Université Libre de Bruxelles

Summary

# Vibration & Acoustics
# MECA-H411

*Author:*
Enes Ulusoy

*Professors:*
Patrick Guillaume
Steve Vanlanduit

Year 2016 - 2017

# Appel à contribution

### Synthèse Open Source

Ce document est grandement inspiré de l'excellent cours donné par Patrick Guillaume et Steve Vanlanduit à l'EPB (École Polytechnique de Bruxelles), faculté de l'ULB (Université Libre de Bruxelles). Il est écrit par les auteurs susnommés avec l'aide de tous les autres étudiants et votre aide est la bienvenue ! En effet, il y a toujours moyen de l'améliorer surtout que si le cours change, la synthèse doit être changée en conséquence. On peut retrouver le code source à l'adresse suivante

`https://github.com/nenglebert/Syntheses`

Pour contribuer à cette synthèse, il vous suffira de créer un compte sur *Github.com*. De légères modifications (petites coquilles, orthographe, ...) peuvent directement être faites sur le site ! Vous avez vu une petite faute ? Si oui, la corriger de cette façon ne prendra que quelques secondes, une bonne raison de le faire !

Pour de plus longues modifications, il est intéressant de disposer des fichiers : il vous faudra pour cela installer LaTeX, mais aussi *git*. Si cela pose problème, nous sommes évidemment ouverts à des contributeurs envoyant leur changement par mail ou n'importe quel autre moyen.

Le lien donné ci-dessus contient aussi un `README` contenant de plus amples informations, vous êtes invités à le lire si vous voulez faire avancer ce projet !

### Licence Creative Commons

Le contenu de ce document est sous la licence Creative Commons : *Attribution-NonCommercial-ShareAlike 4.0 International (CC BY-NC-SA 4.0)*. Celle-ci vous autorise à l'exploiter pleinement, compte- tenu de trois choses :

1. *Attribution* ; si vous utilisez/modifiez ce document vous devez signaler le(s) nom(s) de(s) auteur(s).
2. *Non Commercial* ; interdiction de tirer un profit commercial de l'œuvre sans autorisation de l'auteur
3. *Share alike* ; partage de l'œuvre, avec obligation de rediffuser selon la même licence ou une licence similaire

Si vous voulez en savoir plus sur cette licence :

`http://creativecommons.org/licenses/by-nc-sa/4.0/`

**Merci !**

# Contents

# Part I

# Vibration

# Chapter 1

# Discrete systems

## 1.1 Introduction

Vibrations are found on everything around us, trains, cars and even human body is subject to vibration. Its effects are disturbing because it causes fatigue, loss of performance, no comfort, ... As vibration source we can find the earthquakes, the interaction with road, the wind, the waves, ... The basic terminology for the course is:

- **The source** $F(\omega)$, this characterizes the dynamic forces
- **The path** $H(\omega)$, this characterizes the structural dynamics
- **The response** $X(\omega)$, such that $X(\omega) = H(\omega)F(\omega)$.

Vibrations cause failure, loss of comfort and is harmful for precision operations. We try to suppress it by damping, isolation and structure design.

We have two different approach for analysing a vibration problem. The one called **Signal analysis** or **Fourier analysis** deals with the case where we only have the response of the system to unknown forces. The one called **System analysis** or **Modal analysis** where we stimulate the system with known forces and measure the response, being able to find $H(s)$ (dynamic forces - transfer function of the system).

#### Basic notions

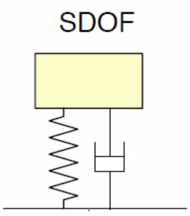


Figure 1.1

Three main forces are acting on bodies:

- the one due to springs, proportional to the displacement: $F = kd$
- the one due to dampers, proportional to the velocity: $F = cv$
- the one due to the mass, proportional to acceleration: $F = ma$.

Notice that we have one resonance frequency for each degree of liberty of each mass.

