Université Libre de Bruxelles

Synthèse

Fluid mechanics II
MECA-H-305

Auteur :
Enes Ulusoy

Professeur :
Gérard Degrez

Année 2015 - 2016

Appel à contribution

Synthèse Open Source

Ce document est grandement inspiré de l'excellent cours donné par Gérard Degrez à l'EPB (École Polytechnique de Bruxelles), faculté de l'ULB (Université Libre de Bruxelles). Il est écrit par les auteurs susnommés avec l'aide de tous les autres étudiants et votre aide est la bienvenue ! En effet, il y a toujours moyen de l'améliorer surtout que si le cours change, la synthèse doit être changée en conséquence. On peut retrouver le code source à l'adresse suivante

`https://github.com/nenglebert/Syntheses`

Pour contribuer à cette synthèse, il vous suffira de créer un compte sur *Github.com*. De légères modifications (petites coquilles, orthographe, ...) peuvent directement être faites sur le site ! Vous avez vu une petite faute ? Si oui, la corriger de cette façon ne prendra que quelques secondes, une bonne raison de le faire !

Pour de plus longues modifications, il est intéressant de disposer des fichiers : il vous faudra pour cela installer LaTeX, mais aussi *git*. Si cela pose problème, nous sommes évidemment ouverts à des contributeurs envoyant leur changement par mail ou n'importe quel autre moyen.

Le lien donné ci-dessus contient aussi le `README` contient de plus amples informations, vous êtes invités à le lire si vous voulez faire avancer ce projet !

Licence Creative Commons

Merci !

Contents

Chapter 1

Generalities

1.1 Fundamental laws

Reminder

Let's first remind the 3 basic principles of *Fluid mechanics I* :

- **Mass conservation :** *The mass of a closed system remains constant in time.*
  This is much a definition of a closed system than a principle. We have to notice that related to Einstein law of relativity, $E = mc^2$, mass must vary with energy. But if we exclude nuclear reactions, our approximation is valid. Indeed, the square of light velocity has a greater impact on energy than the mass term. If the energy exchange is huge like in nuclear reaction, mass vary, but in smaller energies domain (combustion for example), the mass can be considered as constant.

- **Newton's law :** *the time rate of change of momentum of a closed system is equal to the sum of the forces applied on the system.*

- **First principle of thermodynamics :** *the time rate of change of the total energy of a closed system is equal to the sum of the power of the forces applied on the system and the thermal power provided to the system.*

Useful equations

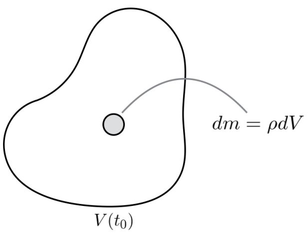

Figure 1.1

Let's consider the integral on a moving volume of a function depending on time and position $f(\vec{x}, t)$. Imagine that Figure 1.1 represents the moving volume at initial time containing mass m. An infinitesimal part of that volume contains an infinitesimal mass $dm = \rho dV$, where ρ is mass density. We deduce the expression of the total mass at any time by that of the initial time

$$M(t_0) = \int_{V(t_0)} \rho(\vec{x}, t_0)\, dV \qquad \Rightarrow M(t) = \int_{V(t)} \rho(\vec{x}, t)\, dV \qquad (1.1)$$

By considering $\rho(\vec{x}, t)$ as $f(\vec{x}, t)$, the derivative of the integral is given by

Reynolds transport theorem

$$\frac{d}{dt}\int_{V(t)} f(\vec{x},t)\,dV = \int_{V(t)} \frac{\partial f}{\partial t}(\vec{x},t)\,dV + \oint_{S(t)=\partial V(t)} f(\vec{x},t)\vec{b}\,\vec{n}\,dS \tag{1.2}$$

where $\vec{b}$ is the surface displacement velocity.

The second equation that will be used in the developement is given by

Gauss theorem

$$\oint_{S=\partial V} \vec{a}\,\vec{n}\,dS = \int_V \nabla\vec{a}\,dV \tag{1.3}$$

1.1.1 Mass conservation equation

If $V(t)$ is the moving volume occupied by the closed system as time varies, then by definition of a closed system $\frac{dM(t)}{dt} = 0$. The corresponding equation using Reynolds transport theorem is

$$M(t) = \int_{V(t)} \rho\,dV \qquad \Rightarrow \frac{dM(t)}{dt} = \int_{V(t)} \frac{\partial \rho}{\partial t}\,dV + \oint_{S(t)=\partial V(t)} \rho\,\vec{b}\,\vec{n}\,dS = 0 \tag{1.4}$$

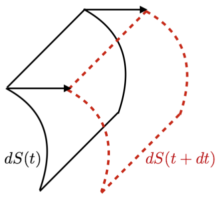

Figure 1.2

We have to express that this volume is not traversed by material. There is no flux of fluid and the particles in the volume are always the same. By definition, the infinitesimal distance traveled by the surface and the fluid are

$$d\vec{x} = \vec{b}\,dt \qquad and \qquad d\vec{x}' = \vec{u}\,dt \tag{1.5}$$

where $\vec{u}$ is the fluid velocity. Under wich condition do we know that the fluid has not traversed the boundary? We have to define the relative dispolacement $d\vec{x}' - d\vec{x}$ of the fluid in regard to the fluid. For a closed system

$$\begin{aligned} (d\vec{x}' - d\vec{x})\cdot\vec{n} = 0 \quad &\Leftrightarrow \quad dt(\vec{u}-\vec{b})\cdot\vec{n} = 0 \quad \Leftrightarrow \quad (\vec{u}-\vec{b})\cdot\vec{n} = 0 \\ &\Rightarrow \quad \vec{b}\vec{n} = \vec{u}\vec{n} \end{aligned} \tag{1.6}$$

Mass conservation equation for closed systems (integral form)

$$\int_{V(t)} \frac{\partial \rho}{\partial t}\,dV + \oint_{S(t)=\partial V(t)} \rho\,\underbrace{\vec{b}\,\vec{n}}_{=\vec{u}\,\vec{n}}\,dS = 0 \tag{1.7}$$

How to write this equation in a different way? Let's consider now a fixed open system composed of fluid particles in the fixed volume $V_0(t) = V(t_0)$. Similarly to the previous point, the mass variation in this fixed volume is expressed like

$$M_0(t) = \int_{V_0(t)} \rho\,dV \qquad \Rightarrow \int_{V_0(t)} \frac{\partial \rho}{\partial t}\,dV + \oint_{S_0(t)=\partial V_0(t)} \rho\,\vec{b}\,\vec{n}\,dS. \tag{1.8}$$

The volume integral expresses the variable mass in the fixed volume and the surface integral is nul due to the nul surface velocity (since the volume is fixed). This relation enables us to write the

Mass conservation equation for fixed open systems (integral form)

$$\frac{dM_0}{dt} + \underbrace{\oint_{S_0(t)=\partial V_0(t)} \rho\,\vec{u}\,\vec{n}\,dS = 0}_{\text{mass flow out of the system}} \tag{1.9}$$

Let's finally consider an arbitrary open system containing fluid particles in a moving volume $V_*(t)$ such that $V_*(t_0) = V(t_0) = V_0$. Similarly we have using the Reynolds transport theorem

$$M_*(t) = \int_{V_*(t)} \rho\,dV \qquad \Rightarrow \frac{dM_*(t)}{dt} = \int_{V_*(t)} \frac{\partial \rho}{\partial t}\,dV + \oint_{S_*(t)=\partial V_*(t)} \rho\,\vec{b}\,\vec{n}\,dS \tag{1.10}$$

Using the definition of the volume at $t = t_0$, we can equalize the volume integral with that of (1.7) to find

Mass conservation equation for arbitrary open systems (integral form)

$$\frac{dM_*(t_0)}{dt} + \oint_{S(t_0)=\partial V(t_0)} \rho(\vec{u} - \vec{b})\,\vec{n}\,dS = 0 \tag{1.11}$$

Let's now take (1.7) again and apply Gauss theorem

$$\begin{gathered}\int_{V(t)} \frac{\partial \rho}{\partial t}\,dV + \oint_{S(t)=\partial V(t)} \rho\,\underbrace{\vec{u}\,\vec{n}}_{\vec{a}}\,dS = 0 \qquad \textit{with} \qquad \oint_{S(t)} \rho\,\underbrace{\vec{u}\,\vec{n}}_{\vec{a}}\,dS = \int_{V(t)} \nabla\rho\vec{u}\,dV \\ \Leftrightarrow \int_{V(t)} \left[\frac{\partial \rho}{\partial t} + \nabla\rho\vec{u}\right] dV = 0\end{gathered} \tag{1.12}$$

For this last equation to be true for all systems, the integrated term must be equal to zero

Mass conservation equation (differential form (1) - divergent form)

$$\frac{\partial \rho}{\partial t} + \nabla\rho\vec{u} = 0 \tag{1.13}$$

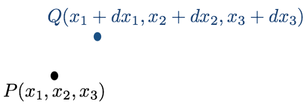

Figure 1.3

In order to find the second differential form, let's consider 2 points Q and P as described in Figure 1.3. The difference of density between the 2 points is

$$\begin{aligned}\rho_Q(t+dt) - \rho_P(t) &= \rho(x_1 + dx_1, x_2 + dx_2, x_3 + dx_3, t + dt) - \rho(x_1, x_2, x_3) \\ &= d\rho = \frac{\partial \rho}{\partial x_1}dx_1 + \frac{\partial \rho}{\partial x_2}dx_2 + \frac{\partial \rho}{\partial x_3}dx_3 + \frac{\partial \rho}{\partial t}dt\end{aligned} \tag{1.14}$$

In general, the fluid particles at $P(t)$ and $Q(t+dt)$ are different. However, if $dx_1 = u_1dt, dx_2 = u_2dt, dx_3 = u_3dt$, then the fluid particles at the 2 points are the same. By making appear these velocities in (1.14),

$$d\rho = \left(\frac{\partial \rho}{\partial x_1}u_1 + \frac{\partial \rho}{\partial x_2}u_2 + \frac{\partial \rho}{\partial x_3}u_3 + \frac{\partial \rho}{\partial t}\right) dt \tag{1.15}$$

Finally, after dividing by dt the 2 members of the equation, we obtain the definition of the time rate of change of density when I follow the fluid $\dot{\rho}$. As (1.13) can be expressed in term of indicial notation like

$$\frac{\partial \rho}{\partial t} + u_i\frac{\partial \rho}{\partial x_i} + \rho\frac{\partial u_i}{\partial x_i} = 0 \tag{1.16}$$

Replacing the sum of first and second term by $\dot{\rho}$ gives the last form

Mass conservation equation (differential form (2) - substancial form)

$$\dot{\rho} + \rho \nabla \vec{u} = 0 \tag{1.17}$$

1.1.2 Newton's second law : Momentum equation

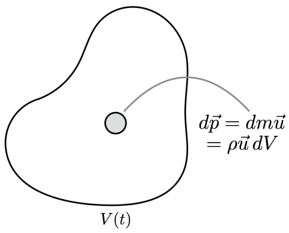

Figure 1.4

Momentum in this course is noted $\vec{P}(t)$. For closed systems,

$$\frac{d\vec{P}(t)}{dt} = \sum \vec{F} = \frac{d}{dt}\int_{V(t)} \rho\vec{u}\,dV \tag{1.18}$$

where $\rho\vec{u}$ is the momentum density. We will spell out the expression of the 2 members. First, the derivative, using the Reynolds transport theorem gives

$$\frac{d\vec{P}}{dt} = \int_{V(t)} \frac{\partial \rho\vec{u}}{\partial t}\,dV + \oint_{S(t)=\partial V(t)} \rho\vec{u}(\vec{u}\vec{n})\,dS \tag{1.19}$$

This written in indicial notation

$$\begin{aligned}\frac{dP_i}{dt} &= \int_{V(t)} \frac{\partial \rho u_i}{\partial t}\,dV + \oint_{S(t)=\partial V(t)} \underbrace{\rho u_i u_j}_{tensor:\,\vec{u}\otimes\vec{u}}\; n_j\,dS \\ &= \int_{V(t)} \frac{\partial \rho u_i}{\partial t}\,dV + \oint_{S(t)=\partial V(t)} \rho(\vec{u}\otimes\vec{u})\vec{n}\,dS\end{aligned} \tag{1.20}$$

and by applying Gauss theorem to the surface integral

$$\frac{d\vec{P}}{dt} = \int_{V(t)} \left[\frac{\partial \rho\vec{u}}{\partial t} + \nabla\rho\vec{u}\otimes\vec{u}\right] dV \qquad and \qquad \frac{dP_i}{dt} = \int_{V(t)} \left[\frac{\partial \rho u_i}{dt} + \frac{\partial}{\partial x_j}(\rho u_i u_j)\right] dV \tag{1.21}$$

Based on the previous forms, we can generalize this for any arbitrary function ϕ

$$\begin{aligned}\frac{d}{dt}\int_{V(t)} \rho\phi\,dV &= \int_{V(t)} \left[\frac{\partial\rho\phi}{\partial t} + \frac{\partial}{\partial x_j}\rho\phi u_j\right] dt \\ &= \int_{V(t)} \left[\rho\frac{\partial\phi}{dt} + \phi\underbrace{\left(\frac{\partial\rho}{\partial t} + \frac{\partial\rho u_j}{\partial x_j}\right)}_{=0\ (1.13)} + \rho u_j\frac{\partial\phi}{\partial x_j}\right] dV \\ &= \int_{V(t)} \rho\left[\frac{\partial\phi}{\partial t} + u_j\frac{\partial\phi}{\partial x_j}\right] dV\end{aligned} \tag{1.22}$$

Similarly to thermodynamics courses, we can introduce an extensive variable Φ and an intensive ϕ to have

General relation for any arbitrary function in closed systems

$$\frac{d\Phi}{dt} = \int_{V(t)} \left[\frac{\partial\rho\phi}{\partial t} + \nabla\rho\phi\vec{u}\right] dV = \int_{V(t)} \rho\dot{\phi}\,dV \tag{1.23}$$

For the specific case where $\mathbf{\Phi} = \vec{P}$ and $\phi = \vec{u}$, we obtain

$$\frac{d\vec{P}}{dt} = \int_{V(t)} \rho \underbrace{\left[\frac{\partial \vec{u}}{\partial t} + \vec{u}\nabla\vec{u}\right]}_{\dot{u}} dV \tag{1.24}$$

We can now express the forces applied on the system. There are 2 main classes :

- **Distance forces (volume)** $\vec{F}_V$:
  This type of force allows a body to influence another without being in contact with.
  - The most present one is gravity which is applied on each fluid particles ($d\vec{F} = dm\vec{g}$). We can imagine that there exists a force density $\vec{f}$ such that

    $$\vec{F}_V = \int_{V(t)} \vec{f}\,dV = \int_{V(t)} \rho\vec{a}\,dV \tag{1.25}$$

    where $\vec{a}$ is a force per unit mass, so an acceleration (gravity : $\vec{f} = \rho\vec{g}$).
  - If we have an electric material, we can talk about electromagnetic forces, which can be modelled as

    $$\vec{f} = \rho_c(\vec{E} + \vec{u} \times \vec{B}) + \vec{J} \times \vec{B} \tag{1.26}$$

    where ρ_c is the charge density $[C/m^3]$ and the second term is the Lorentz force. Indeed, if we have a lot of particles, we can talk of an average velocity $\vec{v}_k = \vec{u} + \vec{C}_k$, where C_k is a particular velocity due to molecular agitation. The force applied on the system is

    $$\vec{F}_k = q_k[\vec{E} + \vec{v}_k \times \vec{B}] \qquad \Leftrightarrow \qquad \underbrace{\frac{\sum \vec{F}_k}{V}}_{\rho_c} = \frac{\sum q_k}{V}(\vec{E} + \vec{u} \times \vec{B}) + \underbrace{\frac{\sum q_k \vec{C}_k}{V}}_{\vec{J}} \times \vec{B} \tag{1.27}$$

    Mollecules are in general neutral, but containing non-neutral regions. Fluids are essentially neutral, $\vec{F}_V = 0$ in most cases. They are called quasi-neutral fluids. Electric influenced fluids will not be considered in that course but they existence has to be known.
  - They are also entrainment and Coriolis forces in rotating frame of references. These forces due to the rotation of Earth are not considered due to the small rotative velocity, unlike pomps and turbines.

- **Contact forces (surface)** $\vec{F}_S$:

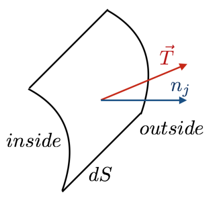

Figure 1.5

These forces results from the contact of an internal and external fluid in regard of a region of surface $dS(t)$. We have

$$d\vec{F}_S = \vec{T}\,dS \qquad \Rightarrow \vec{F}_S = \oint_S \vec{T}\,dS \tag{1.28}$$

$\vec{T}$ is a force per unit area, a continuous function of space depending on location $\vec{x}$ and also linearly on the infinitesimal surface orientation $\vec{n}$. If $\vec{T}$ is the force per unit area for a surface element normal to the unit vector in the j direction e_j, $\vec{T}(\vec{x}) = \vec{T}_j n_j$. (1.28) becomes

$$\vec{F}_S = \oint_S \vec{T}_j n_j\,dS \qquad \textit{and} \qquad F_{S_i} = \oint_S \underbrace{\tau_{j,i}}_{\sigma_{ji}} n_j\,dS \tag{1.29}$$

where σ_{ji} is the stress tensor.

