CHAPTER 9

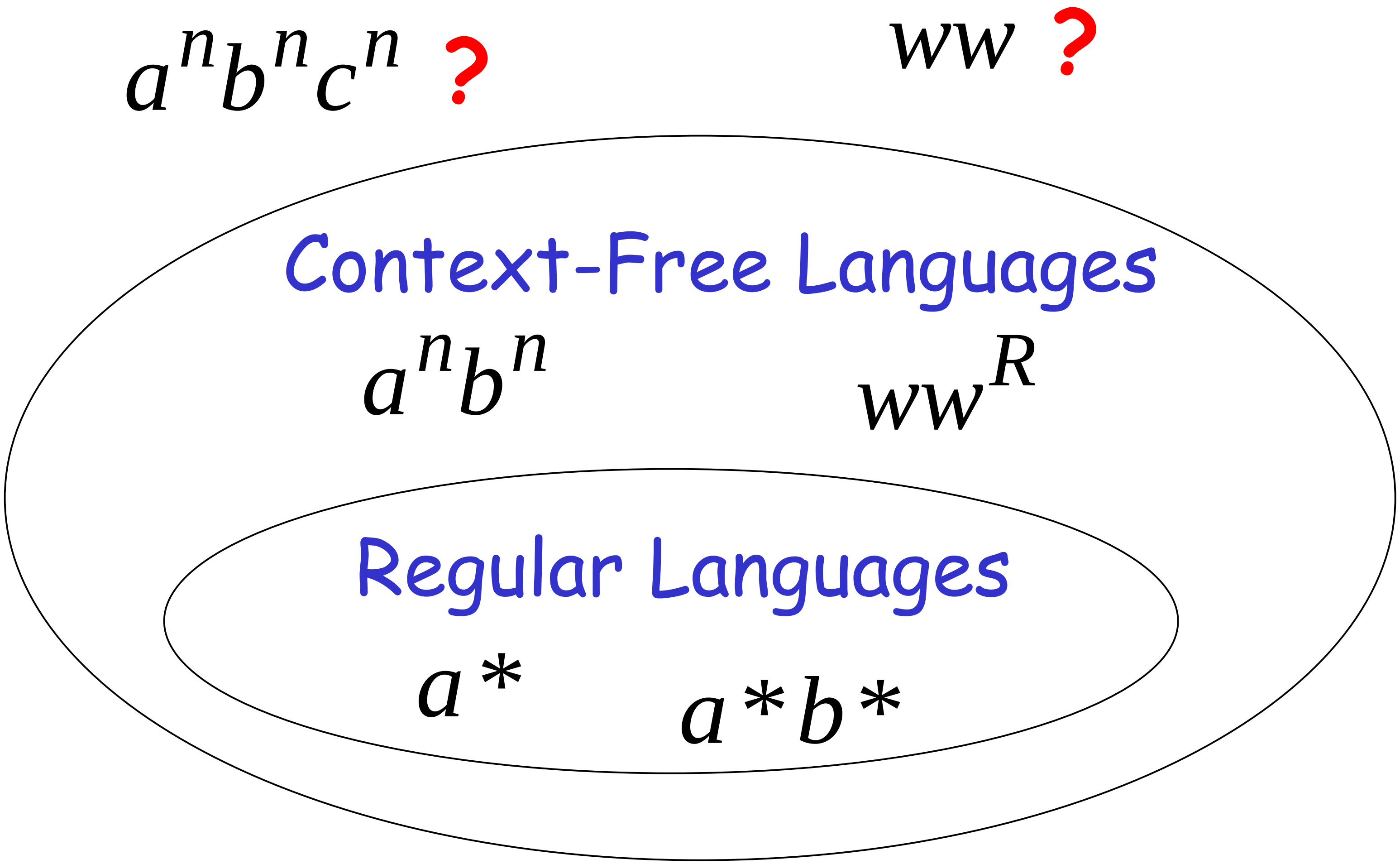
Turing Machines

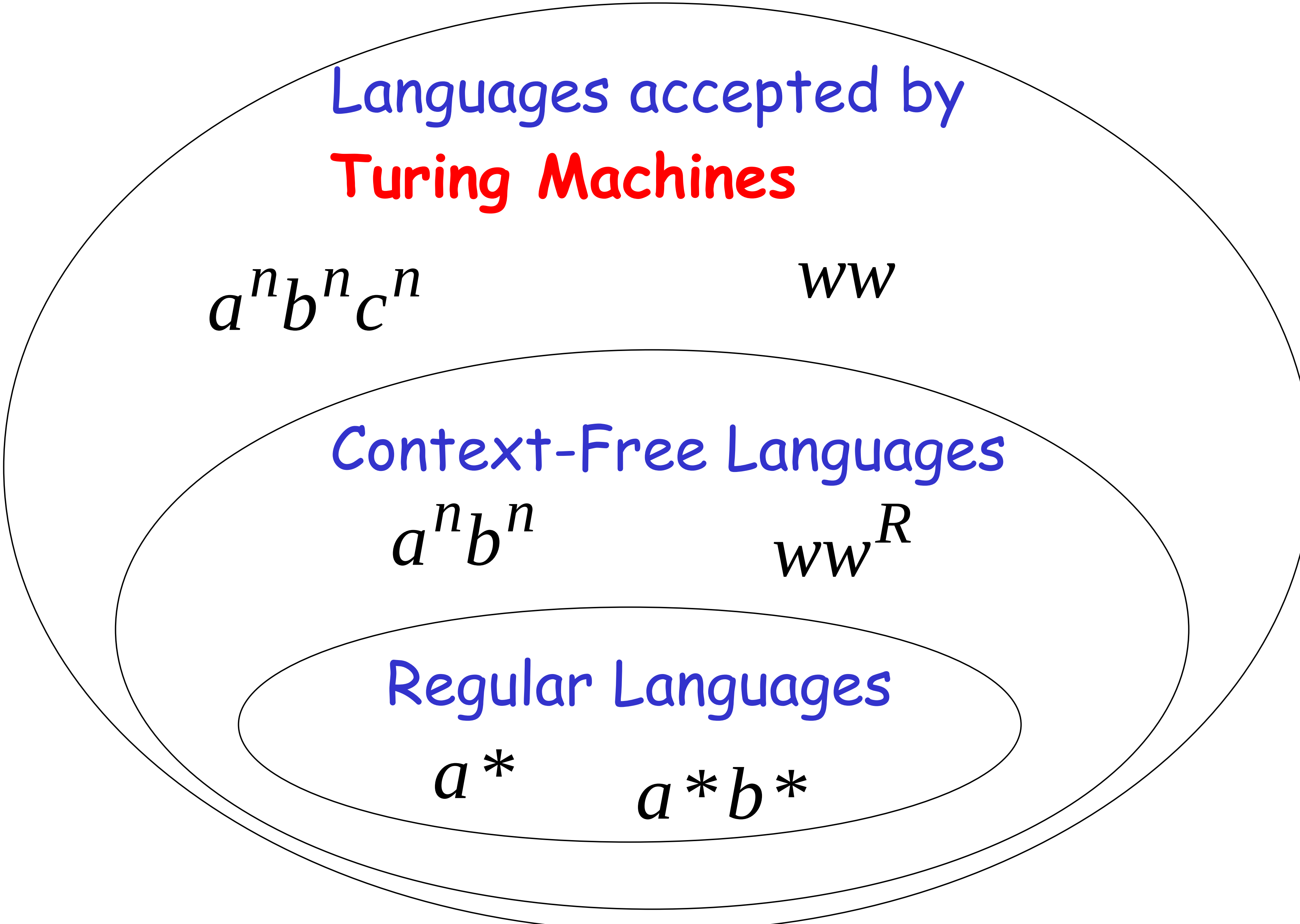
By R.Ameri

Today's Lecture

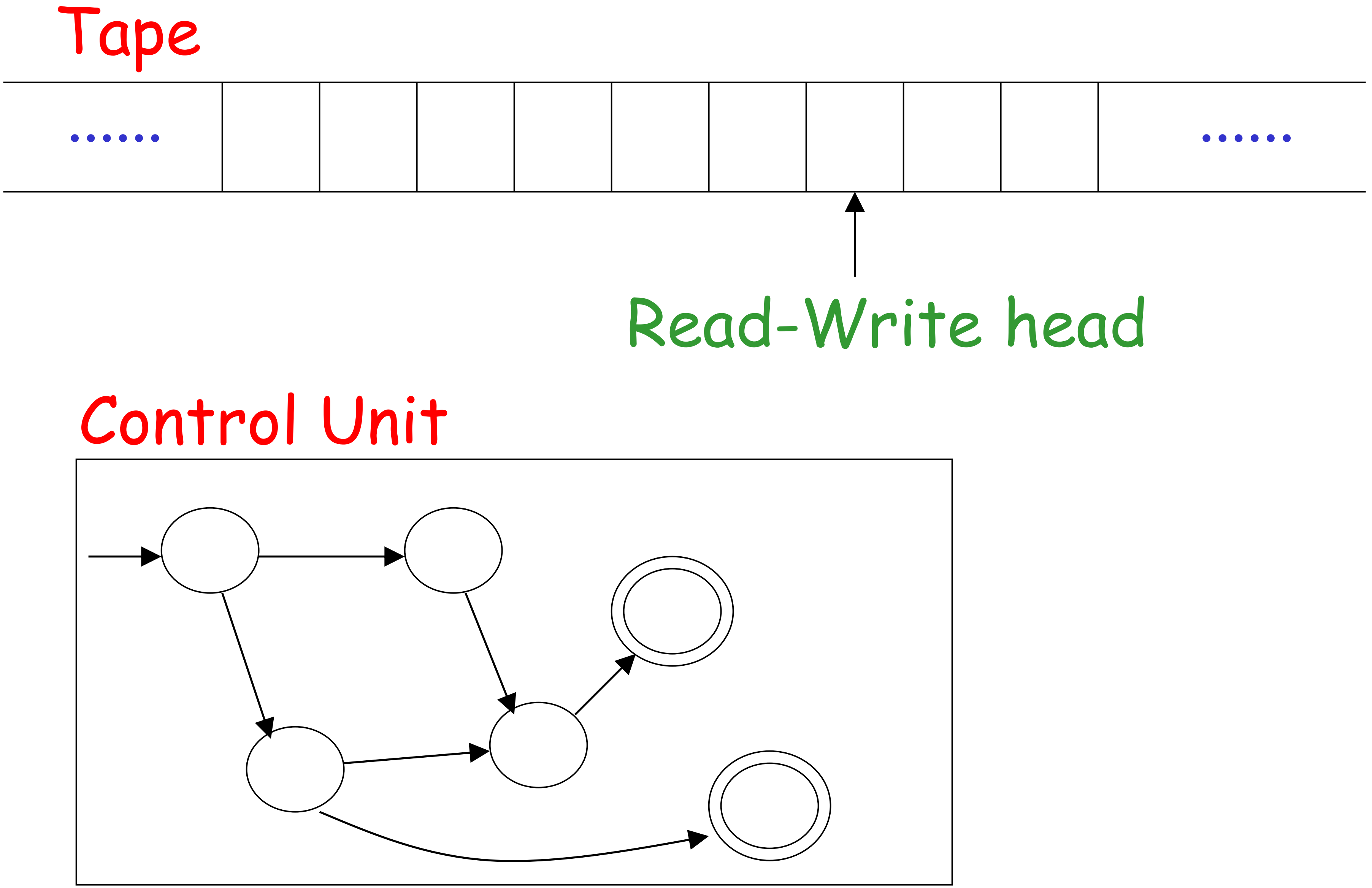
- Introducing the Turing machine
- Formal Definitions for Standard Turing Machines
- Computing Functions with Turing Machines
- Turing's Thesis

The Language Hierarchy



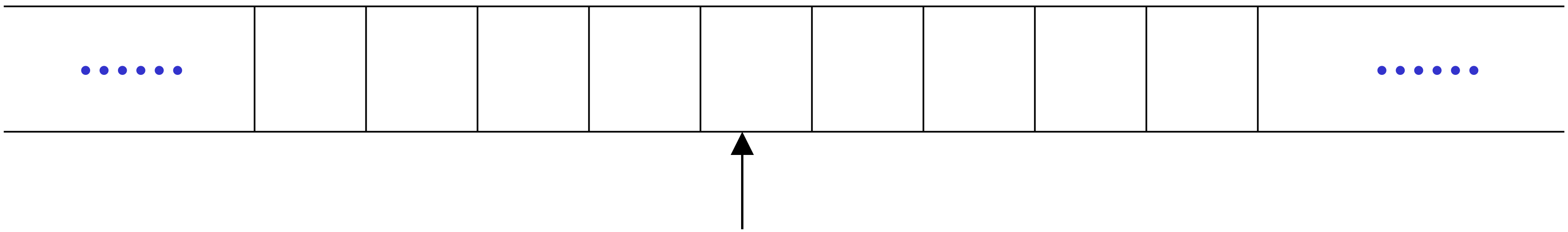

Languages accepted by
Turing Machines
$a^n b^n c^n$
ww
Context-Free Languages
$a^n b^n$
ww^R
Regular Languages
a^*
$a^* b^*$

A Turing Machine



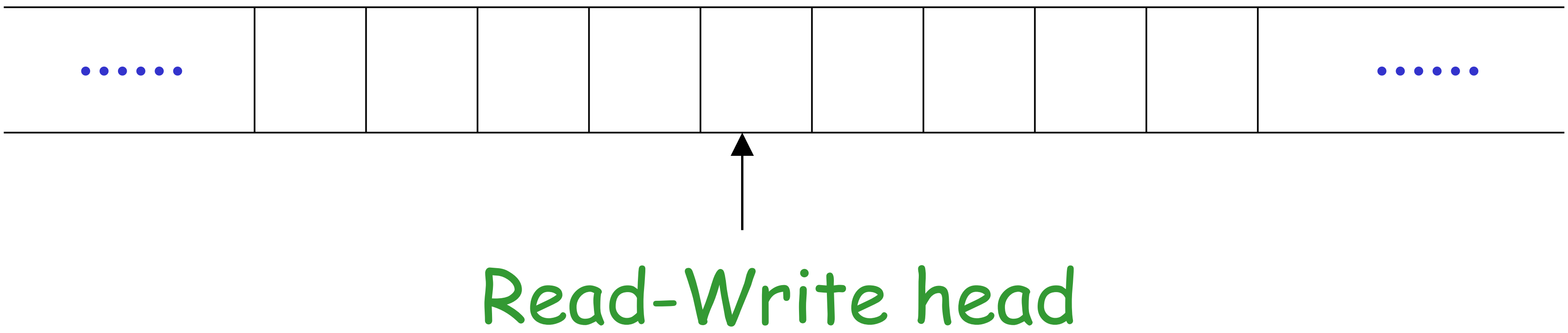
The Tape

No boundaries -- infinite length



Read-Write head

The head moves Left or Right

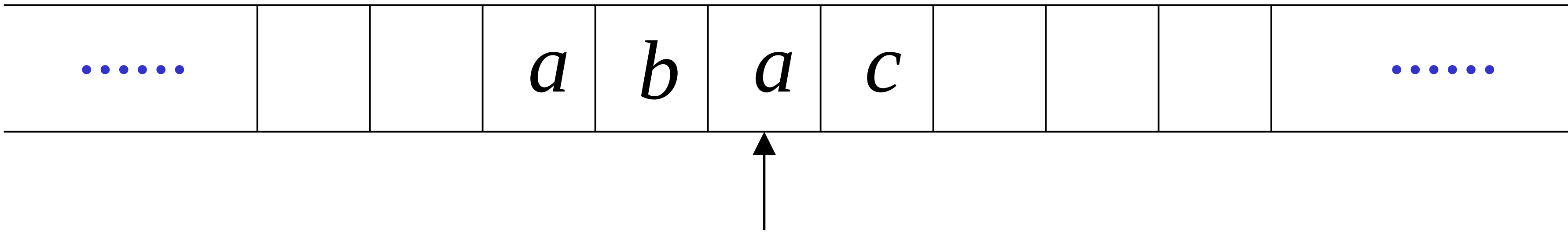


The head at each time step:

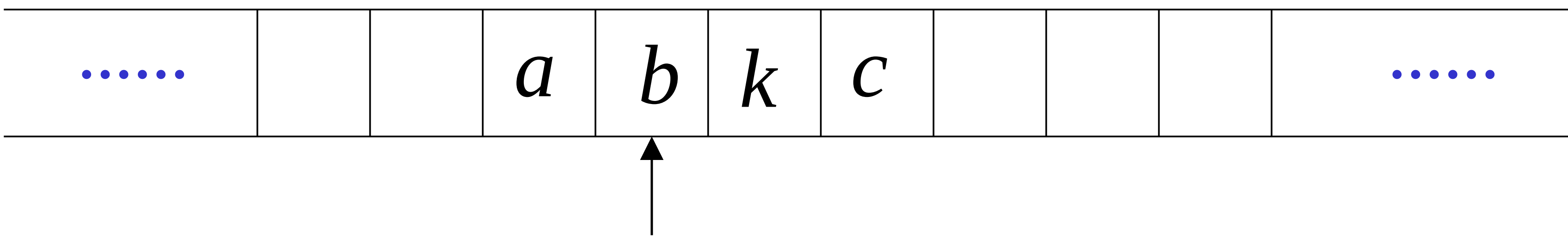
1. Reads a symbol
2. Writes a symbol
3. Moves Left or Right

Example:

Time 0

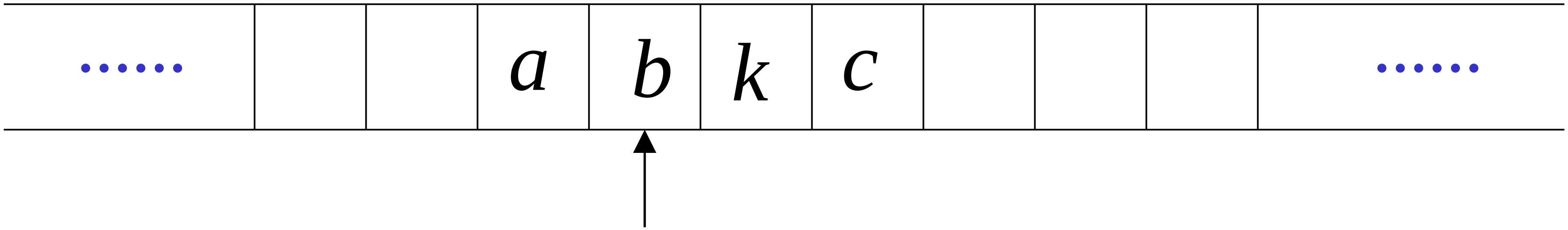


Time 1

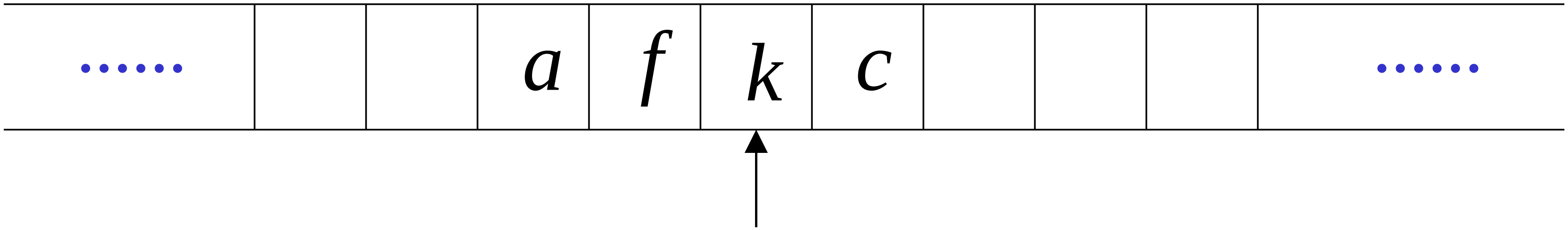


1. Reads a
2. Writes k
3. Moves Left

Time 1



Time 2

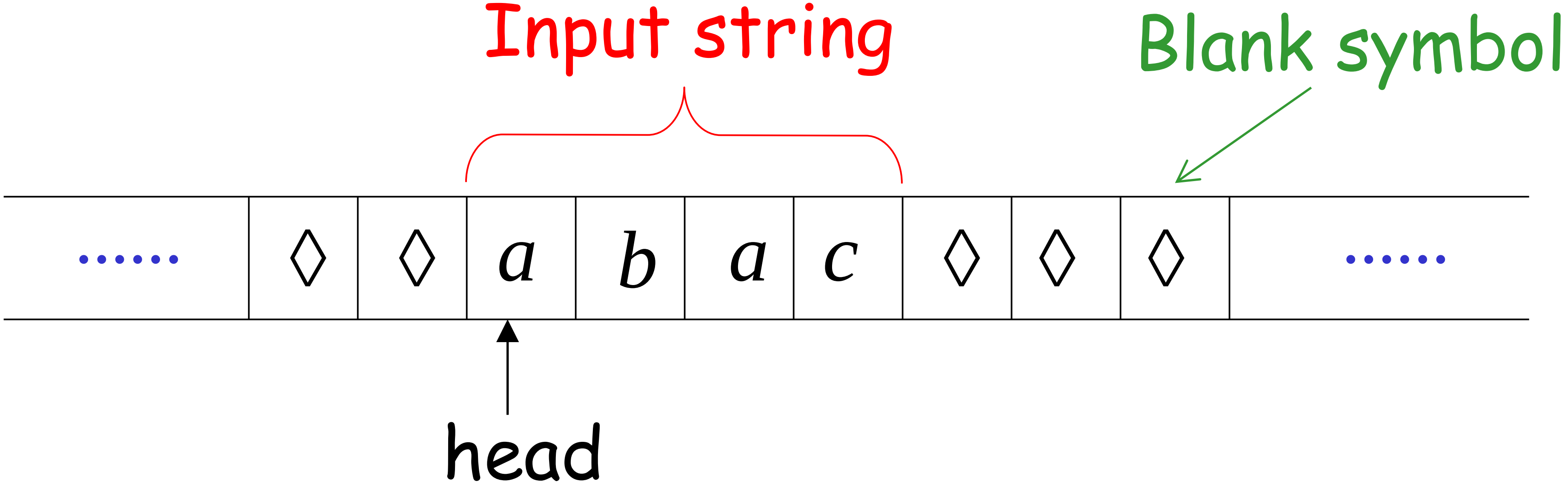


1. Reads b

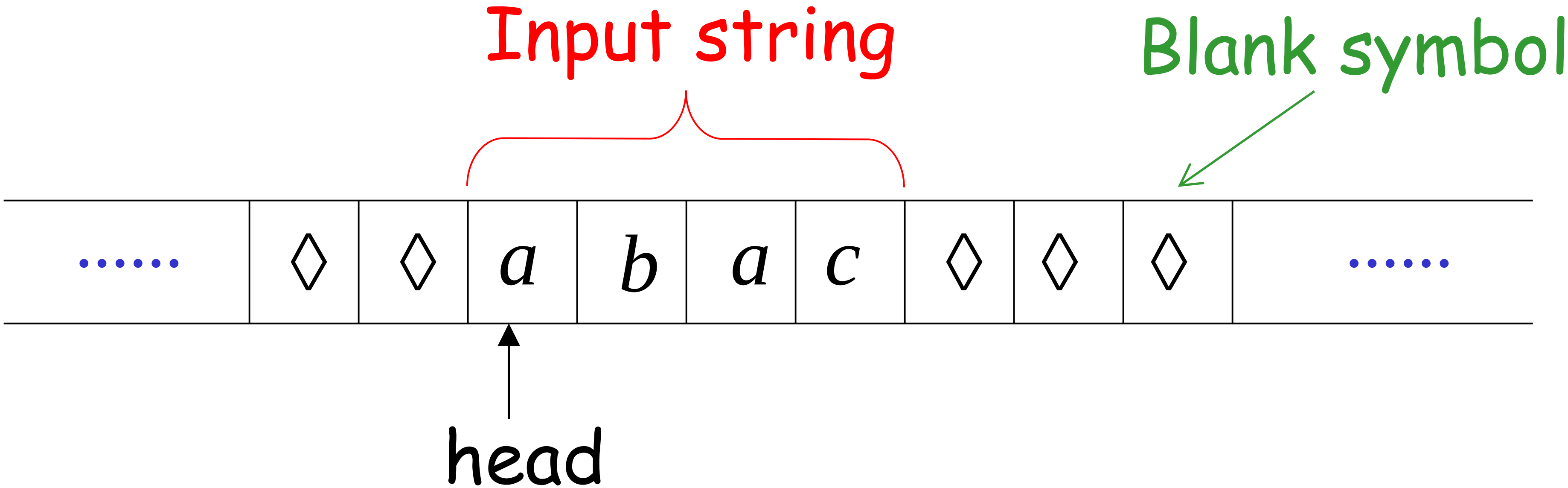
2. Writes f

3. Moves Right

The Input String

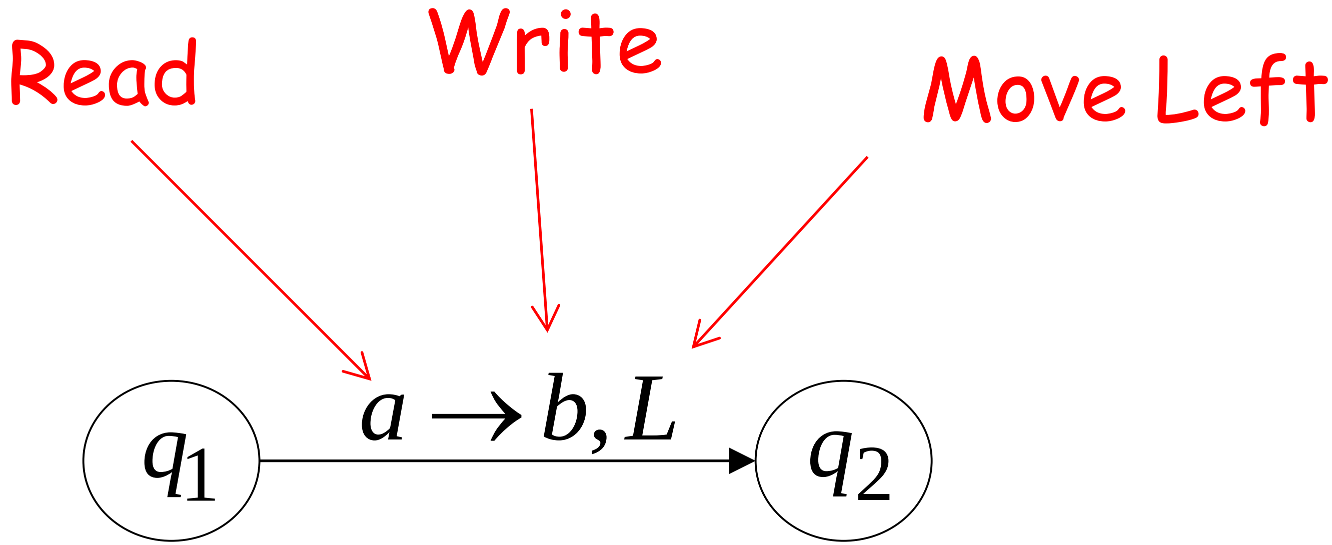


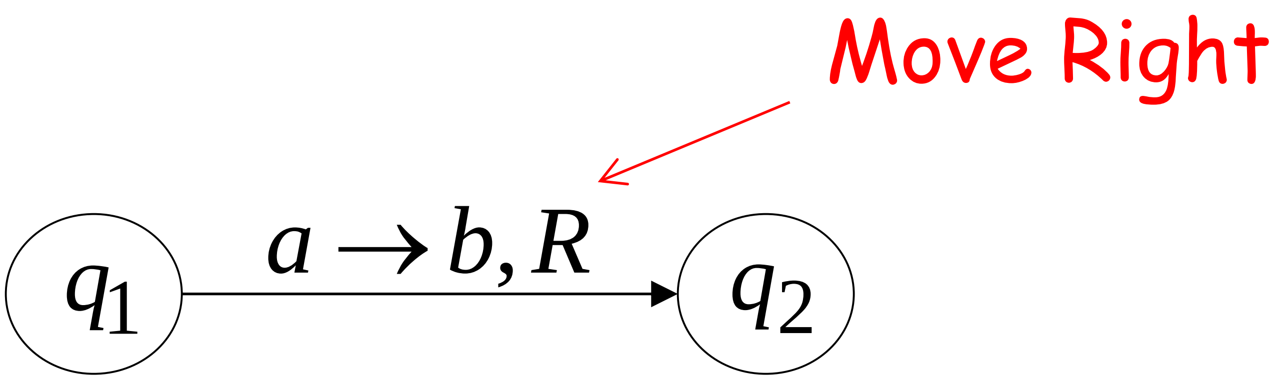
Head starts at the leftmost position of the input string



Remark: the input string is never empty

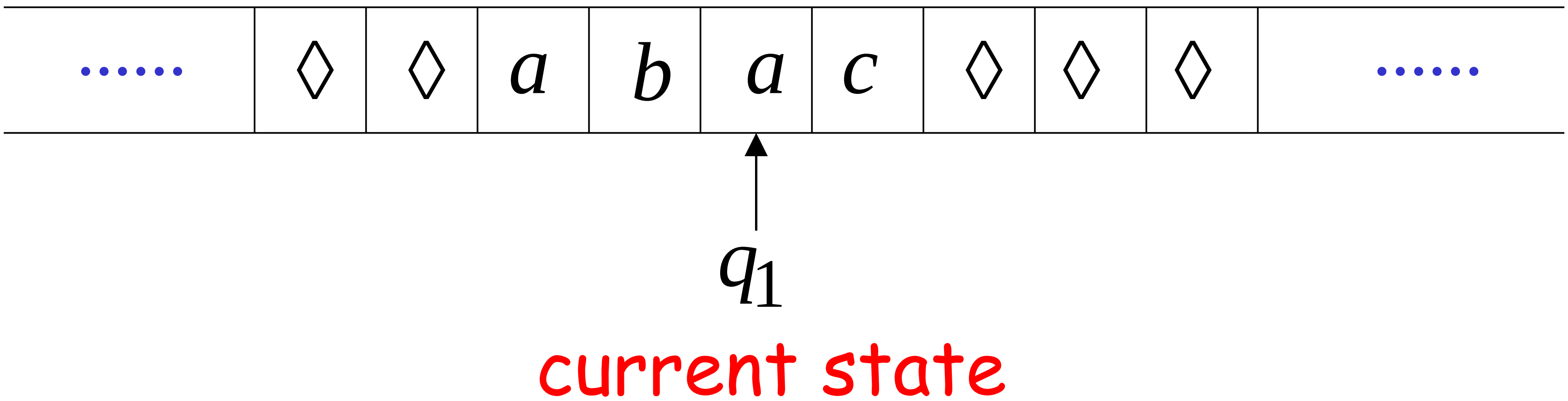
States & Transitions

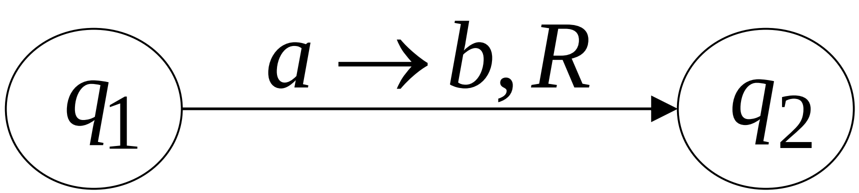




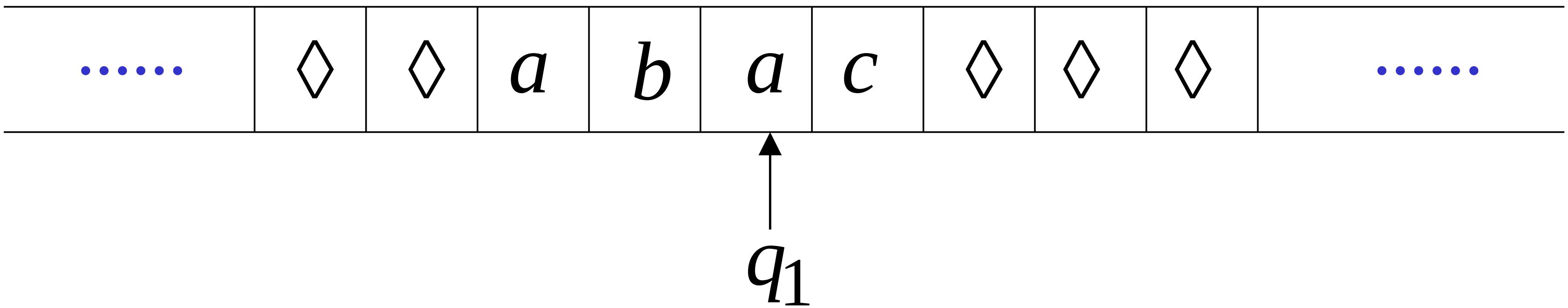
Example:

