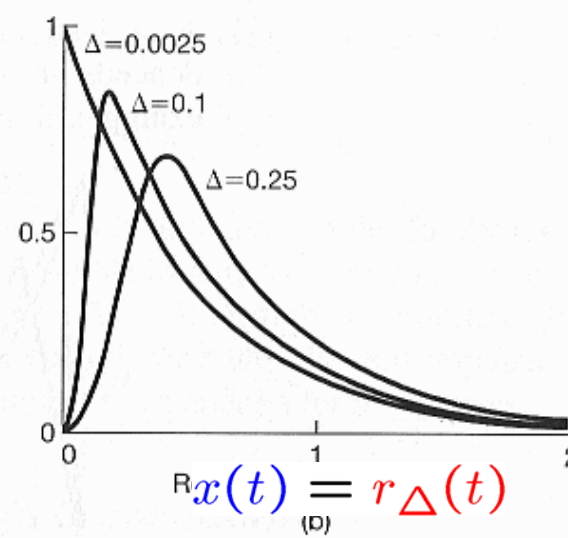
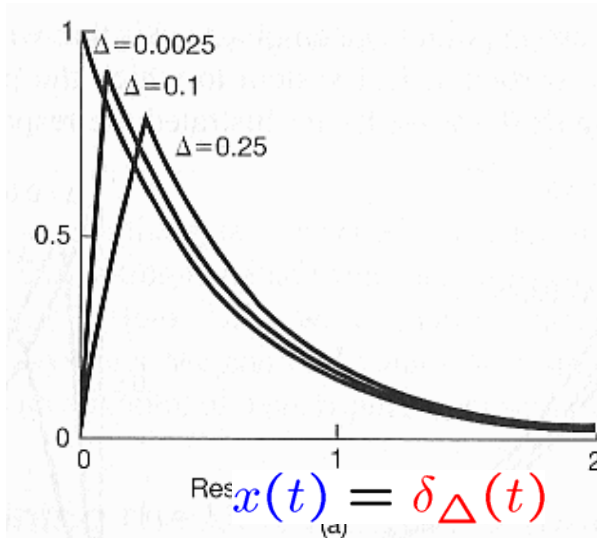
$$x(t) = \delta_{\Delta}(t)$$

$$x(t) = r_{\Delta}(t)$$

$$x(t) = \delta_{\Delta}(t) * r_{\Delta}(t)$$

$$x(t) = r_{\Delta}(t) * r_{\Delta}(t)$$

$$x(t) = \delta(t)$$





■ Example 2.16  $\frac{d}{dt}y(t) + 20y(t) = x(t)$

with initial-rest condition

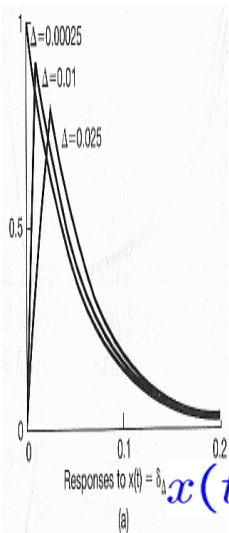
$$x(t) = \delta_{\Delta}(t)$$

$$x(t) = r_{\Delta}(t)$$

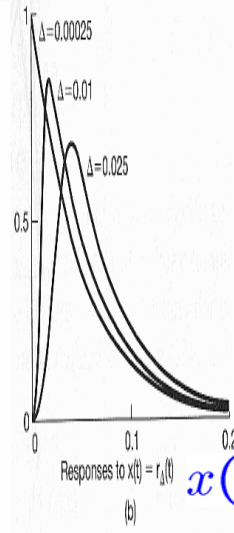
$$x(t) = \delta_{\Delta}(t) * r_{\Delta}(t)$$

