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From Solvability to Symmetry; Using group theory to solve differential equations

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Contents

Abstract

There are mainly two parts in this study. First, solvability, where we question ourselves about the qualitative aspects of the differential equations such as existence and uniqueness of solutions of the equations. In which, we also try to show a uniqueness theorem for a general first order differential equation following the proof in the similar proceedings of the well known Picard-Lindelof theorem. Second, symmetry, in which we exploit the symmetries of solutions of differential equations and with which we try to solve them by building much more theory that includes One- parameter Lie Groups, canonical coordinates etc. and we end by stating basic definitions and results of Lie algebras in this context.

Chapter 1

Existence and Uniqueness theorems

1.1 Review of basic definitions and theorems

Definition 1 (Continuity)**.** Let, $J \subset \mathbb{R}$, $f : J \to \mathbb{R}$ fix $a \in J$. f is **continuous** at a of for any given $\epsilon > 0$, $\exists\ \delta > 0$ such that

$$x \in J \text{ and } |x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon.$$

Definition 2 (Uniform Continuity)**.** Let, $J \subset \mathbb{R}$, $f : J \to \mathbb{R}$. f is **uniform continuous** on J if for each $\epsilon > 0$, $\exists$ $\delta > 0$ such that

$$x_1, x_2 \in J \text{ with } |x_1 - x_2| < \delta \Rightarrow |f(x_1) - f(x_2)| < \epsilon.$$

Definition 3 (Lipschitz)**.** Let, $J \subset \mathbb{R}$, $f : J \to \mathbb{R}$. f is **Lipschitz** on J if $\exists\ L > 0$ such that

$$|f(x) - f(y)| \leq L|x - y|\, \forall\, x, y\, \in J.$$

Definition 4 ($\alpha-$Holder continuity)**.** Let, $J \subset \mathbb{R}$, $f : J \to \mathbb{R}$. f satisfies $\alpha-$**Holder condition** (or is Holder continuous) on J if $\exists\ C > 0$ such that

$$|f(x) - f(y)| \leq C|x - y|^{\alpha}\, \forall\, x, y\, \in J$$

Definition 5 (Absolutely continuous)**.** Let, $J \subset \mathbb{R}$, $f : J \to \mathbb{R}$. f is **absolutely continuous** on J if for each $\epsilon > 0$, $\exists\ \delta > 0$ such that whenever a finite sequence of pairwise disjoint sub-intervels $(x_k, y_k) \subset J$ satisfies

$$\sum_k (y_k - x_k) < \delta, \text{ then } \sum_k |f(y_k) - f(x_k)| < \epsilon.$$

Definition 6 (Bounded variation)**.** Let f be defined on $[a, b]$. If $\mathcal{P} = \{x_0, x_1, \cdots, x_n\}$ is a partition of $[a, b]$, further let $\Delta f_k \coloneqq f(x_k) - f(x_{k-1})$ for $k = 1, 2, \cdots, n$. If there exists a real $M > 0$ such that

$$\sum_{k=1}^{n} |\Delta f(x_k)| \leq M$$

for all partitions of $[a, b]$ then f is said to be of **bounded variation** on $[a, b]$

Definition 7 (Equicontinuous)**.** Let $(X, d), (Y, d)$ be two metric spaces and $a \in X$. A family of functions $\mathcal{A}$ from $X \to Y$ is said to be **equicontinuous** at a if for any $\epsilon > 0$, $\exists\ \delta > 0$ such that

$$d(x, a) < \delta \Rightarrow d(f(x), f(a)) < \epsilon\ \forall\ f \in \mathcal{A}$$

Remark. We say that $\mathcal{A}$ is **uniformly equicontinuous** on X if for given any $\epsilon > 0$, $\exists\ \delta > 0$ such that

$$d(x_1, x_2) < \delta \Rightarrow d(f(x_1), f(x_2)) < \epsilon\ \forall\ x_1, x_2 \in X, f \in \mathcal{A}$$

Definition 8 (Differentiablity)**.** Let, $J \subset \mathbb{R}$, $f : J \to \mathbb{R}$ fix $a \in J$. f is **differentiable** at a if $\exists$ a real number $\alpha > 0$ such that,

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \alpha$$

Theorem 1 ((First) Fundamental Theorem of Calculus)**.** Let f be a continuous real-valued function defined on a closed interval $[a, b]$. Let F be the function defined, for all x in $[a, b]$, by $F(x) = \int_a^x f(t)dt$. Then, F is uniformly continuous on $[a, b]$ and differentiable on the open interval (a, b), and $F'(x) = f(x)$ for all x in (a, b).

Theorem 2 (Arzela-Ascoli)**.** Let X be a compact metric space. Let $C(X, \mathbb{K})$ be given the sup norm metric. ($\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$) If a sequence $\{f_n\}_{n \in N}$ in $C(X, \mathbb{K})$ is uniformly bounded and uniformly equicontinuous then it has a uniformly convergent subsequence.

1.2 Examples and properties

- Any Lipschitz function is continuous. *(For any given $\epsilon > 0$, choose $\delta = \frac{\epsilon}{L}$, where L is the Lipschitz constant)*
- f is Lipschitz $\iff$ $S \coloneqq \{s : s = \frac{f(x)-f(y)}{x-y}, x, y \in J\}$ is bounded v.i.z., f is Lipschitz iff the set of slooes of all possible chords on the graph of f is bounded in $\mathbb{R}$.
- Let, $\phi \neq A \subset \mathbb{R}$ then, $d_A(x) \coloneqq glb\{|x - a| : a \in A\}$ is Lipschitz on $\mathbb{R}$.

  Proof. Let, $x, y \in \mathbb{R}$. For any $a \in A$, we have $|x - a| \leq |x - y| + |y - a|$. Let, $M = \{|x - a| : a \in A\}$, $N = |x - y| + |y - a| : a \in A \Rightarrow glb(M) \leq glb(N) \Rightarrow d_A(x) \leq |x - y| + d_A(y) \Rightarrow d_A(x) - d_A(y) \leq |x - y|$ $\square$

  - For $A = \{0\}$, $d_A(x) = |x|$ is Lipschitz on $\mathbb{R}$
- $f : J \to \mathbb{R}$ be differentiable with bounded derivative then f is Lipschitz.
  - $f(x) = \sin(x)$, $g(x) = \cos(x)$, $h(x) = \arctan(x)$ are Lipschitz on $\mathbb{R}$.
- Any Lipschitz function is uniform continuous.
  - $f(x) = \sqrt{x}, x \in [0, 1]$ is uniformly continuous but not Lipschitz.
  - $f(x) = x^2, g(x) = \sin(x^2)$, $x \in \mathbb{R}$ are not lipscitz continuous.
- If $\alpha = 1$, then the $\alpha-$Holder condition is equivalent to Lipschitz condition. For any $\alpha > 0$, the $\alpha-$Holder condition implies the function is uniformly continuous.
- Over a closed, bounded interval on $\mathbb{R}$,
  - continuously differentiable $\subset$ Lipschitz continuous $\subset$ $\alpha-$Holder continuous
  - lipchitz continuous $\subset$ absolutely continuous $\subset$ uniformly continuous
  - continuously differentiable $\subset$ Lipschitz continuous $\subset$ absolutely continuous $\subset$ bounded variation $\subset$ differentiable almost everywhere

1.3 Inequalities

Theorem 3 (Gronwall's Inequality)**.** Let, K be a non-negative constant and let f, g be continuous, non-negative functions on some interval $a \leq t \leq b$ satisfying the inequality

$$f(t) \leq K + \int_a^t f(s)g(s)ds \text{ for } a \leq t \leq b.$$

Then,

$$f(t) \leq K \exp\left(\int_a^t g(s)ds\right) \quad a \leq t \leq b.$$

Proof. Let, $h(t) = K + \int_a^t f(s)g(s)ds$. So, $h(a) = K$ and $f(t) \le h(t)$, $a \le t \le b$. Since, $f \cdot g \in C[a,b]$ from the fundamental theorem of calculus, h is uniformly continuous on $a \le t \le b$ and $h'(t) = f(t)g(t)$, $a < t < b$, and from $g \ge 0$ and $f(t) \le h(t)$ we have,

$$h'(t) = f(t)g(t) \le h(t)g(t)$$

$$\exp\left(-\int_a^t g(s)ds\right) h'(t) \le \exp\left(-\int_a^t g(s)ds\right) h(t)g(t)$$

$$\exp\left(-\int_a^t g(s)ds\right) h'(t) - h(t)\exp\left(-\int_a^t g(s)ds\right) g(t) \le 0$$

$$\frac{d}{dt}\left[\exp\left(-\int_a^t g(s)ds\right) h(t)\right] \le 0,\ a < t < b.$$

Hence, $\exp\left(-\int_a^t g(s)ds\right) h(t)$ is bounded above by $h(a) = K \Rightarrow \exp\left(-\int_a^t g(s)ds\right) h(t) \le K \Rightarrow h(t) \le K \exp\left(\int_a^t g(s)ds\right)$. Since, $f(t) \le h(t)$ theorem follows. □

1.3.1 Maximal-Minimal solutions

Definition 9. Fix $R = \{(t,y) : t_0 \le t \le t_0 + a, |y - y_0| \le b\}$. A solution $r(t)\,[\rho(t)]$ of 1.1 which exists on an interval (say) J is **maximal [minimal]** if for an arbitrary solution $y(t)$ of 1.1 in J we have

$$y(t) \le r(t)\ [\rho(t) \le y(t)]\ \forall\, t \in J$$

Lemma 4. If $f(t,y) \in C(R)$, then there exists maximal and minimal solutions for 1.1 in $[t_0, t_0 + \alpha]$ for some $\alpha > 0$

1.3.2 Dini derivatives

Definition 10. Suppose $y(t)$ is continuous over some $J \subset \mathbb{R}$, then

- $D^+y(t) = \limsup_{h\to 0^+} \frac{y(t+h) - y(t)}{h}$
- $D_+y(t) = \liminf_{h\to 0^+} \frac{y(t+h) - y(t)}{h}$
- $D^-y(t) = \limsup_{h\to 0^-} \frac{y(t+h) - y(t)}{h}$
- $D_-y(t) = \liminf_{h\to 0^-} \frac{y(t+h) - y(t)}{h}$

Remark. *If $D^+y(t) = D_+y(t)$, then we say right derivative of $y(t)$ exists and is denoted by $y'_+(t)$.*

Lemma 5. Let $f(t,y)$ be continuous on some domain D. Let $r(t)$ be maximal solution of 1.1 over $J_1 = [t_0, t_0 + a)$ and let $\rho(t)$ be minimal solution of 1.1 over $J_2 = (t_0 - a, t_0]$, further

- if $y(t)$ satisfies $D^+y(t) \le f(t, y(t))$ then

$$y(t_0) \le y_0 \Rightarrow y(t) \le r(t),\ \forall\, t \in J_1$$

- if $y(t)$ satisfies $D_-y(t) \le f(t, y(t))$ then

$$y_0 \le y(t_0) \Rightarrow \rho(t) \le y(t),\ \forall\, t \in J_2$$

1.4 Existence and Uniqueness theorems

Consider a system of d first order differential equations and an initial condition.

$$y' = f(t,y), \qquad y(t_0) = y_0, \tag{1.1}$$

where, $y : \mathbb{R} \to \mathbb{R}^d$, $y(t) = (y^1(t), y^2(t), \cdots, y^d(t))$, and each $y^i : \mathbb{R} \to \mathbb{R}$. $y'(t) = (y^{1'}(t), y^{2'}(t), \cdots, y^{d'}(t))$. For, $E \subset \mathbb{R} \times \mathbb{R}^d$ we have, $f : E \to \mathbb{R}^d$

In this case, $y = y(t)$ defined on a t-interval J containing $t = t_0$ is called *solution* of the initial value problem (1.1) if $y(t_0) = y_0$, $(t, y(t)) \in E$, $y(t)$ is differentiable, and $y'(t) = f(t, y(t))$ for $t \in J$. For the most part, we assume f to be continuous, so it is clear that $y(t)$ has a continuous derivative.

Now, consider the following *integral equation*.

$$y(t) = y_0 + \int_{t_0}^{t} f(s, y(s))ds, \tag{1.2}$$

for $t \in J$.

Theorem 6. The solutions of the initial value problem (1.1) – if any exist – are precisely the continuous solutions of the integral equation (1.2).

Proof.

- Suppose $y(t)$ is a solution of (1.1). Existence of the derivative y' guarentees the continuity of y, so $y(t)$ is continuous and right side of
$$y'(t) = f(t, y(t))$$
is a continuous function of t and on integrating from t_0 to t and using the initial condition of $y(t_0) = y_0$, we get (1.2). Hence, any solution of (1.1) is continuous solution of (1.2)

- Suppose $y(t)$ is a continuous solution of (1.2, then at $t = t_0$, since the integral vanishes, we have $y(t_0) = y_0$. Moreover, since, f is continuous, from the second funcdamental theorem of calculus/generalised stokes' theorem, the RHS of (1.2) is differentiable w.r.t. t and the derivative is equal to $f(t, y(t))$. Hence, $y'(t) = f(t, y(t))$

□

Remark. One can automatically obtain a solution for (1.1) if we can construct a continuous solution for (1.2)

Given a initial value problem, we consider the following questions,

1. (**local existence**) does (1.1) have a solution $y(t)$ defined for t near t_0 ?
2. on what $t-$ ranges does a solution of (1.1) exist? We consider the following scalar case of IVP, where $f(t, y)$ is defined for all (t, y),
$$y' = y^2, \qquad y(0) = c(> 0).$$
We see that, $y = \frac{c}{1-ct}$ is solution for the above, but this solution exists only on the range $-\infty < t < \frac{1}{c}$ which depends on the initial condition.
3. uniqueness of solutions

Theorem 7 (Picard-Lindelof)**.** Let, $f, y \in \mathbb{R}^d$, f is continuous on parallelepiped $R = \{t, y : |t - t_0| \leq a, |y - y_0| \leq b\}$ and uniformly Lipschitz continuous with respect to y. Let, $|f(t, y)| \leq M$ on R and $\alpha := \min(a, b/M)$ then, (1.1) has a unique solution $y = y(t)$ on $|t - t_0| \leq \alpha$.