We can now take (1.18) and replace the terms using (1.24), (1.25) and (1.29) to obtain the

Momentum equation (integral form)

$$\frac{dP_i}{dt} = \int_{V(t)} \left[\frac{\partial \rho u_i}{\partial t} + \nabla \rho u_i \vec{u}\right] dV = \int_{V(t)} \rho \dot{u}\, dV = \int_{V(t)} \rho a_i\, dV + \oint_{S(t)} \sigma_{ji} n_j\, dS \tag{1.30}$$

We can see that σ_{ji} and $\rho u_i u_j$ have the same mathematical nature. This is not surprising because in fact these forces result from molleculare agitation in fluids. Let's discuss it. We said that $\vec{v}_k = \vec{u} + \vec{C}_k$. Let's consider a surface element and make the hypothesis that the fluid is in rest, so the average velocity $\vec{u} = 0$. It doesn't mean that the particles are immobile, but that if all particles have the same mass (pure fluid) and if a certain number of particles are going from right to left with velocity $\vec{c}$, there are the same number of particles going from left to right with velocity $-\vec{c}$. There is no global mass flux. So for n particles going in one direction, the mass flux

$$2nm\vec{u} = nm\vec{c} + 2nm(-\vec{c}) = 0 \tag{1.31}$$

To obtain the momentum in direction x_1, we have to multiply the mass flow in this direction by the velocity in this direction

$$nm(\vec{c} \cdot \vec{e_1})c_1 + nm(-\vec{c_1} \cdot \vec{e_1})(-c_1) = 2nmc_1^2 \tag{1.32}$$

The global momentum flux traversing the unit surface is so positive going out of the volume. We need so a balance force in the opposite direction to keep the mass in. This explains the presence and nature of σ_{ji} which is a momentum flux.
Let's finally establish the differential form of the momentum equation, applying Gauss theorem to the second right side of (1.30)

$$\int_{V(t)} \rho a_i\, dV + \oint_{S(t)} \sigma_{ji} n_j\, dS = \int_{V(t)} \left[\rho a_i + \frac{\partial \sigma_{ji}}{\partial x_j}\right] dV \tag{1.33}$$

and by considering the whole equation

$$\int_{V(t)} \left[\frac{\partial \rho u_i}{\partial t} + \frac{\partial \rho u_i u_j}{\partial x_j} - \rho a_i - \frac{\partial \sigma_{ji}}{\partial x_j}\right] dV = 0 = \int_{V(t)} \left[\rho \dot{u}_i - \rho a_i - \frac{\partial \sigma_{ji}}{\partial x_j}\right] dV \tag{1.34}$$

and for this to be true for all systems we consider, we obtain

Momentum equation (differential form (1) - divergent form)

$$\frac{\partial \rho u_i}{\partial t} + \frac{\partial \rho u_i u_j}{\partial x_j} = \rho a_i + \frac{\partial \sigma_{ji}}{\partial x_j} \qquad \textit{and} \qquad \frac{\partial \rho \vec{u}}{\partial t} + \nabla \rho \vec{u} \otimes \vec{u} = \rho \vec{a} + \nabla \bar{\bar{\sigma}} \tag{1.35}$$

Momentum equation (differential form (2) - substancial form)

$$\rho a_i + \frac{\partial \sigma_{ji}}{\partial x_j} = \rho \dot{u}_i \qquad \textit{and} \qquad \rho \vec{a} + \nabla \bar{\bar{\sigma}} = \rho \dot{\vec{u}} \tag{1.36}$$

1.1.3 Angular momentum equation

This is a corollary of the momentum equation that states that *the time rate of change of the angular momentum of a closed system is equal to the sum of the torques applied to the system.* There is no additional information except that the stress tensor should be symetric

$$\sigma_{ji} = \sigma_{ij} \tag{1.37}$$

1.1.4 Energy equation - First principle of thermodynamics

If we note $\mathcal{E}$ the total energy of the system, the first principle tells that

$$\frac{d\mathcal{E}}{dt} = \dot{W} + \dot{Q} \tag{1.38}$$

where $\dot{W}$ is the mechanical power provided by the forces applied on the system and $\dot{Q}$ the thermal power provided to the system. We will proceed like the previous equation expressing first the left side then the right side. If we note E the total energy per unit mass, e the internal energy per unit mass and k the kinetic energy per unit mass (potential energy is not considered in order not to take into account power coming from potential forces)

$$\mathcal{E} = \int_{V(t)} E\, dm = \int_{V(t)} \rho E\, dV = \int_{V(t)} \rho(e+k)\, dV \qquad with \quad k = \frac{\vec{u}\vec{u}}{2} \tag{1.39}$$

The time derivative of the energy using the Reynolds transport theorem, then the Gauss theorem is

$$\begin{aligned} \frac{d\mathcal{E}}{dt} &= \int_{V(t)} \frac{\rho(e+k)}{dt}\, dV + \oint_{S(t)} \rho(e+k)\vec{u}\vec{n}\, dS \\ &= \int_{V(t)} \left[\frac{\rho(e+k)}{dt} + \nabla\rho(e+k)\vec{u}\right] dV = \int_{V(t)} \rho(e \dot{+} k)\, dV \end{aligned} \tag{1.40}$$

Let's now go on with with the mechanical power expression. We expressed in (1.30) that there are volume and surface forces. These multiplied by the velocity and using Gauss gives

$$\dot{W} = \int_{V(t)} \rho a_i u_i\, dV + \oint_{S(t)} \sigma_{ji} u_i n_j\, dS = \int_{V(t)} \left[\rho a_i u_i + \frac{\partial}{\partial x_j}\sigma_{ji}u_i\right] dV \tag{1.41}$$

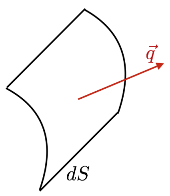

Figure 1.6

For the thermal power expression, we need to introduce a new concept that is the heat flux vector $\vec{q}$ which qualifies a thermal power per unit area leaving the surface. Physically, there is only two heat transport mecanism which are radiation and conduction. Indeed, convection is a specific conduction case where the temperature gradient region becomes thinner and favorises the exchange. The thermal power is

$$\dot{Q} = -\oint_{S(t)} \vec{q}\vec{n}\, dS = -\int_{V(t)} \nabla\vec{q}\, dV \tag{1.42}$$

Replacing the terms of (1.38) by (1.40), (1.41) and (1.42) gives

Total energy equation (integral form)

$$\int_{V(t)} \frac{\rho(e+k)}{dt}\, dV + \oint_{S(t)} \rho(e+k)\vec{u}\vec{n}\, dS = \int_{V(t)} \rho\vec{a}\vec{u}\, dV + \oint_{S(t)} (\bar{\bar{\sigma}}\vec{n})\vec{u}\, dS - \oint_{S(t)} \vec{q}\vec{n}\, dS \tag{1.43}$$

The differantial form is obtained using Gauss theorem for the two sides and regrouping all the terms into one integral

$$\begin{aligned} & \int_{V(t)} \left[\frac{\rho(e+k)}{dt} + \nabla\rho(e+k)\vec{u} - \rho\vec{a}\vec{u} - \nabla\bar{\bar{\sigma}}\vec{u} + \nabla\vec{q}\right] dV = 0 \\ \Leftrightarrow \quad & \int_{V(t)} \left[\rho(e \dot{+} k) - \rho\vec{a}\vec{u} - \nabla\bar{\bar{\sigma}}\vec{u} + \nabla\vec{q}\right] dV = 0 \end{aligned} \tag{1.44}$$

And considering the fact that this has to be true for all systems, we obtain the two last forms

Total energy equation (differential form (1) - divergent form)

$$\frac{\rho(e+k)}{dt} + \nabla\rho(e+k)\vec{u} = \rho\vec{a}\vec{u} + \nabla\bar{\bar{\sigma}}\vec{u} - \nabla\vec{q} \tag{1.45}$$

Total energy equation (differential form (2) - substantial form)

$$\rho(\dot{\overline{e+k}}) = \rho(\dot{e}+\dot{k}) = \rho\vec{a}\vec{u} + \nabla\bar{\bar{\sigma}}\vec{u} - \nabla\vec{q} \tag{1.46}$$

Let's finally establish the distribution of the forces in the different energies. If we multiply (1.36) by velocity $\vec{u}$ and if we observe that $\dot{k} = \frac{\dot{u}_i u_i + u_i \dot{u}_i}{2} = u_i\dot{u}_i$, we obtain

Kinetic - Mechanical energy equation

$$\vec{u}\left(\rho a_i + \frac{\partial\sigma_{ji}}{\partial x_j} = \rho\dot{u}_i\right) \qquad \Leftrightarrow \qquad \rho\underbrace{u_i\dot{u}_i}_{\dot{k}} = \rho u_i a_i + \frac{u_i\partial\sigma_{ji}}{\partial x_j} \tag{1.47}$$

The difference between total energy (1.46) and kinetic energy (1.47) gives the internal energy

Internal energy equation

$$\rho\dot{e} = 0 + \sigma_{ji}\frac{\partial u_i}{\partial x_j} - \nabla\vec{q} \tag{1.48}$$

We see that volume forces only contributes to the kinetic energy, heat flux only to the internal energy and the suface forces to both.

1.1.5 Summary - Complementary equation

Let's make the inventory of the 3 substancial equations that we found. How many equations and unknowns do we have?

- In continuity equation (1.17), ρ and u_i are 4 unknowns in 3D.
- In momentum equation (1.36), a_i is an external applied force so is known, σ_{ji} consists in 6 unknowns (symetric matrix).
- In internal energy equation (1.48), e and $\vec{q}$ are 4 most unknowns.

The total unknowns number is 14. The number of disponible equations is 5, 1 thanks to the energy, 1 thanks to the continuity and 3 thanks to the vectorial momentum equation. In this stage, we haven't made any assumption on the nature of the material we're considering. These equations are valid for an elastic solid as a fluid. The main difference is that solids resist to a deformation whereas fluid doesn't. But fluid resists to a rate of deformation. The way that stress tensor σ_{ji} is related to the displacement field is called the constitutuve equations.

Constitutive relations

For a fluid, the stress tensor depends on the fluid rate of deformation (rate of strain). To express σ_{ji}, we have to find a quantity in the field of motion of the fluid that represents the rate of strain. If the velocity field $\vec{u}(\vec{x},t)$ was uniform, not depending on $\vec{x}$, the fluid will be moving as a bulk and there is no rate of deformation. The rate of strain must be somehow related to the velocity gradient tensor $\nabla \otimes \vec{u}$. We know that all tensors can be decomposed in an antisymetric and symetric part like

$$\nabla \otimes \vec{u} = \frac{\partial u_j}{\partial x_i} = \Omega_{ji} + S_{ij} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} - \frac{\partial u_i}{\partial x_j}\right) + \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right). \tag{1.49}$$

For a constant gradient velocity field, the velocity field is linear in the coordinates

$$u_j = \frac{\partial u_j}{\partial x_i} x_i = \Omega_{ji} x_i + S_{ij} x_i \tag{1.50}$$

Let's look to the mathematical nature of the antisymetric part. If we express using Kronecker δ, we have

$$\Omega_{ji} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} - \frac{\partial u_i}{\partial x_j}\right) = \frac{1}{2}\delta_{kij}\delta_{kqp}\frac{\partial u_p}{\partial x_q} \qquad with \quad \delta_{kij}\delta_{kqp} = \delta_{iq}\delta_{jp} - \delta_{ip}\delta_{jq} \tag{1.51}$$

Knowing that $(a \times b)_k = \delta_{kpq} a_p b_q$, we can introduce the **curle** (rotationnel) of velocity called vorticity $\vec{\omega}$

$$\delta_{kqp}\frac{\partial u_p}{\partial x_q} = (\nabla \times \vec{u})_k \qquad \Rightarrow \nabla \times \vec{u} = \vec{\omega} \tag{1.52}$$

Let's look to the way this is linked to (1.50)

$$u_j^{AS} = \Omega_{ji} x_i = \frac{1}{2}\delta_{jki}\omega_k x_i = \frac{1}{2}(\vec{\omega} \times \vec{x})_j \qquad \Leftrightarrow \qquad \vec{u}^{AS} = \frac{1}{2}\vec{\omega} \times \vec{x} \tag{1.53}$$

In conclusion, we see that the antisymetric part consists in a pure rotation velocity field, a rigid body motion of angular velocity $\frac{1}{2}\vec{\omega}$ without strain. $\vec{\omega}$ is twice the angular velocity of fluid particles around themselves. The quantity representative of the fluid rate of strain can only be the symetric part of the velocity gradient tensor called the rate of strain tensor. For a fluid, $\sigma_{ij} = f(S_{ij})$.

To determine the nature of this relationship, we will assume that σ_{ij} is a linear function of S_{pq}. This is called

Newton's assumption for stresses

$$\sigma_{ij} = a_{ij} + b_{ijpq} S_{pq}. \tag{1.54}$$

In this equation, b_{ijpq} is a tensor with four indices, but we know that it's symetric with respect to pq and ij because S is symetric with respect to pq and ij, leading to $6 \times 6 = 36$ coefficients. Symetric tensor a_{ij} counts 6 coefficients, for a total of 42 coefficients.
If we assume that the fluid is **isotropic**, meaning that the fluid react in the same way whatever the sollicitation direction. For example, let's take a case of S_{ij} where all coefficients are null except the S_{11} term. Diagonal terms reprensent a rate of elongation/stretch while the off-diagonal terms represent an angular deformation between two perpendicular direction. The assumption means that if the rate of stress is not in 1 direction but 2, the fluid reaction will be the same. In other words, if we make a rotation of coordinates, the relation in the rotated frame of reference must be the same. In that case, the relation reduces to

$$\sigma_{ij} = a\delta_{ij} + bS_{ij} + c\delta_{ij}S_{kk} \tag{1.55}$$

where only 3 coefficient must be found. It is natural to think that air and water have no preferential direction unlike certain solid as wood that has a preferential direction related to the

orientation of fibers. Blood or dissolved polymer chains are examples of non isotropic fluids. We will from now consider the fluid to be isotropic.
In (1.55) a is a constant that represents the stress present when the fluid is at rest. The surface force associated to that component is purely normal

$$\sigma_{ij}n_j = a\delta_{ij}n_j = an_i \tag{1.56}$$

This constant corresponds to the pressure exerted by the fluid at rest. Because of its application in the opposite direction to the normal, it's negative. The two other coefficients represents the 2 coefficients of viscosity

$$a = -p \qquad b = 2\mu \qquad c = \lambda \tag{1.57}$$

The stress tensor equation can so be written with a pressure stress and a viscous stress part like

$$\sigma_{ij} = -p\delta_{ij} + \tau_{ij} \qquad with \qquad \tau_{ij} = 2\mu S_{ij} + \lambda\delta_{ij}S_{kk} \tag{1.58}$$

An alternative form to that is the following

$$\tau_{ij} = 2\mu \underbrace{\left(S_{ij} + \frac{1}{3}\delta_{ij}S_{kk}\right)}_{\equiv S_{ij}^S} + \underbrace{\left(\lambda + \frac{2\mu}{3}\right)}_{\equiv \mu_V} \delta_{ij}S_{kk} \tag{1.59}$$

This notation is necessary to make appear the part of the strain tensor which has no trace S_{ij}^S, called the rate of shear. Indeed

$$S_{ii}^S = S_{ii} - \frac{1}{3}\delta_{ii}S_{kk} = 0 \tag{1.60}$$

This means that S_{ij}^S represents the trace less part of the rate of **strain** tensor called the sheer rate tensor. What is now S_{kk} ?

$$S_{kk} = \frac{1}{2}\left(\frac{\partial u_k}{\partial x_k} + \frac{\partial u_k}{\partial x_k}\right) = \frac{\partial u_k}{\partial x_k} = \nabla\vec{u} \tag{1.61}$$

The divergence of the velocity is related to the rate of dilatation of the fluid, the change of volume. We decomposed the rate of strain in a part representing the deformation without change of volume (pure deformation) and another with change of volume (μ_V is the bulk viscosity). Another expression for τ_{ij} with a final net gain of 3 unknowns is

$$\tau_{ij} = 2\mu S_{ij}^S + \mu_V\delta_{ij}\nabla\vec{u} \tag{1.62}$$

At this stage, we have to determine still 6 unknowns from the 9 at the beginning.

Heat flux

We discussed about the fact that heat flux propagates using 2 physical mechanism : conduction and radiation. In the energy equation it's the divergence of the heat fluc that appears. In most application, the radiative effect does not imply heat accumulation or loss. The fluids are so transparent to radiative heat flux $\nabla\vec{q}^{rad} = 0$. We only have conduction and the Fourier law says that

$$\vec{q} \propto \nabla T = d\nabla T = -\kappa\nabla T \qquad \Leftrightarrow \qquad q_i = -\kappa\frac{\partial T}{\partial x_i} \tag{1.63}$$

The negative sign comes from the fact that heat goes from hot to cold (decrease of T). We have a net gain of 1 unknown with this equation.

Thermodynamics

At this stage, we are using 4 thermodynamics intensive variables which are ρ, e, p, T. We know that for a single phase fluid, the variance is 2, meaning that we can use 2 thermodynamics equations of state (EoS) relating them. For example, for a alorically and thermally perfect gas, we will have

$$p = \rho RT \qquad and \qquad e = c_v T \tag{1.64}$$

We have a net gain of 2 unknows, so there remains 2 unknowns.

Transport coefficients

The remaining variables are the shear viscosity μ, the bulk viscosity μ_V and thermal conductivity κ. These are functions of the thermodynamic state. For example for gases we have the relations

$$\mu = f(T) \qquad and \qquad Pr = \frac{\mu c_p}{\kappa} = cst \Leftrightarrow \kappa = \frac{\mu(T)c_p(T)}{Pr} \tag{1.65}$$

The bulk viscosity is more difficult to determine, but it can be shown that for monoatomic gases (no internal degres of freedom) $\mu_V = 0$. For diatomic gases it's much more delcate to measure, but it has been shown that for many flows, the flows is insensitive to the variation of value of bulk viscosity. In fact, for fluids without divergence of velocity, we don't care about μ_V because there is no variation of volume (1.62). We will so make the following assumption

Stokes assumption

$$\mu_V = 0 \quad \text{even for other gases.} \tag{1.66}$$

We're done, we have as many equations as variables. We mentioned the first principle of thermodynamics but not the second. Let's analyse that.

Second principle of thermodynamics

We will reuse the internal energy equation (1.48), replace σ_{ij} by it's expression in (1.58) and use the fact that $\frac{\partial u_i}{\partial x_j} = \Omega_{ij} + S_{ij}$

$$\rho\dot{e} = \sigma_{ij}\frac{\partial u_i}{\partial x_j} - \frac{\partial q_i}{\partial x_i} = -p\delta_{ij}\frac{\partial u_i}{\partial x_j} + (2\mu S^S_{ij} + \mu_V\delta_{ij}\nabla\vec{u})(S_{ij} + \cancel{\Omega_{ij}}) - \frac{\partial q_i}{\partial x_i} \tag{1.67}$$

where Ω_{ij} doesn't contribute because the contraction of the symetric tensor by the antisymetric tensor is equal to 0. We have the relation

$$S^S_{ij} = S_{ij} - \frac{1}{3}\delta_{ij}\nabla\vec{u} \qquad \Leftrightarrow \qquad S_{ij} = S^S_{ij} + \frac{1}{3}\delta_{ij}\nabla\vec{u} \tag{1.68}$$

Combined to the fact that $\delta_{ij}\frac{\partial u_i}{\partial x_j} = \frac{\partial u_i}{\partial x_i} = \nabla\vec{u}$, we obtain

$$\rho\dot{e} = -p\nabla\vec{u} + (2\mu S^S_{ij} + \mu_V\delta_{ij}\nabla\vec{u})\left(S^S_{ij} + \frac{1}{3}\delta_{ij}\nabla\vec{u}\right) - \nabla\vec{q} \tag{1.69}$$

The mass conservation equation tells us that we can write the divergence as

$$\dot{\rho} + \rho\nabla\vec{u} = 0 \qquad \Leftrightarrow \qquad \nabla\vec{u} = -\frac{\dot{\rho}}{\rho} = -\rho\left(\frac{\dot{\rho}}{\rho^2}\right) = \rho\left(\dot{\frac{1}{\rho}}\right) = \rho\dot{v} \tag{1.70}$$

If we replace the divergence in the previous equation, we have

$$\rho\dot{e} = -\rho p\dot{v} + (2\mu S_{ij}^S + \mu_V \delta_{ij}\nabla\vec{u})\left(S_{ij}^S + \frac{1}{3}\delta_{ij}\nabla\vec{u}\right) - \nabla\vec{q} \tag{1.71}$$

The first term here looks like the reversible work $-pdv$ in thermodynamics and is so the reversible contribution to the internal energy. Let's make this appear by bringing this to the left side. We make appear $\rho[\dot{e} + p\dot{v}]$, but we have the famous Gibbs relation $de = Tds - pdv \Leftrightarrow \dot{e} = T\dot{s} - p\dot{v}$. We have now

$$\rho\dot{s} = \frac{(2\mu S_{ij}^S + \mu_V \delta_{ij}\nabla\vec{u})\left(S_{ij}^S + \frac{1}{3}\delta_{ij}\nabla\vec{u}\right)}{T} - \frac{\nabla\vec{q}}{T} \tag{1.72}$$

