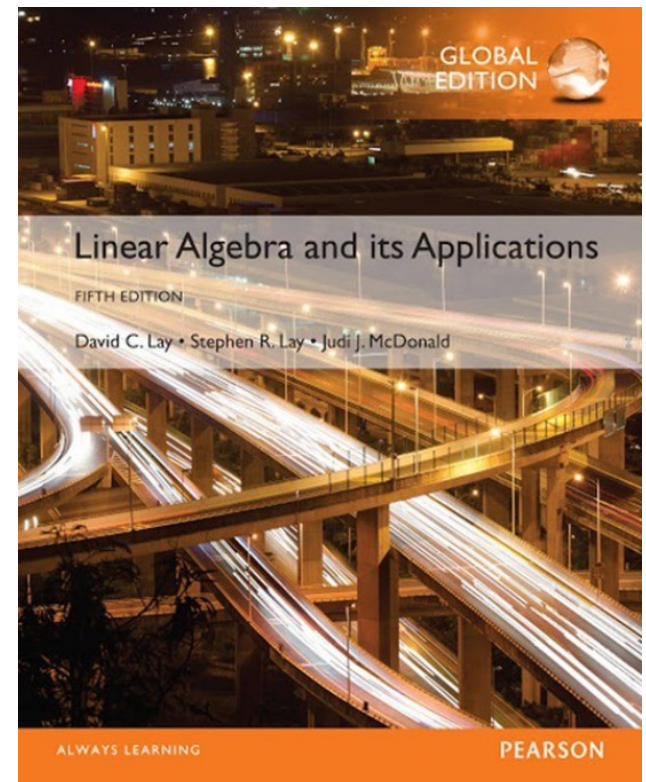


2

Matrix Algebra

2.5

MATRIX FACTORIZATIONS



MATRIX FACTORIZATIONS

- A *factorization* of a matrix A is an equation that expresses A as a product of two or more matrices.
- Whereas matrix multiplication involves a *synthesis* of data (combining the effects of two or more linear transformations into a single matrix), matrix factorization is an *analysis* of data.

THE LU FACTORIZATION

Suppose that A is an $m \times n$ matrix that can be reduced to an echelon form, using only **row replacements** that add a multiple of one row to another **below** it.

- Then A can be written in the form $A = LU$, where L is an $m \times m$ lower triangular matrix with 1's on the diagonal and U is an $m \times n$ echelon form of A .
- For instance, see Fig. 1 below. Such a factorization is called an **LU factorization** of A . The matrix L is invertible and is called a unit lower triangular matrix.

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ * & 1 & 0 & 0 \\ * & * & 1 & 0 \\ * & * & * & 1 \end{bmatrix} \begin{bmatrix} \blacksquare & * & * & * & * \\ 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

L
 U

THE LU FACTORIZATION

- Before studying how to construct L and U , we should look at why they are so useful.
- When $A = LU$, the equation $Ax = b$ can be written as $L(Ux) = b$.
- Writing y for Ux , we can find x by solving the pair of equations

$$\begin{array}{l} Ly = b \\ Ux = y \end{array}$$

- First solve $Ly = b$ for y , and then solve $Ux = y$ for x .
- See Fig. 2 on the next slide. Each equation is easy to solve because L and U are triangular.

THE LU FACTORIZATION

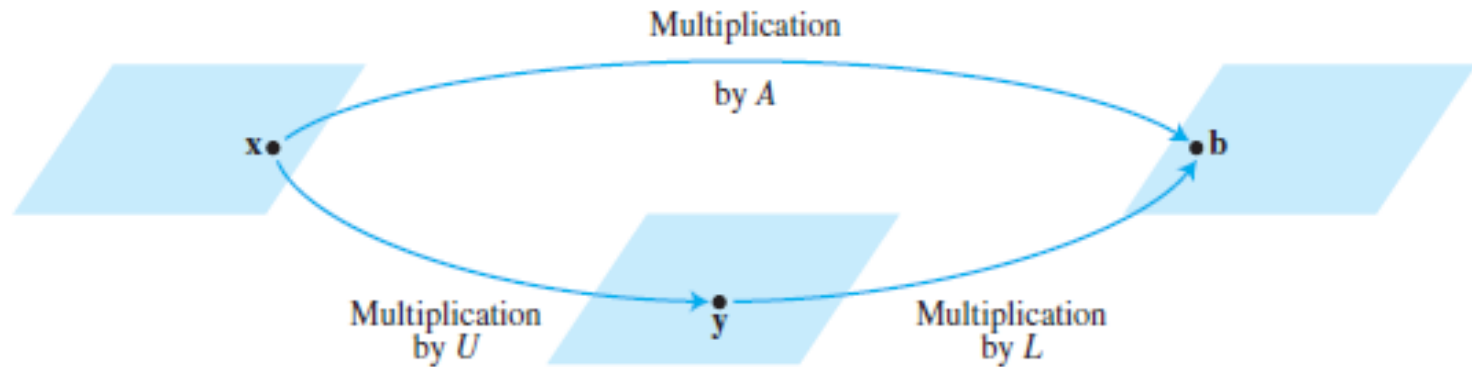


FIGURE 2 Factorization of the mapping $x \mapsto Ax$.