Time 1

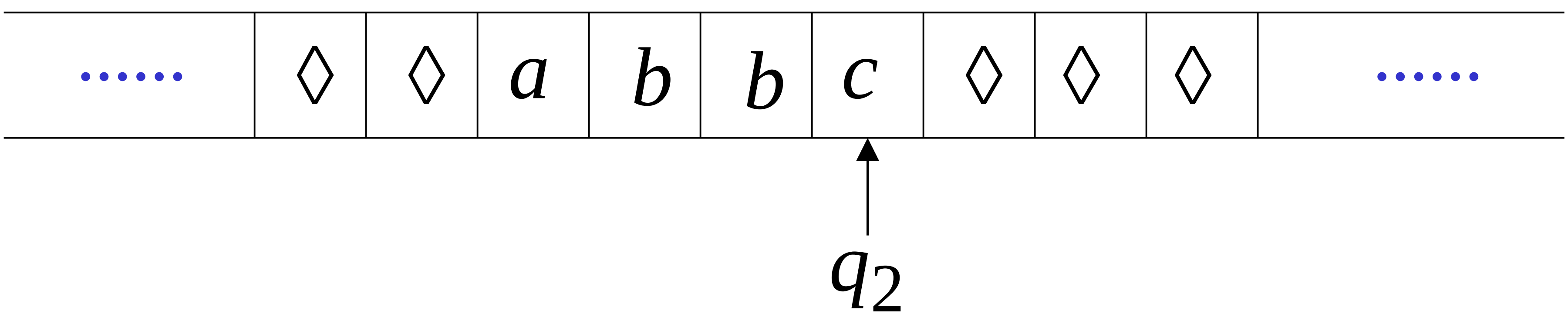




Time 1



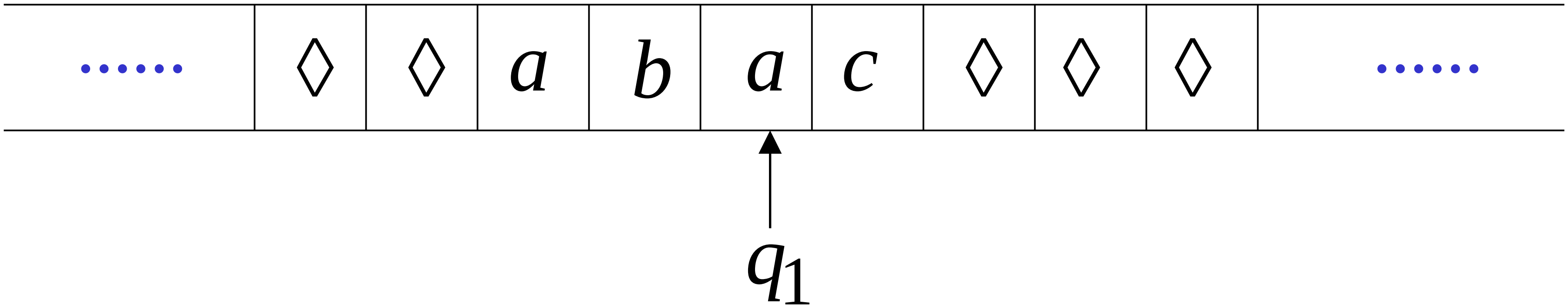
Time 2



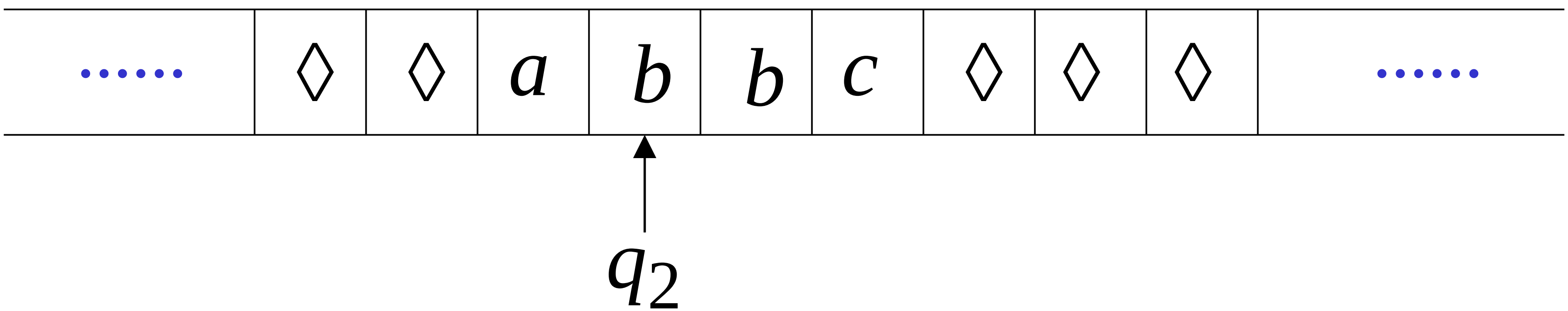
$$q_1 \xrightarrow{a \rightarrow b, R} q_2$$

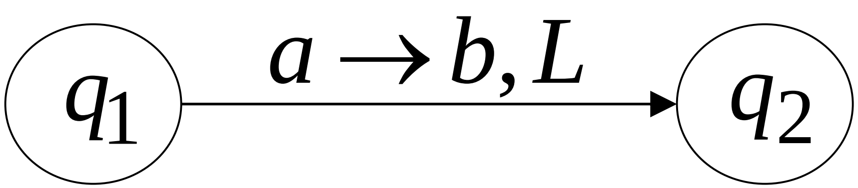
Example:

Time 1

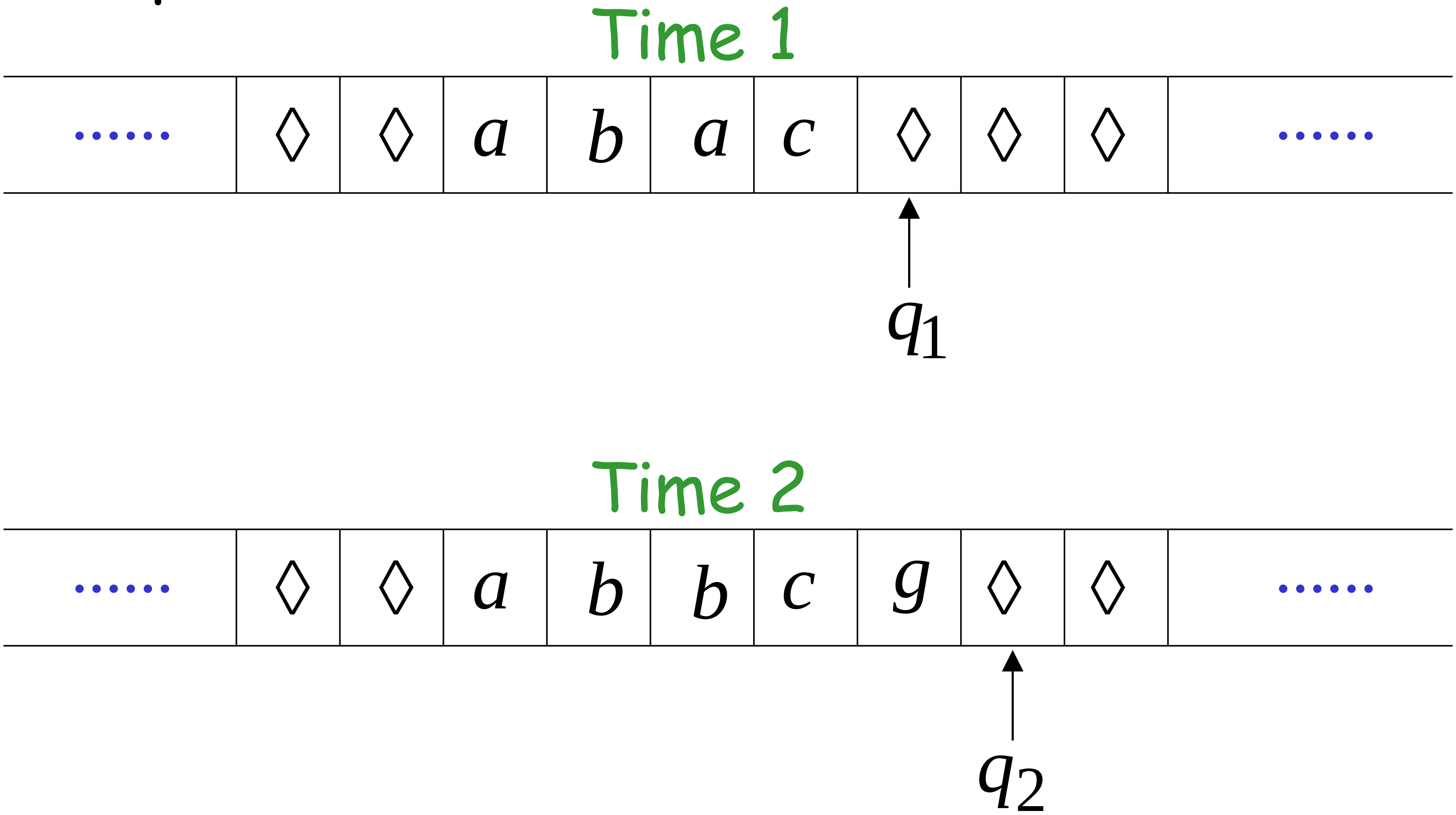


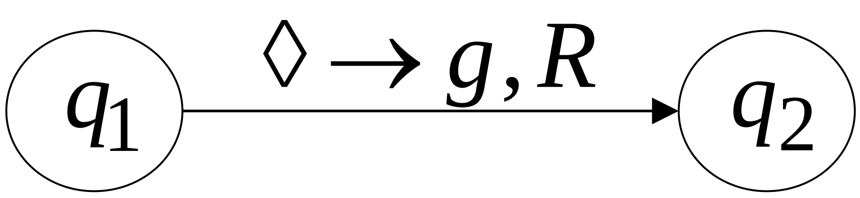
Time 2





Example:

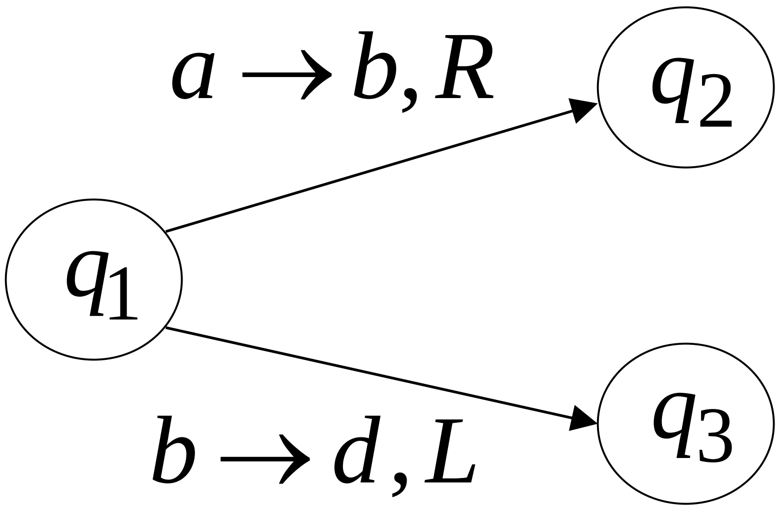




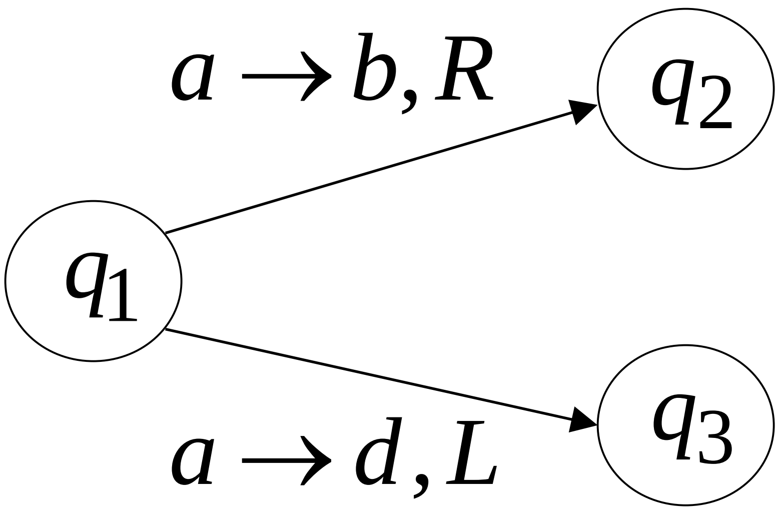
Determinism

Turing Machines are deterministic

Allowed



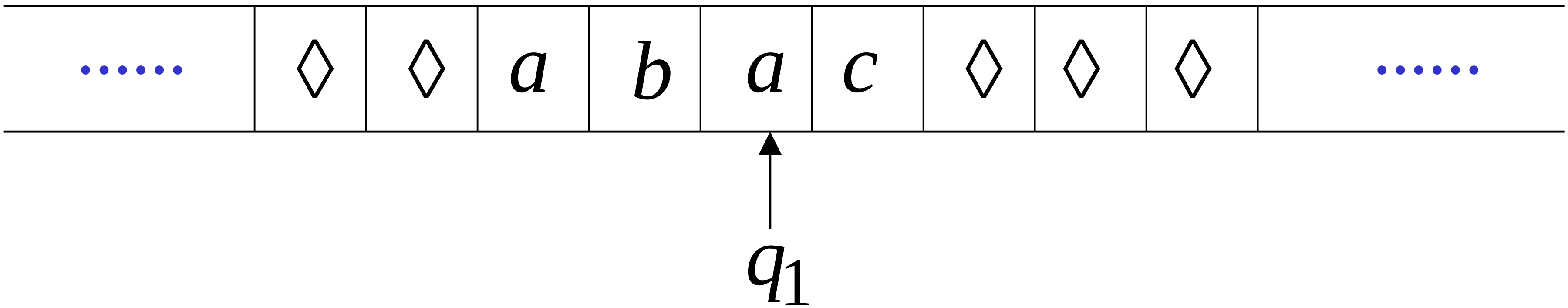
Not Allowed

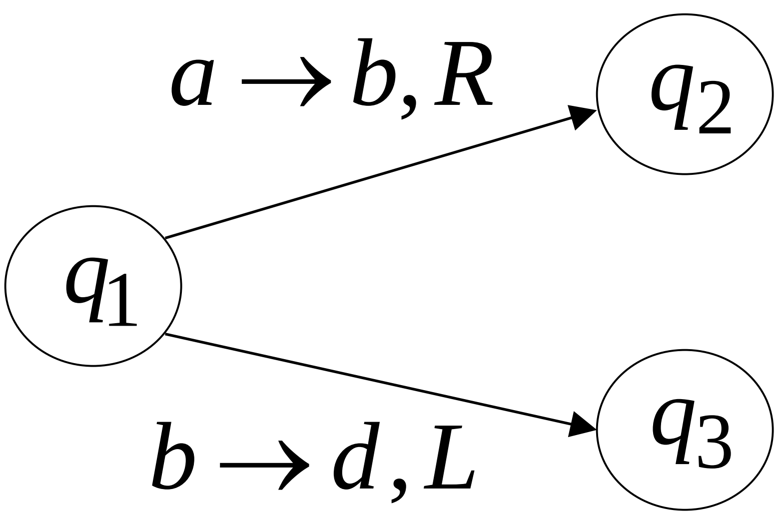


No lambda transitions allowed

Partial Transition Function

Example:





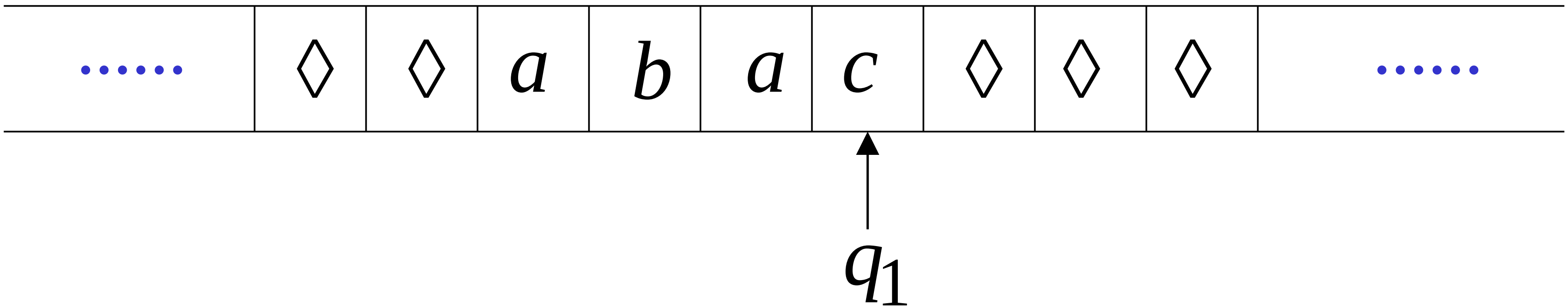
<u>Allowed:</u>

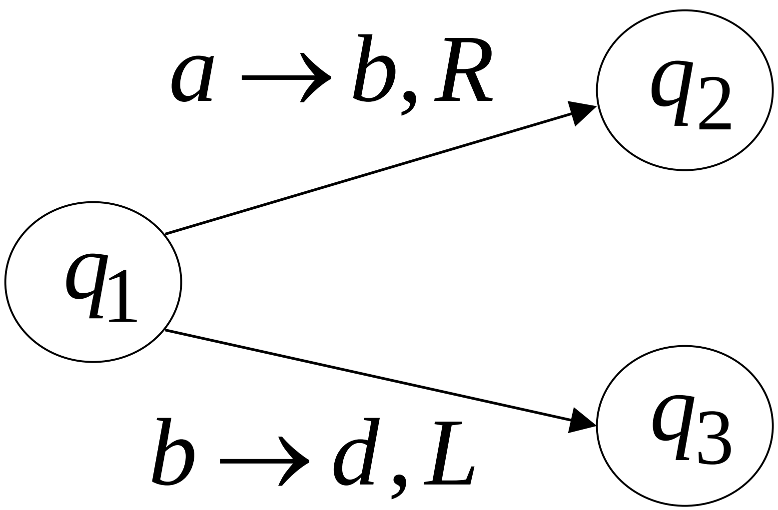
No transition
for input symbol c

Halting

The machine ***halts*** if there are no possible transitions to follow

Example:

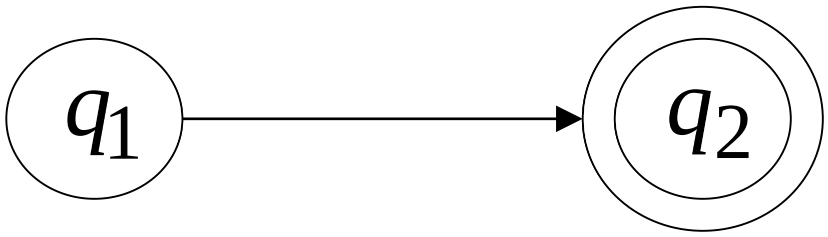




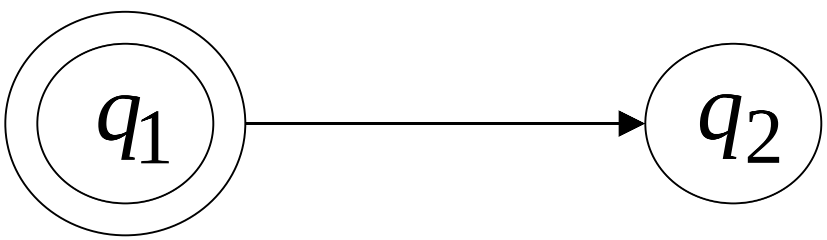
No possible transition

HALT!!!

Final States



Allowed



Not Allowed

- Final states have no outgoing transitions

- In a final state the machine halts

Acceptance

Accept Input → If machine halts in a final state

Reject Input → If machine halts in a non-final state

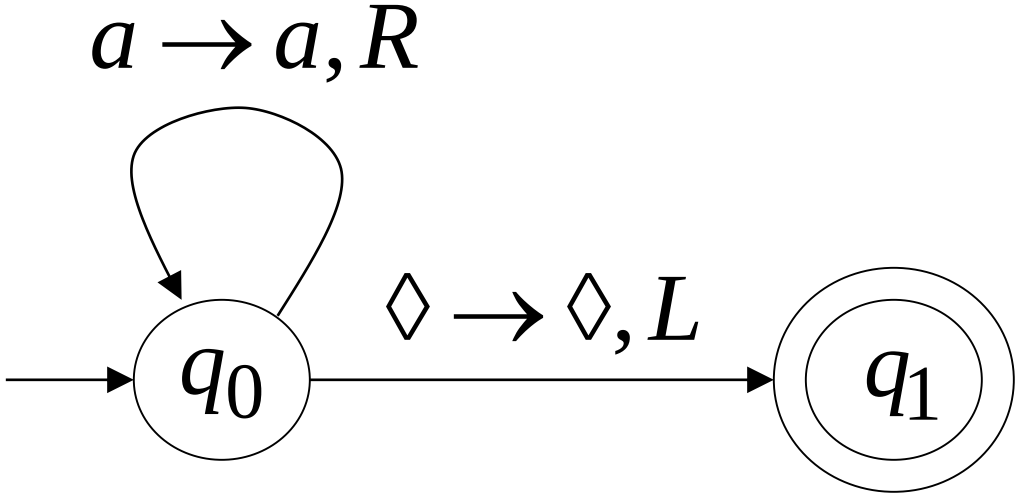
or

If machine enters an *infinite loop*

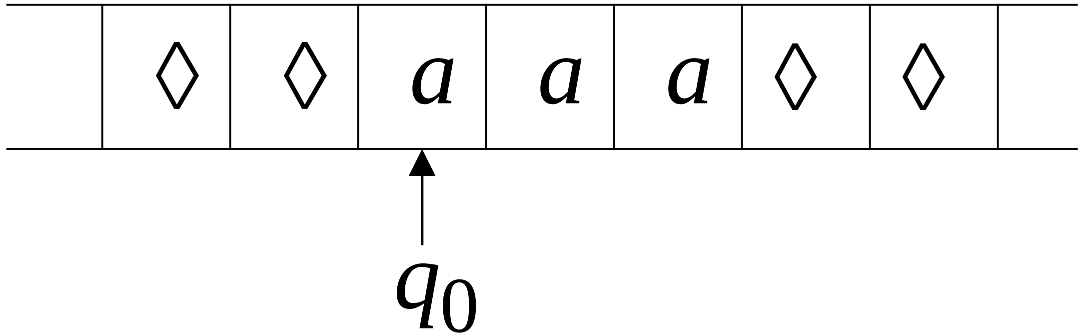
Turing Machine Example

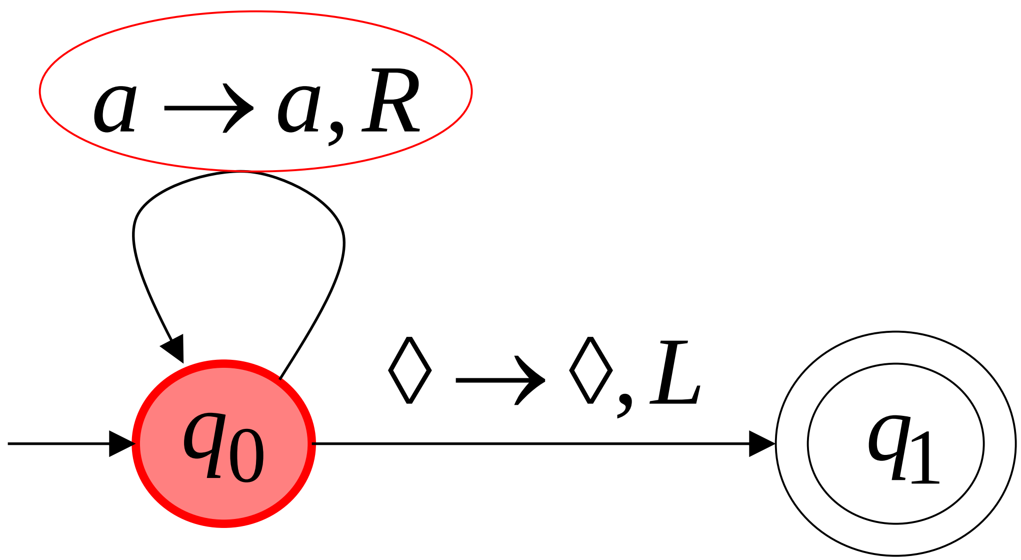
A Turing machine that accepts the language:

$$aa^*$$

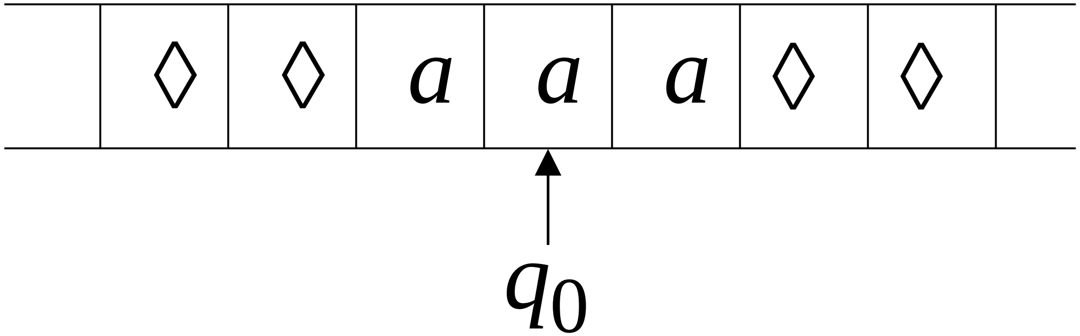


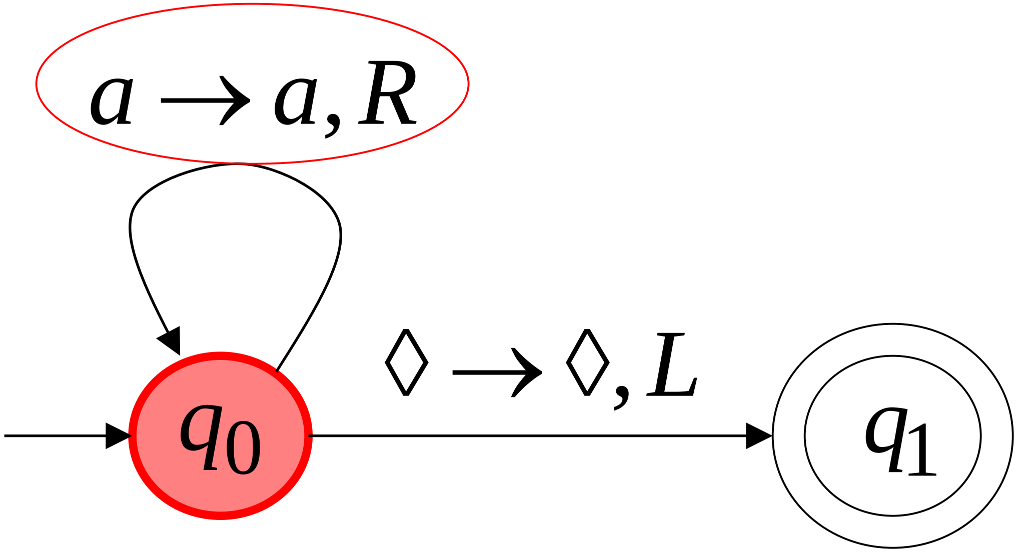
Time 0



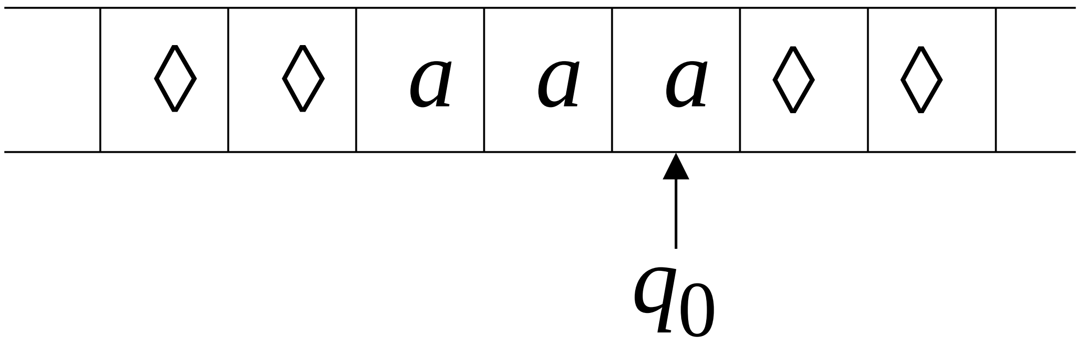


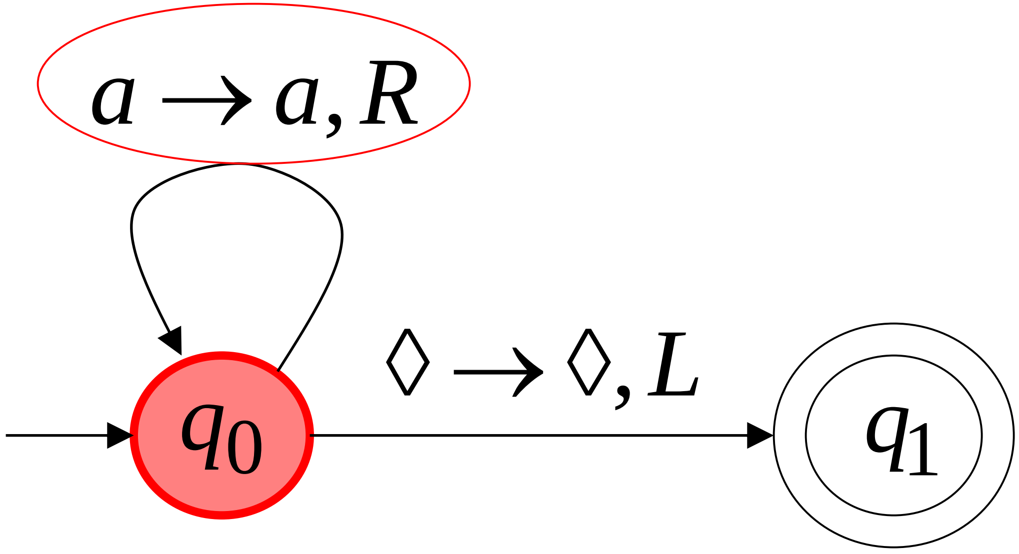
Time 1



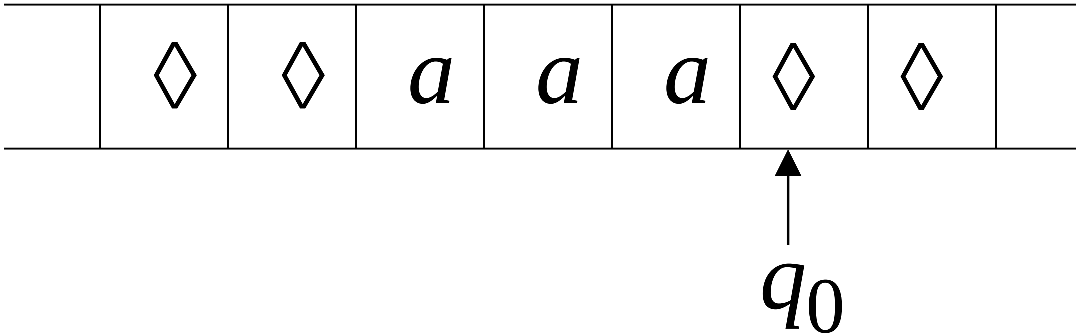


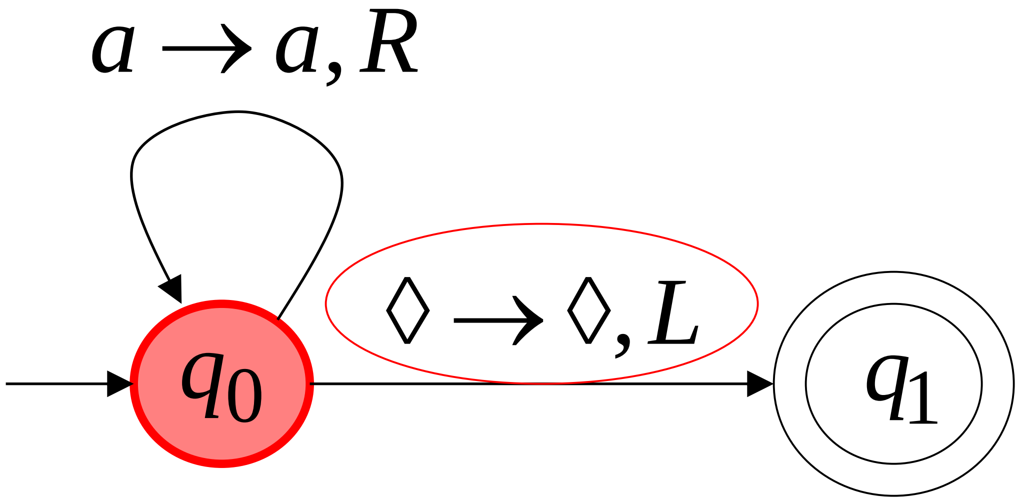
Time 2



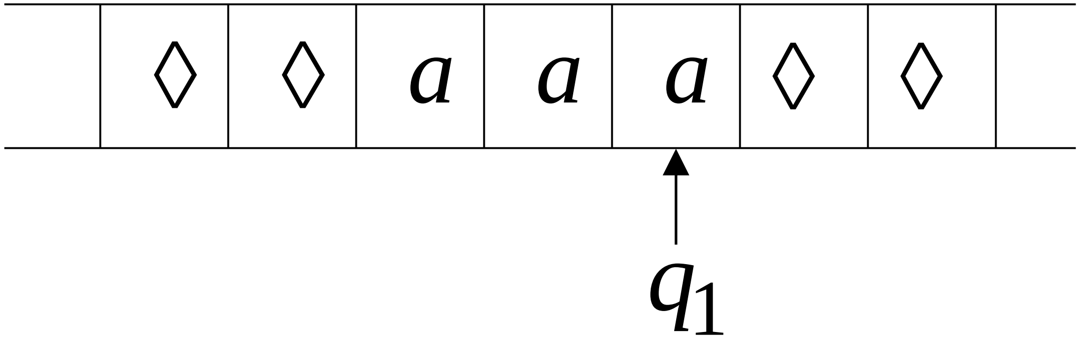


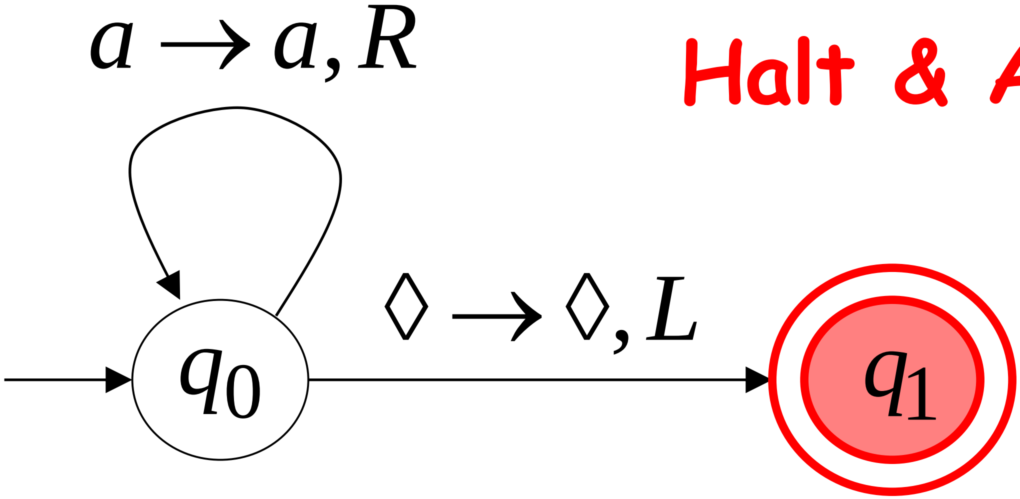
Time 3





Time 4

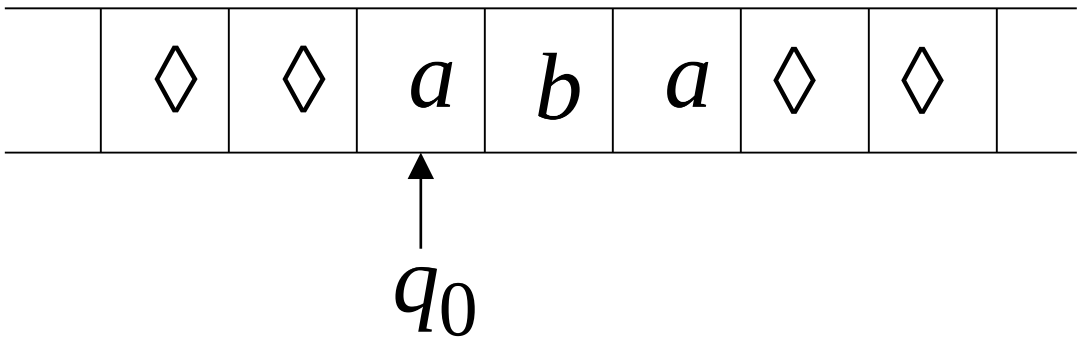


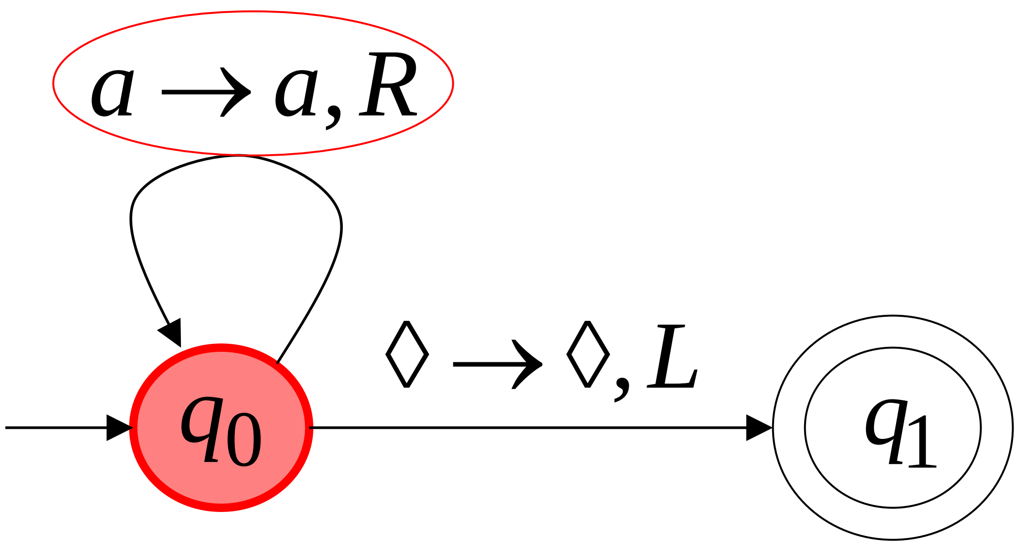


Halt & Accept

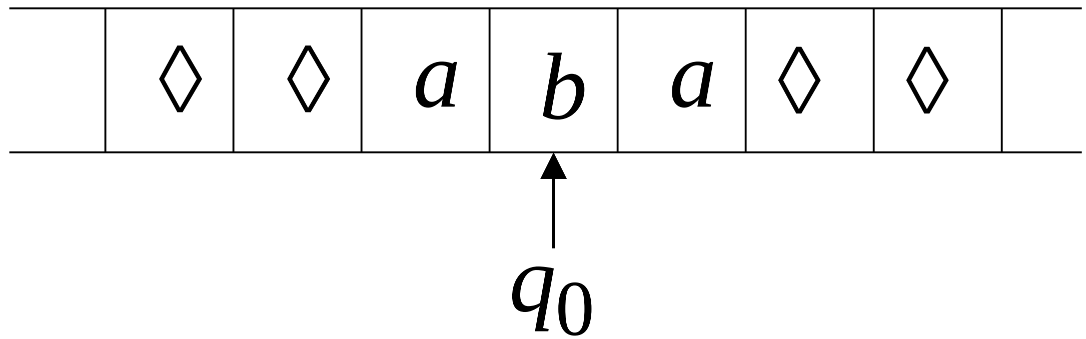
Rejection Example

Time 0



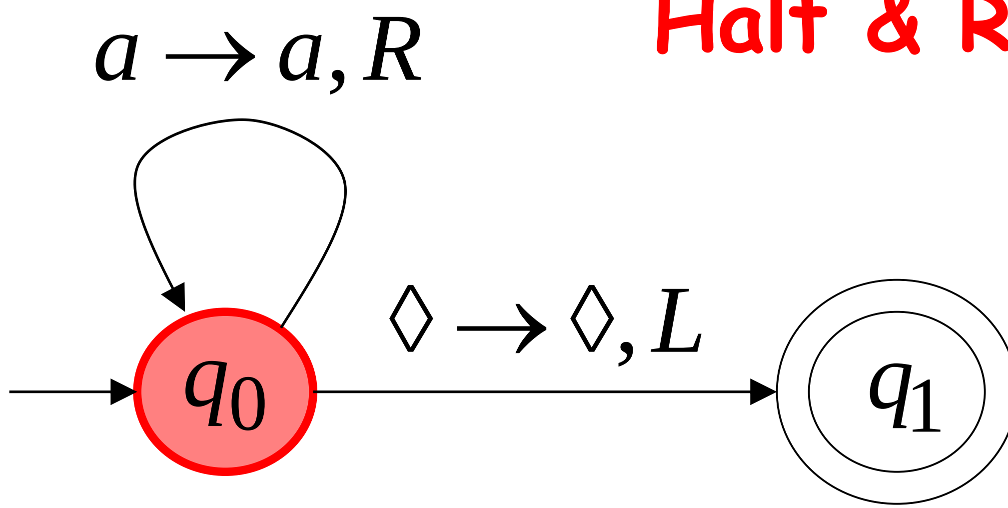


Time 1



No possible Transition

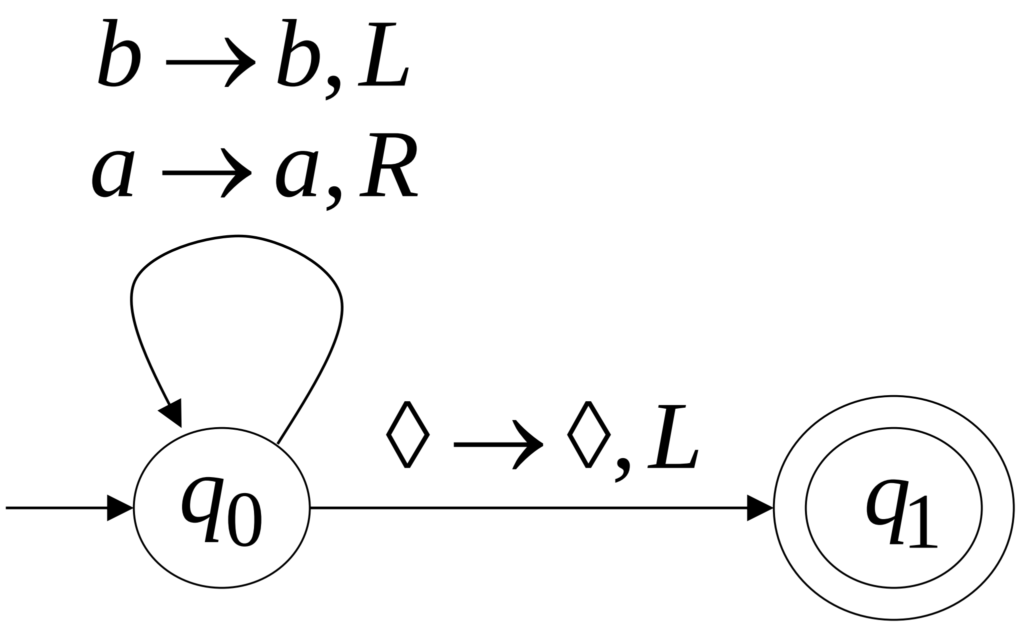
Halt & Reject



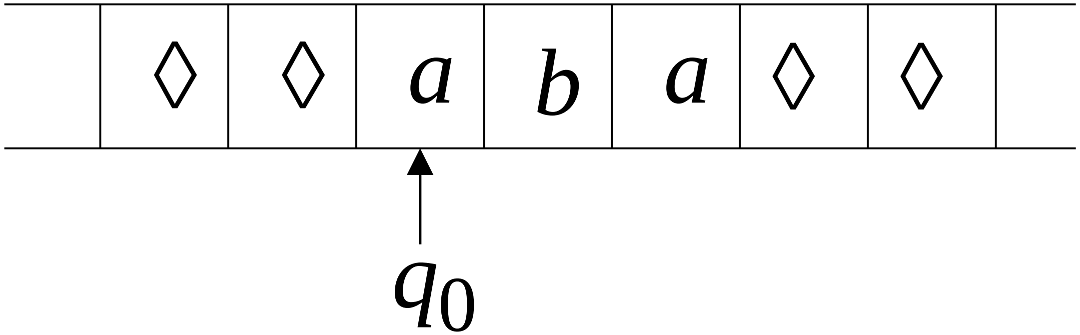
Infinite Loop Example

A Turing machine

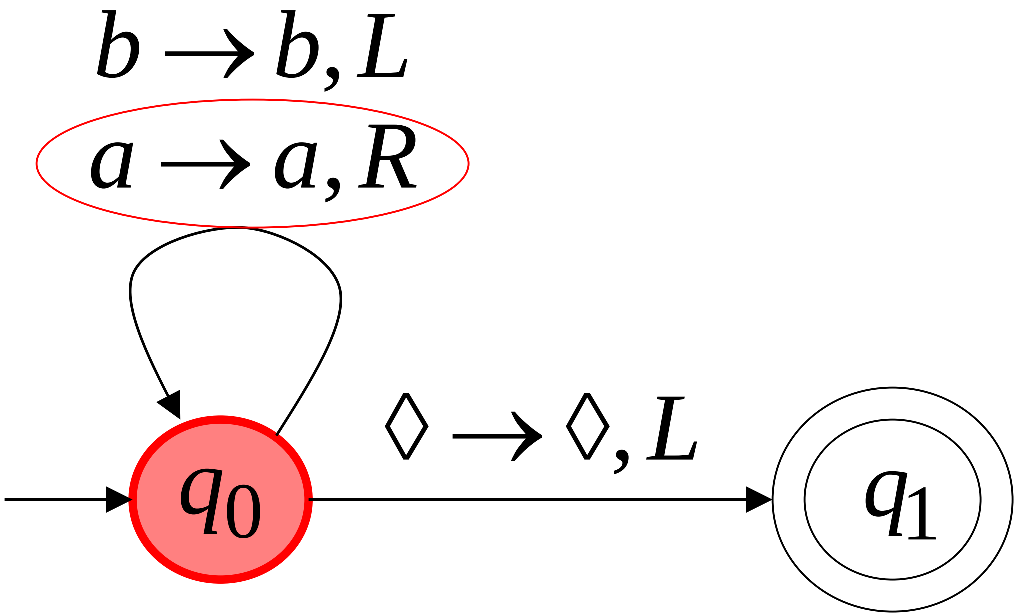
for language $aa^*+b(a+b)^*$



Time 0

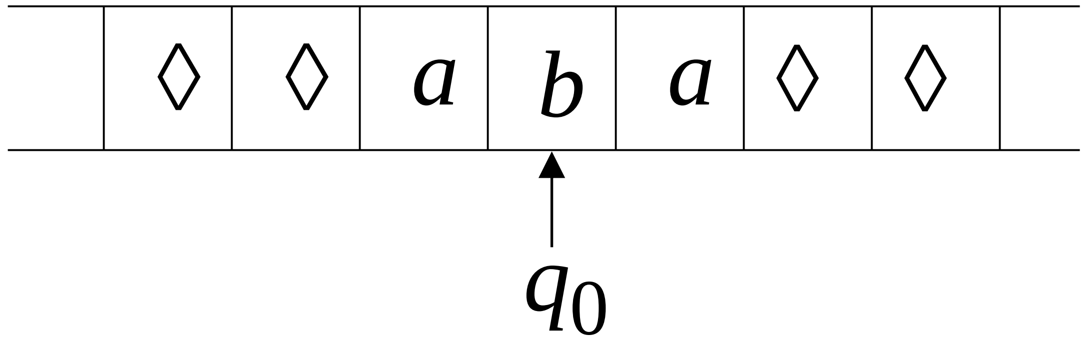


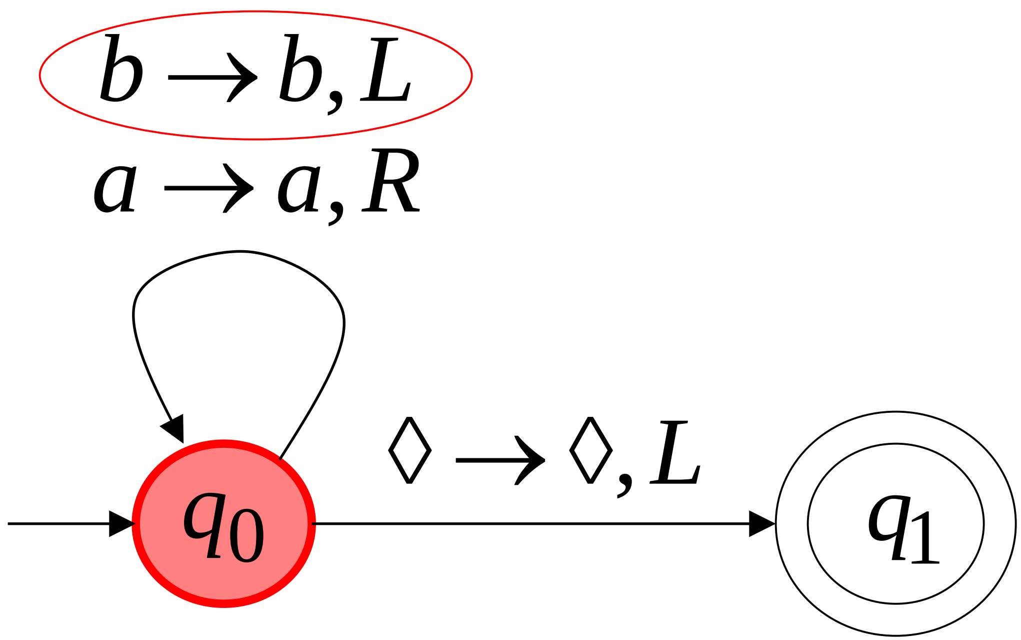
◊
◊
a
b
a
◊
◊
q_0



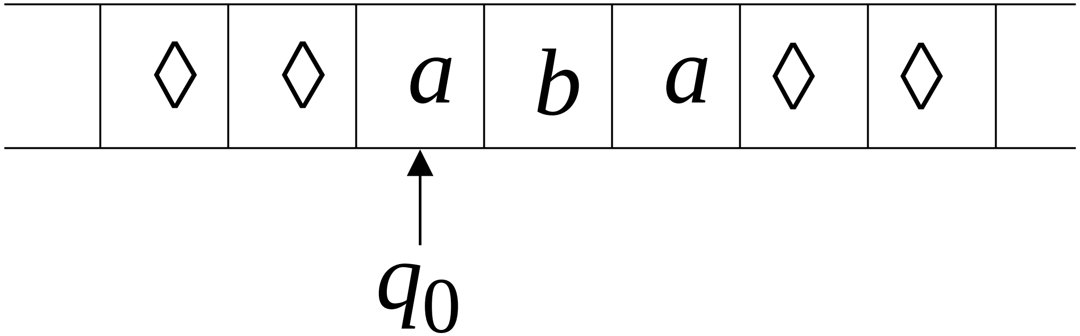
$b \rightarrow b, L$
$a \rightarrow a, R$
$\lozenge \rightarrow \lozenge, L$
q_0
q_1

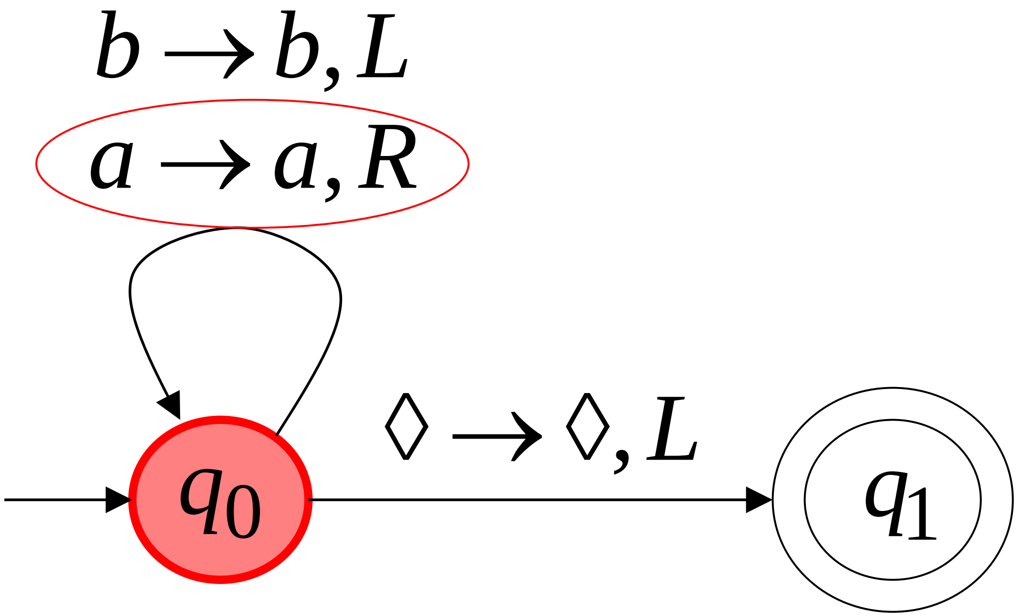
Time 1


◊ ◊ a b a ◊ ◊
q_0

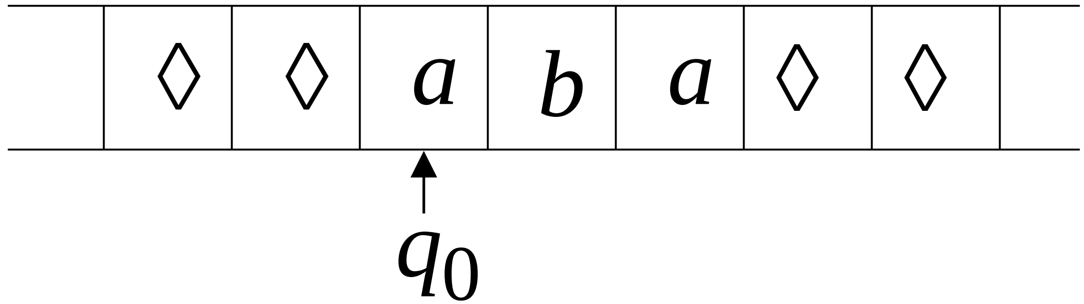

$b \rightarrow b, L$
$a \rightarrow a, R$
$\Diamond \rightarrow \Diamond, L$
q_0
q_1

Time 2

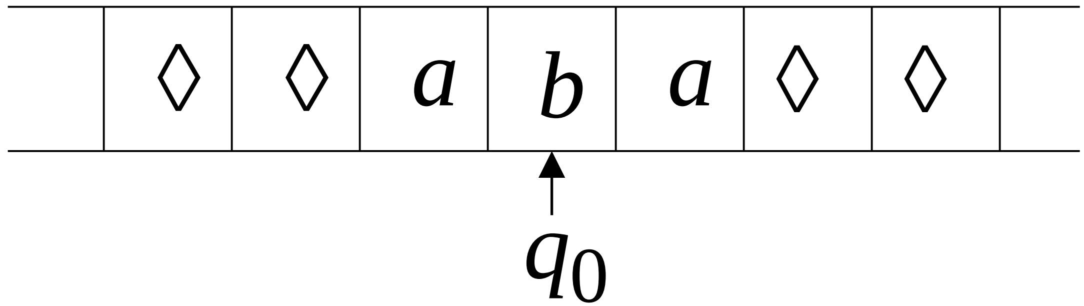




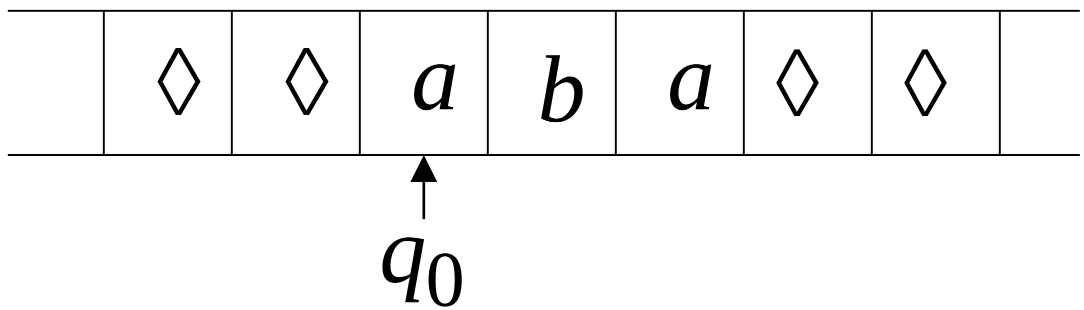
Time 2



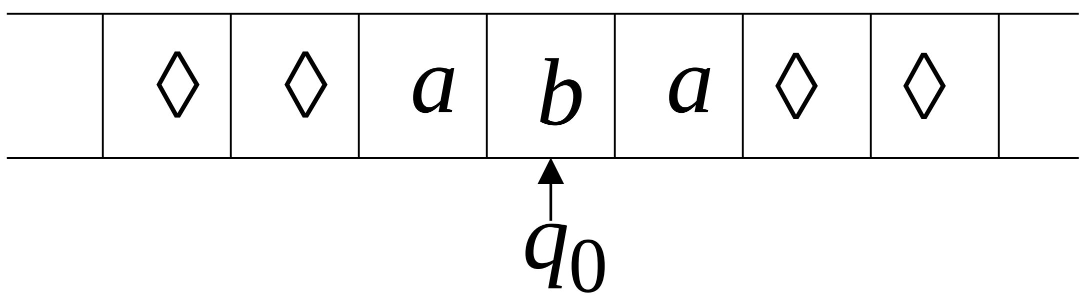
Time 3



Time 4



Time 5



Infinite loop

Because of the infinite loop:

- The final state cannot be reached

- The machine never halts

- The input is not accepted

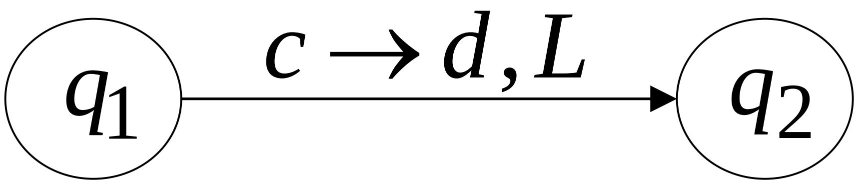
Formal Definitions
for
Turing Machines

Transition Function

$$q_1 \xrightarrow{a \rightarrow b, R} q_2$$

$$\delta(q_1, a) = (q_2, b, R)$$

Transition Function



$$\delta(q_1, c) = (q_2, d, L)$$

Turing Machine:

States

Input alphabet

Tape alphabet

$$M = (Q, \Sigma, \Gamma, \delta, q_0, \Diamond, F)$$

Transition function

Initial state

blank

Final states

Turing Machine:

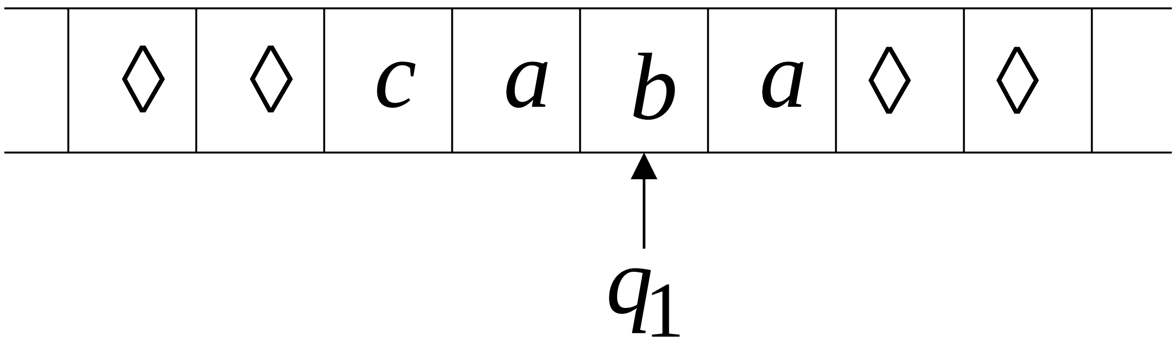
$\square \in \Gamma$

$q0 \in Q$

$F \subseteq Q$

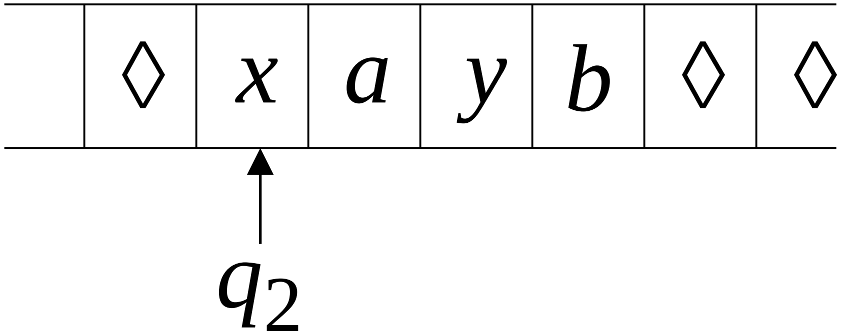
$\delta : Q \times \Gamma \rightarrow Q \times \Gamma \times \{L, R\}$

Configuration

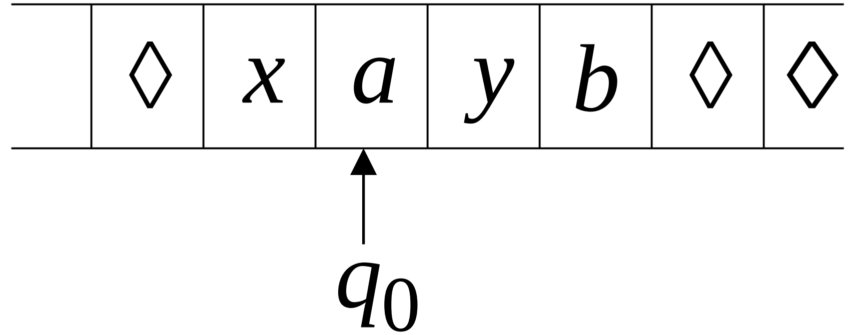


Instantaneous description: $ca\ q_1\ ba$

Time 4

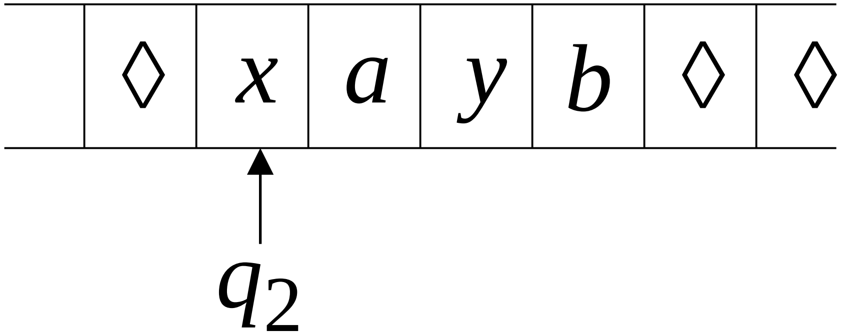


Time 5

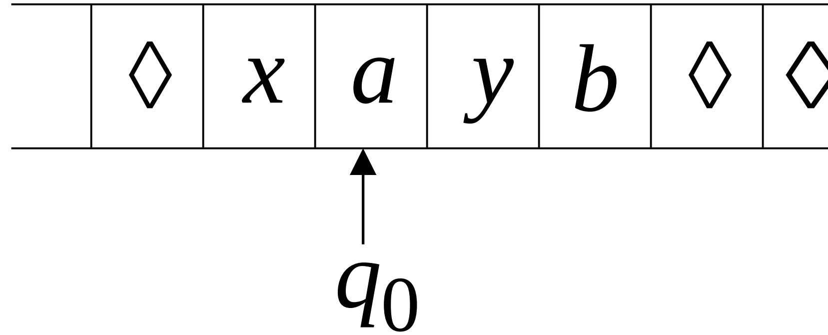


A Move: $q_2\ xayb \succ x\ q_0\ ayb$

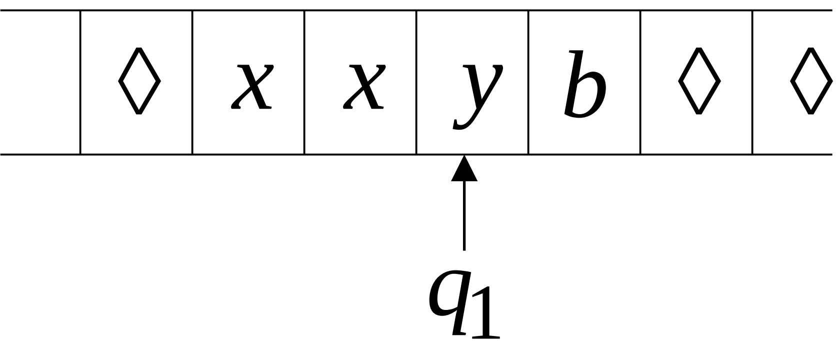
Time 4



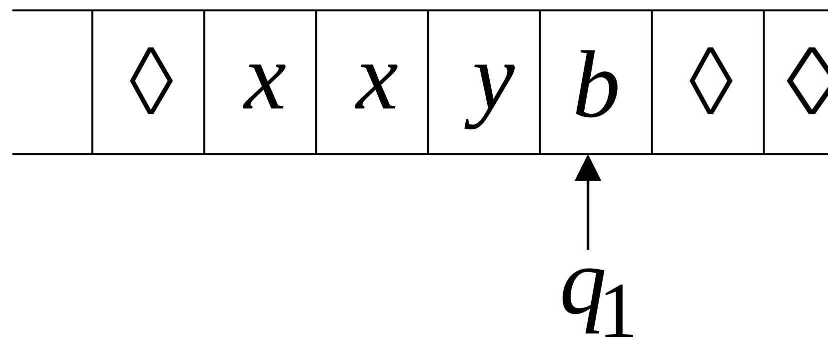
Time 5



Time 6



Time 7



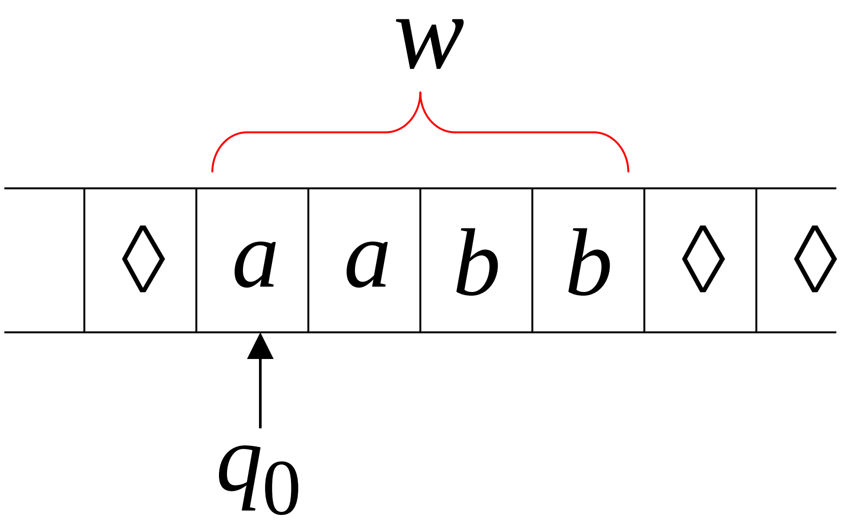
$$q_2\ xayb \succ x\ q_0\ ayb \succ xx\ q_1\ yb \succ xxy\ q_1\ b$$

$$q_2\ xayb \succ x\, q_0\ ayb \succ xx\, q_1\ yb \succ xxy\, q_1\, b$$

Equivalent notation: $$q_2\ xayb \overset{*}{\succ} xxy\, q_1\, b$$

Initial configuration: $q_0\ w$

Input string



The Accepted Language

For any Turing Machine M

$$L(M) = \{w : \quad q_0 \, w \overset{*}{\succ} x_1 \, q_f \, x_2\}$$

Initial state

Final state

Standard Turing Machine

The machine we described is the standard:

- Deterministic
- Infinite tape in both directions
- Tape is the input/output file

