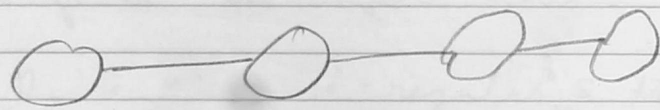
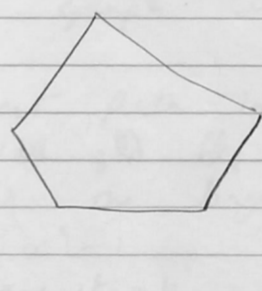
Problem-1

A self complementary graph is a graph which is isomorphic to its complement. The simplest non-trivial self-complementary graphs are the 4-vertex path graph and the 5-vertex cycle graph.

(a) An n-vertex self complementary graph has exactly half number edges of the complete graph. i.e. $\left[\frac{n(n-1)}{4}\right]$ edges, and if there is more than one vertex it must have diameter either 2 or 3. Since $[n(n-1)]$ must be divisible by 4, n must be congruent to $[0 \text{ or } 1 \bmod 4]$; for instance a 6-vertex graph cannot be self complementary.

(b) 

Path graph 4-vertex



Cycle graph with 5-vertices

(c) Since $[n(n-1)]$ must be divisible by 4

$\therefore$ $n = 4k$ or $n-1 = 4k$

$\Rightarrow$ $n = 4k$ or $n = 4k+1$

Problem - 2

(c) We need to find no of edges in Q_8.

For $n \in N$ The hypercube Q_8 is an n-regular loop free undirected graph with 2^n vertices

Q_8 has $\left(\frac{1}{2}\right) n 2^n = n 2^{n-1}$ edges

The hypercube Q_8 is a 8-regular loop free undirected graph with 2^n vertices.

Using the above eq^n the no of edges in

$$Q_8 = 8 \cdot 2^{8-1}$$
$$= 8 \cdot 2^7$$
$$= 1024$$

Hence no of edges in Q_8 is $= 1024$

(b) We need to find maximum distance between pairs of vertices in Q_8 and given an example of one such pair which achieves this distance.

The hypercube Q_8 is an 8-regular loop free undirected graph.

For a hypercube Q_8 for $n \in N_8$ The maximum distance between any two vertices of Q_8 is n.

For example the distance between the vertices 00000000 and 11111111 Q_8 is 8.

Hence the max distance between pairs of vertices is $Q_8 = 8$.

(c) We need to find length of longest path in Q_8 for $n \in N$. The hypercube Q_8 is an n-regular loop free undirected graph with x-vertices.

So Q_8 is a 8-regular loop free undirected graph with 2^4 vertices.

A longest path in Q_8 has contain all the vertices in Q_8 that is it will contain 2^4 vertices. The length of any path having 2^4 vertices is $2^4 - 1$.

Thus the longest path in Q_8 has length $= 2^4 - 1$

Problem - 3

We need to find the total number of distinct though isomorphic paths of length 2 in an n-dimensional hypercube Q_n for $n \in Z^+$.

The hypercube Q_n is n-regular loop-free undirected graph with 2^n vertices.
The typical path of length 2 uses two edges of the form $\{a, b\}, \{b, c\}$

We can select the vertex b as any vertex of Q_n. As there are 2^n vertices in an n-dimensional hypercube, there are 2^n choices for b.

The vertex b is adjacent to n other vertices in Q_n and out of these n vertices we can choose two vertices in ${}^nC_2$ ways.

Hence there are $\left[{}^nC_2 \times 2^n\right]$ paths of length 2 in Q_n.

Problem-4

The objective of the problem is to prove that for each $n \in \mathbb{Z}^+$ there exists a loop free connected undirected graph $G = (V, E)$, and $|V| = 2n$ and which two vertices of degree i for every $1 \le i \le n$.

Proof by mathematical induction.

Prove result is true for $n=1$.

For $n=1$, put $V = \{a_1, b_1\}$ and $E = \{\{a_1, b_1\}\}$.

Then, $\deg a_1 = \deg b_1 = 1$.