We can already get some definition, let's consider the free vibration assumed to be always in resonance. We can then define the **period of resonance** $T_n$, the **resonance frequency** $f_n = \frac{1}{T_n}$ and the **resonance pulsation**:

$$\omega_n = 2\pi f_n = \sqrt{\frac{k}{m}}. \tag{1.1}$$

Notice that if we increase mass, the frequency decreases.

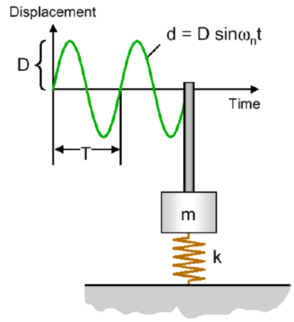


Figure 1.2

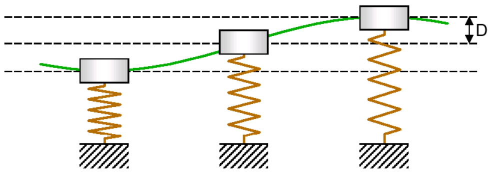


Figure 1.3

Then we have the **energy transfer** between the kinetic energy and the potential energy. Indeed we see that velocity is null on extreme position and max when at the middle. This is written as:

$$\frac{1}{2}mV^2 = \frac{1}{2}kD^2 \tag{1.2}$$

where $V = 2\pi f_n D$. Replacing by this:

$$m(2\pi f_n D)^2 = kD^2 \qquad \Rightarrow f_n = \frac{1}{2\pi}\sqrt{\frac{k}{m}}. \tag{1.3}$$

We find a waited result. Now, let's show that **increasing damping reduces amplitudes over time**. Take the general newton equation for a linear free system and multiply by $\dot{x}(t)$:

$$m\ddot{x}\dot{x} + b\dot{x}\dot{x} + kx\dot{x} = 0 \qquad \Rightarrow \frac{d}{dt}\left(\frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2\right) = -b\dot{x}^2 \leq 0 \tag{1.4}$$

## 1.2 Single degree of freedom oscillator

Given the single degree system here, its free response is given by:

$$m\ddot{x}\dot{x} + b\dot{x}\dot{x} + kx\dot{x} = 0. \tag{1.5}$$

We assume that this differential equation admits a solution of type $x = Ae^{st}$. We can then write the characteristic equation and its eigenvalues as:

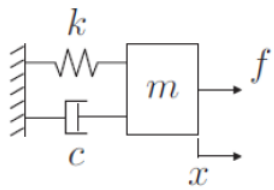


Figure 1.4

$$ms^2 + cx + k = 0, \qquad s = -\frac{c}{2m} \pm j\sqrt{\frac{k}{m} - \frac{c^2}{4m^2}}. \tag{1.6}$$

By defining two new quantity, the **natural pulsation** $\omega_n^2 = \frac{k}{m}$ and the **damping ratio** $\xi$ such that $\xi\omega_n = \frac{c}{2m}$, we can rewrite:

$$s = -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}. \tag{1.7}$$

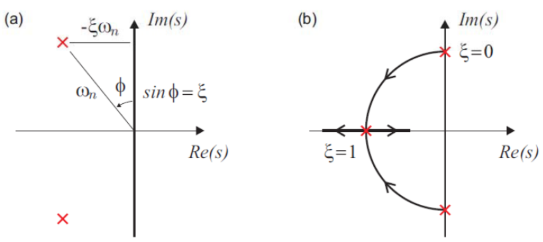


Figure 1.5

We can introduce the **damping pulsation** $\omega_d = \omega_n\sqrt{1-\xi^2}$. This explicitly makes appear the real and imaginary part of $s$ that we plot on a diagram. Notice that the norm and the angle of the complex number are $\omega_n$ and $\arcsin\xi$.

The right figure is the **Nyquist diagram**. Last, the final expression for $x$ is:

$$x = e^{-\xi\omega_n t}\left(Ae^{j\omega_d t} + Be^{-j\omega_d t}\right) = e^{-\xi\omega_n t}\left(A_1\cos(\omega_d t) + B_1\sin(\omega_d t)\right) \tag{1.8}$$

where $A, B, A_1, B_1$ depends on initial conditions.