If we remind the relation we demonstrated before for any variable $\dot{\Phi} = \int_V \rho\dot{\phi}dV$, we can make the analogy here to say that when this is integrated over a volume, it gives the time rate of change of the entropy of the closed system that's initially inside this volume. We have to identify the reversible part in this equation. We know that the reversible entropy rate of exchange for a uniform system and its integral over a closed surface is given by

$$\frac{\vec{q}dS}{T} \qquad \Rightarrow \qquad \oint_S \frac{\vec{q}}{T}(-\vec{n})\, dS = -\int_V \nabla\frac{\vec{q}}{T}\, dV. \tag{1.73}$$

We see that we have to make appear a this in the last equation. But we know that

$$\frac{\nabla\vec{q}}{T} = \nabla\frac{\vec{q}}{T} - \vec{q}\nabla\left(\frac{1}{T}\right) = \nabla\frac{\vec{q}}{T} + \vec{q}\frac{\nabla T}{T^2} \tag{1.74}$$

And by introducing this into the relation (1.72), we make appear the reversible entropy rate of exchange

$$\rho\dot{s} = -\nabla\frac{\vec{q}}{T} - \frac{\vec{q}\nabla T}{T^2} + \frac{(2\mu S_{ij}^S + \mu_V \delta_{ij}\nabla\vec{u})\left(S_{ij}^S + \frac{1}{3}\delta_{ij}\nabla\vec{u}\right)}{T} \tag{1.75}$$

We also know that $\vec{q} = -\kappa\nabla T$, making appear $(\nabla T)^2$

$$\rho\dot{s} = -\nabla\frac{\vec{q}}{T} + \frac{\nabla T\nabla T}{T^2} + \frac{(2\mu S_{ij}^S + \mu_V \delta_{ij}\nabla\vec{u})\left(S_{ij}^S + \frac{1}{3}\delta_{ij}\nabla\vec{u}\right)}{T} \tag{1.76}$$

If we imagine a fluid at rest with only a heat exchange operating on it, the third term $= 0$, the first term is reversible so anyway the sign and the second term must be positive. This implies $\kappa \geq 0$ due to the square of the other varaibles (the heat has to go from hot to cold). Let's expand the third term

$$\rho\dot{s} = -\nabla\frac{\vec{q}}{T} + \frac{\nabla T\nabla T}{T^2} + \frac{1}{T}\left[2\mu S_{ij}^S S_{ij}^S + \cancel{\mu_V\nabla\vec{u}\delta_{ij}S_{ij}^S} + \cancel{2\mu S_{ij}^S\frac{\delta_{ij}}{3}}\nabla\vec{u} + \mu_V\frac{\delta_{ij}\delta_{ij}}{3}(\nabla\vec{u})^2\right] \tag{1.77}$$

In this last equation, the second and third terms are nul because $S_{ii}^S = 0$. Let's imagine that we have a fluid with only dilation and no shear S_{ij}^S, the last term must be positive and so μ_V has to be positive (≥ 0). In the other hand, for the first term, we have a quadratic form (sum of squares ≥ 0), so μ has to be positive. To verify the second principle, we have to verify these 3 inequalities. In fluid mechanics, we don't have to worry about the second principle, it's built in the equations as long as the transport coefficient are positive.

1.1.6 Boundary conditions

We have now to establish the boundary conditions which makes the difference between the flow cases. First of all, we have two main categories of flows :

- **External flows** (unbounded domain)
  For example, a flow over a wing, assuming that atmosphere extends to infinity. In that case we have far field boundary conditions, what happens far from the body ($u \to u_\infty, p \to p_\infty, T \to T_\infty$).
- **Internal flows** (bounded domain)
  For example, a flow in a pipe or a fluid in a rotating machine like a pump. In that case we don't have the far field conditions but the inlet and outlet boundary conditions but this problem is not discussed here.

Solid surfaces

In both case we have solid surfaces, we have to make a distinction. We wrote the equation for the general case of a viscous flow, but there is flows where the viscous stresses can be neglected (not = 0 !) leading to what we call the **inviscid flows**. Let's analyse the two cases.

Viscous flows

Viiscosity is associated to the exchange of momentum between neighboring fluid layers due to molecular agitation. If we have a molecule coming from a low velocity region to a high velocity region, it slows down the molecule there and inversely. The same occurs when a fluid particle enter in contact with solid surfaces, it exchange momentum. The result is that velocity and temperature fields must be continuous

$$\vec{u}_{fluid} = \vec{u}_{wall} \qquad and \qquad T_{fluid} = T_{wall} \tag{1.78}$$

In particular, for a surface at rest, the fluid must be at rest on the solid surface as well. This is called the **no-slip condition**.

Inviscid flows

For inviscid flows, this mecanism doesn't exist, the fluid may slip. The boundary condition is that the fluid can't go throw the solid

$$\vec{u}_{fluid}\vec{n} = \vec{u}_{wall}\vec{n} \tag{1.79}$$

This is called the **slip/no penetration condition**. The previous condition is stronger because in fact $\vec{u} = \vec{u}_n\vec{n} + \vec{u}_t$ includes the tangential condition too.

1.2 Special cases

1.2.1 General case

The generale equations are the following :

- Mass conservation equation

$$\dot{\rho} + \rho\nabla\vec{u} = 0 \tag{1.80}$$

- Momentum equation

$$\rho\dot{\vec{u}} = -\nabla p + \nabla\bar{\bar{\tau}} + \rho\vec{F} \tag{1.81}$$

- Energy equation

$$\rho\dot{e} = -p\nabla\vec{u} + \underbrace{\bar{\bar{u}}..\nabla\otimes\vec{u}}_{\epsilon_V} - \nabla\vec{q} \tag{1.82}$$

  where $\nabla\otimes\vec{u}$ can be replace by the symetric part, rate of strain tensor $\bar{\bar{S}}$ and ϵ_V is the viscous dissipation.

- Constituve relation

$$\begin{aligned}\tau_{ij} &= 2\mu\left(S_{ij} - \frac{1}{3}\delta_{ij}\nabla\vec{u}\right) + \mu_V S_{ij}\nabla\vec{u} \\ &= \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\nabla\vec{u}\right) + \mu_V S_{ij}\nabla\vec{u}\end{aligned} \tag{1.83}$$

- Conductive heat flux

$$\vec{q} = -\kappa\nabla T \tag{1.84}$$

1.2.2 Steady flow

This is caracterized by the fact that $\frac{\partial}{\partial t} = 0$ and implies for example that

$$\dot{\rho} = \frac{\partial\rho}{\partial t} + \vec{u}\nabla\rho = \vec{u}\nabla\rho \tag{1.85}$$

and for the others.

1.2.3 Inviscid flows

They are defined as flows in which vicous stresses and conduction heat flux can be neglected. We are talking about flows and not fluids because there is no fluid with $\mu = 0$ or $\kappa = 0$. This happens for superfluids but we don't care. When we look at the fluid properties tables, in SI units, water and air have very small μ but we can't say that there are negligible because it depends on the system of refference used. If something is negligible it is with respect to something else. Let's start with the viscous stresses. Momentum equation can be written as

$$\rho\dot{\vec{u}} = \frac{\partial\rho\vec{u}}{\partial t} + \nabla\rho\vec{u}\otimes\vec{u} = -\nabla p + \nabla\bar{\bar{\tau}} + \rho\vec{F} \tag{1.86}$$

where the viscous stress tensor is a tensor as the momentum flux tensor. They correspond to the same physical phenomenon but at different scales, the viscous stress tensor is due to the molecular agitation whereas the momentum flux tensor is for the macroscopic scale, the average scale. So it makes sense to compare the order of magnitude of the two ones.

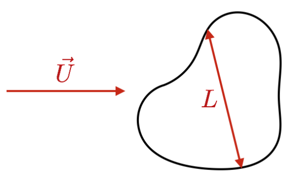

Figure 1.7

Let's consider a fluid flows of far field velocity $\vec{U}$ around a solid body of characteristic lenght L, if we consider the momentum flux tensor, we know that the velocity around the body will vary between 0 and U so the order of magnitude will be $\theta(\rho U^2)$. What about τ? We see that in (1.83) appears the velocity gradient, derivative. What is the order of magnitude of the derivative of a function?

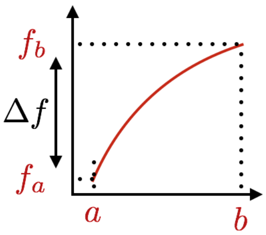

Figure 1.8

Let's consider a function $f(x)$ represented on Figure 1.8. If the function is smooth, so if the function doesn't vary much in the interval, its derivative keeps a constant order of magnitude in the integral. We see it in the figure, the slope varies between a and b, can be twice the slope at the center but keeps the same order of magnitude. So for a smooth function, the order of magnitude of f' remains the same over the interval $f' = \theta\left(\frac{\Delta f}{\Delta x}\right)$. Let's use this to have an approximation for the velocity gradient tensor

$$\nabla \otimes \vec{u} = \theta\left(\frac{U}{L}\right) \qquad \Rightarrow \bar{\bar{\tau}} = \theta\left(\mu\frac{U}{L}\right) \tag{1.87}$$

The relative order of magnitude of viscous stresses with respect to momentum flow is

$$\frac{\mu\frac{\cancel{U}}{L}}{\rho U^{\cancel{2}}} = \frac{\mu}{\rho U L} = \frac{1}{Re_L} \tag{1.88}$$

We conclude that viscous stresses can be neglected in the case of high Reynolds number.

Now we have to verify the assumption that velocity is a smooth function of the coordinates. Examples of not smooth functions are represented on Figure 1.9 where the green curve is smooth close to the limits but not smooth in a small interval and the yellow one is a periodic function with the characteristic wave length. In these cases, Δx is not the appropriate length scale to determine the order of magnitude of the derivative.

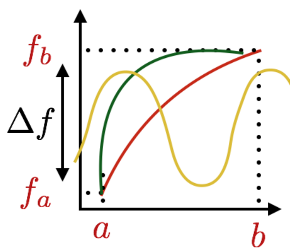

Figure 1.9

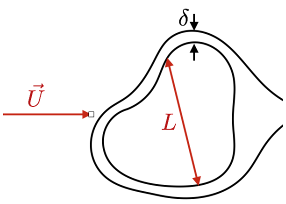

Figure 1.10

Now what about the velocity field? Let's assume that velocity is smooth and the fluid inviscid, the boundary conditions to respect are the slip conditions. But we know that μ is not strictly 0, so the velocity must be 0 at the wall and therefore, there must exist a region of another caracteristic scale δ of rapidly changing velocity close to the wall. In this case the study of order of magnitude made is incorrect, U/L must be replaced by U/δ. Due to the smaller scale of δ compared to L, the viscous stress is more important than outside this region and can be of comparable size to the momentum flux tensor. We conclude that the flow can be decomposed into two regions : a region outside of this viscous layer, a distal region where the viscous stresses and heat conduction term can be neglected and a proximal or inner region where viscous stresses may not be neglected. We complete the definition with high Reynolds number by adding "**except close to solid bodies and in their wake**[1]".

1.2.4 Inviscid flows equations

They are the same as the general case, except that the viscous and heat flux terms are neglected.

- Mass conservation equation

$$\dot{\rho} + \rho\nabla\vec{u} = 0 \tag{1.89}$$

- Momentum equation

$$\rho\dot{\vec{u}} = -\nabla p + \cancel{\nabla\bar{\bar{\tau}}} + \rho\vec{F} \tag{1.90}$$

1. Sillage : refermement de la couche visqueuse à droite de la Figure 1.10.

- Energy equation

$$\rho\dot{e} = -p\nabla\vec{u} + \cancel{\bar{\bar{u}}..\nabla\otimes\vec{u}} - \cancel{\nabla\vec{q}} \qquad \Leftrightarrow \qquad \rho\dot{e} + p\nabla\vec{u} = 0 \tag{1.91}$$

  We already analyzed this expression before, with (1.70) we can conclude that

$$\rho(\dot{e} + p\dot{v}) = 0 = \rho T\dot{s} \qquad \Rightarrow \dot{s} = 0 \tag{1.92}$$

  Entropy per unit mass is constant along trajectories. Only viscous term and heat flux are responsible of irreversible entropy variations. So all the particles keeps constant entropy and if the incoming fluid particles are uniform it means that the entropy will be constant across the whole flow.

Let's specify the terminology, when we speek about uniform quantity it means that $\nabla q = 0$ and steady q means that it doesn't vary with time $\frac{\partial q}{\partial t} = 0$. Let's now see what happens with the momentum equation

$$\begin{aligned}\rho\dot{\vec{u}} &= -\nabla p + \rho\vec{F} = \rho\left[\frac{\partial u}{\partial t} + (\vec{u}\nabla)\vec{u}\right]\\ \rho\dot{u}_i &= -\frac{\partial p}{\partial x_i} + \rho F_i = \rho\left[\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial u_j}\right]\end{aligned} \tag{1.93}$$

But if we say that, by adding and removing the needed term

$$u_j\frac{\partial u_i}{\partial x_j} = u_j\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) + u_j\frac{\partial u_j}{\partial x_i} \tag{1.94}$$

But what's the last term? We know that the gradient of kinetic energy corresponds to that because

$$u_j\frac{\partial u_j}{\partial x_i} = \frac{\partial\frac{u_ju_j}{2}}{\partial x_i} = \frac{\partial k}{\partial x_i} \qquad \text{additionally} \qquad \frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} = \delta_{kji}\omega_k \tag{1.95}$$

Now if we replace this in (1.93), we find the

Lamb's form of the momentum equation

$$\begin{aligned}&-\frac{\partial p}{\partial x_i} + \rho F_i = \rho\left[\frac{\partial u_i}{\partial t} + \delta_{kji}\omega_k u_j + \frac{\partial k}{\partial x_i}\right]\\ \Leftrightarrow &-\frac{\partial p}{\partial x_i} + \rho F_i = \rho\left[\frac{\partial u_i}{\partial t} + (\vec{\omega}\times\vec{u})_i + \frac{\partial k}{\partial x_i}\right]\\ \Leftrightarrow &-\nabla p + \rho\vec{F} = \rho\left[\frac{\partial\vec{u}}{\partial t} + \vec{\omega}\times\vec{u} + \nabla k\right]\end{aligned} \tag{1.96}$$

1.2.5 Barotropic flows - Force deriving from a potential

Like the previous one, there are barotropic flows but no barotropic fluids. Let's rewrite the Lamb's equation

$$\frac{\partial\vec{u}}{\partial t} + \vec{\omega}\times\vec{u} + \nabla k = -\frac{\nabla p}{\rho} + \vec{F} \qquad with\vec{F} = \vec{a} \tag{1.97}$$

In general we know that the thermodynamic state of a pure fluid in single phase is determined by 2 thermodynamic variables. This means that in general, p and ρ are independant variables. But when another thermodynamic variable is constant, uniform, then it exists a relation between p

and ρ and hence it exists a certain function $P(p)$ such that $\frac{dP}{dp} = \frac{1}{\rho(p)}$. This implies that the gradient

$$\nabla P = \frac{dP}{dp}\nabla p = \frac{\nabla p}{\rho} \tag{1.98}$$

allowing us to replace this quantity in (1.97). Two examples of constant variables :

- **Constant density flows** : $\rho(p) = \rho = cst$ and so $\frac{dP}{dp} = \frac{1}{\rho} \Rightarrow P = \frac{p}{\rho}$.
- **Isentropic flows** : we have the Gibbs relation $dh = Tds + vdp$ simplifying in $dh = \frac{dp}{\rho}$ and so $P = h$.

If now in addition to the barotropic flow assumption we assume that $\vec{F}$ derives from a potential, we will make other assumptions. This means that the curl $\nabla \times \vec{F} = 0$, allowing to write $\vec{F}$ as the gradient of a certain potential energy per unit mass $\vec{F} = -\nabla\Phi$. Then we can write (1.97) as

$$\frac{\partial \vec{u}}{\partial t} + \vec{\omega} \times \vec{u} = -\nabla(P + k + \Phi) \tag{1.99}$$

Making the additional assumption that we have a steady flow and multiplying by $\vec{u}$ the two members leads to

Bernouilli's equation (1)

$$\vec{u}(\omega \vec{\times} \vec{u}) = -\vec{u}\nabla(P + k + \Phi) \qquad \Leftrightarrow \qquad P + k + \Phi = e_m \tag{1.100}$$

telling that mecanical energy is constant along streamlines (but can vary between streamlines).

1.2.6 Irrotational flows

This points to flows such that $\vec{\omega} = 0$. In that case we see that

$$-\nabla(P + k + \Phi) = 0 \qquad \Rightarrow P + k + \Phi = cst \tag{1.101}$$

Everywhere in the domain! When the flow is irrotational $\vec{\omega} = \nabla \times \vec{u} = 0$, $\vec{u}$ can be experssed as a velocity potential $\nabla\phi$. Now if we consider an unsteady, barotropic, irrotational inviscid flows with irrotational body forces

$$\frac{\partial \vec{u}}{\partial t} = \frac{\partial \nabla\phi}{\partial t} = \nabla\frac{\partial \phi}{\partial t} = -\nabla(P + k + \Phi) \qquad \Leftrightarrow \qquad \frac{\partial \phi}{\partial t} + P + k + \phi = cst \tag{1.102}$$

Everywhere in the domain.

1.2.7 Incompressible/quasi incompressible flows

Let's consider an inviscid steady flow in absence of body forces (and uniform S due to inlet conditions). Then the momentum equation (1.93) tells that

$$\rho(\vec{u}\nabla)\vec{u} = -\nabla p = -\left.\frac{dp}{d\rho}\right|_S \nabla\rho \tag{1.103}$$

where $\frac{dp}{d\rho}|_S = a^2$, with a the speed of sound.

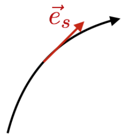

Figure 1.11

If $\vec{e}_s$ is the unit vector along the following streamline. We have that

$$\begin{aligned} \vec{u} = u\vec{e}_s \qquad & \Rightarrow (\vec{u}\nabla)\vec{u} = u\frac{d}{ds}(u\vec{e}_s) = -\frac{a^2}{\rho}\nabla\rho = -\frac{a^2}{\rho}\left[\frac{d\rho}{ds}\vec{e}_s + \frac{d\rho}{dn}\vec{e}_n\right] \\ & \Leftrightarrow u\frac{du}{ds}\vec{e}_s + u^2\frac{d\vec{e}_s}{ds} = -\frac{a^2}{\rho}\left[\frac{d\rho}{ds}\vec{e}_s + \frac{d\rho}{dn}\vec{e}_n\right] \end{aligned} \tag{1.104}$$

We will not discuss the derivative of the unit vector, so if we look to the streamline's direction component, we have

$$u\frac{du}{ds} = -\frac{a^2}{\rho}\frac{d\rho}{ds} \quad \Leftrightarrow \quad \frac{u^2}{a^2}\frac{du}{u} = -\frac{d\rho}{\rho} \qquad \Rightarrow \frac{d\rho}{\rho} = -M^2\frac{du}{u} \tag{1.105}$$

where $M = \frac{u}{a}$ is the **Mach number** and compares the local velocity to the speed of sound. The conclusion is that when M is small, density variations are small. We can so make the approximation of constant density. Even for compressible fluids (like gases), in the conditions provided by the assumptions, density almost does not vary as long as the Mach number is much smaller than 1 (smaller than 0.3 in practice).
A first counter example is the sound waves because they create unsteady flows and natural convection where the density variation is due to temperature variation (we only considered pressure variation here). In these cases density variation is not negligible even for small Mach number.