Another Turing Machine Example

Turing machine for the language $\{a^n b^n \mid n \geq 1\}$

$$\underline{a\,a\,a\,a\,b\,b\,b\,b}$$

$$\cancel{a}\,a\,a\,a\,\cancel{b}\,b\,b\,b$$

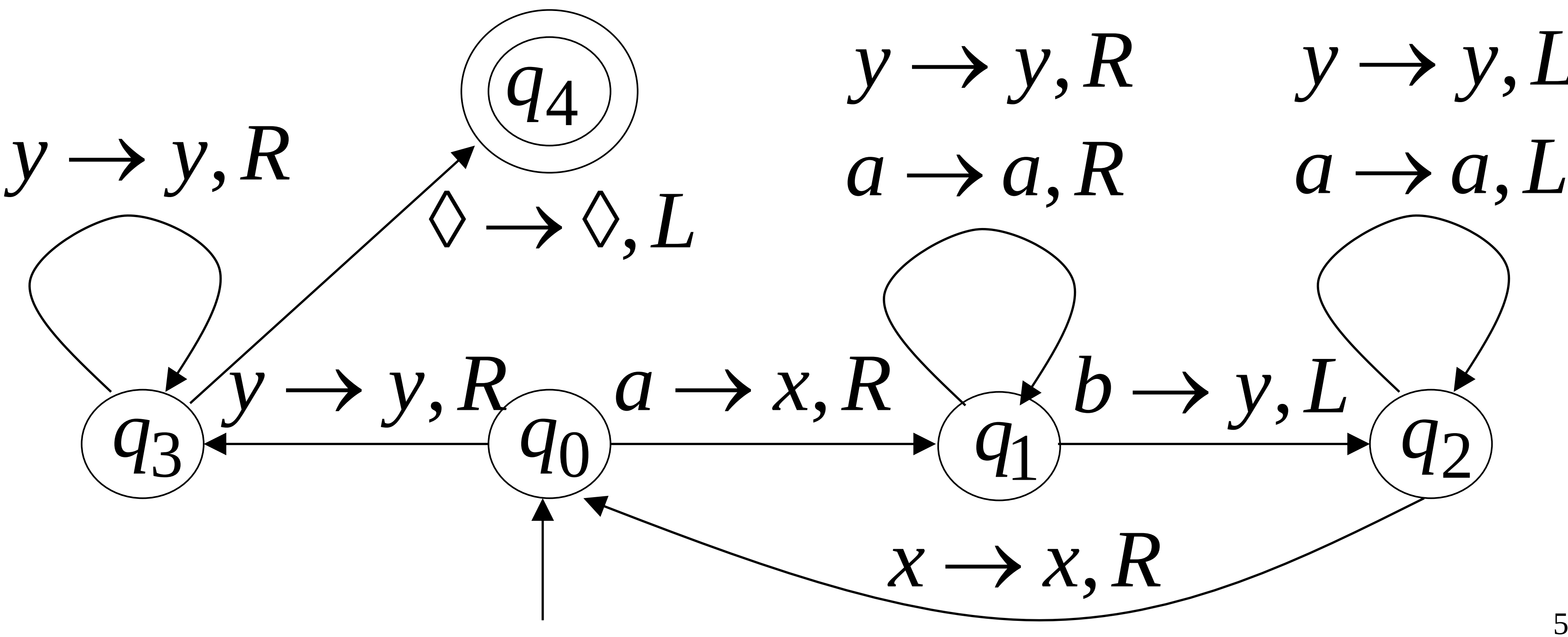
$$\cancel{a}\,\cancel{a}\,a\,a\,\cancel{b}\,\cancel{b}\,b\,b$$

$$\cancel{a}\,\cancel{a}\,\cancel{a}\,a\,\cancel{b}\,\cancel{b}\,\cancel{b}\,b$$

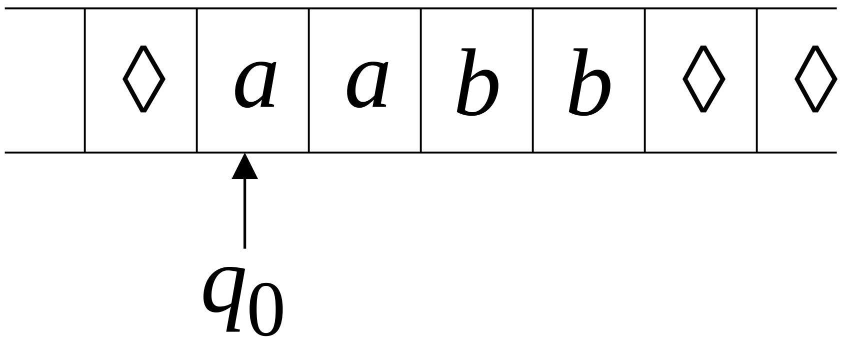
$$\cancel{a}\,\cancel{a}\,\cancel{a}\,\cancel{a}\,\cancel{b}\,\cancel{b}\,\cancel{b}\,\cancel{b}$$

Another Turing Machine Example

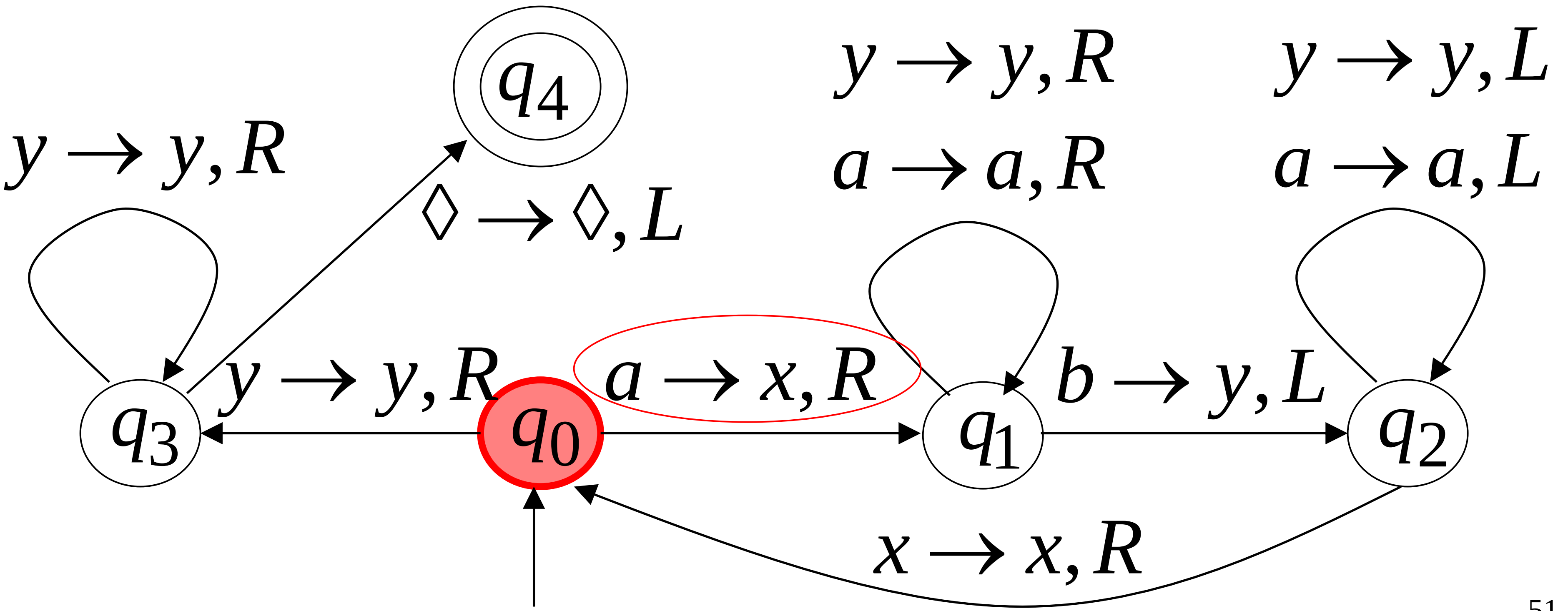
Turing machine for the language $\{a^n b^n\}$



Time 0

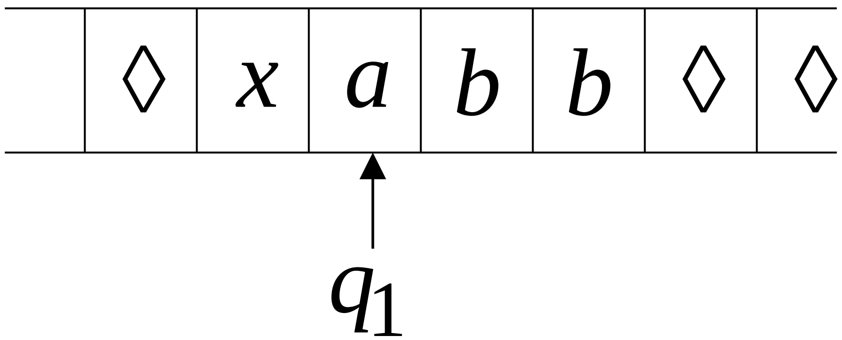


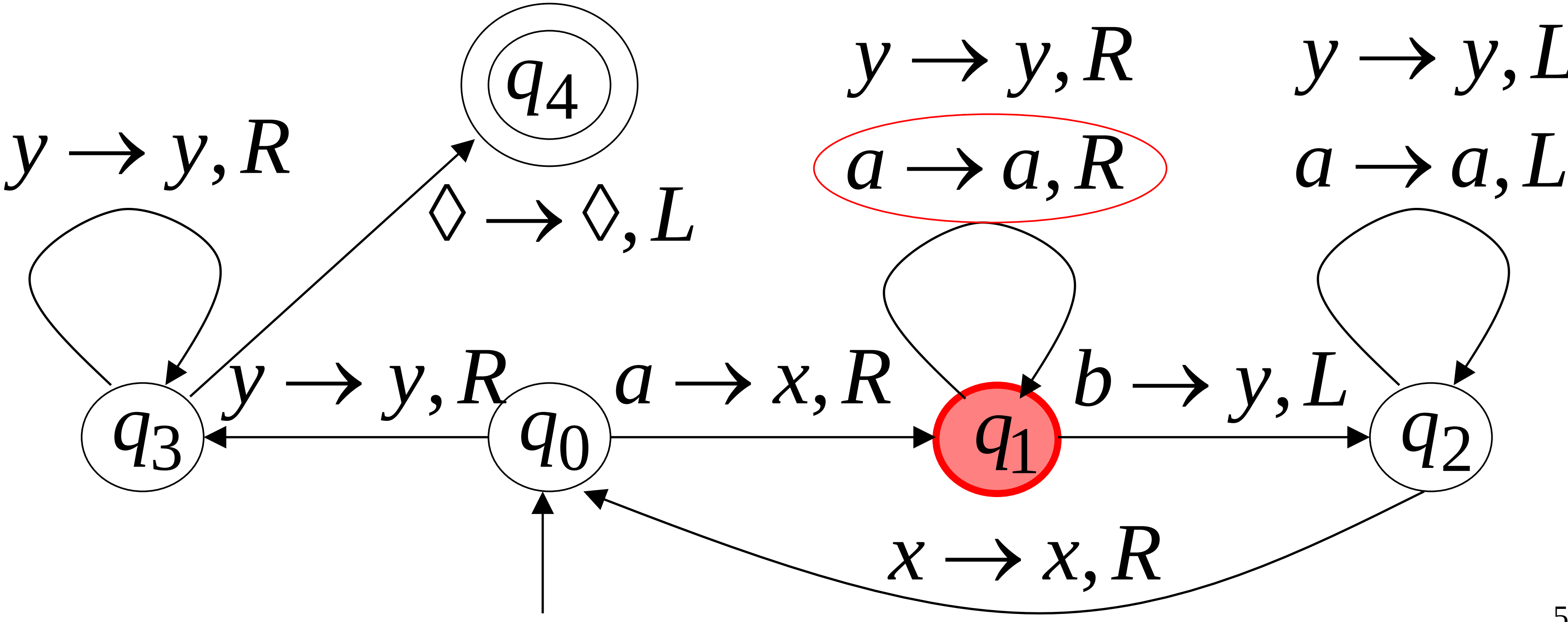
◊
a
a
b
b
◊
◊
q_0



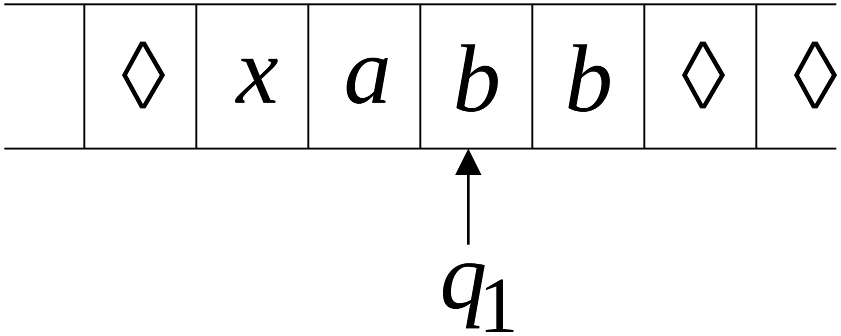
q_4
$y \to y, R$
$\lozenge \to \lozenge, L$
$y \to y, R$
$a \to a, R$
$y \to y, L$
$a \to a, L$
q_3
$y \to y, R$
q_0
$a \to x, R$
q_1
$b \to y, L$
q_2
$x \to x, R$

Time 1

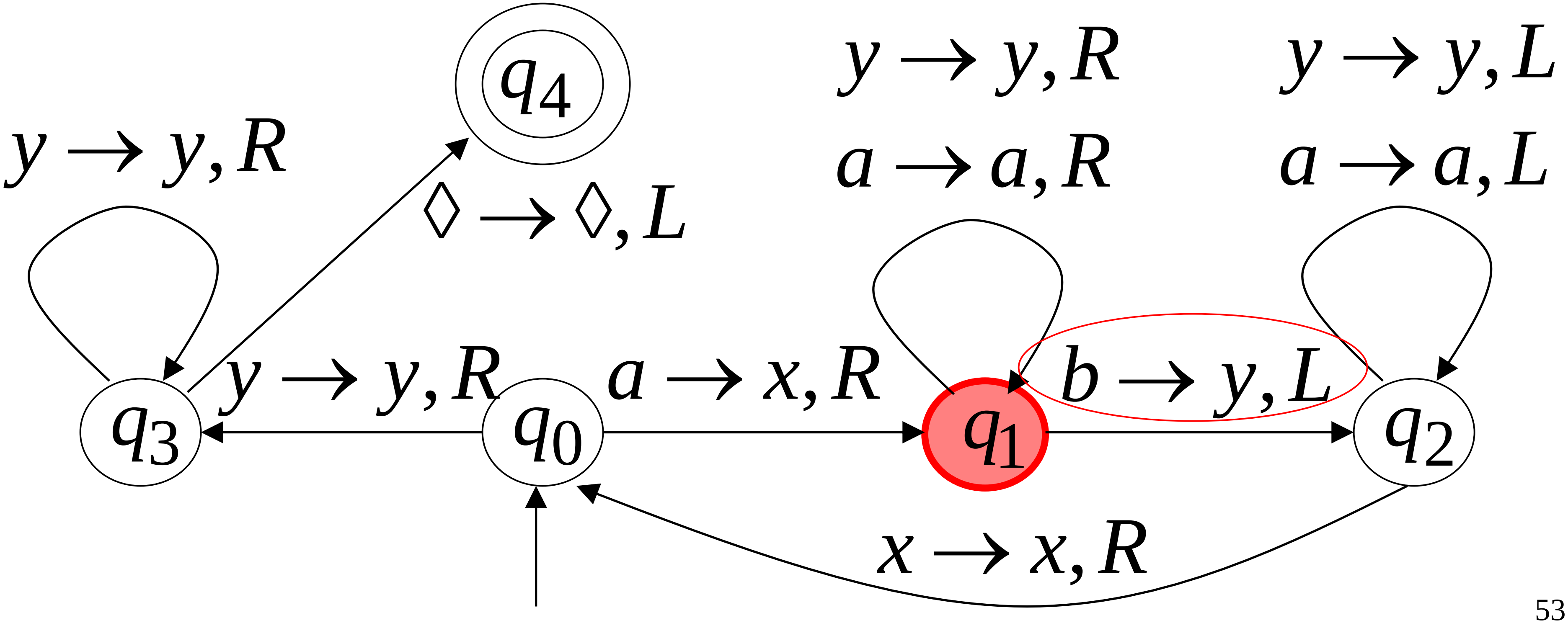

◊
x
a
b
b
◊
◊
q_1


q_4
$y \to y, R$
$a \to a, R$
$y \to y, L$
$a \to a, L$
$y \to y, R$
$\Diamond \to \Diamond, L$
$y \to y, R$
$a \to x, R$
$b \to y, L$
q_3
q_0
q_1
q_2
$x \to x, R$

Time 2

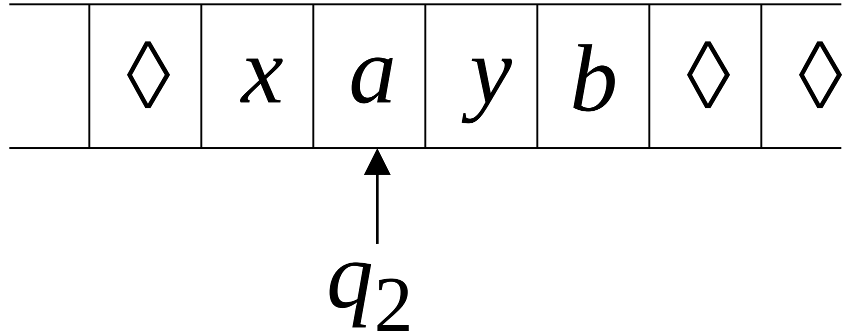


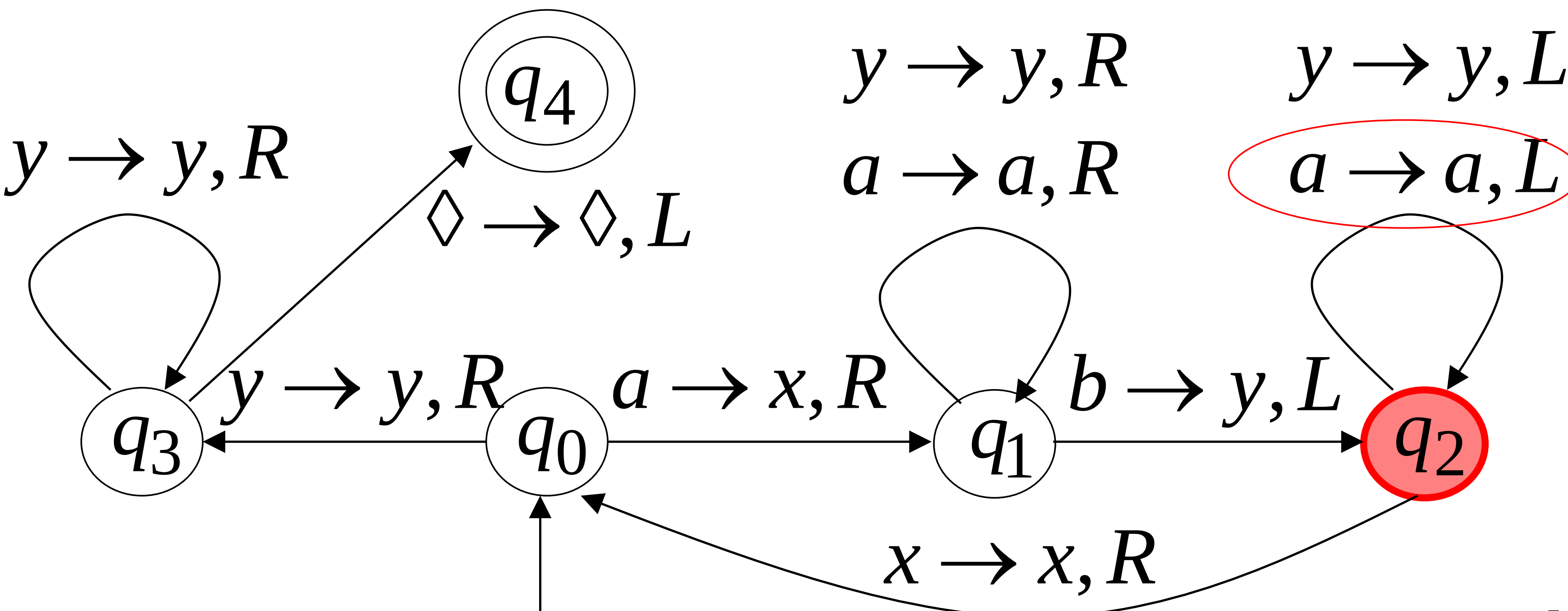
◊
x
a
b
b
◊
◊
q_1



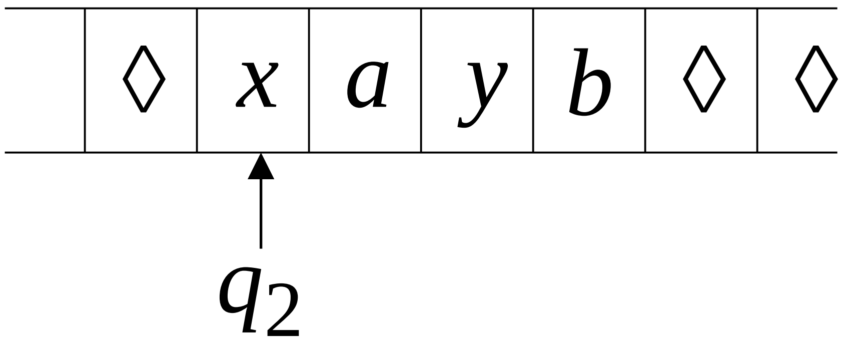
q_4
$\Diamond \rightarrow \Diamond, L$
$y \rightarrow y, R$
q_3
$y \rightarrow y, R$
q_0
$a \rightarrow x, R$
$y \rightarrow y, R$
$a \rightarrow a, R$
q_1
$b \rightarrow y, L$
$y \rightarrow y, L$
$a \rightarrow a, L$
q_2
$x \rightarrow x, R$

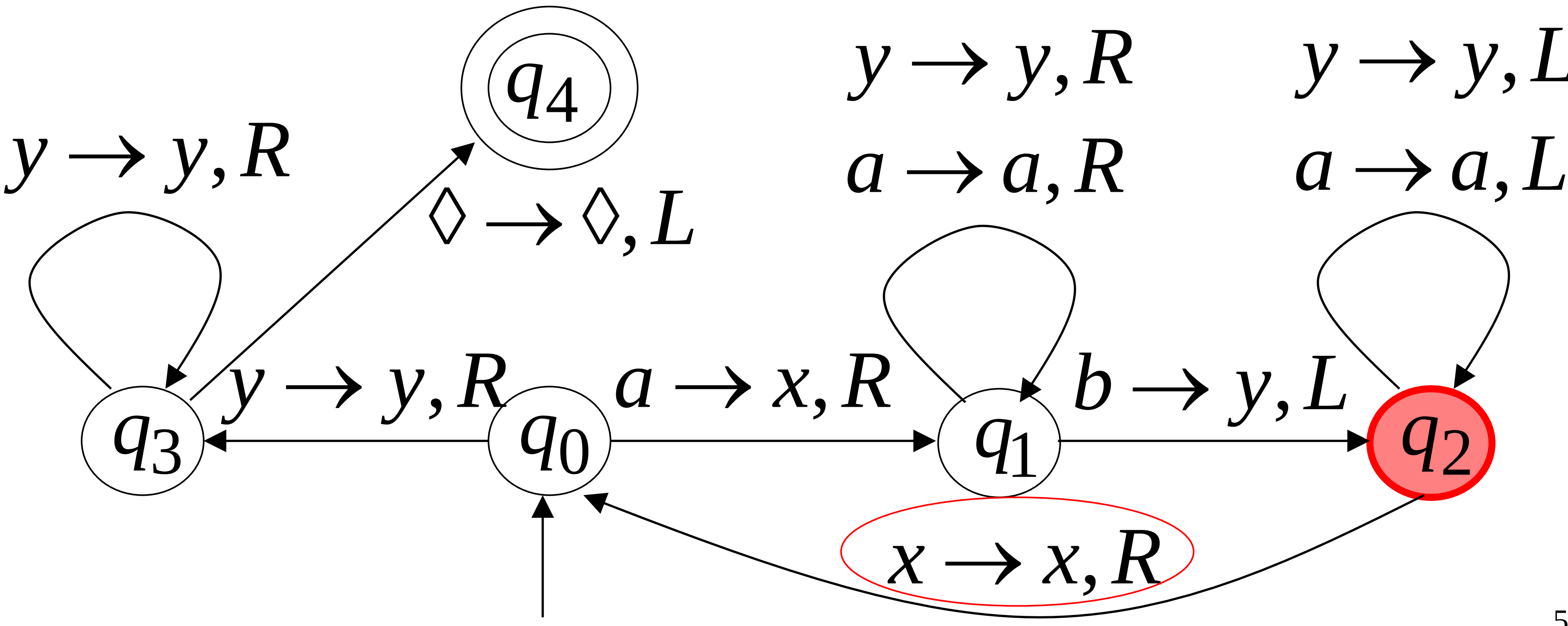
Time 3


◊
x
a
y
b
◊
◊
q_2

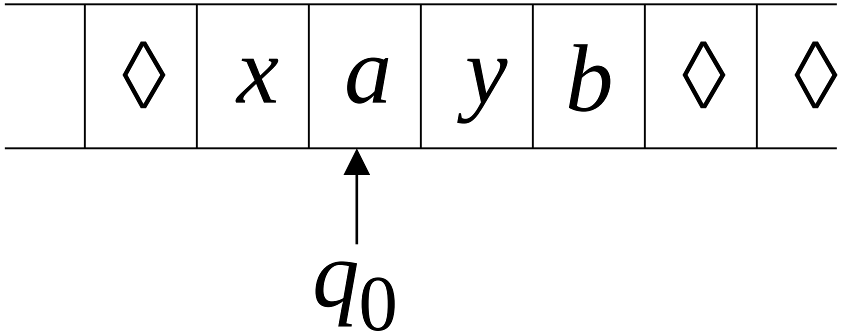

q_4
$y \to y, R$
$\Diamond \to \Diamond, L$
$y \to y, R$
$a \to a, R$
$y \to y, L$
$a \to a, L$
q_3
$y \to y, R$
q_0
$a \to x, R$
q_1
$b \to y, L$
q_2
$x \to x, R$

Time 4

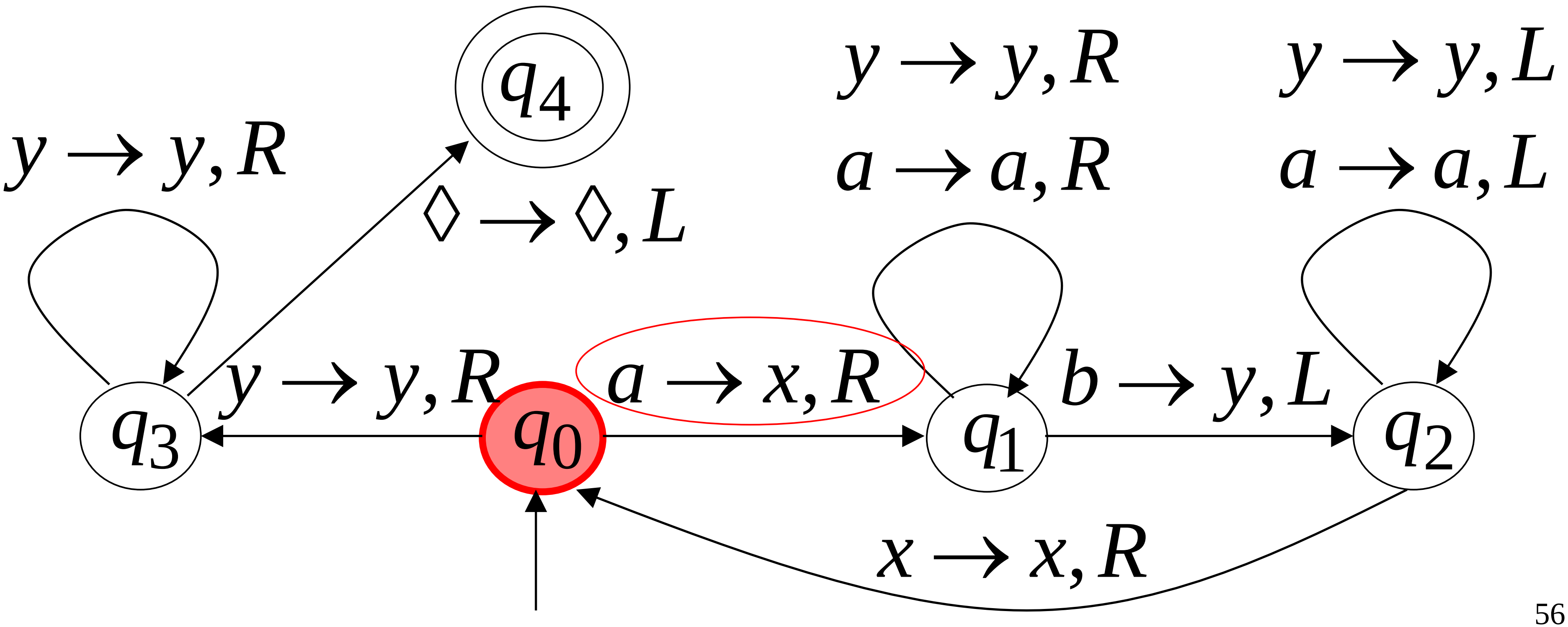

◊ x a y b ◊ ◊
q_2


q_4
$\Diamond \rightarrow \Diamond, L$
$y \rightarrow y, R$
q_3
$y \rightarrow y, R$
q_0
$a \rightarrow x, R$
$y \rightarrow y, R$
$a \rightarrow a, R$
q_1
$b \rightarrow y, L$
$y \rightarrow y, L$
$a \rightarrow a, L$
q_2
$x \rightarrow x, R$

Time 5

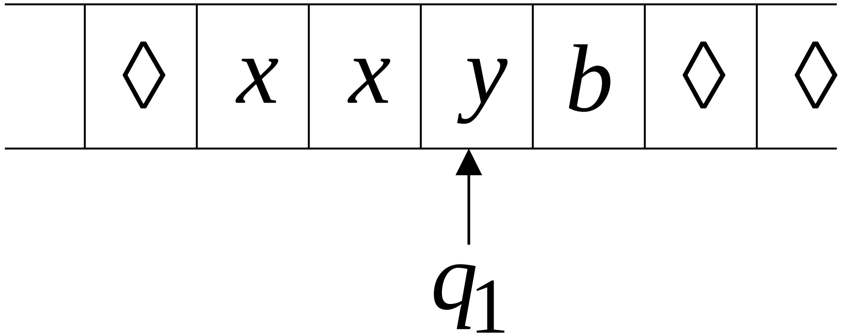


◊
x
a
y
b
◊
◊
q_0

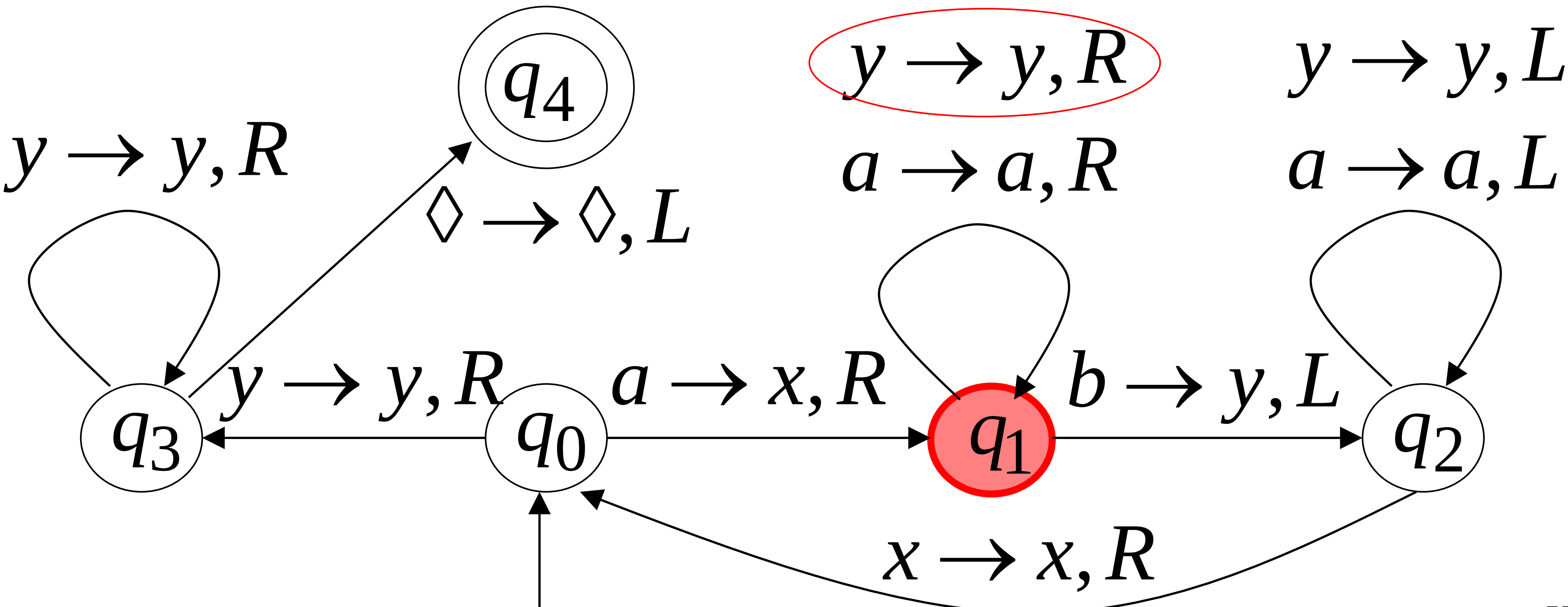


q_4
$y \rightarrow y, R$
$\Diamond \rightarrow \Diamond, L$
$y \rightarrow y, R$
$a \rightarrow a, R$
$y \rightarrow y, L$
$a \rightarrow a, L$
q_3
$y \rightarrow y, R$
q_0
$a \rightarrow x, R$
q_1
$b \rightarrow y, L$
q_2
$x \rightarrow x, R$