$$x(t) = r_{\Delta}(t) * r_{\Delta}(t)$$

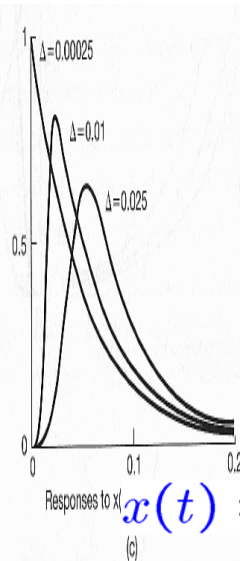
$$x(t) = \delta(t)$$



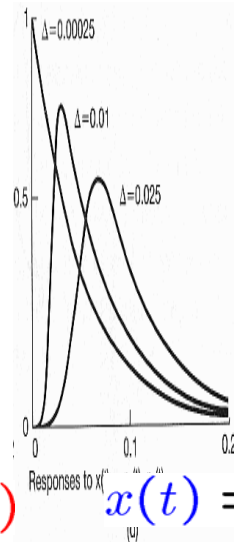
$$x(t) = \delta_{\Delta}(t)$$



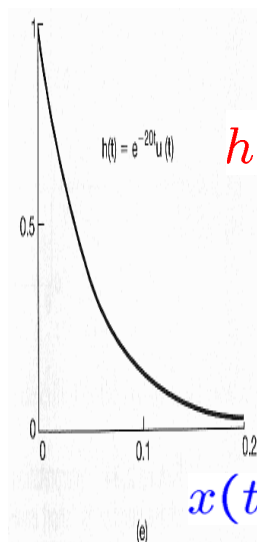
$$x(t) = r_{\Delta}(t)$$



$$x(t) = \delta_{\Delta}(t) * r_{\Delta}(t)$$



$$x(t) = r_{\Delta}(t) * r_{\Delta}(t)$$



$$x(t) = \delta(t)$$

## ■ Defining the Unit Impulse through Convolution:

$$x(t) = x(t) * \delta(t)$$

- Let  $x(t) = 1$ ,

$$1 = x(t) = x(t) * \delta(t) = \delta(t) * x(t)$$

$$= \int_{-\infty}^{\infty} \delta(\tau) x(t - \tau) d\tau = \int_{-\infty}^{\infty} \delta(\tau) d\tau$$

- So that the unit impulse has unit area

## ■ Defining Unit Impulse through Convolution:

- Alternatively, consider an arbitrary signal  $g(t)$ ,

$$g(-t) = g(-t) * \delta(t) = \int_{-\infty}^{\infty} g(\tau - t) \delta(\tau) d\tau$$

$$g(0) = \int_{-\infty}^{\infty} g(\tau) \delta(\tau) d\tau$$

- Define  $x(t - \tau) = g(\tau)$

$$\begin{aligned} x(t) = g(0) &= \int_{-\infty}^{\infty} g(\tau) \delta(\tau) d\tau \\ &= \int_{-\infty}^{\infty} x(t - \tau) \delta(\tau) d\tau = x(t) * \delta(t) \end{aligned}$$

## ■ Defining Unit Impulse through Convolution:

- Consider the signal  $f(t)\delta(t)$

$$\int_{-\infty}^{\infty} g(\tau) f(\tau) \delta(\tau) d\tau = g(0) f(0)$$

- On the other hand, consider the signal  $f(0)\delta(t)$

$$\int_{-\infty}^{\infty} g(\tau) f(0) \delta(\tau) d\tau = g(0) f(0)$$

- Therefore,

$$f(t)\delta(t) = f(0)\delta(t)$$

## ■ Unit Doublets of Derivative Operation:

- A system: Output is the derivative of input

$$y(t) = \frac{d}{dt}x(t)$$

⇒ The unit impulse response of the system  
is the derivative of the unit impulse,  
which is called the unit doublet  $u_1(t)$

- That is, from  $x(t) = x(t) * \delta(t)$ , we have

$$\frac{d}{dt}x(t) = x(t) * u_1(t)$$

## ■ Unit Doublets of Derivative Operation:

- Similarly,

$$\frac{d^2}{dt^2}x(t) = x(t) * u_2(t)$$

- But,

$$\frac{d^2}{dt^2}x(t) = \frac{d}{dt} \left( \frac{d}{dt}x(t) \right) = \left( x(t) * u_1(t) \right) * u_1(t)$$

- Therefore,

$$u_2(t) = u_1(t) * u_1(t)$$

- In general,

$u_k(t)$ ,  $k > 0$ , the  $k$ th derivative of  $\delta(t)$

$$u_k(t) = u_1(t) * \cdots * u_1(t)$$

## ■ Unit Doublets of Integration Operation:

- A system: Output is the **integral** of input

$$y(t) = \int_{-\infty}^{\infty} x(\tau) d\tau$$

- Therefore,

$$u(t) = \int_{-\infty}^{\infty} \delta(\tau) d\tau$$

- Hence, we have

$$x(t) * u(t) = \int_{-\infty}^t x(\tau) d\tau$$

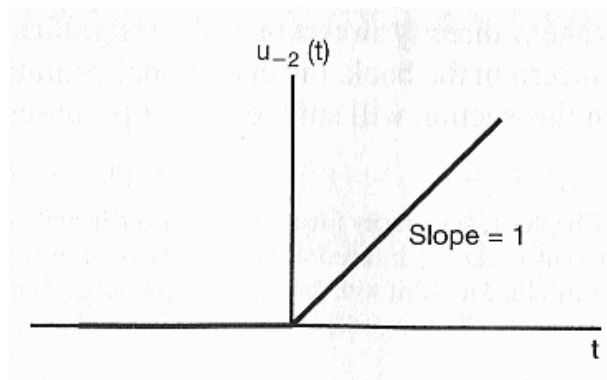
## ■ Unit Doublets of Integration Operation:

- Similarly,

$$u_{-2}(t) = u(t) * u(t) = \int_{-\infty}^t u(\tau) d\tau$$

- That is,

$$u_{-2}(t) = t u(t) \quad \text{the unit ramp function}$$





## ■ Unit Doublets of Integration Operation:

- Moreover,

$$\begin{aligned}x(t) * u_{-2}(t) &= x(t) * u(t) * u(t) \\&= \left( \int_{-\infty}^t x(\sigma) d\sigma \right) * u(t) \\&= \int_{-\infty}^t \left( \int_{-\infty}^{\tau} x(\sigma) d\sigma \right) d\tau\end{aligned}$$