1.2.8 Two-dimensional (planar) flows

They are essentially flows over cylindrical geometries. A general cylilnder is a body made of straightlines parallel to each other and wich lie upon a two dimensional curl. When we take a surface and draw infinite straightlines, there is no reason for the solution to vary in the infinite direction, all the derivative are 0. This means, according to Figure 1.12, that $\frac{\partial}{\partial x_3} = 0$. Moreover, in general there is no velocity component in this direction but it isn't necessary $u_3 = 0$.

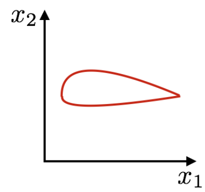

Figure 1.12

In that case, for **steady** flows, the continuity equation becomes

$$\dot{\rho} + \rho\nabla\vec{u} = \frac{\partial\cancel{\rho}}{\cancel{\partial t}} + \nabla\rho\vec{u} = 0 \qquad \Leftrightarrow \frac{\partial(\rho u_1)}{\partial x_1} + \frac{\partial(\rho u_2)}{\partial x_2} = 0 \tag{1.106}$$

This equation can be made satisfied by introducing an auxiliary funciton ψ (called streamfunction), such that

$$\begin{cases} \rho u_1 = & \rho_0\dfrac{\partial\psi}{\partial x_2} \\ \rho u_2 = & -\rho_0\dfrac{\partial\psi}{\partial x_1} \end{cases} \qquad \Rightarrow \qquad \underbrace{\frac{\partial\rho u_1}{\partial x_1}}_{\rho_0\frac{\partial^2\psi}{\partial x_1\partial x_2}} + \underbrace{\frac{\partial^2\rho u_2}{\partial x_2}}_{-\rho_0\frac{\partial^2\psi}{\partial x_2\partial x_1}} = 0 \tag{1.107}$$

We see that continuity equation is satisfied for this function. We can replace the two velocity variables by this function and reduce the unknowns from 2 to 1.

Physical meaning of the streamfunction

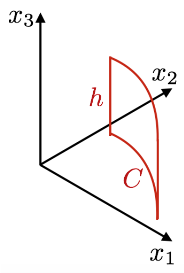

Figure 1.13

If we are in 3D with x_3 the direction of homogeneity and a surface composed of straightlines (height h=1) lying upon a curl C in x_1x_2 plan. The mass flow over the surface is ($\vec{b}=0$)

$$\dot{m} = \int_S \rho \vec{u}\vec{n}\, dS = \int_0^1 dx_3 \int_C \dot{m}\vec{u}\vec{n}\, ds \tag{1.108}$$

Let's look to the curl from top. We will make a little bit geometry on the magnifier of the curl C.

In order to determine the normal to the circuit, we have to close it following an anticlockwise fashion (A to B). If we write the normal following x_1 and x_2 direction, according to Figure 1.14 we have ($dx_1 < 0$)

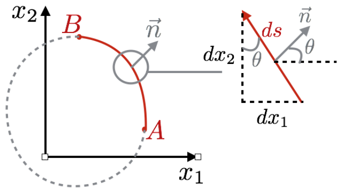

Figure 1.14

$$\left.\begin{aligned} n_1 &= \cos\theta = \frac{dx_2}{ds} \\ n_2 &= \sin\theta = -\frac{dx_1}{ds} \end{aligned}\right\} \Rightarrow \begin{aligned} \rho\vec{u}\vec{n}\, ds &= \rho(u_1n_1 + u_2n_2)\, ds \\ &= \rho\left[u_1\frac{dx_2}{ds} - u_2\frac{dx_1}{ds}\right] ds \\ &= \rho(u_1dx_2 - u_2dx_1) \end{aligned} \tag{1.109}$$

And now if we remplace in (1.108) and use the definition of the streamline function, we have

$$\dot{m} = \int_C \rho(u_1dx_2 - u_2dx_1) = \int_C \rho_0 \underbrace{\left[\frac{\partial\psi}{\partial x_2}dx_2 - \left(-\frac{\partial\psi}{\partial x_1}\right)dx_1\right]}_{d\psi} = \rho_0(\psi_B - \psi_A) \tag{1.110}$$

So, the physical meaning is that ψ on a certain point is the mass flow between this point and a reference point where $\psi = 0$. And because of a steady flow, it's the same mass flow over the surface from A to B whatever the curl used for. If 2 points A and B are on the same streamline then $\psi_A = \psi_B$, so lines with $\psi = cst$ are streamlines.

Streamfunction equation (constant-density flow)

We are interested in finding an equation the streamfunction has to satisfied. The assumption of constant density allow us to consider $\rho = \rho_0$ as it's not a function of space in (1.107)

$$\begin{cases} u_1 = & \dfrac{\partial\psi}{\partial x_2} \\ u_2 = & -\dfrac{\partial\psi}{\partial x_1} \end{cases} \Rightarrow \nabla\vec{u} = 0 \text{ (continuity equation)} \tag{1.111}$$

If we compute the vorticity vector

$$\vec{\omega} = \nabla\times\vec{u} = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \cancel{\frac{\partial}{\partial x_3}} \\ u_1 & u_2 & \cancel{u_3} \end{vmatrix} = \vec{e}_3 \underbrace{\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right)}_{\omega_3} \tag{1.112}$$

Now if we replace the velocity components by the streamfunction equivalence, we found the

Streamfunction equation

$$\omega_3 = -\frac{\partial^2\psi}{\partial x_1^2} - \frac{\partial^2\psi}{\partial x_2^2} \qquad \Leftrightarrow \qquad -\omega_3(x_1,x_2) = \nabla^2\psi \tag{1.113}$$

An equation for vorticity can be obtained by taking the curl of the momentum equation. This last equation compatible with our assumptions is

$$\rho\dot{\vec{u}} = \rho\left[\frac{\partial\vec{u}}{\partial t} + (\vec{u}\nabla)\vec{u}\right] = -\nabla p + p\vec{F} + \nabla\bar{\bar{\tau}} \tag{1.114}$$

Combined with (1.111), we have a set of 4 equations and 4 unknowns in 3D ($\vec{u}$ and p). So the continuity and momentum equations can be solved independently of the energy equation, the thermal problem is dissociated from the hydrodynamic problem. Let's finally point that for irrotational flows we have

$$\nabla^2\psi = 0 \qquad \text{(Laplace's equation)} \tag{1.115}$$

Chapter 2

Similarity and dimensional analysis

This was hystorically first developped by engineers. We derived the governing equations for fluids flows but in practice we can not everytime find an analytical solution. The main weapon of engineers is testing. But it's not easy to do tests on full scale prototypes. We will do tests on a scaled model. But there rises the question of the representativity of the tests.

2.1 Experimental testing - Similarity

Under wich conditions are experiments carried out over a scaled model (m) representative of the flow over the actual prototype (p) ? The response is that the flow has to verify 3 different similarity conditions.

Geometrical similarity

The first and easy one. There exists a unique constant C_x such that space coordinates at analogous points in the model and the prototype are related by a proportionality relation

$$x_{i,m} = C_x x_{i,p} \tag{2.1}$$

This implies that the model and the prototype are **homothetic** $L_m = C_x L_p$. It seems easy but in reality when we have to manufacture something, there are irregularities in molecular level. All machined surfaces are characterized by a certain roughness ϵ. So, the roughness should also be proportional. We have in terms of relative roughness

$$\epsilon_m = C_x \epsilon_p \qquad \Leftrightarrow \qquad \left(\frac{\epsilon}{L}\right)_m = \left(\frac{\epsilon}{L}\right)_p \tag{2.2}$$

We see here that for a model x time smaller than the prototype, the roughness must be x time smaller in order to have the same relative roughness, which is very difficult.

Kinematic similarity

The relation that relates the analogous points of the model and the prototype is (2.1). Similarly, for analogous points in time or analogous time, we have

$$t_m = C_t t_p \tag{2.3}$$

Time is not necesseraly the same. For example, for the humming bird testing, we need to slow down the movement to do measurements leading to more time to accomplish the same movement as the prototype. The kinematic similarity says that velocities at analogous points and times are related by a unique proportionality constant

$$u_{j,m}(x_{i,m}t_m) = C_u u_{j,p}(x_{i,p}, t_{i,p}) = C_u u_{j,p}\left(\frac{x_{i,m}}{C_x}, \frac{t_m}{C_t}\right) \tag{2.4}$$

Dynamic similarity

Forces per unit volue of area at analogous points and times should also be related by a unique proportionality constant. For example if we consider the pressure, we must have

$$p_m = C_p p_p \left(\frac{x_{i,m}}{C_x}, \frac{t_m}{C_t}\right) \tag{2.5}$$

If the dimensional similarity is easy to check, it's not the case for the txo others for which we must study the equaitions of motion.

2.2 Non-dimensional form of the governing equations - Similarity conditions

We deduces from the previous similarities that at analogous points

$$\left.\begin{aligned} x_{i,m} &= C_x x_{i,p} \\ L_m &= C_x L_p \end{aligned}\right\} \qquad \Rightarrow \frac{x_{i,m}}{L_m} = \frac{x_{i,p}}{L_p} \tag{2.6}$$

At this stage we can define non-dimensional coordinates and times like

$$\tilde{x}_i = \frac{x_i}{L} \quad \Rightarrow \tilde{x}_{i,m} = \tilde{x}_{i,p} \qquad and \qquad \tilde{t} = \frac{t}{T} \quad \Rightarrow \tilde{t}_m = \tilde{t}_p \tag{2.7}$$

Analogous points are caracterized by the fact that they have the same non-dimensional coordinates and time value. We have to define a non-dimensional velocity like

$$\tilde{u}_j = \frac{u_j}{U} \qquad \Rightarrow \tilde{u}_{j,m} = \tilde{u}_{j,p} \tag{2.8}$$

For a kinematically similar flows, the non-dimensional velocity fields should be the same for the model and the prototype. How can we verify these idenity? This can only be achieved if the non-dimensional equations are the same for the model and the prototype.

2.2.1 Continuity equation

The dimensional and index form of the continuity equation were

$$\frac{\partial \rho}{\partial t} + \nabla \rho u = 0 = \frac{\partial \rho}{\partial t} + \frac{\partial \rho u_j}{\partial x_j} \tag{2.9}$$

We have to introduce the following independent non-dimensional variables

$$\tilde{t} = \frac{t}{T} \qquad \tilde{x}_j = \frac{x_j}{L} \qquad \tilde{u}_j = \frac{u_j}{U} \qquad \tilde{\rho} = \frac{\rho}{\rho_0} \tag{2.10}$$

And by replacing the variables in general equation and dividing by convective term coefficient, we have

$$\frac{\rho_0}{T}\frac{\partial \tilde{\rho}}{\partial \tilde{t}} + \frac{\rho_0 U}{L}\frac{\partial \tilde{\rho}\tilde{u}_j}{\tilde{x}_j} = 0 \qquad \Leftrightarrow \qquad \frac{L}{UT}\frac{\partial \tilde{\rho}}{\partial \tilde{t}} + \frac{\partial \tilde{\rho}\tilde{u}_j}{\partial \tilde{x}_j} = 0 \tag{2.11}$$

There appears a non-dimensional number that we define as **Strouhal number** $St = \frac{L}{UT}$. L is the caracteristic lenght of the body and UT the lenght travelled in a caracteristic time scale. So St is the ratio of the two length.

Non-dimensional continuity equation

$$St\frac{\partial\tilde{\rho}}{\partial\tilde{t}} + \frac{\partial\tilde{\rho}\tilde{u}_j}{\partial\tilde{x}_j} = 0 \tag{2.12}$$

This equation is the same for the model and the prototype. For the solution to be the same, Strouhal numbers must be the same $St_m = St_p$. It is not a very strict condition to verify because the caracteristic time can be chosen the way to verify the identity.

2.2.2 Momentum equation

The procedure is exactly the same. The divergence form was (considering first the gravitation force)

$$\frac{\partial\rho u_i}{\partial t} + \frac{\partial\rho u_i u_j}{\partial x_j} = \rho g\alpha_i - \frac{\partial p}{\partial x_i} + \frac{\partial\tau_{ji}}{\partial x_j} \tag{2.13}$$

where α_i is the orientation of the gravity vector. Here we have to express the viscous stress tensor using **Stokes hypothesis** ($\mu_V = 0$)

$$\tau_{ji} = 2\mu S_{ij}^S = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\frac{\partial u_k}{\partial x_k}\right) \tag{2.14}$$

There we only have to introduce a non-dimensional viscosity $\tilde{\mu} = \frac{\mu}{\mu_0}$. We have so

$$\tau_{ji} = \frac{\mu_0 U}{L}\tilde{\mu}\left(\frac{\partial\tilde{u}_i}{\partial\tilde{x}_j} + \frac{\partial\tilde{u}_j}{\partial\tilde{x}_i} - \frac{2}{3}\delta_{ij}\frac{\partial\tilde{u}_k}{\partial\tilde{x}_k}\right) = \frac{\mu_0 U}{L}\tilde{\tau}_{ji} \tag{2.15}$$

We have also to introduce a non-dimensional pressure but if we rewrite left side of (2.13) like

$$\rho\left[\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j}\right] = \rho g\alpha_i - \frac{\partial p}{\partial x_i} + \frac{\partial\tau_{ji}}{\partial x_j} \tag{2.16}$$

where p only appears in differentiated form while ρ and u appear also in non-differentiated form. This implies that we define a relative non-dimensional pressure

$$\tilde{p} = \frac{p - p_0}{\Delta p} \tag{2.17}$$

where p_0 is a reference pressure and Δp a caracteristic pressure variation scale. So we're done, we can replace the variables

$$\frac{\rho_0 U}{T}\frac{\partial\tilde{\rho}\tilde{u}_i}{\partial\tilde{t}} + \frac{\rho_0 U^2}{L}\frac{\partial\tilde{\rho}\tilde{u}_i\tilde{u}_j}{\partial\tilde{x}_j} = \rho_0\tilde{\rho}g\alpha_i - \frac{\Delta p}{L}\frac{\partial\tilde{p}}{\partial\tilde{x}_i} + \frac{\mu_0 U}{L^2}\frac{\partial\tilde{\tau}_{ji}}{\partial\tilde{x}_j} \tag{2.18}$$

Dividing by the convective term coefficient $\frac{\rho_0 U^2}{L}$, we have

$$\underbrace{\frac{L}{UT}}_{St}\frac{\partial\tilde{\rho}\tilde{u}_i}{\partial\tilde{t}} + \frac{\partial\tilde{\rho}\tilde{u}_i\tilde{u}_j}{\partial\tilde{x}_j} = \underbrace{\frac{gL}{U^2}}_{\frac{1}{Fr^2}}\tilde{\rho}\alpha_i - \underbrace{\frac{\Delta p}{\rho_0 U^2}}_{Eu}\frac{\partial\tilde{p}}{\partial\tilde{x}_i} + \underbrace{\frac{\mu_0}{\rho_0 LU}}_{\frac{1}{Re_L}}\frac{\partial\tilde{\tau}_{ji}}{\partial\tilde{x}_j} \tag{2.19}$$

where $Fr = \frac{U}{\sqrt{gL}}$ is the **Froude number**, the ratio of the caracteristic velocity with another wich is the velocity of propagation of waves on shallow water (in a pond for example). There is also the **Euler number** $Eu = \frac{\Delta p}{\rho_0 U^2}$.

Non-dimensional momentum equation

$$St\frac{\partial\tilde{\rho}\tilde{u}_i}{\partial\tilde{t}} + \frac{\partial\tilde{\rho}\tilde{u}_i\tilde{u}_j}{\partial\tilde{x}_j} = \frac{1}{Fr^2}\tilde{\rho}\alpha_i - Eu\frac{\partial\tilde{p}}{\partial\tilde{x}_i} + \frac{1}{Re_L}\frac{\partial\tilde{\tau}_{ji}}{\partial\tilde{x}_j} \tag{2.20}$$

Therefor, for the similarity we need to have additionally

$$Fr_m = Fr_m \qquad Eu_m = Eu_p \qquad Re_m = Re_p \tag{2.21}$$

2.2.3 Energy equation

We will go from the total energy equation wich was

$$\begin{aligned} &\frac{\partial\rho E}{\partial t} + \frac{\partial\rho Eu_j}{\partial x_j} = \rho g\alpha_i u_i - \frac{\partial pu_j}{\partial x_j} + \frac{\partial\tau_{ji}u_i}{\partial x_j} - \frac{\partial q_i}{\partial x_i} \\ \Leftrightarrow\qquad &\frac{\partial\rho E}{\partial t} + \frac{\partial(\rho E + p)u_j}{\partial x_j} = \rho g\alpha_i u_i + \frac{\partial\tau_{ji}u_i}{\partial x_j} - \frac{\partial q_i}{\partial x_i} \end{aligned} \tag{2.22}$$

We remember that

$$\rho E + p = \rho(e+k) + p = \rho(e+pv) + \rho k = \rho(h+k) = \rho H \tag{2.23}$$

where H is defined as the total enthalpy per unit mass. If we replace $E = H - p$ and the others

$$\frac{\partial\rho H}{\partial t} + \frac{\partial\rho Hu_j}{\partial x_j} - \frac{\partial p}{\partial t} = \rho g\alpha_i u_i + \frac{\partial\tau_{ji}u_i}{\partial x_j} - \frac{\partial q_i}{\partial x_i} \tag{2.24}$$

We have to introduce 2 non-dimensional variables, one for h and the other for q. Again, the enthalpy appears only in derivated form, so we introduce

$$\tilde{h} = \frac{h-h_0}{\Delta h} \quad and \quad k = \frac{u_iu_j}{2} = U^2\underbrace{\frac{\tilde{u}_i\tilde{u}_j}{2}}_{\tilde{k}} \qquad \Rightarrow \begin{cases} \partial H = \Delta h\partial\tilde{h} + U^2\partial\tilde{k} \\ \qquad = \Delta h\left(\partial\tilde{h} + \dfrac{U^2}{\Delta h}\partial\tilde{k}\right) \end{cases} \tag{2.25}$$

Where the coefficient $Ec = \frac{U^2}{\Delta h}$ is the **Eckert number**. Let's attack the heat flux and remind that $Pr = \frac{\mu c_p}{\kappa}$

$$q_i = -\kappa\frac{\partial T}{\partial x_i} = -\frac{\kappa}{c_p}\frac{c_p\partial T}{\partial x_i} = -\frac{\mu}{Pr}\frac{\partial h}{\partial x_i} = \frac{\mu_0\Delta h}{PrL}\underbrace{-\left(\tilde{\mu}\frac{\partial\tilde{h}}{\partial\tilde{x}_i}\right)}_{\tilde{q}_i} \tag{2.26}$$