#### Impulse response

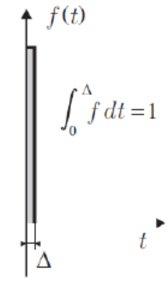


Figure 1.6

Let's now apply an impulse on the system and let's analyse when it is applied during the infinitesimal time $\Delta$, given the initial conditions $x = 0, \dot{x} = 0$. If we integrate the newton equation:

$$\int_0^\Delta m\ddot{x}\,dt = \int_0^\Delta f\,dt - \cancel{\int_0^\Delta c\dot{x}\,dt} - \cancel{\int_0^\Delta kx\,dt} = 1 \tag{1.9}$$

where the spring and damping forces cancel as they are finite (infinitesimal integral), the impulse integral =1 by definition. Taking the limit we find new initial conditions:

$$\lim_{\Delta\to 0} m\dot{x}(\Delta) = m\dot{x}(0^+) = 1 \qquad \Rightarrow x(0^+) = 0, \dot{x}(0^+) = \frac{1}{m}. \tag{1.10}$$

The resolution of (1.8) gives the **impulse response**:

$$x(t) = h(t) = \frac{1}{m\omega_d}e^{-\xi\omega_n t}\sin(\omega_d t) \tag{1.11}$$

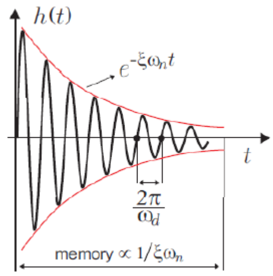


Figure 1.7

## 1.3 Convolution integral

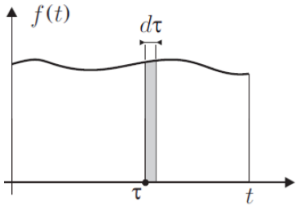


Figure 1.8

Consider a transfer function $h(t)$ of a system and the decomposition shown on the figure. The output of the system will be computed with the convolution integral:

$$x = \int_0^t h(t-\tau)f(\tau)\,d\tau. \tag{1.12}$$

where $h(t)$ is the **impulse response**. In particular, for a **causal** system:

$$x(t) = \int_{-\infty}^{\infty} h(t-\tau)f(\tau)\,d\tau = \int_{-\infty}^{\infty} h(\tau)f(t-\tau)\,d\tau = h(t) * f(t). \tag{1.13}$$

### 1.3.1 Harmonic response

Consider an undamped system to which we apply an harmonic force:

$$m\ddot{x} + kx = Fe^{j\omega t}. \tag{1.14}$$

By considering the Fourier transform $x(t) = X(\omega)e^{j\omega t}$, we get:

$$-\omega^2 X(j\omega) + \frac{k}{m}X(j\omega) = \frac{F(j\omega)}{m} \qquad \Rightarrow X = \frac{F}{k}\frac{1}{1-(\omega/\omega)^2} = \frac{F}{k}D(\omega) \tag{1.15}$$

where $D(\omega)$ is the **dynamic amplification**. For the damped case, we only have to know that $\xi\omega_n = c/2m$ and we get in the same way as previously:

$$X = \frac{F}{k}\frac{1}{1-(\omega/\omega)^2+2j\omega/\omega_n} = \frac{F}{k}D(\omega). \tag{1.16}$$

The two dynamic amplifications are plotted on the figures below, the second on a Bode diagram where we can clearly see the **Quality factor** defined as $Q = 1/2\xi$.