Remark. The definition of α is natural since, on one hand for t to be in R, $\alpha \leq a$ is necessary. On other hand, if $y = y(t)$ is a solution of (1.1), then $|y'(t)| \leq M \;\Rightarrow\; |y(t) - y_0| \leq M|(t - t_0)|$ which should(and doesn't) exceed b if $|t - t_0| \leq b/M$.

Proof. We prove by method of succesive approximations. Let,

$$y_0(t) = y_0 \qquad\qquad y_n(t) = y_0 + \int_{t_0}^{t} f(s, y_{n-1}(s))ds$$

We now show that the sequence $\{y_n\}$ of successive approximations converges(*uniformly*) on the interval $I := \{t : |t - t_0| \leq \alpha\}$ to a solution $y = y(t)$ of (1.1). From the remark we have that all y_n are defined on the interval I. We now observe that

$$y_j(t) = y_0(t) + \sum_{m=1}^{j-1} [y_{m+1}(t) - y_m(t)]$$

We shall estimate $r_j(t) := |y_{j+1}(t) - y_j(t)|$, $j \geq 0$ in two cases.

Case 1. for $t_0 \leq t \leq t_0 + \alpha$, we have

$$\begin{aligned} r_j(t) := |y_{j+1}(t) - y_j(t)| &= \left| \int_{t_0}^{t} [f(s, y_j(s))ds - f(s, y_{j-1}(s))]ds \right| \\ &\leq \int_{t_0}^{t} |f(s, y_j(s))ds - f(s, y_{j-1}(s))|\, ds \\ &\leq K \int_{t_0}^{t} |y_j(s) - y_{j-1}(s)|\, ds \quad \ldots \text{(Lipschitz in } y\text{)} \end{aligned}$$

$\Rightarrow r_j(t) \leq K \int_{t_0}^{t} r_{j-1}(s)ds$ for $j \geq 1$. For, j=0, we have,

$$r_0(t) = |y_1(t) - y_0(t)| = \left| \int_{t_0}^{t} f(s, y_0(s))ds \right| \leq \int_{t_0}^{t} |f(s, y_0(s))|\, ds \leq M(t - t_0)$$

From induction we shall prove that

$$r_j(t) \leq \frac{MK^j (t - t_0)^{j+1}}{(j+1)!},\ j \geq 0,\ t_0 \leq t \leq t_0 + \alpha$$

Suppose it is true for some $j = p - 1$, then

$$r_p(t) \leq K \int_{t_0}^{t} r_{p-1}(s) \leq \int_{t_0}^{t} \frac{MK^p (s - t_0)^{p-1}}{p!} = \frac{MK^p}{p!} \int_{t_0}^{t} (s - t_0)^p ds = \frac{MK^p}{(p+1)!} \left[(s - t_0)^{p+1}\right]_{t_0}^{t}$$

the hypothesis follows.

Case 2. for $t_0 - \alpha \leq t \leq t_0$, since (t, t_0) is the interval now, we have

$$r_0(t) := \left| \int_{t_0}^{t} f(s, y_0(s))ds \right| = \left| \int_{t}^{t_0} f(s, y_0(s))ds \right| \leq \int_{t}^{t_0} |f(s, y_0(s))|\, ds \leq M(t_0 - t)$$

and for the same reason we have $r_j(t) \leq K \int_{t}^{t_0} r_{j-1}(s)ds$ for $j \geq 1$. Similarly arguing as above, from induction we get

$$r_j(t) \leq \frac{MK^j (t_0 - t)^{j+1}}{(j+1)!},\ j \geq 0,\ t_0 - \alpha \leq t \leq t_0$$

from the two cases we have that,

$$r_j(t) \leq \frac{MK^j |t - t_0|^{j+1}}{(j+1)!} = \frac{M|K(t - t_0)|^{j+1}}{K(j+1)!} \leq \frac{M(K\alpha)^{j+1}}{K(j+1)!},\ \ j \geq 0,\ t \in I \tag{1.3}$$

Set, $W_j := \dfrac{M(K\alpha)^{j+1}}{K(j+1)!}$ and we now observe that

- $|y_{j+1}(t) - y_j(t)| \leq W_j$ for $j \geq 0$, $t \in I$
- $\sum_{j=0}^{\infty} W_j = \sum_{j=0}^{\infty} \frac{M(K\alpha)^{j+1}}{K(j+1)!} = \frac{M}{K} \sum_{j=0}^{\infty} \frac{(K\alpha)^{j+1}}{(j+1)!} = \frac{M}{K}(e^{K\alpha} - 1)$ converges.

hence, from the Weierstass M-test the series $\sum_{j=0}^{\infty}[y_{j+1}(t)-y_j(t)]$ converges (absolutely) and uniformly over I.

Since, $y_j(t) = y_0(t) + \sum_{m=1}^{j-1}[y_{m+1}(t)-y_m(t)]$, we have the convergence of the sequence $\{y_j(t)\}$ for every $t \in I$ to some function of t say $y(t)$. We shall now show that $y(t)$ is continuous and satisfies (1.2).

So, for $t \in I$, we have

$$y(t) = y_0(t) + \sum_{m=1}^{\infty}[y_{m+1}(t)-y_m(t)] \Rightarrow y(t)-y_j(t) = \sum_{m=j}^{\infty}[y_{m+1}(t)-y_m(t)]$$

and, from (1.3)

$$\begin{aligned}|y(t)-y_j(t)| = \left|\sum_{m=j}^{\infty}[y_{m+1}(t)-y_m(t)]\right| \leq \sum_{m=j}^{\infty}|y_{m+1}(t)-y_m(t)| = \sum_{m=j}^{\infty} r_m(t) &\leq \frac{M}{K}\sum_{m=j}^{\infty}\frac{(K\alpha)^{m+1}}{(m+1)!} \\ &= \frac{M}{K}\frac{(K\alpha)^{j+1}}{(j+1)!}\sum_{m=0}^{\infty}\frac{(K\alpha)^m}{m!}\end{aligned}$$

$\Rightarrow\ |y(t)-y_j(t)| \leq \dfrac{M}{K}\dfrac{(K\alpha)^{j+1}}{(j+1)!}e^{K\alpha}$, $t \in I$.

Set, $\epsilon_j = \dfrac{(K\alpha)^{j+1}}{(j+1)!}$. We fix an integer n such that, $n \geq K\alpha$. This gives, $n(n+1)\cdots(j-1) \geq (K\alpha)^{j-n}$, $\forall\ j \geq n$. Further,

$$0 \leq \frac{(K\alpha)^j}{j!} \leq \frac{(K\alpha)^n n(n+1)\cdots(j-1)}{j!} = \frac{(K\alpha)^n}{(n-1)!j}$$

As $j \to \infty$, $\frac{(K\alpha)^n}{(n-1)!j} \to 0$. Hence by squeeze theorem, $\epsilon_j \to 0$ as $j \to \infty$.

Now,

$$\begin{aligned}|y(t+h)-y(t)| &= |y(t+h)-y_j(t+h)+y_j(t+h)-y_j(t)+y_j(t)-y(t)| \\ &\leq |y(t+h)-y_j(t+h)| + |y_j(t+h)-y_j(t)| + |y_j(t)-y(t)| \\ &\leq 2\epsilon_j + |y_j(t+h)-y_j(t)| \quad \ldots\text{(as estimated before)}\end{aligned}$$

On choosing sufficiently large j and small h and continuity of y_j with $\lim_{j\to 0}\epsilon_j = 0$, one can make $|y(t+h)-y(t)| < \epsilon$, hence the continuity of $y(t)$ follows.

We now show that $y(t)$ satisfies (1.2). Since, $y_j(s) \to y(s)$ (uniformly) and f is continuous over compact space and as composition of continuous mapping on compact space preserves uniform convergence, we have that $f(s, y_j(s)) \to f(s, y(s))$ (uniformly). Hence, we have

$$\lim_{j\to\infty}\int_{t_0}^{t} f(s, y_j(s)) = \int_{t_0}^{t}\lim_{j\to\infty} f(s, y_j(s)) = \int_{t_0}^{t} f(s, \lim_{j\to\infty} y_j(s)) = \int_{t_0}^{t} f(s, y(s))$$

So, we have constructed a continuous solution for (1.2). Hence, from (**Theorem 6**), the $y(t)$ is a solution of (1.1).

We now show the uniqueness of solution. Let, y_1, y_2 be two solutions of (1.1) over some common interval $J \ni t_0$. They satisfy the integral equation (1.2). So, for $t \in J$

$$y_1(t) = y_0 + \int_{t_0}^{t} f(s, y_1(s))ds \qquad\qquad y_2(t) = y_0 + \int_{t_0}^{t} f(s, y_2(s))ds$$

$$\Rightarrow |y_1(t)-y_2(t)| = \left|\int_{t_0}^{t} [f(s,y_1(s)) - f(s,y_2(s))]\, ds\right| \leq \left|\int_{t_0}^{t} |f(s,y_1(s)) - f(s,y_2(s))| ds\right|$$
$$\leq K\left|\int_{t_0}^{t} |y_1(s)-y_2(s)| ds\right|$$

From (**Theorem 3**), $|y_1(t)-y_2(t)| \leq 0 \Rightarrow |y_1(t)-y_2(t)| = 0 \Rightarrow y_1 \equiv y_2$ over J. The uniqueness of the solutions follows. ☐

Theorem 8. Consider any first order differential equation

$$g(t,y,y') = 0,$$

where, $g : \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ and $f \coloneqq g + y'$ is continuous on $R = \{(t,u,v) : |t-t_0| \leq a, |u-y_0| \leq b, |v| \leq M'\}$ and uniformly Lipschitz continuous w.r.t v and u with Lipschitz constants K_1, K_2 respectively such that $K_1 \in [0,1)$. Let, $|f(t,u,v)| \leq M$ over R and $\alpha \coloneqq \min(a, b/M, (1-K_1)/K_2)$ then,

$$y' = f(t,y,y'),\ y(t_0) = y_0 \tag{1.4}$$

has unique solution $y = y(t)$ over $I \coloneqq \{t : |t-t_0| \leq \alpha\}$.

Remark. *If $y(t)$ is a solution to (1.4) then $|y'(t)| \leq \min(M, M')$ over I.*

Proof. For $y = y(t)$ defined on a t-interval $J \ni t_0$, we consider the following *integral equation*

$$y(t) = y_0 + \int_{t_0}^{t} f(s,y(s),y'(s))ds, \tag{1.5}$$

for $t \in J$. From a slightly modified version of Theorem (6), we can say that the solutions of the initial value probelm (1.4) - if any exist - are precisely the continuous solutions of the integral equation (1.5). We consider the following approximations,

$$y_0(t) \equiv y_0 \qquad\qquad y_n(t) = y_0 + \int_{t_0}^{t} f(s, y_{n-1}(s), y'_{n-1}(s))ds$$

whence, $y'_n(t) = f(t, y_{n-1}(t), y'_{n-1}(t))$. We now show that the sequence $\{y_n\}$ and $\{y'_n\}$ of successive approximations converges(*uniformly*) on the interval $I \coloneqq \{t : |t-t_0| \leq \alpha\}$ to a solution $y = y(t)$ of (1.4) and $y'(t)$ respectively. From the remark we have that all y_n and y'_n are defined on the interval I We now observe that

$$y_j(t) = y_0(t) + \sum_{m=1}^{j-1} [y_{m+1}(t) - y_m(t)], \qquad\qquad y'_j(t) = \sum_{m=0}^{j-1} [y'_{m+1}(t) - y'_m(t)]$$

Let, $a_j(t) \coloneqq |y_{j+1}(t) - y_j(t)|$ and $b_j(t) \coloneqq |y'_{j+1}(t) - y'_j(t)|$, $j \geq 0$. We have,

$$\begin{aligned}
a_j(t) &= \left|\int_{t_0}^{t} [f(s,y_j(s),y'_j(s)) - f(s,y_{j-1}(s),y'_{j-1}(s))]ds\right| \\
&\leq \left|\int_{t_0}^{t} \left|f(s,y_j(s),y'_j(s)) - f(s,y_{j-1}(s),y'_{j-1}(s))\right| ds\right| \\
&= \left|\int_{t_0}^{t} \left|f(s,y_j(s),y'_j(s)) - f(s,y_j(s),y'_{j-1}(s)) + f(s,y_j(s),y'_{j-1}(s)) - f(s,y_{j-1}(s),y'_{j-1}(s))\right| ds\right| \\
&\leq \left|\int_{t_0}^{t} \left|f(s,y_j(s),y'_j(s)) - f(s,y_j(s),y'_{j-1}(s))\right| ds + \int_{t_0}^{t} \left|f(s,y_j(s),y'_{j-1}(s)) - f(s,y_{j-1}(s),y'_{j-1}(s))\right| ds\right|
\end{aligned}$$

$$\leq \left| \int_{t_0}^{t} \left| f(s, y_j(s), y_j'(s)) - f(s, y_j(s), y_{j-1}'(s)) \right| ds \right| + \left| \int_{t_0}^{t} \left| f(s, y_j(s), y_{j-1}'(s)) - f(s, y_{j-1}(s), y_{j-1}'(s)) \right| ds \right|$$

$$\leq \left| K_1 \int_{t_0}^{t} \left| y_j'(s) - y_{j-1}'(s) \right| ds \right| + \left| K_2 \int_{t_0}^{t} \left| y_j(s) - y_{j-1}(s) \right| ds \right| \qquad \ldots\ldots \text{(Lipschitz)}$$

$$= K_1 \left| \int_{t_0}^{t} b_{j-1}(s) ds \right| + K_2 \left| \int_{t_0}^{t} a_{j-1}(s) ds \right|$$

$$\Rightarrow a_j(t) \leq K_1 \left| \int_{t_0}^{t} b_{j-1}(s) ds \right| + K_2 \left| \int_{t_0}^{t} a_{j-1}(s) ds \right| \tag{1.6}$$

