`Macaulay2` workshop Leipzig 2018 – project on Tensors

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The main goal of this project is to create a package helpful for researchers interested in *tensors* and their *decompositions*. In the literature, there are several algorithms (some of them already written in `Macaulay2`) and new lines of research that should be addressed with tools collected in a well-structured package to be efficiently used. See [Lan12] for an extensive survey on tensors and their uses in algebraic geometry and applications. Here is a list of projects we would like to suggest as points of departure.

Basic constructions of tensors

Constructions

(i) Give an efficient way to handle tensors in *Macaulay2*, e.g., how to input a tensor, how to go from a (multi-)homogeneous polynomial and a (partially) symmetric tensor

Manipulations

(i) Give functions to construct new tensors from old ones, e.g., Kronecker products, Hadamard products, etc...

Apolarity Theory

Apolar ideal

(i) Define a function that, given a homogeneous polynomial f, returns the apolar ideal of f.

Sylvester's Algorithm

(i) Define a function that, given a binary form f returns the ideal of a minimal reduced set of points apolar to f.

(ii) Define a function that, given a form f in two essential variables returns the ideal of a minimal reduced set of points apolar to f.

Apolarity Lemma

(i) Define a function that, given a homogeneous polynomial f and an ideal I, checks if I defines a 0-dim (reduced) scheme apolar to f.

(ii) Can we define a function that, given a homogeneous polynomial f and an ideal defining a minimal set of reduced points apolar to f, returns a decomposition of f using the corresponding linear forms?

OBS.: in general, this might require to work over fields which are not precise...

Generalizations of Apolarity Lemma

(i) Apolarity Lemma has been generalized to any *toric variety*; see [Gał16, Tei14]. Implement a procedure that, given a toric variety X, a closed subscheme R and a non-zero global section F, checks if R is apolar to F.

OBS.: as testing cases, we should try to use partially symmetric and non-symmetric tensors.

Flattenings: equations of secant varieties and decomposition algorithms

Flattenings

(i) Define a procedure to construct (symmetric) flattenings of a given (symmetric) tensor. OBS.: as testing cases, we should get the equations defining some secant varieties to some Veronese varieties; see the table at page 2 of [LO13].

Exact decomposition algorithms

(i) Implement the *Catalecticant algorithm*; see [Lan12, Section 12.4.1];

(ii) Implement the *Koszul flattenings algorithm*; see [Lan12, Section 12.4.2] and [OO13]. The aim of Koszul flattenings is to go beyond lower bounds to border ranks for tensors $T \in A \otimes B \otimes C$ given by rank conditions on flattenings; namely, beyond the equations $\wedge^{r+1} A^* \otimes \wedge^{r+1}(B \otimes C)^*$. The idea is to use an augmented version of our tensor T. Consider $\mathrm{Id}_A \otimes T : A \otimes B^* \to A \otimes A \otimes C$. We show that the flattenings of $\mathrm{Id}_A \otimes T$ have relations with the border rank of T. We have the two canonical projections $T_A^\wedge : A \otimes B^* \to \wedge^2 A \otimes C$ and $T_A^s : A \otimes B^* \to S^2 A \otimes C$.
Suppose $\dim A = \mathbf{a} = 3$. Choosing bases of A, B, C, we may write T as

$$T = a_1 \otimes X_1 + a_2 \otimes X_2 + a_3 \otimes X_3,$$

where $X_i : B^* \to C$. In terms of matrices, we have

$$T_A^\wedge = \begin{pmatrix} 0 & X_2 & X_3 \\ -X_2 & 0 & -X_1 \\ -X_3 & X_1 & 0 \end{pmatrix}.$$

If the border rank of T satisfies $\underline{\mathbf{R}}(T) \leq r$, then $\mathrm{rank}(T_A^\wedge) \leq r(\mathbf{a}-1)$; hence $r \geq \frac{\mathrm{rank}(T_A^\wedge)}{(\mathbf{a}-1)}$. More generally, let A, B, C be vector spaces with $\mathbf{a} = 2p+1 \leq \mathbf{b} \leq \mathbf{c}$. Consider the map

$$T_A^{\wedge p} : \wedge^p A \otimes B^* \to \wedge^{p+1} A \otimes C.$$

If T has rank one, then $\mathrm{rank}(T_A^{\wedge p}) = \binom{2p}{p}$. If T is generic, then $\mathrm{rank}(T_A^{\wedge p}) = \binom{2p+1}{p} \cdot \mathbf{b}$. Thus the size $(r+1)\binom{2p}{p}$ minors of $T_A^{\wedge p}$ furnish equations for the rth secant variety $\hat{\sigma}_r$ up to $r = \frac{2p+1}{p+1} \cdot \mathbf{b} - 1$.

OBS.: other procedures to decompose low rank symmetric tensors have been given in [MO18]. A package "ApolarLowRank.m2" is also given.

(iii) Implement the algorithm to find the unique tensor decomposition of a *general* element in $\mathbb{C}^2 \otimes \mathbb{C}^a \otimes \mathbb{C}^b$; see [Lan12, Section 12.4.3].

OBS.: when these exact algorithms fail to give an exact decomposition, bounds on the ranks of tensors should be obtained. In this direction, e.g., see also [Lan12, Theorem 3.8.2.4].

Useful constructions to study tensors

Symmetry groups of tensors

Let $T \in V_1 \otimes \cdots \otimes V_d$ be a tensor. Its *symmetry* or *isotropy* group is defined to be

$$G_T = \{g \in \mathrm{GL}(V_1) \times \cdots \times \mathrm{GL}(V_d)/(\mathbb{C}^*)^{d-1} \mid gT = T\}.$$

Here the $(d-1)$-dimensional torus comes from the trivial action on every tensor given by $\prod_{i=1}^{d} \lambda_i = 1$, for $\lambda_i \in \mathbb{C}^*$. Since the action is rational, the symmetry group is an algebraic group and thus an algebraic variety. Given a tensor T, we determine the dimension of its symmetry group G_T. Moreover, we use `GAP` to analyze further properties of this group related to its Lie algebra $\mathfrak{g}_T$. For instance, the Levi-Malcev decomposition of $\mathfrak{g}_T$ and the structure of its semisimple part.

Eigenvectors of tensors

The *E-eigenvalues* and *E-eigenvectors* of tensors (the "E" stands for "Euclidean") were proposed independently by Lek-Heng Lim and Liqun Qi in [Lim05, Qi05]. There are different types of eigenvectors and eigenvalues in the literature, see [CS13, HHLQ13, NQWW07, Qi07, QL17]. The notions of E-eigenvalues and E-eigenvectors of tensors arise mainly in the context of approximation of tensors, which deals usually with real tensors. Some excellent references about the *best rank k approximation problem* for tensors are [DOT17, FO14]. Possibility to experiment the reality of eigenvectors. Some reality issues of eigenvalues and eigenvectors are studied in [Mac16, Koz17].

Hyperderminants

For definitions and constructions, we refer to [GKZ08, Chapter 14].

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