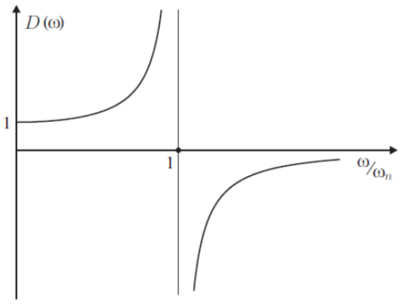


Figure 1.9

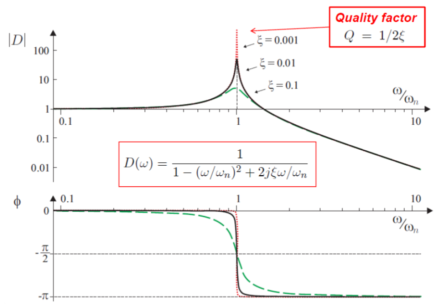


Figure 1.10

### 1.3.2 Frequency response function

Let's remember that the response is given by the convolution integral (1.13), where we will define now $h(t)$. Assuming the applied force to be harmonic $Fe^{i\omega t}$, then proceeding to the Fourier transform of x, we get:

$$Xe^{i\omega t} = \int_{-\infty}^{\infty} h(t)Fe^{i\omega(t-\tau)}\,d\tau \qquad \Rightarrow \frac{X(\omega)}{F(\omega)} = \int_{-\infty}^{\infty} h(t)e^{-i\omega\tau}\,d\tau = H(\omega) \tag{1.17}$$

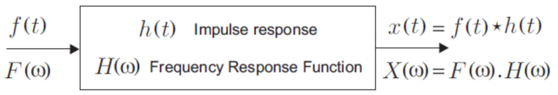


Figure 1.11

where we end up with the fact that **the frequency response function is the Fourier transform of the impulse response**. We can thus avoid the convolution to have a simple multiplication after a Fourier transform. Here is also a useful theorem where $|F(\omega)/2\pi|$ is the **energy spectrum** of $f(t)$:

**Parseval theorem**

$$\int_{-\infty}^{\infty} f^2(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2\,d\omega. \tag{1.18}$$

### 1.3.3 Discrete Fourier Transform

Consider a signal x sampled in N samples, the equivalence of the continuous Fourier transform in discrete domain and the inverse transform are:

$$X_k = \sum_{n=0}^{N-1} x_n e^{-i\frac{2k\pi}{N}n} \qquad \text{and} \qquad X_n = \frac{1}{N}\sum_{k=0}^{N-1} X_k e^{i\frac{2k\pi}{N}n} \tag{1.19}$$

The equivalent Parseval theorem is:

$$\sum_{n=0}^{N-1} |x_n|^2 = \frac{1}{N}\sum_{k=0}^{N_1} |X_k|^2 \tag{1.20}$$

And as last definition, we have the root-mean-square value defined as

$$RMS = \sqrt{\frac{1}{N}\sum_{n=0}^{N-1}|x_n|^2} = \sqrt{\frac{1}{N^2}\sum_{k=0}^{N-1}|X_n|^2} \tag{1.21}$$

## 1.4 Transient response (Beat)

Consider an undamped oscillator excited by an harmonic force $F\cos\omega t$ and of impulse response:

$$h(t) = \frac{1}{m\omega_n}\sin\omega_n t \tag{1.22}$$

Applying the convolution integral to this problem and integrating by part we find:

$$x(t) = \int_0^t F\cos(\omega t)h(t - /tau)\,d\tau = \frac{F}{m}\frac{\cos(\omega t) - \cos(\omega_n t)}{\omega_n^2 - \omega^2} \tag{1.23}$$

