Université Libre de Bruxelles

Summary

Aerodynamics: Typical Questions
MECA-Y402

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Appel à contribution

Synthèse Open Source

Ce document est grandement inspiré de l'excellent cours donné par Herman Deconinck et Chris Lacor à l'EPB (École Polytechnique de Bruxelles), faculté de l'ULB (Université Libre de Bruxelles). Il est écrit par les auteurs susnommés avec l'aide de tous les autres étudiants et votre aide est la bienvenue ! En effet, il y a toujours moyen de l'améliorer surtout que si le cours change, la synthèse doit être changée en conséquence. On peut retrouver le code source à l'adresse suivante

`https://github.com/nenglebert/Syntheses`

Pour contribuer à cette synthèse, il vous suffira de créer un compte sur *Github.com*. De légères modifications (petites coquilles, orthographe, ...) peuvent directement être faites sur le site ! Vous avez vu une petite faute ? Si oui, la corriger de cette façon ne prendra que quelques secondes, une bonne raison de le faire !

Pour de plus longues modifications, il est intéressant de disposer des fichiers : il vous faudra pour cela installer LaTeX, mais aussi *git*. Si cela pose problème, nous sommes évidemment ouverts à des contributeurs envoyant leur changement par mail ou n'importe quel autre moyen.

Le lien donné ci-dessus contient aussi un `README` contenant de plus amples informations, vous êtes invités à le lire si vous voulez faire avancer ce projet !

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1 Explain how lift can generated around an airfoil for inviscid incompressible flow.

See Q2 for mathematical explanation.

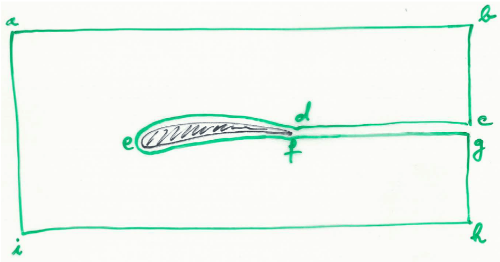

Figure 1

When looking at an airfoil, we have a pressure that is applied around it. This pressure is non uniform, due to the presence of the airfoil. The integral of pressures over the surface gives a force expressed as the product of the incoming speed and the vorticity vector: the lift.

1.1 Explain the origin of the bound vortex which exists around a profile which generates lift. Explain using the Kelvin theorem

In inviscid case, the Kelvin theorem states that there cannot be vorticity, so no lift. If we take an arbitrary contour around the airfoil we will have no circultion. In inviscid case we can never get a lift → D'alembert paradox. At the trailing edge, if the flow wants to continue on the other corner from below, the velocity must be infinity so that the flow separates. But this is not the case in reality.

To satisfy the Kutta condition (the flow has to leave the airfoil smoothly), there needs to be circulation if we take a contour that contains the airfoil, but for all contour that does not contain the airfoil it is null. Γ varies with the stagnation point position, but only one corresponds to the Kutta condition.

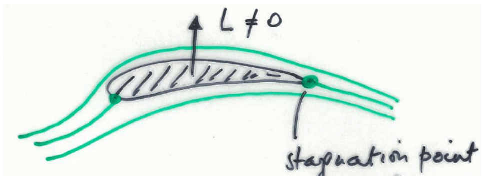

Figure 2

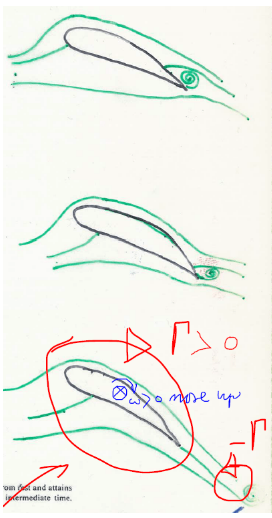

Figure 3

The bound vortex is the vortex around the airfoil. It is compensated by a starting vortex that detaches from the airfoil.
We can show that every contour containing the airfoil has a non 0 circulation. Let's proof that a contour that doesn't contain the airfowl has $\Gamma = 0$:

$$\oint_C \vec{v}\, d\vec{l} = \oint_{\text{airfoil}} \vec{v}\, d\vec{l} + \int_{cd} \vec{v}\, d\vec{l} + \int_{fg} \vec{v}\, d\vec{l} = 0. \tag{1}$$

As the contour elements are exactly opposed to each other, the result is null.

1.2 What is the start-up vortex, what happens with it when the flow reaches a steady state

What happens is that initially we have the first kind of flow, then the formation of the starting vortex due to viscous effects (separation) which is compensated by a **bound vortex** around the airfoil (to respect Kelvin theorem of irrotational flow) that makes $\Gamma \neq 0$. Then the vortex goes away to infinity. Indeed if we take $R = \rho v_\infty \Gamma$, $\Gamma \neq 0$, so we have lift.

1.3 Explain the Kutta condition

The Kutta condition states that the flow can only have a stagnation point at the trailing edge. If this was not the case, the underwing flow would need to accelerate "going backwards" to meet the upperwing flow, see figure 2.

2 Derive an expression for the aerodynamic lift of on airfoil for inviscid incompressible flow (i.e. neglecting viscous effects).

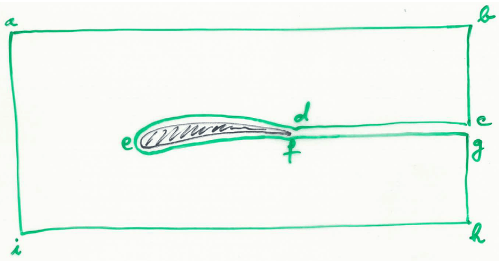

Figure 4

The fundamental integral form of the mass conservation equation is:

$$\frac{d}{dt}\int_V \rho\, dV + \oint_S \rho\vec{v}\, d\vec{S} = 0. \tag{2}$$

By applying Gauss theorem $\oint_S \vec{a}.\vec{n}\, dS = \int_V \nabla\vec{a}\, dV$:

$$\int_V \left[\frac{d\rho}{dt} + \nabla.(\rho\vec{v})\right] dV = 0. \tag{3}$$

True for all volumes:

Continuity equation

$$\frac{d\rho}{dt} + \nabla.(\rho\vec{u}) = 0 \tag{4}$$

General form of the momentum equation is:

$$\rho\dot{\vec{v}} = \frac{\partial\rho\vec{v}}{\partial t} + \rho(\vec{v}\nabla)\vec{v} = -\nabla p + \nabla\bar{\bar{\tau}}. \tag{5}$$

Steady state, time derivative goes away. If we consider the x component of the velocity, we can expend the derivative to the whole left term as:

$$\rho(\vec{v}\nabla)v_x = \nabla(\rho\vec{v}v_x) - v_x\cancel{\nabla(\rho\vec{v})} \tag{6}$$

where the last term is null related to (4) in steady state. Integrating both sides around the volume contained in the closed surface S (abcdefghi on figure) in (6), and applying Gauss theorem, we obtain:

$$\oint_S \vec{v}(\rho\vec{v}\vec{n})\, dS = -\oint_S p\, d\vec{S} + \oint_S \bar{\bar{\tau}}\, d\vec{S}. \tag{7}$$