Time 6

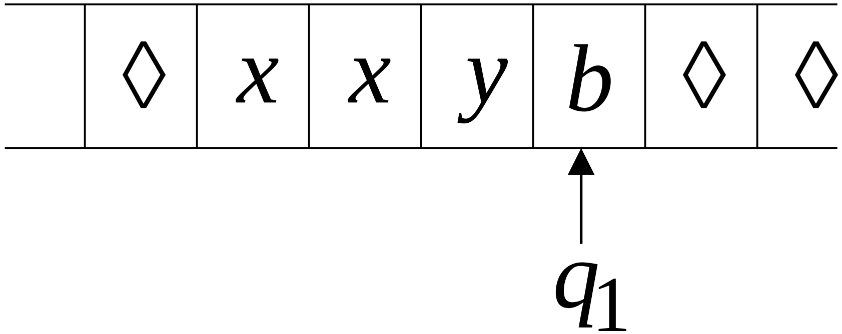


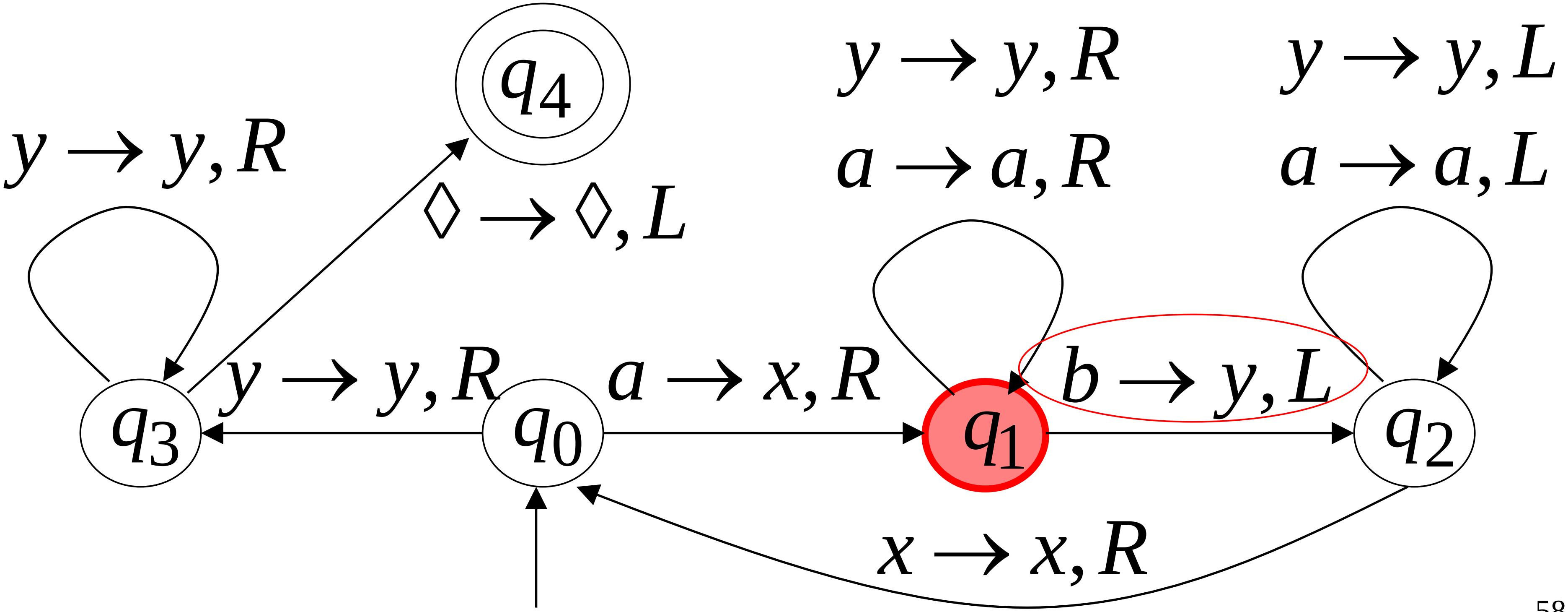
◊
x
x
y
b
◊
◊
q_1



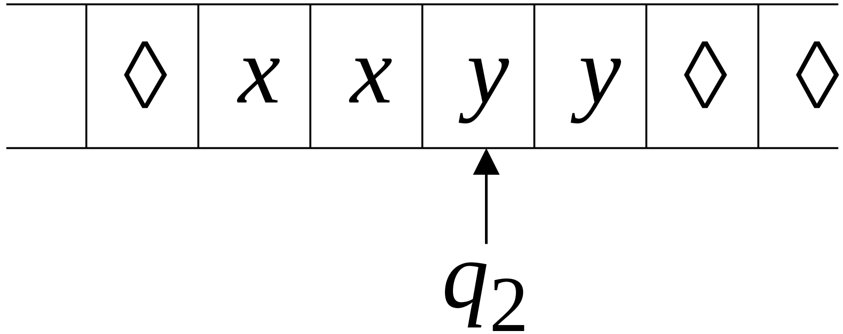
q_4
$y \rightarrow y, R$
$\Diamond \rightarrow \Diamond, L$
q_3
$y \rightarrow y, R$
q_0
$a \rightarrow x, R$
$y \rightarrow y, R$
$a \rightarrow a, R$
q_1
$b \rightarrow y, L$
$y \rightarrow y, L$
$a \rightarrow a, L$
q_2
$x \rightarrow x, R$

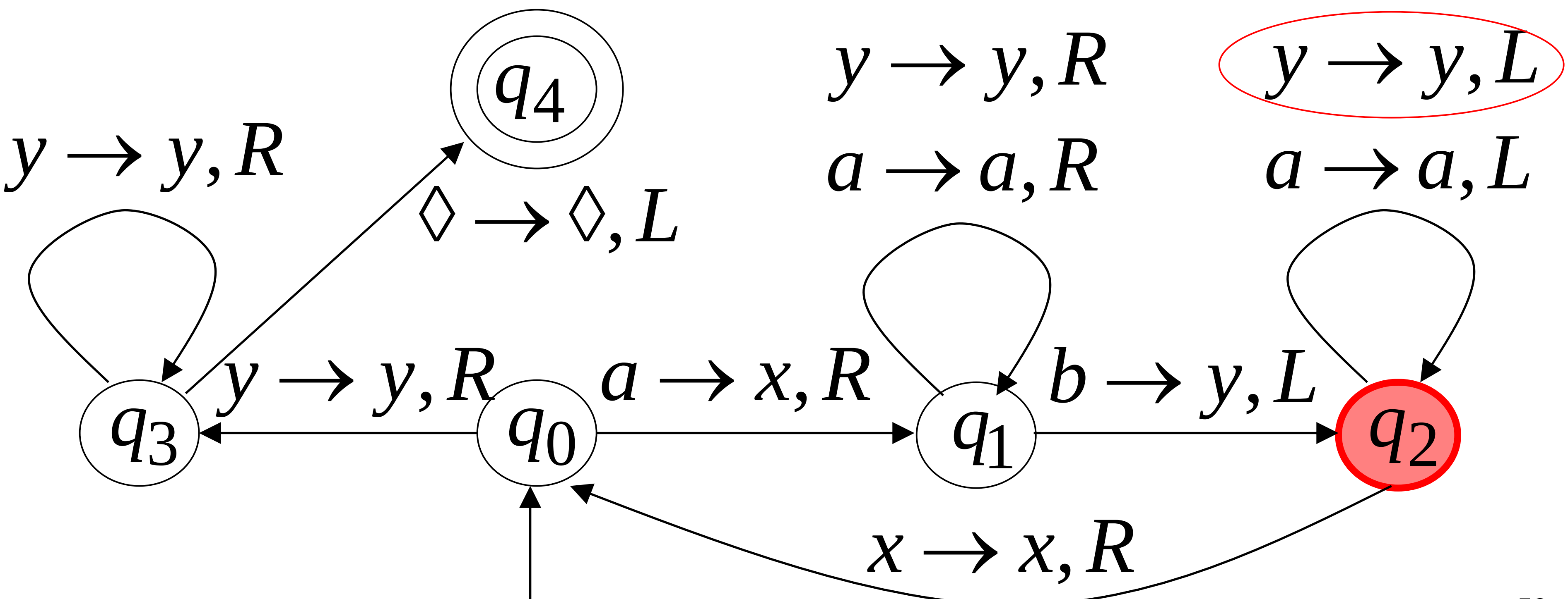
Time 7



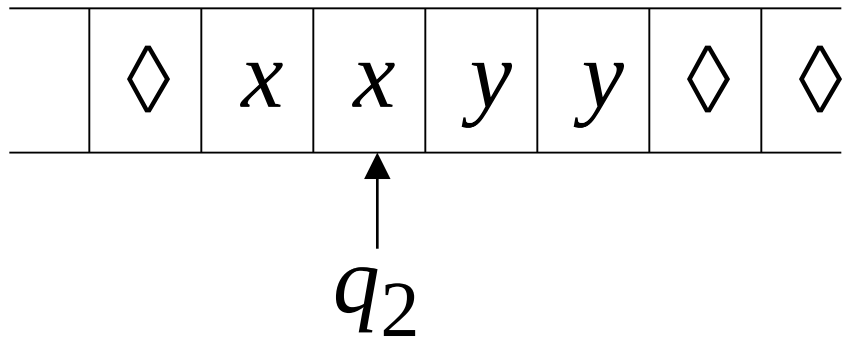


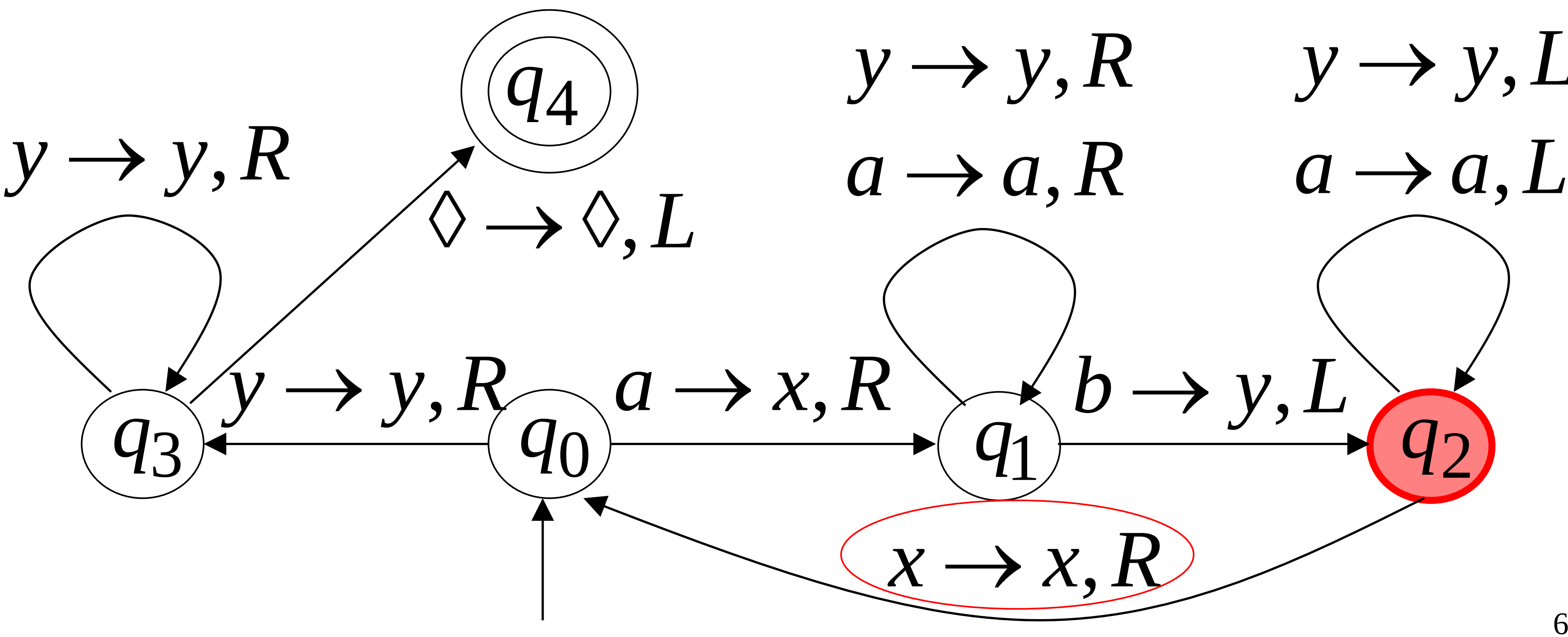
Time 8


◊
x
x
y
y
◊
◊
q_2

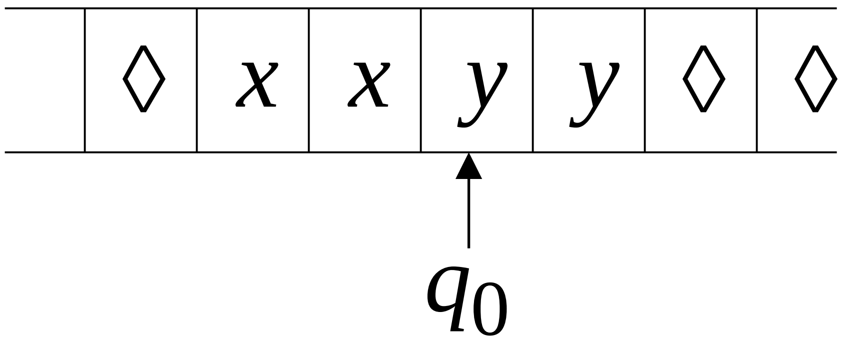

q_4
$\lozenge \rightarrow \lozenge, L$
$y \rightarrow y, R$
$a \rightarrow a, R$
$y \rightarrow y, L$
$a \rightarrow a, L$
$y \rightarrow y, R$
q_3
$y \rightarrow y, R$
q_0
$a \rightarrow x, R$
q_1
$b \rightarrow y, L$
q_2
$x \rightarrow x, R$

Time 9

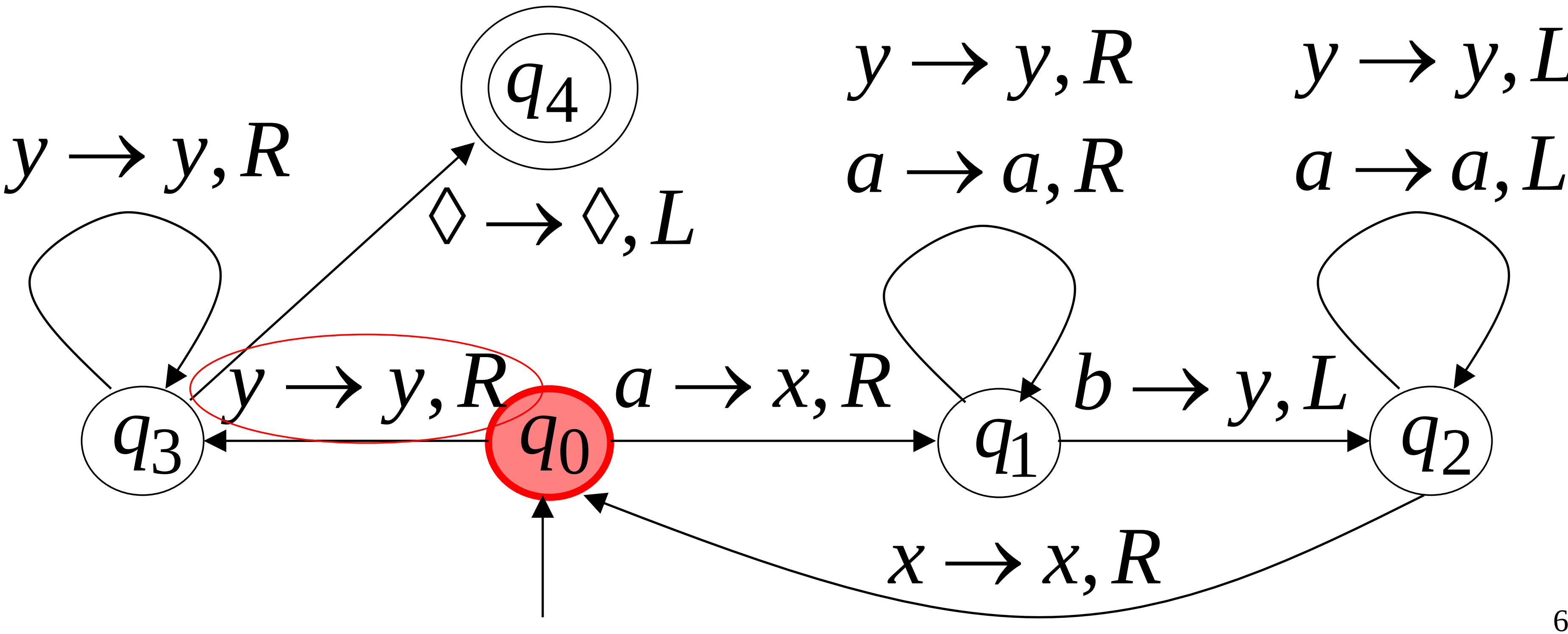

◊ x x y y ◊ ◊
q_2


q_0
q_1
q_2
q_3
q_4
$a \rightarrow x, R$
$b \rightarrow y, L$
$x \rightarrow x, R$
$y \rightarrow y, R$
$a \rightarrow a, R$
$y \rightarrow y, L$
$a \rightarrow a, L$
$y \rightarrow y, R$
$\lozenge \rightarrow \lozenge, L$

Time 10

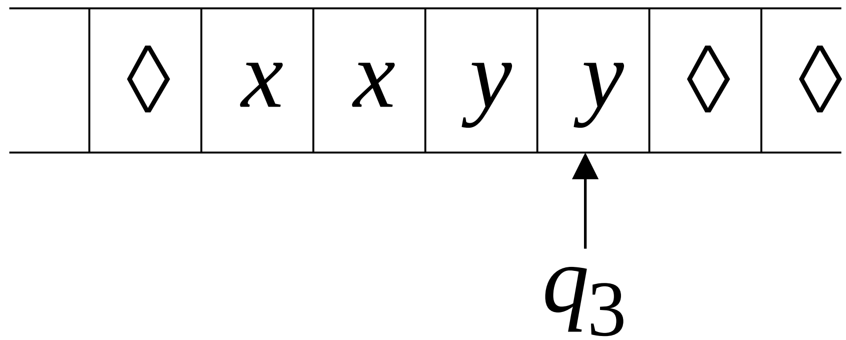


◊
x
x
y
y
◊
◊
q_0

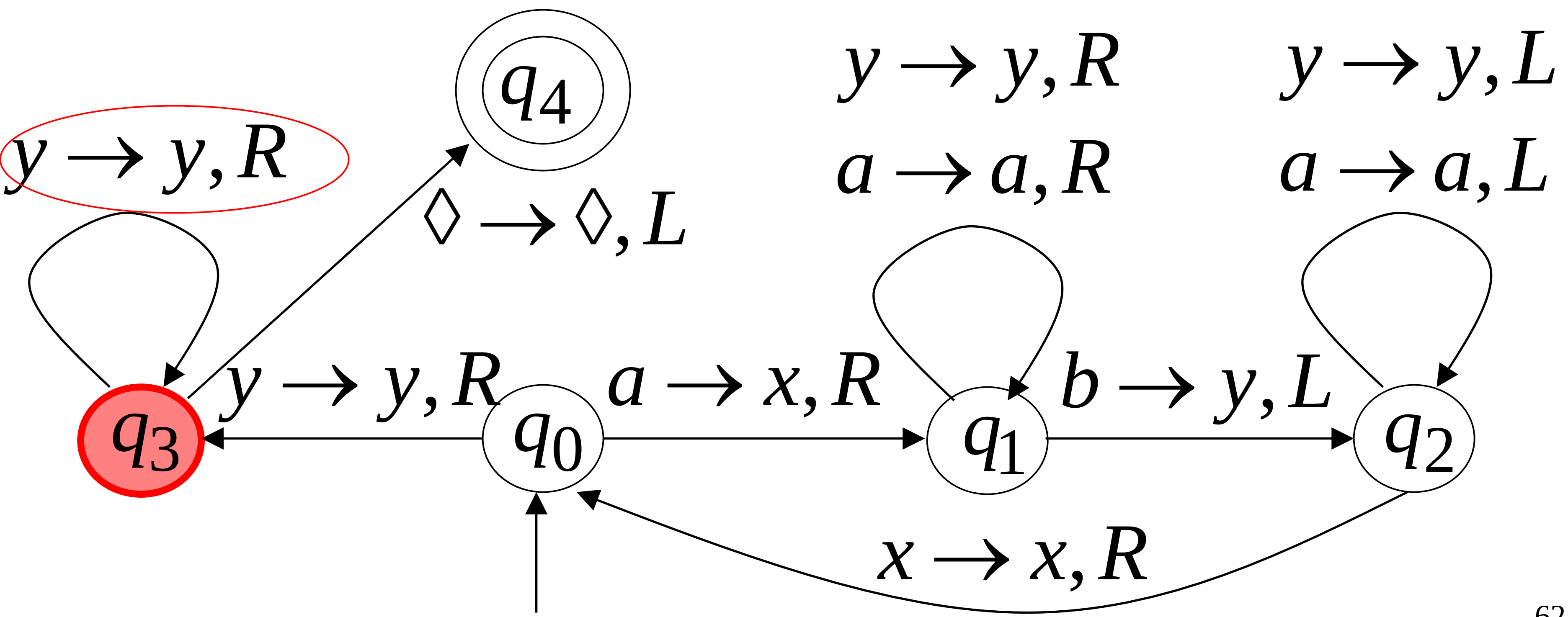


q_4
$y \rightarrow y, R$
$\Diamond \rightarrow \Diamond, L$
$y \rightarrow y, R$
$a \rightarrow a, R$
$y \rightarrow y, L$
$a \rightarrow a, L$
$y \rightarrow y, R$
$a \rightarrow x, R$
$b \rightarrow y, L$
q_3
q_0
q_1
q_2
$x \rightarrow x, R$

Time 11

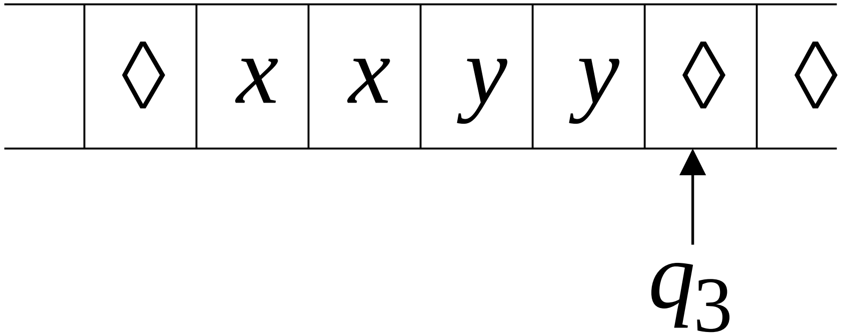


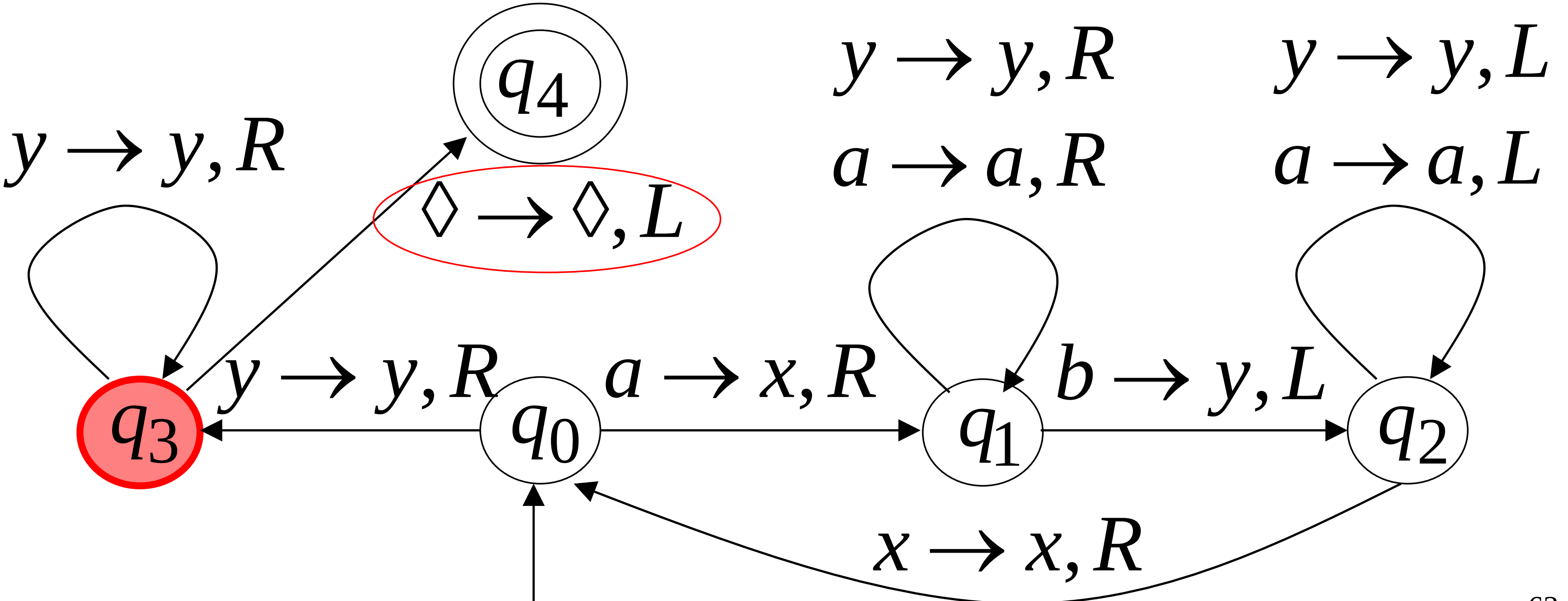
◊
x
x
y
y
◊
◊
q_3



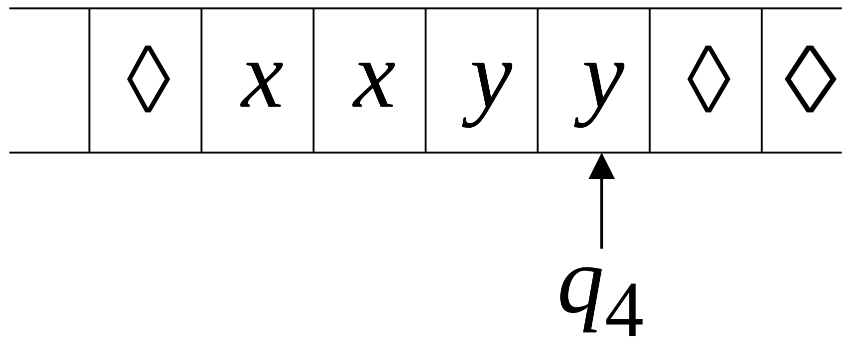
q_4
$y \rightarrow y, R$
$\Diamond \rightarrow \Diamond, L$
$y \rightarrow y, R$
$a \rightarrow a, R$
$y \rightarrow y, L$
$a \rightarrow a, L$
$y \rightarrow y, R$
$a \rightarrow x, R$
$b \rightarrow y, L$
q_3
q_0
q_1
q_2
$x \rightarrow x, R$

Time 12

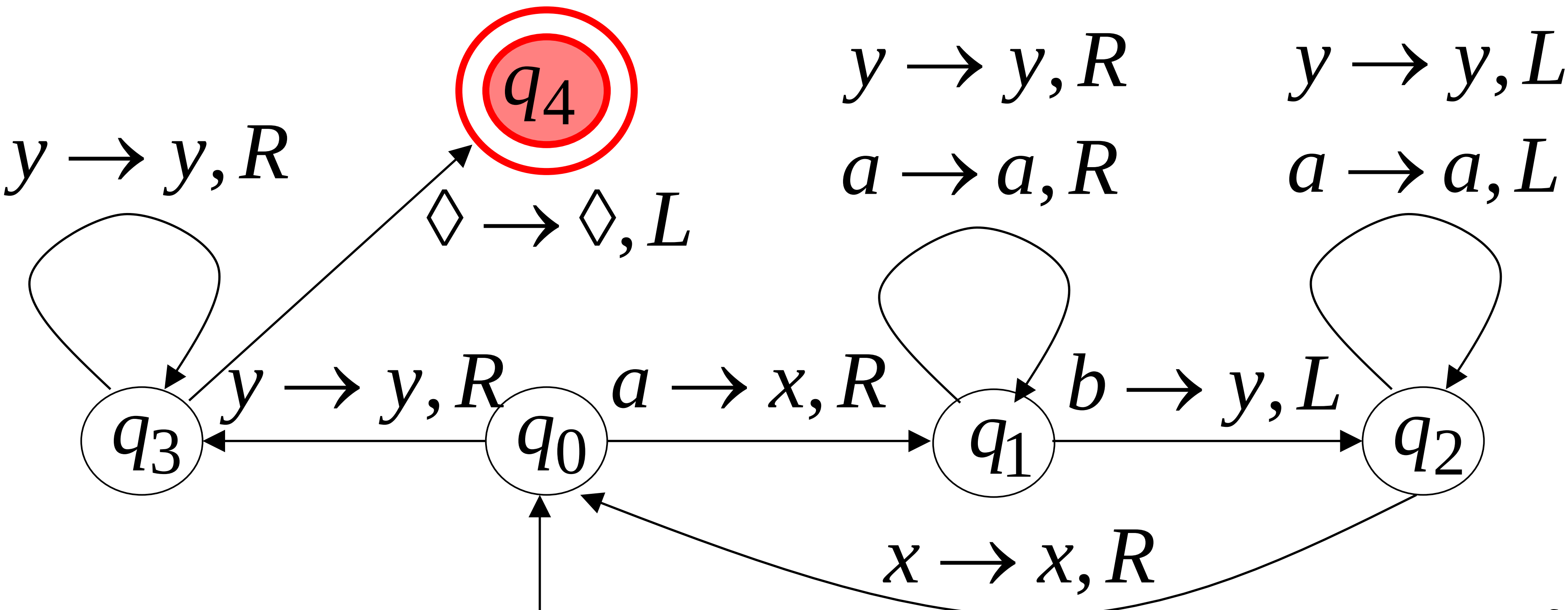

◊
x
x
y
y
◊
◊
q_3


q_4
$y \rightarrow y, R$
$\Diamond \rightarrow \Diamond, L$
$y \rightarrow y, R$
$a \rightarrow a, R$
$y \rightarrow y, L$
$a \rightarrow a, L$
$y \rightarrow y, R$
$a \rightarrow x, R$
$b \rightarrow y, L$
q_3
q_0
q_1
q_2
$x \rightarrow x, R$

Time 13



Halt & Accept



Observation:

If we modify the machine for the language $\{a^n b^n\}$

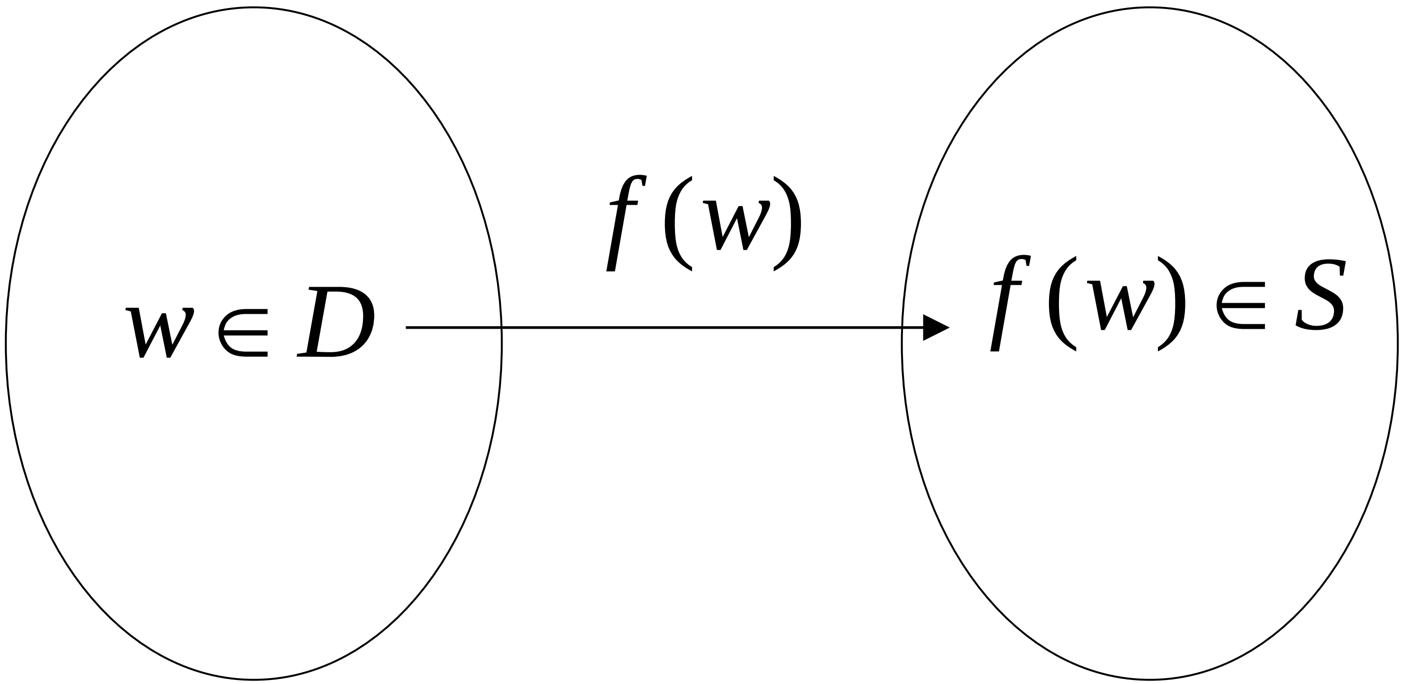
we can easily construct a machine for the language $\{a^n b^n c^n\}$

Computing Functions with Turing Machines

A function $f(w)$ has:

Domain: D

Result Region: S



A function may have many parameters:

Example: Addition function

$$f(x, y) = x + y$$

Integer Domain

Decimal: 5

Binary: 101

Unary: 11111

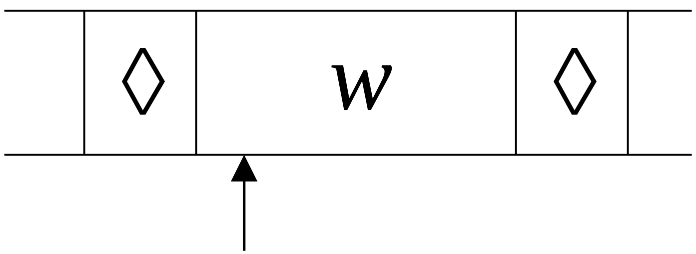
We prefer **unary** representation:

easier to manipulate with Turing machines

Definition:

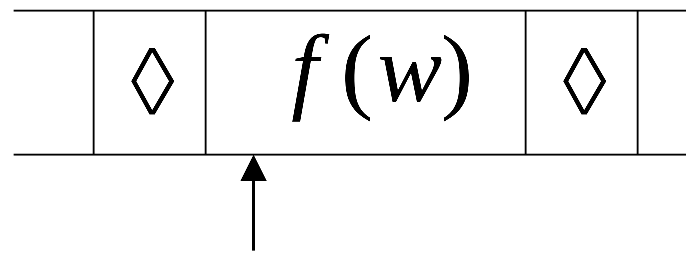
A function f is computable if there is a Turing Machine M such that:

Initial configuration



q_0 initial state

Final configuration



q_f final state

For all $w \in D$ Domain

In other words:

A function f is computable if
there is a Turing Machine M such that:

$$q_0\, w \overset{*}{\succ} q_f\, f(w)$$

Initial Configuration

Final Configuration

For all $w \in D$ Domain

Example

The function $f(x, y) = x + y$ is computable

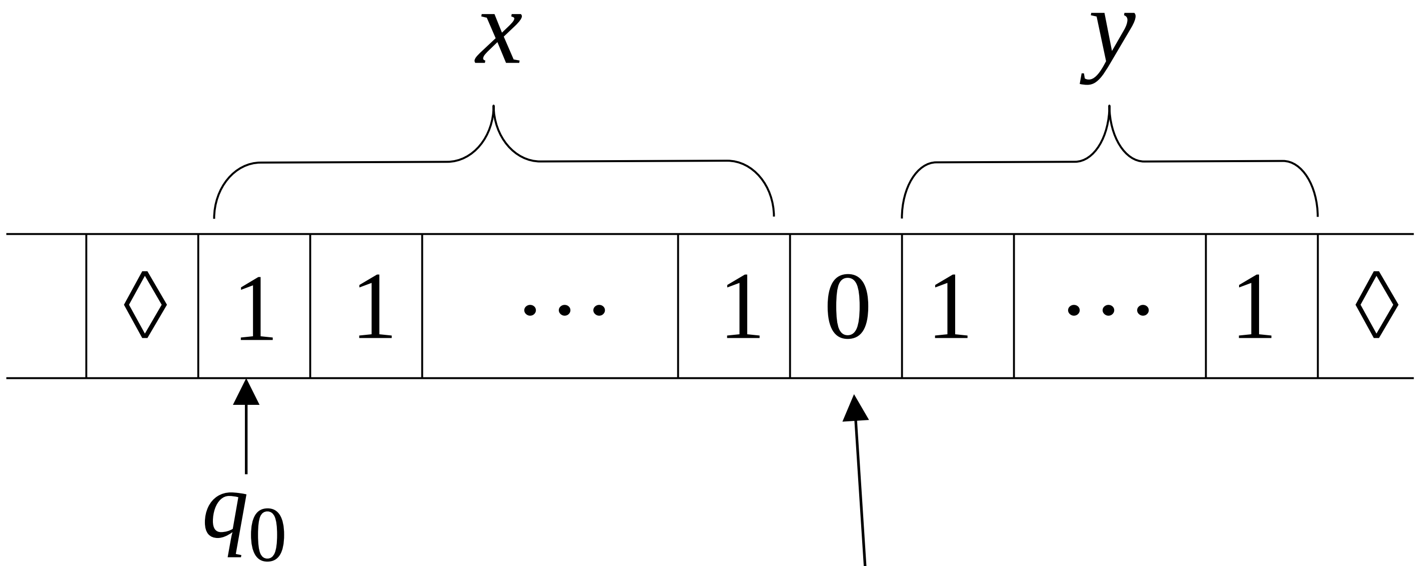
x, y are integers

Turing Machine:

Input string: $x0y$ unary

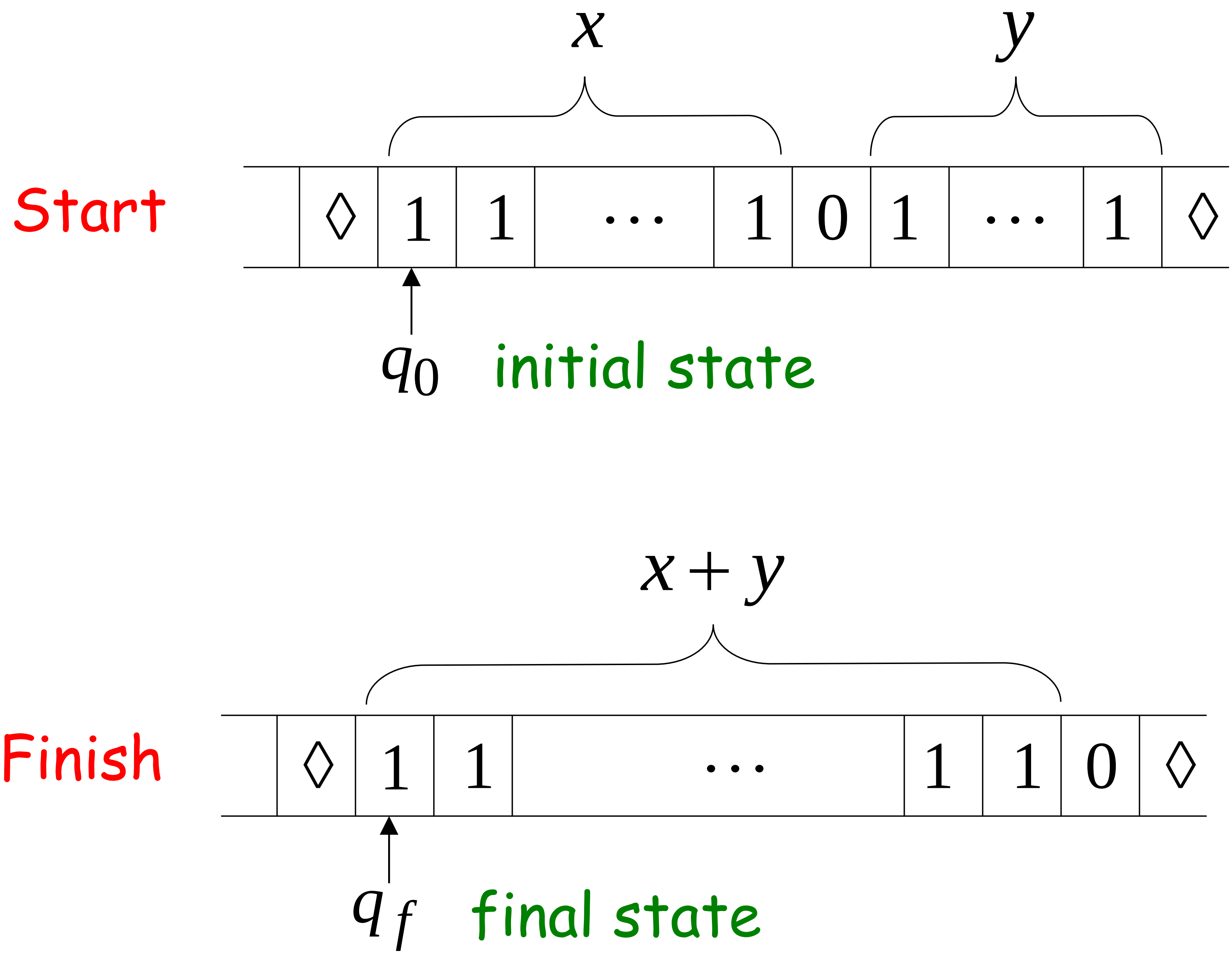
Output string: $xy0$ unary

Start



initial state

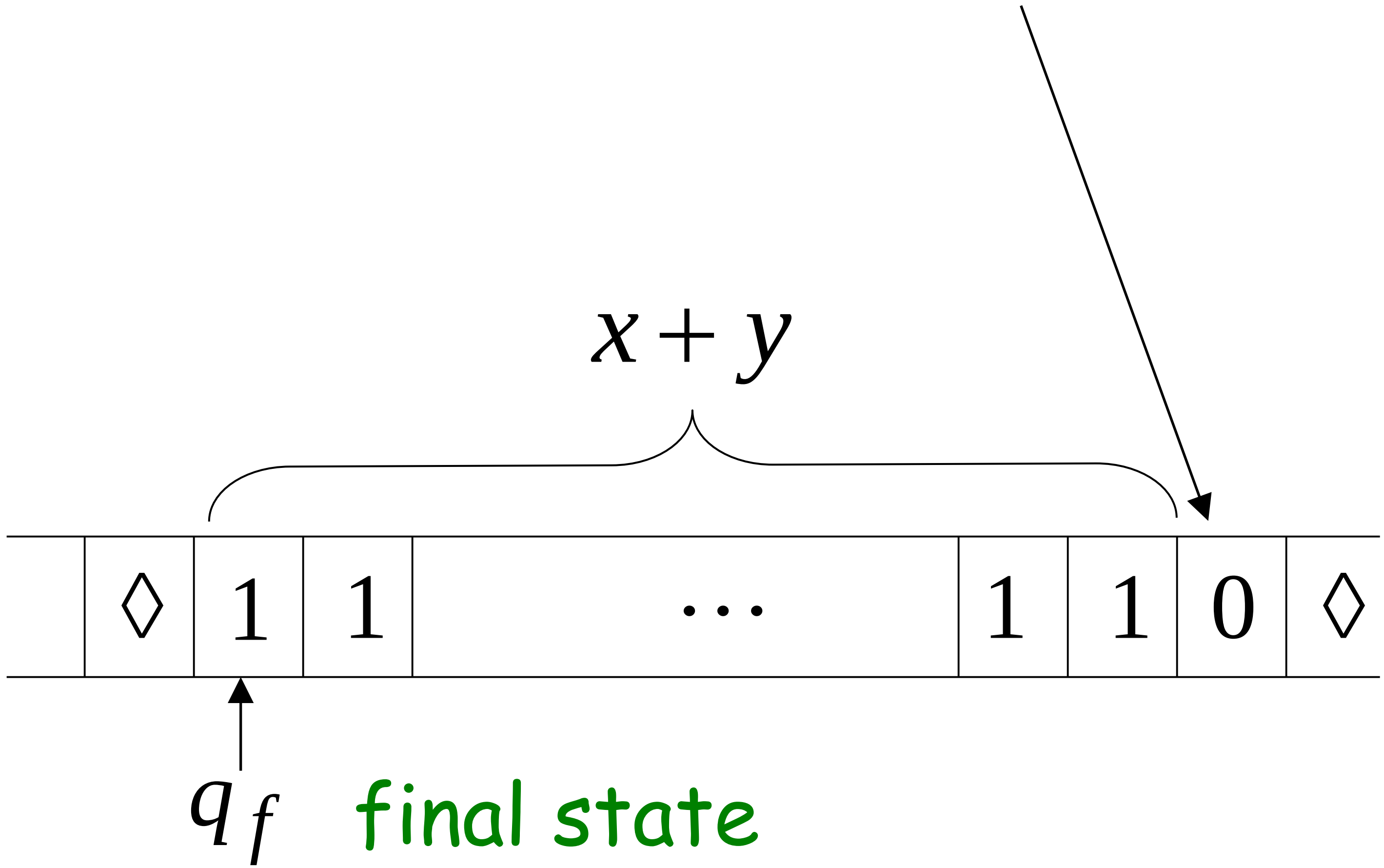
The 0 is the delimiter that separates the two numbers



Start
x
y
◊ 1 1 ⋯ 1 0 1 ⋯ 1 ◊
q_0
initial state
Finish
$x+y$
◊ 1 1 ⋯ 1 1 0 ◊
q_f
final state

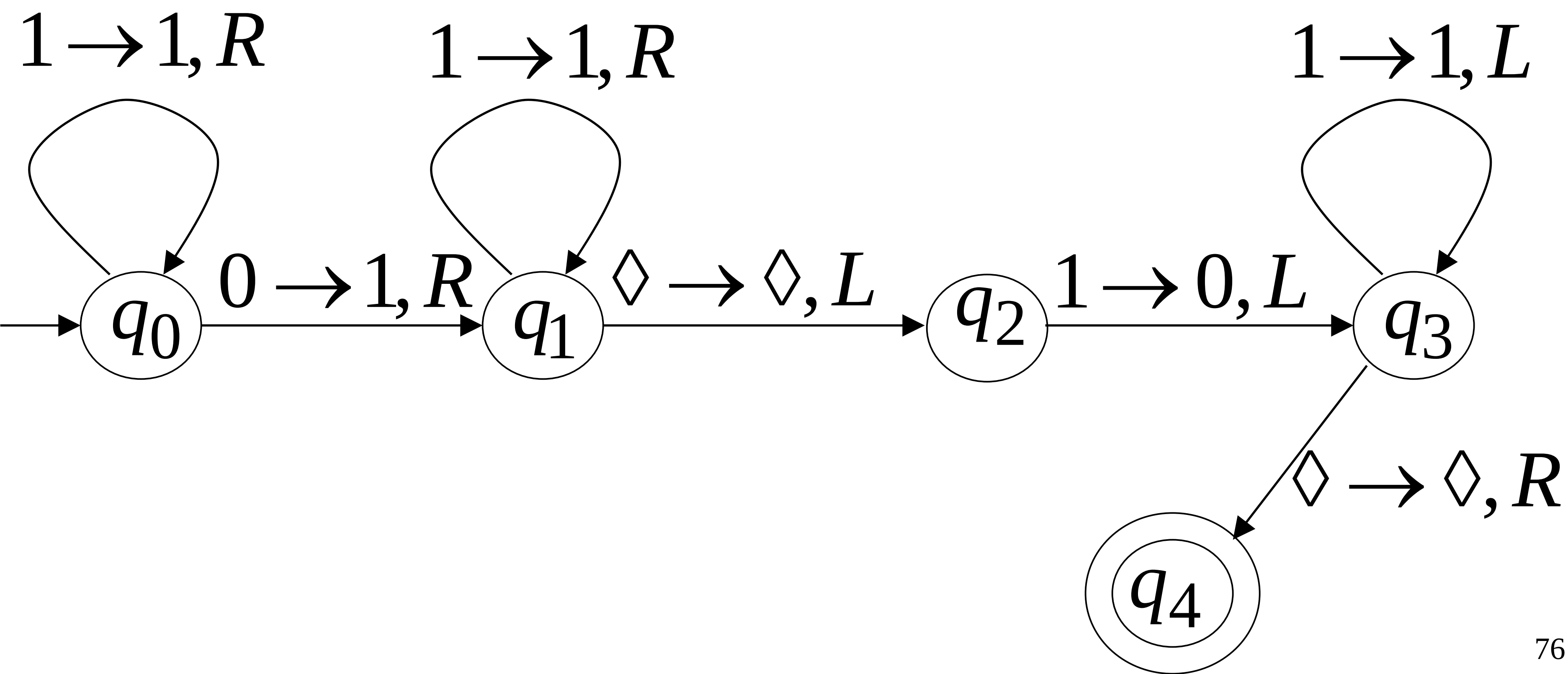
The 0 helps when we use
the result for other operations

Finish



final state

Turing machine for function $f(x, y) = x + y$

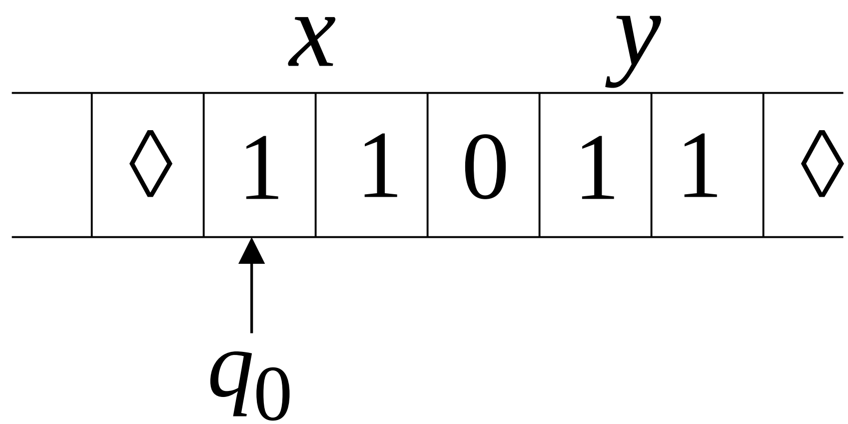


Execution Example:

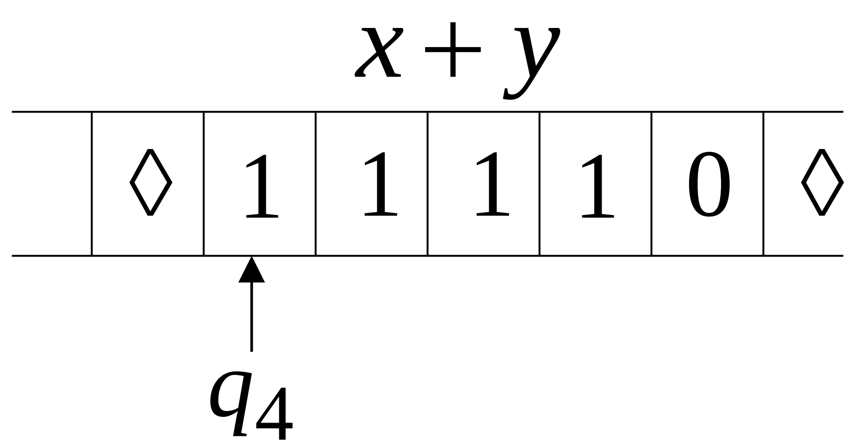
$x = 11$ (2)

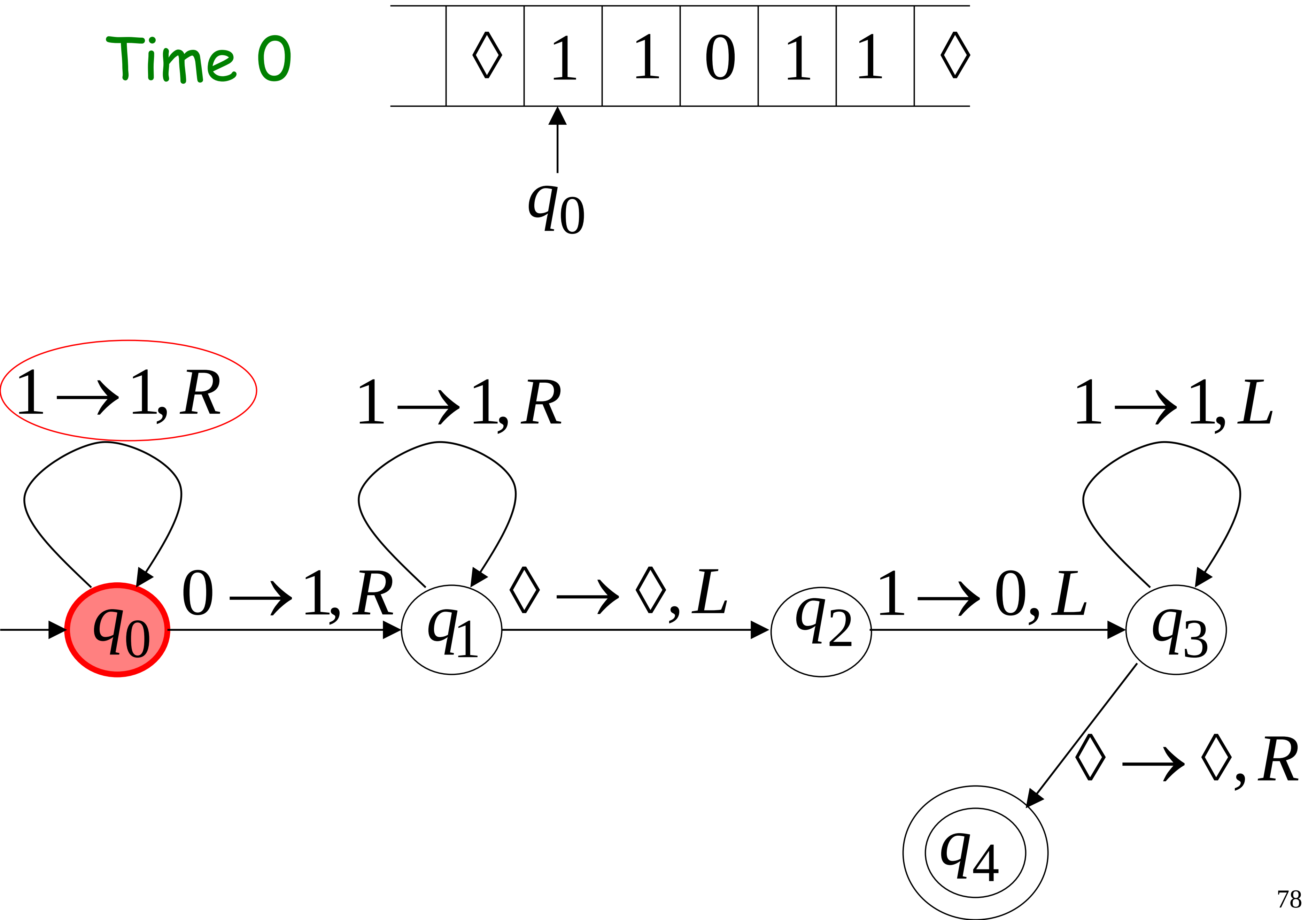
$y = 11$ (2)

Time 0



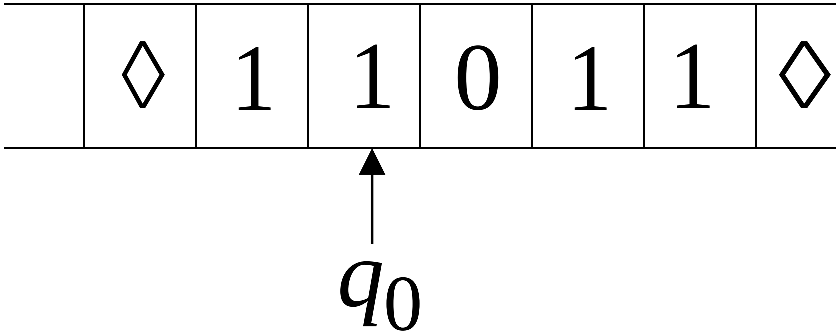
Final Result



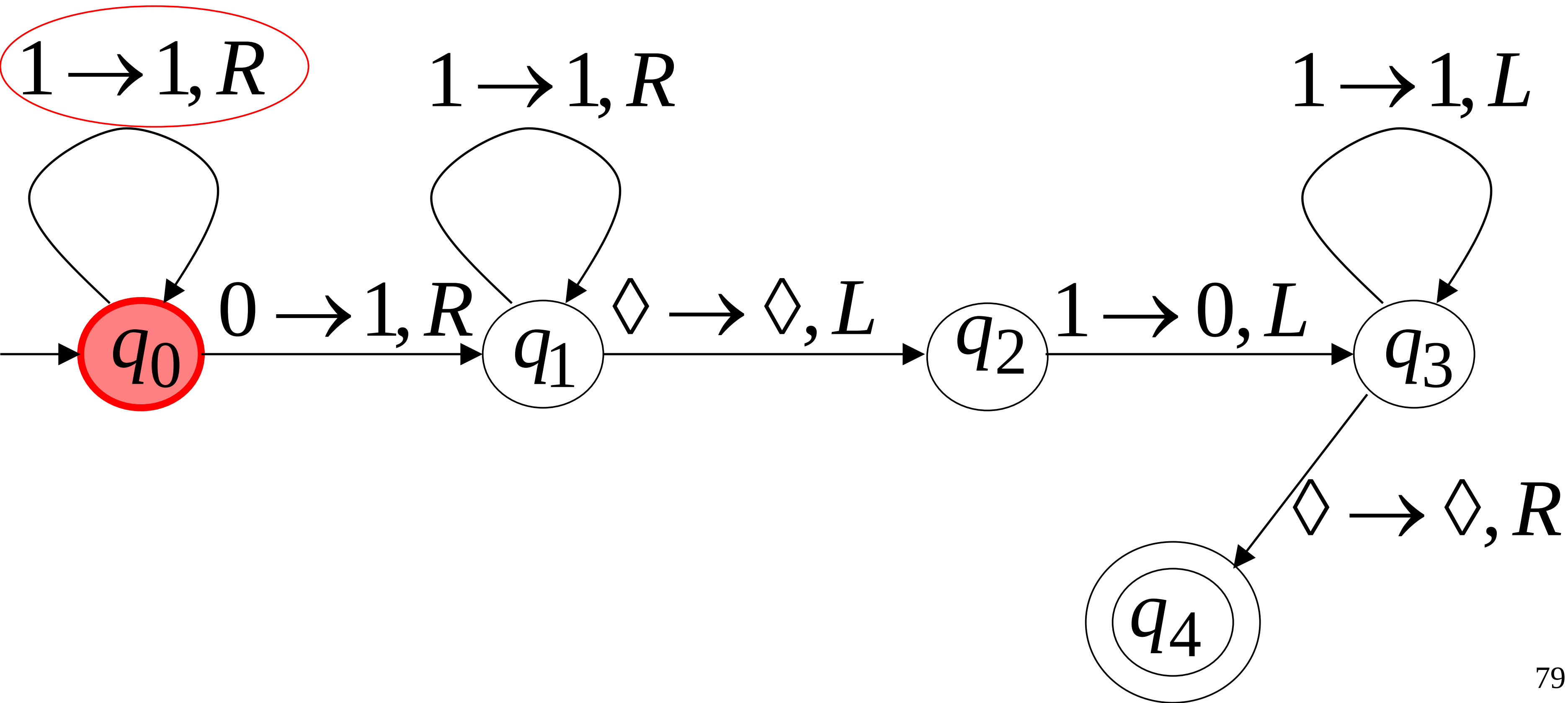


Time 0
◊ 1 1 0 1 1 ◊
q_0
$1 \to 1, R$
$1 \to 1, R$
$1 \to 1, L$
$0 \to 1, R$
$◊ \to ◊, L$
$1 \to 0, L$
$◊ \to ◊, R$
q_0
q_1
q_2
q_3
q_4

Time 1

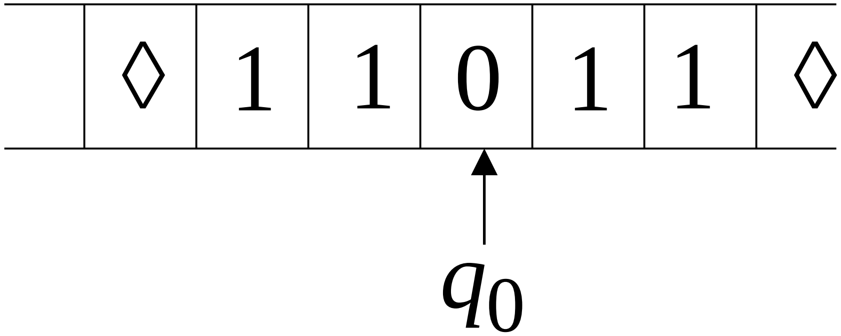


◊
1
1
0
1
1
◊
q_0

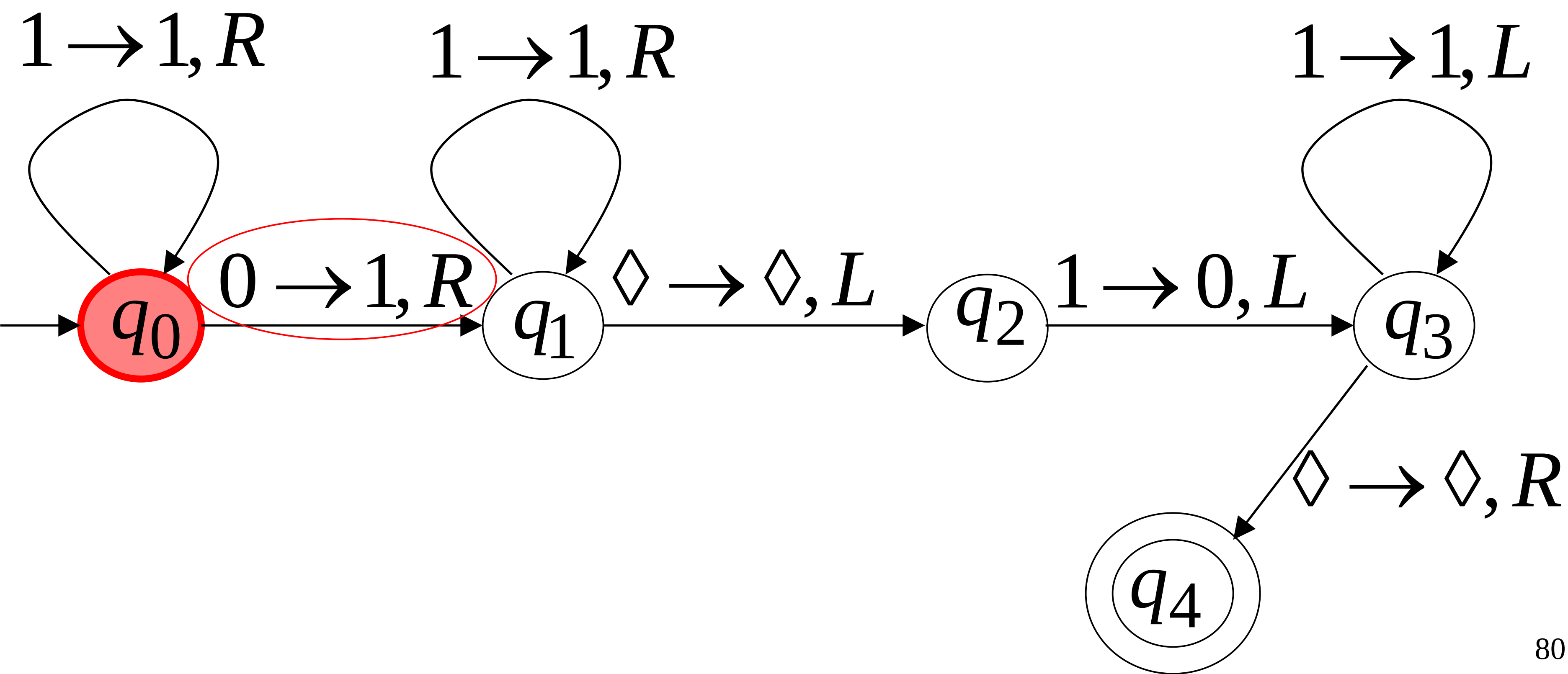


$1 \rightarrow 1, R$
$1 \rightarrow 1, R$
$1 \rightarrow 1, L$
$0 \rightarrow 1, R$
$\lozenge \rightarrow \lozenge, L$
$1 \rightarrow 0, L$
$\lozenge \rightarrow \lozenge, R$
q_0
q_1
q_2
q_3
q_4

Time 2

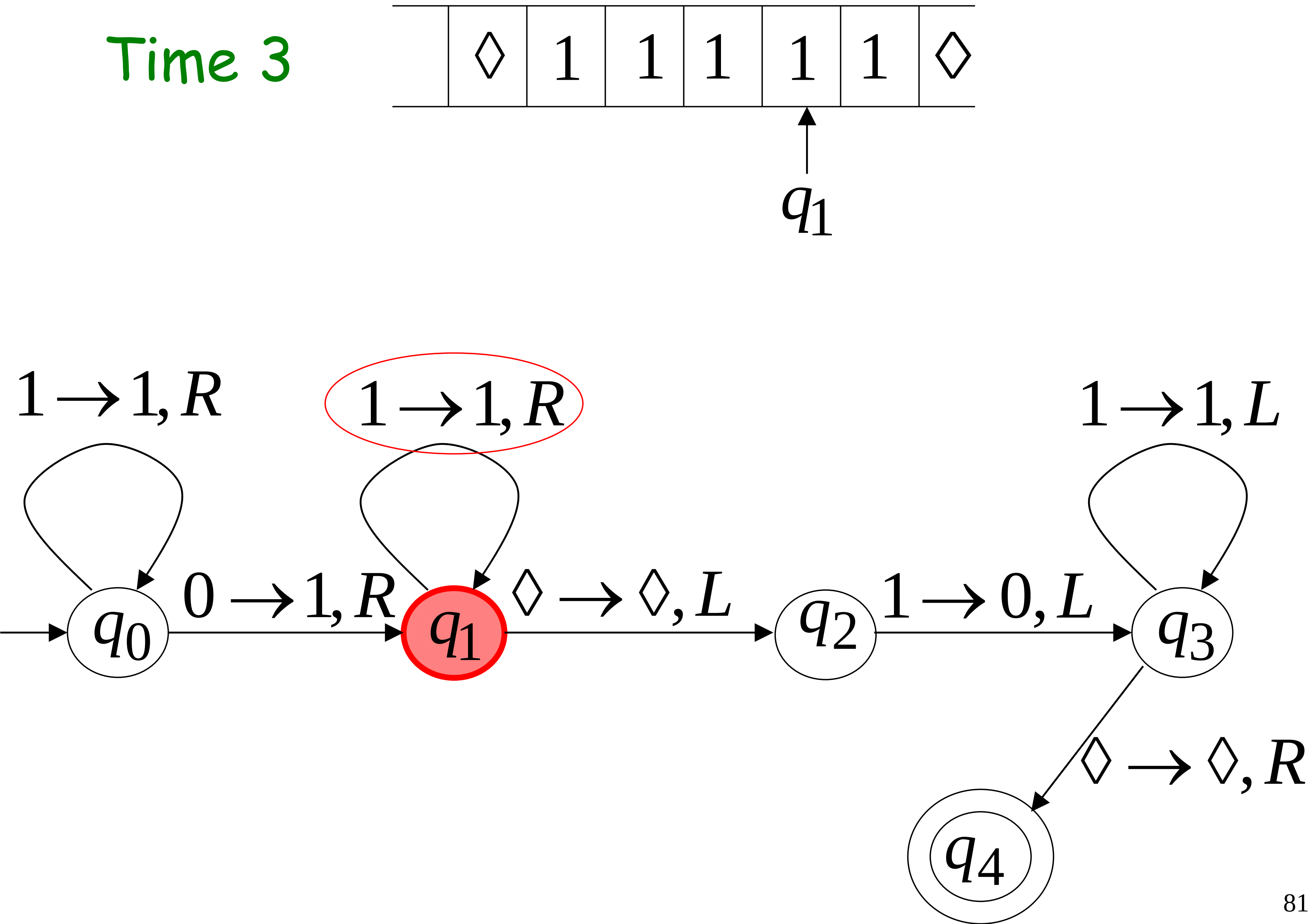


◊
1
1
0
1
1
◊
q_0



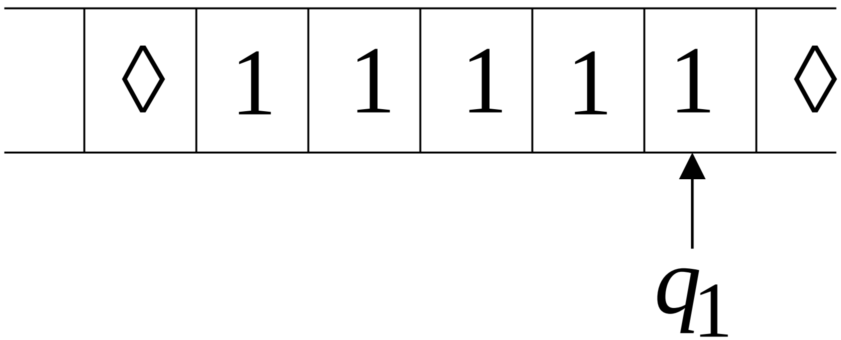
$1 \rightarrow 1, R$
$1 \rightarrow 1, R$
$1 \rightarrow 1, L$
$0 \rightarrow 1, R$
$\Diamond \rightarrow \Diamond, L$
$1 \rightarrow 0, L$
$\Diamond \rightarrow \Diamond, R$
q_0
q_1
q_2
q_3
q_4