Similarly,

$$\begin{aligned} b_j(t) &= \left| f(t, y_j(t), y_j'(t)) - f(t, y_{j-1}(t), y_{j-1}'(t)) \right| \\ &\leq |f(t, y_j(t), y_j'(t)) - f(t, y_j(t), y_{j-1}'(t))| + |f(t, y_j(t), y_{j-1}'(t)) - f(t, y_{j-1}(t), y_{j-1}'(t))| \\ &\leq K_1 |y_j'(t) - y_{j-1}'(t)| + K_2 |y_j(t) - y_{j-1}(t)| \\ &= K_1 b_{j-1}(t) + K_2 a_{j-1}(t) \end{aligned}$$

$$\Rightarrow b_j(t) \leq K_1 b_{j-1}(t) + K_2 a_{j-1}(t) \tag{1.7}$$

From (1.6) and (1.7), we have

$$\begin{aligned} a_j(t) &\leq K_2 \left| \int_{t_0}^{t} a_{j-1}(s) ds \right| + K_1 \left| \int_{t_0}^{t} K_1 b_{j-2}(s) + K_2 a_{j-2}(s) ds \right| \\ &\leq K_2 \left| \int_{t_0}^{t} a_{j-1}(s) ds \right| + K_1^2 \left| \int_{t_0}^{t} b_{j-2}(s) \right| + K_1 K_2 \left| \int_{t_0}^{t} a_{j-2}(s) ds \right| \end{aligned}$$

continuing similarly, we obtain

$$a_j(t) \leq \sum_{i=0}^{j-1} \left[K_2 K_1^{j-i-1} \left| \int_{t_0}^{t} a_i(s) ds \right| \right] + K_1^j \left| \int_{t_0}^{t} b_0(s) ds \right|$$

Since $b_0(t) \coloneqq |y_1'(t) - 0| = |y_1'(t)| \leq M$, we have

$$a_j(t) \leq \sum_{i=0}^{j-1} \left[K_2 K_1^{j-i-1} \left| \int_{t_0}^{t} a_i(s) ds \right| \right] + M K_1^j \, |t - t_0| \tag{1.8}$$

We seek to show that, $a_j(t) \leq M|t - t_0|(K_1 + K_2|t - t_0|)^j$. So, we first show that, by induction, $\forall j \geq 0$,

$$a_j(t) \leq \sum_{i=0}^{j} \frac{M}{K_2} \binom{j}{i} \frac{(K_2|t - t_0|)^{j+1-i} (K_1)^i}{(j+1-i)!} \tag{1.9}$$

- For $j = 0$, $a_0(t) \coloneqq |y_1(t) - y_0| = \left| \int_{t_0}^{t} f(s, y_0, 0) ds \right| \leq M|t - t_0|$ and from above, $a_0(t) \leq \frac{M}{K_2} K_2 |t - t_0| = M|t - t_0|$. So (1.9) is true for $j = 0$.
- We assume that (1.9) is true for all $1 \leq j \leq l$ for some $l \in \mathbb{N}$. So $\forall\ 1 \leq j \leq l$,

$$a_j(t) \leq \sum_{i=0}^{j} \frac{M}{K_2} \binom{j}{i} \frac{(K_2|t - t_0|)^{j+1-i} (K_1)^i}{(j+1-i)!}$$

$$
\begin{aligned}
\Rightarrow \left|\int_{t_0}^{t} a_j(s)ds\right| &\le \left|\int_{t_0}^{t}\sum_{i=0}^{j}\frac{M}{K_2}\binom{j}{i}\frac{(K_2|s-t_0|)^{j+1-i}(K_1)^i}{(j+1-i)!}ds\right| \\
&= \left|\sum_{i=0}^{j}\frac{M}{K_2}\binom{j}{i}\frac{(K_2)^{j+1-i}(K_1)^i}{(j+1-i)!}\int_{t_0}^{t}(|s-t_0|)^{j+i-1}ds\right| \\
&\le \sum_{i=0}^{j}\frac{M}{K_2}\binom{j}{i}\frac{(K_2)^{j+1-i}(K_1)^i}{(j+1-i)!}\left|\int_{t_0}^{t}(|s-t_0|)^{j+i-1}ds\right| \\
&= \begin{cases} \sum_{i=0}^{j}\frac{M}{K_2}\binom{j}{i}\frac{(K_2)^{j+1-i}(K_1)^i}{(j+1-i)!}\left(\int_{t_0}^{t}(s-t_0)^{j+i-1}ds\right), & t_0\le t \\ \sum_{i=0}^{j}\frac{M}{K_2}\binom{j}{i}\frac{(K_2)^{j+1-i}(K_1)^i}{(j+1-i)!}\left(\int_{t_0}^{t}(t_0-s)^{j+i-1}ds\right), & t<t_0 \end{cases} \\
&= \begin{cases} \sum_{i=0}^{j}\frac{M}{K_2}\binom{j}{i}\frac{(K_2)^{j+1-i}(K_1)^i}{(j+1-i)!}\left(\frac{(t-t_0)^{j+2-i}}{j+2-i}\right), & t_0\le t \\ \sum_{i=0}^{j}\frac{M}{K_2}\binom{j}{i}\frac{(K_2)^{j+1-i}(K_1)^i}{(j+1-i)!}\left(\frac{(t_0-t)^{j+2-i}}{j+2-i}\right), & t<t_0 \end{cases} \\
&= \sum_{i=0}^{j}\frac{M}{K_2}\binom{j}{i}\frac{(K_2)^{j+1-i}(K_1)^i}{(j+1-i)!}\frac{|t-t_0|^{j+2-i}}{j+2-i} \\
&= \frac{M}{K_2^2}\sum_{i=0}^{j}\binom{j}{i}\frac{(K_2|t-t_0|)^{j+2-i}(K_1)^i}{(j+2-i)!}.
\end{aligned}
$$

So, for $1\le j\le l$

$$
\left|\int_{t_0}^{t} a_j(s)ds\right| \le \frac{M}{K_2^2}\sum_{i=0}^{j}\binom{j}{i}\frac{(K_2|t-t_0|)^{j+2-i}(K_1)^i}{(j+2-i)!}. \tag{1.10}
$$

Consider $a_{l+1}(t)$, from (1.8)

$$
\begin{aligned}
a_{l+1}(t) &\le \sum_{i=0}^{l}\left[K_2K_1^{l-i}\left|\int_{t_0}^{t}a_i(s)ds\right|\right] + MK_1^{l+1}\,|t-t_0| \\
&\le \sum_{i=0}^{l}\left[K_2K_1^{l-i}\frac{M}{K_2^2}\sum_{r=0}^{i}\binom{i}{r}\frac{(K_2|t-t_0|)^{i+2-r}(K_1)^r}{(i+2-r)!}\right] + MK_1^{l+1}\,|t-t_0| \\
&= \frac{M}{K_2}\sum_{i=0}^{l}\left[\sum_{r=0}^{i}\binom{i}{r}\frac{(K_2|t-t_0|)^{i+2-r}(K_1)^{l-(i-r)}}{(i+2-r)!}\right] + MK_1^{l+1}\,|t-t_0| \\
(set\ m=i-r)\quad &= \frac{M}{K_2}\sum_{i=0}^{l}\left[\sum_{m=0}^{i}\binom{i}{m}\frac{(K_2|t-t_0|)^{m+2}(K_1)^{l-m}}{(m+2)!}\right] + MK_1^{l+1}\,|t-t_0| \\
&= \frac{M}{K_2}\left[\sum_{m=0}^{0}\binom{0}{m}\frac{(K_2|t-t_0|)^{m+2}(K_1)^{l-m}}{(m+2)!} + \sum_{m=0}^{1}\binom{1}{m}\frac{(K_2|t-t_0|)^{m+2}(K_1)^{l-m}}{(m+2)!} + \right. \\
&\qquad \left. \cdots + \sum_{m=0}^{l}\binom{l}{m}\frac{(K_2|t-t_0|)^{m+2}(K_1)^{l-m}}{(m+2)!}\right] + MK_1^{l+1}\,|t-t_0|
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{M}{K_2}\Bigg[\binom{0}{0}\frac{(K_2|t-t_0|)^2(K_1)^l}{2!} + \\
&\qquad \binom{1}{0}\frac{(K_2|t-t_0|)^2(K_1)^l}{2!} + \binom{1}{1}\frac{(K_2|t-t_0|)^3(K_1)^{l-1}}{3!} + \\
&\qquad \binom{2}{0}\frac{(K_2|t-t_0|)^2(K_1)^l}{2!} + \binom{2}{1}\frac{(K_2|t-t_0|)^3(K_1)^{l-1}}{3!} + \binom{2}{2}\frac{(K_2|t-t_0|)^4(K_1)^{l-2}}{4!} \\
&\qquad \qquad \vdots \qquad\qquad\qquad\qquad \vdots \qquad\qquad\qquad\qquad \vdots \\
&\qquad \binom{l}{0}\frac{(K_2|t-t_0|)^2(K_1)^l}{2!} + \binom{l}{1}\frac{(K_2|t-t_0|)^3(K_1)^{l-1}}{3!} + \cdots + \binom{l}{l}\frac{(K_2|t-t_0|)^{l+2}}{(l+2)!}\Bigg] \\
&\quad + MK_1^{l+1}\,|t-t_0|
\end{aligned}
$$

Recalling, $\binom{n}{k} = 0$ if $n < k$, we appeal to the following famous Hockey-stick identity,

$$
\sum_{i=r}^{n}\binom{i}{r} = \binom{n+1}{r+1}, \text{ for } n, r \in \mathbb{N},\, n \geq r
$$

$$
\begin{aligned}
a_{l+1}(t) &\leq \frac{M}{K_2}\Bigg[\binom{l+1}{0}\frac{K_1^{l+1}K_2|t-t_0|}{1!} + \binom{l+1}{1}\frac{(K_1)^l(K_2|t-t_0|)^2}{2!} + \binom{l+1}{2}\frac{(K_1)^{l-1}(K_2|t-t_0|)^3}{3!} + \\
&\qquad \cdots + \binom{l+1}{l+1}\frac{(K_2|t-t_0|)^{l+2}}{(l+2)!}\Bigg] \\
&= \frac{M}{K_2}\Bigg[\binom{l+1}{l+1}\frac{K_1^{l+1}K_2|t-t_0|}{1!} + \binom{l+1}{l}\frac{(K_1)^l(K_2|t-t_0|)^2}{2!} + \binom{l+1}{l-1}\frac{(K_1)^{l-1}(K_2|t-t_0|)^3}{3!} + \\
&\qquad \cdots + \binom{l+1}{0}\frac{(K_2|t-t_0|)^{l+2}}{(l+2)!}\Bigg] \\
&= \frac{M}{K_2}\sum_{i=0}^{l+1}\binom{l+1}{i}\frac{(K_2|t-t_0|)^{l+2-i}K_1^i}{(l+2-i)!}
\end{aligned}
$$

which implies (1.9) is true for $j = l+1$. Hence, from the principle of induction (1.9) is true for all $\mathbb{N}$.

For $j \geq 0$,

$$
\begin{aligned}
a_j(t) \leq \sum_{i=0}^{j}\frac{M}{K_2}\binom{j}{i}\frac{(K_2|t-t_0|)^{j+1-i}(K_1)^i}{(j+1-i)!} &\leq \sum_{i=0}^{j}\frac{M}{K_2}\binom{j}{i}(K_2|t-t_0|)^{j+1-i}(K_1)^i \\
&= M|t-t_0|\sum_{i=0}^{j}\binom{j}{i}(K_2|t-t_0|)^{j-i}(K_1)^i \\
&= M|t-t_0|(K_2|t-t_0| + K_1)^j
\end{aligned}
$$

Hence, over I, we have $\forall j \geq 0$

$$
a_j(t) \leq M\alpha(K_1 + K_2\alpha)^j \tag{1.11}
$$

Set, $W_j := M\alpha(K_1 + K_2\alpha)^j$ and we now observe that

- $|y_{j+1}(t) - y_j(t)| \leq W_j$ for $j \geq 0$, $t \in I$
- As $\alpha = \min(a, b/M, (1-K_1)/K_2)$, we have

$$
0 < \alpha < (1-K_1)/K_2 \Rightarrow K_1 + K_2\alpha \in (0,1)
$$

Hence, $\sum_{j=0}^{\infty} W_j = \sum_{j=0}^{\infty} M\alpha(K_1+K_2\alpha)^j = M\alpha\sum_{j=0}^{\infty}(K_1+K_2\alpha)^j < \infty$ (geometric series).

Now, from the Weierstass M-test the series $\sum_{j=0}^{\infty}[y_{j+1}(t)-y_j(t)]$ converges (absolutely) and uniformly over I.

