Comp 339

Assignment-1

Ex-1.1 & 1.2

Q-10

So there are 15 different books $b_1, b_2, b_3, \dots b_{15}$ that can be arranged in $\boxed{15!}$ ways in different order.

Order

1 2 3 ... 15

↓ ↓ ↓ ↓

15 books 14 books 13 books 1 book

As it says now we need to arrange these books in two shelves such that there is at least one book on each shelf.

$b_1, b_2, b_3, \dots b_{15}$

Lets consider the divider between these shelves to separate books.

can't X because at least one book

$\_\, b_1 \,\_\, b_2 \,\_\, b_3 \,\_\, \dots \,\_\, b_{14} \,\_\, b_{15} \,\_$

X can't because at least one book on each shelf.

14 choices

So for each 14 choices ~~to~~ to place the divider there at 15! ways to arrange the books.

$\boxed{\text{So the total possible ways} = 14 \times 15!}$

Q-38

Committee of 15 people 9 women & 6 men seating. Ways to find them to seat in circular table such that no two men sit together.

So for circular arrangement of n objects in a round ~~table~~ table there are $\frac{n!}{n}$ ways

So for 9 women there $\frac{9!}{9} = 8!$ ways

Keeping six empty alternating seats.

Each ~~o~~ such arrangement provides nine spaces between women where a man can be placed. We can select six of these places and situate a man in each of them in $\binom{9}{6} 6! = 9 \times 8 \times 7 \times 6 \times 5 \times 4$ ways

This arrangement is for each women arrangement so product of rule.

$$\boxed{(8!)\binom{9}{6} 6! = 2438553600}$$

Ex1.3

Q-8

(a) So we want five cards of same suit.

So we have one option from 4 suit to chose $\binom{4}{1}$

And for each suit we have $\binom{13}{5}$ ways to chose

5 cards from 13 cards.

So total possible ways $= \binom{4}{1}\binom{13}{5}$

(b) Five cards in which 4 aces and 1 any card

So we have only 4 aces so there are $\binom{4}{4}$ ways

and for this case the last card can be chosen in

$\binom{48}{1}$ ways because 48 cards left and 1 card to

chose.

So total possible ways $= \binom{4}{4}\binom{48}{1}$

(c) Four card of same kind means same number or Kings, etc.

So there are 13 different values so $\binom{13}{1}$ ways

and for this four of a kind means $\binom{4}{4}$ and for each of this last card can be chosen in $\binom{48}{1}$ ways.

So total possible ways $= \binom{13}{1}\binom{4}{4}\binom{48}{1}$

(d) Three aces & 2 Jacks

Three aces can be chosen in $\binom{4}{3}$ ways and for each of this option we have $\binom{4}{2}$ ways to chose jacks (2 jacks from the jacks).
So total possible ways $= \binom{4}{3}\binom{4}{2}$

(e) Three aces and pair

Aces can be chosen in $\binom{4}{3}$ ways. For each of this case we can chose a pair from 12 cards so $\binom{12}{1}$ to chose a kind and to choose a pair $\binom{4}{2}$ ways

So total possible ways $= \binom{4}{3}\binom{12}{1}\binom{4}{2}$

(f) 3 of a kind & pair

To chose 1 card $\binom{13}{1}$ ways and to chose 3 of a same kind $\binom{4}{3}$ ways. For each of this case to chose pair card there is $\binom{12}{1}$ ways to chose card and a pair of that is $\binom{4}{2}$ ways.

So total possible ways $= \binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2}$

(g) 3 of a kind
To chose 1 card $\binom{13}{1}$ ways and of same kind $\binom{4}{3}$ ways. After this card we can select other card cards from 12 in $\binom{12}{2}$ ways. Each card has two pair (4 cards), then two different cards can be selected in $\binom{4}{1}\binom{4}{1}$ ways.

So total possible ways $= \binom{13}{1}\binom{4}{3}\binom{12}{2}\binom{4}{1}\binom{4}{1}$

(h) Two pairs

2 Different card from one suit can be chose in $\binom{13}{2}$ ways and two different pair can be chosen in $\binom{4}{2}\binom{4}{2}$ ways.

For each of this case the last card can be chosen in $\binom{44}{1}$ ways because can't chose the same card as other four cards.