Now assume that the result is true for n and prove that the result is true for $n+1$. Suppose now that such a graph G exists for some $n \in \mathbb{N}$ and let the set of vertices be $V = \{a_1, a_2, a_3 \ldots, a_n, b_1, b_2, b_3, \ldots b_n\}$

where $\deg a_i = \deg b_i = i$ for every i.

Now add two more vertices a_{n+1} and b_{n+1}. Then, connect a_{n+1} with a_i for every i and connect a_{n+1} and b_{n+1}.

The new graph, consisting of $[2(n+1)]$ vertices

is connected, because G is connected. Also,
$\deg b_1 = \deg b_{n+1} = 1$, $\deg a_{i-1} = \deg b_i = i$ for

every $i \in \{1, 2, 3, \ldots, n\}$ and $\deg a_n = \deg a_{n+1} = n+1$.

This proves that there exists such a graph for $n+1$.
Thus by the principle of mathematical induction, such graph exists for every $n \in N$.

Hence, for each $n \in Z^+$ there exists a loop free connected undirected graph $G = (V, E)$, and $|V| = 2n$ and which two vertices of degree i for every $1 \le i \le n$.

Problem-5

We need to prove that loop free connected undirected graph $G = (V, E)$ contains a path of length k where k is a fixed positive integer and $\deg(v) \ge k$ for all $v \in V$.

A loop free connected undirected graph is a graph in which

(i) There exists a path between any two vertices of graph

(ii) There is no loop

(iii) There is no concern about the direction of edges.

For any undirected connected loop free graph $G = (V, E)$ having $v \in V$, the degree of the vertex v, $\deg(v)$ is defined as the total number of edges incident on that vertex.

Now for the graph $G = (V, E)$, let us consider that $v_1, v_2, v_3, v_4, \ldots v_k$ $k \in V$ are the vertices of the graph.

If $v_1, v_2 \in V$ then there must be an edge $\{v_1, v_2\}$ since $V \neq 0$ and $\deg(v) \geq k \geq 1$ for all $v \in V$. This condition is true for $k = 1$.

Now if $k > 1$, suppose we have selected vertices $v_1, v_2, \ldots, v_k \in V$ with edges $\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}, \ldots \{v_{k-1}, v_k\} \in E$

Since $\deg(v_k) \geq k$, there exists $v_{k+1} \in V$, where $v_{k+1} \neq v_i$ for $1 \leq i \leq k-1$

and $\{v_k, v_{k+1}\} \in E$.

Then $\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{k-1}, v_k\}, \{v_k, v_{k+1}\}$

provides a path of length k.

Hence a loop-free connected undirected graph $G = (V, E)$ contain a path of length k with k being a fixed positive integer and $\deg(v) \geq k$ for all $v \in V$.

Problem - 6

(a) $K_{1,4}$ length of longest path

A graph $G = (V, E)$ is called bipartite if $V = V_1 \cup V_2$ with $V_1 \cap V_2 = \phi$ and every edge of G is of the form $\{a, b\}$ with $a \in V_1$ and $b \in V_2$. If each vertex in V_1 is joined with every vertex in V_2, we have complete bipartite graph.

In this case, if $|V_1| = m$, $|V_2| = n$, the graph is denoted by $K_{m,n}$.

For an undirected path $G = (V, E)$ if no vertex of the $x-y$ walk occurs more than once then the walk is called a $x-y$ path.

In any complete bipartite graph $K_{m,n}$, where $m \leq n$ and $|V_1| = m$, $|V_2| = n$, each vertex in V_1 is used only once in the longest path (because no vertex can be repeated in the path). Since each vertex in V_1 must be joined with every vertex of V_2. A vertex in V_1 contribute a length of 2 in the longest path.

Therefore, length of the longest path in $K_{m,n}$ with $m \leq n$ is $2 \times$ (total no. of vertices in m) that is $2m$.

We have

$m = 1$

Hence, for $K_{1,4}$ length of longest path is $\boxed{2}$

(b) Explanation same as above

length $= 2m$

$m = 3$

Hence for $K_{3,7}$ length of longest path is

$\boxed{6}$.

(c) $K_{7,12}$

length $= 2m$.