On other hand, we seek to show that, $b_j(t)\le M(K_1+K_2|t-t_0|)^j$. So, we first show that, by induction, $\forall j\ge 0$,

$$b_j(t)\le M\sum_{i=0}^{j}\binom{j}{i}\frac{K_1^i(K_2|t-t_0|)^{j-i}}{(j-i)!} \tag{1.12}$$

- For $j=0$, $b_0(t)\le M$ and from above $b_0(t)\le M$. So, (1.12) is true for $j=0$.
- We assume that (1.12) is true for some $l\in\mathbb{N}$. So,

$$b_l(t)\le M\sum_{i=0}^{l}\binom{l}{i}\frac{K_1^i(K_2|t-t_0|)^{l-i}}{(l-i)!}$$

From (1.7), we have

$$\begin{aligned}
b_{l+1}(t) &\le K_1 b_l(t)+K_2 a_l(t)\\
(from\ (1.9)) &\le K_1\,M\sum_{i=0}^{l}\binom{l}{i}\frac{K_1^i(K_2|t-t_0|)^{l-i}}{(l-i)!}+K_2\sum_{i=0}^{l}\frac{M}{K_2}\binom{l}{i}\frac{K_1^i(K_2|t-t_0|)^{l+1-i}}{(l+1-i)!}\\
&= M\left[\sum_{i=0}^{l}\binom{l}{i}\frac{K_1^{i+1}(K_2|t-t_0|)^{l-i}}{(l-i)!}+\sum_{i=0}^{l}\binom{l}{i}\frac{K_1^i(K_2|t-t_0|)^{l+1-i}}{(l+1-i)!}\right]\\
&= M\left[\binom{l}{0}\frac{K_1(K_2|t-t_0|)^l}{l!}+\binom{l}{1}\frac{K_1^2(K_2|t-t_0|)^{l-1}}{(l-1)!}+\cdots+\binom{l}{l-1}\frac{K_1^l(K_2|t-t_0|)}{(1)!}+\binom{l}{l}K_1^{l+1}+\right.\\
&\left.\binom{l}{0}\frac{(K_2|t-t_0|)^{l+1}}{(l+1)!}+\binom{l}{1}\frac{K_1(K_2|t-t_0|)^l}{(l)!}+\binom{l}{2}\frac{K_1^2(K_2|t-t_0|)^{l-1}}{(l-1)!}+\cdots+\binom{l}{l}\frac{K_1^l(K_2|t-t_0|)}{(1)!}\right]\\
&= M\left[\binom{l}{0}\frac{(K_2|t-t_0|)^{l+1}}{(l+1)!}+\left[\binom{l}{0}+\binom{l}{1}\right]\frac{K_1(K_2|t-t_0|)^l}{(l)!}+\left[\binom{l}{1}+\binom{l}{2}\right]\frac{K_1^2(K_2|t-t_0|)^{l-1}}{(l-1)!}\right.\\
&\left.\quad+\cdots+\left[\binom{l}{l-1}+\binom{l}{l}\right]\frac{K_1^l(K_2|t-t_0|)}{(1)!}+\binom{l}{l}K_1^{l+1}\right]
\end{aligned}$$

We now appeal to the Pascal's identity,

$$\binom{n}{k-1}+\binom{n}{k}=\binom{n+1}{k}\quad \text{for}\, n,k\in\mathbb{Z}_{\ge 0},\ \text{with}\ 1\le k\le n$$

$$\begin{aligned}
b_{l+1}(t) &\le M\left[\binom{l+1}{0}\frac{(K_2|t-t_0|)^{l+1}}{(l+1)!}+\binom{l+1}{1}\frac{K_1(K_2|t-t_0|)^l}{(l)!}+\right.\\
&\left.\quad\cdots+\binom{l+1}{l}\frac{K_1^l(K_2|t-t_0|)}{(1)!}+\binom{l+1}{l+1}K_1^{l+1}\right]\\
&=\sum_{i=0}^{l+1}\binom{l+1}{i}\frac{K_1^i(K_2|t-t_0|)^{l+1-i}}{(l+1-i)!}
\end{aligned}$$

which implies (1.12) is true for $j=l+1$. Hence, from the principle of induction (1.12) is true for all $\mathbb{N}$.

For $j\ge 0$,

$$\begin{aligned}
b_j(t)\le M\sum_{i=0}^{j}\binom{j}{i}\frac{(K_2|t-t_0|)^{j-i}(K_1)^i}{(j-i)!} &\le M\sum_{i=0}^{j}\binom{j}{i}(K_2|t-t_0|)^{j+1-i}(K_1)^i\\
&= M(K_2|t-t_0|+K_1)^j
\end{aligned}$$

Hence, over I, we have $\forall j \geq 0$

$$b_j(t) \leq M(K_1 + K_2\alpha)^j \tag{1.13}$$

Set, $W_j' \coloneqq M(K_1 + K_2\alpha)^j$ and we now observe that

- $|y_{j+1}'(t) - y_j'(t)| \leq W_j'$ for $j \geq 0$, $t \in I$
- As $\alpha = \min(a, b/M, (1 - K_1)/K_2)$, we have

$$0 < \alpha < (1 - K_1)/K_2 \Rightarrow K_1 + K_2\alpha \in (0, 1)$$

  Hence, $\sum\limits_{j=0}^{\infty} W_j' = \sum\limits_{j=0}^{\infty} M(K_1 + K_2\alpha)^j = M \sum\limits_{j=0}^{\infty} (K_1 + K_2\alpha)^j < \infty$ (geometric series).

Now, from the Weierstass M-test the series $\sum\limits_{j=0}^{\infty} \left[y_{j+1}'(t) - y_j'(t)\right]$ converges (absolutely) and uniformly over I.

Since, $y_j(t) = y_0(t) + \sum\limits_{m=1}^{j-1} [y_{m+1}(t) - y_m(t)]$ and $y_j'(t) = \sum\limits_{m=1}^{j-1} [y_{m+1}'(t) - y_m'(t)]$, we have the convergence of the sequences $\{y_j(t)\}$ and $\{y_j'(t)\}$ for every $t \in I$ to some function of t say $y(t)$ and $y'(t)$ respectively. We shall now show that $y(t)$ is continuous and satisfies (1.5). So, for $t \in I$, we have

$$y(t) = y_0(t) + \sum_{m=1}^{\infty} [y_{m+1}(t) - y_m(t)] \;\Rightarrow\; y(t) - y_j(t) = \sum_{m=j}^{\infty} [y_{m+1}(t) - y_m(t)]$$

and, from (1.11)

$$\begin{aligned}
|y(t) - y_j(t)| = \left| \sum_{m=j}^{\infty} [y_{m+1}(t) - y_m(t)] \right| \leq \sum_{m=j}^{\infty} |y_{m+1}(t) - y_m(t)| = \sum_{m=j}^{\infty} a_m(t) &\leq M\alpha \sum_{m=j}^{\infty} (K_1 + K_2\alpha)^m \\
&= M\alpha (K_1 + K_2\alpha)^j \sum_{m=0}^{\infty} (K_1 + K_2\alpha)^m
\end{aligned}$$

$\Rightarrow\; |y(t) - y_j(t)| \leq \dfrac{M\alpha}{1 - K_1 - K_2\alpha} (K_1 + K_2\alpha)^j \; t \in I.$

Set, $\epsilon_j = \dfrac{M\alpha}{1 - K_1 - K_2\alpha} (K_1 + K_2\alpha)^j$. Since $K_1 + K_2\alpha \in (0, 1)$, as $j \to \infty$, $\epsilon_j \to 0$.

Now,

$$\begin{aligned}
|y(t + h) - y(t)| &= |y(t + h) - y_j(t + h) + y_j(t + h) - y_j(t) + y_j(t) - y(t)| \\
&\leq |y(t + h) - y_j(t + h)| + |y_j(t + h) - y_j(t)| + |y_j(t) - y(t)| \\
&\leq 2\epsilon_j + |y_j(t + h) - y_j(t)| \quad \ldots \text{(as estimated before)}
\end{aligned}$$

On choosing sufficiently large j and small h and continuity of y_j with $\lim_{j \to 0} \epsilon_j = 0$, we can make $|y(t+h) - y(t)| < \epsilon$, hence the continuity of $y(t)$ follows.

We now show that $y(t)$ satisfies (1.5). Since, $y_j(s) \to y(s)$ (uniformly) and f is continuous over compact space and as composition of continuous mapping on compact space preserves uniform convergence, we have that

$$f(s, y_j(s), y_j'(s)) \to f(s, y(s), y'(s)) \text{ (uniformly)}.$$

Hence, we have

$$
\begin{aligned}
y(t) = \lim_{j\to\infty} y_j(t) = \lim_{j\to\infty}\left[y_0 + \int_{t_0}^{t} f(s, y_{j-1}(s), y'_{j-1}(s))\right] &= y_0 + \lim_{j\to\infty}\int_{t_0}^{t} f(s, y_{j-1}(s), y'_{j-1}(s)) \\
(\textit{uniform convergence}) &= y_0 + \int_{t_0}^{t} \lim_{j\to\infty} f(s, y_{j-1}(s), y'_{j-1}(s)) \\
(\textit{continuity}) &= y_0 + \int_{t_0}^{t} f(s, \lim_{j\to\infty} y_{j-1}(s), \lim_{j\to\infty} y'_{j-1}(s)) \\
&= y_0 + \int_{t_0}^{t} f(s, y(s), y'(s))
\end{aligned}
$$

Hence, we have constructed a continuous solution for (1.5). From Lemma (6), the $y(t)$ is a solution of (1.4).

We now show the uniqueness of solution. Let, y_1, y_2 be two solutions of (1.4) over some common interval $J \ni t_0$. They satisfy the integral equation (1.5). So, for $t \in J$

$$
y_1(t) = y_0 + \int_{t_0}^{t} f(s, y_1(s))ds \qquad\qquad y_2(t) = y_0 + \int_{t_0}^{t} f(s, y_2(s))ds
$$

$$
\begin{aligned}
\Rightarrow |y_1(t) - y_2(t)| &= \left|\int_{t_0}^{t} \left[f(s, y_1(s), y_1'(s)) - f(s, y_2(s), y_2'(s))\right] ds\right| \\
&= \left|\int_{t_0}^{t} \left[f(s, y_1(s), y_1'(s)) - f(s, y_1(s), y_2'(s)) + f(s, y_1(s), y_2'(s)) - f(s, y_2(s), y_2'(s))\right] ds\right| \\
&\le \left|\int_{t_0}^{t} |f(s, y_1(s), y_1'(s)) - f(s, y_1(s), y_2'(s))|ds\right| + \left|\int_{t_0}^{t} |f(s, y_1(s), y_2'(s)) - f(s, y_2(s), y_2'(s))|ds\right| \\
&\le K_1 \left|\int_{t_0}^{t} |y_1'(s) - y_2'(s)|ds\right| + K_2 \left|\int_{t_0}^{t} |y_1(s) - y_2(s)|ds\right| \qquad (1.14)
\end{aligned}
$$

We have,

$$
\begin{aligned}
|y_1'(s) - y_2'(s)| &= |f(t, y_1(s), y_1'(s)) - f(s, y_2(s), y_2'(s))| \\
&\le K_1|y_1'(s) - y_2'(s)| + K_2|y_1(s) - y_2(s)| \\
\Rightarrow (1 - K_1)|y_1'(s) - y_2'(s)| &\le K_2|y_1(s) - y_2(s)| \\
\Rightarrow |y_1'(s) - y_2'(s)| &\le \frac{K_2}{1 - K_1}|y_1(s) - y_2(s)| \qquad (1.15)
\end{aligned}
$$

On substituting (1.15) in (1.14), we get

$$
\begin{aligned}
|y_1(t) - y_2(t)| &\le K_1 \left|\int_{t_0}^{t} \frac{K_2}{1 - K_1}|y_1(s) - y_2(s)|ds\right| + K_2 \left|\int_{t_0}^{t} |y_1(s) - y_2(s)|ds\right| \\
&= \frac{K_2}{1 - K_1}\left|\int_{t_0}^{t} |y_1(s) - y_2(s)|ds\right|
\end{aligned}
$$

From Lemma (3), $|y_1(t) - y_2(t)| \le 0 \Rightarrow |y_1(t) - y_2(t)| = 0 \Rightarrow y_1 \equiv y_2$ over J. The uniqueness of the solutions follows. □

Remark. *Here, $K_1 < 1$ corresponds to a* ***contraction map****. The considered function f is a contraction with respect to y'.*

Example 1. Consider the function $f(t, y, y') = \dfrac{t\sin(y)\sin(y')}{4}$ over a cuboid $R = \{(t, u, v) : |t| \le 1, |u| \le 2, |v| \le \frac{1}{4}\}$. Clearly f is continuous over R. Hence, it is bounded and $|f(t, y, y')| = \left|\dfrac{t\sin(y)\sin(y')}{4}\right| \le \dfrac{t}{4} \le \dfrac{1}{4}$ over R. Since, $|y'| \le \dfrac{1}{4}$, the equality $y' = f(t, y, y')$ is well-defined. We consider the following initial value problem

$$y' = \frac{t\sin(y)\sin(y')}{4}, \qquad y(0) = 0. \tag{1.16}$$

Here, $a = 1, b = 2, M = 1/4$. Over R, we have

$$\begin{aligned}
|f(t, y, y_1') - f(t, y, y_2')| = \left|\frac{t\sin(y)\sin(y_1')}{4} - \frac{t\sin(y)\sin(y_1')}{4}\right| &= \left|\frac{t\sin(y)[\sin(y_1') - \sin(y_2')]}{4}\right| \\
&\le \frac{|t||\sin(y)||\sin(y_1') - \sin(y_2')|}{4} \\
&\le \frac{|\sin(y_1') - \sin(y_2')|}{4} \\
\textit{(Lagrange mean value theorem)} &= \frac{|\cos(y_3')||y_1' - y_2'|}{4}, \text{ for some } y_3' \in (y_1', y_2') \\
&\le \frac{1}{4}|y_1' - y_2'|
\end{aligned}$$

Hence, $K_1 = 1/4$. Similarly, for the same reasons as above we have $K_2 = 1/4$ which implies $\alpha := \min(a, b/M, (1 - K_1)/K2) = \min(1, 2/(1/4), (3/4)/(1/4)) = \min(1, 8, 3) = 1$. From the Theorem (1.4), we can say that (1.16) has an unique solution over the set $\{t : |t| \le 1\}$.

Example 2. Consider the function $f(t, y, y') = \frac{3}{4}(t^2 + yt)\arctan((y')^2)$ over $R = \{(t, u, v) : |t| \le 1/2, |u| \le 1/4, |v| \le 1\}$. Here, $a = 1/2, b = 1/4$. Over R,

- f is continuous and

$$\begin{aligned}
|f| = \frac{3}{4}|t^2 + yt||\arctan((y')^2)| &\le \frac{3}{4}|t^2 + yt|\arctan(1) \text{ ......(for justification, see Figure (1.1)))} \\
&\le \frac{3}{4}\frac{\pi}{4}(|t|^2 + |y||t|) \\
&\le \frac{3\pi}{16}\left(\frac{1}{4} + \frac{1}{4}\cdot\frac{1}{2}\right) = \frac{9\pi}{128} \Rightarrow \boxed{M = \frac{9\pi}{128}}
\end{aligned}$$

-

$$\begin{aligned}
|f(t, y, y_1') - f(t, y, y_2')| &= \left|\frac{3}{4}(t^2 + yt)[\arctan((y_1')^2) - \arctan((y_2')^2)]\right| \\
\textit{(Lagrange mean value theorem)} &\le \frac{3}{4}(|t|^2 + |y||t|)\frac{2y_3'}{(y_3')^4 + 1}|y_1' - y_2'| \text{ for some } y_3' \in (y_1', y_2') \\
&\le \frac{3}{2}\left(\frac{1}{4} + \frac{1}{4}\cdot\frac{1}{2}\right)\frac{y_3'}{(y_3')^4 + 1}|y_1' - y_2'| \\
&\le \frac{9}{16}|y_1' - y_2'| \text{ ......(for justification, see Figure (1.1)))} \\
&\Rightarrow \boxed{K_1 = \frac{9}{16}}
\end{aligned}$$

-

$$|f(t, y_1, y') - f(t, y_2, y')| = \left|\frac{3}{4}t\,(y_1 - y_2)\arctan((y')^2)\right| \le \frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{4}|y_1 - y_2| = \frac{3\pi}{32}|y_1 - y_2|$$

$$\Rightarrow \boxed{K_2 = \frac{3\pi}{32}}$$

$\alpha \coloneqq \min(a, b/M, (1-K_1)/K2) = \min(1/2, (1/4)/(9\pi/128), (1-9/16)/(3\pi/32)) = \min(1/2, 32/9\pi, 14/3\pi) = 1/2$. From the Theorem (1.4), we can say that the following IVP

$$y' = \frac{3}{4}(t^2 + yt)\arctan((y')^2), \qquad\qquad y(0) = 0$$

has an unique solution over the set $\{t : |t| \leq 1/2\}$.

