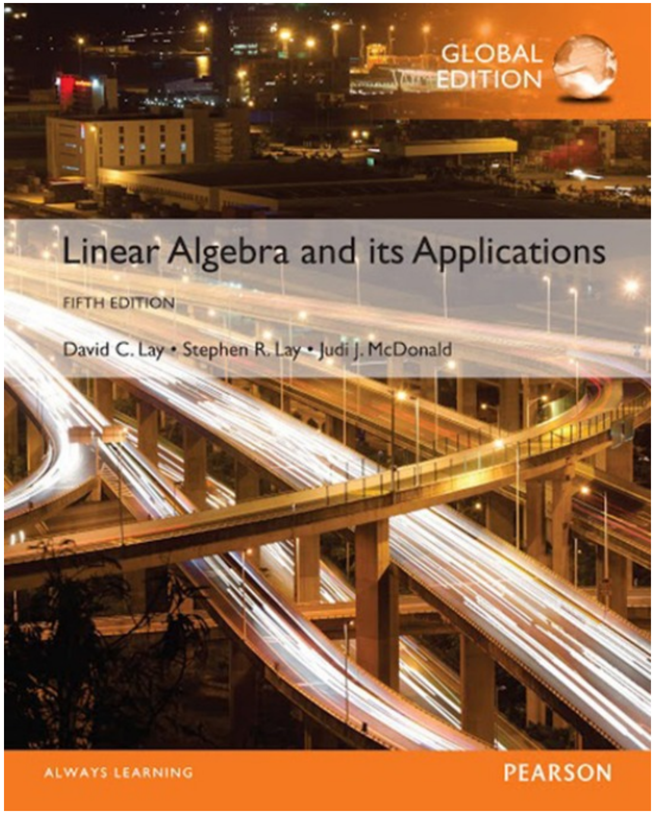
1 Linear Equations in Linear Algebra

1.7

LINEAR INDEPENDENCE



LINEAR INDEPENDENCE

- **Definition:** An indexed set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is said to be **linearly independent** if the vector equation

$$x_1\mathrm{v}_1 + x_2\mathrm{v}_2 + \ldots + x_p\mathrm{v}_p = 0$$

has only the trivial solution. The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is said to be **linearly dependent** if there exist weights $c_1, \ldots, c_p$, not all zero, such that

$$c_1\mathrm{v}_1 + c_2\mathrm{v}_2 + \ldots + c_p\mathrm{v}_p = 0 \qquad (2)$$

LINEAR INDEPENDENCE

- Equation (2) is called a **linear dependence relation** among $\mathbf{v}_1, \ldots, \mathbf{v}_p$ when the weights are not all zero.
- An indexed set is linearly dependent if and only if it is not linearly independent.
- **Example 1:** Let $\mathrm{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \mathrm{v}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, and $\mathrm{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$.

LINEAR INDEPENDENCE

a. Determine if the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.

b. If possible, find a linear dependence relation among $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

- **Solution:** We must determine if there is a nontrivial solution of the equation on the previous slide.

LINEAR INDEPENDENCE

- Row operations on the associated augmented matrix show that

$$\begin{bmatrix} 1 & 4 & 2 & 0 \\ 2 & 5 & 1 & 0 \\ 3 & 6 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 2 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

- x_1 and x_2 are basic variables, and x_3 is free.
- Each nonzero value of x_3 determines a nontrivial solution of (1).
- Hence, $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ are linearly dependent.

LINEAR INDEPENDENCE

b. To find a linear dependence relation among $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$, completely row reduce the augmented matrix and write the new system:

$$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{aligned} x_1 - 2x_3 &= 0 \\ x_2 + x_3 &= 0 \\ 0 &= 0 \end{aligned}$$

- Thus, $x_1 = 2x_3$, $x_2 = -x_3$, and x_3 is free.
- Choose any nonzero value for x_3—say, $x_3 = 5$.
- Then $x_1 = 10$ and $x_2 = -5$.

LINEAR INDEPENDENCE

- Substitute these values into equation (1) and obtain the equation below.

$$10\mathrm{v}_1 - 5\mathrm{v}_2 + 5\mathrm{v}_3 = 0$$

- This is one (out of infinitely many) possible linear dependence relations among $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

LINEAR INDEPENDENCE OF MATRIX COLUMNS

- Suppose that we begin with a matrix $A = \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{bmatrix}$ instead of a set of vectors.