- In general,

$$u_{-k}(t) = u(t) * \cdots * u(t) = \int_{-\infty}^t u_{-(k-1)}(\tau) d\tau$$

$$u_{-k}(t) = \frac{t^{k-1}}{(k-1)!} u(t)$$

## ■ In Summary

$$\delta(t) = u_0(t)$$

$$u(t) = u_{-1}(t)$$

$$u_k(t)$$

$$k > 0,$$

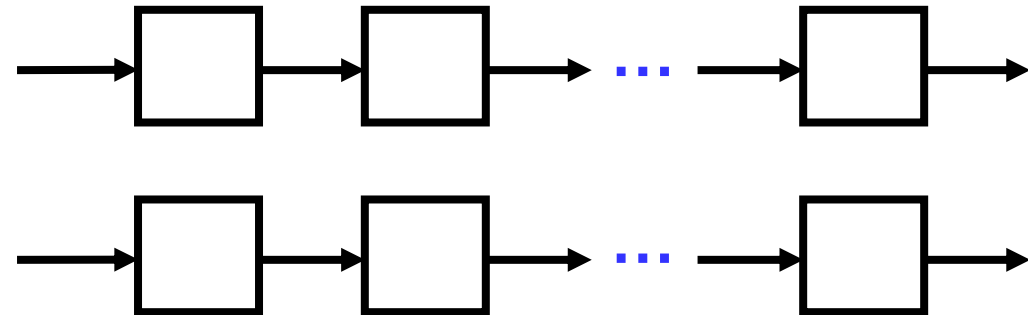
Impulse response of a cascade of  $k$  differentiators

$$k < 0,$$

Impulse response of a cascade of  $|k|$  integrators

$$u(t) * u_1(t) = \delta(t) \quad \text{or,} \quad u_{-1}(t) * u_1(t) = u_0(t)$$

$$\Rightarrow u_k(t) * u_r(t) = u_{k+r}(t)$$



## ■ Discrete-Time Linear Time-Invariant Systems

### • The convolution sum

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k] \quad y[n] = x[n] * h[n]$$

## ■ Continuous-Time Linear Time-Invariant Systems

### • The convolution integral

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau \quad y(t) = x(t) * h(t)$$

## ■ Properties of Linear Time-Invariant Systems

### 1. Commutative property

$$x(t) * h(t) = h(t) * x(t)$$

### 2. Distributive property

$$x(t) * (h_1(t) + h_2(t)) = x(t) * h_1(t) + x(t) * h_2(t)$$

### 3. Associative property

$$a(t) * (b(t) * c(t)) = (a(t) * b(t)) * c(t)$$

### 4. With or without memory

$$h(t) = 0 \text{ for } t \neq 0 \quad h(t) = 0, \text{ for } t < 0$$

### 5. Invertibility

### 6. Causality

### 7. Stability

$$h_2(t) * h_1(t) = \delta(t) \quad \text{if } \int_{-\infty}^{+\infty} |h(\tau)|d\tau < \infty$$

### 8. Unit step response

## ■ Causal Linear Time-Invariant Systems

Described by Differential & Difference Equations

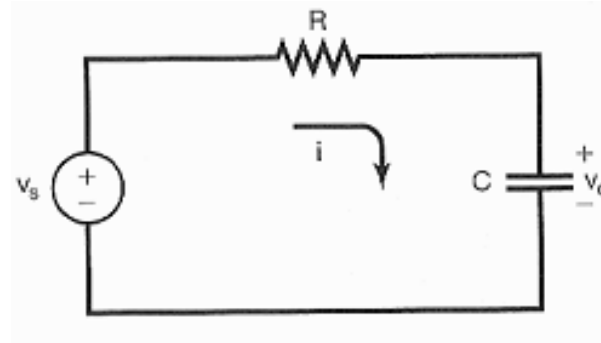
## ■ Singularity Functions

$$u_k(t) * u_r(t) = u_{k+r}(t)$$

## ■ Linear Constant-Coefficient Differential Equations

- e.x., RC circuit

Input signal:  $v_s(t)$



Output signal:  $v_c(t)$

$$\Rightarrow \frac{dv_c(t)}{dt} + \frac{1}{RC}v_c(t) = \frac{1}{RC}v_s(t)$$

$$x(t) \rightarrow \text{RC Circuit} \rightarrow y(t) \Rightarrow \frac{d}{dt}y(t) + a y(t) = b x(t)$$

- Provide an **implicit specification** of the system
- You have learned **how to solve** the equation in **Diff Eqn**