New closed contour $S* = S - \text{airfoil} - cd - fg$ (previous abhi in fact). (7) becomes:

$$\begin{aligned}&\oint_{S^*} \vec{v}(\rho\vec{v}\, d\vec{S}) + \cancel{\oint_{\text{airfoil}} \vec{v}(\rho\vec{v}\, d\vec{S})} + \cancel{\oint_{cd+fg} \vec{v}(\rho\vec{v}\, d\vec{S})}\\ &= -\oint_{S^*} p\, d\vec{S} - \oint_{\text{airfoil}} p\, d\vec{S} - \cancel{\oint_{cd+fg} p\, d\vec{S}} + \oint_{S^*} \bar{\bar{\tau}}\, d\vec{S} + \oint_{\text{airfoil}} \bar{\bar{\tau}}\, d\vec{S} + \cancel{\oint_{cd+fg} \bar{\bar{\tau}}\, d\vec{S}}\end{aligned} \tag{8}$$

Forces applied to a wing:

$$\vec{R} = \oint_{\text{airfoil}} p\, d\vec{S} - \oint_{\text{airfoil}} \bar{\bar{\tau}}\, d\vec{S} \tag{9}$$

$$\oint_{S^*} \vec{v}(\rho \vec{v}\, d\vec{S}) = -\oint_{S^*} p\, d\vec{S} + \cancel{\oint_{S^*} \bar{\bar{\tau}}\, d\vec{S}} - \vec{R}. \tag{10}$$

Pressure effects induced by the body remains at a long distance from the body. We have to analyse the **non uniform** p along S*. In order to apply Bernouilli equation $p + \frac{1}{2}\rho v^2 = cst$, let's add the constants p_∞ and v_∞ to (10), as $\oint p_\infty\, d\vec{S} = p_\infty \oint d\vec{s} = 0$:

$$\vec{R} = -\oint_{S^*} (p - p_\infty)\, d\vec{S} - \oint_{S^*} (\vec{v} - \vec{v}_\infty)\, d\dot{m} \tag{11}$$

Let's express $\vec{v} = \vec{v}_\infty + \vec{\delta}_c$ with $\vec{\delta}_c$ a perturbation. Introducing this in Bernouilli equation:

$$\begin{gathered} p_\infty + \frac{1}{2}\rho \cancel{\vec{v}_\infty^2} = p + \frac{1}{2}\rho(\vec{v}_\infty + \vec{\delta}_c)^2 = p + \frac{1}{2}\rho(\cancel{\vec{v}_\infty^2} + 2\vec{v}_\infty\vec{\delta}_c + \cancel{\vec{\delta}_c^2}) \\ \Rightarrow p - p_\infty = -\rho\vec{v}_\infty\vec{\delta}_c. \end{gathered} \tag{12}$$

If we replace this result in (11), we find:

$$\vec{R} = \oint_{S^*} \rho[(\vec{v}_\infty\vec{\delta}_c)\, d\vec{S} - \vec{\delta}_c[(\vec{v}_\infty.d\vec{S})] \tag{13}$$

by using a vector property $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}\vec{b})\vec{c} - (\vec{a}\vec{c})\vec{b}$:

$$= \rho\vec{v}_\infty \times \oint_{S^*} \vec{\delta}_c \times d\vec{S} = \rho\vec{v}_\infty \times \left[\oint_{S^*} \vec{v} \times d\vec{S} - \cancel{\oint_{S^*} \vec{v}_\infty \times d\vec{S}}\right] \tag{14}$$

and by applying Stokes theorem $\oint_S \vec{a} \times d\vec{S} = \int_V \nabla \times \vec{a}\, dV$:

$$= \rho\vec{v}_\infty \times \int (\nabla \times \vec{v})\, dV = \rho\vec{v}_\infty \times \int \vec{\omega}\, dV \tag{15}$$

where $\vec{w}$ is the **vorticity vector** of direction $\vec{1}_z$ (pointing in the paper):

$$\vec{\omega} = \begin{vmatrix} \vec{1}_x & \vec{1}_y & \vec{1}_z \\ \partial_x & \partial_y & 0 \\ v_x & v_y & 0 \end{vmatrix} = [\partial_x v_y - \partial_y v_x]\vec{1}_z \tag{16}$$

This shows that the lift force is always perpendicular to the flow!

2.1 Express conservation of momentum over a closed surface S far away from the wing, neglecting viscous forces

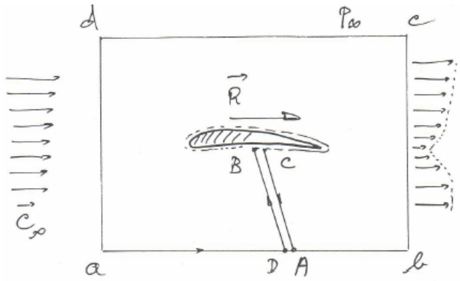

Figure 5

By considering this (assumption of far field), we can compute the force only by knowing the far field parameters. Indeed, uniform pressure implies null surface integral, so that (10) becomes:

$$\vec{R} = -\oint_{S^*} \vec{v}(\rho\vec{v}\, d\vec{S}). \tag{17}$$

The velocity term remains, as by experience we know that there is a **wake** making the velocity profile non-uniform. Let's now consider that the velocity is horizontal so that $\vec{R} = R.\vec{1}_x$, at the inlet we have $\vec{v}$ and $\vec{n}$ are opposed while at the outlet they are in the same direction:

$$\vec{R} = \int_a^d \vec{v}\, d\dot{m} - \int_b^e \vec{v}\, d\dot{m} > 0 \tag{18}$$

showing that there is only **drag** force.

2.2 Show that the lift force is linked to the vorticity through the closed surface S

See above.

2.3 Show that for the 2D airfoil the lift force is linked to the circulation around the airfoil, derive the Kutta-Joukowski formula for the lift generated by a 2D profile

We will now introduce the circulation $\Gamma = -\oint \vec{v}\, d\vec{l} > 0$ around a body. The convention is to take the anticlockwise direction for $d\vec{l}$ and so for Γ to be > 0 we must have $\vec{v}$ in the clockwise direction. There is a link between the lift force and the circulation. Let's introduce **Stokes theorem**:

$$\oint \vec{a} d\vec{l} = \int_S (\nabla \times \vec{a})\, d\vec{S} \qquad \Rightarrow -\Gamma = \int_S \vec{\omega} d\vec{S}. \tag{19}$$

We remember that:

$$\begin{aligned} &\vec{R} = \rho \vec{v}_\infty \times \int \vec{\omega}\, dV = \rho \vec{v}_\infty \times \int l\vec{\omega}\, dS \qquad \Leftrightarrow \frac{\vec{R}}{l} = \rho \vec{v}_\infty \times \int \vec{\omega}\, dS \\ &\frac{\vec{R}}{l} = \rho \vec{v}_\infty \times \int \vec{\omega}\, (d\vec{S}.\vec{1}_z) = \rho \vec{v}_\infty \times (-\Gamma)\vec{1}_z = \rho v_\infty \Gamma\, \vec{1}_y \end{aligned} \tag{20}$$

to finally obtain a very good approximation of the lift:

Kutta formula for lift 2D airfoil

$$|R| = \rho v_\infty \Gamma \tag{21}$$

3 Characteristics of a 2D airfoil:

3.1 Define lift and drag, give the principle components of lift and drag force in normal operation and in case of separation

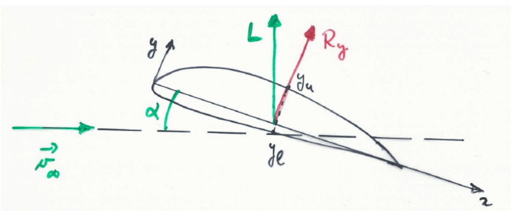

Figure 6

Force applied on the wing:

$$\vec{R} = -\oint p\, d\vec{S} + \oint \bar{\bar{\tau}}\, d\vec{S} \tag{22}$$

with an external normal to the airfoil. The angle of attack is represented on Figure 6.
The pressure term is responsible for lift and the friction term is responsible for drag. Friction forces work tangential to the airfoil and the pressure forces are perpendicular, if there is **no separation** in the flow. The drag created by the stress is called the **skin** or **friction** drag. Note that in a subsonic inviscid incompressible flow, we have the paradox of d'Alembert because we

have no drag. This shows that the pressure only contributes to lift.