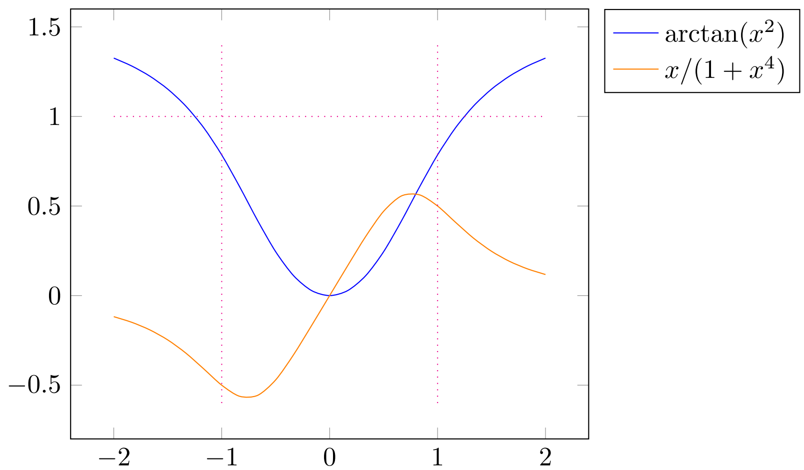

Figure 1.1: Plot of $\arctan(x^2)$ and $x/(1+x^4)$

Theorem 9 (Peano)**.** Let, $f, y \in \mathbb{R}^d$, f is continuous on parallelepiped $R = \{t, y : t_0 \leq t \leq a, |y - y_0| \leq b\}$. Let, $|f(t, y)| \leq M$ on R and $\alpha \coloneqq \min(a, b/M)$ then, (1.1) has atleast one solution $y = y(t)$ on $[t_0, t_0 + \alpha]$.

Proof. We construct the following sequences called *Tonelli sequences*. For each $k \in \mathbb{N}$, let $y_k : [t_0, t_0 + \alpha] \to \mathbb{R}^d$ defined as

$$y_k(t) = \begin{cases} y_0, & \text{if } t_0 \leq t \leq t_0 + 1/k \\ y_0 + \int_{t_0}^{t-1/k} f(s, y_k(s))ds, & \text{if } t_0 + 1/k \leq t \leq t_0 + \alpha \end{cases}$$

First of all by the choice of α it is clear that $\{y_k(t)\}$ is well defined over the domain (see remark under Theorem 7) and $y_k \in C[t_0, t_0 + \alpha]$. We observe that

1. for $t \in [t_0, t_0 + \alpha]$,

$$\begin{aligned} |y_k(t)| \leq |y_0| + \left| \int_{t_0}^{t-1/k} f(s, y_k(s))ds \right| \leq |y_0| + \int_{t_0}^{t-1/k} |f(s, y_k(s))|\, ds &\leq |y_0| + M(t - t_0 - 1/k) \\ &\leq |y_0| + M(\alpha - 1/k) \\ &\leq |y_0| + M(\alpha) \\ &\leq |y_0| + M(b/M) = |y_0| + b \end{aligned}$$

   which shows that $\{y_k\}$ is uniformly bounded.

2. for $t_1, t_2 \geq t_0 + 1/k$,

$$|y_k(t_1) - y_k(t_2)| \leq \left| \int_{t_1}^{t_2} f(s, y_k(s))ds \right| \leq \left| \int_{t_1}^{t_2} |f(s, y_k(s))|\, ds \right| \leq M|t_1 - t_2|$$

   since $|y_k(t_1) - y_k(t_2)| = 0$ for $t_1, t_2 < t_0 + 1/k$, we have $|y_k(t_1) - y_k(t_2)| \leq M|t_1 - t_2|, \forall\, t_1, t_2 \in [t_0 + t_0 + \alpha]$. So, for any given $\epsilon > 0$, we choose $\delta = \epsilon/M$ such that $\{y_k\}$ is uniformly equicontinuous.

Now, from **Theorem 2**, there exists a subsequence, say y_{k_l} of y_k that converges uniformly, say to y over $[t_0, t_0 + \alpha]$. Since, f is continuous over compact space and as composition of continuous mapping on compact space preserves uniform convergence, we have that $f(s, y_{k_l}(s)) \to f(s, y(s))$ (uniformly) over $[t_0, t_0 + \alpha]$. Hence, we have

$$
\begin{aligned}
\lim_{l\to\infty} y_{k_l}(t) &= \begin{cases} \lim\limits_{l\to\infty} y_0, & \text{if } t_0 \le t \le t_0 + \lim\limits_{l\to\infty} 1/k_l \\ \lim\limits_{l\to\infty} \left[y_0 + \int_{t_0}^{t-1/k_l} f(s, y_{k_l}(s)) ds \right], & \text{if } t_0 + \lim\limits_{l\to\infty} 1/k_l \le t \le t_0 + \alpha \end{cases} \\
&= \begin{cases} y_0, & \text{if } t = t_0 \\ y_0 + \lim\limits_{l\to\infty} \left[\int_{t_0}^{t-1/k_l} f(s, y_{k_l}(s)) ds \right], & \text{if } t_0 \le t \le t_0 + \alpha \end{cases} \\
&= \begin{cases} y_0, & \text{if } t = t_0 \\ y_0 + \lim\limits_{l\to\infty} \left[\int_{t_0}^{t} f(s, y_{k_l}(s)) ds - \int_{t-1/k_l}^{t} f(s, y_{k_l}(s)) ds \right], & \text{if } t_0 \le t \le t_0 + \alpha \end{cases}
\end{aligned}
$$

we have,

$$
0 \le \left| \int_{t-1/k_l}^{t} f(s, y_{k_l}(s)) ds \right| \le \int_{t-1/k_l}^{t} |f(s, y_{k_l}(s))| \, ds \le \frac{M}{k_l} \Rightarrow \lim_{l\to\infty} \int_{t-1/k_l}^{t} f(s, y_{k_l}(s)) ds = 0 \quad \text{(squeeze theorem)}
$$

So,

$$
y(t) = \lim_{l\to\infty} y_{k_l}(t) = \begin{cases} y_0, & \text{if } t = t_0 \\ y_0 + \int_{t_0}^{t} \lim\limits_{l\to\infty} f(s, y_{k_l}(s)) ds) ds, & \text{if } t_0 \le t \le t_0 + \alpha \end{cases}
$$

$$
\Rightarrow y(t) = \begin{cases} y_0, & \text{if } t = t_0 \\ y_0 + \int_{t_0}^{t} f(s, y(s)) ds & \text{if } t_0 \le t \le t_0 + \alpha \end{cases}
$$

$y(t)$ satisfies (1.2) for $t \in [t_0 + t_0 + \alpha]$. We have constructed a continuous solution for (1.2). Hence, from (**Theorem 6**), the $y(t)$ is a solution of (1.1) on $[t_0 + t_0 + \alpha]$. ☐

Definition 11 (Solution in the extended sense)**.** A function y is called a **solution in the *extended sense*** on the IVP 1.1 if y is absolutely continuous, y satisfies differential equation almost everywhere and y satisfies initial condition.

Theorem 10 (Caratheodory)**.** Let, f be defined on the rectangle $R = \{(t, y) || t - t_0| \le a, |y - y_0| \le b\}$. If f satisfies

1. $f(t, y)$ is continuous in y for each fixed t.
2. $f(t, y)$ is measurable in t for each fixed y.
3. there exists a lebesgue integrable function $m : [t_0 - a, t_0 + a] \to [0, \infty)$ such that $|f(t, y)| \le m(t) \ \forall \ (t, y) \in \mathbb{R}$

then 1.1 has a solution in the *extended sense* in a neighbourhood of the initial condition.

Theorem 11 (Perron's Uniqueness theorem)**.** Assume that,

1. $g(t, z)$ is continuous and non-negative in $R = \{(t, y) : t_0 \le t \le t_0 + a, 0 \le z \le 2b\}$. For every $t_1 \in (t_0, t_0 + a)$, $z(t) \equiv 0$ is the only differentiable function in $[t_0, t_1)$ which satisfies
$$z'(t) = g(t, z(t)),\ t_0 \le t \le t_1,\ z(t_0) = 0$$
2. $f(t, y)$ is continuous over R and $\forall (t, y_1), (t, y_2) \in R$,
$$|f(t, y_1) - f(t, y_2)| \le g(t, |y_1 - y_2|)$$

then 1.1 has atmost one solution in $[t_0, t_0 + a]$.

Proof. Let, $y_1(t), y_2(t)$ are two solutions of 1.1 in $[t_0, t_0 + a]$. Further let, $\phi(t) := |y_1(t) - y_2(t)|$. $\Rightarrow \phi(t_0) = 0$. We observe that

$$
\begin{aligned}
D^+\phi(t) \le |y_1'(t) - y_2'(t)| &= |f(t, y_1) - f(t, y_2)| \\
&\le g(t, |y_1 - y_2|) \\
&= g(t, \phi(t))
\end{aligned}
$$

from Lemma 5 for any $t_1 \in (t_0, t_0 + a)$, we have $\phi(t) \le r(t) \, \forall \, t \in [t_0, t_1)$ where $r(t)$ is the maximal solution of 1.1. But, from the hypothesis of the theorem we have that $r(t) \equiv 0$ hence, $\phi(t) = 0$ over $[t_0, t_1)$ from which the result follows. ☐

Chapter 2

Symmetric analysis of differential equations

2.1 Set-up and examples

Definition 12 (Symmetry of DE)**.** We consider ODEs of the form

$$y^{(n)} = w(x, y, y', \cdots, y^{(n-1)}), \qquad y^{(k)} = \frac{d^k y}{dx^k} \tag{2.1}$$

where, it is assumed that w is a (locally) smooth function over all of its arguments. A *symmetry* of a differential equation is an invertible point transformation that maps set of solutions of the DE to itself.

Example 3. Consider the equation, $\frac{dy}{dx} = x$. The solution curves are $y(x) = \frac{x^2}{2} + c$ any $c \in \mathbb{R}$. For any $\epsilon \in \mathbb{R}$, the transformation $(x^*, y^*) = (x, y + \epsilon)$ is a symmetry since it maps a solution $y_c(x) = \frac{x^2}{2} + c$ to another solution $y_{c-\epsilon} = \frac{x^2}{2} + c - \epsilon$. Moreover, this is an example of *one parameter lie group symmetry.* We will see more about this further.

Any diffeomrophism $\Gamma : (x, y) \mapsto (x^*, y^*)$ maps smooth planar curves to smooth planar curves. This action of Γ on the plane induces an action on the derivatives $y^{(k)}$ as

$$\Gamma : (x, y, \cdots, y^{(n)}) \mapsto (x^*, y^*, \cdots, y^{*(n)}), \qquad y^{*(k)} = \frac{d^k y^*}{dx^{*(k)}}$$

This map is called n^{th} prolongation of Γ and

$$y^{*(k)} = \frac{dy^{*(k-1)}}{dx^*} = \frac{D_x y^{*(k-1)}}{D_x x^*}, \qquad y^{*(0)} = y^* \tag{2.2}$$

where, $D_x \coloneqq \partial_x + y'\partial_y + y''\partial_{y'} + \cdots$ which is the *total derivaite operator* w.r.t x (the above equality is justified from *chain rule.*) So, if (2.1) holds then the symmetric condition for the ODE is

$$y^{*(n)} = w(x^*, y^*, \cdots, y^{*(n-1)}) \tag{2.3}$$

Example 4. We consider the transformation

$$(x^*, y^*) = \left(\frac{1}{x}, \frac{y}{x}\right)$$

We show that this is a symmetry of the second order ODE $y'' = 0, x > 0$. From above, we have

$${y^*}' = \frac{D_x(y/x)}{D_x(1/x)} = \frac{-y/x^2 + y'/x}{-1/x^2} = y - xy'$$

and hence,

$${y^*}'' = \frac{D_x\left({y^*}'\right)}{D_x(1/x)} = \frac{D_x(y - xy')}{-1/x^2} = \frac{-y' + y' + y''(-x)}{-1/x^2} = x^3 y''$$

So, $y'' = 0 \Rightarrow {y^*}'' = 0$ which implies that symmetric condition is satisfied. We observe that

$$(x^{**}, y^{**}) = (1/x^*, y^*/x^*) = (1/(1/x), (y/x)/(1/x)) = (x, y),$$

the inverse of the transformation is itself. So, we say that this corresponds to a discrete group of order 2. We now define what a group is.