- The matrix equation $A\mathbf{x} = \mathbf{0}$ can be written as

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n = \mathbf{0}.$$

- *Each linear dependence relation among the columns of A corresponds to a nontrivial solution of* $A\mathbf{x} = \mathbf{0}$

- The columns of matrix A are linearly independent if and only if the equation $A\mathbf{x} = \mathbf{0}$ has *only* the trivial solution.

SETS OF ONE OR TWO VECTORS

- A set containing only one vector – say, **v** – is linearly independent if and only if **v** is not the zero vector.
- This is because the vector equation $x_1\mathbf{v} = \mathbf{0}$ has only the trivial solution when $\mathbf{v} \neq \mathbf{0}$.
- The zero vector is linearly dependent because $x_1\mathbf{0} = \mathbf{0}$ has many nontrivial solutions.

SETS OF ONE OR TWO VECTORS

- A set of two vectors $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly dependent if at least one of the vectors is a multiple of the other.

- The set is linearly independent if and only if neither of the vectors is a multiple of the other.

SETS OF TWO OR MORE VECTORS

THEOREM 7

Characterization of Linearly Dependent Sets

An indexed set $S = \{v_1, \ldots, v_p\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in S is a linear combination of the others. In fact, if S is linearly dependent and $v_1 \neq 0$, then some $\mathbf{v}_j$ (with $j > 1$) is a linear combination of the preceding vectors, $\mathbf{v}_1, \ldots, v_{j-1}$.

SETS OF TWO OR MORE VECTORS

- **Proof:** If some $\mathbf{v}_j$ in S equals a linear combination of the other vectors, then $\mathbf{v}_j$ can be subtracted from both sides of the equation, producing a linear dependence relation with a nonzero weight (-1) on $\mathbf{v}_j$.
- [For instance, if $\mathrm{v}_1 = c_2\mathrm{v}_2 + c_3\mathrm{v}_3$, then $0 = (-1)\mathrm{v}_1 + c_2\mathrm{v}_2 + c_3\mathrm{v}_3 + 0\mathrm{v}_4 + \ldots + 0\mathrm{v}_p$.]
- Thus S is linearly dependent.
- Conversely, suppose S is linearly dependent.
- If $\mathbf{v}_1$ is zero, then it is a (trivial) linear combination of the other vectors in S.

SETS OF TWO OR MORE VECTORS

- Otherwise, $\mathbf{v}_1 \neq \mathbf{0}$, and there exist weights $c_1, \ldots, c_p$, not all zero, such that
$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_p\mathbf{v}_p = \mathbf{0}.$$
- Let j be the largest subscript for which $c_j \neq 0$.
- If $j = 1$, then $c_1\mathbf{v}_1 = \mathbf{0}$, which is impossible because $\mathbf{v}_1 \neq \mathbf{0}$.

SETS OF TWO OR MORE VECTORS

- So $j > 1$, and

$$c_1\mathbf{v}_1 + \ldots + c_j\mathbf{v}_j + 0\mathbf{v}_j + 0\mathbf{v}_{j+1} + \ldots + 0\mathbf{v}_p = 0$$

$$c_j\mathbf{v}_j = -c_1\mathbf{v}_1 - \ldots - c_{j-1}\mathbf{v}_{j-1}$$

$$\mathbf{v}_j = \left(-\frac{c_1}{c_j}\right)\mathbf{v}_1 + \ldots + \left(-\frac{c_{j-1}}{c_j}\right)\mathbf{v}_{j-1}.$$