## ■ Linear Constant-Coefficient Differential Equations

- For a general CT LTI system, with N-th order,

$$x(t) \rightarrow \boxed{\text{CT LTI}} \rightarrow y(t)$$

$$a_N \frac{d^N}{dt^N} y(t) + a_{N-1} \frac{d^{N-1}}{dt^{N-1}} y(t) + \cdots + a_1 \frac{d}{dt} y(t) + a_0 y(t)$$

$$= b_M \frac{d^M}{dt^M} x(t) + b_{M-1} \frac{d^{M-1}}{dt^{M-1}} x(t) + \cdots + b_1 \frac{d}{dt} x(t) + b_0 x(t)$$

$$\Rightarrow \sum_{k=0}^N a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^M b_k \frac{d^k}{dt^k} x(t)$$

$$\Rightarrow h(t) = ?$$

## ■ Linear Constant-Coefficient Difference Equations

- For a general DT LTI system, with N-th order,

$$x[n] \rightarrow \boxed{\text{DT LTI}} \rightarrow y[n]$$

$$a_0 y[n] + a_1 y[n-1] + \cdots + a_{N-1} y[n-N+1] + a_N y[n-N]$$

$$= b_0 x[n] + b_1 x[n-1] + \cdots + b_{M-1} x[n-M+1] + b_M x[n-M]$$

$$\Rightarrow \sum_{k=0}^N a_k y[n-k] = \sum_{k=0}^M b_k x[n-k]$$

$$\Rightarrow h[n] = ?$$

## ■ Recursive Equation:

$$a_0 y[n] + a_1 y[n-1] + \cdots + a_{N-1} y[n-N+1] + a_N y[n-N]$$
$$= b_0 x[n] + b_1 x[n-1] + \cdots + b_{M-1} x[n-M+1] + b_M x[n-M]$$

$$\sum_{k=0}^N a_k y[n-k] = \sum_{k=0}^M b_k x[n-k]$$

$$\Rightarrow y[n] = \frac{1}{a_0} \left\{ \sum_{k=0}^M b_k x[n-k] - \sum_{k=1}^N a_k y[n-k] \right\}$$

## Recursive Equation:

- For example, (Example 2.15)



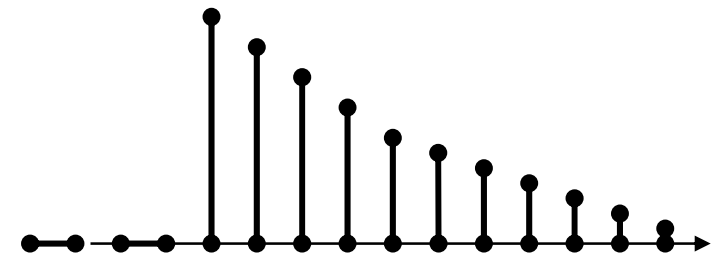
$$y[n] - \frac{1}{2}y[n-1] = x[n]$$

$$y[n] = 0, \quad \text{for } n \leq -1$$

$$x[n] = K \delta[n]$$

$$\Rightarrow \left\{ \begin{array}{ll} y[0] = x[0] + \frac{1}{2}y[-1] & = K \\ y[1] = x[1] + \frac{1}{2}y[0] & = K \frac{1}{2} \\ y[2] = x[2] + \frac{1}{2}y[1] & = K \left(\frac{1}{2}\right)^2 \\ \vdots & \\ y[n] = x[n] + \frac{1}{2}y[n-1] & = K \left(\frac{1}{2}\right)^n \end{array} \right.$$

$$\Rightarrow h[n] = \left(\frac{1}{2}\right)^n u[n]$$



$\Rightarrow$  an Infinite Impulse Response (IIR) system



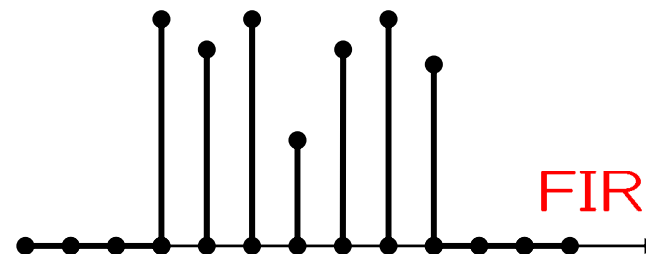
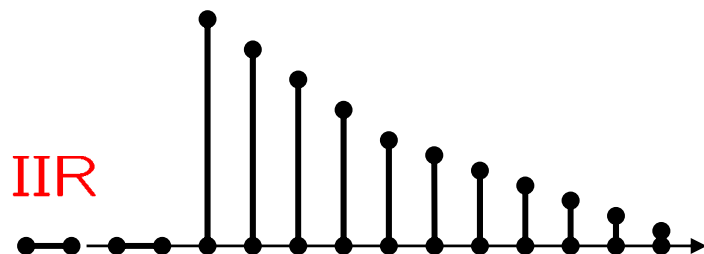
- Nonrecursive Equation:  $\sum_{k=0}^N a_k y[n-k] = \sum_{k=0}^M b_k x[n-k]$
- When  $N = 0$ ,