Separation: region above the airfoil where $p - p_\infty \approx 0 \Rightarrow$high pressure below $p \gg p_\infty$ that slows down the wing. This implies that the applied force is higher than the case without separation and due to the attack angle, the drag force too. This phenomenon is called **pressure drag** (form drag), and here the pressure contributes to drag.

3.2 Explain the non-dimensional force coefficients(C_l, C_d, C_m) and which are the similarity parameters influencing these coefficients

Let's look to the non-dimensional parameters that will influence the lift, the drag and the momentum. We have to define some reference quantities:

$$\begin{array}{llll} L_{ref} = C & v_{ref} = v_\infty & t_{ref} = L_{ref}/v_{ref} & \rho_{ref} = \rho_\infty \\ t' = t/t_{ref} & p_{ref} = \rho_{ref}\frac{v_{ref}^2}{2} & \text{Mach} = V_{ref}/a_{ref} & a_{ref} = \gamma\pi T_{ref} \\ \gamma = c_p/c_v & & Re_{ref} = \frac{\rho_{ref} v_{ref} L_{ref}}{\mu_{ref}} & \end{array} \tag{23}$$

where a is the speed of sound. By replacing all these in the mass, momentum and energy equations, we obtain the non-dimensional ones (see Fluid Mechanics II):

$$\begin{aligned} &\bullet \frac{\partial \rho'}{\partial t'} + \nabla\left(\rho'\vec{v}'\right) = 0 \\ &\bullet \rho'\frac{d\vec{v}'}{dt'} = -\frac{1}{\gamma M_{ref}^2}\nabla p' + \frac{1}{\mathrm{Re}_{ref}}\nabla \bar{\bar{\tau}}' \\ &\bullet \frac{d}{dt'}(\rho' e') + \frac{\gamma(\gamma-1)}{2}M_{ref}^2\frac{d}{dt'}(\rho'\vec{v}'^2) \\ &= \frac{\gamma}{Pr_{ref}Re_{ref}}\nabla(k'\nabla T') - (\gamma-1)\nabla(p'\vec{v}') + \gamma(\gamma-1)\frac{M_{ref}^2}{Re_{ref}}\nabla(\bar{\bar{\tau}}'\vec{v}') \end{aligned} \tag{24}$$

We can see that a solution can only be function of 4 parameters: $M, Re, Pr = \frac{c_p\mu}{k}, \gamma$, but we know that the geometry and the angle of attack α have a role by means of the boundary conditions. Then, we assume that the fluid is air ($\gamma = 1.4$) and that we can neglect heat effects (no influence of Pr, incompressible and so low speed flows). The non-dimensional lift, drag and moment are thus function of M, Re, geometry and α. We can define **lift**, **drag** and **moment coefficient** as (we forget about compressibility $\to$ M, and Re effects are low for $C_L and C_M$):

$$\begin{aligned} C_L(\cancel{M}, \cancel{Re}, geometry, \alpha) &= \frac{L}{\frac{1}{2}\rho_{ref}v_{ref}^2 S} \\ C_D(\cancel{M}, Re, geometry, \alpha) &= \frac{D}{\frac{1}{2}\rho_{ref}v_{ref}^2 S} \\ C_M(\cancel{M}, \cancel{Re}, geometry, \alpha) &= \frac{M}{\frac{1}{2}\rho_{ref}v_{ref}^2 Sc} \end{aligned} \tag{25}$$

where L, D, M are the **dimensional** forces, c the mean chord (S/b) and S a reference surface (3D wing $\to$ total wing surface, 2D wing $\to$ S = c). We can experimentally show that the lift increases mainly linearly with α and the drag force is caused by friction effects and pressure differences involving with α. This gives the following equations (lower case for 2D):

$$c_l = m(\alpha - \alpha_{L_0}) \qquad c_d = c_{d_0} + kc_l^2 \tag{26}$$

where $m \approx 2\pi$ theoretically and 5.7 practically, k is a constant of order of magnitude 0.01.

3.3 Explain:

center of pressure CP: It's the x value on the chord where the carrier of the force $\vec{R}$ intersects the chord. It's function of the angle of attack. Indeed, if alpha increases, the suction peak will be higher, this induces that the center of pressure move forward (participation of the forward pressure more important).

Note that the center of pressure is not a fixed point. Indeed, it varies with the angle of attack: if $\alpha \nearrow$, the pressure peak on the LE is more important making the x_p move upstream, and the contrary for $\alpha \searrow$. This notion will be completed by the **zero lift angle** α_0.
equivalent force system in an arbitrary point Q on the chord of the profile:

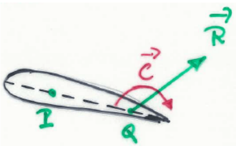

Figure 7

The force at the pressure center P is equivalent to another force in point Q, but by adding the moment to compensate the one added by moving the force. This moment is:

$$\vec{C_Q} = -\vec{PQ} \times \vec{R}. \tag{27}$$

aerodynamic center AC: Suppose that there is a point Q where the couple C_Q is independent of the angle of attack (because the pressure center changes with alpha). This point is called the aerodynamic center.
The moment at this point is always nose-down and the point is situated upstream to the center of pressure.
the graph of the momentum coefficient c_m at a point Q versus the lift coefficient c_l for Q located at the trailing edge, at the leading edge and at the aerodynamic center of the profile: It is shown experimentally that:

$$c_m(Q) = c_{m_0} + kc_l \tag{28}$$

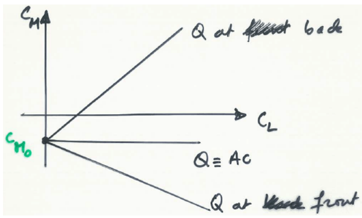

Figure 8

k is a constant that is related to the reference point chosen. If Q is taken on the LE for example, increasing α will produce an increase of the lift and make the center of pressure move upstream. The L increase will compensate the moving x_p such that the moment becomes even more noose-down (more negative following $\vec{1}_z$) $\Rightarrow k < 0$ for a decrease in (28). The same reasoning applied on the trailing edge gives $k > 0$.

3.4 Compute the Location of the center of pressure CP as a function of the angle of attack alpha

$$\begin{array}{llll} 1) & c_m = c_{m_0} + kc_l & 2) & M_{ac} = (x_{ac} - x_{cp})N \\ 3) & m_{ac} = M_{AC} = M_0 < 0 & 4) & N = n(\alpha - \alpha_0) \end{array} \tag{29}$$

The AC being always upstream the CP the difference in 2) is < 0. In 4), $n > 0$. By using equation 3,4 and 2, we can compute:

$$M_0 = -(x_{cp} - x_{ac}).n(\alpha - \alpha_0) \quad \Leftrightarrow \quad -\frac{M_0}{n} = (x_{cp} - x_{ac}).(\alpha - \alpha_0) \tag{30}$$

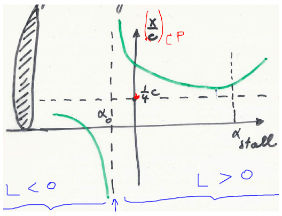

Figure 9

This is the equation of an **hyperbola**. To see it, we only have to compute the limits of:

$$x_{cp} = x_{ac} - \frac{M_0}{n}\frac{1}{\alpha-\alpha_0}$$

$$\lim_{\alpha\to\pm\infty} x_{cp} = x_{ac} \qquad \lim_{\alpha\to\alpha_0>0} x_{cp} = +\infty \qquad \lim_{\alpha\to\alpha_0<0} x_{cp} = -\infty \tag{31}$$

Let's finally say that commonly, $x_{ac} = cst \approx \frac{1}{4}C$.