SETS OF TWO OR MORE VECTORS

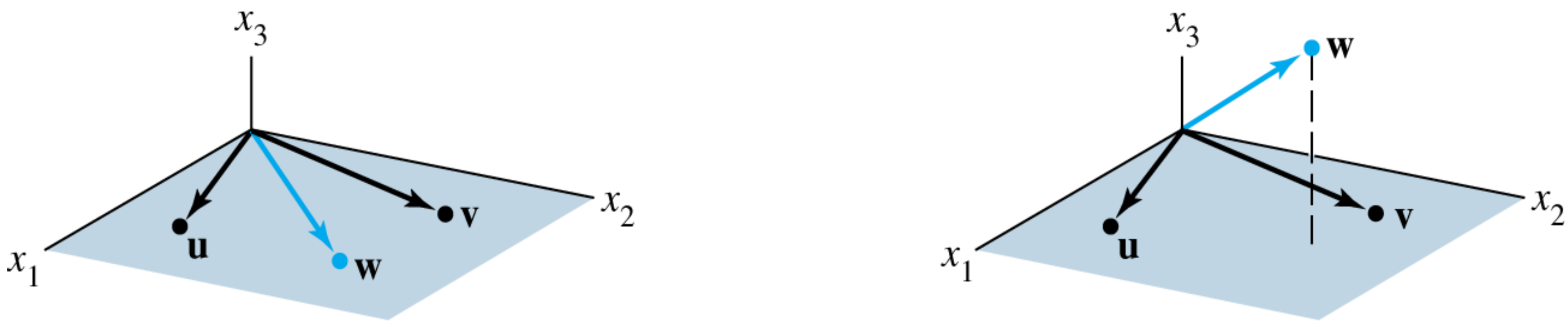
- Theorem 7 does *not* say that *every* vector in a linearly dependent set is a linear combination of the preceding vectors.
- A vector in a linearly dependent set may fail to be a linear combination of the other vectors.
- **Example 4:** Let $\mathbf{u} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 6 \\ 0 \end{bmatrix}$. Describe the set spanned by **u** and **v**, and explain why a vector **w** is in Span {**u**, **v**} if and only if {**u**, **v**, **w**} is linearly dependent.

SETS OF TWO OR MORE VECTORS

- **Solution:** The vectors **u** and **v** are linearly independent because neither vector is a multiple of the other, and so they span a plane in $\mathbb{R}^3$.
- Span {**u**, **v**} is the x_1x_2-plane (with $x_3 = 0$).
- If **w** is a linear combination of **u** and **v**, then {**u**, **v**, **w**} is linearly dependent, by Theorem 7.
- Conversely, suppose that {**u**, **v**, **w**} is linearly dependent.
- By theorem 7, some vector in {**u**, **v**, **w**} is a linear combination of the preceding vectors (since $\mathbf{u} \neq \mathbf{0}$).
- That vector must be **w**, since **v** is not a multiple of **u**.

SETS OF TWO OR MORE VECTORS

- So **w** is in Span {**u**, **v**}. Fig. 2 below



Linearly dependent,
w in Span{**u**, **v**}

Linearly independent,
w *not* in Span{**u**, **v**}

- Example 4 generalizes to any set {**u**, **v**, **w**} in $\mathbb{R}^3$ with **u** and **v** linearly independent.
- The set {**u**, **v**, **w**} will be linearly dependent if and only if **w** is in the plane spanned by **u** and **v**.

SETS OF TWO OR MORE VECTORS

THEOREM 8

If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. That is, any set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is linearly dependent if $p > n$.

- **Proof:** Let $A = \begin{bmatrix} \mathrm{v}_1 & \cdots & \mathrm{v}_p \end{bmatrix}$.
- Then A is $n \times p$, and the equation $A\mathrm{x} = 0$ corresponds to a system of n equations in p unknowns.
- If $p > n$, there are more variables than equations, so there must be a free variable.

SETS OF TWO OR MORE VECTORS

- Hence $A\mathbf{x} = 0$ has a nontrivial solution, and the columns of A are linearly dependent.
- See the figure below for a matrix version of this theorem.

$$\begin{array}{cc} & \begin{array}{ccccc} & & p & & \end{array} \\ n & \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix} \end{array}$$

If $p > n$, the columns are linearly dependent.

- Theorem 8 says nothing about the case in which the number of vectors in the set does *not* exceed the number of entries in each vector.

SETS OF TWO OR MORE VECTORS

THEOREM 9

If a set $S = \{\mathrm{v}_1, \ldots, \mathrm{v}_p\}$ in $\mathbb{R}^n$ contains the zero vector, then the set is linearly dependent.

- **Proof:** By renumbering the vectors, we may suppose $\mathrm{v}_1 = 0$.
- Then the equation $1\mathrm{v}_1 + 0\mathrm{v}_2 + \ldots + 0\mathrm{v}_p = 0$ shows that S in linearly dependent.