1)

a) X orthogonal set spans $\mathbb{R}^n$.

b) ✓ $\langle Ax, Ay \rangle = x^t A^t A y \stackrel{!}{=} x^t y$

c) ✓ $\boxed{\dfrac{y\,(cv)}{(cv)(cv)}\, cv = \dfrac{c^2(y.v)}{c^2(v.v)} v = \mathrm{proj}_v\, y}$

d) ✓ $Q^T = Q^{-1}$

e) ✓ if $U = \{u_1 — u_n\}$ is Basis for w

$z = y - \hat{y}$ $z.u_1 = 0$

$z.u_2 = 0$

$\vdots$

$z.u_n = 0$

$\Rightarrow y - \mathrm{proj}\, y \longrightarrow$ orthogonal

f) X $y = \hat{y}_1 + z_1$

$\hat{y}_1 + z_1 = \hat{y} + z \Rightarrow \hat{y} - \hat{y}_1 = z_1 - z$

$v = \hat{y} - \hat{y}_1 \Rightarrow \langle v, v \rangle = 0 \Rightarrow v = 0 \Rightarrow \hat{y}_1 = \hat{y}$

y) ✓ $$y = \hat{y} + \underbrace{z}_{0} \Rightarrow y = \hat{y}$$ ✓

2)

$$\|au + bv\| \leq \underbrace{\|au\|}_{\|} + \underbrace{\|bv\|}_{\|}$$

$$\Rightarrow \|bu + av\| \leq \|av\| + \|bu\|$$

$$\left.\begin{aligned} &\Rightarrow \|bu + av\| \leq \|av\| + \|bv\| \\ &\|au + bv\| \leq \|av\| + \|bv\| \end{aligned}\right\}$$

$$\Rightarrow \|bu + av\| = \|au + bv\|$$

y=5

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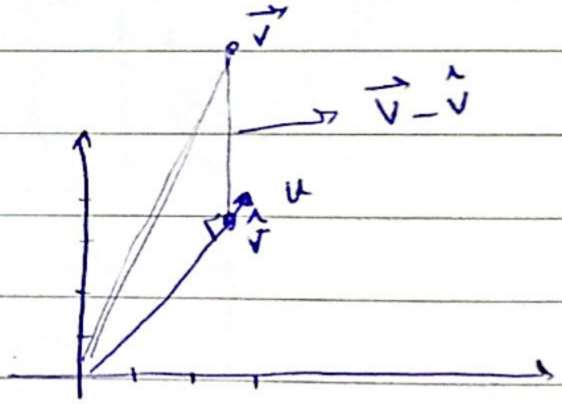
3) $A = [2q_1 \;.\; 3q_2 \;\; 5q_3]$

$M = [q_1 \;\; q_2 \;\; q_3] \Rightarrow \det M = \pm 1$

$MM^T = I \rightarrow \det MM^T = 1 \Rightarrow \det M = \pm 1$

thus, if we scale a column of a matrix, det of that matrix will scale.

$\Rightarrow \det A = 2 \times 3 \times 5 (\pm 1) = \pm 30$



4) $\vec{u} = (3, 4)$

$\|\hat{v}\| = 1$

$\|\vec{v}\| = \alpha = \sqrt{x^2 + y^2}$

$\|\vec{v} - \hat{v}\| = \|v\| + \|\hat{v}\| - 2\|v\hat{v}u\|$

$\sqrt{\|\vec{v} - \hat{v}\|} = x^2 + y^2 + 1 - 2\left(\frac{4x + 3y}{5}\right) + 1$

$\Rightarrow \cancel{x^2 + y^2} = \cancel{x^2 + y^2} + 1 - 2\,\frac{4x + 3y}{5} + 1$

$\Rightarrow \frac{8}{5}x + \frac{6}{5}y = 1 + 1 = 2 \rightarrow \frac{4}{5}x + \frac{3}{5}y = 1$

$\Rightarrow 5 = 4x + 3y \Rightarrow V = \left[\begin{matrix} x \\ y \end{matrix}\right] \Big|\; x, y \in 4x + 3y = 5$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \end{bmatrix}$$

5) $v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$

$$v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{2}{3} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} \frac{2}{3} \\ 0 \\ \frac{2}{3} \\ \frac{2}{3} \end{bmatrix} = \begin{bmatrix} -\frac{2}{3} \\ 1 \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} \qquad v_2' = \begin{bmatrix} -2 \\ 3 \\ 1 \\ 1 \end{bmatrix}$$

$$u_1 = \frac{v_1}{\|v_1\|} = \frac{v_1}{\sqrt{3}} = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ 0 \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$u_2 = \frac{v_2'}{\|v_2'\|} = \frac{v_2'}{15} = \begin{bmatrix} -2/\sqrt{15} \\ 3/\sqrt{15} \\ 1/\sqrt{15} \\ 1/\sqrt{15} \end{bmatrix}$$

