1 Limits

You have seen enough limits to be ready for a definition. It is true that we have survived this far without one, and we could continue. But this seems a reasonable time to define limits more carefully. The goal is to achieve rigor without rigor mortis.

First you should know that limits of $\Delta y/\Delta x$ are by no means the only limits in mathematics. Here are five completely different examples. They involve $n \to \infty$, not $\Delta x \to 0$:

1. $a_n = (n-3)/(n+3)$ (for large n, ignore the 3's and find $a_n \to 1$)
2. $a_n = \frac{1}{2}a_{n-1} + 4$ (start with any a_1 and always $a_n \to 8$)
3. $a_n =$ probability of living to year n (unfortunately $a_n \to 0$)
4. $a_n =$ fraction of zeros among the first n digits of n ($a_n \to \frac{1}{10}$?)
5. $a_1 = .4$, $a_2 = .49$, $a_3 = .493$, ... No matter what the remaining decimals are, the a's converge to a limit. Possibly $a_n \to .493000\ldots$, but not likely.

2 Applications of the Derivative

2.1 MAXIMA AND MINIMA

Which x makes $f(x)$ as large as possible? Where is the smallest $f(x)$? Without calculus we are reduced to computing values of $f(x)$ and comparing. With calculus, the information is in df/dx.

Suppose the maximum or minimum is at a particular point x. It is possible that the graph has a corner—and no derivative. *But if df/dx exists, it must be zero.* The tangent line is level. The parabolas in Figure 3.3 change from decreasing to increasing. The slope changes from negative to positive. At this crucial point *the slope is zero.*

Theorem 1 (Local Maximum or Minimum)**.** *Suppose the maximum or minimum occurs at a point x inside an interval where $f(x)$ and df/dx are defined. Then $f'(x) = 0$.*

The word *"local"* allows the possibility that in other intervals, $f(x)$ goes higher or lower. *We only look near x*, and we use the definition of df/dx.

Start with $f(x + \Delta x) - f(x)$. If $f(x)$ is the maximum, this difference is negative or zero. The step Δx can be forward or backward:

$$\text{if } \Delta x > 0 : \frac{f(x+\Delta x) - f(x)}{\Delta x} = \frac{\text{negative}}{\text{positive}} \leq 0 \text{ and in the limit } \frac{df}{dx} \leq 0.$$

$$\text{if } \Delta x < 0 : \frac{f(x+\Delta x) - f(x)}{\Delta x} = \frac{\text{negative}}{\text{negative}} \geq 0 \text{ and in the limit } \frac{df}{dx} \geq 0.$$

Both arguments apply. Both conclusions $df/dx \leq 0$ and $df/dx \geq 0$ are correct. Thus $df/dx = 0$.

3 Integrals

3.1 THE SIGMA NOTATION

In a section about sums, there has to be a decent way to express them. Consider $1^2+2^2+3^2+4^2$. The individual terms are $v_j = j^2$. Their sum can be written in ***summation notation***, using the capital Greek letter Σ (pronounced sigma):

$$1^2+2^2+3^2+4^2 \text{ is written } \sum_{j=1}^{4} j^2.$$

Spoken aloud, that becomes "***the sum of*** j^2 ***from*** $j=1$ ***to 4***." It equals 30. The limits on j (written below and above Σ) indicate where to start and stop:

$$v_1+\cdots+v_n=\sum_{j=1}^{n} v_j \quad \text{and} \quad v_3+\cdots+v_9=\sum_{j=1}^{9} v_j.$$

4 The Definite Integral

The integral of $v(x)$ is an antiderivative $f(x)$ plus a constant C. This section takes two steps. First, we choose C. Second, we construct $f(x)$. The object is ***to define the integral***—in the most frequent case when a suitable $f(x)$ is not directly known.

The indefinite integral contains "$+C$." The constant is not settled because $f(x)+C$ has the same slope for every C. When we care only about the derivative, C makes no difference. When the goal is a number—***a definite integral***—C can be assigned a definite value at the starting point.

For mileage traveled, ***we subtract the reading at the start***. This section does the same for area. Distance is $f(t)$ and area is $f(x)$—while the definite integral is $f(b)-f(a)$. Don't pay attention to t or x, pay attention to the great formula of integral calculus:

$$\int_a^b v(t)\,dt = \int_a^b v(x)\,dx = f(b)-f(a). \tag{1}$$

5 Maxima, Minima, and Saddle Points

5.1 STATIONARY POINT → HORIZONTAL TANGENT → ZERO DERIVATIVES

Suppose f has a minimum at the point (x_0, y_0). This may be an ***absolute minimum*** or only a ***local minimum***. In both cases $f(x_0,y_0) \le f(x,y)$ near the point. For an absolute minimum, this inequality holds wherever f is defined. For a local minimum, the inequality can fail far away from (x_0,y_0). The bottom of your foot is an absolute minimum, the end of your finger is a local minimum.

We assume for now that (x_0, y_0) is an interior point of the domain off. At a boundary point, we cannot expect a horizontal tangent and zero derivatives.

Main conclusion: At a minimum or maximum (absolute or local) a nonzero derivative is impossible. The tangent plane would tilt. In some direction f would decrease. Note that the minimum point is (x_0, y_0), and the minimum value is $f(x_0, y_0)$.

Theorem 2. *If derivatives exist at an interior minimum or maximum, they are zero:*

$$\partial f/\partial x = 0 \quad \textit{and} \quad \partial f/\partial y = 0 \quad \textbf{(together this is } \operatorname{grad} f = 0\text{)}. \tag{2}$$

For a function $f(x, y, z)$ of three variables, add the third equation $\partial f/\partial z = 0$.

Example 1 Minimize the quadratic $f(x, y) = x^2 + xy + y^2 - x - y + 1$.

To locate the minimum (or maximum), set $f_x = 0$ and $f_y = 0$:

$$f_x = 2x + y - 1 = 0 \quad \text{and} \quad f_y = x + 2y - 1 = 0. \tag{3}$$

Notice what's important: ***There are two equations for two unknowns*** x ***and*** y. Since f is quadratic, the equations are linear. Their solution is $x_0 = \frac{1}{3}$, $y_0 = \frac{1}{3}$ (the stationary point). This is actually a minimum, but to prove that you need to read further.

The constant 1 affects the minimum value $f = \frac{2}{3}$—but not the minimum point. The graph shifts up by 1. The linear terms $-x - y$ affect f_x and f_y. They move the minimum away from $(0, 0)$. The quadratic part $x^2 + xy + y^2$ makes the surface curve upwards. Without that curving part, a plane has its minimum and maximum at boundary points.

Example 2 Find the derivative of $f(x) = 1/x$ at the point $x = 2$.

Solution. The derivative is the limit of the difference quotient, so we look at

$$f'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}.$$

Using the formula for f and simplifying gives

$$f'(2) = \lim_{h \to 0} \frac{1}{h}\left(\frac{1}{2+h} - \frac{1}{2}\right) = \lim_{h \to 0} \frac{2 - (2+h)}{2h(2+h)} = \lim_{h \to 0} \frac{-h}{2h(2+h)}.$$

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