Each equation is easy to solve because L and U are **triangular**

AN LU FACTORIZATION ALGORITHM

- Suppose A can be reduced to an echelon form U using only row replacements that add a multiple of one row to another below it.

- In this case, there exist unit lower triangular elementary matrices E_1, \dots, E_p such that

$$E_p \dots E_1 A = U$$

- Then (3)

$$A = (E_p \dots E_1)^{-1} U = LU$$

- where

$$L = (E_p \dots E_1)^{-1}. \quad (4)$$

- It can be shown that products and inverses of unit lower triangular matrices are also unit lower triangular. Thus L is unit lower triangular.

AN LU FACTORIZATION ALGORITHM

- Note that row operations in equation (3), which reduce A to U , also reduce the L in equation (4) to I , because $E_p \dots E_1 L = (E_p \dots E_1)(E_p \dots E_1)^{-1} = I$. This observation is the key to *constructing* L .

Algorithm for an LU Factorization

1. Reduce A to an echelon form U by a sequence of row replacement operations, if possible.
2. Place entries in L such that the *same sequence of row operations* reduces L to I .

AN LU FACTORIZATION ALGORITHM

- Step 1 is not always possible, but when it is, the argument above shows that an LU factorization exists.
- Example 2 on the followings slides will show how to implement step 2. By construction, L will satisfy

$$(E_p \dots E_1)L = I$$

- using the same E_p, \dots, E_1 as in equation (3). Thus L will be invertible, by the Invertible Matrix Theorem, with $(E_p \dots E_1) = L^{-1}$. From (3), $L^{-1}A = U$, and $A = LU$. So step 2 will produce an acceptable L .

AN LU FACTORIZATION ALGORITHM

- **Example 2** Find an LU factorization of

$$A = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{bmatrix}$$

- **Solution** Since A has four rows, L should be 4×4 . The first column of L is the first column of A divided by the top pivot entry:

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & & 1 & 0 \\ -3 & & & 1 \end{bmatrix}$$

AN LU FACTORIZATION ALGORITHM

- **Example 2** Find an LU factorization of

$$A = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{bmatrix}$$

A_{11} points to the first row, first column.
 A_{21} points to the second row, first column.

$$A_{21} + k A_{11} = 0$$

$$k = -A_{21} / A_{11}$$

- **Solution** Since A has four rows, L should be 4×4 . The first column of L is the first column of A divided by the top pivot entry:

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ -3 & 0 & 0 & 1 \end{bmatrix}$$

L_{11} points to the first row, first column.
 L_{21} points to the second row, first column.

$$L_{21} + k L_{11} = 0$$

$$L_{21} + k = 0$$

$$L_{21} = -k = A_{21}/A_{11}$$

AN LU FACTORIZATION ALGORITHM

- Compare the first columns of A and L . *The row operations that create zeros in the first column of A will also create zeros in the first column of L .*
- To make this same correspondence of row operations on A hold for the rest of L , watch a row reduction of A to an echelon form U . That is, *highlight the entries* in each matrix that are used to determine the sequence of row operations that transform A onto U .

$$\begin{aligned}
 A = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{bmatrix} &\sim \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & -9 & -3 & -4 & 10 \\ 0 & 12 & 4 & 12 & -5 \end{bmatrix} = A_1 \\
 &\sim A_2 = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 4 & 7 \end{bmatrix} \sim \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix} = U
 \end{aligned} \tag{5}$$

AN LU FACTORIZATION ALGORITHM

- The highlighted entries above determine the row reduction of A to U . At each pivot column, divide the highlighted entries by the pivot and place the result onto L :

$$\begin{array}{cccc}
 \begin{bmatrix} 2 \\ -4 \\ 2 \\ -6 \end{bmatrix} & \begin{bmatrix} 3 \\ -9 \\ 12 \end{bmatrix} & \begin{bmatrix} 2 \\ 4 \end{bmatrix} & [5] \\
 \div 2 & \div 3 & \div 2 & \div 5 \\
 \downarrow & \downarrow & \downarrow & \downarrow \\
 \begin{bmatrix} 1 & & & \\ -2 & 1 & & \\ 1 & -3 & 1 & \\ -3 & 4 & 2 & 1 \end{bmatrix} & \text{and } L = & \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & -3 & 1 & 0 \\ -3 & 4 & 2 & 1 \end{bmatrix}
 \end{array}$$

- An easy calculation verifies that this L and U satisfy $LU = A$.