So total possible ways $= \binom{13}{2}\binom{4}{2}\binom{4}{2}\binom{44}{1}$

Q-18

(a) No of strings of length 10 (4 0's, 3 1's, 3 2's)

So of kind 0000111222

So there 10! ways to arrange it differently

but there is repetition of 4 0's, 3 1's, 3 2's

So total possible ways $= \dfrac{10!}{4!\,3!\,3!}$

(b) At least 8 1's so there are 3 cases

8 1's $= \dfrac{10!}{8!} \times 2^2$ → (2 space left with choice 0 or 2)

9 1's $= \dfrac{10!}{9!} \times 2$ → (1 space left with choice 0 or 2)

10 1's $= \dfrac{10!}{10!}$

Adding all these cases because they are independent of each other

$$\text{Total ways} = \left(\frac{10!}{8!} \times 2^2\right) + \left(\frac{10!}{9!} \times 2\right) + \left(\frac{10!}{10!}\right)$$

ⓒ Weight ~~four~~ 4

So there are 3 cases here too

four 1's & six 0's $= \binom{10}{4} \rightarrow$ (1111000000) All arrangements

two 1's, one 2's & seven 0's $= \binom{10}{2}\binom{8}{1} \rightarrow$ (1120000000)

two 2's, eight 0's $= \binom{10}{2} \rightarrow$ (2200000000)

So adding all these cases as they are independent of each other

$$\text{Total possible ways} = \binom{10}{4} + \binom{10}{2}\binom{8}{1} + \binom{10}{2}$$

Q-26

For positive integers n, t, the coefficient of $x_1^{n_1} x_2^{n_2} x_3^{n_3} \dots x_t^{n_t}$ in the expansion of $(x_1 + x_2 + x_3 + \dots + x_t)^n$ is

$$\frac{n!}{n_1!\, n_2! \dots n_t!}$$

$w^2 x^2 y^2 z^2$ in all (a), (b), (c)

(a) $(w+x+y+z+1)^{10}$

So the coefficient of our term is

$$\binom{10}{2,2,2,2,2} = \frac{10!}{2!\,2!\,2!\,2!\,2!}$$

this is for one 1

$(1) = 10-2-2-2-2$

$= 2$

$= 113400$

(b) $(2w-x+3y+z-2)^{12}$

$w^2 x^2 y^2 z^2$ coefficient.

so degree of (-2) is $= 12-2-2-2-2$

$= 4$

$$\cancel{\binom{12}{2,2,2,2,4} = \frac{12!}{2!\,2!\,2!\,2!\,4!}} =$$

$$\binom{12}{2,2,2,2,4}(2w)^2(-x)^2(3y)^2(z)^2(-2)^4$$

$$= \binom{12}{2,2,2,2,4} 2^2 (-1)^2 (3)^2 (1)^2 (-2)^4 \quad w^2x^2y^2z^2$$

$$= 1247400 \times 2^2(-1)^2(3)^2(1)^2(-2)^4 \times w^2x^2y^2z^2$$

$$= 718502400 \; w^2x^2y^2z^2$$

ⓒ $(v+w-2x+y+5z+3)^{12}$

$w^2x^2y^2z^2$ so there is no term of v
so degree of $v=0$

degree of 3 = $12-2-2-2-2-0$
$=4$

$$\binom{12}{0,2,2,2,2,4}(v)^0(w)^2(-2x)^2(y)^2(5z)^2(3)^4$$

$$=\binom{12}{0,2,2,2,2,4}1^2(-2)^2(1)^2(5)^2(3)^4\quad w^2x^2y^2z^2$$

$= 10103940000 \quad w^2x^2y^2z^2$

Q-28

ⓐ $$\sum_{i=0}^{n}\frac{1}{i!(n-i)!}$$

$$I=\sum_{i=0}^{n}\frac{1}{i!(n-i)!}$$

$$=\frac{1}{n!}\sum_{i=0}^{n}\frac{n!}{i!(n-i)!}$$ (multiply & divide by $n!$)

$$=\frac{1}{n!}\sum_{i=0}^{n}\binom{n}{i}$$ $\left(\frac{n!}{r!(n-r)!}=\binom{n}{r}\right)$

$$= \frac{1}{n!}\left[\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n}\right]$$

