1)

$$A = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix} \longrightarrow A - \lambda I = \begin{bmatrix} 2-\lambda & 2 & 2 \\ 2 & 2-\lambda & 2 \\ 2 & 2 & 2-\lambda \end{bmatrix}$$

$$\det(A-\lambda I) = -\lambda^3 + 6\lambda^2 = \lambda^2(6-\lambda) \Rightarrow \boxed{\begin{matrix}\lambda_1 = 0 \\ \lambda_2 = 0 \ \times \\ \lambda_3 = 6\end{matrix}}$$

$\lambda_1 = 0$

$$A - \lambda_1 I = A = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = R$$

$$Rx = 0 \longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} x_1 + x_2 + x_3 = 0 \\ x_1 = -x_2 - x_3 \end{matrix} \Rightarrow x = \begin{bmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\Rightarrow x = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} x_3$$

$$\boxed{\begin{matrix} x_2 = 0 \\ x_3 = 1 \end{matrix} \rightarrow \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \qquad \begin{matrix} x_2 = 1 \\ x_3 = 1 \end{matrix} \longrightarrow \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}}$$

$\lambda_2 = 6$

$$\Rightarrow A - \lambda_2 I = \begin{bmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{-1}{2} & \frac{-1}{2} \\ 2 & -4 & 2 \\ 2 & 2 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{-1}{2} & \frac{-1}{2} \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{-1}{2} & \frac{-1}{2} \\ 0 & 1 & -1 \\ 0 & 3 & -3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow Rx = 0 \longrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{matrix} x_1 - x_3 = 0 \rightarrow x_1 = x_3 \\ x_2 - x_3 = 0 \rightarrow x_2 = x_3 \end{matrix} \Rightarrow x = \begin{bmatrix} x_3 \\ x_3 \\ x_3 \end{bmatrix}$$

$$\Rightarrow x = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} x_3$$

$$\boxed{x_3 = 1 \longrightarrow \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}$$

2)

a) $(A-\lambda I) = (A^T - \lambda I) \leftarrow I$ is symmetric

$\Rightarrow \det(A^T - \lambda I) = \det((A-\lambda I)^T) = \det(A - \lambda I)$.

$\Rightarrow \sigma(A) = \sigma(A^T)$

b) let A be lower triangular

thus A^t is upper triangular.

also $\sigma(A) = \sigma(A^t)$

thus σ (lower triangular) = σ (upper triangular)

c) $V = (1, 1, 1, \cdots, 1, 1)^T$

$AV = \begin{bmatrix} S \\ S \\ \vdots \\ S \end{bmatrix} = S \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$ is an <u>eigenvector</u> for A

when $\lambda = \underline{\underline{S}}$

$Ax = \lambda x$

also $\sigma(A) = \sigma(A)^T$

thus S is an eigenvalue of $\underline{\underline{A^T}}$

3) $U = E_2 + 2\begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix}$

$x_5 = M^4 U = 2^4 E_2 + (-1)^4 \begin{bmatrix} 2 \\ 4 \\ 2 \\ 2 \end{bmatrix} + (-1)^4 \begin{bmatrix} 3 \\ 3 \\ 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 16 \\ 16 \\ 16 \\ 16 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \\ 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 3 \\ 3 \\ 6 \end{bmatrix} = \underline{\underline{\begin{bmatrix} 21 \\ 23 \\ 21 \\ 24 \end{bmatrix}}}$

4)

a) $A = PDP^{-1}$ $\quad A^{-1} = (PDP^{-1})^{-1}$

$\xRightarrow[\text{است}]{D \text{ معکوس پذیر}} A^{-1} = (P^{-1})^{-1} D^{-1} P^{-1} = PDP^{-1}$

b)

اگر مجموع تعداد بعدهای eigen basis به 7 نرسد این ماتریس قابل قطری سازی نیست.