3.5 Explain the lift, drag and moment curves as a function of the angle of attack alpha

Starting from what is said in 3.2, experiment showed that lift is increasing linearily with α, and drag goes proportionnaly to c_l^2. The moment is proportionnal to c_l, we thus expect it to be linear with respect to α.

3.6 Explain what is stall and critical angle of attack. Explain different types of stall.

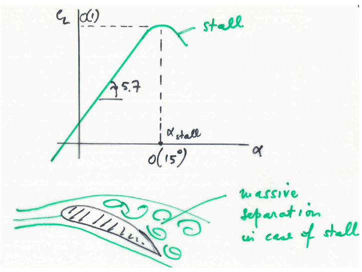

Figure 10

At a certain angle of attack ($\approx 15^\circ$), the lift suddenly drops. This is due to massive separation on the suction side (reverse pressure gradient too high) and happens at the **critical angle of attack**. This phenomenon is called **stall**. In the separated part, the pressure will no longer decrease and will form a pressure plateau.

We have to make the difference between leading-edge stall and trailing-edge stall. For **leading-edge stall**, the massive separation occurs suddenly near the LE resulting in a strong and sudden drop of lift, when at maximum lift. This especially occurs to thin airfoils with cross-sections between 10 and 16% of the chord. For the **trailing-edge stall**, the point of separation gradually goes upstream with increasing angle of attack resulting in a more gradual drop of lift (more thick airfoils). The comparison is done on the right figure. We can also see a third type of stall called **thin airfoil stall** with the example of a flat plat.

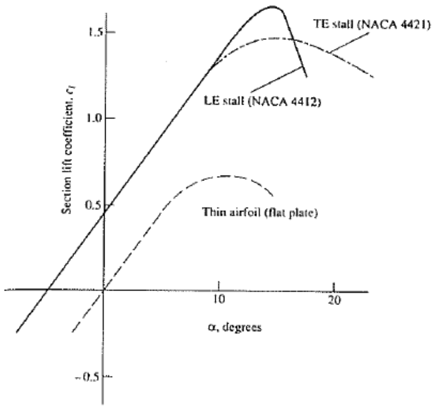

Figure 11

In conclusion, the LE must be sufficiently rounded to have a good maximum lift. In fact the profile may nor be too thick nor too thin. The figure on the left shows the influence of the thickness on the lift. We notice that the optimum thickness is situated around 12% of the chord.The maximum lift increases with RE, indeed higher the Re, higher is the ratio of speed versus viscosity. So we can better oppose to separation. Unlike the Re number, the roughness has great effects on the maximum lift. Finally, let's notice that the camber have also an effect on maximum lift, the best is a camber of 8 up to 10%.

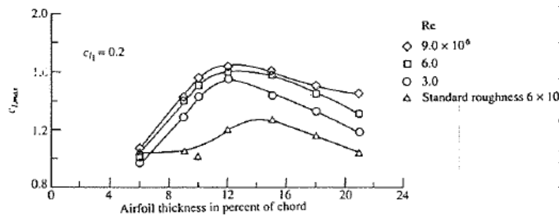

Figure 12

$$L = C_L \frac{1}{2} \rho_{ref} v_{ref}^2 S. \tag{32}$$

The lift force must always at least be equal to the weight of the plane. This implies that for low speed (take-off and landing), the C_L must be large. This is accomplished with large α and slats or flaps. The minimum speed where the lift can still balance the weight (C_L maximum) is called **stall speed** and from (32) we find:

$$v_{stall} = \sqrt{\frac{W}{C_{L_{max}} \frac{1}{2} \rho_{ref} S}} \tag{33}$$

3.7 Discuss the polar curve of the airfoil, i.e. lift coefficient as a function of drag coefficient, show the glide ratio on this plot

The curve that represents C_L in function of C_D is the **polar curve** of the wing. The ratio $\frac{C_L}{C_D}$ is the **glide ratio** or **finesse** and is like an efficiency parameter.

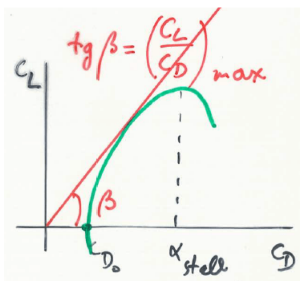

Figure 13

3.8 What is the importance of the glide ratio, discuss the case of a gliding plane (engine off situation)

The best parameter is obtained using the graph by calculating β such that:

$$\tan \beta = \left(\frac{C_L}{C_D} \right)_{max} \tag{34}$$

This point is important for the quality of the wing because if we plot the thrust, the lift, the drag and the weight of a plane describing a horizontal flight (Figure 14), the thrust is given by:

$$T = \frac{L}{\tan\beta} = \frac{W}{\tan\beta} \tag{35}$$

where we see that when $\tan\beta$ (so the glide ratio) increases, T decreases. Another interpretation can be given when we have no thrust (Figure 15). In this case the gliding ratio has to be adapted to travel the larger distance knowing that:

$$\frac{C_L}{C_D} = \frac{\text{distance travelled}}{\text{height loss}} \tag{36}$$

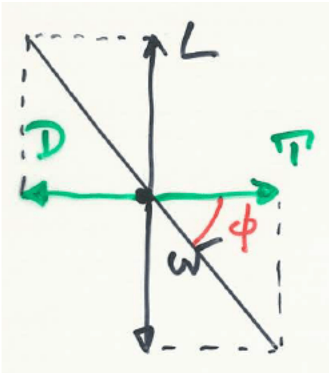

Figure 14

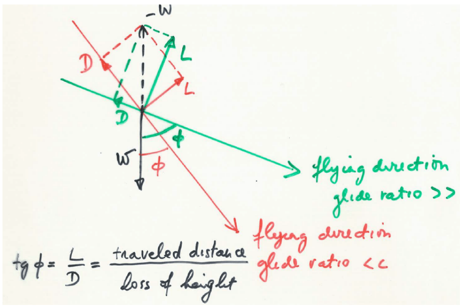

Figure 15

4 Computation of inviscid irrotational flow around a 2D airfoil using conformal mapping - Joukowski profiles

4.1 Explain methods based on complex potential function, explain the connection with usual stream function and potential function

We will begin here with steady, inviscid irrotational flows. This gives for the mass conservation equation:

$$\frac{\partial\rho}{\partial t} + \nabla(\rho\vec{v}) = 0 \qquad \Rightarrow \nabla\vec{v} = 0 = \partial_x u + \partial_y v \tag{37}$$

In the other hand, we have the assumption of irrotational flow:

$$\vec{\omega} = 0 \qquad \Rightarrow \partial_x v - \partial_y u = 0. \tag{38}$$