But as per textbook

By (Corollary 1.1(a)) $\binom{n}{0}+\binom{n}{1}+\cdots+\binom{n}{n} = 2^n$

follows the binomial theorem when set $x = y = 1$

Therefore

$$= \frac{1}{n!} \times 2^n$$

$$= \frac{2^n}{n!}$$

$$\therefore \sum_{i=0}^{n} \frac{1}{i!\,(n-i)!} = \frac{2^n}{n!}$$

(b) $$I = \sum_{i=0}^{n} \frac{(-1)^i}{i!\,(n-i)!}$$

$$= \frac{1}{n!} \sum_{i=0}^{n} \frac{(-1)^i\,(n!)}{i!\,(n-i)!}$$ (multiply & divide by $n!$)

$$= \frac{1}{n!} \sum_{i=0}^{n} (-1)^i \binom{n}{i}$$ $\left(\frac{n!}{r!\,(n-r)!} = \binom{n}{r}\right)$

$$= \frac{1}{n!}\left[\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \dots + (-1)^n\binom{n}{n}\right]$$

But as per textbook

By [Corollary 1.1 (b)]

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \dots + (-1)^n\binom{n}{n} = 0$$

follow Binomial theorem when $x = -1$, $y = 1$

$$= \frac{1}{n!}(0)$$

$$= 0$$

$$\therefore \sum_{i=0}^{n} \frac{(-1)^i}{i!\,(n-i)!} = 0$$

<u>Ex 1.4</u> <u>Q-12</u>

$$x_1 + x_2 + x_3 + x_4 + x_5 < 40$$

$x_i \geq 0, \quad 1 \leq i \leq 5$

Transform the eq^n to

$x_1+x_2+x_3+x_4+x_5+x_6 = 40$ with $x_i \geq 0, 1 \leq i \leq 5,$
$x_6 > 0$

As $x_6 > 0$ So take $y_6 = x_6 - 1$ and $y_i = x_i$
for $1 \leq i \leq 5$

$\therefore y_1+y_2+y_3+y_4+y_5+y_6 = 39$

So the solution to the above eq^n is

$$= \binom{39+6-1}{39}$$

$$= \binom{44}{39} = 1086008$$

So there are 1086008 sol^ns for given equation
$x_i \geq 0, 1 \leq i \leq 5$

(b) $x_1 + x_2 + x_3 + x_4 + x_5 < 40$

$x_i \geq -3, \quad 1 \leq i \leq 5$

Let take $y_i = x_i + 3$

$y_1 - 3 + y_2 - 3 + y_3 - 3 + y_4 - 3 + y_5 - 3 < 40$

$y_1 + y_2 + y_3 + y_4 + y_5 < 55$

Transform the equation to

$y_1 + y_2 + y_3 + y_4 + y_5 + y_6 = 55$ where $y_i \geq 0$, $1 \leq i \leq 5$, $y_6 \geq 0$

let $z_6 = y_6 - 1$ and $z_i = y_i$ $1 \leq i \leq 5$

$z_1 + z_2 + z_3 + z_4 + z_5 + z_6 = 54$

So the solⁿ is

$$= \binom{54 + 6 - 1}{54}$$

$$= \binom{59}{54}$$

$$= 5006386$$

Therefore there are 5006386 solⁿs for given equation when $x_i \geq -3$, $1 \leq i \leq 5$

Q-16

For what n positive integer

$$x_1+x_2+x_3+\dots+x_{19}=n \quad (1)$$

$$y_1+y_2+y_3+\dots+y_{64}=n \quad (2)$$

$$(1) = (2)$$

will have same no of positive integer solutions

No of sol^ns for (1)

means $\boxed{x_i > 0}$ for both equations

$\therefore x_1+x_2+x_3+\dots+x_{19}=n-19$

has

$$\cancel{(n-16)+} \binom{(n-19)+19-1}{n-19}$$

sol^ns

implies both

No of sol^ns for (2)

$$y_1+y_2+y_3+\dots+y_{64}=n-64$$

has $\binom{(n-64)+64-1}{(n-64)}$ sol^ns

$$\therefore \binom{(n-19)+19-1}{(n-19)} = \binom{(n-64)+64-1}{(n-64)}$$

$$\therefore \binom{n-1}{n-19} = \binom{n-1}{n-64}$$

$$\binom{n-1}{(n-1)-(n-19)} = \binom{n-1}{n-64}$$

$$\binom{n}{r} = \binom{n}{n-r}$$

$$\binom{n-1}{18} = \binom{n-1}{n-64}$$

$$\therefore n-64 = 18$$

$$\therefore \boxed{n = 82}$$

Thus for $n = 82$ both eq^n has no. of positive integer sol^{n}s

Q-18

$x_1 + x_2 + x_3 = 6$ (1)