$$\Rightarrow y[n] = \sum_{k=0}^M \left( \frac{b_k}{a_0} \right) x[n-k]$$

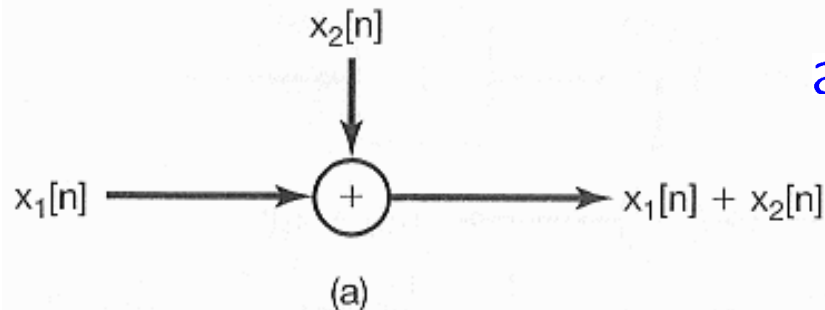
$$y[n] = \sum_{k=-\infty}^{+\infty} h[k] x[n-k]$$

$$\Rightarrow h[n] = \begin{cases} \frac{b_n}{a_0}, & 0 \leq n \leq M \\ 0, & \text{otherwise} \end{cases}$$

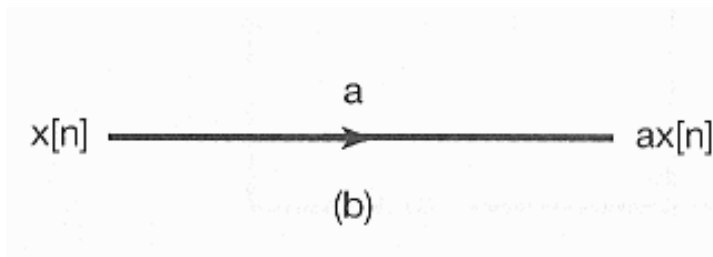
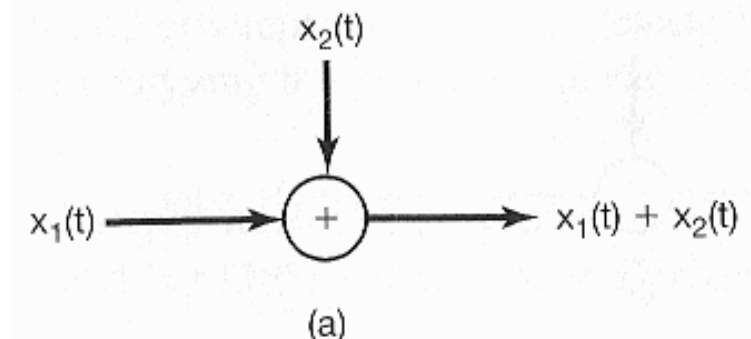
$\Rightarrow$  a **Finite Impulse Response (FIR)** system



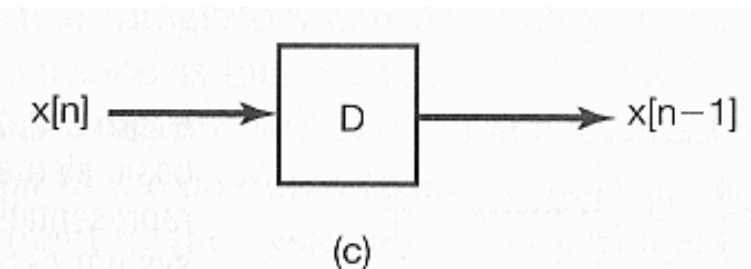
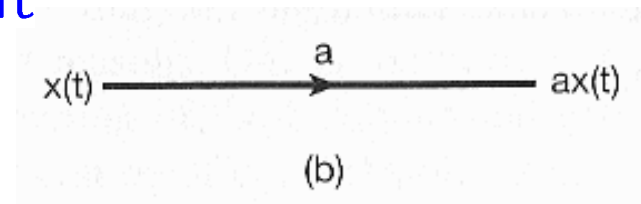
## ■ Block Diagram Representations:



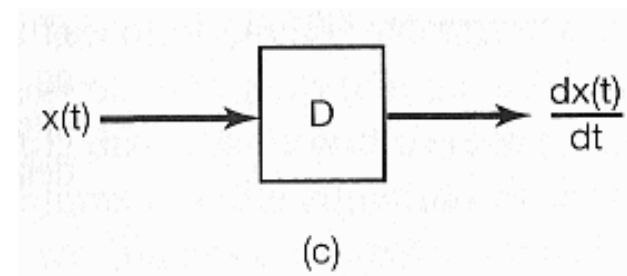
an adder



multiplication  
by a coefficient



a unit delay/  
differentiator



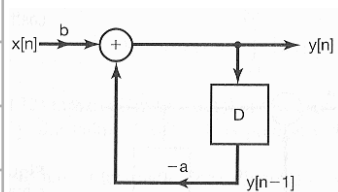
## ■ Block Diagram Representations:

$$y[n] + ay[n-1] = bx[n]$$

$$\frac{d}{dt}y(t) + ay(t) = bx(t)$$

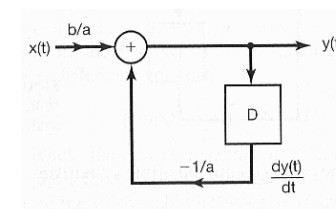
$$y[n] = -ay[n-1] + bx[n]$$

$$y(t) = -\frac{1}{a} \frac{d}{dt}y(t) + \frac{b}{a}x(t)$$



$$D \iff z^{-1}$$

$$D \iff s$$

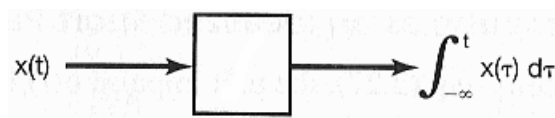


## ■ Block Diagram Representations:

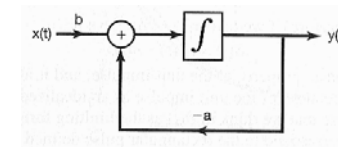
$$\frac{d}{dt}y(t) = bx(t) - ay(t)$$

$$\Rightarrow y(t) = \int_{-\infty}^t [bx(\tau) - ay(\tau)] d\tau$$

$$\Rightarrow y(t) = y(t_0) + \int_{t_0}^t [bx(\tau) - ay(\tau)] d\tau$$



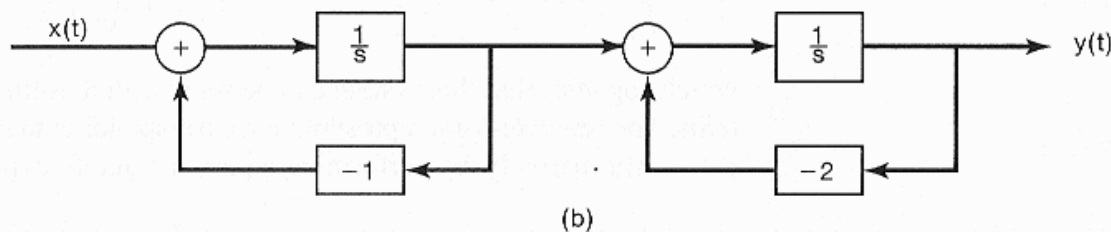
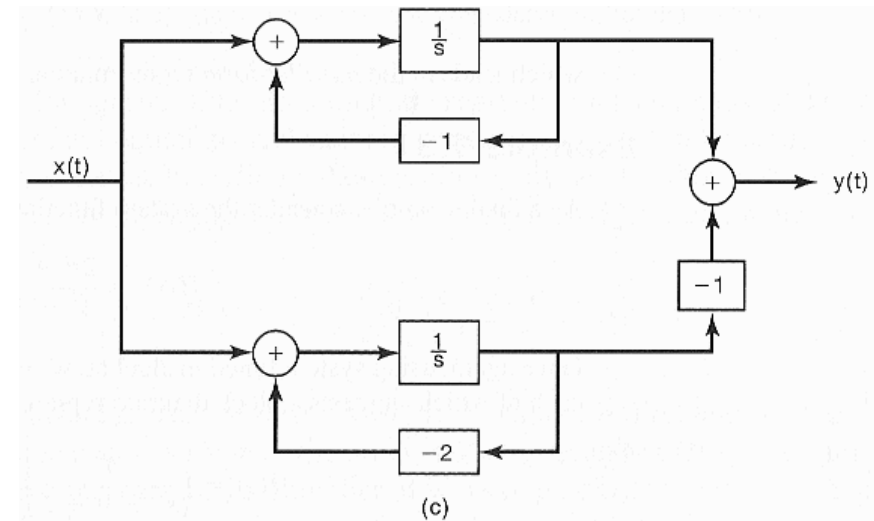
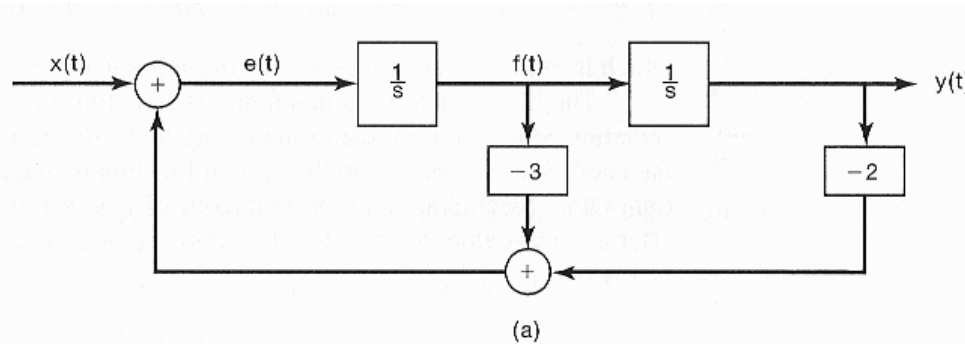
$$\int \Longleftrightarrow \frac{1}{s}$$