$m = 7$

$K_{7,12}$ length of longest path is $\boxed{14}$

(d) $K_{m,n}$ $m, n \in \mathbb{Z}^+$.

Explanation as in (c)

length of longest path is $\boxed{2m}$.

Problem - 7

(a) Two complete bipartite graphs on n-vertices are non-isomorphic if and only if their partite set are of different sizes.

For $|V| = 6$ we need to find the no of ways in which 6 can be written as a sum of two non-negative numbers in which the order of summands doesn't matter.

Thus, with $|V| = 6$ we can have 3 complete bipartite graphs with different bipartite and these are :-
(1, 5), (2, 4), (3, 3)

So there are 3 non-isomorphic complete bipartite graphs with $|V| = 6$ and these are $K_{1,5}$, $K_{2,4}$ and $K_{3,3}$.

(6) Two complete bipartite graphs on n-vertices are non-isomorphic if and only if their bipartite sets are of different sizes and to find the bipartite set for n-vertices, we need to find the number of ways in which n can be written as a sum of two non-negative numbers in which the order of summands does not matter.

Now, if n is even, then there are $\frac{n}{2}$ complete bipartite graphs that are non-isomorphic. For example, for, $n=6$, there are 3 $(K_{1,5}, K_{3,3}, K_{2,4})$ complete bipartite nonisomorphic graphs.

For $n=$ odd, there are $\frac{n-1}{2}$ graphs that are

non-isomorphic (in odd graphs there is no case where both partitions have $\frac{n}{2}$ vertices).

Thus, there are $\left\lfloor \frac{n}{2} \right\rfloor$ graphs (rounding down) $G=(V,E)$

for any $n \geq 2$ that satisfy $|V|=n$.

Therefore, the total no. of nonisomorphic complete bipartite graphs on n vertices is $\left\lfloor \frac{n}{2} \right\rfloor$ with

$|V|=n$

Problem - 8

(a) A graph G is called planar if G can be drawn in the plane with its edges intersecting only at vertices of G.

Let $G = (V, E)$ with $|V| = 11$. Then, $\bar{G} = (V, E_1)$ where $\{a, b\} \in E_1$ if $\{a, b\} \notin E$

Let $e = |E|$, $e_1 = |E_1|$

If $G = (V, E)$ is a loop free connected planar graph with $|V| = v$, $|E| = e > 2$, and r region, then

$3r \leq 2e$ and $e \leq 3v - 6$

If both G and $\bar{G}$ are planar then

$$
\begin{aligned}
e &\leq 3 \cdot |V| - 6 \\
&= 3 \cdot 11 - 6 \\
&= 27
\end{aligned}
$$

and

$$
\begin{aligned}
e_1 &\leq 3|V| - 6 \\
&= 3 \times 11 - 6 \\
&= 27.
\end{aligned}
$$

But with $|V| = 11$, there are

$$\binom{11}{2} = \frac{11!}{2!\,9!} = 55$$

edges in complete graph K_{11}, so $|E| + |E_1| = 55$

This means that either $e \geq 28$ or $e_1 \geq 28$

Hence, one of G or $\bar{G}$ must be non-planar.

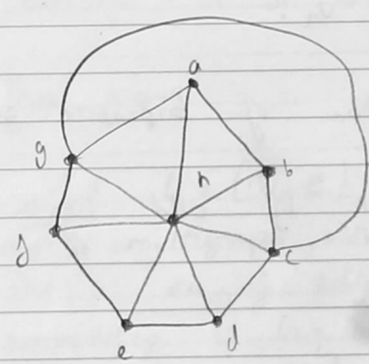
(b) We need to give an example of a graph G where G and $\bar{G}$ both are planar, if $G = (V, E)$ be a loop-free connected graph with $|V| \geq 8$.

A graph G (or a multigraph) G is called planar, if G can be drawn in the plane with its edges intersecting only at vertices of G.