Definition 13 (Group)**.** A group G is a set of a set of elements with a law of composition ϕ between elements satisfying the following axioms:

1. (**Closure**) For any elements $a, b \in G$, $\phi(a, b) \in G$
2. (**Associativity**) For any elements $a, b, c \in G$, $\phi(a, \phi(b, c)) = \phi(\phi(a, b), c)$
3. (**Identity**) there exists a unique identity element $e \in G$ such that for any element $g \in G$, $\phi(e, g) = \phi(g, e) = g$
4. (**Inverse**) for any element $g \in G$ there exists a unique inverse element $g^{-1} \in G$ such that, $\phi(g, g^{-1}) = \phi(g^{-1}, g) = e$

Example 5. $\mathbb{Z}$ with $\phi(a, b) = a + b$ forms a group. Here, $e = 0$ and $a^{-1} = -a$

Example 6. $\mathbb{Z}$ with $\phi(a, b) = a - b$ doesn't form a group since associativity fails. $[1 - (2 - 3) \neq (1 - 2) - 3]$

Definition 14 (One-parameter group of transformations)**.** Let $x = (x_1, x_2, \cdots, x_n) \in D \subset \mathbb{R}^n$. The set of transformations

$$x^* = X(x; \epsilon) \tag{2.4}$$

defined for each $x \in D$ and parameter ϵ in set $S \subset \mathbb{R}$, with $\phi(\epsilon, \delta)$ defining a law of composition of parameters ϵ and δ in S, forms a one-parameter group of transformations on D if the following hold

1. for each $\epsilon \in S$ the transformations are one-to-one onto D ($\Rightarrow x^* \in D$)
2. (S, ϕ) forms a group (say) G
3. for each $x \in D$, $x^* = x$ when $\epsilon = \epsilon_0$ which corresponds to identitiy element of G. So, $X(x; \epsilon_0) = x$
4. if $x^* = X(x; \epsilon), x^{**} = X(x^*; \delta)$ then, $x^{**} = X(x; \phi(\epsilon, \delta))$

Definition 15 (One-parameter Lie groups of transformations)**.** A one-parameter group of transformations defines a one-parameter Lie group of transformations if, in addition to satisfying axioms (1)–(4) of Definition 14, the following hold

5. ϵ is a continuous parameter, i.e., S is an intervel (connected subset) in $\mathbb{R}$. Without loss of generality, $\epsilon = 0$ corresponds to the identity element.
6. X is infinitely differentiable with respect to $x \in D$ and an analytic function of ϵ in S
7. $\phi(\epsilon, \delta)$ is an analytic function of $\epsilon \in S$ and $\delta \in S$

Example 7. Consider the group of translations, $(x^*, y^*) = (x + \epsilon, y)$, $x, y, \epsilon \in \mathbb{R}$. Here, $\phi(\epsilon, \delta) = \epsilon + \delta$. This group corresponds to motions parallel to the x-axis. Fix $p = (x_0, y_0) \in \mathbb{R}^2$. Under this transformation the path traced by x_0 $\forall \epsilon \in \mathbb{R}$ is the line $y = x_0$.

Example 8. Consider the group of scalings, $(x^*, y^*) = (\alpha x, \alpha^2 y)$, $x, y \in \mathbb{R}$, $\alpha \in \mathbb{R}^+$. Here, $\phi(\alpha, \beta) = \alpha\beta$ where the identity element corresponds to $\alpha = 1$. Fix $p = (x_0, y_0) \in \mathbb{R} \setminus 0 \times \mathbb{R} \setminus 0$. Under this transformation the path traced by p $\forall \epsilon \in \mathbb{R}$ is parabola $y = \frac{y_0}{x_0^2} x^2$. We later properly define this notion of *path*.

Remark. Lie group transformations with non-zero Jacobian are considered for the notion of Lie symmetries. The Lie group symmetries we are considering are symmetries under a *local* group i.e., the action may not be defined over the whole plane for instance, $(x^*, y^*) = (x/(1 - \lambda x), y/(1 - \lambda x))$ is defined only if $\lambda < 1/x$, $x > 0$ and $\lambda > 1/x$, $x > 0$.

Definition 16. For a given point $x_0 \in \mathbb{R}^n$, $S \subset \mathbb{R}$ and a group of transformation X, the set $\{X(x_0, \epsilon) : \epsilon \in S\}$ is called *Orbit* of x_0.

Remark (1)**.** When the X is one parameter lie group of transformation, since S is an interval i.e., a connected subset of $\mathbb{R}$, we get a smooth curve as an orbit.

Remark (2)**.** There can be one or more *invariant points*, each of which is mapped to itself by lie symmetries. An *invariant point* is a zero dimensional orbit of the lie group.

Remark (3)**.** For a given point $x_0 \in \mathbb{R}^n$, S be interval in $\mathbb{R}$, and ϕ is the law of composotion where (S, ϕ) forms a group G, let X be one parameter lie group of transformation, orbit of x_0 be $\gamma := \{X(x_0, \epsilon) : \epsilon \in S\}$, choose any point $y \in \gamma$, there exists some $\delta_0 \in S$ such that $X(x_0, \delta_0) = y$. So, for any $\delta \in S$,

$$x_0^{**} = y^* = X(X(x_0, \epsilon), \delta_0) = X(x_0, \phi(\delta_0, \delta))$$

where third equality is from Definitions (14) and (15). Since, $\phi(\delta_0, \delta) \in S$, we have $X(x_0, \phi(\delta_0, \delta)) = x_0^{**} \in \gamma$. From this we understand that action of a lie group maps every point on an orbit to a point on the same orbit v,i,z., every orbit is invariant under the action of the lie groups.

2.2 Planar Symmetries

We now study action of one parameter lie group of symmetries (x^*, y^*) on points in the plane considering the differential equation

$$y' = w(x, y) \tag{2.5}$$

Definition 17 (Tangent vector)**.** Consider the orbit through a non-invariant point $(x, y) \in \mathbb{R}^2$. The *tangent vector* to the orbit at a point (x^*, y^*) is $(\xi(x^*, y^*), \eta(x^*, y^*))$ where,

$$\frac{dx^*}{d\epsilon} = \xi(x^*, y^*), \qquad\qquad \frac{dy^*}{d\epsilon} = \eta(x^*, y^*). \tag{2.6}$$

Remark (1)**.** In particular since, $\epsilon = 0$ corresponds to $x^* = x$ and $y^* = y$, the tangent vector at (x, y) is

$$(\xi(x, y), \eta(x, y)) = \left(\left.\frac{dx^*}{d\epsilon}\right|_{\epsilon=0}, \left.\frac{dy^*}{d\epsilon}\right|_{\epsilon=0} \right)$$

From definition (15), lie group action is an analytic function of $\epsilon \in S$. Hence, we have the following taylor series,

$$x^* = x + \epsilon\xi(x, y) + O(\epsilon^2), \qquad y^* = y + \epsilon\eta(x, y) + O(\epsilon^2).$$

So, (x, y) is invariant iff the tangent vector is *zero*, i.e., $\xi(x, y) = \eta(x, y) = 0$.

Example 9. Under symmetry

$$(x^*, y^*) = (x, y + \epsilon)$$

every orbit have same tangent vector at every point namely,

$$(\xi(x, y), \eta(x, y)) = \left(\left.\frac{dx^*}{d\epsilon}\right|_{\epsilon=0}, \left.\frac{dy^*}{d\epsilon}\right|_{\epsilon=0} \right) = \left(\left.\frac{dx}{d\epsilon}\right|_{\epsilon=0}, \left.\frac{d(y+\epsilon)}{d\epsilon}\right|_{\epsilon=0} \right) = \left(0|_{\epsilon=0}, 1|_{\epsilon=0}\right) = (0, 1)$$

Remark (2)**.** Since, everything is nice and smooth, tangent vectors vary smoothly, the set of tangent vectors for a particular lie group is an example of a *smooth vector field*.

Remark (3)**.** One can think ϵ as *time*, the tangent vector at a point as *velocity*, orbit describing the *path* of the particle, whence (2.6) describes a *steady flow* of the particles on the plane. Invariant points are the *fixed points* of the flow.

Consider any curve $C \in \mathbb{R}^2$. Suppose any orbit γ crosses C transversely at a point (x, y). Since, under the lie symmetries (x, y) is mapped to some point on the orbit γ (which is clear from definition 16 and remarks under it), so it is getting mapped to points that are not in C. As we established already that each orbit is invariant under symmetries, we can say that curve is invariant iff no orbit crosses it. In other words, C is invariant iff the tangent to C at each point (x, y) is parallel to the tangent vector $(\xi(x, y), \eta(x, y))$. We now state this condition by formally defining a *characteristic*.

Definition 18 (Characteristic)**.** We define *characteristic* Q as,

$$Q(x, y, y') = \eta(x, y) - y'\xi(x, y)$$

If C is the smooth curve $y = y(x)$, the tangent to C at $(x, y(x))$ is $(1, y'(x))$. This is parallel to $(\xi(x, y), \eta(x, y))$ iff $y' \cdot \eta(x, y) = 1 \cdot \xi(x, y) \iff Q(x, y, y') = 0$ on C. Considering the equation $y' = w(x, y)$, on these solutions, we have

$$Q(x, y, w(x, y)) = \eta(x, y) - w(x, y)\xi(x, y)$$

We define $\tilde{Q}(x, y) = Q(x, y, w(x, y))$ and call it *reduced characteristic*. Hence, any solution curve $y = f(x)$ is invariant iff $\tilde{Q}(x, y) = 0$ where $y = f(x)$.

2.3 Canonical coordinates

Lemma 12. A differential equation with a translational symmetry in the dependent variable is separable.

Proof. $X(x, y, \epsilon) = (x, y + \epsilon)$ be a symmetry of the ODE (2.5). Then,

$$w(y, x + \epsilon) = w(x^*, y^*) = \frac{dy^*}{dx^*} = \frac{d(y + \epsilon)}{dx} = \frac{dy}{dx} = w(x, y)$$

$\Rightarrow w(x, y + \epsilon) = w(x, y)\ \forall\, \epsilon \in \mathbb{R}$ which implies, w is independent of y and the function $w(x, y)$ can be relabelled as $w(x)$. Hence, (2.5) changes to

$$\frac{dy}{dx} = w(x)$$

which is separable and hence integrable. □

Remark. Any differential equation which is separable trivially have translations as their symmetry.

Example 10. We now consider the differential equation

$$\frac{dy}{dx} = \frac{y^3 + x^2 y - x - y}{x^3 + xy^2 - x + y}$$

which at first glance may seem quite difficult to solve. However transforming cartiesian coordinates to polar coordinates

$$x = r\cos\theta, \qquad y = r\sin\theta$$

results in the transfomed equation

$$\frac{dr}{d\theta} = r(1 - r^2).$$

So, clearly these solutions are invariant under the conitnuous group of transformations $(r, \theta) \mapsto (r, \theta + \epsilon)$ that represents solutions about the origin.

These suggest that a first order ODE can be transformed into a separable equation if its set of solution curves is invarianr under translation in some coordinate system. So, for now, given a first order ODE, our goal is to find a general method to determine this coordinate system such that simplified equation can be integrated.

Definition 19 (Canonical coordinates)**.** Given any one parameter Lie group of transformations and a differential equation, we introduce *canonical coordinates* $(r, s) = (r(x, y), s(x, y))$ where the equation becomes separable such that,

$$(r^*, s^*) = (r(x^*, y^*), s(x^*, y^*)) = (r, s + \epsilon).$$

From example (9),

$$\left.\frac{dr^*}{d\epsilon}\right|_{\epsilon=0} = 0, \qquad \left.\frac{ds^*}{d\epsilon}\right|_{\epsilon=0} = 1$$

by chain rule we have,

$$r_x \left.\frac{dx^*}{d\epsilon}\right|_{\epsilon=0} + r_y \left.\frac{dy^*}{d\epsilon}\right|_{\epsilon=0} = 0$$
$$s_x \left.\frac{dx^*}{d\epsilon}\right|_{\epsilon=0} + s_y \left.\frac{dy^*}{d\epsilon}\right|_{\epsilon=0} = 1$$

which implies,

$$\begin{aligned} \xi(x, y) r_x + \eta(x, y) r_y &= 0 \\ \xi(x, y) s_x + \eta(x, y) s_y &= 1 \end{aligned} \tag{2.7}$$

Moreover, in order to recover the solution in terms of original coordinates the transformation $(x, y) \mapsto (r, s)$ must be invertible, v.i.z.,

$$r_x s_y - r_y s_x \neq 0.$$

In (2.7), we have a system of first order linear partial differential equations. We solve them using the famous method of characteristics. Here, we have

$$\frac{dx}{\xi(x,y)} = \frac{dy}{\eta}, \qquad \frac{dx}{\xi(x,y)} = \frac{dy}{\eta(x,y)} = \frac{ds}{1}$$

We have,

- if $\xi(x,y) \neq 0$, then

$$r_x + \frac{\eta(x,y)}{\xi(x,y)} r_y = 0$$

  so, on solving

$$\frac{dy}{dx} = \frac{\eta(x,y)}{\xi(x,y)}$$

  we can get $r = \phi(x,y)$ v.i.z., $y = y(x,r)$ where r is the constant of the integration. s is obtained as follows,

$$s(r,x) = \left(\int \frac{dx}{\xi(x, y(x,r))} \right) \Bigg|_{r=r(x,y)}$$

  here the integral is evaluated with r being treated as a constant.