$\{u_1, u_2\}$ orthogonalisBasis

6)

$$v_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} 2 \\ 1 \\ 4 \\ -4 \\ 2 \end{bmatrix} - \frac{\overset{-5}{\cancel{2-1-4-4+2}}}{\underset{5}{1+1+1+1+1}} v_1 = x_2 + v_1 = \begin{bmatrix} 3 \\ 0 \\ 3 \\ -3 \\ 3 \end{bmatrix}$$

$$v_3 = x_3 - \text{proj}_{v_1} x_3 - \text{proj}_{v_2} x_3 = \begin{bmatrix} 5 \\ -4 \\ -3 \\ 7 \\ 1 \end{bmatrix} - \frac{5+4+3+7+1}{5} v_1 - \frac{10-4-12-28+2}{4+1+16+16+4} v_2$$

$$= \begin{bmatrix} 2 \\ 0 \\ 2 \\ 2 \\ -2 \end{bmatrix}$$

$$u_1 = \frac{1}{\sqrt{5}} v_1 \qquad u_2 = \frac{1}{6} v_2 \qquad u_3 = \frac{1}{4} v_3$$

$$\Rightarrow Q = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{1}{2} & 1/2 \\ \frac{-1}{\sqrt{5}} & 0 & 0 \\ -1/\sqrt{5} & 1/2 & 1/2 \\ 1/\sqrt{5} & -1/2 & 1/2 \\ 1/\sqrt{5} & 1/2 & -1/2 \end{bmatrix}$$

$$R = Q^T A = \begin{bmatrix} 1/\sqrt{5} & -1/\sqrt{5} & -1/\sqrt{5} & 1/\sqrt{5} & 1/\sqrt{5} \\ 1/2 & 0 & 1/2 & -1/2 & 1/2 \\ 1/2 & 0 & 1/2 & 1/2 & -1/2 \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & 2 & 5 \\ -1 & 1 & -4 \\ -1 & 4 & -3 \\ 1 & -4 & 7 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} \sqrt{5} & -\sqrt{5} & 4\sqrt{5} \\ 0 & 0 & -2 \\ 0 & 0 & 4 \end{bmatrix}$$

7) $z = ax + by + c$

x	0	1	0	1	0
y	0	1	1	0	-1
z	1.1	2	-.1	3	2

$+ \quad + c = 1.1$

$a + b + c = 2$

$+ b + c = -.1$

$a + \quad + c = 3$

$- b + c = 2$

$$\underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a \\ b \\ c \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 1.1 \\ 2 \\ -.1 \\ 3 \\ 2 \end{bmatrix}}_{b}$$

$Ax = b$ incostistant

=> least squares

$$A^T A x = A^T b \Rightarrow (A^T A)^{-1} A^T b = X$$

$$A^T A = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix} =$$

$$= \underbrace{\begin{bmatrix} 2 & 1 & 2 \\ 0 & 3 & 1 \\ 3 & 1 & 5 \end{bmatrix}}_{M}$$

$$[M \; I] \sim [I \; M^{-1}]$$

$$\Rightarrow \begin{bmatrix} 2 & 1 & 2 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 & 1 & 0 \\ 3 & 1 & 5 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & \frac{14}{13} & \frac{-3}{13} & \frac{-5}{13} \\ 0 & 1 & 0 & 3/13 & 4/13 & -2/13 \\ 0 & 0 & 1 & -9/13 & \frac{1}{13} & 6/13 \end{bmatrix}$$

$$\Rightarrow (A^TA)^{-1} = \begin{bmatrix} 14/13 & -3/13 & -5/13 \\ 3/13 & 4/13 & -2/13 \\ -9/13 & 1/13 & 6/13 \end{bmatrix}$$

$$A^Tb = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1.01 \\ 2 \\ -0.1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ -.1 \\ 8 \end{bmatrix}$$

$$(A^TA)^{-1} A^Tb = \begin{bmatrix} 14/13 & -3/13 & -5/13 \\ 3/13 & 4/13 & -2/13 \\ -9/13 & 1/13 & 6/13 \end{bmatrix} \begin{bmatrix} 5 \\ -0.1 \\ 8 \end{bmatrix}$$

$$= \begin{bmatrix} -1097/130 \\ -7/65 \\ 29/130 \end{bmatrix} = X$$

$$\Rightarrow a = \frac{-1097}{130} \qquad b = \frac{-7}{65} \qquad c = \frac{29}{130}$$

8) $\text{Row } A^T = \text{null} A \Rightarrow \text{Row } A^T = \{y \in \mathbb{R}^n : x \cdot y = 0$ for all $x \in \text{Row } A\}$

suppose $x \in \text{null} A$ and $y \in \text{Row } A$

$Ax = 0$

$\Rightarrow (Ax)^T = 0 \Rightarrow x^T A^T = 0$

$x^T (A^T x) = 0$

$x^T y = 0$

$x \cdot y = 0$

Thus any element of null A is orthogonal to any element of Row A

now let $Ax = b$ be consistent and $x = p + u$ where $p \in \text{Row } A$ and $u \in \text{null } A$

$Ap = A(x - u)$

$= Ax - Au$

$= b - 0$

$= b$

Thus Ax has a solution for all b.

let p and q are the solutions of $Ax = b$ where p and q $\in$ Row A,

$A(p - q) = Ap - Aq = b - b = 0$

Thus $p-q$ is in the null A also $p-q$ is in Row A since $\text{null} A \cap \text{Row } A = \{0\}$ Thus $p - q = 0 \Rightarrow p = q$