Assignment 3

Submit your assignments to the ENCS's Electronic Assignment Submission (EAS) system: https://fis.encs.concordia.ca/eas

Problem 1

a) Let G be an undirected graph with n vertices. If G is isomorphic to its own complement $\overline{G}$, how many edges must G have? (Such a graph is called *self-complementary*.)

b) Find an example of a self-complementary graph on four vertices and one on five vertices.

c) If G is a self-complementary graph on n vertices, where $n > 1$, prove that $n = 4k$ or $n = 4k + 1$, for some $k \in \mathbf{Z}^+$.

Problem 2

a) Find the number of edges in Q_8.

b) Find the maximum distance between pairs of vertices in Q_8. Give an example of one such pair that achieves this distance.

c) Find the length of a longest path in Q_8.

Problem 3

For $n \in \mathbf{Z}^+$, how many distinct (though isomorphic) paths of length 2 are there in the n-dimensional hypercube Q_n?

Problem 4

Prove that for each $n \in \mathbf{Z}^+$ there exists a loop-free connected undirected graph $G = (V, E)$, where $|V| = 2n$ and which has two vertices of degree i for every $1 \leq i \leq n$.

Problem 5

Let k be a fixed positive integer and let $G = (V, E)$ be a loop-free undirected graph, where $\deg(v) \geq k$ for all $v \in V$. Prove that G contains a path of length k.

Problem 6

What is the length of a longest path in each of the following graphs?

a) $K_{1,4}$ **b)** $K_{3,7}$ **c)** $K_{7,12}$

d) $K_{m,n}$, where $m, n \in \mathbf{Z}^+$ with $m < n$.

Problem 7

a) Find all the nonisomorphic complete bipartite graphs $G = (V, E)$, where $|V| = 6$.

b) How many nonisomorphic complete bipartite graphs $G = (V, E)$ satisfy $|V| = n \geq 2$?

Problem 8

a) Let $G = (V, E)$ be a loop-free connected graph with $|V| \geq 11$. Prove that either G or its complement $\overline{G}$ must be nonplanar.

b) The result in part (a) is actually true for $|V| \geq 9$, but the proof for $|V| = 9, 10$, is much harder. Find a counterexample to part (a) for $|V| = 8$.

Problem 9

Can a bipartite graph contain a cycle of odd length? Explain.

Problem 10

Let $G = (V, E)$ be a loop-free connected planar graph. If G is isomorphic to its dual and $|V| = n$, what is $|E|$?

Problem 11

If $G = (V,E)$ is a connected graph with $|E| = 17$ and $deg(v) > 2$ for *all* vertices of graph G, what is the maximum value for $|V|$.

Problem 12

Prove that any subgraph of a bipartite graph is bipartite.

Problem 13

Let $G = (V,E)$ be an undirected connected loop-free planar graph. Suppose G determines 53 regions. If, for some planar embedding of G, each region has at least five edges in its boundary, prove that $|V| > 81$.

Problem 14

(a) If graph G is self-complementary (see Problem 1) (i) determine $|E|$ if $|V| = n$; (ii) Prove that G is connected. b) Let $n = 4k$ or $n = 4k + 1$ for non- negative number k. Prove that there exist a self-complementary graph $G = (V,E)$, where $|V| = n$.