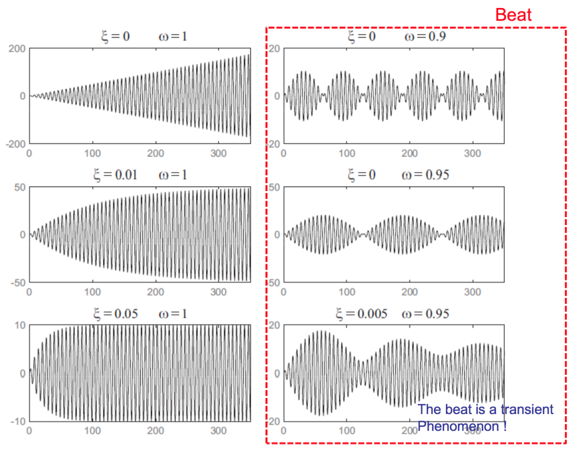


Figure 1.12

By using Simpson's formula, defining $\omega + \omega_n = 2\omega_0$ and $\omega - \omega_n = 2\Delta$, we get:

$$x(t) = \frac{F}{m}\frac{\sin(\omega_0 t)\sin(\Delta t)}{2\omega_0\Delta} \tag{1.24}$$

Remark that for $\Delta \to 0$, $\frac{\sin\Delta t}{\Delta} = t$. This term is called the **modulating function** as it grows the response amplitude while the other sinus is confined in $[-1, 1]$. We get thus for resonance:

$$x(t) = \frac{F}{m}\frac{\sin(\omega_n t)}{2\omega_n}t. \tag{1.25}$$

The figure shows that the phenomenon known as beat is **in fact** a transient phenomenon.

## 1.5 Multiple degree of freedom

Before going throw the real subject, let's introduce what's the **state space model** (yes CSD). It consists in writing any differential equation in this form of first order equation:

$$\dot{x} = Ax + Bu \tag{1.26}$$

where A, B are matrices and x, u vectors. Let's apply this for an oscillator:

$$\ddot{x} = \frac{f}{m} - 2\xi\omega_n - \omega_n^2 \tag{1.27}$$

We choose as state variable $x_1 = x$ and $x_2 = \dot{x}$, then we only have to rewrite the definition of $\dot{x}_1$ and $\dot{x}_2$ as:

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & 2\xi\omega \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{Bmatrix} 0 \\ \frac{1}{m} \end{Bmatrix} f \tag{1.28}$$

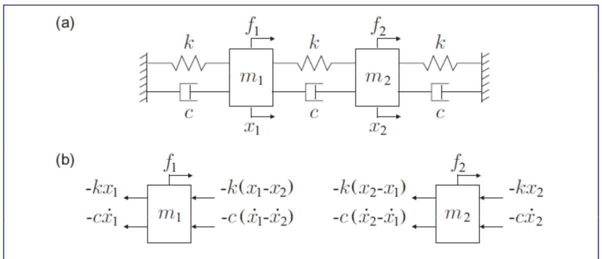


**Fig. 2.1.** (a) Two degree-of-freedom lumped mass system. (b) Free body diagrams.

$$m_1\ddot{x}_1 = f_1 + c(\dot{x}_2 - \dot{x}_1) + k(x_2 - x_1) - kx_1 - c\dot{x}_1$$

$$m_2\ddot{x}_2 = f_2 + c(\dot{x}_1 - \dot{x}_2) + k(x_1 - x_2) - kx_2 - c\dot{x}_2$$

'n matrix form:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 2c & -c \\ -c & 2c \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix}$$

Figure 1.13

Let's now look at the multiple degree case on Figure 1.13. The first thing to do is to put everything in matrix form. We see that we have something in the form:

$$M\ddot{x} + C\dot{x} + Kx = f. \tag{1.29}$$

The matrices M, C, K are symmetric and semi-positive definite: $K = K^T$ and $x^T Kx \geq 0, \ \forall x$. In particular, we can compute the kinetic energy and the strain energy as:

$$\frac{1}{2}\dot{x}^T M\dot{x} \qquad \text{and} \qquad \frac{1}{2}x^T Kx \tag{1.30}$$

## 1.6 Eigenvalue problem

The method to solve this problem is to first consider the free response of the conservative system ($C = 0$): $M\ddot{x} + Kx = 0$. A non trivial solution exists if:

$$(K + s^2 M)\phi = 0 \tag{1.31}$$

The eigenvalues $s$ are solution of $\det(K + s^2 M) = 0$ Because K and M are symmetric and semi-positive definite, the eigenvalues are purely imaginary: $s = \pm j\omega$, this gives:

$$(K - \omega_i M)\phi_i = 0 \tag{1.32}$$

where $\omega_i$ are the natural pulsations and $\phi_i$ the mode shapes.