## ■ Block Diagram Representations:

$$\frac{d^2}{dt^2}y(t) + 3\frac{d}{dt}y(t) + 2y(t) = x(t)$$

$$\int \Longleftrightarrow \frac{1}{s}$$

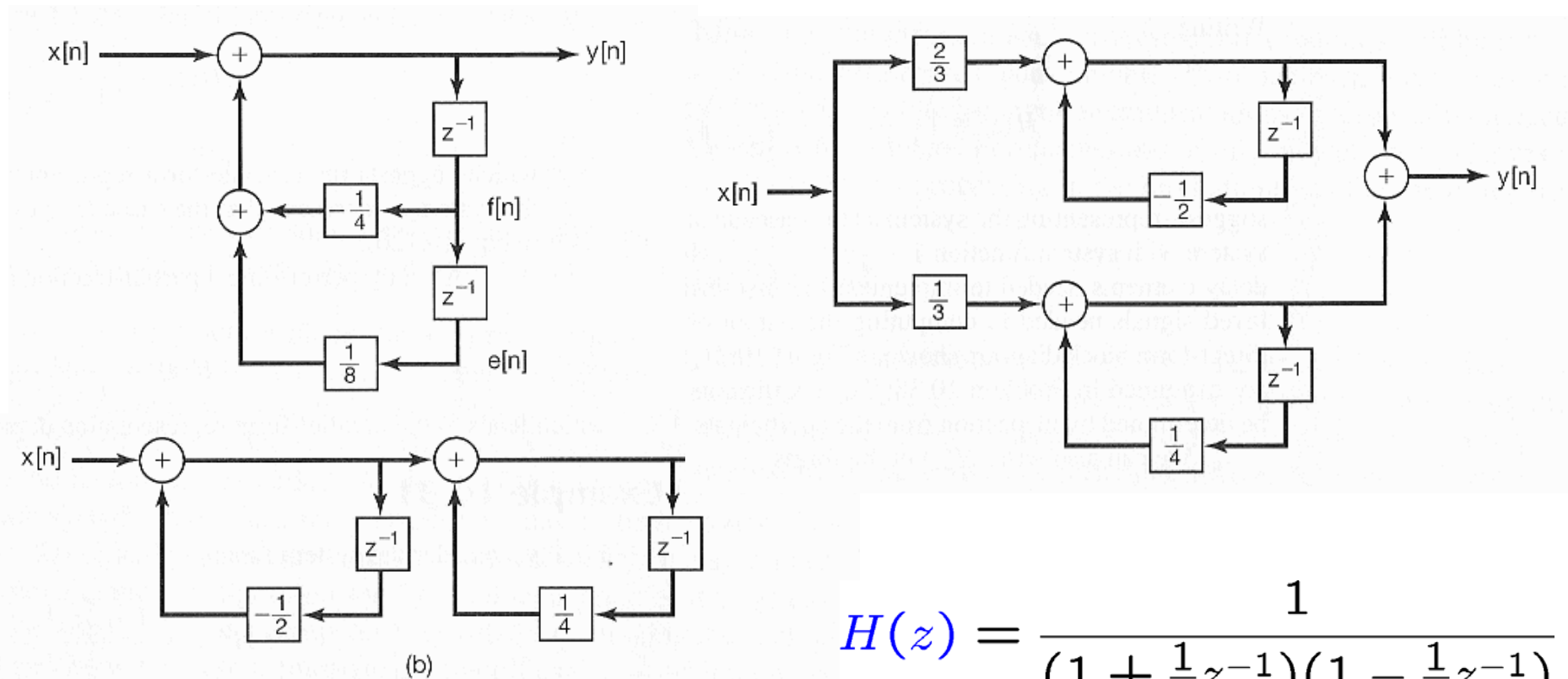


■ Example 9.30 (pp.711)

$$H(s) = \frac{1}{(s+1)(s+2)}$$

## Block Diagram Representations:

$$y[n] + \frac{1}{4}y[n-1] - \frac{1}{8}y[n-2] = x[n]$$



$$H(z) = \frac{1}{(1 + \frac{1}{2}z^{-1})(1 - \frac{1}{4}z^{-1})}$$

Example 10.30 (pp.786)

