Section – 7.7 Vector Spaces

Span and Linear Independence

We are given a set (v1, v2, v3 . . . . ., vn) of vectors. Any vector that can be expressed in the form of linear combination of vectors, it is said to be in the span of the vectors v1, v2 . . ., vn. Whether b is in the span of vector set can be determined by determining whether the equation mx = b has a solution, where m is a matrix with vectors (v1, v2, v3 . . . vn) as its columns.

Is vector b = (1, 2, 3) in the span of the vectors v1 = (10, 4, 5), v2 = (4, 4, 7) and v3 = (8, 1, 0)?

In[1]:=
```
Clear[v1, v2, v, b, m, c];
v1 = {10, 4, 5};
v2 = {4, 4, 7};
v3 = {8, 1, 0};
b = {1, 2, 3};
m = Transpose[{v1, v2, v3}];
c = LinearSolve[m, b]
```

Out[7]= $\left\{\frac{3}{2}, -\frac{9}{14}, -\frac{10}{7}\right\}$

Checking whether b can be written in terms of linear combination of vectors?

In[11]:=
```
c[[1]] v1 + c[[2]] v2 + c[[3]] v3
```

Out[11]= {1, 2, 3}

Here b is in the span of vectors as it can be written in the form of Linear combination as c1v1 + c2v2 + c3v3 = b.

A set of vectors is said to be independent when, in the linear combination a1v1 + a2v2 + a3v3 . . . . . anvn, ai = 0. if atleast one ai ≠ 0, then set of vectors is linearly dependent. To check the independency of the set of vectors, the equation mx = 0 should have only the trivial solution and to show that nullspace[] is used .

In[12]:=
```
NullSpace[m]
```

Out[12]= {}

Yes, the vectors are independent.

The set of vectors is linearly independent when the determinant is not zero.

In[13]:=
```
Det[{v1, v2, v3}]
```

Out[13]= 14

Basis

A basis for a vector space is a set of linearly independent vectors whose span includes every vector in the vector space.

In[21]:=
```
Clear[v1, v2, v3, v4, m, a, b, c];
v1 = {2, 1, 15, 10, 6};
v2 = {2, -5, -3, -2, 6};
v3 = {0, 5, 15, 10, 0};
v4 = {2, 6, 18, 8, 6};
m = {v1, v2, v3, v4};
RowReduce[m] // MatrixForm
```

Out[27]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & -2 & 3 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Here the non - zero rows forms the basis for the space spanned by the vector set (v1, v2, v3, v4). It is called row space of the matrix. Basis can be find by rowreducing the matrix whose columns are the vector set and then the column containing non zero entries will form the basis.

In[28]:=
```
Clear[v1, v2, v3, v4, m, a, b, c];
v1 = {2, 1, 15, 10, 6};
v2 = {2, -5, -3, -2, 6};
v3 = {0, 5, 15, 10, 0};
v4 = {2, 6, 18, 8, 6};
m = Transpose[{v1, v2, v3, v4}];
RowReduce[m] // MatrixForm
```

Out[34]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & \frac{5}{6} & 0 \\ 0 & 1 & -\frac{5}{6} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

In[35]:=
```
NullSpace[Transpose[{v1, v2, v4}]]
```

Out[35]= {}

Here, v1, v2, v4 form basis for the space spanned by the set {v1, v2, v3, v4}, It is also linearly independent. The number of vectors in any basis for that vector space is called the dimension of the vector space.

Rank and Nullity

Dimension of matrix is called nullity of matrix. Length is used to find nullity of matrix.
Rank of the matrix is the common dimension of the row space and column space.

In[36]:=
```
Length[NullSpace[m]]
```
Out[36]= 1

In[37]:=
```
MatrixRank[m]
```
Out[37]= 3

Orthonormal bases

A collection of vectors is orthogonal if vectors are mutually perpendicular,
the set is orthonormal if in addition,
the norm of each vector is 1. Normalize command is used to find unit vector,
vector with length 1. Orthogonalize command is used to find orthonormal basis.

In[38]:=
```
Clear[v1, v2, v3, u1, w1, w2, w3];
v1 = {2, 3, -4, 1, 0};
v2 = {1, 5, -6, 10, -3};
v3 = {7, -2, 1, 1, 1};
Norm[v1]
```
Out[42]= $\sqrt{30}$

In[43]:=
```
u1 = Normalize[v1]
```
Out[43]= $\left\{\sqrt{\frac{2}{15}}, \sqrt{\frac{3}{10}}, -2\sqrt{\frac{2}{15}}, \frac{1}{\sqrt{30}}, 0\right\}$

In[44]:=
```
Norm[u1]
```
Out[44]= 1

In[45]:=
```
NullSpace[Transpose[{v1, v2, v3}]]
```
Out[45]= {}

In[46]:=
```
{w1, w2, w3} = Orthogonalize[{v1, v2, v3}];
{w1, w2, w3} // MatrixForm
```
Out[47]//MatrixForm=

$$\begin{pmatrix} \sqrt{\frac{2}{15}} & \sqrt{\frac{3}{10}} & -2\sqrt{\frac{2}{15}} & \frac{1}{\sqrt{30}} & 0 \\ -4\sqrt{\frac{6}{1405}} & -\frac{1}{\sqrt{8430}} & 4\sqrt{\frac{2}{4215}} & \frac{83}{\sqrt{8430}} & -\sqrt{\frac{30}{281}} \\ \frac{5368}{\sqrt{38\,274\,729}} & -706\sqrt{\frac{3}{12\,758\,243}} & \frac{1489}{\sqrt{38\,274\,729}} & \frac{1574}{\sqrt{38\,274\,729}} & 176\sqrt{\frac{3}{12\,758\,243}} \end{pmatrix}$$

In[48]:=
```
{w1.w2, w2.w3, w3.w1}
```
Out[48]= {0, 0, 0}

In[49]:=
```
{Norm[w1], Norm[w2], Norm[w3]}
```

Out[49]=
```
{1, 1, 1}
```

Dot product is 0,
thus vectors are orthogonal also lenth of each vector is 1 thus orthonormal.