Then we define the **complex potential function w**:

$$w = \phi + I\psi \tag{39}$$

where ϕ is the **potential function** (satisfies $w = 0$ by construction) such that:

$$\begin{cases} u = \partial_x\phi \\ v = \partial_y\phi \end{cases} \qquad \nabla\phi = \vec{v} = \partial_x\phi\vec{1}_x + \partial_y\phi\vec{1}_y \tag{40}$$

We must satisfy the mass conservation equation:

$$\nabla(\nabla\phi) = 0 \qquad \Rightarrow \Delta\phi = 0 \tag{41}$$

coupled with boundary conditions, we can find a solution $\phi(x,y)$. The **stream function** satisfies the mass conservation by construction:

$$\begin{cases} u = \partial_y \psi \\ v = -\partial_x \psi \end{cases} \Rightarrow \partial_x u + \partial_y v = 0 \Leftrightarrow \partial_x(\partial_y \psi) + \partial_y(-\partial_x \psi) = 0 \tag{42}$$

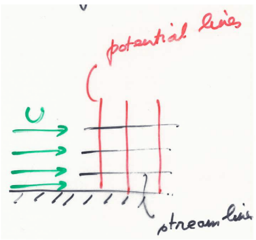

Figure 16

We still have to verify the $\omega = 0$ condition:

$$\partial_x v - \partial_y u = 0 \qquad \Rightarrow \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \qquad \Delta \psi = 0 \tag{43}$$

A streamline and a potential line are perpendicular to each other:

$$\nabla \psi . \nabla \phi = \partial_x \psi \partial_x \phi + \partial_y \psi \partial_y \phi = -vu + uv = 0. \tag{44}$$

Analytical means differentiable. This consist in defining a function $f(z)$ analytical such that:

$$w = f(z) \qquad z, \omega \in \mathbb{C} \qquad \Rightarrow w = \phi + i\psi \qquad \begin{cases} z = x + iy \\ \phi = \phi(x,y) \in \mathbb{R} \\ \psi = \psi(x,y) \in \mathbb{R} \end{cases} \tag{45}$$

If this is differentiable everywhere, $\Delta \phi = \Delta \psi = 0$.

4.2 Explain how the velocity can be computed from the complex potential function

The complex velocity (velocity field):

$$\frac{dw}{dz} = \frac{df}{dz} = A + iB \qquad A = \frac{\partial \phi}{\partial x} = -\frac{\partial \psi}{\partial y} = u \quad B = \frac{\partial \psi}{\partial x} = -\frac{\partial phi}{\partial y} = -v \tag{46}$$

A property of this $f(z)$ is the superposition principle: $w_1 = f_1(z), w_2 = f_2(z)$ so $w_1 + w_2 = f_1(z) + f_2(z)$.

4.3 Derive the complex potential function for uniform flow, source/sink flow, free vortex

4.3.1 Uniform flow

$$w = Uz \qquad \frac{dw}{dz} = U = u + iv \qquad \Rightarrow u = U; v = 0 \tag{47}$$

4.3.2 Source / Sink

In this case, using the cylindrical coordinates, the complex potential is defined as (Λ being the volumetric flow):

$$\begin{aligned} w &= \frac{\Lambda}{2\pi} \ln z = \frac{\Lambda}{2\pi} \ln(re^{i\theta}) = \frac{\Lambda}{2\pi}(\ln r + i\theta) \\ &\Rightarrow \phi = \frac{\Lambda}{2\pi} \ln r, \psi = \frac{\Lambda}{2\pi}\theta. \end{aligned} \tag{48}$$

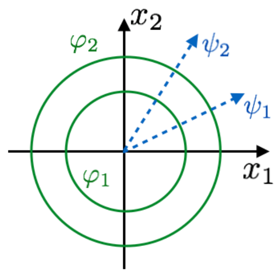

Figure 17

We see that complex lines corresponds to $r = cst$ so are circles and streamlines $\theta = cst$ are line of constant angle. $\oint \vec{v}\, d\vec{l} = 0$ as velocity is everywhere tangent to any circular contour. Let's compute the derivative for the velocity field:

$$\frac{dw}{dz} = \frac{\Lambda}{2\pi z} = \frac{\Lambda(x-iy)}{2\pi(x^2+y^2)} = \frac{\Lambda}{2\pi r}(\cos\theta - i\sin\theta). \tag{49}$$

We see that the velocity decreases in $1/r$, this is due to the constant mass flow, so if the surface increases with r the velocity has to decrease to keep $\dot{m} = \rho v S$ constant.

4.3.3 Free vortex

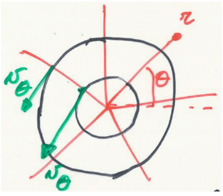

Figure 18

We do the same as the other cases:

$$\begin{aligned} w &= \frac{i\Gamma}{2\pi}\ln z = \frac{i\Gamma}{2\pi}\ln(re^{i\theta}) = \frac{i\Gamma}{2\pi}(\ln r + i\theta) = -\frac{\Gamma}{2\pi}\theta + \frac{i\Gamma}{2\pi}\ln r \\ \phi &= -\frac{\Gamma}{2\pi}\theta, \psi = \frac{\Gamma}{2\pi}\ln r \end{aligned} \tag{50}$$

We see that this is the inverse case of the previous one, streamlines are circles oriented in negative rotational motion around z-axis (z entering in the sheet) so clockwise. We can compute the velocity field by deriving among z and we find that:

$$u = \frac{\Gamma\sin\theta}{2\pi r} \qquad v = -\frac{\Gamma\cos\theta}{2\pi r} \tag{51}$$

Let's specify that $v_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -\frac{\Gamma}{2\pi r}$, and that we have a vortex singularity in the center because $\Gamma = 0.\infty$.

4.4 Compute flow around Rankine body and around a cylinder

Let's make a combination of a uniform flow and a source + sink as shown on the figure. The combination gives:

$$w = Uz + \frac{\Lambda}{2\pi}\ln\frac{z+a}{z-a} = Uz + \frac{\Lambda}{2\pi}\ln\frac{1+a/z}{1-a/z}. \tag{52}$$

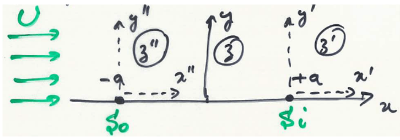

Figure 19

To have the flow around a cylinder we need to compute the limit $a \to 0$, and will need the Taylor expansion of ln:

$$\ln\frac{1+\epsilon}{1-\epsilon} \approx 2\epsilon + \cancel{o(\epsilon^3)} \qquad \Rightarrow \lim_{a\to 0} w = \lim_{a\to 0}\left[Uz + \frac{\Lambda}{2\pi}2\frac{a}{z}\right] \tag{53}$$

by defining $\mu = 2\Lambda a$ we find the **flow around a cylinder**:

$$w = Uz + \frac{\mu}{2\pi z}. \tag{54}$$

If we replace $z = x + iy$ to find ϕ and ψ we find:

$$\phi = Ux + \frac{\mu}{2\pi}\frac{x}{r^2} \qquad \psi = Uy - \frac{\mu}{2\pi}\frac{y}{r^2}. \tag{55}$$

In this flow a closed streamline exists forming the so called **Rankine body** and which describes a cylinder in the case $a \to 0$. Indeed it is possible to find an exact solution for $\psi = 0$. This configuration has a symmetry according to x and y-axis when taking the center of the cylinder as origin. This implies that $\vec{F} = -\oint_{cyl} p\, d\vec{S} = 0$. This is the so called **paradox of d'Alembert** because we expect to find at least a drag force. A lift force can be find on the cylinder by adding a vortex. We conclude by saying that we can rewrite (54) as (R the radius of the cylinder):

$$w = U\left(z + \frac{R^2}{z}\right) \qquad with\ R^2 = \frac{\mu}{2\pi U}. \tag{56}$$

