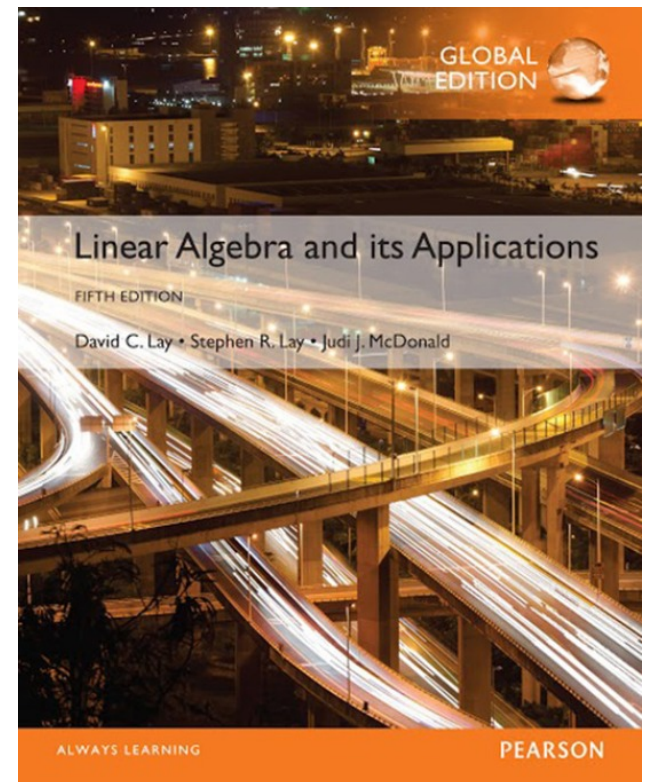


6

Orthogonality and Least Squares

6.7

Inner Product Spaces



INNER PRODUCT SPACES

- **Definition** An **inner product** on a vector space V is a function that, to each pair of vectors \mathbf{u} and \mathbf{v} in V , associates a real number $\langle \mathbf{u}, \mathbf{v} \rangle$ and satisfies the following axioms, for all $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in V and all scalars c :
 1. $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$
 2. $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$
 3. $\langle c\mathbf{u}, \mathbf{v} \rangle = c\langle \mathbf{u}, \mathbf{v} \rangle$
 4. $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$ and $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ if and only if $\mathbf{u} = \mathbf{0}$
- A vector space with an inner product is called an **inner product space**.

INNER PRODUCT SPACES

- **Example 1** Fix any two positive numbers—say, 4 and 5—and for vectors $u = (u_1, u_2)$ and $v = (v_1, v_2)$ in \mathbb{R}^2 , set

$$\langle u, v \rangle = 4u_1v_1 + 5u_2v_2 \quad (1)$$

- Show that equation (1) defines an inner product.
- **Solution** Certain Axiom 1 is satisfied, because $\langle u, v \rangle = 4u_1v_1 + 5u_2v_2 = 4u_1v_1 + 5u_2v_2 = \langle v, u \rangle$.

INNER PRODUCT SPACES

- If $w = (w_1, w_2)$, then

$$\begin{aligned}\langle u + v, w \rangle &= 4(u_1 + v_1)w_1 + 5(u_2 + v_2)w_2 \\ &= 4u_1w_1 + 5u_2w_2 + 4v_1w_1 + 5v_2w_2 \\ &= \langle u, w \rangle + \langle v, w \rangle\end{aligned}$$

- This verifies Axiom 2. For Axiom 3, compute

$$\langle cu, v \rangle = 4(cu_1)v_1 + 5(cu_2)v_2 = c(4u_1v_1 + 5u_2v_2) = c\langle u, v \rangle$$

INNER PRODUCT SPACES

- For Axiom 4, note that $\langle u, u \rangle = 4u_1^2 + 5u_2^2 \geq 0$, and $4u_1^2 + 5u_2^2 = 0$ only if $u_1 = u_2 = 0$, that is, if $u = 0$.
- Also, $\langle 0, 0 \rangle = 0$. So (1) defines an inner product on \mathbb{R}^2 .