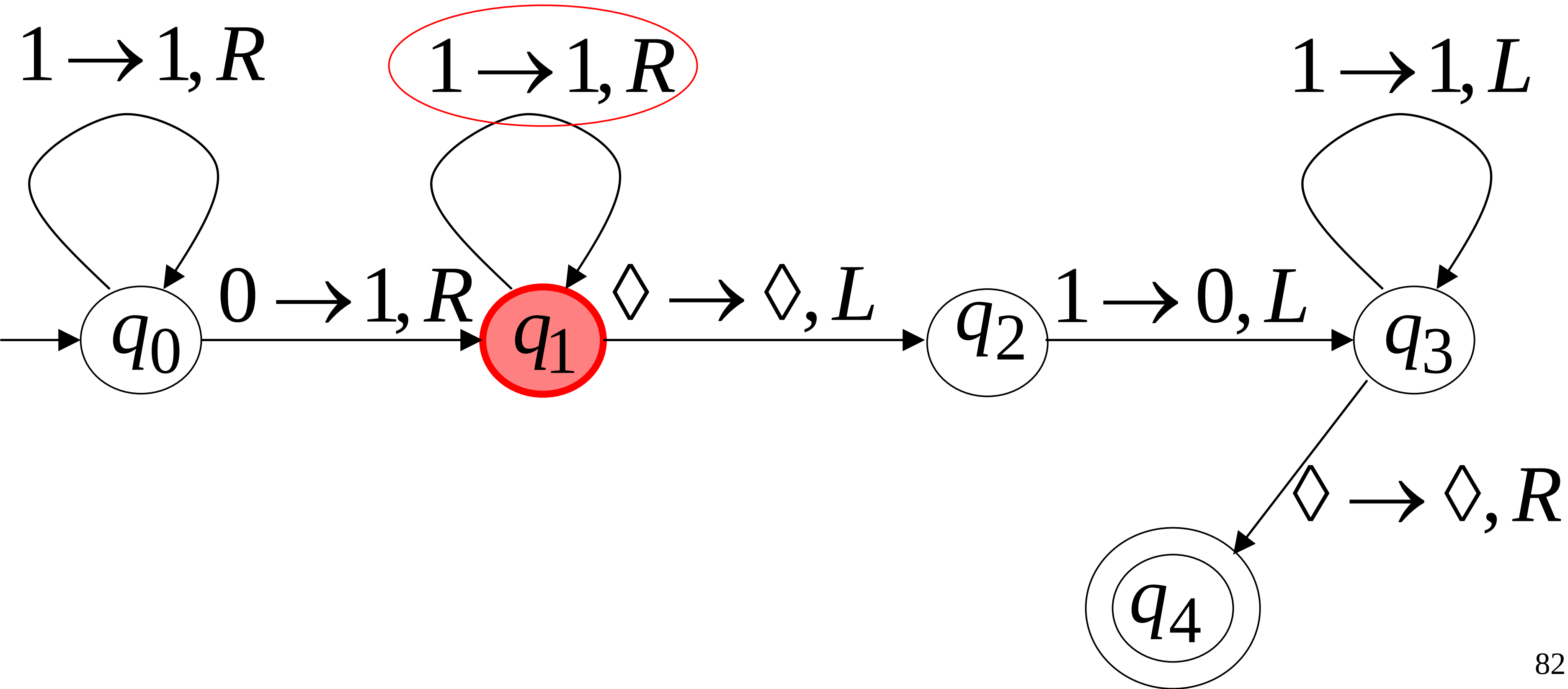
Time 3



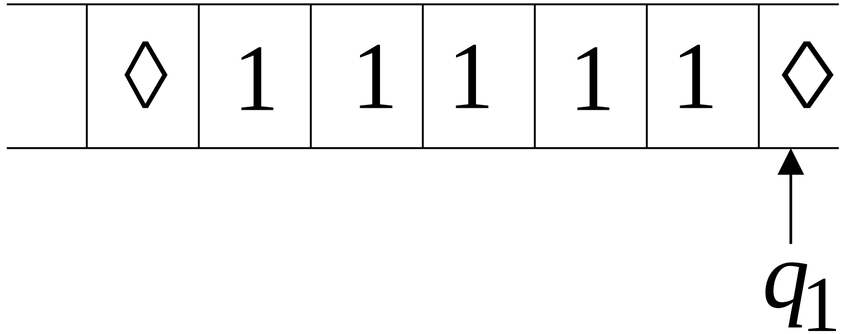
◊
1
1
1
1
1
◊
q_1
$1 \rightarrow 1, R$
$1 \rightarrow 1, R$
$1 \rightarrow 1, L$
$0 \rightarrow 1, R$
$\lozenge \rightarrow \lozenge, L$
$1 \rightarrow 0, L$
q_0
q_1
q_2
q_3
$\lozenge \rightarrow \lozenge, R$
q_4

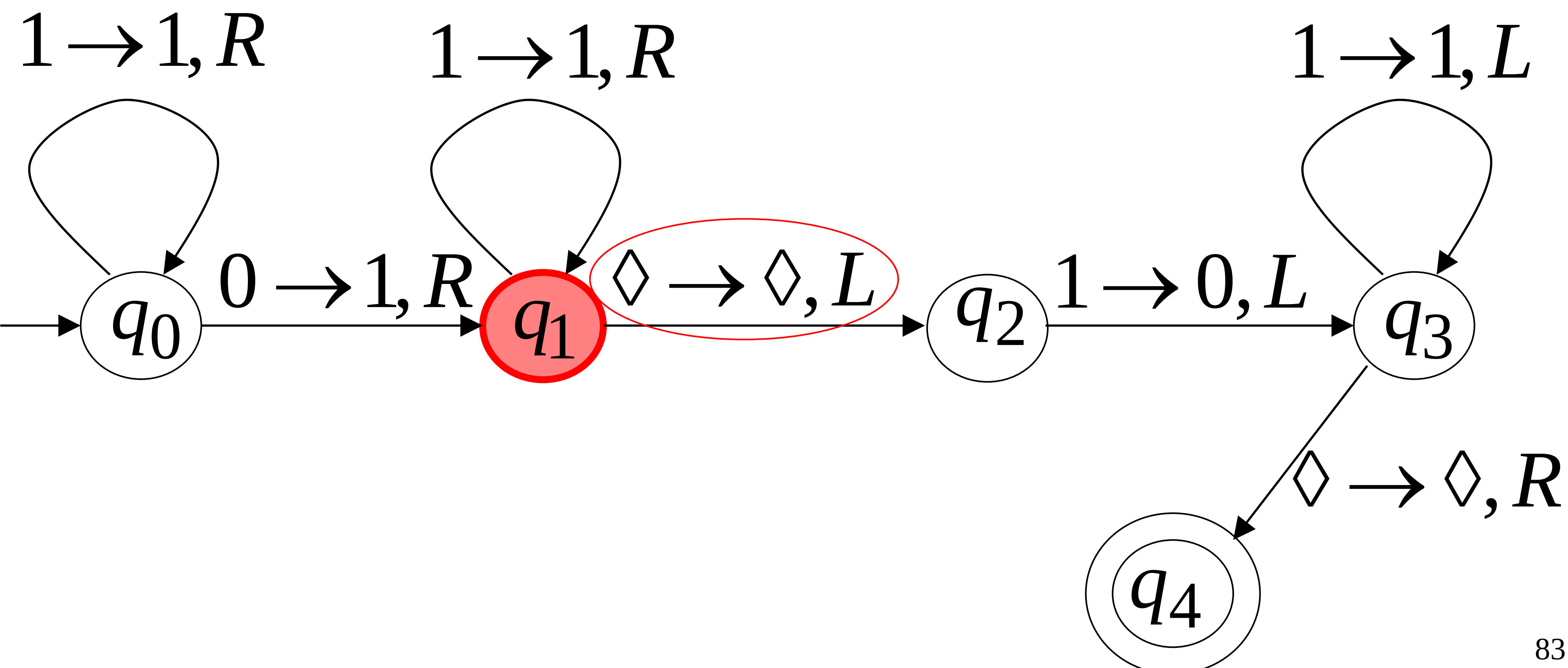
Time 4


◊ 1 1 1 1 1 ◊
q_1

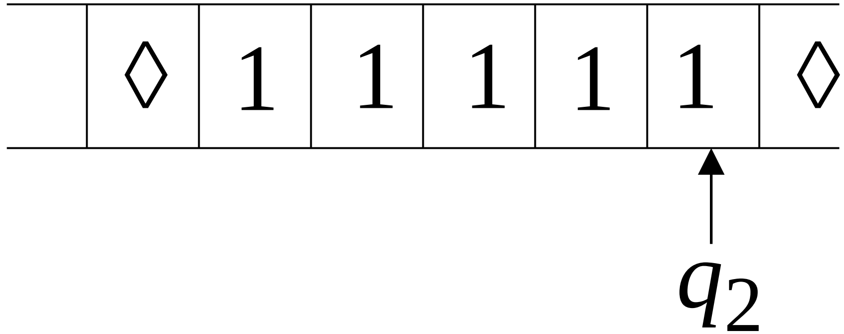

$1 \rightarrow 1, R$
$1 \rightarrow 1, R$
$1 \rightarrow 1, L$
q_0
$0 \rightarrow 1, R$
q_1
$\lozenge \rightarrow \lozenge, L$
q_2
$1 \rightarrow 0, L$
q_3
$\lozenge \rightarrow \lozenge, R$
q_4

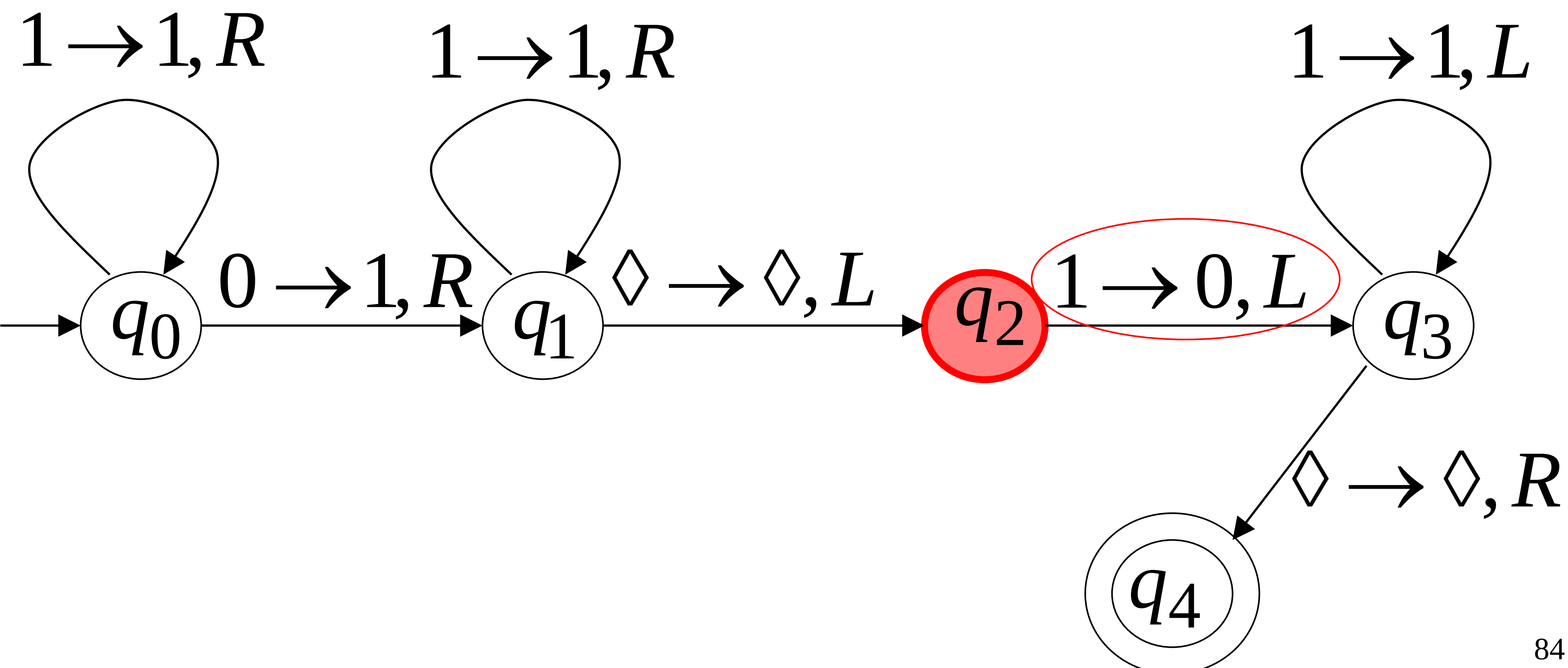
Time 5


◊ 1 1 1 1 1 ◊
q_1

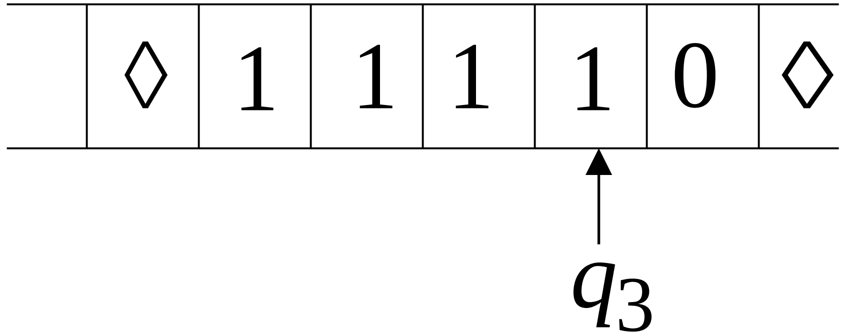

$1 \to 1, R$
$1 \to 1, R$
$1 \to 1, L$
q_0
$0 \to 1, R$
q_1
$\Diamond \to \Diamond, L$
q_2
$1 \to 0, L$
q_3
$\Diamond \to \Diamond, R$
q_4

Time 6

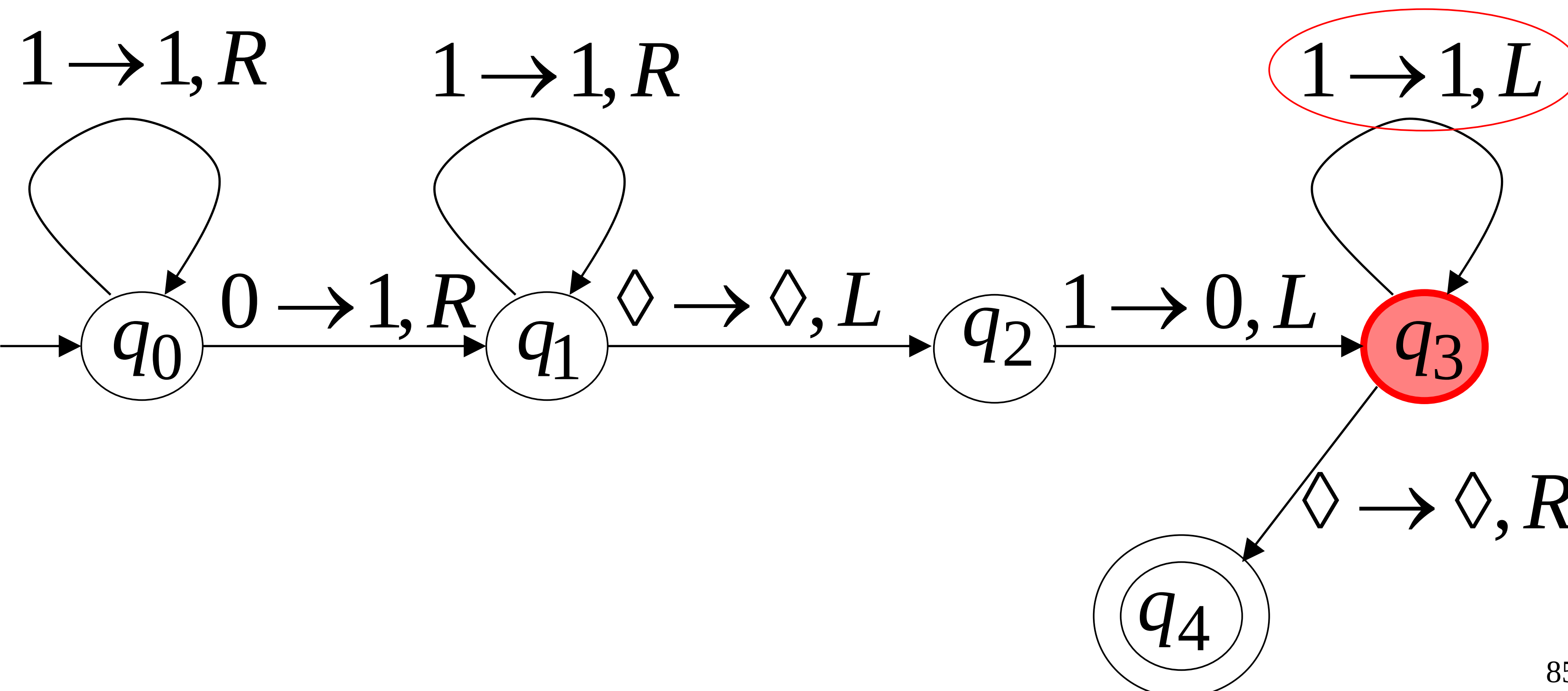

◊ 1 1 1 1 1 ◊
q_2


$1 \rightarrow 1, R$
$1 \rightarrow 1, R$
$1 \rightarrow 1, L$
$0 \rightarrow 1, R$
$\lozenge \rightarrow \lozenge, L$
$1 \rightarrow 0, L$
$\lozenge \rightarrow \lozenge, R$
q_0
q_1
q_2
q_3
q_4

Time 7

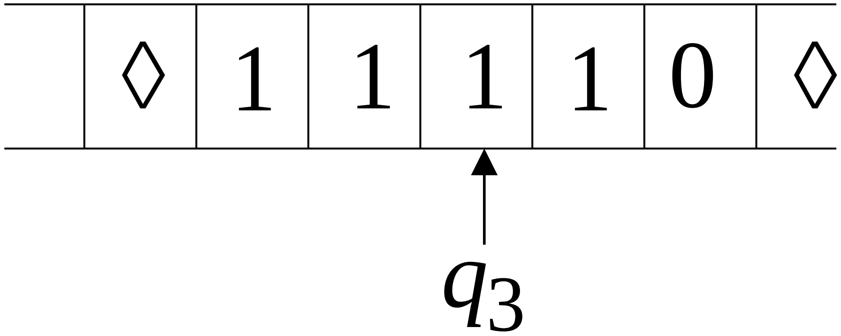


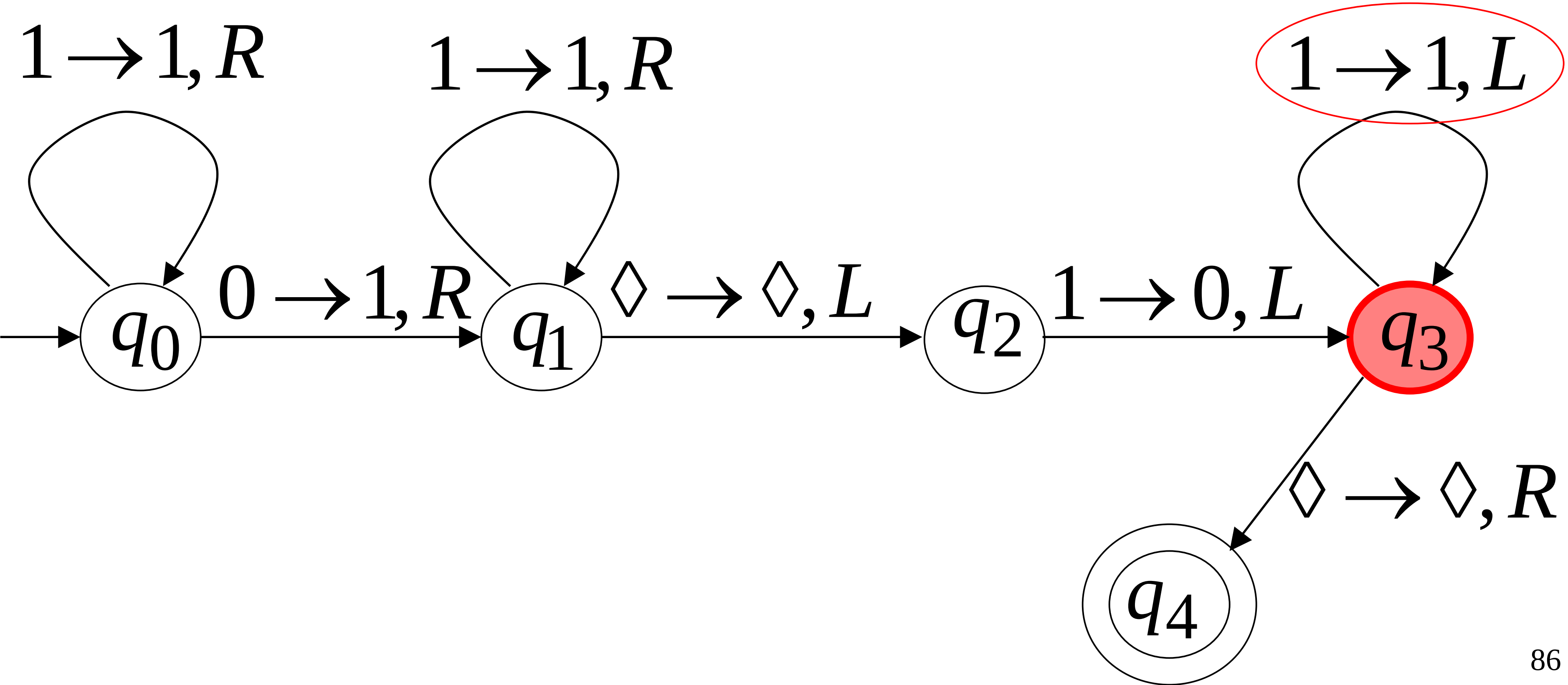
◊ 1 1 1 1 0 ◊
q_3



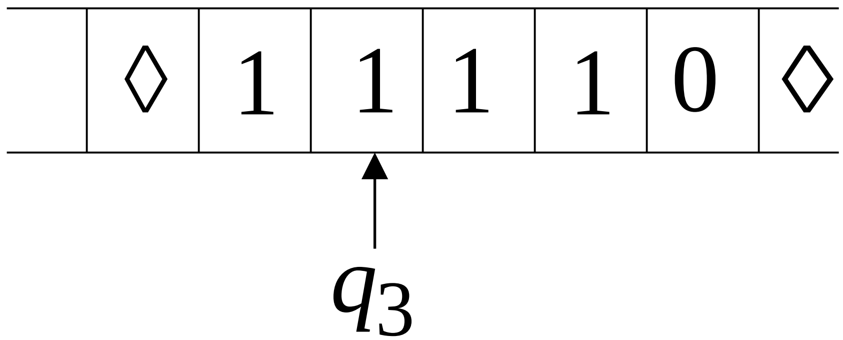
$1 \rightarrow 1, R$
$1 \rightarrow 1, R$
$1 \rightarrow 1, L$
$0 \rightarrow 1, R$
$\Diamond \rightarrow \Diamond, L$
$1 \rightarrow 0, L$
$\Diamond \rightarrow \Diamond, R$
q_0
q_1
q_2
q_3
q_4

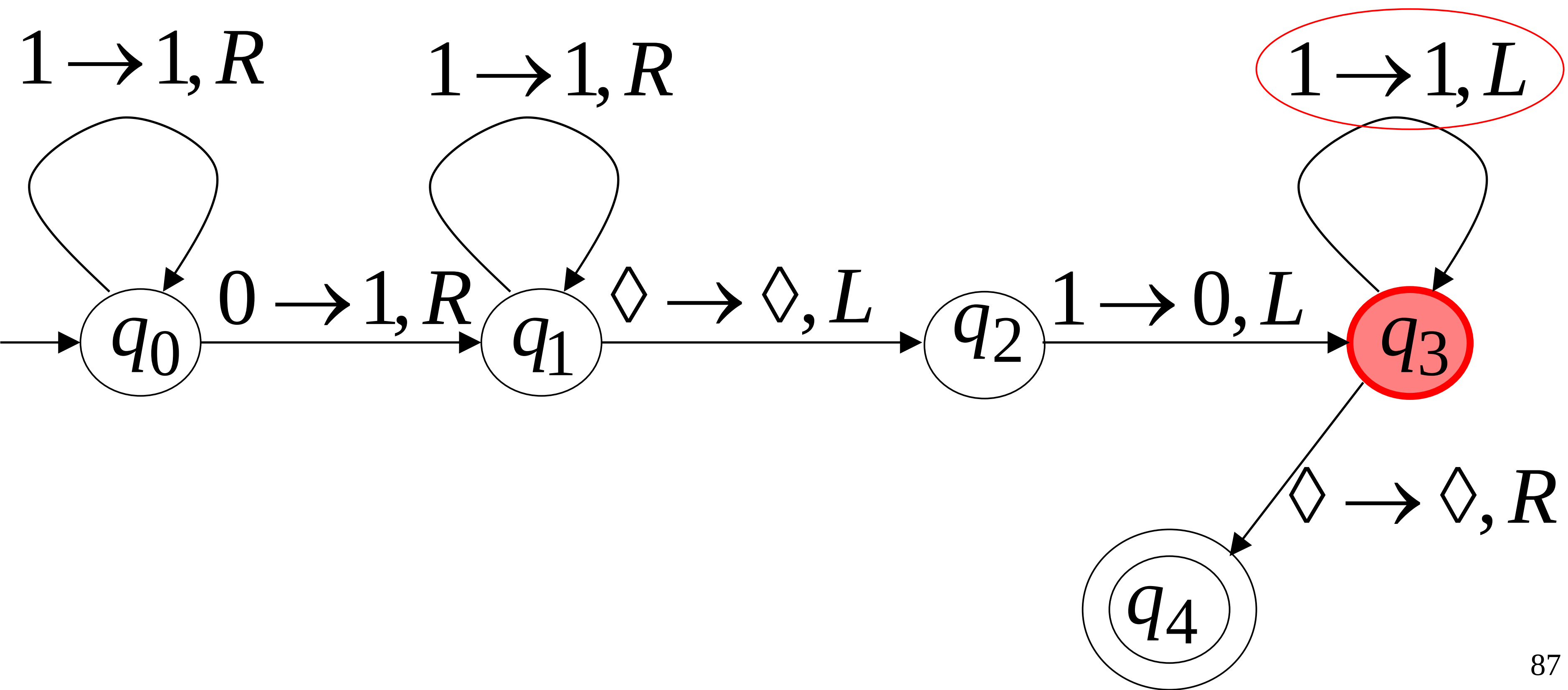
Time 8


◊
1
1
1
1
0
◊
q_3

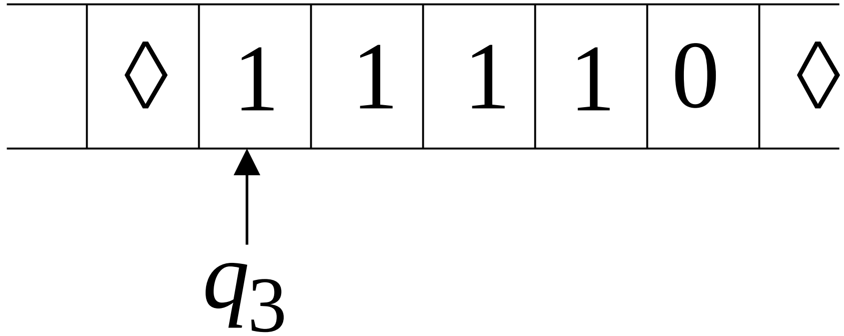

$1 \rightarrow 1, R$
$1 \rightarrow 1, R$
$1 \rightarrow 1, L$
$0 \rightarrow 1, R$
$\Diamond \rightarrow \Diamond, L$
$1 \rightarrow 0, L$
$\Diamond \rightarrow \Diamond, R$
q_0
q_1
q_2
q_3
q_4

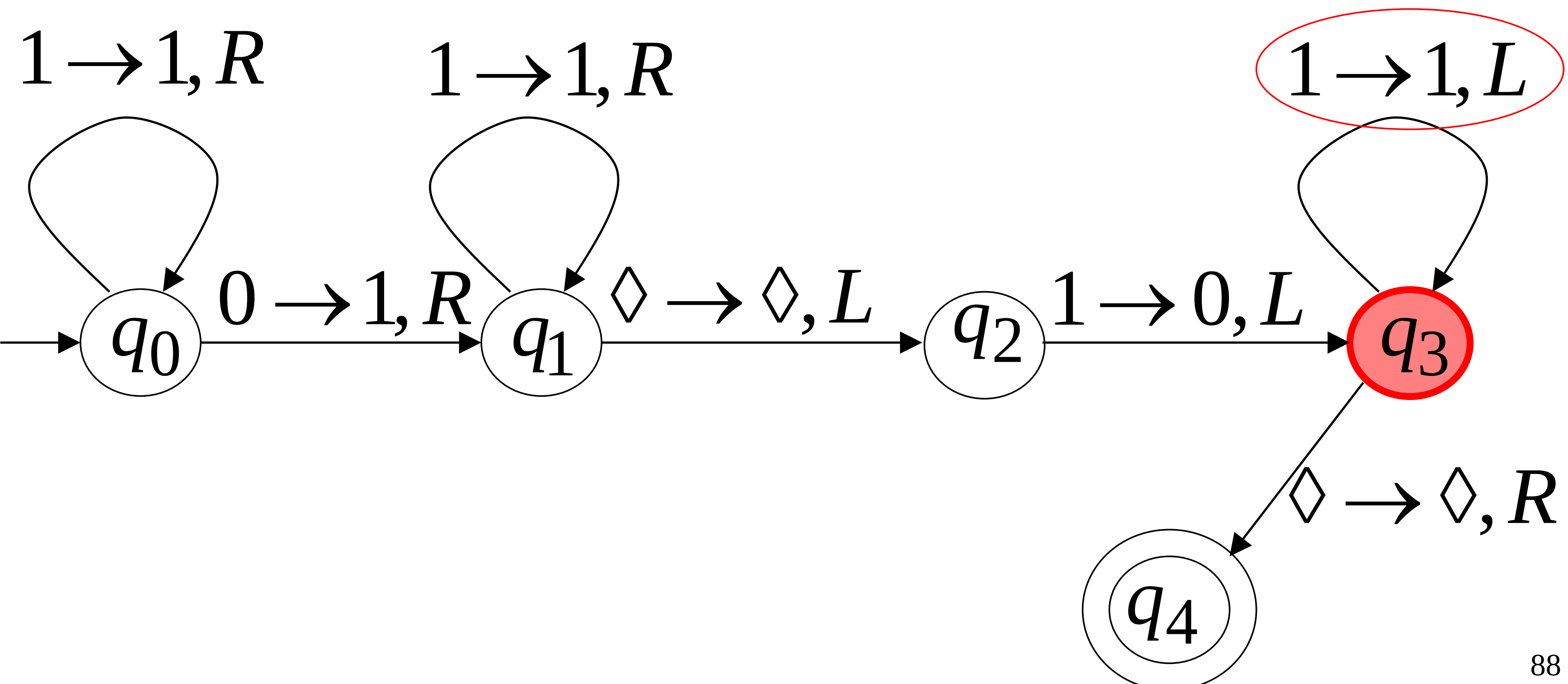
Time 9


◊
1
1
1
1
0
◊
q_3

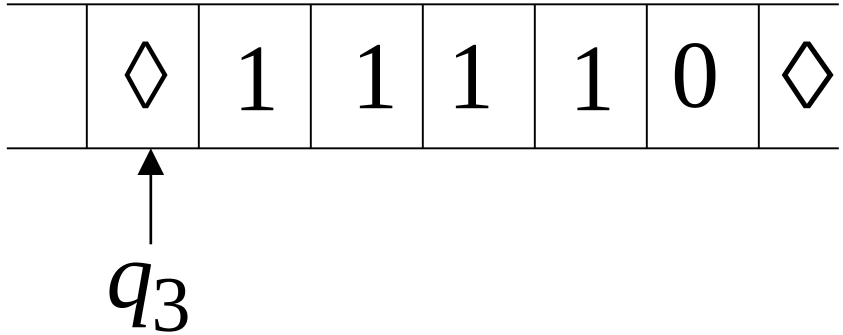

$1 \rightarrow 1, R$
$1 \rightarrow 1, R$
$1 \rightarrow 1, L$
q_0
$0 \rightarrow 1, R$
q_1
$\Diamond \rightarrow \Diamond, L$
q_2
$1 \rightarrow 0, L$
q_3
$\Diamond \rightarrow \Diamond, R$
q_4