### Orthogonality of the mode shapes

Let's demonstrate that (substracting with the i,j permuted equation):

$$(K - \omega_i M)\phi_i = 0 \qquad \Rightarrow \begin{array}{c} \phi_j^T K\phi_i = \omega_i \phi_j^T M\phi_i \\ \underline{-\phi_i^T K\phi_j = \omega_j \phi_i^T M\phi_j} \\ 0 = (\omega_i^2 - \omega_j^2)\phi_j^T M\phi_i \end{array} \qquad \Rightarrow \phi_j^T M\phi_i = 0 \qquad (\omega_i \neq \omega_j) \tag{1.33}$$

The conclusion is that the mode shapes corresponding to distinct natural frequencies are orthogonal with respect to M and K. We then have the:

**Orthogonality relationships**

$$\phi_i^T M\phi_j = \mu_i \delta_{ij} \qquad \textit{and} \qquad \phi_i^T K\phi_j = \mu_i \omega_i^2 \delta_{ij}, \qquad \omega_i^2 = \frac{\phi_i^T K\phi_i}{\phi_i^T M\phi_i} \tag{1.34}$$

where $\mu_i$ is the modal mass and $\omega_i$ the Rayleigh coefficient

.

And in matrix form with $\Phi = (\phi_1, \phi_2, ..., \phi_n)$:

$$\boxed{\Phi^T M \Phi = diag(\mu_i) \qquad \Phi^T K \Phi = diag(\mu_i \omega_i^2)} \quad (1.35)$$

To complete, two remarks:

- If several modes have the same natural frequency, they form a subspace and any vector in this subspace is also solution of the eigenvalue problem.

- Rigid body modes, they have no strain energy so $u_i^T K u_i = 0$. They also satisfies $Ku_i = 0$ such that their are also solution of the eigenvalue problem with $\omega_i = 0$.

**Free response from initial conditions**

As the eigenvalues are imaginary, we have:

$$x = \sum_{i=1}^{n} (A_i \cos \omega_i t + B_i \sin \omega_i t) \phi_i \quad (1.36)$$

We have so 2n constants to determine using the orthogonality conditions.

# Part II

# Acoustics

# Chapter 1

# Fundamental principles of acoustics

## 1.1 Definition and origin of sound

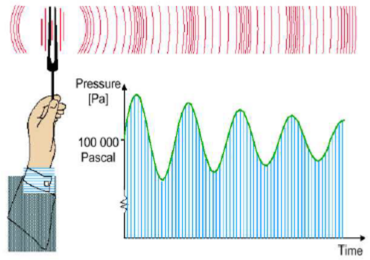


Figure 1.1

Vibration when a mechanical excitation is applied on a material, air or water is the main source of sound. We speak about sound when the vibrations in air are perceptible by ear. The example on Figure 2.1 illustrates the vibration of a tuning fork that induces over- and under-pressure in the air around (order of magnitude small compared to ATM). Air particles are moving and describe a wave, a longitudinal wave, meaning that the particles displacement is parallel to the wave direction. Man can hear sound frequencies between 20Hz-20kHz, below we speak about infra-sound and above the limit, about ultra-sound.

## 1.5 Sound levels

### 1.5.1 The effective sound pressure

The sound perceived with a constant loudness may be both a pure sine tone and a stochastic sound generated by a source with constant parameter: p(t) is extremely complicated, and yet the human ear have the impression of a constant loudness. The ear seems to be sensitive to the energy of sound waves. This led to the consideration of the **effective–** or **Root-Mean-Square (RMS)** value of the sound pressure, over a certain time interval, as an measure of intensity:

$$p_{eff} = \sqrt{\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} p^2(t)\, dt} \tag{1.1}$$