We are ready to write the non dimensional form of (2.24)

$$\frac{\rho_0\Delta h}{T}\tilde{\rho}\left[\frac{\partial\tilde{h} + Ec\partial\tilde{k}}{\partial\tilde{t}} + \frac{UT}{L}\tilde{u}_j\frac{\partial\tilde{h} + Ec\partial\tilde{k}}{\partial\tilde{x}_j}\right] - \frac{\Delta p}{T}\frac{\partial\tilde{p}}{\partial\tilde{t}} = \rho_0 gU\alpha_i\tilde{\rho}\tilde{u}_i + \frac{\mu_0U^2}{L^2}\frac{\partial\tilde{\tau}_{ji}\tilde{u}_i}{\partial\tilde{x}_j} - \frac{\mu_0\Delta h}{PrL^2}\frac{\partial\tilde{q}_i}{\partial\tilde{x}_i} \tag{2.27}$$

Again, if we divide by the coefficient of the convective term $\frac{\rho_0\Delta h}{T}\frac{UT}{L}$

$$\begin{aligned} &\underbrace{\frac{L}{UT}}_{St}\tilde{\rho}\frac{\partial\tilde{h} + Ec\partial\tilde{k}}{\partial\tilde{t}} + \tilde{\rho}\tilde{u}_j\frac{\partial\tilde{h} + Ec\partial\tilde{k}}{\partial\tilde{x}_j} - \underbrace{\frac{L}{UT}\frac{\Delta p}{\rho_0U^2}\frac{U^2}{\Delta h}}_{StEuEc}\frac{\partial\tilde{p}}{\partial\tilde{t}} \\ &= \underbrace{\frac{glU^2}{U^2\Delta h}}_{\frac{Ec}{Fr^2}}\alpha_i\tilde{\rho}\tilde{u}_i + \underbrace{\frac{\mu_0U^2}{L\rho_0\Delta hU}}_{\frac{Ec}{Re_L}}\frac{\partial\tilde{\tau}_{ji}\tilde{u}_i}{\partial\tilde{x}_j} - \underbrace{\frac{\mu_0}{PrL\rho_0U}}_{\frac{1}{PrRe_L}}\frac{\partial\tilde{q}_i}{\partial\tilde{x}_i} \end{aligned} \tag{2.28}$$

Non-dimensional form of the energy equation

$$St\tilde{\rho}\frac{\partial\tilde{h}+Ec\partial\tilde{k}}{\partial\tilde{t}}+\tilde{\rho}\tilde{u}_j\frac{\partial\tilde{h}+Ec\partial\tilde{k}}{\partial\tilde{x}_j}-StEuEc\frac{\partial\tilde{p}}{\partial\tilde{t}}=\frac{Ec}{Fr^2}\alpha_i\tilde{\rho}\tilde{u}_i+\frac{Ec}{Re_L}\frac{\partial\tilde{\tau}_{ji}\tilde{u}_i}{\partial\tilde{x}_j}-\frac{1}{PrRe_L}\frac{\partial\tilde{q}_i}{\partial\tilde{x}_i} \tag{2.29}$$

At this stage what we have a last condition which is $Ec_m = Ec_p$. For the tests over the scaled model to be representative of the actual flow over the prototype, all these dimensionless parameters should be the same for the experiment and for the true configuration. This is called complete similarity. This is very difficult, impossible to achieve. It is why we have to study the relax we can give to the parameters. What is the interpretation we can give to the dimensionless numbers? For those who appear in the momentum equation, ρg is a force per unit volume, so all terms in the dimentional momentum equation are force density. And these dimensionless numbers represents the ratio of the divers force densities. For example, St number represents the relative magnitude of the inertial force density to the convective force density. We can also give other interpretations like previously with the definition of the numbers.

2.2.4 Partial similarity

We will not consider Prandtl number because is relatively constant for most fluids.

Strouhal number

For flows where there are intrinsic time scales, for example the flapping bird has a certain period of flapping. This is imposed by the problem. For flows there is no imposed period, for example steady flows. For those we can choose

$$T=\frac{L}{U} \qquad \Rightarrow St=1 \tag{2.30}$$

for both model and prototype. The fact that there is no caracteristic time scale doesn't mean that the flow is steady. The example of that is the flow at low speed around a cylinder of diameter D. When $Re > 40$, the flow becomes unsteady naturally. Even though the cylinder is fiwed, even the velocity of the fluid constant in time, the flow develops natural unsteadiness by its own. In fact, we have a shedding of vortices alternatively on the upper and lower side (Karman vortex street - Aelion times). The oscillation takes place at a very specific frequency and the non-dimensional period is

$$\tilde{T}_{osc}=\frac{T_{osc}}{T}=\frac{T_{osc}U}{L} \tag{2.31}$$

and is the same for both model and prototype. We have also $St_{osc}=\frac{f_{osc}L}{U} \Rightarrow St_{osc}=f(Re_L)$ that is function of the other parameters of the oscillation like the Re number.

Euler number

We know that the governing equations must be supplemented by some additional equations. We never spoke about the thermodynamic equations of states that must be the same for the model and prototype. We will assume a flow of thermically and calorically perfect gas

$$\left.\begin{aligned} &p=\rho RT\\ &h=c_pT=\frac{\gamma}{\gamma-1}RT\end{aligned}\right\} \quad\Leftrightarrow\quad h=\frac{\gamma}{\gamma-1}\frac{p}{\rho} \quad\Leftrightarrow\quad \rho=\frac{\gamma}{\gamma-1}\frac{p}{h} \tag{2.32}$$

By introducing non-dimentional variables

$$\rho_0\tilde{\rho} = \frac{\gamma}{\gamma-1}\frac{p_0+\Delta p\tilde{p}}{h_0+\Delta h\tilde{h}} \quad \Leftrightarrow \quad \rho_0\tilde{\rho} = \frac{\gamma}{\gamma-1}\frac{p_0\left(1+\frac{\Delta p\tilde{p}}{p_0}\right)}{h_0\left(1+\frac{\Delta h\tilde{h}}{h_0}\right)} \quad \Leftrightarrow \quad \tilde{\rho} = \frac{1+\frac{\Delta p\tilde{p}}{p_0}}{1+\frac{\Delta h\tilde{h}}{h_0}} \tag{2.33}$$

In the other hand, if we remind that $a^2 = \gamma RT = \gamma\frac{p}{\rho}$, we have

$$\frac{\Delta p}{p_0} = \frac{\Delta p}{\rho_0 U^2}\frac{\rho_0 U^2}{p_0} = \underbrace{\frac{\Delta p}{\rho_0 U^2}}_{Eu}\gamma\underbrace{\frac{U^2}{a^2}}_{Ma^2} \tag{2.34}$$

And if we replace in previous relation

$$\tilde{\rho} = \frac{1+\gamma EuMa^2\tilde{p}}{1+\frac{\Delta h\tilde{h}}{h_0}} \tag{2.35}$$

There is no imposed scale for pressure, so if we choose $\Delta p = \rho_0 U^2 \Rightarrow Eu = 1$, we don't care about. In that case

$$\tilde{\rho} = \frac{1+\gamma Ma^2\tilde{p}}{1+\frac{\Delta h\tilde{h}}{h_0}} \tag{2.36}$$

So here we have to satisfy the Mach number similarity, we didn't gain anything, we can replace Euler number similarity by Mach number similarity. We see that if Ma is negligible, the term is small compared to 1 and so the Ma number disappears. So we will not have to take into account that similarity in the last case

$$\tilde{\rho} = \frac{1}{1+\frac{\Delta h\tilde{h}}{h_0}} \tag{2.37}$$

Froude number

We're going to speak about the governing equation. In fact, for flows in which there is no free surface (for example a pipe where the fluid is contained), we can integrate the gravity term with the pressure term (hydrostatic pressure) in momentum equation. The reason this is not possible in free surface cases is because the pressure does not vary on the surface. Consider a flow of a liquid of a gas without free surface. The corresponding terms for pressure gradient and gravity are respectively

$$-\frac{\partial p}{\partial x_i} + \rho g\alpha_i \qquad \Rightarrow \qquad -\frac{\partial}{\partial x_i}(p+\rho g\alpha_i x_i) \tag{2.38}$$

We will define $\delta p = p - (p_0 + \rho_0 g\alpha_i x_i)$ where appears the hydrostatic pressure field, due to the fact that hydrostatic pressure force in (2.38) can be expressed by $-\rho\frac{\partial}{\partial x_i}(g\alpha_i x_i)$, derivative of the potential energy. It comes that

$$\frac{\partial\delta p}{\partial x_i} = \frac{\partial p}{\partial x_i} - \frac{\partial}{\partial x_i}(\rho_0 g\alpha_i x_i) \tag{2.39}$$

Making δp appear in (2.38), we have

$$-\frac{\partial}{\partial x_i}\underbrace{(p-p_0-\rho_0 g\alpha_i x_i)}_{\delta p} + (\rho-\rho_0)\frac{\partial}{\partial x_i}g\alpha_i x_i = -\frac{\partial\delta p}{\partial x_i} + \underbrace{(\rho-\rho_0)}_{\delta\rho}g\alpha_i \tag{2.40}$$

The corresponding terms in non-dimensional equation are

$$-Eu\frac{\partial\delta\tilde{p}}{\partial\tilde{x}_i}+\frac{1}{Fr^2}\underbrace{(\tilde{\rho}-1)}_{\delta\tilde{\rho}}\alpha_i \tag{2.41}$$

Let's consider a low Mach number in order not to worry about the Mach number similarity ($p=\rho_0U^2 \leftarrow Eu=1$). Let's look to the contribution of gravity with Froude number. Using (2.37), we know that

$$\tilde{\rho}-1=\delta\tilde{\rho}=\frac{1}{1+\frac{\Delta h\tilde{h}}{h_0}}-1\approx-\frac{\Delta h\tilde{h}}{h_0} \tag{2.42}$$

and we see that for small temperature variations in the flow, it means that the density variations are small and the term with Fr number in (2.41) can be neglected. The gravity has no influence on the flow. And so we don't have to worry about Froude number when temperature variations are small. Liquids have a thermal expansion coefficient making them sensible to the temperature variation (natural convection can take place).

As there is no intrinsic velocity scale for natural convection flows, we will see if we can make the same trick considering a velocity in order to have $Fr=1$ and not wonder about the similarity. If we take the Fr term in (2.41), what $\delta\tilde{\rho}$ relative density variation? When there are small density variations we can write this and make a first order expansion

$$\frac{\delta\rho}{\rho_0}=-\beta\Delta T \qquad\Leftrightarrow\qquad \rho=\rho_0(1-\beta\Delta T) \qquad\Leftrightarrow\qquad -\beta\rho_0=\left.\frac{\partial\rho}{\partial T}\right|_0 \tag{2.43}$$

where β is the thermal expansion parameter $\beta=-\frac{1}{\rho_0}\left.\frac{\partial\rho}{\partial T}\right|_0$. The minus sign describes the decrease of density with temperature. Fr term in (2.41) becomes

$$\frac{1}{Fr^2}\delta\tilde{\rho}\alpha_i=-\frac{gl}{U^2}\beta\Delta T\tilde{\Delta t}\alpha_i \tag{2.44}$$

where $\tilde{\Delta t}$ is the non-dimensional local temperature variation. For natural convection, we choose $U^2=gL\beta\Delta T$ in order to not worry about Froude number similarity. If we do that, there is an interresant corollary for Reynolds number

$$Re_L=\frac{UL}{\nu_0}=\sqrt{\frac{U^2L^2}{\nu_0^2}}=\sqrt{\frac{g\beta\Delta TL^3}{\nu_0^2}}\equiv\sqrt{Gr} \tag{2.45}$$

The Grashoff number is nothing else but the square root of Reynolds number in the case where U is chosen like Fr $=1$. The conclusion is that the only case where we have to consider the Froude number is the **flows with free surfaces**.

Eckert number

If we look to energy equation for inviscid flows, we are not worrying about Froude number and are not considering free surfaces, making all the right side of the energy equation disappear. It reduces to

$$\frac{\partial H}{t}+\vec{u}\nabla H-\frac{\partial p}{\partial t}=0 \tag{2.46}$$

We see that for steady flows flows $H=cst=h+\frac{u^2}{2}$ so the inviscide enthalpy variation scale $\Delta h^{inv}\approx\frac{u^2}{2}=H_0-h_0$. If solid bodies are heated, then there is another thermal enthalpy variation scale $\Delta h^{th} = H_0 - h(T_w)$. What is the appropriate scale? The actual Δh will be the maximum of the 2 and so the Eckert number

$$Ec = \frac{U^2}{\Delta h} = \min\left(\underbrace{\frac{U^2}{\Delta h^{inv}}}_{2}, \frac{U^2}{\Delta h^{th}}\right) \tag{2.47}$$

So Ec is always smaller than 2. There are instances in which $Ec \ll 1$ and in which case the energy equation can be significantly simplified. We can look at the thermal part, reminding that $(\gamma - 1)h = a^2$ (2.32)

$$\frac{U^2}{\Delta h^{th}} = \frac{U^2}{h_0}\frac{h_0}{\Delta h^{th}} = \frac{(\gamma-1)Ma^2}{\frac{\Delta h^{th}}{h_0}} \tag{2.48}$$

We see that if Ma is smaal, Eckert number is small, unless the denominator is small too. So this is when $Ma^2 \ll \frac{\Delta h^{th}}{h_0}$, energy equation can be simplified. The final condition for similarity, in the equation of state we have the ratio $\Delta h/h_0$ that has to be the same for the model and the prototype. This reduces the condition to

$$\frac{\Delta h^{th}}{h_0} = \frac{H_0}{h_0} - \frac{h_w}{h_0} = 1 + \frac{\gamma-1}{2}Ma^2 - \frac{h_w}{h_0} \tag{2.49}$$

So the last term must be the same, implying that $\frac{T_w}{T_0}$ must be the same.

Reynolds number

Experimentally, it has been observed that, for many configurations, flow quantities became insensitive to Reynolds number beyond a certain critical value. In this range, we don't have to worry about Reynolds number similarity as long as we are in the insensitivity range.

2.3 Dimensional analysis - Vashy Buckingam π theorem

Assuming that there exists a relationship between n physical variables $f(q_1, q_2, \ldots, q_3) = 0$ involving j physical dimensions, then their exists a relationship between $n - j$ non-dimensional groups Π_k : $g(\Pi_1, \Pi_2, \ldots, \Pi_{n-j}) = 0$.

Construction of dimensionless groups : methods of repeating variables

1. Among the n physical variables, pick j that involve all the physical dimensions $[q_n, q_{n-1}, \ldots, q_{n-j+1}]$. These are the repeating variables.
2. Construct
$$\Pi_k = \frac{q_k}{q_{n-j+1}^{\alpha_1} q_{n-j+2}^{\alpha_2} \cdots q_n^{\alpha_j}} \tag{2.50}$$
3. Adjust the exponents $\alpha_1, \alpha_2, \ldots, \alpha_j$ so that Π_j is dimensionless.

Let's take the example of the drag of a sphere (called D). We must first assume what are the involving variables in the drag. It depends on $D = f(d, U, \mu, \rho)$. We have 5 physical quantities here and 3 physical dimensions, so there is 2 dimensionless groups. The repeating variables will be d, U, ρ. The first group is

$$\Pi_1 = \frac{D}{d^{\alpha_1} U^{\alpha_2} \rho^{\alpha_1}} = \left[\frac{MLT^{-2}}{L^{\alpha_1}(LT^{-1})^{\alpha_2}(ML^{-3})^{\alpha_3}}\right] \qquad \Rightarrow \begin{cases} M : 1 - \alpha_3 = 0 \Rightarrow \alpha_3 = 1 \\ L : 1 - \alpha_1 - \alpha_2 + 3\alpha_3 = 0 \Rightarrow \alpha_1 = 2 \\ T : -2 + \alpha_2 = 0 \Rightarrow \alpha_2 = 2 \end{cases} \tag{2.51}$$

Finally, $\Pi_1 = \frac{D}{\rho U^2 d^2}$. When the same is applied for μ $[ML^{-1}T^{-1}]$, we obtain $\Pi_2 = \frac{\mu}{\rho U d} = \frac{1}{Re_d}$. We have so that $\Pi_1 = f(Re_D)$ but rather than using Π_1, we have a similar non-dimensional number (we only make appear other non-dimensional numbers) which is the drag coefficient $C_D = \frac{D}{\frac{1}{2}\rho U^2 S^2}$. We conclude that $C_D = f(Re_D)$ and make vary this two variables only.

Conclusive example : ship hull resistance (Froude)

In order to have similarity between 2 flows, we must have the same non-dimentional numbers. The example is clearly a free surface flow so the numbers are :

- ~~**Strouhal number**~~ : for a steady flow we take $T = L/U$, which make the similarity satisfied.
- ~~**Euler number**~~ : we are working with liquids so there is no density variation and we don't care about this number. Low speed flow $\Delta p = \rho U^2$.
- **Froude number**
- ~~**Reynolds number**~~ : Re number is hugh because dimensions of the ship are hugh, so in general we are in the range of insensitivity of Re number.
- ~~**Eckert number**~~ : we are not concerned about thermal effects so it doesn't matter.

1. Symilarity analysis

We see that the only similarity of the problem to be respected is Froude number similarity

$$Fr_m = Fr_p \qquad \Leftrightarrow \qquad \frac{U_m}{\sqrt{gL_m}} = \frac{U_p}{\sqrt{gL_p}} \qquad \Leftrightarrow \qquad U_m = \sqrt{\frac{L_m}{L_p}} U_p \tag{2.52}$$

What about the resistance? We know that since the fluids are the same, the pressure fields relation is

$$p_m = C_p p_p \qquad \Leftrightarrow \qquad \rho U_m^2 = C_p \rho U_p^2 \qquad \Leftrightarrow \qquad C_p = \left(\frac{U_m}{U_p}\right)^2 \tag{2.53}$$

As the drag force is proportional to the pressure $D_m \propto S_m p_m$, using (2.52) we conclude that

$$D_p = \left(\frac{L_m}{L_p}\right)^2 \left(\frac{U_m}{U_p}\right)^2 D_p = \left(\frac{L_m}{L_p}\right)^3 D_p \tag{2.54}$$

2. Dimensional analysis

If we don't make a reference to the equations of motion. We say that $D = f(\rho, U, L, g)$ which gives 5 quantities and 3 dimensions. And we found the same conclusions

$$\frac{D}{\rho U^2 L^2} = \varphi\left(\frac{U}{\sqrt{gL}}\right) \qquad \Rightarrow \frac{U_m}{U_p} = \sqrt{\frac{L_m}{L_p}} \qquad \Rightarrow \frac{D_m}{D_p} = \left(\frac{U_m L_m}{U_p L_p}\right)^2 = \left(\frac{L_m}{L_p}\right)^3 \tag{2.55}$$

Chapter 3

Inviscid incompressible potential flows

3.1 Introduction

3.1.1 Governing equations

Let's remind that we have a set of 4 equations and 4 unknowns. It's inviscid so there are no stresses. These equations are decoupled from the energy equation, for constant density flows, the hydrodynamic problem gets decoupled from the thermal problem. So when we speak about compressible fluid we have to couple them.

$$\rho = cst \qquad \nabla \vec{u} = 0 \qquad \rho \left[\frac{\partial \vec{u}}{\partial t} + \vec{u} \nabla \vec{u} \right] = -\nabla p + \rho \vec{F} \tag{3.1}$$

3.1.2 Bernouilli equation

The flow is barotropic, in additoin let's consider that the flow is steady and that the force derives from a potential $\vec{F} = -\nabla \Phi$ (F = 0 for most applications). We have the Bernouilli equation

$$\epsilon_m = \frac{p}{\rho} + k + \cancel{\Phi} = cst = \frac{p}{\rho} + \frac{u^2}{2} \tag{3.2}$$

The mechanical energy is constant on a streamline. We can give an interpretation to the constant by writing it as $\frac{p_t}{\rho} = \frac{p^0}{\rho}$. Physically, p^0 is the pressure where u = 0 and is called **stagnation pressure**. The constant differs from streamline to streamline. Indeed we found that $\vec{\omega} \times \vec{u} = -\nabla \epsilon_m$, so if the fluid is rotational ϵ_m will differ from streamline to another, meaning that the stagnation pressure differs.