4.5 Explain how a the class of Joukowski profiles is obtained by conformal mapping of a circle

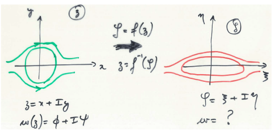

Figure 20

Let's do a mapping, a transformation, to try to find our airfoil based on simple geometries. Let's as first example apply the transformation $Z = z + \frac{R^2}{z}$ to the cylinder of radius R. Let's first remark that the cylinder will become a flat plate:

$$Z = z + \frac{R^2}{z} = Re^{i\theta}\frac{R^2}{Re^{i\theta}} = 2R\cos\theta \tag{57}$$

Indeed, as $\cos\theta \in [-1, 1]$ and the result is real, we have a flat plate between -2R and 2R in the x-axis. The flow Z is directly found: $W(Z) = UZ$. The second example will be the application of the same transformation on a cylinder of this time radius $r > R$. In this case the circle becomes an ellipse:

$$Z = re^{i\theta} + \frac{R^2}{r^2}e^{-i\theta} = \left(r + \frac{R^2}{r}\right)\cos\theta + i\left(r - \frac{R^2}{r}\right)\sin\theta. \tag{58}$$

Let's also compute the velocity field using the chain rule:

$$\frac{dW}{dZ} = \frac{dw}{dZ} = \frac{dw}{dz}\frac{dz}{dZ} = \left(1 - \frac{r^2}{z^2}\right)\left(\frac{1}{1 - \frac{R^2}{z^2}}\right). \tag{59}$$

We see that the expression becomes infinite when $z^2 = R^2$. The reason is that the transformation is not analytical in these points so they must not be in the flow.
The examples are summarized in the figures below

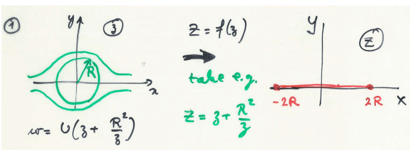

Figure 21

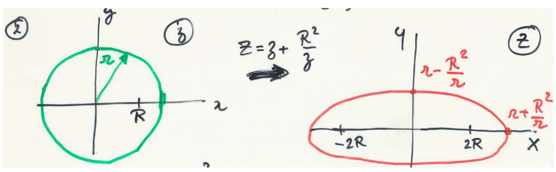

Figure 22

Now suppose that we place no longer the center of the cylinder at the origin, but on the real axis. The mapping of the cylinder now takes the shape of a **symmetrical wing profile**. We see that there is two remarkable points that are H and A corresponding to the points H_1 and A_1 of the black and red circles, our profile is in between them. Now to give camber we only have to move the center of the cylinder on the y-axis. Please reffer to figures below.

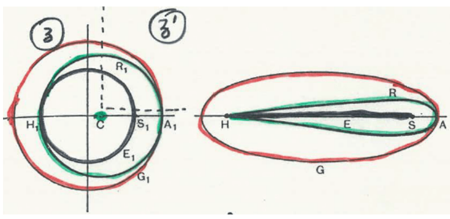

Figure 23

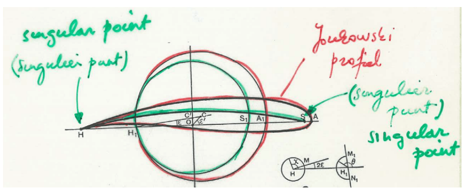

Figure 24

Note that for the green circle in first figure, the complex potential becomes:

$$w = U\left(z - z_c + \frac{r^2}{z - z_c}\right) \tag{60}$$

As last remark, let's remind that we had singularities in the second example. These points corresponds here to H_1 and S_1. The mapping of H_1 is always H the trailing edge, the velocity is there infinitely large. This was the discussion we've previously done with the stagnation point that has to move on the trailing edge otherwise $v = \infty$ because of the sharp edge. We can solve this by adding a vortex. This methods gives a limited amount of airfoils.

5 Computation of inviscid irrotational flow around a thin airfoil based on a continuous distribution of vortices

5.1 Explain what is a free vortex

A free vortex is an irrotationnal vortex where the flow velocity u is inversely proportional to the distance r.

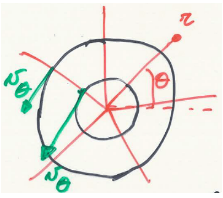

Figure 25

Streamlines are circles oriented in negative rotational motion around z-axis (z entering in the sheet) so clockwise. We can compute the velocity field by deriving among z and we find that:

$$u = \frac{\Gamma \sin\theta}{2\pi r} \qquad v = -\frac{\Gamma \cos\theta}{2\pi r} \tag{61}$$

Let's specify that $v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -\frac{\Gamma}{2\pi r}$, and that we have a vortex singularity in the center because $\Gamma = 0.\infty$.

5.2 Explain what is a continuous distribution of free vortices on a line

A way to represent thin airfoils is to use only their camberline and to retrieve the flow using an infinite line of free vorticies (or sources). A infinite line of vorticies thus represents an infinite thin airfoil.

5.3 Explain principle of the method of free vortex distribution applied thin airfoils, limitations and hypotheses

We will suppose infinitely thin airfoil and small angle of attack, so that the airfoil is represented by the camber line. This means also small camber about 2-3% of the chord and $\alpha < 8\%$. We can try to retrieve the flow by superposition principle by using infinite number of elementary sources or elementary vorticies. The potential function for the sources is:

$$\phi = \frac{\Lambda}{2\pi}\ln r \qquad d\phi = \frac{d\Lambda}{2\pi}\ln r \tag{62}$$

We then describe the source distribution by the source intensity $\lambda = \frac{d\Lambda}{ds}$ on a part ds of the wing so that the last equation becomes:

$$d\phi = \frac{\lambda}{2\pi}\ln r\, ds. \tag{63}$$

We will use the second method presented now which is using the vorticies:

$$\phi = -\frac{\Gamma}{2\pi}\theta \qquad \vec{v} = \nabla\phi = \underbrace{\frac{\partial\phi}{\partial r}\vec{1}_r}_{=v_r=0} + \underbrace{\frac{1}{r}\frac{\partial\phi}{\partial\theta}\vec{1}_\theta}_{=v_\theta} \qquad \Rightarrow v_\theta = -\frac{\Gamma}{2\pi}\frac{1}{r}. \tag{64}$$

In the same way as the other we can define a **vortex intensity** to characterize the vortex distribution on a part ds $\gamma = \frac{d\Gamma}{ds}$, the derivative of ϕ and the elementary velocity are then:

$$d\phi = -\frac{\gamma}{2\pi}\theta\, ds \qquad dv_\theta = -\frac{\gamma ds}{2\pi r}. \tag{65}$$