- if $\xi(x,y) = 0$, then $\eta(x,y) \neq 0$ implies,

$$r = x, \qquad s = \left(\int \frac{dy}{\eta(r,y)} \right) \Bigg|_{r=x}$$

2.4 Solving ODEs with Lie symmetries

Consider (2.5), we have

$$\frac{ds}{dr} = \frac{D_x s}{D_x r} = \frac{s_x + s_y(dy/dx)}{r_x + r_y(dy/dx)} = \frac{s_x + s_y w(x,y)}{r_x + r_y w(x,y)}$$

under any invertible transformation $(x,y) \mapsto (r,s)$. So we have

$$\frac{ds}{dr} = F(r,s), \text{ for some function } F/.$$

If (r,s) are canonical coordinates, then $(r^*, s^*) = (r, s+\epsilon)$ and from Lemma (12)

$$\frac{ds}{dr} = F(r).$$

Hence general solution is given by, $s(r) = \int F(r)\, dr + c$ which can be inverted ans (x,y) can be found.
It is now left to find symmetries of a given ODE. For (2.5) if $(x,y) \mapsto (x^*, y^*)$ is a symmetry, then

$$\frac{dy^*}{dx^*} = \frac{D_x y^*}{D_x x^*} = \frac{y^*_x + w(x,y) y^*_y}{x^*_x + w(x,y) x^*_y} = w(x^*, y^*)$$

But, unfortunately, this is a complicated non-linear partial differential equation and is difficult to solve. So, we exploit the properties of Lie symmetries and shall find a better way.
We have, for fixed (x,y) and smooth ξ, η,

$$x^* = x + \epsilon\xi(x,y) + O(\epsilon^2), \qquad y^* = y + \epsilon\eta(x,y) + O(\epsilon^2)$$

substituting this in the above equation, we have

$$\Rightarrow w(x + \epsilon\xi(x,y) + O(\epsilon^2), y + \epsilon\eta(x,y) + O(\epsilon^2)) = \frac{\epsilon\eta_x + O(\epsilon^2) + w(x,y)(1 + \epsilon\eta_y)}{1 + \epsilon\eta_x + O(\epsilon^2) + w(x,y)(\epsilon\eta_y)}$$

$$w(x + \epsilon\xi(x,y) + O(\epsilon^2), y + \epsilon\eta(x,y) + O(\epsilon^2)) = \frac{w(x,y) + \epsilon(\eta_x + w(x,y)\eta_y) + O(\epsilon^2)}{1 + \epsilon(\eta_x + w(x,y)\eta_y) + O(\epsilon^2)}$$

$$w(x + \epsilon\xi(x,y) + O(\epsilon^2), y + \epsilon\eta(x,y) + O(\epsilon^2)) = [w(x,y) + \epsilon(\eta_x + w(x,y)\eta_y) + O(\epsilon^2)][1 + \epsilon(\eta_x + w(x,y)\eta_y) + O(\epsilon^2)]^{-1}$$

as, w is (locally) smooth which implies *analytic*, we expand around (x, y) and similarly the right side around $\epsilon = 0$ to obtain

$$\begin{aligned}w(x,y) + w_x(x,y)(\epsilon\xi + O(\epsilon^2)) + w_y(x,y)(\epsilon\eta + O(\epsilon^2)) &= [w(x,y) + \epsilon(\eta_x + w(x,y)\eta_y) + O(\epsilon^2)] \\ &\quad [1 - \epsilon(\eta_x + w(x,y)\eta_y) + O(\epsilon^2)] \\ w + (w_x\xi + w_y\eta)\epsilon + O(\epsilon^2) &= w + (\eta_x + w\eta_y - w\xi_x - w^2\eta_y)\epsilon + O(\epsilon^2)\end{aligned}$$

on equating coefficient of ϵ we get something known as *linearized symmetry condition*

$$\eta_x + w(\eta_y - \xi_x) - w^2\eta_y = w_x\xi + w_y\eta \tag{2.8}$$

Now to solve (2.25) it is necessary to use an appropriate *ansatz* that is to place some additional constraint on ξ, η.

Example 11 (Integrating factor)**.** Consider the first oder linear differential equation

$$\tag{2.9}$$

where, F,G are continuous. The standard approach to solve this is to multiply by the integratinf factor and integrate to obtain

$$y = e^{-\int_0^x F dt} \int e^{\int_0^x F dt} G(x) dx$$

We shall now see how this solution arises using symetries. We have the corresponding homogenous equation as $y' + F(x)y = 0$, which is separable and hence can be directly integrated to get

$$y_h = e^{-\int_0^x F dt}$$

as the solution. Since, the equation (2.9) is linear, we know that if y_p is a particular solution of (2.9) then $y_p + y_h$ is also a solution. Hence, clearly the transformation $(x, y) \mapsto (x, y + ty_h(x))$ is a symmetry of (2.9). We have the tangent vector field as, $(\xi, \eta) = (0, y_h(x))$. Here, $\xi = 0$ which implies

$$\boxed{r = x}, \qquad s = \int \frac{dy}{\eta(r,y)}\bigg|_{r=x} = \int \frac{dy}{y_h(r)}\bigg|_{r=x} = \frac{y}{y_h(x)} \Rightarrow \boxed{y = y_h(x)s(x)}$$

$$\begin{aligned}\Rightarrow \frac{ds}{dr} = \frac{D_x s}{D_x r} = \frac{s_x + s_y(dy/dx)}{r_x + r_y(dy/dx)} &= \frac{-y(y_h'(x))/(y_h^2(x)) + (G(x) - yF(x)/y_h(x)}{1+0} \\ &= \frac{y(y_h(x)F(x))}{y_h^2(x)} + \frac{G(x) - yF(x)}{y_h(x)} \\ &= \frac{G(x)}{y_h(x)}\end{aligned}$$

$$\Rightarrow s(r) = s(x) = \int e^{\int_0^x F dt} G(x) dx$$

$$\Rightarrow \boxed{y = y_h(x)s(x) = e^{-\int_0^x F dt} \int e^{\int_0^x F dt} G(x) dx}$$

Example 12 (Reduction of order)**.** We can use symmetry to reduce the order od a second order ODE. We consider the homogenous equation

$$y'' + Fy' + Gy = 0 \tag{2.10}$$

for continuous F, G. Say, y is a solution to (2.10), then for any function f, clearly $(x, y) \mapsto (x, f(\epsilon)y)$ is a symmetry of 2.10 as it can be factored out. The tangent vector field is $(0, f'(0)y)$. For simplicity let $f(\epsilon) = e^\epsilon \Rightarrow f'(0) = 1$. Hence,

$$\boxed{r = x}, \qquad s = \int \frac{dy}{y} = \ln y \Rightarrow \boxed{y = e^s}$$

which implies,

$$y' = e^s s_x, \qquad y'' = e^s s_x + e^s s_{xx} = e^s(s_x^2 + s_{xx})$$

on substituting these in (2.10), we get

$$\begin{aligned} e^s(s_x^2 + s_{xx}) + Fe^s s_x + Ge^s = 0 \\ \Rightarrow s_x^2 + s_{xx} + Fs_x + G = 0 \end{aligned}$$

Since, s doesn't appear explicitly in this transformed equation, we let $z = s_x$ to obtain the following first order ODE

$$z_x + z^2 + F(x)z + G(x) = 0.$$

2.5 Extension to higher order

We define a concise notation that can be easily expanded to handle differential equations of any order, with any number of dependent and independent variables.

Definition 20 (Infinitesimal generator)**.** Suppose that a first-order ODE has a one-parameter Lie group of symmetries, whose tangent vector at (x, y) is (ξ, η). Then the partial differential operator

$$X = \xi(x, y)\partial_x + \eta(x, y)\partial_y \tag{2.11}$$

is called *infinitesimal generator* of the Lie group.

Recall equation (2.7), it can be rewritten as

$$Xr = 0, \qquad Xs = 1. \tag{2.12}$$

We ask ourselves the following question " How does an arbitrary locally-smooth function $F(x, y)$ vary along the orbit of a 1-parameter local Lie group? "

Suppose that (u, v) are the new coordinates, consider the transformation $(x, y) \mapsto (u, v)$ then by chain rule

$$\begin{aligned} XF(u, v) = XF(u(x, y), v(x, y)) &= \xi(x, y)\partial_x F(u(x, y), v(x, y)) + \eta(x, y)\partial_y F(u(x, y), v(x, y)) \\ &= \xi(x, y)(F_u u_x + F_v v_x) + \eta(x, y)(F_u u_y + F_v v_y) \\ &= (Xu)F_u + (Xv)F_v \\ \Rightarrow XF = (Xu)\partial_u F + (Xv)\partial_v F & \end{aligned}$$

Since, $F(u, v)$ is arbitrary and therefore, in terms of the new coordinates, the infinitesimal generator is

$$X \equiv (Xu)\partial_u + (Xv)\partial_v \tag{2.13}$$

If the transformation was to canonial coordinates (r, s), from (2.12) we have

$$X = (Xr)\partial_r + (Xs)\partial_s = \partial_s \tag{2.14}$$

Suppose, $G(r, s)$ is a smooth function and let

$$F(x, y) = G(r(x, y), s(x, y)).$$

At any non-invariant point (x, y), under Lie symmetry we have

$$F(x, y) \mapsto F(x^*, y^*) = G(r(x^*, y^*), s(x^*, y^*)) = G(r^*, s^*) = G(r, s + \epsilon)$$

On expanding around $\epsilon = 0$, we get

$$\begin{aligned} G(r, s+\epsilon) &= G(r, s) + \epsilon \left.\frac{\partial G(r, s+\epsilon)}{\partial \epsilon}\right|_{\epsilon=0} + \frac{\epsilon^2}{2!}\left.\frac{\partial^2 G(r, s+\epsilon)}{\partial \epsilon^2}\right|_{\epsilon=0} + \cdots \\ &= G(r, s) + \epsilon \left.\left(\frac{\partial G(r, s+\epsilon)}{\partial r}(0) + \frac{\partial G(r, s+\epsilon)}{\partial s}(1)\right)\right|_{\epsilon=0} + \frac{\epsilon^2}{2!}\left.\left(\frac{\partial^2 G(r, s+\epsilon)}{\partial r^2}(0) + \frac{\partial^2 G(r, s+\epsilon)}{\partial s^2}(1)\right)\right|_{\epsilon=0} + \cdots \\ &= \sum_{j=0}^{\infty} \frac{\epsilon^j}{j!}\frac{\partial^j G}{\partial s^j}(r, s) = \sum_{j=0}^{\infty} \frac{\epsilon^j}{j!} X^j G(r, s) \end{aligned}$$

On reverting to original coordinates (x, y) we have the following equation which is called *Lie series* of F about (x, y) if the series in the right hand side converges.

$$\begin{aligned} F(x^*, y^*) &= \sum_{j=0}^{\infty} \frac{\epsilon^j}{j!} X^j F(x, y) \\ F(x^*, y^*) &= e^{\epsilon X} F(x, y) \end{aligned} \tag{2.15}$$

In particular, since $F(x, y) = x$ and $F(x, y) = y$ are smooth, we have

$$x^* = e^{\epsilon X} x, \qquad y^* = e^{\epsilon X} y,$$

due to which we have the following identity

$$F(e^{\epsilon X} x, e^{\epsilon X} y) = e^{\epsilon X} F(x, y) \tag{2.16}$$

The infinitesimal generator provides a coordinate independent way of characterizing the action of Lie symmetries on functions.

2.5.1 On to any number of variables

Everything in this section generalizes to any number of variables. Suppose that there are n variables, $z^1, z^2, \cdots, z^n$ and that the Lie symmetries are

$$z^{*s}(z^1, z^2, \cdots, z^n, \epsilon) = z^s + \epsilon \zeta^s(z^1, z^2, \cdots, z^n) + O(\epsilon^2), \qquad s = 1, 2, \cdots, n \tag{2.17}$$

Then the infinitesimal generator of the one-parameter Lie group is

$$X = \sum_{s=1}^{n} \zeta^s(z^1, z^2, \cdots, z^n) \frac{\partial}{\partial z^s} \tag{2.18}$$

Lie symmetries can be reconstructed from the Lie series as follows,

$$z^{*s} = e^{\epsilon X} z^s, \qquad s = 1, 2, \cdots, n \tag{2.19}$$

and hence for a smooth function F,

$$F(e^{\epsilon X} z^1, e^{\epsilon X} z^2, \cdots, e^{\epsilon X} z^n) = e^{\epsilon X} F(z^1, z^2, \cdots, z^n) \tag{2.20}$$

2.5.2 Examples of some known symmetries

The following Table 2.1 has some first order and second order equations with known symmetries, where F, P, Q are nice, smooth functions in their erspective domains.

2.5.3 Prolongation formula

Recalling equation (2.3), the linearized symmetry condition for Lie symmetries is derived by the same method that we used for first-order ODEs. The trivial symmetry corresponding to $\epsilon = 0$ leaves every point unchanged. Hence, for ϵ sufficiently close to zero, the prolonged Lie symmetries are of the form

$$\begin{aligned} x^* &= x + \epsilon \xi + O(\epsilon^2) \\ y^* &= y + \epsilon \eta + O(\epsilon^2) \\ y^{*(k)} &= y^{(k)} + \epsilon \eta^{(k)} + O(\epsilon^2), \qquad k \geq 1 \end{aligned} \tag{2.21}$$

where, the subscript $\eta^{(k)}$ doesn't denote a derivative of η it's just an index. On substituting equation (2.21) into equation (2.3) and expanding around $\epsilon = 0$ we get

$$\begin{aligned} y^{*(n)} &= w(x^*, y^*, \cdots, y^{*(n-1)}) \\ &= w(x + \epsilon\xi + O(\epsilon^2), y + \epsilon\eta + O(\epsilon^2), \cdots, y^{(n-1)} + \epsilon\eta^{(n-1)} + O(\epsilon^2)) \\ y^{(n)} + \epsilon\eta^{(n)} + O(\epsilon^2) &= w(x, y, \cdots, y^{(n-1)}) + \epsilon(w_x \xi + w_y \eta + \cdots + w_{y^{(n-1)}} \eta^{(n-1)}) + O(\epsilon^2) \end{aligned}$$