3.1.3 Irrotational (potential) flow

In that case we have that $\omega = 0$ so $\vec{u} = \nabla \varphi$ (φ beeing a velocity potential) and p^0 is the same for all streamlines : $\nabla p^0 = 0$. For incompressible flows, we have the definition

$$p^0 = p + \frac{\rho u^2}{2} \tag{3.3}$$

where the two terms are respectively the **static pressure** and the **dynamic pressure**. The definition $\vec{u} = \nabla \varphi$ can be introduced in (3.1) to have

$$\nabla \vec{u} = \nabla \nabla \varphi = \nabla^2 \varphi \tag{3.4}$$

which is **Laplace's equation**. His linearity makes it more solvable than Navier-Stokes equations and we are unable to get $\varphi \to \vec{u} \to p$ and we can use **superposition principle**. So if φ_1 and φ_2 are two solutions, then any linear combination $a\varphi_1 + b\varphi_2$ is also a solution.

3.2 Elementary planar (2D) flows

3.2.1 Complex potential

We start from the previous general equation

$$\vec{u} = \nabla\varphi \qquad \Rightarrow \qquad \begin{cases} u_1 = \dfrac{\partial\varphi}{\partial x_1} = \dfrac{\partial\psi}{\partial x_2} \\ u_2 = \dfrac{\partial\varphi}{\partial x_2} = -\dfrac{\partial\psi}{\partial x_1} \end{cases} \tag{3.5}$$

where the second equality comes from the streamfuntion (1.111). This is called the **Cauchy-Riemann equations** that a complex analytic function has to satisfy. So $\varphi + i\psi \equiv \chi(x_1 + ix_2) = \chi(z)$ is one of that and we call it the **complex potential**. For ψ we have (1.113) which says, in case of irrotational flows

$$\nabla\psi = -\omega = 0. \tag{3.6}$$

As χ is analytic, we can compute its derivative

$$\frac{\partial\chi}{\partial x_1} = \frac{\partial\chi}{\partial z}\underbrace{\frac{\partial z}{\partial x_1}}_{=1} \qquad \Rightarrow \qquad \frac{\partial\chi}{\partial z} = \frac{\partial}{\partial x_1}(\varphi + i\psi) = \frac{\partial\varphi}{\partial x_1} + i\frac{\partial\psi}{\partial x_1} = u_1 - iu_2 \equiv w, \tag{3.7}$$

where w is the **complex velocity**. Let's compute the integral over some curl of w

$$\int_C w\,dz = \int_C (u_1 - iu_2)\,(dx_1 + idx_2) = \int_C \underbrace{u_1\,dx_1 + u_2\,dx_2}_{\vec{u}\,d\vec{s}} + i\int_C u_1\,dx_2 - u_2\,dx_1 \tag{3.8}$$

where $\vec{u}\,d\vec{s}$ is the work of the velocity. This beeing true for all curls, it's in particular true for a closed curl

$$\oint w\,dz = \underbrace{\oint \vec{u}\,d\vec{s}}_{\Gamma} + i\underbrace{\oint u_1\,dx_2 - u_1\,dx_2}_{\dot{V}} \tag{3.9}$$

where Γ is the **circulation**. For the second term, let's remind the discussion in (1.109) where we found $\vec{n} = \frac{dx_2}{ds}\vec{e}_1 - \frac{dx_1}{ds}\vec{e}_2 \Rightarrow \vec{u}\vec{n}\,ds = u_1\,dx_2 - u_2\,dx_1$. This is exactly the second term integrated in (3.9) where $\dot{V}$ corresponds to the **volume flow out of the closed contour**. A property of analytic functions is that its derivative is analytic and if it's analytic **everywhere inside the contour**,

$$\oint_C w\,dz = \Gamma + i\dot{V} \neq 0 \tag{3.10}$$

only if there are singularities inside the contour.

3.2.2 Uniform flow

This is the first elementary flow we will study is the case of a uniform flow along x_1 axis

$$\left.\begin{matrix} u_1 = u_\infty \\ u_2 = 0 \end{matrix}\right\} \to w = u_\infty \qquad \Rightarrow \qquad \chi = u_\infty z \to \begin{cases} \varphi = u_\infty x_1 \\ \psi = u_\infty x_2 \end{cases}. \tag{3.11}$$

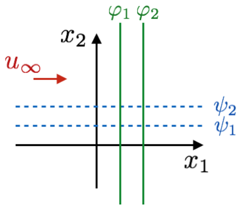

Figure 3.1

This means that streamlines are lines with constant x_2 and equi-potential lines with constant x_1 as represented on Figure 3.1. In fact, if we remind (3.5) the scalar

$$\nabla\varphi \cdot \nabla\psi = \frac{\partial\varphi}{\partial x_1}\frac{\partial\psi}{\partial x_1} + \frac{\partial\varphi}{\partial x_2}\frac{\partial\psi}{\partial x_1} = -u_1u_2 + u_2u_1 = 0. \tag{3.12}$$

This means that the equi-potential lines and streamlines must be perpendicular. If we have a uniform flow with an angle α with respect to x_1, then

$$\begin{gathered}\left.\begin{aligned} u_1 &= u_\infty \cos\alpha \\ u_2 &= u_\infty \sin\alpha \end{aligned}\right\} \to w = u_\infty(\cos\alpha - i\sin\alpha) = u_\infty e^{-i\alpha} \\ \Downarrow \\ \chi = u_\infty z e^{-i\alpha} \to \begin{cases} \varphi = \mathrm{Re}(\chi) = u_\infty(x_1\cos\alpha + x_2\sin\alpha) \\ \psi = \mathrm{Im}(\chi) = u_\infty(x_2\cos\alpha - x_1\sin\alpha) \Leftrightarrow x_2 = x_1\tan\alpha + \dfrac{\psi}{u_\infty} \end{cases}\end{gathered} \tag{3.13}$$

where we see that the equation for ψ is a line with angle α.

3.2.3 Source flow

Let's imagine two plates separated by a fluid and a pipe on both surfaces allowing to inject fluid at some point. In reality fluid has a certain dimension but let's imagine that we word in 0 dimension. The point of impact can be represented as in red on Figure 3.2 and the symetry considerations can be made :

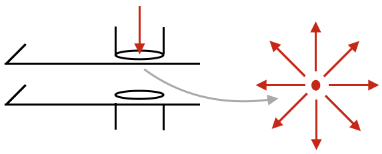

Figure 3.2

- The flow is radial : $\vec{u} = u_r\vec{e}_r$
- Azimuthal symetry : $\frac{\partial u_r}{\partial\theta} = 0 \qquad \Rightarrow u_r = f(r)$

If we define θ the angle with respect to horizontal axis, we have

$$\vec{u} = f(r)\vec{e}_r = f(r)(\cos\theta\vec{e}_1 + \sin\theta\vec{e}_2) \Rightarrow \begin{cases} u_1 = f(r)\cos\theta \\ u_2 = f(r)\sin\theta \end{cases} \Rightarrow w = f(r)e^{-i\theta}. \tag{3.14}$$

If we consider a closed curl of radius r around the source, there is no circulation because streamlines are perpendicular. Considering polar coordinates, $z = re^{i\theta}$ and $dz = ire^{i\theta}\,d\theta$

$$\oint_C w dz = i\dot{V} = \int_0^{2\pi} f(r)e^{-i\theta}ire^{i\theta}\,d\theta = irf(r)2\pi. \tag{3.15}$$

There is a volume flux going out of the source and the function $f(r)$ is given by (with $\log z = \ln r + i\theta$)

$$\begin{gathered} f(r) = \frac{\dot{V}}{2\pi r} \qquad \Rightarrow w = \frac{\dot{V}}{2\pi r}e^{-i\theta} = \frac{\dot{V}}{2\pi z} \qquad \Rightarrow \chi = \frac{\dot{V}}{2\pi}\log z \\ \Rightarrow \qquad \varphi = \mathrm{Re}(\chi) = \frac{\dot{V}}{2\pi}\ln r \qquad \textit{and} \qquad \psi = \mathrm{Im}(\chi) = \frac{\dot{V}}{2\pi}\theta \end{gathered} \tag{3.16}$$

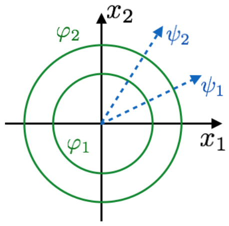

Figure 3.3

We see that streamlines are lines with constant θ and equi-potential lines are circles of constant radius r centered at the origin (Figure 3.3). Notice that they are perpendiculat as the previous case. The source flow corresponds to **Green function** for fluids. Let's finally consider the case where the source is not located at the center but at a point z_0. We only have to make a shift of coordinates

$$\chi = \frac{\dot{V}}{2\pi}\log(z - z_0). \tag{3.17}$$

3.2.4 Concentrated vortex flow

We will make an analogy with electricity. We know that vorticity and the current density are defined as

$$\vec{\omega} = \nabla \times \vec{u} \qquad \vec{J} = \nabla \times \vec{H} \tag{3.18}$$

where $\vec{H}$ is the magnetic field. $\vec{J}$ is nothing else but the current in amperes divided by the surface of the electrical wire within the current circulates. With application of Stoke's theorem, we have

$$I = \int_S \vec{J}\vec{n}\, dS = \oint_C \vec{H}\, d\vec{S}. \tag{3.19}$$

We can, similarly to the current tube, define a vortex tube with

$$\int_S \vec{\omega}\vec{n}\, dS = \oint_C \vec{u}\, d\vec{S} = \Gamma \tag{3.20}$$

where we find the **circulation** Γ. Actually when we model a wire, we do not consider a 2D section but concentrate the current over a line. We make the same with Γ and look for the velocity field associated to the concentrated vortex tube (same as looking for $\vec{H}$ associated to concentrated I). For a concentrated current, we have an azimuthal magnetic field with an azimuthal symetry (circles), so this is the same for $\vec{u}$

$$\vec{H} = H_\theta \vec{e}_\theta \qquad \Rightarrow \vec{u} = \underbrace{u_\theta(r)}_{f(r)}\, \vec{e}_\theta \tag{3.21}$$

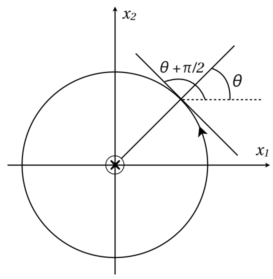

Figure 3.4

where $f(r)$ will be determine as in the previous section but θ becomes $\theta + \pi/2$ due to $\vec{e}_r \perp \vec{e}_\theta$

$$\left.\begin{aligned} u_1 &= f(r)\cos(\theta + \pi/2) = -f(r)\sin\theta \\ u_2 &= f(r)\sin(\theta + \pi/2) = f(r)\cos\theta \end{aligned}\right\} \quad \Rightarrow \quad \begin{aligned} w &= u_1 - iu_2 = f(r)(i^2\sin\theta - i\cos\theta) \\ &= -if(r)(\cos\theta - i\sin\theta) = -if(r)e^{-i\theta} \end{aligned} \tag{3.22}$$

And if we integrate over a closed curl and equalize to Γ ($\dot{V} = 0$)

$$\oint_C w\, dz = -i\, irf(r)2\pi = \Gamma \qquad \Rightarrow \qquad f(r) = \frac{\gamma}{2\pi r}. \tag{3.23}$$

And so

$$w = -i\frac{\Gamma}{2\pi r}e^{-i\theta} = -i\frac{\Gamma}{2\pi z} \qquad \Rightarrow \chi = -\frac{\Gamma}{2\pi}\log z \qquad \Rightarrow \varphi = \frac{\Gamma}{2\pi}\theta \qquad \psi = -\frac{\Gamma}{2\pi}\ln r \tag{3.24}$$

As conclusion, we see that streamlines are lines of constant r (circles) and equi-potential lines lines of constant θ. It is the same as Figure 3.3 except that we have an exchange between φ and ψ. This exchange is due to the $-i$ factor.

3.2.5 Fluid dipole - doublet

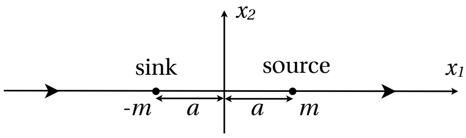

Figure 3.5

The idea is to build more complex designs. If we have a source of volume flow $\dot{V}$ located on $+a$ and a source with $-\dot{V}$ (a sink) at $-a$. We let then $a \to 0$, but if $a = 0$, the sources are superposed and there is no flow. To remediate, we let $\dot{V} \to \infty$ as $a \to 0$, such that $\dot{V} \times 2a = \mu = cst$. For a finate a

$$\begin{aligned}\chi = \chi_{source} + \chi_{sink} &= \frac{-\dot{V}}{2\pi}\log(z-a) + \frac{\dot{V}}{2\pi}\log(z+a) = \frac{\dot{V}}{2\pi}(\log(z-a) - (\log(z+a)) \\ &= \frac{\dot{V}}{2\pi}\log\frac{z-a}{z+a} = \frac{\dot{V}}{2\pi}\log\left(1 + \frac{2a}{z+a}\right)\end{aligned} \tag{3.25}$$

Because of $a \to 0$, we can make an expansion knowing that $\frac{1}{1-t}$ is the sum of a geometric series

$$\ln(1-t) = \int \frac{1}{1-t}\,dt = -\left(\int (1 + t + t^2 + ...)\,dt\right) = -t - \frac{t^2}{2} - \frac{t^3}{3} + ... \tag{3.26}$$

Replacing this in (3.25)

$$\chi = \frac{\dot{V}}{2\pi}\left(-\frac{2a}{z+a} - \frac{4a^2}{2(z+a)^2} + ...\right) = -\frac{\mu}{2\pi(z+a)} - \frac{\mu a}{(z+a)^2} + ... \tag{3.27}$$

In the limit $a \to 0$

$$\chi_{dipole} = -\frac{\mu}{2\pi z} \qquad and \qquad w = \frac{\mu}{2\pi z^2}. \tag{3.28}$$

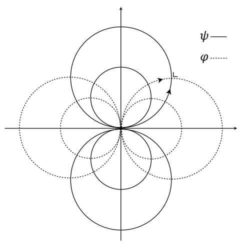

Figure 3.6

This makes sense because $w \geq 0$ means that the source pushes the fluid to the right and so the sink sucks the fluid from left to right. So when z is a real, w should be positive. We see that a dipole has a certain direction to the contrary of the vortices and sources. Before computing ψ and φ, let's remind that

$$\frac{1}{z} = \frac{x_1 - ix_2}{x_1^2 + x_2^2} \qquad \Rightarrow \psi = \mathrm{Im}(-\frac{\mu}{2\pi z}) = \frac{\mu}{2\pi}\frac{x_2}{x_1^2 + x_2^2} \qquad \Leftrightarrow x_1^2 + x_2^2 - \frac{\mu}{2\pi\psi}x_2 = 0 \tag{3.29}$$

This last equation corresponds to a circle going throw the origin and since x_2 is in the linear part, the center is on x_2 axis. We do the same for φ with taking

$$\varphi = \mathrm{Re}(\chi) = -\frac{\mu}{2\pi}\frac{x_1}{x_1^2 + x_2^2} \qquad \Leftrightarrow \qquad x_1^2 + x_2^2 + \frac{\mu}{2\pi\varphi}x_1 = 0. \tag{3.30}$$

We see that $(0, 0) \in \varphi$ too and now the center is on x_1 axis (Figure 3.6).

3.3 Force and torque on a body in an incompressible plana (2D) potential flow

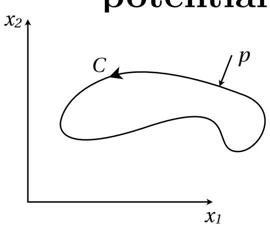

Figure 3.7

We will create more complex flows by assumbling these elementary solutions but we need a theorem useful to compute forces on a solid body in a potential flow. Let's consider the surface forces on a solid body. For

invicid flows, the only component is the **pressure**. The elementary force given by $p\,ds(-\vec{n})$ [1], we have [2]

$$\vec{F} = -\oint_C p\vec{n}\,ds \qquad with \qquad \begin{cases} F_1 = -\oint_C pn_1\,ds = -\oint p\,dx_2 \\ F_2 = -\oint_C pn_2\,ds = \oint p\,dx_1 \end{cases} \tag{3.31}$$

It's interesting to compute a complex number like previously

$$F_1 - iF_2 = -\oint_C p(-i^2 dx_2 + idx_1) = -i\oint_C p\,\underbrace{(dx_1 - idx_2)}_{d\bar{z}}. \tag{3.32}$$

We know that the pressure is linked to the velocity by Bernoulli's theorem

$$p + \rho\frac{u^2}{2} = p + \rho\frac{u_1^2 + u_2^2}{2} = p + \rho\frac{w\bar{w}}{2} = cst = p_t \qquad \Rightarrow p = p_t - \rho\frac{w\bar{w}}{2} \tag{3.33}$$

where p_t is called the **total or stagnation pressure**. By replacing this in previous equation, we have

$$F_1 - iF_2 = -i\oint_C \left(p_t - \rho\frac{w\bar{w}}{2}\right) d\bar{z}. \tag{3.34}$$

Let's check the contribution of each term. I say that contribution of p_t is null, the proof :

- Mathematical proof :
  Whe have the integral over a closed contour of an exact differantial, so $p_t \oint dx_1 = 0$.
- Physical proof :
  If we have a pressure applied somewhere, it exists another pressure exactly opposed somewhere else that cancels this first one for a cst p_t.