## Discrete-Time Linear Time-Invariant Systems

- The convolution sum

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

$$y[n] = x[n] * h[n]$$

## Continuous-Time Linear Time-Invariant Systems

- The convolution integral

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

$$y(t) = x(t) * h(t)$$

## Properties of Linear Time-Invariant Systems

- Commutative property
- Distributive property
- Associative property
- With or without memory
- Invertibility
- Causality
- Stability
- Unit step response

$$x(t) * h(t) = h(t) * x(t)$$

$$x(t) * (h_1(t) + h_2(t)) = x(t) * h_1(t) + x(t) * h_2(t)$$

$$a(t) * (b(t) * c(t)) = (a(t) * b(t)) * c(t)$$

$$h[n] = 0 \text{ for } n \neq 0$$

$$h(t) = 0, \text{ for } t < 0$$

$$h_2(t) * h_1(t) = \delta(t) \quad \text{if } \int_{-\infty}^{+\infty} |h(\tau)|d\tau < \infty$$

## Causal Linear Time-Invariant Systems

Described by Differential & Difference Equations

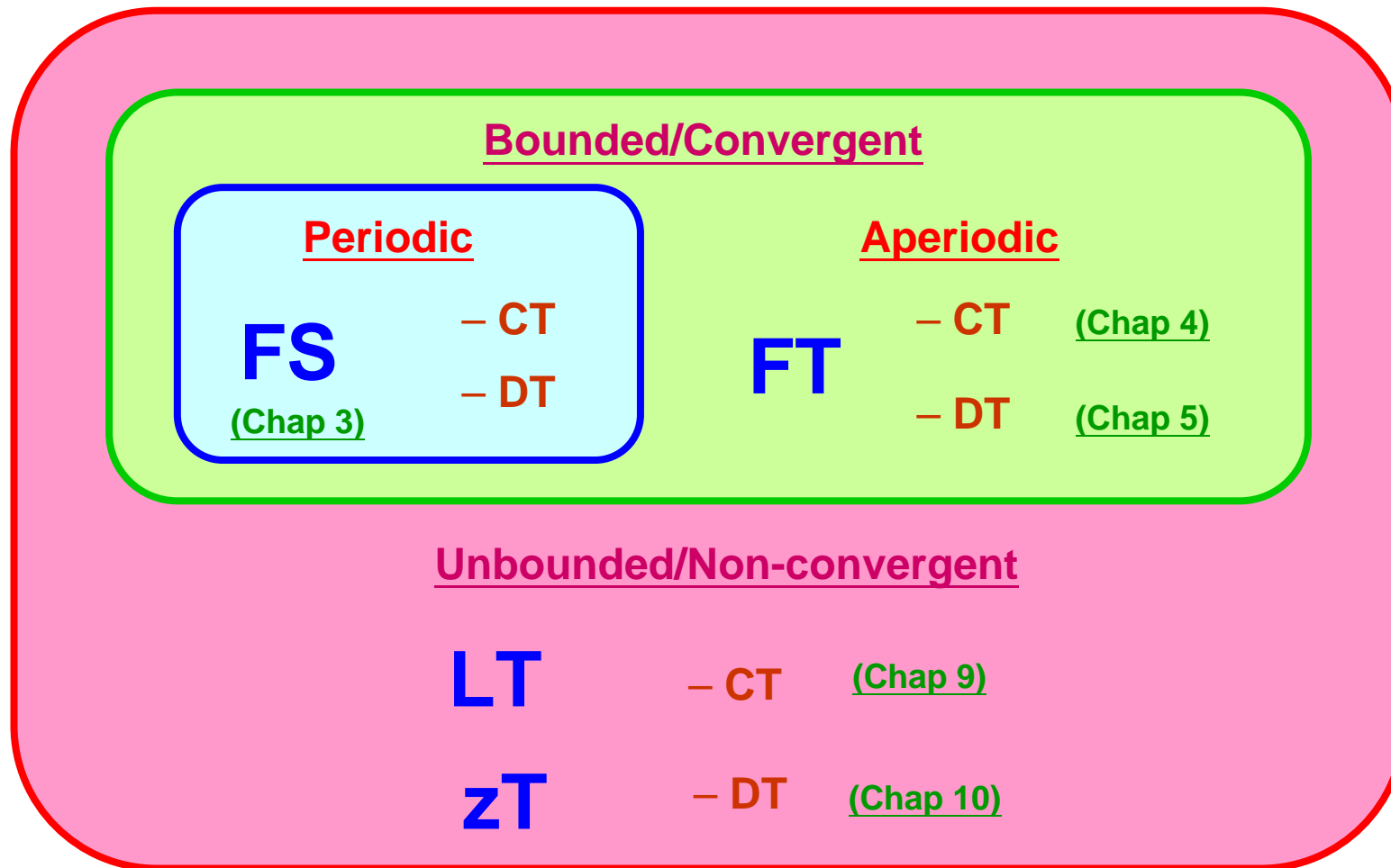
$$h[n] = \left(\frac{1}{2}\right)^n u[n]$$

## Singularity Functions

$$u_k(t) * u_r(t) = u_{k+r}(t)$$

Signals & Systems [\(Chap 1\)](#)

LTI & Convolution [\(Chap 2\)](#)



Time-Frequency [\(Chap 6\)](#)

Communication [\(Chap 8\)](#)

CT-DT [\(Chap 7\)](#)

Control [\(Chap 11\)](#)