5)

a) $A = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix} \longrightarrow A-\lambda I = \begin{bmatrix} -\lambda & -1 & -1 \\ 1 & 2-\lambda & 1 \\ -1 & -1 & -\lambda \end{bmatrix}$

$\det(A-\lambda I) = -\lambda(-\lambda(2-\lambda)+1) = -\lambda(\lambda-1)^2$

$\lambda_1 = 0 \quad \times 1$

$\lambda_2 = +1 \quad \times 2$

$\lambda_1 = 0 \rightarrow A-\lambda_1 I = A \Rightarrow \begin{bmatrix} 0 & -1 & -1 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & +1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow x = \begin{bmatrix} x_3 \\ -x_3 \\ x_3 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} x_3 \longrightarrow x_3 = 1 \quad \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

$\lambda_2 = +1 \Rightarrow A - \lambda_2 I = \begin{bmatrix} -1 & -1 & -1 \\ 1 & 1 & 1 \\ -1 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow x = \begin{bmatrix} -x_2-x_3 \\ x_2 \\ x_3 \end{bmatrix}$

$\Rightarrow x = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} x_3 \longrightarrow \begin{matrix} x_2 = 0 \\ x_3 = 1 \end{matrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \qquad \begin{matrix} x_3 = 0 \\ x_2 = 1 \end{matrix} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

$C = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

$[C\ I] = \left[\begin{array}{ccc|ccc} 1 & -1 & -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 2 & 1 \end{array}\right] = [I\ C^{-1}]$

$D = C^{-1}AC \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & 0 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 0 & -1 & -1 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & -1 & -1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \Rightarrow C^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$

$\Rightarrow D = \begin{bmatrix} 0 & 0 & 0 \\ -1 & -1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & -1 & -1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\Rightarrow A = C D C^{-1}$

b)

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 5 & -2 \\ 1 & 3 & 1 \end{bmatrix} \rightarrow A - \lambda I = \begin{bmatrix} 1-\lambda & 2 & -3 \\ 2 & 5-\lambda & -2 \\ 1 & 3 & 1-\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = -\lambda \cdot \left(\lambda + \frac{i\sqrt{15} - 7}{2}\right)\left(\lambda - \frac{i\sqrt{15} + 7}{2}\right) = 0$$

$$\lambda_1 = 0$$
$$\lambda_2 = \frac{-i\sqrt{15} + 7}{2}$$
$$\lambda_3 = \frac{i\sqrt{15} + 7}{2}$$

$$\lambda_1 = 0 \rightarrow A - \lambda I = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 5 & -2 \\ 1 & 3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -11 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow x = \begin{bmatrix} 11x_3 \\ -4x_3 \\ x_3 \end{bmatrix}$$

$$\Rightarrow x = \begin{bmatrix} 11 \\ -4 \\ 1 \end{bmatrix} x_3 \longrightarrow \begin{bmatrix} 11 \\ -4 \\ 1 \end{bmatrix}$$

$$\lambda_2 = \frac{-i\sqrt{15} + 7}{2} \rightarrow A - \lambda I = \begin{bmatrix} \frac{i\sqrt{15} - 5}{2} & 2 & -3 \\ 2 & \frac{i\sqrt{15} + 3}{2} & -2 \\ 1 & 3 & \frac{i\sqrt{15} - 5}{2} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \frac{i\sqrt{15} + 1}{8} \\ 0 & 1 & \frac{i\sqrt{15} - 7}{8} \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow x = \begin{bmatrix} \frac{-i\sqrt{15} + 1}{8} \\ \frac{i\sqrt{15} - 7}{8} \\ 1 \end{bmatrix} x_3 \rightarrow \begin{bmatrix} \frac{-i\sqrt{15} + 1}{8} \\ \frac{i\sqrt{15} - 7}{8} \\ 1 \end{bmatrix}$$

$$\lambda_3 = \frac{i\sqrt{15} + 7}{2} \rightarrow A - \lambda I = \begin{bmatrix} \frac{-i\sqrt{15} - 5}{2} & 2 & -3 \\ 2 & \frac{-i\sqrt{15} + 3}{2} & -2 \\ 1 & 3 & \frac{-i\sqrt{15} - 5}{2} \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & \frac{-i\sqrt{15} + 1}{8} \\ 0 & 1 & \frac{-i\sqrt{15} - 7}{8} \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow x = \begin{bmatrix} \frac{-i\sqrt{15} + 1}{8} \\ \frac{-i\sqrt{15} - 7}{8} \\ 1 \end{bmatrix} x_3 \rightarrow \begin{bmatrix} \frac{-i\sqrt{15} + 1}{8} \\ \frac{-i\sqrt{15} - 7}{8} \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 11 & \frac{-i\sqrt{15} + 1}{8} & \frac{-i\sqrt{15} + 1}{8} \\ -4 & \frac{i\sqrt{15} - 7}{8} & \frac{i\sqrt{15} - 7}{8} \\ 1 & 1 & 1 \end{bmatrix} \rightarrow \det(C) = 0 \longrightarrow C \text{ is not invertable}$$

$\Rightarrow$ A is not diagonalizable

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6)

a)