$\underbrace{x_1 + x_2 + x_3}_{} + x_4 + x_5 + x_6 + x_7 = 37$

$6 + x_4 + x_5 + x_6 + x_7 = 37$

$x_4 + x_5 + x_6 + x_7 = 31$ — (2)

No of sol^n for (1)

$\binom{6+3-1}{6} = \binom{8}{6}$

For each of this sol^n (2) eq^n has

$\binom{31+4-1}{31}$ sol^ns

$= \binom{34}{31}$

Total no of sol^ns $= \binom{8}{6}\binom{34}{31}$ sol^ns

(b) $x_1, x_2, x_3 > 0$

$\therefore \quad x_1 + x_2 + x_3 = 6 \qquad x_i \geq 0, \; 1 \leq i \leq 3$

$x_1 + x_2 + x_3 = 3 \qquad x_i > 0, \; 1 \leq i \leq 3$

So no of sol^n $\binom{3+3-1}{3} = \binom{5}{3} = \binom{5}{3}$

For each of this sol^n eq^n (2) has

$\binom{34}{31}$ sol^n

$\therefore$ Total no. of sol^n $= \binom{5}{3}\binom{34}{31}$

when $x_1, x_2, x_3 > 0$

Ex 4.2

Q-10

For all $x \in \mathbb{R}$

$$|x| = \sqrt{x^2} = \begin{cases} x & \text{if } x > 0 \\ -x & \text{if } x < 0 \end{cases}$$

and $-|x| \le x \le |x|$

We need to prove that for $n \in \mathbb{Z}^+$, $n \geq 2$ and $x_1, x_2, \ldots x_n \in \mathbb{R}$

Then
$|x_1 + x_2 + \ldots + x_n| \le |x_1| + |x_2| + \ldots + |x_n|$

Let $S(n) : |x_1 + x_2 + \ldots + x_n| \le |x_1| + |x_2| + \ldots + |x_n|$

For $n = 2$

$S(2) : |x_1 + x_2| \le |x_1| + |x_2|$

$$|x_1 + x_2|^2 = x_1^2 + 2x_1x_2 + x_2^2$$
$$\le x_1^2 + 2|x_1||x_2| + x_2^2$$
$$= |x_1|^2 + 2|x_1||x_2| + |x_2|^2$$
$$= (|x_1| + |x_2|)^2$$

There $|x_1 + x_2|^2 \le (|x_1| + |x_2|)^2$

Thus $|x_1 + x_2| \le |x_1| + |x_2|$ for all $x, y \in \mathbb{R}$

Hence $S(n)$ is true for $n = 2$

We assume that truth of $S(k)$

that is $|x_1+x_2+\dots+x_k| \le |x_1|+|x_2|+\dots+|x_k|$

Now lets prove for $S(k+1)$

$$|x_1+x_2+\dots+x_{k+1}| \le |x_1+x_2+\dots+x_k| + |x_{k+1}|$$

$$\le |x_1|+|x_2|+\dots+|x_k| + |x_{k+1}|$$

$\therefore$ $S(k+1)$ is true

$$|x_1+x_2+\dots+x_n| \le |x_1|+|x_2|+\dots+|x_n| \quad \forall n \ge 2$$

Q-12

Proof by Mathematical induction

We find that $F_0 = \sum_{i=0}^{0} F_i = 0 = 1-1 = F_2 - 1$

So the given statement holds in this first case and thus provides base step of the proof.

For inductive step

We assume the truth of the statement when $n = k$ (≥ 0) that is

$$\sum_{i=0}^{k} F_i = F_{k+2} - 1$$

Now we consider what happens $n = k+1$

$$\sum_{i=0}^{k+1} F_i = \left(\sum_{i=0}^{k} F_i\right) + F_{k+1}$$

$$= (F_{k+2} - 1) + F_{k+1}$$

$$= (F_{k+2} + F_{k+1}) - 1$$

$$= F_{k+3} - 1$$

So the truth of statement at $n = k$ Implies the truth at $n = k+1$.

Consequently $\sum_{i=0}^{n} F_i = F_{n+2} - 1$ for all $n \in \mathbb{N}$ -

by the Principle of Mathematical Induction.