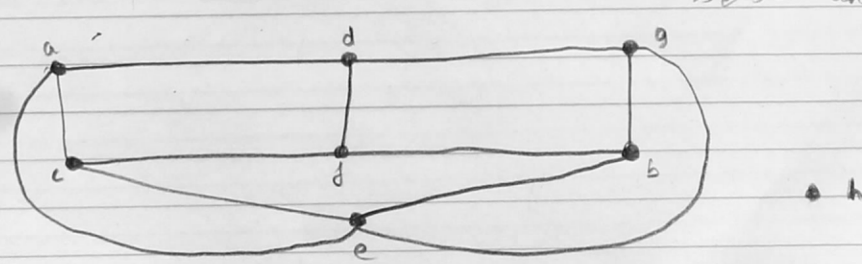
Let G be a loop free undirected graph on n vertices

The complement of G, denoted $\bar{G}$ is the subgraph of K_n consisting of the n-vertices in G and all the edges that are not in G.

The graph G with $|V| = 8$ is shown below and G is planar.



The complement $\bar{G}$ of G is shown below and it is also planar.



Problem-9

A graph G is called bipartite if $V = V_1 \cup V_2$ with $V_1 \cap V_2 = \phi$, and every edge of G is of the form $\{a, b\}$ with $a \in V_1$ and $b \in V_2$.

For any x-y walk in the graph $G = (V, E)$ if no vertex occurs more than once then the walk is called a path and when $x = y$, it is called a cycle.

If G has a cycle, then it is of the form

$a \to b \to c \to a$ (a cycle of length 3). In this cycle, there is an edge $\{a, a\}$, where $a, a \in V_1$ or $a, a \in V_2$

But in a bipartite graph every edge must be of the form $\{a, b\}$ with $a \in V_1$ and $b \in V_2$.

Thus, this contradicts the definition of bipartite graph.

Hence, there cannot be a cycle of odd length in a bipartite graph.

Problem - 10

We need to find $|E|$ for the connected loop-free planar graph $G = (V, E)$ if G is isomorphic to its dual and $|V| = n$.

The dual graph of a given planar graph G is a graph that has a vertex corresponding to each plane region of G, and an edge joining two neighbour regions for each edge in G, for a certain embedding of G.

If G^d is a dual graph of G then, by above definition the number of vertices in G^d is r where r is the number of regions in a planar depiction of G.

Since G is isomorphic to G^d then, the no. of ~~edges~~ vertices in G is equal to the no. of vertices in G^d.

Thus $r = n$

Let $G = (V, E)$ be a connected ~~graph~~ planar graph or ~~multig~~ multigraph with $|V| = v$ and $|E| = e$. Let r be the no. of edges in the plane determined by a planar embedding of G; one of these regions has infinite area and is called the infinite region. Then $v - e + r = 2$

On substituting $v = n$, $r = n$ and $e = |E|$ in the equation $v - e + r = 2$, we get

$n - |E| + n = 2$

$|E| = 2n - 2$

Therefore, $|E| = 2n - 2$ for the connected loop-free planar graph $G = (V, E)$ if G is isomorphic to its dual and $|V| = n$.

Problem - 11

For any graph G, if $G = (V, E)$ is an undirected graph. Then the graph is G connected if there is path between any two distinct vertices of G and the following holds:

$$\sum_{v \in V} \deg(v) = 2|E| \quad -(1)$$

Same as $\deg(v) > 2$

Substitute $|E| = 17$ and $\deg(v) \geq 3$ in the above equation and obtain the following:

$$\sum_{v \in V} \deg(v) = 2 \times 17 = 34$$

Given that $\deg(v) \geq 3$

Use this in above step and proceed as follows:-

$$34 = \sum_{v \in V} \deg(v) \geq 3|V|$$

$$|V| \leq \frac{34}{3}$$

$$|V| \leq 11.33$$

$$|V| = 11$$

Therefore the maximum value of $|V|$ is $\boxed{11}$

Problem - 12

A graph $G = (V, E)$ is called bipartite if $V = V_1 \cup V_2$ with $V_1 \cap V_2 = \phi$ and every edge of G is of the form $\{a, b\}$ with $a \in V_1$, $b \in V_2$.

Let $G = (V, E)$ be bipartite with set of vertices V, partitioned as $V_1 \cup V_2$, so that each edge in the set of edges E, is of the form $\{a, b\}$ where $a \in V_1$ and $b \in V_2$.