The aim now is to make that infinitely thin airfoil a streamline. We assume because of superposition that the flow is a uniform flow $\vec{U}_\infty$, that we have an angle of attack α. Because of the vorticies, we have a velocity perturbation $\vec{v}$ such that the total velocity is:

$$\vec{V}_\infty = \vec{U}_\infty + \vec{v}. \tag{66}$$

5.4 Establish the integral equation which allows to compute the vortex distribution for a given profile

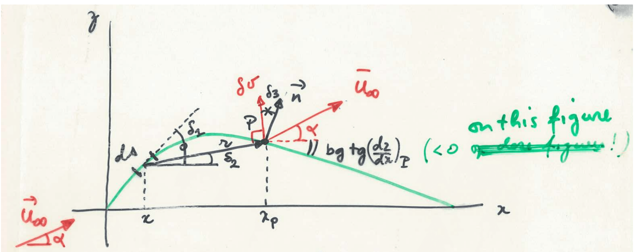

Figure 26

We must now choose γ such that $\vec{V}$ is tangential to the airfoil everywhere (we want the camber line to be a streamline). In other words, $\forall P$ the normal component of the velocity should be null $V_{nP} = U_{\infty_{nP}} + v_{nP} = 0$. Let's determine these components by projection. First, for $U_{\infty_{nP}}$ we can remark the sum of angle α and the camber line slope $\tan\beta = \left(-\frac{dz}{dx}\right)_p \Rightarrow \beta = -\arctan\left(\frac{dz}{dx}\right)_p$, the projection is (camber line: $z = f(x)$):

$$U_{\infty_{nP}} = U_\infty \sin\left[\alpha - \arctan\left(\frac{dz}{dx}\right)_P\right]. \tag{67}$$

Now for v_{np}, we consider an elementary vortex on a point x on the airfoil that creates an elementary perturbation δv_n on point P. This velocity direction is θ in a (r, θ) axis with origin at x, so perpendicular to r on the figure. If the angle with the normal is δ_3, the projection will be:

$$\delta v_n = -\frac{\gamma(x)ds}{2\pi r}\cos\delta_3. \tag{68}$$

Now we have infinite number of contribution of the infinite vorticies, as γ, r and δ_3 depend on position P, we have to integrate over the whole airfoil:

$$v_n = -\frac{1}{2\pi}\int_0^c \frac{\gamma(x)ds}{r}\cos\delta_3 \tag{69}$$

where c is the chord length. We can express both r and ds in function of x as:

$$r = \frac{x_P - x}{\cos\delta_2} \qquad ds = \frac{dx}{\cos\delta_1} \tag{70}$$

which gives

$$v_n = -\frac{1}{2\pi}\int_0^c \frac{\gamma(x)dx}{x_P - x}\frac{\cos\delta_2}{\cos\delta_1}\cos\delta_3. \tag{71}$$

We are able to reconsider the condition (67) by replacing our results:

$$\boxed{\frac{1}{2\pi}\int_0^c \frac{\gamma(x)dx}{x_P - x}\frac{\cos\delta_2}{\cos\delta_1}\cos\delta_3 = U_\infty \sin\left[\alpha - \arctan\left(\frac{dz}{dx}\right)_P\right]. \qquad (72)}$$

This is a relatively complicated equation, we can simplify it by **assuming a small camber** (in practice 2% of the chord), which allows to say that $\delta_1 \approx \delta_2 \approx \delta_3 \approx 0$ and $\arctan\left(\frac{dz}{dx}\right)_P = \left(\frac{dz}{dx}\right)_P$. By considering α small, $\sin\alpha \approx \alpha$:

$$\frac{1}{2\pi}\int_0^c \frac{\gamma(x)\,dx}{x_P - x} = U_\infty\left[\alpha - \left(\frac{dz}{dx}\right)_P\right]. \qquad (73)$$

We will introduce a new variable θ, considering $x = \frac{1}{2}c(1-\cos\theta)$ and $dx = \frac{1}{2}c\sin\theta\,d\theta$:

$$\frac{1}{2\pi}\int_0^\pi \frac{\gamma(\theta)\sin\theta\,d\theta}{\cos\theta - \cos\theta_P} = U_\infty\left[\alpha - \left(\frac{dz}{dx}\right)_P\right]. \qquad (74)$$

5.5 Show how to solve this equation using a spectral method, i.e. by expressing the vortex distribution as a truncated Fourier series

Let's express $\gamma(\theta)$ in series:

$$\gamma(\theta) = 2U_\infty\left(A_0\coth\frac{\theta}{2} + \sum_{n=1}^{\infty})A_n\sin(n\theta)\right). \qquad (75)$$

This is in fact a solution of the last equation.
Now we can replace this definition on the previous equation, knowing that $\coth(\theta/2)\sin\theta = 1+\cos\theta$ and writing $\theta_P = \theta'$, we get:

$$\frac{1}{2\pi}\int_0^\pi \frac{\gamma(\theta)\sin\theta\,d\theta}{\cos\theta-\cos\theta_P} = \frac{U_\infty}{\pi}\left[\int_0^\pi \frac{A_0(1+\cos\theta)\,d\theta}{\cos\theta-\cos\theta'} + \sum_n A_n\int_0^\pi \frac{\sin(n\theta)\sin\theta\,d\theta}{\cos\theta-\cos\theta'}\right] \qquad (76)$$

By using the equality here and the **Glauert integral**:

$$\sin(n\theta)\sin\theta = -\frac{1}{2}[\cos[(n+1)\theta] - \cos[(n-1)\theta]] \qquad \int_0^\pi \frac{\cos(n\theta)\,d\theta}{\cos\theta-\cos\theta'} = \pi\frac{sin(n\theta')}{sin\theta'} \qquad (77)$$

The integral becomes:

$$\frac{U_\infty}{\pi}\left[A_0.0 + A_0.\pi - \frac{\pi}{2}\sum_n A_n\frac{\sin[(n+1)\theta'] - \sin[(n-1)\theta']}{\sin\theta'}\right] = U_\infty\left[A_0 - \sum_n A_n\cos(n\theta')\right] \qquad (78)$$

where we used the Simpson equation. The (74) becomes:

$$\boxed{A_0 - \sum_n A_n\cos(n\theta') = \alpha - \left(\frac{dz}{dx}\right)' \qquad (79)}$$

This equation must be valid $\forall P$ on the airfoil. To find the coefficients A_i, let's integrate first this for $0 \leq \theta' \leq \pi$ in order to compute A_0:

$$A_0\pi - \cancel{\sum_n A_n \int_0^\pi \cos(n\theta')d\theta'} = \alpha\pi - \int_0^\pi \frac{dz}{dx}d\theta \qquad \Rightarrow A_0 = \alpha - \frac{1}{\pi}\int_0^\pi \frac{dz}{dx}\,d\theta. \tag{80}$$

For the A_n, we multiply the same equation by $\cos(m\theta')$ before integrating (we will drop the '). Let's see that $\int_0^\pi \cos(n\theta)\cos(m\theta)d\theta = 0$ if $m \neq n$ and $= \pi/2$ if $n = m$. We finally get:

$$A_n = \frac{2}{\pi}\int_0^\pi \frac{dz}{dx}\cos(m\theta)\,d\theta. \tag{81}$$

We can note that for A_n only the camber plays a role, the angle of attack does not appear. Only A_0 is influenced by α. We are now able to compute any vorticity distribution $\gamma(\theta)$, for example for a flat plate $A_n = 0$ and $A_0 = \alpha$.

5.6 How is the Kutta-Youkowski condition imposed (allowing to compute the lift)

The above respects the Kutta condition that states that there is no vortex allowed on the trailing edge. Indeed, $\gamma(\pi) = 0$ which means no contribution by vortex. We can also state that at the leading edge, the stagnation point is in the pressure side at the front. Indeed, for $\theta = 0$, $\coth\theta = \infty = \gamma(\pi)$ which means that we have a singularity at the TE and that the velocity is infinite due to the turning on the LE.