LENGTHS, DISTANCES, AND ORTHOGONALITY

- Let V be an inner product space, with the inner product denoted by $\langle u, v \rangle$. Just as in \mathbb{R}^n , we define the length, or norm, of a vector v to be the scalar

$$\|v\| = \sqrt{(v, v)}$$

- Equivalently, $\|v\|^2 = \langle v, v \rangle$.
- A **unit vector** is one whose length is 1. The **distance between u and v** is $\|u - v\|$. Vectors u and v are **orthogonal** if $\langle u, v \rangle = 0$.

TWO INEQUALITIES

- Given a vector \mathbf{v} in an inner product space V and given a finite-dimensional subspace W , we may apply the Pythagorean Theorem to the orthogonal decomposition of \mathbf{v} with respect to W and obtain

$$\|v\|^2 = \|\text{proj}_W v\|^2 + \|v - \text{proj}_W v\|^2$$

- See Fig 2 on the next slide. In particular, this shows that the norm of the projection of \mathbf{v} onto W does not exceed the norm of \mathbf{v} itself. This simple observation leads to the following important inequality.

TWO INEQUALITIES

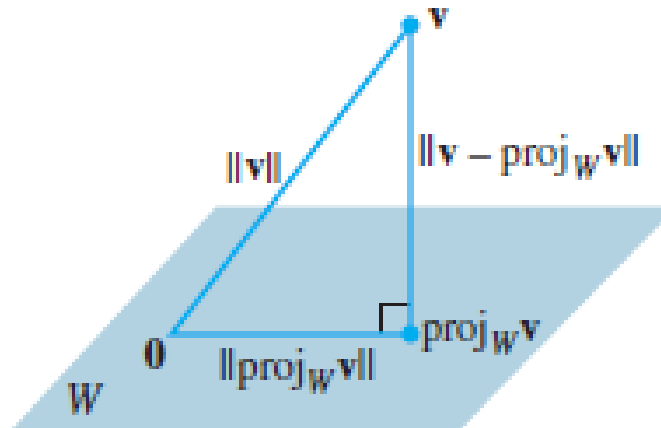


FIGURE 2

The hypotenuse is the longest side.

- **Theorem 16 The Cauchy-Schwarz Inequality:** For all u, v in V ,

$$|(u, v)| \leq \|u\| \|v\| \quad (4)$$

TWO INEQUALITIES

- **Proof** If $u = 0$, then both sides of (4) are zero, and hence the inequality is true in this case.
- If $u \neq 0$, let W be the subspace spanned by u .
- Recall that $\|cu\| = |c| \|u\|$ for any scalar c . Thus
$$\|proj_W v\| = \left\| \frac{\langle v, u \rangle}{\langle u, u \rangle} u \right\| = \frac{|\langle v, u \rangle|}{|\langle u, u \rangle|} \|u\| = \frac{|\langle v, u \rangle|}{\|u\|^2} \|u\| = \frac{|\langle u, v \rangle|}{\|u\|}$$
- Since $\|proj_W v\| \leq \|v\|$, we have $\frac{|\langle u, v \rangle|}{\|u\|} \leq \|v\|$, which gives (4).

TWO INEQUALITIES

- **Theorem 17 The Triangle Inequality:** For all u, v in V ,

$$\|u + v\| \leq \|u\| + \|v\|$$

- **Proof** $\|u + v\|^2 = \langle u + v, u + v \rangle = \langle u, u \rangle + 2\langle u, v \rangle + \langle v, v \rangle$
 $\leq \|u\|^2 + 2|\langle u, v \rangle| + \|v\|^2$
 $\leq \|u\|^2 + 2\|u\|\|v\| + \|v\|^2$
 $= (\|u\| + \|v\|)^2$
- The triangle inequality follows immediately by taking square roots of both sides.