Let H be a subgraph of G and W denote the set of vertices of H.

Then we have

$$W = W \cap V.$$

Substituting ~~$V = (V_1 \cap$~~ $V = (V_1 \cup V_2)$ we get

$$W = W \cap (V_1 \cup V_2)$$

$$= (W \cap V_1) \cup (W \cap V_2)$$

We have

$$(W \cap V_1) \cap (W \cap V_2) = \phi$$

Since H is a subgraph of G and if $\{x, y\}$ is an edge in H then $\{x, y\}$ is an edge in G where, $x \in V_1$, $y \in V_2$ then $x \in W_1$, $y \in W_2$.

Thus the subgraph H also satisfies the condition of a bipartite graph.

Therefore H is a bipartite graph.

Hence, a sub graph of bipartite graph is bipartite.

Problem - 13

A graph is planar only if it can be drawn on a plane and also its edges are such that they will intersect only at the vertices of the graph.

For a graph which is planar as well as loop free and connected where $|V| = v$, $|E| = e$

Suppose the no. of regions is r then

$$3r \leq 2e, \quad e \leq 3v - 6$$

Also by theorem, using the same notation for variables, there will be an infinite area in any one of the region which can be given by the formula:-

$$v - e + r = 2$$

Since every region must have minimum of five edges, the expression can be written as below:-

$$2|E| \geq 5r$$

Therefore using

$$2|E| \geq 5(53)$$

$$|E| \geq \frac{1}{2}(5)(53)$$

Using the theorem result from above

$|v| = |E| - 53 + 2$

$|v| = |E| - 51$

$|v| \geq \frac{1}{2}(5)(53) - 51$

Simplifying the expression above:

$|V| \geq \frac{265}{2} - 51$

$|v| \geq \frac{163}{2}$

$|V| \geq 82$

Thus, the required result is proved.

Problem - 14

(a) (i) & (ii) are combined

Let $G = (V, E)$ be a loop free undirected graph. G is called self complementary if G and $\bar{G}$ are isomorphic. If G is self complementary we need to determine $|E|$ if $|V| = n$, also prove G is connected.

Let G be a loop free undirected graph on n-vertices. The complement of G, denoted $\bar{G}$, is the subgraph of K_n consisting of the n-vertices in G and all the edges that are not in G.

If both G and $\bar{G}$ are isomorphic then G is called a self complementary graph.

An n-vertex self complementary graph has exactly half number of edges of the complete graph and the no. of edges in a complete graph K_n is $\binom{n}{2}$

Thus the no. of edges in a self complementary graph (G) on n vertices is $\frac{1}{2}\binom{n}{2}$

For any undirected graph G if G is not connected then $\bar{G}$, complement of G is connected. Since G is self complementary, so

$$G \cong \bar{G}$$

Thus, G is connected.

Therefore no. of edges in G, the self complementary graph G are $\boxed{|E| = \frac{1}{2}\binom{n}{2}}$ for $|V| = n$ and G is connected.

(b)

Let $n \in \mathbb{Z}^+$, where $n = 4k$ ($k \in \mathbb{Z}^+$) or $n = 4k+1$ ($k \in \mathbb{N}$); we need to prove that there exists a self complementary graph $G = (V, E)$, where $|V| = n$.

A graph G is called self complementary if G and $\bar{G}$ are isomorphic.

For $n=1$, we have K_1.

For $n=4$, the graph with 4 vertices which is path on 4 vertices is an example of a self complementary graph.

And the graph which is a cycle on 5 vertices is an example of self complementary graph with $n=5$.

Now suppose there is a graph $G = (V, E)$.

We construct another graph $G_1 = (V_1, E_1)$ where $V_1 = V \cup \{a, b, c, d\}$ (none of a, b, c, d is in V)

and $E_1 = E \cup \{\{a,b\}, \{b,c\}, \{c,d\}\} \cup \{\{v,a\} \mid v \in V\}$
$\cup \{\{v,d\} \mid v \in V\}$

Then G_1 is self complementary and
$|V_1| = |V| + 4$

Thus there exists a self complementary graph $G = (V, E)$, where $|V| = n$ and $n = 4k$ or $n = 4k+1$.