5.7 Starting from the expressions for the Fourier coefficients, compute the circulation and lift coefficient as a function of angle of attack, explain the result

5.7.1 Calculation of the total circulation

To get Γ we only have to compute the integral over the whole airfoil:

$$\begin{aligned}
\Gamma &= \int_0^c \gamma(x)\,dx = \frac{1}{2}c\int_0^c \gamma(\theta)\sin\theta\,d\theta \\
&= \frac{1}{2}c\left[2U_\infty\int_0^\pi A_0(1+\cos\theta)\,d\theta + 2U_\infty\sum_{n=1}^{\infty}\int_0^\pi A_n\sin(n\theta)\sin\theta\,d\theta\right] \\
&= U_\infty c\left[A_0\pi + 2U_\infty + \int_0^\pi \sin^2(\theta)\,d\theta - \cancel{\frac{1}{2}\sum_{n=2}^{\infty}\int_0^\pi A_n\cos[(n+1)\theta - \cos(n-1)\theta]\,d\theta}\right] \\
&= U_\infty c[A_0\pi + A_1\pi/2].
\end{aligned} \tag{82}$$

We see that the circulation only depends on two coefficients.

5.7.2 Calculation of the lift coefficient

We can now compute the lift using the kutta formula $L = \rho_\infty U_\infty \Gamma$. We are interested in the c_l and not the lift itself. In 2D we have to divide by the chord so:

$$c_l = \frac{L}{\frac{1}{2}\rho_\infty U_\infty^2 C} = \frac{2\Gamma}{U_\infty c} = \pi(2A_0 + A_1) \tag{83}$$

We can now replace by definition of the coefficients:

$$c_l = 2\pi \left(\alpha - \underbrace{\frac{1}{\pi} \int_0^\pi \frac{dz}{dx}(1 - \cos\theta)\, d\theta}_{\alpha_0} \right) = 2\pi(\alpha - \alpha_0) \tag{84}$$

where α_0 is the **zero lift angle of attack**. We see that we have a linear relation with respect to α. We have the theoretical model the same for every profile, only α_0 changes with the profile. Remark that the lift is also the integral of the pressure on the lower and upper side:

$$L = \int_0^c (p_l - p_u)\, dx = \rho_\infty U_\infty \int_0^c \gamma\, dx \qquad \Rightarrow p_l - p_u = \rho_\infty U_\infty \gamma(x). \tag{85}$$

5.8 Starting from the expression for the momentum coefficient at the leading edge c_m(LE), show that it's a linear function of the lift coefficient c_l

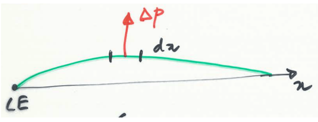

Figure 27

The contribution of the elementary parts of the airfoil gives:

$$\begin{aligned} dM_{LE} &= -(\Delta p\, dx)x \\ \Rightarrow M_{LE} &= -\int_0^c \Delta p x\, dx = -\rho_\infty U_\infty \int_0^c \gamma x\, dx \end{aligned} \tag{86}$$

After some manipulations (not detailed):

$$c_{m_{LE}} = \frac{M_{LE}}{\frac{1}{2}\rho_\infty U_\infty^2 c^2} = -\frac{\pi}{4}(2A_0 + 2A_1 - A_2) = -\frac{1}{4}c_l - \frac{\pi}{4}(A_1 - A_2) \tag{87}$$

where we used (83) for the last expression.

5.9 Compute the location of the aerodynamic center AC and the momentum coefficient in the aerodynamic center

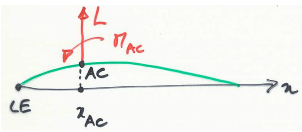

Figure 28

The moment on the LE is related to the moment anywhere:

$$M_{LE} = M_{ac} - x_{ac}L \qquad \Rightarrow c_{m_{LE}} = c_{m_{ac}} - \frac{x_{ac}}{c}c_l \tag{88}$$

By using the last result of the previous section, we get:

$$c_{m_{ac}} = \left(\frac{x_{ac}}{c} - \frac{1}{4}\right)c_l - \frac{\pi}{4}(A_1 - A_2). \tag{89}$$

We see that, for this relation to be independent of the angle of attack, we must have $x_{ac} = \frac{c}{4}$ so that:

$$c_{m_{ac}} = \frac{\pi}{4}(A_2 - A_1). \tag{90}$$

Remark that for symmetrical wings $\frac{dz}{dx} = 0 \Rightarrow A_1 = A_2 = 0 \Rightarrow c_{m_{ac}} = 0$.

5.10 Compute the location of the center of pressure

The formula used in the previous section is valid, we replace ac by cp, and since the moment should be null at this point:

$$c_{m_{cp}} = c_{m_{LE}} + \frac{x_{cp}}{c}c_l = 0 \qquad \Rightarrow \frac{x_{cp}}{c} = \frac{1}{4} + \frac{\frac{\pi}{4}(A_1 - A_2)}{c_l} = \frac{1}{4} - \frac{c_{m_{ac}}}{c_l}. \tag{91}$$

Some remarks:

- the center of pressure is not fixed and varies with the lift
- at 0 lift, $x_{cp} \to \infty$ (for symmetric wing $x_{cp} = x_{ac} = c/4$ fixed)
- cp always downstream to ac because $c_{m_{ac}} < 0$.

6 Computation of inviscid irrotational flow around a thin airfoil based on a continuous distribution of vortices

Discuss the influence of a flap located at the trailing edge of the airfoil According to the definition of the zero lift angle in (84), the effect of the shape becomes greater when $\theta \approx 180^\circ$ (trailing edge). By making the zero lift angle more negative we can produce more lift before the critical angle of attack that decreases a bit.
The effect is evaluated by taking a flat plate as camber line with a deflection near the TE, starting at E% of the chord and slope η. E in function of θ_E is:

$$E = \frac{1}{2}(1 + \cos\theta_E). \tag{92}$$

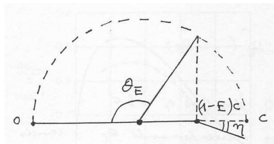

Figure 29

In this case, A_0 and A_n can be rewritten as:

$$\begin{aligned} A_0 &= \alpha - \frac{1}{\pi}\int_0^\pi \frac{dz}{dx} d\theta \approx \alpha - \frac{\eta}{\pi}(\pi - \theta_E) \\ A_n &= \frac{2}{\pi}\int_0^\pi \frac{dz}{dx}\cos(n\theta) d\theta \approx -\frac{2\eta}{\pi}\frac{1}{n}\sin(n\theta_E) \end{aligned} \tag{93}$$

such that the lift coefficient becomes:

$$c_l = \pi(2A_0 + A_1) = \underbrace{2\pi\alpha}_{\text{without flaps}} \underbrace{-2\eta(\pi - \theta_E + \sin\theta)}_{\Delta c_l > 0 \text{ since } \eta < 0}. \tag{94}$$

This seams to be like $c_l = 2\pi(\alpha - \alpha_0)$ allowing the definition for the zero lift angle:

$$\alpha_0 = \frac{\eta}{\pi}(\pi - \theta_E + \sin\theta) \tag{95}$$

which indicates an increase (decrease since $\eta < 0$) of α_0 since it is null for the flat plate. For the moment at the ac (90)

$$\Delta c_{m_{ac}} = \frac{\eta}{2}\sin\theta_E(1 - \cos\theta_E) \tag{96}$$

which also indicates a decrease in the momentum which is 0 for the symmetric wing.

Determine the effect of the flap on:

the lift curve: from the above, we see that lift goes up with flaps

the zero lift angle of attack: It is increased, see above

the moment at the aerodynamic center: It decreases.

7 title

7.1 title

7.2 title

7.3 title

8 title

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