Therefore

$$F_1 - iF_2 = \frac{\rho}{2} i \oint_C w\overline{w}\,d\bar{z}. \tag{3.35}$$

The contour has to be taken on the solid and there we have the tangential conditions, the fluid has to flow tangentially to the body. Let's compute $w\,dz$

$$\begin{aligned} w\,dz &= (u_1 - iu_2)(dx_1 + idx_2) \\ &= u_1dx_1 + u_2dx_2 + i \underbrace{(u_1dx_2 - u_2dx_1)}_{\vec{u}.\vec{n}\,ds=0 \text{ (no penetration)}} \\ &= u_1dx_1 + u_2dx_2 \end{aligned} \tag{3.36}$$

This beeing a pure real, it's equal to its conjugate $\bar{w}\,d\bar{z}$. We finally have the

Blasius formula for forces

$$F_1 - iF_2 = \frac{\rho i}{2}\oint_C w^2\,dz \tag{3.37}$$

Now, what about the torque of the force? It's defined as

$$d\vec{C} = \vec{x} \times (-p\vec{n}\,ds) = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ x_1 & x_2 & 0 \\ -pn_1 & -pn_2 & 0 \end{vmatrix} = dC_3 = (x_1dx_1 + x_2dx_2)\,p = \Re\,(pz\ d\bar{z}) \tag{3.38}$$

1. s here is not a surface because we have 2D flows, we speak about "per unit span/length" (a distance normal to the plan of the flow).
2. $n_1\,ds = dx_2, n_2\,ds = dx_1$ by (1.109)

By integrating this along a contour and recognizing $x_1 dx_1 + x_2 dx_2 = \frac{d(x_1^2+x_2^2)}{2} = \frac{d(z\bar{z})}{2} = \frac{z\,d\bar{z}+\bar{z}\,dz}{2} = \mathrm{Re}(\bar{z}\,dz)$, we have

$$C_3 = \oint p\,\mathrm{Re}(\bar{z}\,dz) = \oint (\cancel{p_\infty} - \rho\frac{w\bar{w}}{2})\,\mathrm{Re}(\bar{z}\,dz) = -\,\mathrm{Re}\left(\oint \rho\frac{w\bar{w}}{2}\bar{z}\,dz\right). \tag{3.39}$$

Finally, $w\,dz = \bar{w}\,d\bar{z}$ for the same reason as $w\,dz$, giving the

Blasius formula for torques

$$C_3 = -\frac{\rho}{2}\,\mathrm{Re}\left(\oint w^2\bar{z}z\,dz\right) \tag{3.40}$$

We will use that to get the expression of force over an arbitrary body immersed in an otherwise uniform flow, consider u_∞ as describing an angle α with x_1 axis. e know from the previous sections that the $w(z)$ is a complex analytical function outside C (inside there is no fluid $\to$ singularity). So $w(z)$ is analytical outside a circle centered at the origin and it can be explanded in Laurent series

$$w(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{m=1}^{\infty}\frac{b_m}{z^m} \tag{3.41}$$

When we take account that the fluid has u_∞ in the far field (uniform), we know that the second part will tend to zero, so the first part must be a cst

$$\lim_{z\to\infty} w(z) = u_\infty e^{-i\alpha} \equiv w_\infty \qquad \Rightarrow a_n = 0\,(n>0) \quad \mathit{and} \quad a_0 = w_\infty. \tag{3.42}$$

We also know that by the Laurent's theorem

$$b_m = \frac{1}{2\pi i}\oint w(z) z^{m-1}\,dz \qquad \Rightarrow \qquad b_1 = \frac{1}{2\pi i}\oint w(z)\,dz = \frac{1}{2\pi i}\left(\Gamma + i\cancel{\dot{V}}\right) = \frac{\Gamma}{2\pi i} \tag{3.43}$$

where $\dot{V} = 0$ because there is no volume flow out, as the body is closed. Therefore,

$$w(z) = w_\infty + \frac{\Gamma}{2\pi i z} + \sum_{m=2}^{\infty}\frac{b_m}{z^m}. \tag{3.44}$$

This allows us to compute w^2 that matters for us, as the product of itself by itself

$$\begin{aligned} w^2(z) &= \left(w_\infty + \frac{\Gamma}{2\pi i z} + \frac{b_2}{z^2} + \cdots\right)\left(w_\infty + \frac{\Gamma}{2\pi i z} + \frac{b_2}{z^2} + \cdots\right) \\ &= w_\infty^2 + \frac{2w_\infty\Gamma}{2\pi i z} + \frac{1}{z^2}\left(-\frac{\Gamma^2}{4\pi^2} + 2b_2 w_\infty\right) + \cdots \\ &= A_0 + \frac{B_1}{z} + \frac{B_2}{z^2} + \cdots \qquad \mathit{with} \qquad B_1 = \frac{2w_\infty\Gamma}{2\pi i} \end{aligned} \tag{3.45}$$

Now if we use Laurent theorem reversed, we find

$$\oint_C w^2(z)\,dz = 2\pi i B_1 = 2w_\infty\Gamma. \tag{3.46}$$

This applied to the Blasius force gives

$$F_1 - iF_2 = \frac{\rho i}{2}2w_\infty\Gamma = \rho i u_\infty e^{-i\alpha}\Gamma = \rho u_\infty\Gamma e^{-i(\alpha-\pi/2)}. \tag{3.47}$$

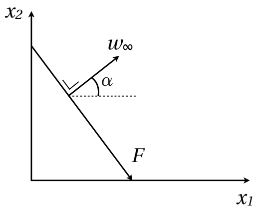

Figure 3.8

This is true independantly of the shape of the body. As first observation, we can see that if u_∞ makes an agnle α with x_1 axis, the force is applied with an angle $\alpha - \pi/2$ meaning that it is $\perp u_\infty$. This means that there is no need of power to move a body in the velocity direction, no resistance. This is called **d'Alembert's paradox**.
The second observation is that the magnitude $|F| = \rho u_\infty \Gamma$, where Γ is the circulation. So we can create a perpendicular force (lifting force) by only generating a circulation, whatever the shape of the body.

3.4 Flow around a circular cylinder

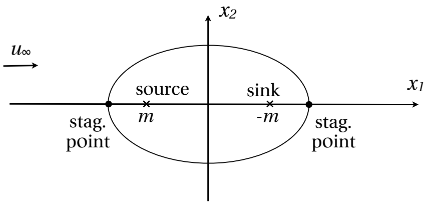

Figure 3.9

Knowing that it is possible to create complex shapes by composing sources and sinks and that a closed streamline can be represented by a solid body (with singularities inside), the limiting case of a doublet flow superposed on a uniform flow can be considered, around a cylindre. We know that the flow generated by the source/sink is radial and the velocity is inversely proportional to the distance to the source/sink. There are 3 velocities to consider as a vector: $u_\infty \rightarrow$, $u_{source} \leftarrow$ and $u_{sink} \rightarrow$ with a magnitude varying with space. We are claiming that there is a point where the **velocity is null** and his symetric point, called the **stagnation point**. So there is a streamline going from one stagnation point to the other and symetrically in the other side of the real axis. This makes sense because there are two singularities inside the contour (not analytic), the flow goes from source to sink, wheras out of the contour, velocity is analytic. We will see how $u_\infty, 2a$ and $\dot{V}$ affect the shape of the body by having $\dot{V}/(u_\infty a)$ non dimensional that controls the shape. The limiting case is when $a \to 0, \dot{V} \to \infty$ and this becomes a doublet. Let's see what's hapening in this limiting case with the complex potential

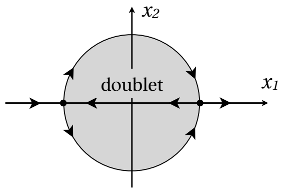

Figure 3.10

$$\chi(z) = \chi_\infty(z) + \chi_{doublet}(z) = u_\infty z + \frac{\mu}{2\pi z} \tag{3.48}$$

where the "+" sign is due to the orientation in $-x_1$ of the doublet and not in x_1, the doublet is facing the flow. The associated velocity and the stagnation point are

$$w(z) = u_\infty - \frac{\mu}{2\pi z^2} \qquad \Rightarrow \qquad w = 0 \qquad \begin{aligned} &\Leftrightarrow u_\infty - \frac{\mu}{2\pi z^2} = 0 \\ &\Leftrightarrow z = \pm\sqrt{\frac{\mu}{2\pi u_\infty}} = \pm a \\ &\Leftrightarrow \mu = 2\pi u_\infty a^2 \end{aligned} \tag{3.49}$$

where $\pm a$ are the two positions of the stagnation points. We can replace in (3.48) to obtain the equation of streamlines

$$\chi = u_\infty \left(z + \frac{a^2}{z}\right) \qquad \Rightarrow \psi = \mathrm{Im}(\chi) = u_\infty \left(x_2 - \frac{a^2 x_2}{x_1^2 + x_2^2}\right) = u_\infty x_2 \left(1 - \frac{a^2}{x_1^2 + x_2^2}\right). \tag{3.50}$$

We see that at the stagnation point ($x_2 = 0$), $\psi(stag) = 0$. In fact there are two solution to this

$$x_2 = 0 \qquad and \qquad 1 - \frac{a^2}{x_1^2 + x_2^2} = 0 \Leftrightarrow x_1^2 + x_2^2 = a^2 \text{ (circle)} \tag{3.51}$$

The streamlines are represented in Figure 3.10 [3]. We have composed the flow over a cylinder. In $w(z)$ there isn't a $1/z$ term so there isn't a B_1 for Laurent series, meaning that circulation $\Gamma = 0$, so there is no force. To explain this, let's analyse the pressure distribution.

Pressure distribution

The velocity over the circle is

$$
\begin{aligned}
& w(z) = u_\infty \left(1 - \frac{a^2}{z^2}\right) \qquad z = ae^{i\theta} \quad 0 \le \theta \le 2\pi \\
\Rightarrow \qquad & w(\theta) = u_\infty \left(1 - \frac{a^2}{a^2\ e^{2i\theta}}\right) = u_\infty e^{-i\theta}\left(e^{i\theta} - e^{-i\theta}\right) = 2u_\infty \sin\theta\, e^{-i\left(\theta - \frac{\pi}{2}\right)}
\end{aligned}
\tag{3.52}
$$

This makes sense because if we consider a point of angle θ, the velocity is tangential to the diameter with angle $\theta - \pi/2$ as it should. We also see that the magnitude is $u = 2u_\infty \sin\theta$, the velocity accelerates from stagnation point, reaches its maximum on $\theta = \pi/2$ and decelarate to 0 till the stagnation point. We can now get the pressure with Bernoulli, but first let's introduce a **non-dimensional pressure** for the special case of **inviscid potential flows**

$$
C_p = \frac{p - p_\infty}{\rho \frac{u_\infty^2}{2}} \qquad \text{(denom : far field dynamic pressure).} \tag{3.53}
$$

Bernouilli's equation is valid in the far field too, so replacing p_t by its value we have

$$
p = p_\infty + \rho\frac{u_\infty^2}{2} - \rho\frac{u^2}{2} \qquad \Rightarrow \frac{p - p_\infty}{\rho\frac{u_\infty^2}{2}} = 1 - \left(\frac{u}{u_\infty}\right)^2 = 1 - 4\sin^2\theta \tag{3.54}
$$

where the final result is obtained by considering u for a circular cylinder. We have a single formula for all cylindres and all velocities (independant). If we take two opposite value of θ, we have the same pressure, also for $\pi - \theta$ (sin). It is symetric to x_1 and x_2 axis, the four forces cancelling each other (normal and tangential components) making sense with d'Alembert's paradox.

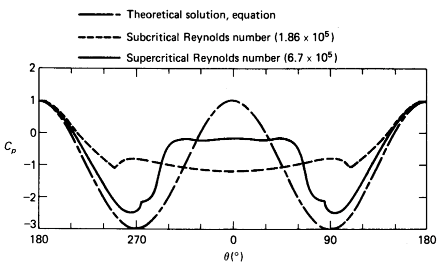

Figure 3.11

The real case only match with this at the instant directly after start of the flows. After a couple of time, there is creation of vorticies at the back that makes the stramlines deviate and the velocity distribution don't correspond to a sin. This effect is large for low Re numbers but tend to disappear for high Re number, making the flow turbulent. For turbulent flows, the separation/deviation appear more on the back ($\approx -\pi/2 < \theta < \pi/2$). This method is used on golf balls by designing dimpels.

Adding a concentrated vortex

Now the question is to determine if the solution (the flow) we found is the only solution. We have already seen that we have circular streamlines for a concentrated vortex. Let's supperimpose to this flow, the flow due to a concentrated vortex (doublet + uniform + vortex). Normally a

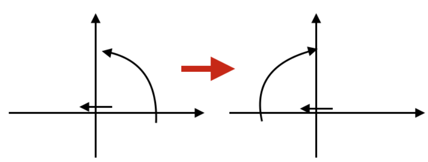

Figure 3.12

3. Don't forget that the flows goes from sink to source.

vortex is represented as left in Figure 3.12 but corresponds to the case of a force with angle $\alpha - \pi/2$ in Figure 3.9. We will consider a lifting force so the clockwise orientation for the vortex and an angle $\alpha + \pi/2$. The new complex potential becomes

$$\chi = u_\infty \left(z + \frac{a^2}{z^2}\right) + \chi_\Gamma = u_\infty \left(z + \frac{a^2}{z^2}\right) + \frac{i\Gamma}{2\pi} \log z. \tag{3.55}$$

If we compute now the streamlines and check their expression on the circle $x_1^2 + x_2^2 = a^2$

$$\begin{aligned} \psi = \mathrm{Im}(\chi) &= u_\infty x_2 \left(1 - \frac{a^2}{x_1^2 + x_2^2}\right) + \mathrm{Im}\left[\frac{i\Gamma}{2\pi}\left(\frac{\ln(x_1^2 + x_2^2) + i\theta}{2}\right)\right] \\ &= u_\infty x_2 \left(1 - \frac{a^2}{x_1^2 + x_2^2}\right) + \frac{\Gamma}{2\pi} \ln\left(\sqrt{x_1^2 + x_2^2}\right) \\ \Rightarrow \qquad \psi_{circ} &= 0 + \frac{\Gamma}{2\pi} \ln a = cst \end{aligned} \tag{3.56}$$

We see that there isn't only one solution admitting the circle as ψ, there are an infinity of solutions dependant of Γ. Now what will be the value of Γ? For a circular cylinder there is no reason to have a circulation as the previous discussion demonstrated, unless we rotate the cylindre in the direction of the potential due to the viscous layer that induces circulation (creates lift). Let's check the stagnation points on the circle $z = ae^{i\theta}$

$$\begin{aligned} w &= u_\infty \left(1 - \frac{a^2}{a^2 e^{i2\theta}}\right) + \frac{i\Gamma}{2\pi a e^{i\theta}} = u_\infty 2ie^{-i\theta} \sin\theta + \frac{i\Gamma}{2\pi a e^{i\theta}} \\ &= e^{-i(\theta - \pi/2)} \left(2u_\infty \sin\theta + \frac{\Gamma}{2\pi a}\right) \quad \Rightarrow w = 0 \Leftrightarrow \sin\theta = -\frac{\Gamma}{4\pi a u_\infty} \end{aligned} \tag{3.57}$$

We see that the previous position is not remaining, the two point are displaced to the lower part of the cylindre but the line joining them remains parallel to the flow. If we reach a $\sin > 1$, the unique stagnation point is outside the circle.

For the pressure distribution we have

$$C_p = 1 - \left(\frac{u}{u_\infty}\right)^2 = 1 - \left(2\sin\theta + \frac{\Gamma}{2\pi a u_\infty}\right)^2. \tag{3.58}$$

Actually for the velocity, we have a larger velocity at the top $\pi/2$ and a lower velocity at $-\pi/2$ because of the positive and then negative contribution of the concentrated vortex. This implies that the pressure is higher below than at the top that creates a lift force. What should be the rotational velocity? The correspondance between angular velocity and the one induced by the vortex is

$$wa = \frac{\Gamma}{2\pi a} \qquad \Leftrightarrow \qquad w = \frac{\Gamma}{2\pi a^2}. \tag{3.59}$$

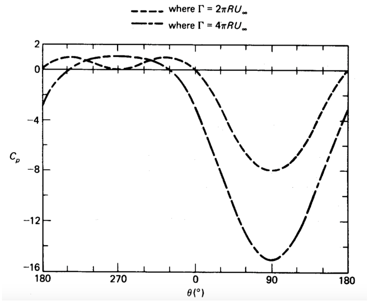

Figure 3.13

This is the theoretical result, in practice we have to rotate twice that velocity. Figure 3.13 confirms our conclusion. Notice that most of the force are not caused by an overpressure on the lower part but by an underpressure on the upper part, leading to consider the air over a wing for example as sucking the wing upper.

Chapter 4

Viscous flows : Laminar and turbulent flows

4.1 Illustrative example : channel flow

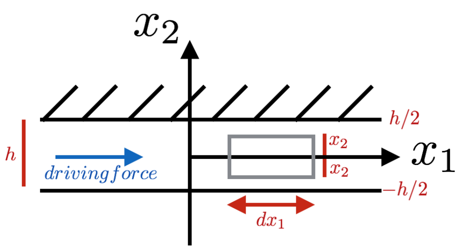

Figure 4.1

It's the flow between two infinite parallel plates whiwh is driven by either a body force or a pression gradient. We will see that they can be linked together. We make the assumption that the flow is a **constant density flow** and **steady**. To analyse this, we need to choose a coordinates system.

Coordinates system We take x_1 in the direction of the driving force ($\rightarrow$) and x_2 normal to the plates, with the origin at the middel of the two plates. This is a planar flow, so

$$\Rightarrow \frac{\partial}{\partial x_3} = 0 \qquad and \qquad u_3 = 0 \text{ assumption.} \tag{4.1}$$

Because of the infinite plates, the origin can be everywhere on x_1 axis and the solution may not depend on it

$$\frac{\partial}{\partial x_1} = 0 \qquad \text{(fully developed flow).} \tag{4.2}$$

This would not be the case if we had an entrance because the region near the entrance is not fully developed, we can see it as a transitory. Now let's make a momentum balance in a small region on x_1.

$\mathbf{x_1}$ momentum balance The time rate of change of the momentum inside a control volume + the net momentum flux going out is equal to 0 because there is no rate of change (steady) and the flow out is equal to the flow in (fully developed)

$$0 = \text{sum of forces in } x_1. \tag{4.3}$$

Mass balance It is the mass flow time rate of change + the mass out - mass in, the second and third term beeing null because velocity is constant

$$0 + \underbrace{\rho u_1 2x_2|_{x_1+dx_1} - \rho u_1 2x_2|_{x_1}}_{=0} + \underbrace{\rho u_2(x_1, x_2) - \rho u_2(x_1, -x_2)}_{=0} = 0. \tag{4.4}$$

The third term teaches us that $2\rho u_2(x_2) = 0 \Rightarrow u_2(x_2) = 0$. The other way to see that is to take the other form of the mass balance

$$\frac{\partial \rho \cancel{u_1}}{\partial x_1} + \frac{\partial \rho u_2}{\partial x_2} = 0 \qquad \Rightarrow \rho u_2 = cst = 0 \text{ (wall condition)}. \tag{4.5}$$

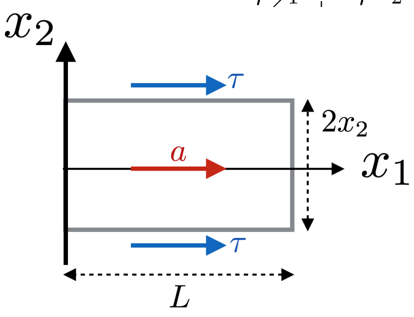

Figure 4.2

Forces There is the body force of acceleration $a \Rightarrow f_1 = am = a2x_2L\rho$, the pressure gradient $f_2 = (p_(0) - p(L))2x_2$ and the shear stress on the walls $f_3 = 2\tau L$, so

$$\begin{aligned} (p(0) - p(L))2x_2 + a2x_2L\rho + 2\tau L = 0 \\ \Rightarrow \tau = -x_2\left(\rho a - \frac{p(L) - p(0)}{L}\right) = -x_2 f_1 \end{aligned} \tag{4.6}$$

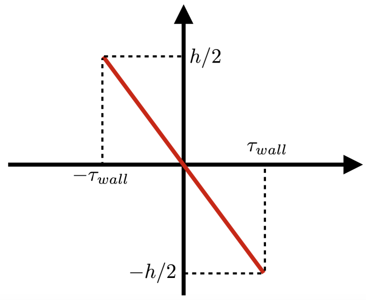

Figure 4.3

where $f_1 = \rho a - \frac{dp}{dx} = -\frac{d\hat{p}}{dx}$ is the driving force (force per unit volume) and $\hat{p} = p - \rho a$ the driving pressure, we see that even the pressure gradient appears in his expression. The evolution of the linear τ is represented on Figure 4.3, shear stress representing the effect of the upper part on the lower part, it seems legit to have $-\tau$ for the upper wall meaning that the wall slows down the fluid (below the fluid drag the wall). The velocity profile can be found as

$$\tau = -x_2 f_1 = \mu\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial \cancel{u_2}}{\cancel{\partial x_1}}\right) \qquad \Rightarrow \mu\frac{\partial u_1}{\partial x_2} = -x_2 f_1 \Leftrightarrow u_1 = \frac{f_1}{2\mu}(-x_2^2 + cst). \tag{4.7}$$

The non slip boundary condition at the wall gives $u_1(\pm\frac{h}{2}) \Rightarrow c = \left(\frac{h}{2}\right)^2$. The velocity profile is

$$u_1 = \frac{f_1}{2\mu}\left(\left(\frac{h}{2}\right)^2 - x_2^2\right) \tag{4.8}$$

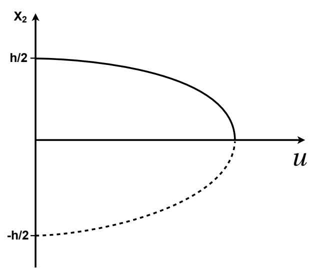

Figure 4.4

which is parabolic as shown on Figure 4.4 with a maximum on x_1 axis of value $u_1 = \frac{f_1 h^2}{8\mu}$. It is also interesting to compute the volume flow per unit span $[m^2/s]$ (x_3)

$$\dot{V} = \int_{-h/2}^{h/2} u_1\, dx_2 = \frac{2}{3}h\frac{f_1 h^2}{8\mu} = \frac{f_1 h^3}{12\mu} \tag{4.9}$$

where the integral has been computed using the definition of the era of the parabole $2/3 \times h \times u_c$.

