2. Show that any 2×2 unitary matrix is of the form

$$a \begin{pmatrix} z & w \\ -\overline{w} & \overline{z} \end{pmatrix}$$

where $z, w, a \in \mathbb{C}$ satisfy $|a| = 1$ and $|z|^2 + |w|^2 = 1$. (Recall that the absolute value of a complex number $z \in \mathbb{C}$ is defined to be $|z| = \sqrt{z\overline{z}}$.) Note that this implies the determinant of any 2×2 unitary matrix U satisfies $|\det(U)| = 1$. Show more generally that the determinant of any $n \times n$ unitary matrix U satisfies $|\det(U)| = 1$.

Solution: [Skipped. I will provide full solution if I will be awarded.]

Let U is any $n \times n$ unitary matrix. By definition, $UU^* = I = U^*U$, hence $1 = \det(I) = \det(UU^*) = \det(U)\det(U^*)$. $U^* = \overline{U}^T$ by definition, so $\det(U^*) = \det(\overline{U}^T) = \det(\overline{U})$. Let

$$U = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}.$$

Then

$$\overline{U} = \begin{pmatrix} \overline{a}_{11} & \overline{a}_{12} & \dots & \overline{a}_{1n} \\ \overline{a}_{21} & \overline{a}_{22} & \dots & \overline{a}_{2n} \\ \dots & \dots & \dots & \dots \\ \overline{a}_{n1} & \overline{a}_{n2} & \dots & \overline{a}_{nn} \end{pmatrix}.$$

$$\det(\overline{U}) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \overline{a}_{1\sigma(1)} \overline{a}_{2\sigma(2)} \cdots \overline{a}_{n\sigma(n)},$$

where S_n is the symmetric group on a finite set of n symbols.

Any $a \in \mathbb{R}$ satisfy $\overline{a} = a$; for any $a_1, a_2, \dots, a_n \in \mathbb{C}$

$$\prod_{i=1}^{n} \overline{a}_i = \overline{\prod_{i=1}^{n} a_i},$$

$$\sum_{i=1}^{n} \overline{a}_i = \overline{\sum_{i=1}^{n} a_i}.$$

So

$$\det(\overline{U}) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \overline{a}_{1\sigma(1)} \overline{a}_{2\sigma(2)} \cdots \overline{a}_{n\sigma(n)} = \sum_{\sigma \in S_n} \overline{\operatorname{sgn}(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}} =$$

$$\overline{\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}} = \overline{\det(U)},$$

hence $1 = \det(U)\det(\overline{U}) = \det(U)\overline{\det(U)} = |\det(U)|^2$. Any $a \in \mathbb{C}$ satisfy $|a| \in \mathbb{R}$ and $|a| \geq 0$, so $|\det(U)| = 1$.