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \rightarrow T(e_1) = b_3$$

$$e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \rightarrow T(e_2) = -b_1 - 2b_2$$

$$e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \rightarrow T(e_3) = 2b_1 + 3b_3$$

b)

$$[T(e_1)]_B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad [T(e_2)]_B = \begin{bmatrix} -1 \\ -2 \\ 0 \end{bmatrix} \qquad [T(e_3)]_B = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$$

c)

$$\begin{bmatrix} 0 & -1 & 2 \\ 0 & -2 & 0 \\ 1 & 0 & 3 \end{bmatrix}$$

7)

$$A = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$$

$$\lambda_1 = 2-i \rightarrow A - \lambda I = \begin{bmatrix} -1+i & -2 \\ 1 & 1+i \end{bmatrix} \sim \begin{bmatrix} 1 & 1+i \\ 0 & 0 \end{bmatrix} \Rightarrow x = \begin{bmatrix} -1-i \\ 1 \end{bmatrix} x_2$$

$$\Rightarrow v_1 = \begin{bmatrix} -1-i \\ 1 \end{bmatrix}$$

$$\lambda_2 = 2+i \rightarrow A - \lambda I = \begin{bmatrix} -1-i & -2 \\ 1 & 1-i \end{bmatrix} \sim \begin{bmatrix} 1 & 1-i \\ 0 & 0 \end{bmatrix} \Rightarrow x = \begin{bmatrix} i-1 \\ 1 \end{bmatrix} x_2$$

$$\Rightarrow v_2 = \begin{bmatrix} i-1 \\ 1 \end{bmatrix}$$

$$P = \begin{bmatrix} -1-i & i-1 \\ 1 & 1 \end{bmatrix} \rightarrow [P\ I] = \begin{bmatrix} -1-i & i-1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \frac{i}{2} & \frac{1+i}{2} \\ 0 & 1 & \frac{-i}{2} & \frac{1-i}{2} \end{bmatrix} = [I\ P^{-1}]$$

$$\Rightarrow P^{-1} = \begin{bmatrix} \frac{i}{2} & \frac{1+i}{2} \\ \frac{-i}{2} & \frac{1-i}{2} \end{bmatrix}$$

$$C = P^{-1} A P = \begin{bmatrix} i/2 & 1+i/2 \\ -i/2 & 1-i/2 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} -1-i & i-1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2-i & 0 \\ 0 & 2+i \end{bmatrix}$$

$$T9 \Rightarrow P = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$$

8)

$$x_0 = \begin{bmatrix} -2 \\ -5 \\ 3 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix} + 2\begin{bmatrix} -3 \\ -3 \\ 7 \end{bmatrix}$$

$$x_k = 3^k \times 2 \times \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + \left(\frac{4}{5}\right)^k \begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix} + \left(\frac{3}{5}\right)^k \times 2 \begin{bmatrix} -3 \\ -3 \\ 7 \end{bmatrix}$$

$$x_{k+1} = A x_k$$

if $k \to \infty$ then $x_k \to \infty$

9)

$$x_1 = A x_0 = \begin{bmatrix} .5 & .2 \\ .4 & .7 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} .5 \\ .4 \end{bmatrix} \to .5 \begin{bmatrix} 1 \\ .8 \end{bmatrix}$$

$$x_2 = A x_1 = \begin{bmatrix} .5 & .2 \\ .4 & .7 \end{bmatrix} \begin{bmatrix} .5 \\ .4 \end{bmatrix} = \begin{bmatrix} .33 \\ .48 \end{bmatrix} \to .48 \begin{bmatrix} .6875 \\ 1 \end{bmatrix}$$

$$x_3 = A x_2 = \begin{bmatrix} .5 & .2 \\ .4 & .7 \end{bmatrix} \begin{bmatrix} .33 \\ .48 \end{bmatrix} = \begin{bmatrix} .261 \\ .468 \end{bmatrix} \to .0467 \begin{bmatrix} .5577 \\ 1 \end{bmatrix}$$

$$x_4 = A x_3 = \begin{bmatrix} .5 & .2 \\ .4 & .7 \end{bmatrix} \begin{bmatrix} .261 \\ .468 \end{bmatrix} = \begin{bmatrix} .2241 \\ .432 \end{bmatrix} \to .4319 \begin{bmatrix} .5188 \\ 1 \end{bmatrix}$$

$$\Rightarrow \lambda = .5 \quad v = \begin{bmatrix} 1 \\ .8 \end{bmatrix}$$