Q-18

(a) No of permutation of 1, 2, 3 where k ascents = 0, 1, 2

k = 0 one permutation 0 ascent = 321

k = 1 four permutation = 132, 213, 231, 312

1 < 3   1 < 3   2 < 3   1 < 2

k = 2 one permutation = 123

(1 < 2 & 2 < 3)

(b)

	Permutations	
k = 0	1	4321
k = 1	11	1432, 2143, 2431, 3142, 3214, 3241, 3421, 4132, 4213, 4231, 4312
k = 2	11	1243, 1324, 1342, 1423, 2134, 2314, 2341, 2413, 3124, 3412, 4123
k = 3	1	1234

(c) 1, 2, 3, 4, 5, 6, 7 that has four ascents

If a permutation of 1, 2, 3 ... m has k ascents for $0 \le k \le m-1$

the no of descents in that permutations is $(m-1)-k$.

So no of descents 1, 2, 3 ..., 7 have four ascents

$$= ~~(m-1)~~ = (m-1)-k$$
$$= (7-1)-4$$
$$= 6-4$$
$$= 2$$

Hence there are 2 descents

(d) If permutation of 1, 2, 3 ... m has k ascents for $0 \le k \le m-1$ then number of descent in that permutation is $(m-1)-k$

For example. Consider the permutation 312 of 1, 2, 3
If a permutation has 1 ascent ($k=1$) & ($m=3$)
no of descent

$$(m-1)-k = 3-1-1$$
$$= 1$$

which is true.
So if no of ascents is k for $0 \le k \le m-1$
then the no of descents is $(m-1)-k$

(e) $p = 12436587$

(i) four ascents

There are 5 locations

In front of 1 $p = 912436587$

four ascents = 12, 24, 36, 56

Between 1, 2 $p = 192436587$

4 ascents = 19, 24, 36, 58

Between 2, 4 $p = 129436587$

4 ascents = 12, 29, 36, 58

Between 3, 6 $p = 124396587$

4 ascents = 12, 24, 39, 58

Between 5, 8 $p = 124365987$

4 ascents = 12, 24, 36, 59

(ii) 5 ascents

There are 4 locations

Between 4, 3 $p = 124936587$

5 ascents = 12, 24, 49, 36, 58

Between 6, 5 $p = 124369587$

5 ascents = 12, 24, 36, 69, 58

Between 8, 7 $\quad p = 124365897$

5 ascents = 12, 24, 36, 58, 89

After 7 $\quad p = 124365 \ 879$

5 ascents = 12, 24, 36, 58, 79

(f) We need to determine how $\pi_{m,k}$ is related to $\pi_{m-1,k-1}$ and $\pi_{m-1,k}$ if $\pi_{m,k}$ denote the number of permutations of $1,2,3,\ldots,m$ with k ascents

$$\pi_{4,2} = 11 = 2(4) + 3(4) = (4-2)\pi_{3,2} + (2+1)\pi_{3,2}$$

The permutation 1432 of 1,2,3,4 has one ascent namely 14 ~~since~~ (since $1<4$). This same permutation also has two descents namely 43 (since $4>3$) and 32 (since $3>2$). The permutation 1423 on the other hand has two ascents at 14 and 23 and one descent 42.

Let $x = x_1, x_2, \ldots x_m$ denote a permutation $1,2,3,\ldots,m$ with k ascents (and $m-k-1$) descents.

There are two case to consider

Case 1 :- If $m = x_m$ or if m occurs in $x_i m x_{i+2}$, $1 \le i \le m-2$, with $x_i > x_{i+2}$ then the total ~~[illegible]~~ removal of m results in permutation of $1, 2, 3, \dots, m-1$ with $k-1$ ascent for a total of

$$[1+(m-k-1)]\,\pi_{m-1,k-1} = (m-k)\,\pi_{m-1,k-1} \quad \text{permutations}$$

Case 2 :- If $m = x_1$ or if m occurs in $x_i m x_{i+2}$, $1 \le i \le m-2$ with $x_i < x_{i+2}$ then the removal of m results in a permutation of $1, 2, 3, \dots, m-1$ with k ascents for a total of $(k+1)\,\pi_{m-1,k}$ permutation.

Since two cases have nothing in common and are mutually exclusive and account for all possibilities, the recursive formula for $\pi_{m,k}$ is:

$$\pi_{m,k} = (k+1)\,\pi_{m-1,k} + (m-1)\,\pi_{m-1,k-1}$$

Therefore $\pi_{m,k}$ is related to $\pi_{m-1,k-1}$ & $\pi_{m-1,k}$ as

$$\boxed{\pi_{m,k} = (k+1)\,\pi_{m-1,k} + (m-1)\,\pi_{m-1,k-1}}$$

— x — x — x — x —