Time 10

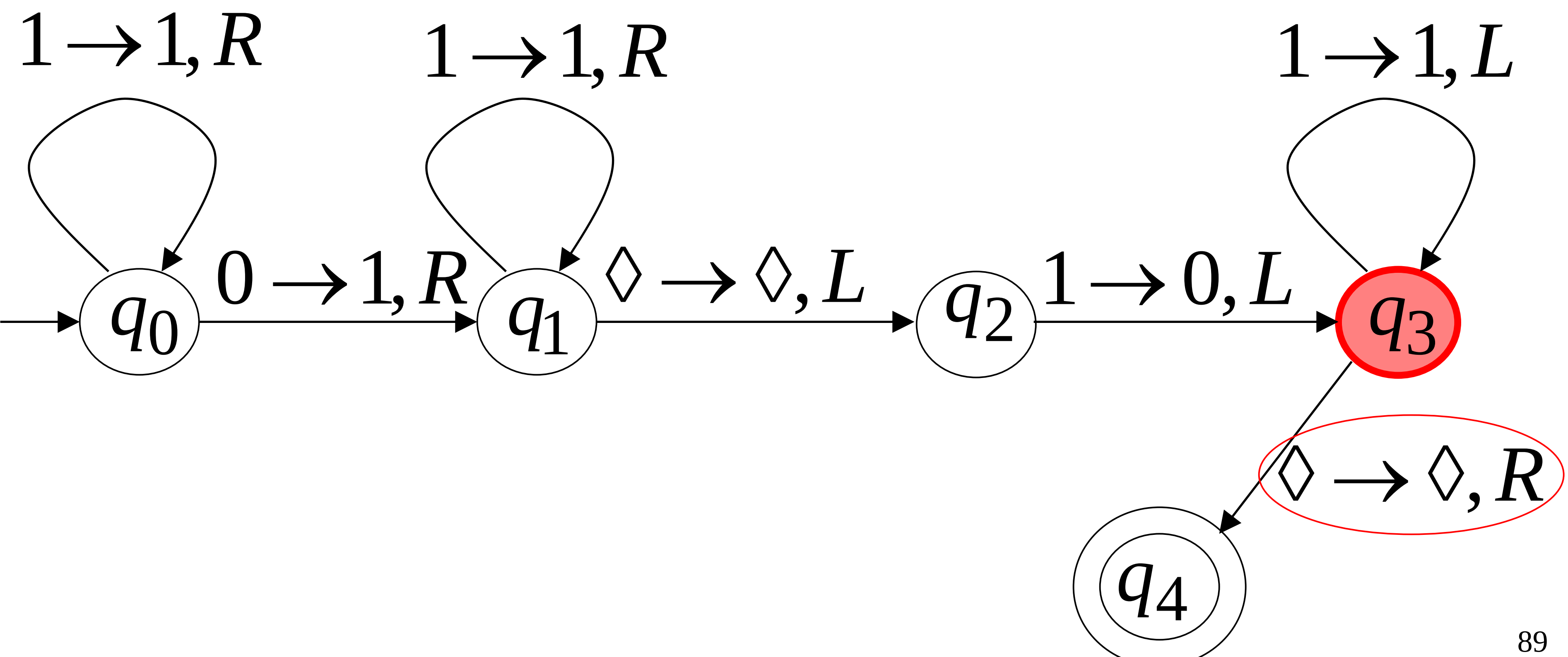

◊
1
1
1
1
0
◊
q_3


$1 \rightarrow 1, R$
$1 \rightarrow 1, R$
$1 \rightarrow 1, L$
$0 \rightarrow 1, R$
$\Diamond \rightarrow \Diamond, L$
$1 \rightarrow 0, L$
$\Diamond \rightarrow \Diamond, R$
q_0
q_1
q_2
q_3
q_4

Time 11

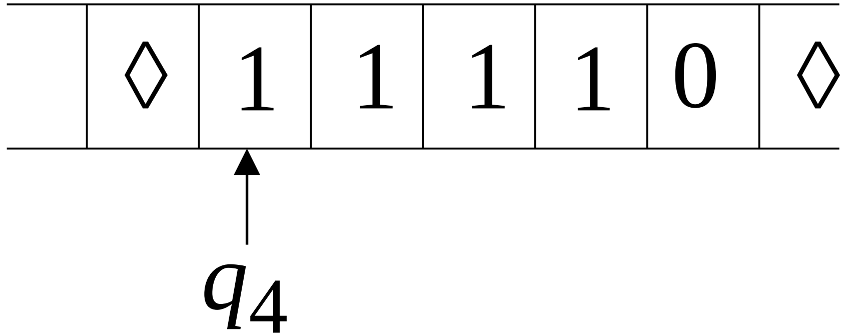


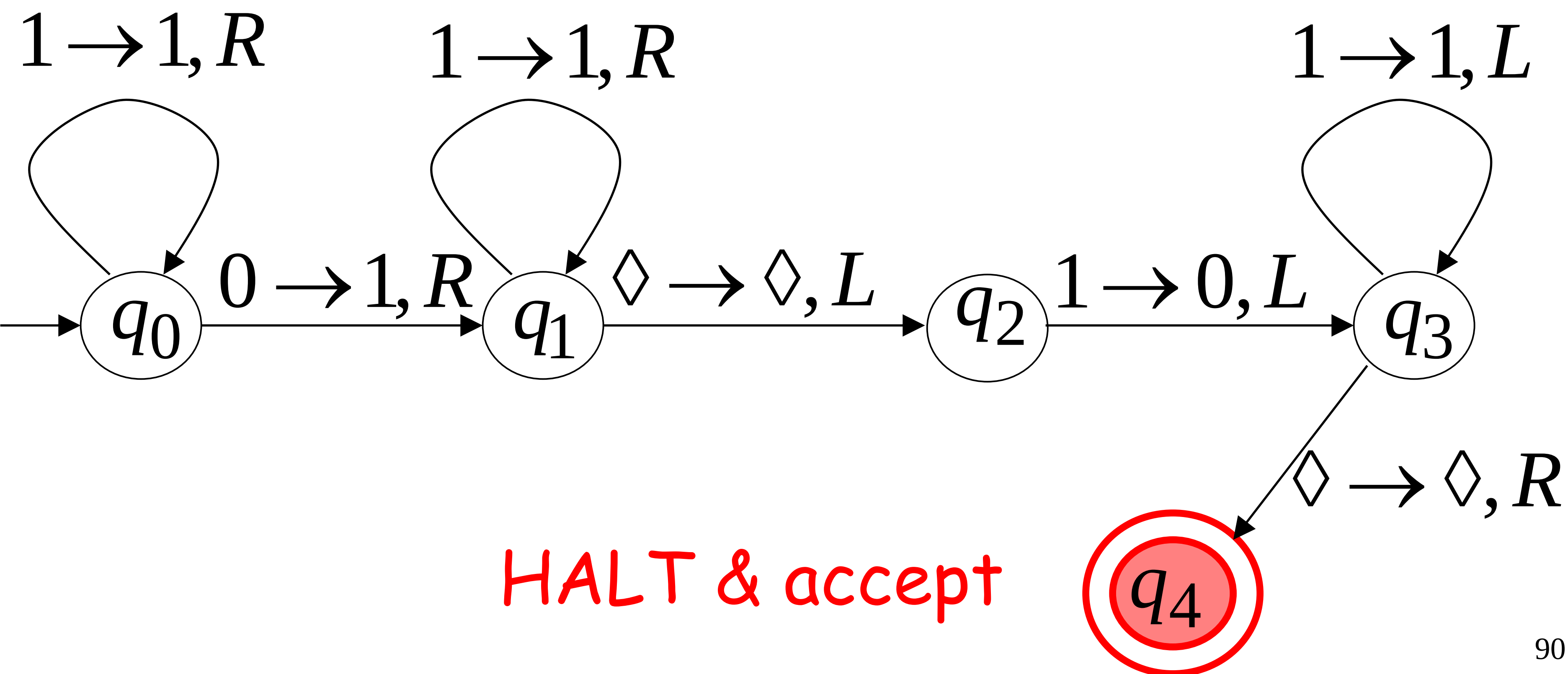
◊
1
1
1
1
0
◊
q_3



$1 \rightarrow 1, R$
$1 \rightarrow 1, R$
$1 \rightarrow 1, L$
q_0
$0 \rightarrow 1, R$
q_1
$◊ \rightarrow ◊, L$
q_2
$1 \rightarrow 0, L$
q_3
$◊ \rightarrow ◊, R$
q_4

Time 12


◊
1
1
1
1
0
◊
q_4


$1 \rightarrow 1, R$
$1 \rightarrow 1, R$
$1 \rightarrow 1, L$
q_0
$0 \rightarrow 1, R$
q_1
$\lozenge \rightarrow \lozenge, L$
q_2
$1 \rightarrow 0, L$
q_3
$\lozenge \rightarrow \lozenge, R$
HALT & accept
q_4

Another Example

The function $f(x) = 2x$ is computable

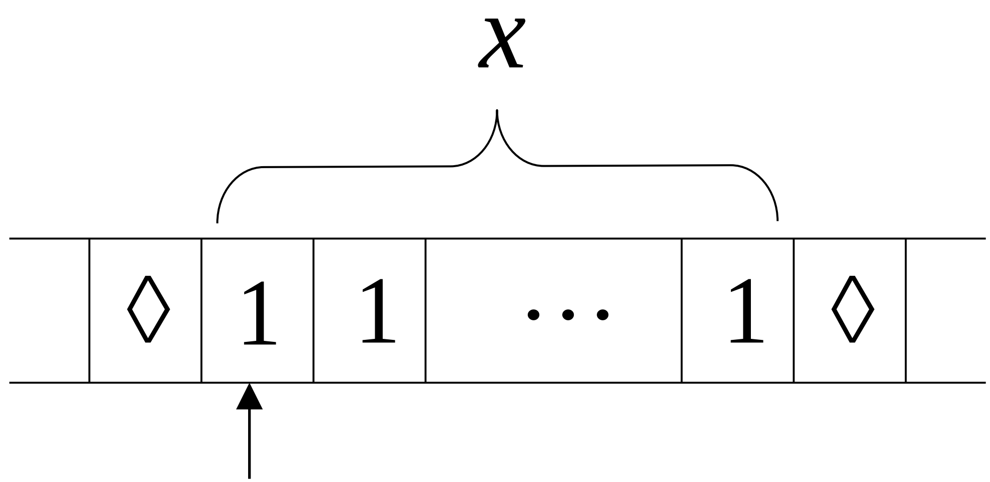
x is integer

Turing Machine:

Input string: x unary

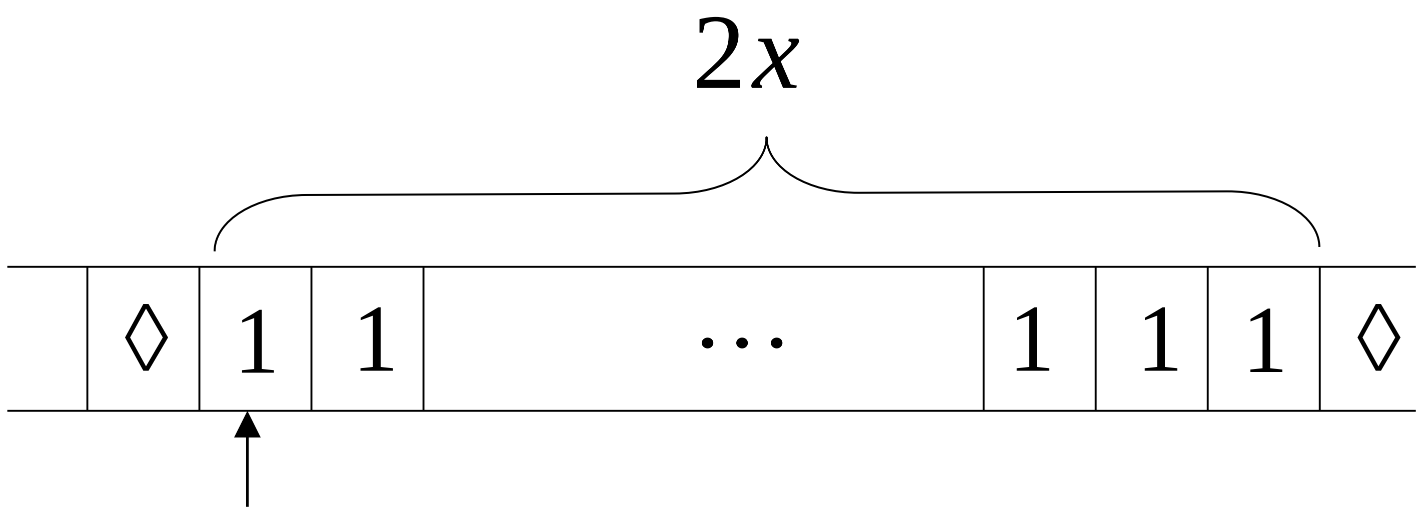
Output string: xx unary

Start



q_0 initial state

Finish



q_f final state

Turing Machine Pseudocode for $f(x) = 2x$

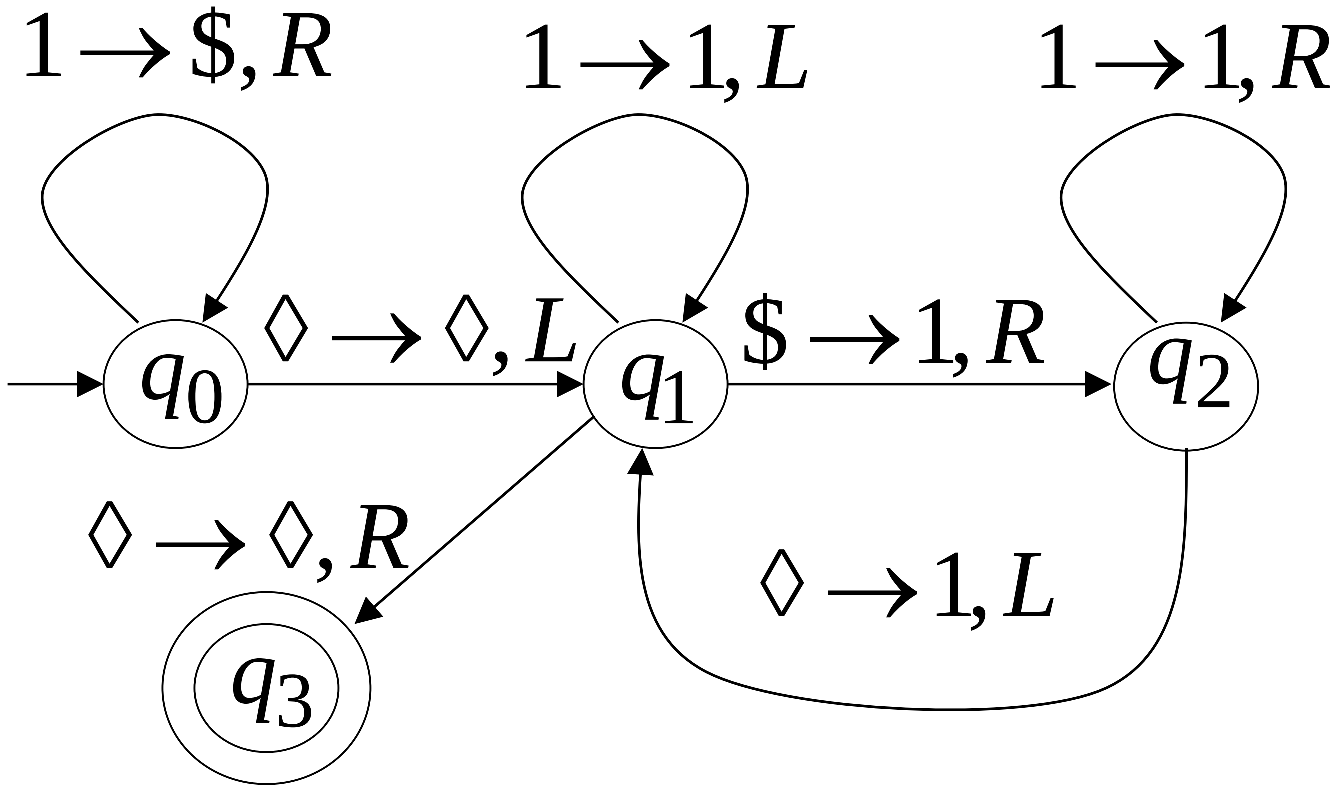
- Replace every 1 with $

- Repeat:
    - Find rightmost $, replace it with 1

    - Go to right end, insert 1

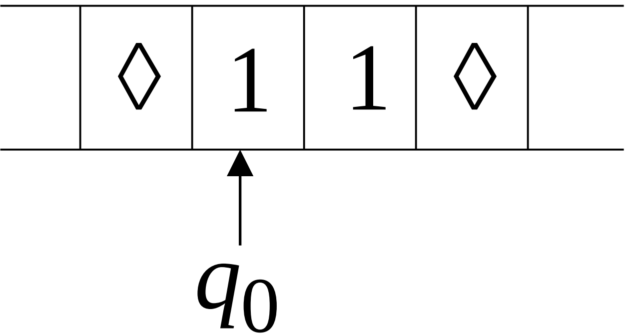
  Until no more $ remain

Turing Machine for $f(x) = 2x$

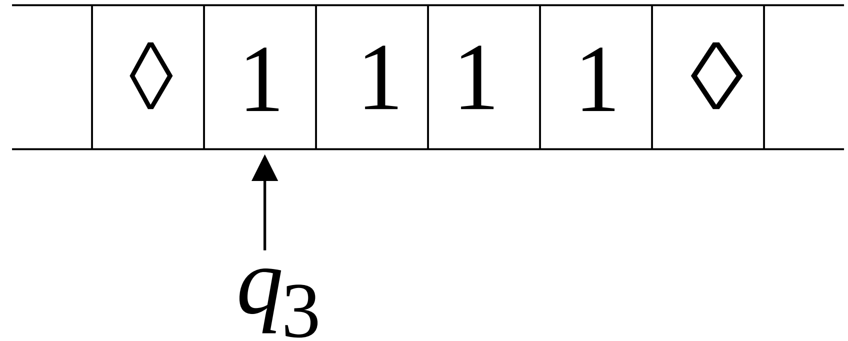


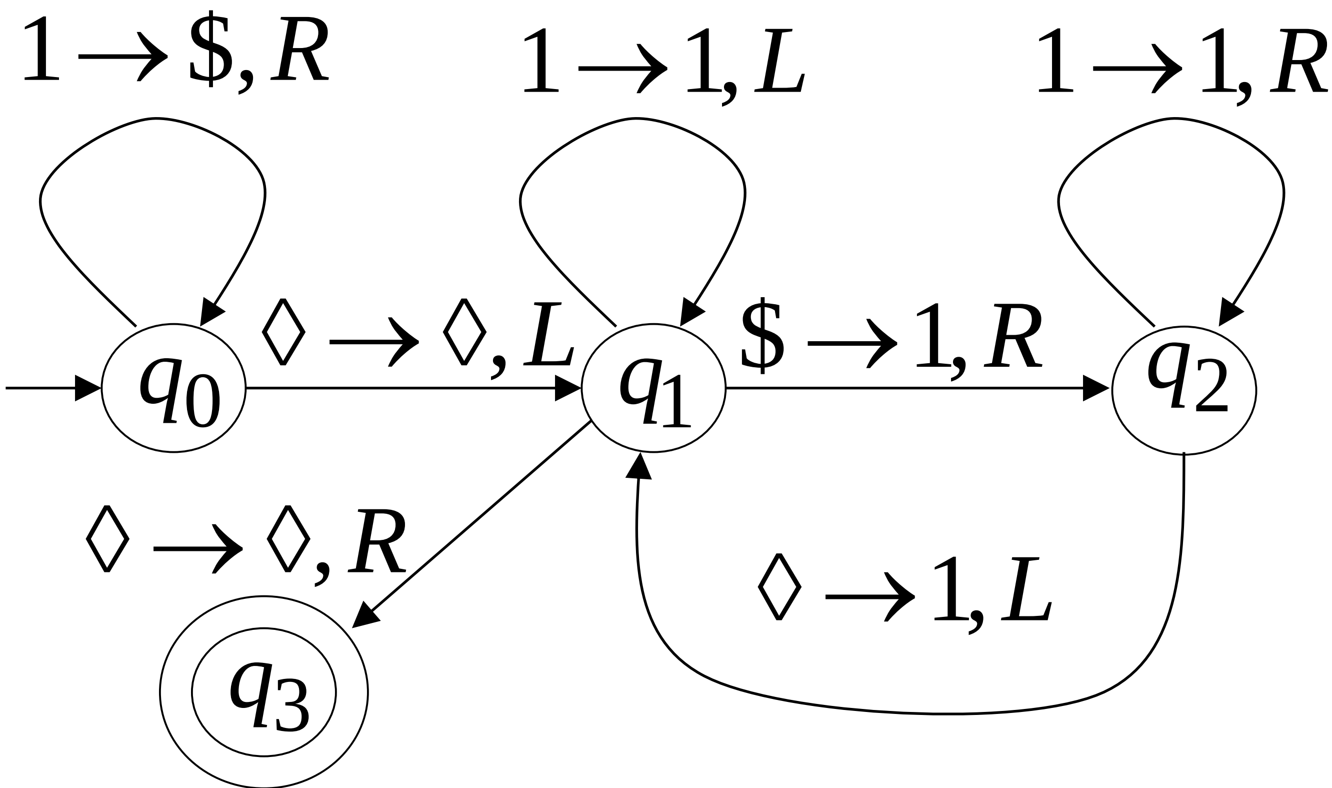
Example

Start



Finish





Another Example

The function

is computable

$$f(x,y) = \begin{cases} 1 & \text{if } x > y \\ 0 & \text{if } x \leq y \end{cases}$$

Turing Machine for

$$f(x,y) = \begin{cases} 1 & \text{if } x > y \\ 0 & \text{if } x \leq y \end{cases}$$

Input: $x0y$

Output: 1 or 0

Turing Machine Pseudocode:

- Repeat

  Match a 1 from x with a 1 from y

  Until all of x or y is matched

- If a 1 from x is not matched

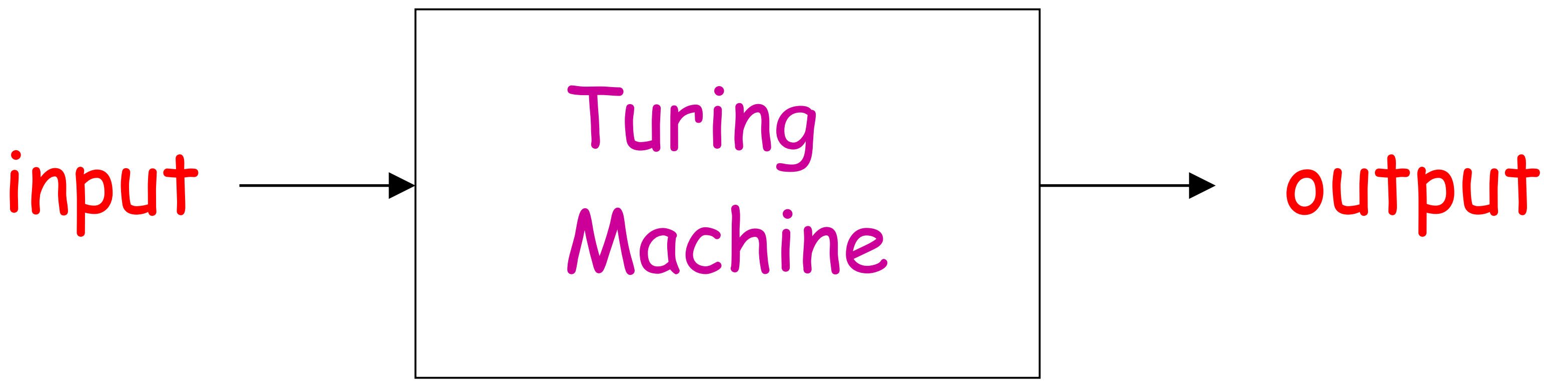
  erase tape, write 1 $(x > y)$

  else

  erase tape, write 0 $(x \leq y)$

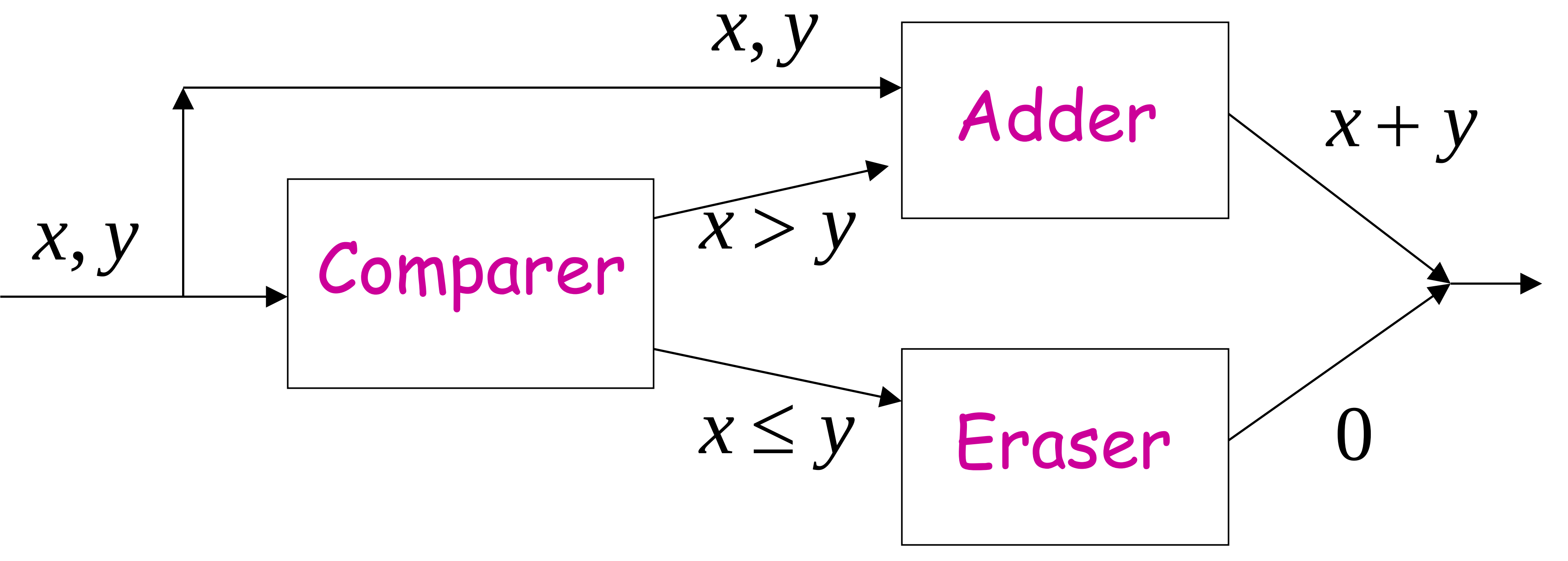
Combining Turing Machines

Block Diagram



Example:

$$f(x,y) = \begin{cases} x+y & \text{if } x > y \\ 0 & \text{if } x \leq y \end{cases}$$



Macro instructions

if a then qj else qk

If the Turing machine reads:

- an *a* → go into state *qj* without changing anything.

- not an *a* → go into state *qk* without changing anything.

Macro instructions

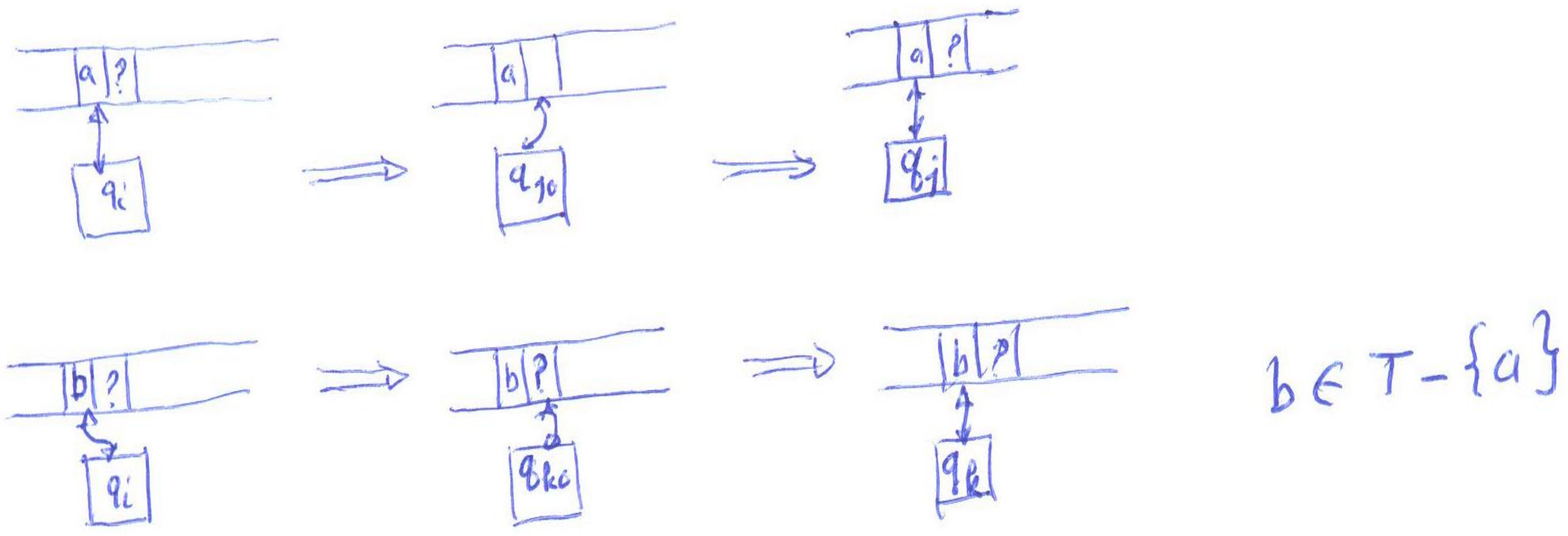
$\delta(q_i, a) = (q_{j0}, a, R)$ For all $q_i \in Q$

$\delta(q_i, b) = (q_{k0}, b, R)$ for all $q_i \in Q$ and all $b \in T - \{a\}$

$\delta(q_{j0}, c) = (q_j, c, L)$ for all $c \in T$

$\delta(q_{k0}, c) = (q_k, c, L)$ for all $c \in T$

Macro instructions



Macro instructions

If $\underline{a}$ then

If $\underline{b}$ then

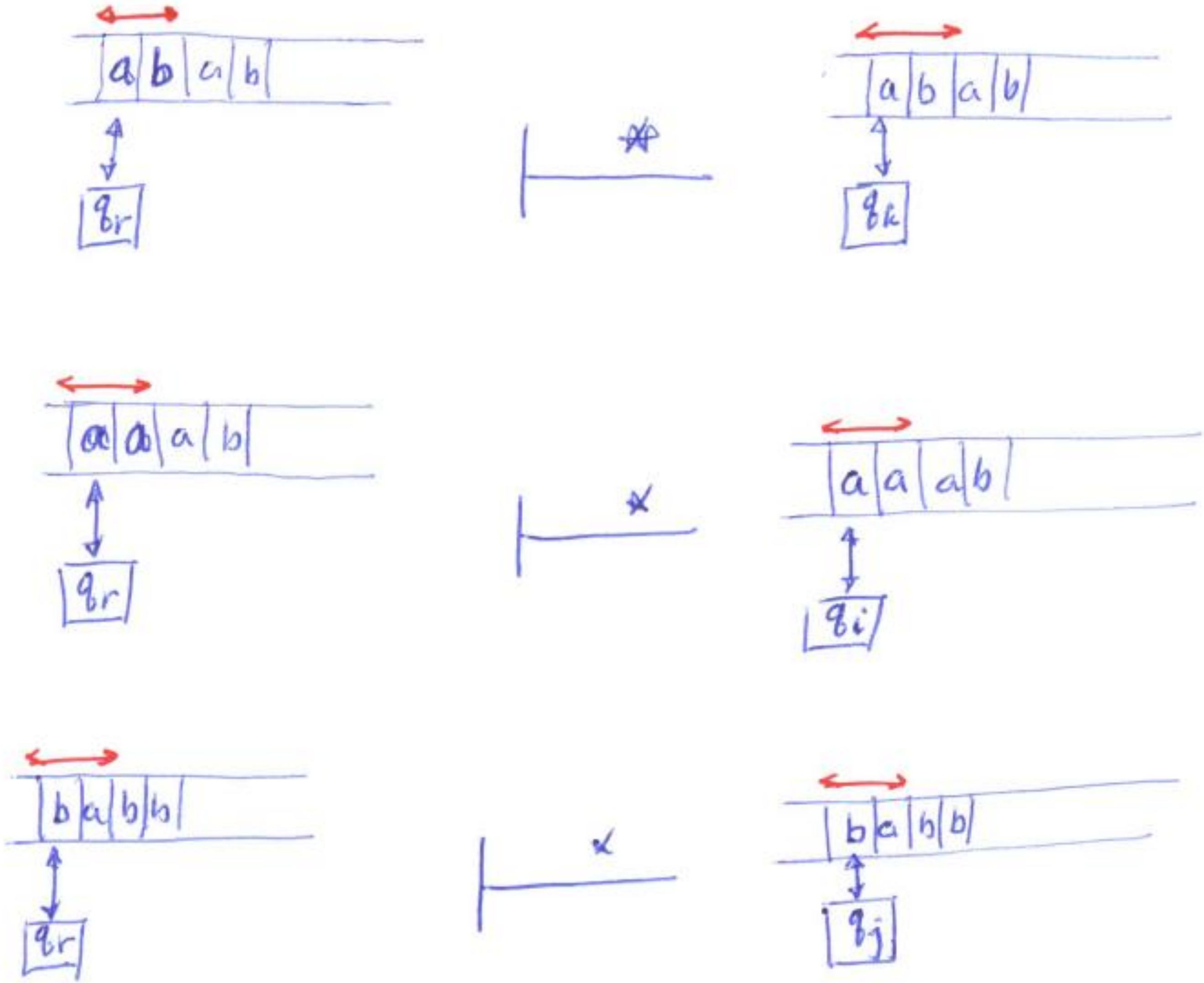
q_k

else

q_i

else

q_j


a b a b
q_r
⊬
a b a b
q_k
a a a b
q_r
⊢
a a a b
q_i
b a b b
q_r
⊢
b a b b
q_j

Turing's Thesis

Turing's thesis:

Any computation carried out
by mechanical means
can be performed by a Turing Machine

(1930)

Computer Science Law:

A computation is mechanical
if and only if
it can be performed by a Turing Machine

There is no known model of computation
more powerful than Turing Machines

Definition of Algorithm:

An algorithm for function $f(w)$
is a
Turing Machine which computes $f(w)$

$$q_0\, w \ \overset{*}{\succ}\ q_f\, f(w)$$

For all $w \in D$ Domain

Algorithms are Turing Machines

When we say:

There exists an algorithm

We mean:

There exists a Turing Machine
that executes the algorithm