Table 2.1: Some symmetries

y'	X	y''	X
$F(kx+ly)$	$l\frac{\partial}{\partial x}-k\frac{\partial}{\partial y}$	$F(kx+ly,y')$	$l\frac{\partial}{\partial x}-k\frac{\partial}{\partial y}$
$\frac{y+xF(\sqrt{x^2+y^2})}{x-yF(\sqrt{x^2+y^2})}$	$y\frac{\partial}{\partial x}-x\frac{\partial}{\partial y}$	$(1+y'^2)^{3/2}F\left(r,\frac{y-xy'}{x+yy'}\right)$	$y\frac{\partial}{\partial x}-x\frac{\partial}{\partial y}$
$F(y/x)$	$x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}$	$y'^3F\left(y,\frac{y-xy'}{y'}\right)$	$y\frac{\partial}{\partial x}$
$x^{k-1}F(y/x^k)$	$x\frac{\partial}{\partial x}+ky\frac{\partial}{\partial y}$	$y'^3\left[F\left(y,\frac{1}{y'}-x\frac{q'(y)}{q(y)}\right)-x\frac{q''(y)}{q(y)}\right]$	$q(y)\frac{\partial}{\partial x}$
$F(xe^{-y})/x$	$x\frac{\partial}{\partial x}+\frac{\partial}{\partial y}$	$F(x,y-xy')$	$x\frac{\partial}{\partial y}$
$yF(ye^{-x})$	$\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}$	$(p''(x)y+F(x,p(x)y'-p'(x)y))/p(x)$	$p(x)\frac{\partial}{\partial y}$
$y/x+xF(y/x)$	$\frac{\partial}{\partial x}+\frac{y}{x}\frac{\partial}{\partial y}$	$(F(x,p(x)y')-p(x)p'(x)y')/p^2(x)$	$p(x)\frac{\partial}{\partial x}$
$(y+F(y/x))/x$	$x^2\frac{\partial}{\partial x}+xy\frac{\partial}{\partial y}$	$F(y/x,y')/x$	$x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}$
$y/(x+F(y/x))$	$xy\frac{\partial}{\partial x}+y^2\frac{\partial}{\partial y}$	$x^{k-2}F(x^{-k}y,x^{1-k}y')$	$x\frac{\partial}{\partial x}+ky\frac{\partial}{\partial y}$
$y/(x+F(y))$	$y\frac{\partial}{\partial x}$	$yF(ye^{-x},y'/y)$	$\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}$
$(y+F(x))/x$	$x\frac{\partial}{\partial y}$	$yF(x,y'/y)$	$y\frac{\partial}{\partial x}$
$y/(\ln x+F(y))$	$xy\frac{\partial}{\partial x}$	$\frac{q'(y)}{q(y)}y'^2+q(y)F\left(x,\frac{y'}{q(y)}\right)$	$q(y)\frac{\partial}{\partial y}$
$y(\ln y+F(x))/x$	$xy\frac{\partial}{\partial y}$	$(y'^2+y^2F(x,(xy'/y)-\ln y))/y$	$xy\frac{\partial}{\partial x}$
$P(x)y+Q(x)$	$[\exp(\int P(x)dx)]\frac{\partial}{\partial y}$	$\frac{y'^3}{x^3}F(y/x,(y-xy')/y')$	$xy\frac{\partial}{\partial x}+y^2\frac{\partial}{\partial y}$
$P(x)y+Q(x)y^n$	$y^n[\exp((1-n)\int P(x)dx)]\frac{\partial}{\partial y}$	$\frac{F(x^{-k}y,xy'-ky)-(1-k)x^{k+1}y'}{x^{k+2}}$	$x^k\left(x\frac{\partial}{\partial x}+ky\frac{\partial}{\partial y}\right)$
$P(x)y$	$y\frac{\partial}{\partial y}$	$\frac{y'^3F(xy^{-k},y/y'-kx)-(k-1)xyy'^2}{xy^2}$	$y^k\left(kx\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)$

On equating the coefficients of ϵ, we get (when equation (2.1) holds)

$$\eta^{(n)}=\xi w_x+\eta w_y+\eta^{(1)}w_{y'}+\cdots+\eta^{(n-1)}w_{y^{(n-1)}}. \tag{2.22}$$

$\eta^{(k)}$ are calculated recursively from equation (2.2), as follows. For $k=1$, we obtain

$$y^{*(1)} = \frac{D_x y^*}{D_x x^*} = \frac{y' + D_x\eta + O(\epsilon^2)}{1+\epsilon D_x \xi + O(\epsilon^2)} = (y' + D_x\eta + O(\epsilon^2))(1+\epsilon D_x\xi + O(\epsilon^2))^{-1} = y' + \epsilon(D_x\eta - y' D_x\xi) + O(\epsilon^2)$$

from equation (2.21) on comparing coefficients of ϵ, we get

$$\eta^{(1)} = D_x\eta - y' D_x\xi \tag{2.23}$$

Similarly, we have

$$y^{*(k)} = \frac{y^{(k)} + \epsilon D_x \eta^{(k-1)} + O(\epsilon^2)}{1+\epsilon D_x\eta + O(\epsilon^2)}$$

hence,

Definition 21. For an ordinary differential equation of order n, the k^{th} prolongation formula is generated by

$$\eta^{(k)}(x, y, y', \cdots, y^{(k)}) = D_x \eta^{(k-1)} - y^{(k)} D_x \eta,$$

for $k = 1, \cdots, n$ and $\eta^0 = \eta$.

To deal with the action of Lie symmetries on derivatives of order n or smaller, we introduce the *prolonged infinitesimal generator*

$$X^{(n)} = \xi\partial_x + \eta\partial_y + \eta^{(1)}\partial_{y'} + \cdots + \eta^{(n)}\partial_{y^{(n)}} \tag{2.24}$$

We can use the prolonged infinitesimal generator to write the linearized symmetry condition (2.22)(when (2.1) holds) in the following form

$$X^{(n)}\left(y^{(n)} - w\left(x, y, y', \cdots, y^{(n-1)}\right)\right) = 0 \tag{2.25}$$

2.6 Lie's integrating factor

Integrating factors used as standard solution method can be generalized using the method of symmetries.

Theorem 13. Consider a first order ordinary differential equation of the form

$$M(x,y)dx + N(x,y)dy = 0. \tag{2.26}$$

If (2.26) have the infinitesimal generator $X(x,y) = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}$ and if $\xi M + \eta N \neq 0$, then the *Lie's integrating factor* (as it is called) of (2.26) is given by

$$\mu(x,y) = \frac{1}{\xi(x,y)M(x,y) + \eta(x,y)N(x,y)} \tag{2.27}$$

Proof. On rewriting (2.26) as follows,

$$y' := \frac{dy}{dx} = -\frac{M(x,y)}{N(x,y)}$$

We use the prolonged infinitesimal generator from (2.24), given by

$$X^{(1)} = \xi\partial_x + \eta\partial_y + \eta^{(1)}\partial_{y'}.$$

From (2.23), on substituting $y' = -M(x,y)/N(x,y)$, we get

$$\begin{aligned} \eta^{(1)} = D_x\eta - y' D_x\xi &= \eta_x + y'\eta_y - y'\xi_x - \left(y'^2\right)\xi_y \\ &= \eta_x - \left(\frac{M(x,y)}{N(x,y)}\right)\eta_y + \left(\frac{M(x,y)}{N(x,y)}\right)\xi_x - \left(\frac{M(x,y)^2}{N(x,y)^2}\right)\xi_y \end{aligned} \tag{2.28}$$

The linearized symmetry condition from (2.25) becomes,

$$X^{(1)}\left(y' + \frac{M(x,y)}{N(x,y)}\right) = 0. \tag{2.29}$$

From (2.29) and (2.28), we obtain (for now we drop (x, y) for sanitary reasons),

$$
\begin{aligned}
(\xi\partial_x + \eta\partial_y + \eta^{(1)}\partial_{y'})\left(y' + \frac{M}{N}\right) &= 0 \\
\xi\partial_x\frac{M}{N} + \eta\partial_y\frac{M}{N} + \eta^{(1)} &= 0 \\
\xi\frac{N\partial_x M - M\partial_x N}{N^2} + \eta\frac{N\partial_y M - M\partial_y N}{N^2} + \eta_x - \left(\frac{M}{N}\right)\eta_y + \left(\frac{M}{N}\right)\xi_x - \left(\frac{M^2}{N^2}\right)\xi_y &= 0 \\
N(\xi M_x + \eta M_y) - M(\eta N_x + \eta N_y) + \eta_x N^2 - \xi_y M^2 - (\eta_y - \xi_x)MN &= 0
\end{aligned}
\tag{2.30}
$$

For $\mu(x, y)$ to be an integrating factor of (2.26), we must have that the following differential equation

$$\mu M dx + \mu N dy = 0$$

is exact. Hence, we must have

$$\frac{\partial}{\partial y}(\mu M) = \frac{\partial}{\partial x}(\mu N). \tag{2.31}$$

We substitute the proposed μ onto the above equation to get,

$$
\begin{aligned}
\frac{\partial}{\partial y}\left(\frac{M}{\xi M + \eta N}\right) &= \frac{\partial}{\partial x}\left(\frac{N}{\xi M + \eta N}\right) \\
(\xi M + \eta N)M_y - M(M\xi_y + \xi M_y + \eta N_y + N\eta_y) &= (\xi M + \eta N)N_x - N(M\xi_x + \xi M_x + \eta N_x + N\eta_x) \\
N(\xi M_x + \eta M_y) - M(\eta N_x + \eta N_y) + \eta_x N^2 - \xi_y M^2 - (\eta_y - \xi_x)MN &= 0.
\end{aligned}
\tag{2.32}
$$

where the last equation is exactly the last equation of (2.30). Hence, the proposed

$$\mu(x, y) = \frac{1}{\xi(x, y)M(x, y) + \eta(x, y)N(x, y)}$$

is an integrating factor of (2.26). □

2.7 Reduction of order using canonical coordinates

Suppose that X is an infinitesimal generator of a one-parameter Lie group of symmetries of the ODE,

$$y^{(n)} = w(x, y, y', \cdots, y^{(n-1)}), \qquad n \geq 2 \tag{2.33}$$

where, w is smooth over all its arguments. Let $(r(x, y), s(x, y))$ be corresponding canonical coordinates, so that

$$X = \partial_s.$$

On writing (2.33) in terms of canonical coordinates, we get

$$s^{(n)} = \Omega(r, s, s', \cdots, s^{(n-1)}), \qquad s^{(k)} = \frac{d^k s}{dr^k} \tag{2.34}$$

for some function Ω. As (2.34) is invariant under the group of translations in the s-direction, from the symmetry conditin we have that

$$\Omega_s = 0$$

. Hence, (2.34) becomes

$$s^{(n)} = \Omega(r, s', \cdots, s^{(n-1)}).$$

On letting $\nu := ds/dr$, the above equation changes to the following

$$\nu^{(n-1)} = \Omega(r, \nu, \cdots, \nu^{(n-2)}) \qquad \nu^{(k)} = \frac{d^{k+1}s}{dr^{k+1}} \tag{2.35}$$

So, by writing (2.33) in terms of canonical coordinates, we have reduced the order (by 1) to $n - 1$ as in 2.35.

Chapter 3

Lie algebras

Let $\mathcal{L}_R$ denote the set of all infinitesimal generators of one-parameter Lie groups of point symmetries of an ODE of order $n \geq 2$. The linearized symmetry condition is linear in ξ and η, and so

$$X_1, X_2 \in \mathcal{L}_R \implies c_1X_1 + c_2X_2 \in \mathcal{L}_R$$

Therefore $\mathcal{L}$ is a vector space. The dimension, R, of this vector space is the number of arbitrary constants that appear in the general solution of the linearized symmetry condition. Every $X \in \mathcal{L}_R$ can be written as

$$X = \sum_{i=1}^{R} c_i X_i, \qquad c_i \in \mathbb{R}$$

where, $\{X_1, X_2, \cdots, X_R\}$ form a basis for $\mathcal{L}_R$ of dimension R.

Definition 22 (Lie bracket)**.** Consider two infinitesimal generators

$$X_1 = \xi_1(x,y)\frac{\partial}{\partial x} + \eta_1(x,y)\frac{\partial}{\partial y}, \qquad X_2 = \xi_2(x,y)\frac{\partial}{\partial x} + \eta_2(x,y)\frac{\partial}{\partial y}.$$

Then the (commutator) *Lie bracket* is defined as

$$[X_1, X_2] = X_1X_2 - X_2X_1$$

equivalently,

$$[X_1, X_2] = (X_1\eta_2 - X_2\eta_1)\frac{\partial}{\partial x} + (X_1\eta_2 - X_2\eta_1)\frac{\partial}{\partial y}$$

Remark. Lie bracket is bilinear, skew-symmetric and satisfies the following Jacobi identity

$$[X_1, [X_2, X_3]] + [X_2, [X_3, X_1]] + [X_3, [X_1, X_2]] = 0.$$

Definition 23 (Lie algebra)**.** The vector space $\mathcal{L}_R$ is called a *Lie algebra* if the Lie bracket $[X, Y] \in \mathcal{L}_R$ when $X, Y \in \mathcal{L}_R$.

Remark (1)**.** Lie in 1893 proved that the dimension R of the symmetry algebra of an ordinary differential equation of order n satisfies

$$R \leq n + 4, \text{ for } n > 2, \qquad R \leq 8, \text{ for } n = 2$$

For the order n, the maximal dimension of the symmetry algebra is reached by the equation $y^{(n)} = 0$

Example 13. The second order ordinary differential equation $y'' = 0$ has a tangent vector field $(\xi(x,y), \eta(x,y))$, where

$$\xi(x,y) = c_1 + c_3x + c_5y + c_7x^2 + c_8xy$$
$$\eta(x,y) = c_2 + c_4y + c_6x + c_7xy + c_8y^2$$

where, $c_i \in \mathbb{R}$. Hence, the infinitesimal generator takes the form

$$X = \sum_{i=1}^{8} c_i X_i$$

where,

$$X_1 = \partial_x,\ X_2 = \partial_y,\ X_3 = x\partial_x,\ X_4 = y\partial_y,\ X_5 = y\partial_x,\ X_6 = x\partial_x,\ X_7 = x^2\partial_x + xy\partial_y,\ X_8 = xy\partial_x + y^2\partial_y$$

Remark (2)**.** Mahmod and Leach in 1988 showed that, that the number of point symmetries which a second order equation can possess is exactly one of 0, 1, 2, 3, or 8 and if the higher order ($n \geq 3$) equation is linear, then $R \in \{n+1, n+2, n+4\}$

As of now, the proofs of the remarks are beyond our scope.

3.1 Examples of some groups

The following Table 3.1 has some second order equations with known Lie groups (basis is given).

Table 3.1: Some Lie groups

y''	Basis
$f(y, y')$	$G_1 :\ X_1 = \frac{\partial}{\partial x}$
$f(y')$	$G_2 :\ X_1 = \frac{\partial}{\partial x},\ X_2 = \frac{\partial}{\partial y}$
$(1/x)f(y')$	$G_2 :\ X_1 = \frac{\partial}{\partial y},\ X_2 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$
$2\,\dfrac{y' + Cy'^{3/2} + y'^2}{y - x}$	$G_3 :\ X_1 = \frac{\partial}{\partial x} + \frac{\partial}{\partial y},\ X_2 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y},\ X_3 = x^2\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}$
Cy'^{-3}	$G_3 :\ X_1 = \frac{\partial}{\partial x},\ X_2 = 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y},\ X_3 = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}$
$Ce^{-y'}$	$G_3 :\ X_1 = \frac{\partial}{\partial x},\ X_2 = \frac{\partial}{\partial y},\ X_3 = x\frac{\partial}{\partial x} + (x + y)\frac{\partial}{\partial y}$
$Cy'^{(k-2)/(k-1)}$	$G_3 :\ X_1 = \frac{\partial}{\partial x},\ X_2 = \frac{\partial}{\partial y},\ X_3 = x\frac{\partial}{\partial x} + ky\frac{\partial}{\partial y}$

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