Dimensional analysis Imagine that we would have liked to determine the velocity profile by dimensional analysis. The veloity dependance is

$$u_1 = f(x_2, f_1, h, \mu, \rho) \tag{4.10}$$

giving us 6 variables and 3 physical dimensions and so 3 dimensionless groups. To make the velocity dimensionless, we computed the velocity at the center, let's use it. The dimensionless velocity will be function of 2 dimensionless groups

$$\frac{u_1}{\frac{f_1 h^2}{8\mu}} = \varphi\left(\frac{2x_2}{h}, Re = \frac{u_c h}{\nu} = \frac{\rho f_1 h^3}{8\mu^2}\right). \tag{4.11}$$

Now if we rewrite (4.8) as

$$u_1 = \frac{f_1 h^2}{8\mu}\left(1 - \left(\frac{2x_2}{h}\right)^2\right). \tag{4.12}$$

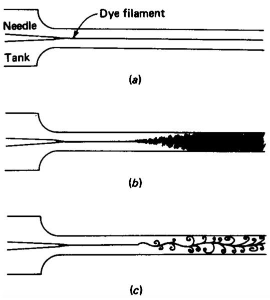

Figure 4.5

When we compare the 2 equations, we see that $\varphi = f(x_2/h)$. That means that when we solve the equation there is no dependance on the Re number. So the dimensional analysis doesn't give the full information, we have to solve. Re number is the ratio between viscous forces and conventional inertial forces. Because of fully developed criteria there is no conventional inertial forces, it is normal so that the flow does not depend on the Re number. This parabolic profile is respected for low velocities but disturb when velocity increases. We made the assumption that the flow is steady, but for a certain velocity the flow is no longer steady, it becomes chaotic (turbulences).

4.2 Macroscopique des cription of turbulent flows - Reynolds decomposition

The interest of this approach is in the mean flow, in average quantities. The idea is to repeat the experiment several times and make a statistical average in order to decompose all variables into an average and a fluctuation. This is called the Reynolds decomposition

$$\begin{gathered} \forall\ quantity\ q : \langle q(x_1,x_2,x_3,t)\rangle = \tfrac{1}{N}\textstyle\sum_{k=1}^{N} q_k(x_1,x_2,x_3,t) \\ and \\ q_k(x_1,x_2,x_3,t) = \underbrace{\langle q_k(\cdots)\rangle}_{average} + \underbrace{q_k(\cdots)}_{fluctuation} \end{gathered} \tag{4.13}$$

where k is the experiment index. We are going to derive equations for the average properties for statistically steady flows (such that $\frac{\partial\langle q\rangle}{\partial t} = 0$). We can consider the time average with a certain period T

$$\bar{q}_T(x_1,x_2,x_3,t) = \frac{1}{T}\int_{t-T/2}^{t+T/2} q(x_1,x_2,x_3,t)\,dt \tag{4.14}$$

which "smooth" the signal by keeping only large time scale fluctuations. For statistically steady flows, the **ergodicity hypothesis** is valid

$$\lim_{T\to\infty} \bar{q}_T = \langle q\rangle. \tag{4.15}$$

For statistically unsteady flows, it is valid only if T is much larger than the turbulent fluctuations time scale and much smaller than the average motion time scale.

Properties of the averaging operator

- **Linearity :**
$$\langle aq_1 + bq_2\rangle = a\langle q_1\rangle + b\langle q_2\rangle \tag{4.16}$$
-
$$\langle\langle q\rangle\rangle = \langle q\rangle \tag{4.17}$$
-
$$\langle q'\rangle = 0 \qquad as \qquad \langle q\rangle = \langle\langle q\rangle + q'\rangle = \langle\langle q\rangle\rangle + \langle q'\rangle = \langle q\rangle + \langle q'\rangle \Rightarrow \langle q'\rangle = 0 \tag{4.18}$$

-
$$\langle\langle q_1\rangle q_2\rangle = \langle q_1\rangle\langle q_2\rangle \tag{4.19}$$

- **Commutativity with differential operators**
$$\left\langle \frac{\partial q}{\partial x_1}\right\rangle = \frac{\partial\langle q\rangle}{\partial x_1} \qquad \left\langle \frac{\partial q}{\partial t}\right\rangle = \frac{\partial\langle q\rangle}{\partial t} \tag{4.20}$$

Average continuity equation

Let's remind that we are considering **constant density** flow. In this case the governing equation for mass is

$$\cancel{\dot\rho} + \rho\nabla\vec{u} = 0 \qquad \Rightarrow \nabla\vec{u} = 0 = \frac{\partial u_i}{\partial x_i} \tag{4.21}$$

Let's average this out

$$\left\langle\frac{\partial u_i}{\partial x_i}\right\rangle = \left\langle\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\right\rangle = \left\langle\frac{\partial u_1}{\partial x_1}\right\rangle + \left\langle\frac{\partial u_2}{\partial x_2}\right\rangle + \left\langle\frac{\partial u_3}{\partial x_3}\right\rangle = 0 \tag{4.22}$$

and using the commutativity with differantial operators, we have the

Average of the continuity equation

$$\frac{\partial\langle u_i\rangle}{\partial x_i} = 0 \tag{4.23}$$

This is exactly what we need because we have only average velocity field.

Average momentum equation

The conservation form of the equation without considering body force is

$$\frac{\partial\rho u_i}{\partial t} + \frac{\partial\rho u_i u_j}{\partial x_j} = \rho\left[\frac{\partial u_i}{\partial t} + \frac{\partial u_i u_j}{\partial x_j}\right] = -\frac{\partial p}{\partial x_i} + \frac{\partial\tau_{ji}}{\partial x_j}. \tag{4.24}$$

Let's average this out

$$\rho\left[\frac{\partial\langle u_i\rangle}{\partial t} + \frac{\partial\langle u_i u_j\rangle}{\partial x_j}\right] = -\frac{\partial\langle p\rangle}{\partial x_i} + \frac{\partial\langle\tau_{ji}\rangle}{\partial x_j} \qquad \textit{with} \qquad \begin{aligned}\tau_{ji} &= \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_j}\right)\\ \langle\tau_{ji}\rangle &= \mu\left(\frac{\partial\langle u_i\rangle}{\partial x_j} + \frac{\partial\langle u_j\rangle}{\partial x_j}\right)\end{aligned} \tag{4.25}$$

This was easy game, let's now concentrate on $\langle u_i u_j\rangle$ by considering the Reynolds decomposition

$$\begin{aligned}\langle u_i u_j\rangle &= \Big\langle(\langle u_i\rangle + u_i')(\langle u_j\rangle + u_j')\Big\rangle\\ &= \langle\langle u_i\rangle\langle u_j\rangle + \langle u_i\rangle u_j' + u_i'\langle u_j\rangle + u_i'u_j'\rangle\\ &= \langle u_i\rangle\langle u_j\rangle + \underbrace{\langle\langle u_i\rangle u_j'\rangle}_{=0} + \underbrace{\langle u_i'\langle u_j\rangle\rangle}_{=0} + \underbrace{\langle u_i'u_j'\rangle}_{\neq 0}\end{aligned} \tag{4.26}$$

The last term is clearly $\neq 0$ as if $i = j$ we have the average of a square wich is never 0 when $u \neq 0$. If now we replace this in (4.25), we have

$$\rho\left[\frac{\partial\langle u_i\rangle}{\partial t} + \frac{\partial\langle u_i\rangle\langle u_j\rangle}{\partial x_j} + \frac{\partial\langle u_i'u_j'\rangle}{\partial x_j}\right] = -\frac{\partial\langle p\rangle}{\partial x_i} + \frac{\partial\langle\tau_{ji}\rangle}{\partial x_j} \tag{4.27}$$

and we see that in fact we still have fluctuations in the average momentum equation. But we see that we have essentially the same equation as for laminar, normal viscous flows and an extra term. Remind that we interpretated $\rho\langle u_i\rangle\langle u_j\rangle$ as beeing the **average momentum flux tensor** and $\frac{\partial\langle\tau_{ji}\rangle}{\partial x_j}$ was the **molecular agitation momentum fluxes**. So the fluctuations results in an additional momentum flux tensor called the **turbulent fluctuation momentum fluxes**. If we bring this term to the right side, we have the

Average momentum equation

$$\rho\left[\frac{\partial\langle u_i\rangle}{\partial t}+\frac{\partial\langle u_i\rangle\langle u_j\rangle}{\partial x_j}\right]=-\frac{\partial\langle p\rangle}{\partial x_i}+\frac{\partial}{\partial x_j}\left(\langle\tau_{ji}\rangle-\rho\langle u_i'u_j'\rangle\right)\tag{4.28}$$

where we can see the term as additional stresses called the **Reynolds stresses** T_{ij}^R.

Channel flow

Let's come back to the channel flow which is statistically steady and fully developed. So, reminding that $u_2=0$ the $i=1$ momentum equation reduces to

$$\frac{\partial\langle u_1\rangle^2}{\partial x_1}=0=\underbrace{-\frac{\partial\langle p\rangle}{\partial x_1}}_{f_1}+\frac{\partial}{\partial x_2}\underbrace{\left(\langle\tau_{21}\rangle-\rho\langle\tau_{21}^R\rangle\right)}_{\tau_{21}^{tot}}\qquad\Rightarrow\tau_{21}^{tot}=-f_1x_2\tag{4.29}$$

which is exactly the same expression as we obtained before at the difference that now we have to consider in addition the new stresses.

4.3 Average velocity profiles in turbulent wall-bounded shear flows

We've seen that for the channel flow in laminar flow we obtained a parabolic velocity profile. It is also the case for a flow in a pipe for laminar flow. Many measure instruments do the averaging process by themself. In the turbulent case, the profiles are much more flatter/uniform even in the channel flow. This can be explained by turbulence. Indeed, the agitations play the role of an agitator, they exchange the momentum neighboring there they do mixing/homogenise so the velocity is more uniform. The consequence is that velocity has to fall down more rapidly close to the wall where the only shear is the molecular shear (no fluctuation), leading to increased friction. Another observation is that contrary to the case of laminar flows, the parabole is no longer independant to the Re number. We will try to express the velocity profile in turbulent flow and for this we will consider a channel flow

4.3.1 Channel flow

Let's start with the average momentum equation (4.28) that simplifies knowing that the flow is statistically steady and fully developed, we have

$$0=-\frac{\partial\langle p\rangle}{\partial x_1}+\frac{\partial}{\partial x_2}\left(\langle\tau_{21}\rangle-\rho\langle u_1'u_2'\rangle\right)\qquad\Rightarrow\tau_{21}^{tot}=-f_1x_2\tag{4.30}$$

leading to the linear relation we found next time for τ_{21}^{tot}.

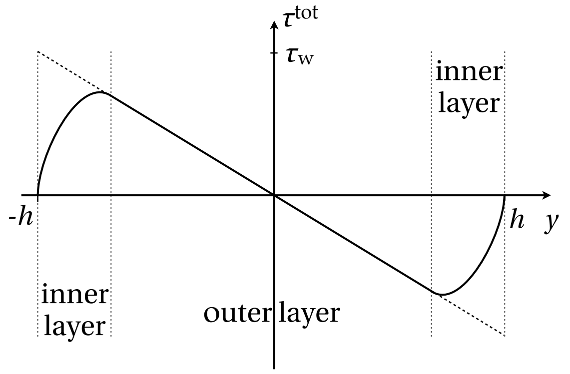

Figure 4.6

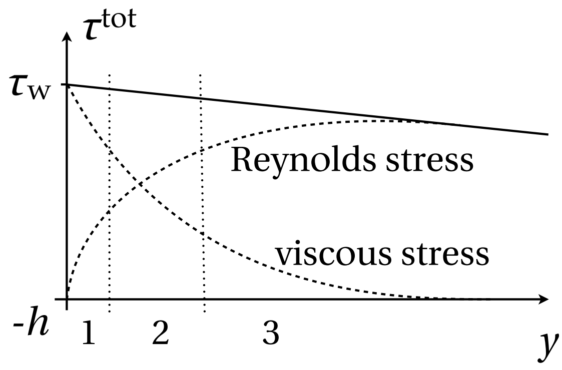

Figure 4.7

The channel can be decomposed in several zones, namely the central zone where the total stress is nearly only the Reynold stress, there is hardly no conrtibution of the viscous stress. Then there is a region close to the wall where this last becomes dominant. The reason why the velocity profile is dependent of the Re number is that these two stresses doesn't vary in the same way with Re. Figure 4.6 represents the Re stress, the totally linear one beeing the total stress, so the viscous stress is null in the center whereas it increases near the wall (where Re stress decreases). We decompose the channel in an **outer zone** where $\tau_{21}^{tot} \approx \tau_{21}^{R} = -\rho \langle u_1' u_2' \rangle$ and an **inner zone** where both stresses are significant. Figure 4.7 consists in a zoom on the left inner zone.

We will now use dimensional analysis to find the velocity profile in the two regions. Let say that the average velocity at a point on the channel is a function of

$$\langle u \rangle = f\left(y, \sqrt{\frac{\tau_{wall}}{\rho}} = u_\tau, \nu, u_c, h\right) \tag{4.31}$$

Depending on the region, this function will change

$$inner : \langle u \rangle = f\left(y, \sqrt{\frac{\tau_{wall}}{\rho}} = u_\tau, \nu, \cancel{u_c}, \cancel{h}\right) \qquad outer : \langle u \rangle = f\left(y, \sqrt{\frac{\tau_{wall}}{\rho}} = u_\tau, \cancel{\nu}, u_c, h\right) \tag{4.32}$$

where u_τ is the friction velocity.

4.3.2 Inner zone (smooth wall)

If we look at our reduced function, this involves 4 quantities and 2 physical dimensions, leading to two dimensionless groups

$$u^+ \equiv \frac{\langle u \rangle}{u_\tau} = f\left(Re = \frac{y u_\tau}{\nu} \equiv y^+\right) \tag{4.33}$$

where u^+ and y^+ are the wall units, notation in litterature for these dimensionless groups. The inner zone can be decomposed into three sublayer as indicated on Figure 4.7 :

- **the viscous sublayer (1) :** very close to the wall, where $\langle \tau_{21}^V \rangle \gg \langle \tau_{21}^R \rangle$
- **the buffer layer (2) :** the transition layer where $\langle \tau_{21}^V \rangle \approx \langle \tau_{21}^R \rangle$
- **the overlap layer (3) :** where $\langle \tau_{21}^V \rangle \ll \langle \tau_{21}^R \rangle$

Viscous sublayer (1)

This layer is so small and close to the wall that

$$
\begin{aligned}
& \tau_{21}^{tot}(y) = \mu \frac{\partial \langle u \rangle}{\partial y} \approx \tau_{wall} && \Leftrightarrow && \frac{\mu}{\rho}\frac{\partial \langle u \rangle}{\partial y} = \frac{\tau_{wall}}{\rho} = u_\tau^2 \\
\Leftrightarrow \quad & \nu \langle u \rangle = u_\tau^2 y && \Leftrightarrow && \frac{\langle u \rangle}{u_\tau} = \frac{u_\tau y}{\nu}
\end{aligned}
\tag{4.34}
$$

This means that here we have the final result

$$
u^+ = y^+ \tag{4.35}
$$

Overlap layer (3)

There viscosity does not play a role. In other words, we expect that

$$
\frac{\partial \langle u \rangle}{\partial y} = f'\left(y, u_\tau, \cancel{\nu}\right) \qquad \Rightarrow \frac{y}{u_\tau}\frac{\partial \langle u \rangle}{\partial y} \approx cst = \frac{1}{\kappa}. \tag{4.36}
$$

So if we make appear the wall notations, we have

$$
\frac{u_\tau^2}{\nu}\frac{\partial u^+}{\partial y^+} = \frac{1}{\kappa}\frac{u_\tau}{y} \qquad \Rightarrow \frac{\partial u^+}{\partial y^+} = \frac{1}{\kappa}\frac{\nu}{y u_\tau} = \frac{1}{\kappa y^+} \qquad \Rightarrow u^+ = \frac{1}{\kappa}\ln y^+ + B. \tag{4.37}
$$

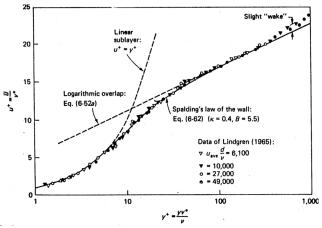

Figure 4.8

This is theory, let's check what theory says about. When we look at the diagram, our theory matches roughly with the linear (in log) for $50 \leq y^+ \leq 500$. For the viscous sublayer represented by the exponential, the curves matches till $y^+ \approx 5$ wall units. That's very small (1/10 - 1/100 of the overloop). There is a smooth transition between the two curves.

Buffer layer (2)

The idea is to rewrite the expression of the two zones but with $y^+ = f(u^+)$

$$
\begin{aligned}
& viscous\ layer : y^+ = u^+ \\
& overloop\ layer : y^+ = \exp(\kappa(u^+ - B)) = \exp(\kappa u^+)\exp(-\kappa B)
\end{aligned}
\tag{4.38}
$$

We can have a good transition between the two by

$$
y^+ = u^+ + \exp(-\kappa B)\left[\exp(\kappa u^+) - \underbrace{\left(1 + \kappa u^+ + \frac{(\kappa u^+)^2}{2} + \frac{(\kappa u^+)^3}{6}\right)}_{taylor\ serie\ expansion\ of\ \exp(\kappa u^+)}\right] \tag{4.39}
$$

Indeed when κu^+ is small, the second part will be negligible regarding u^+ and for high, we found the overlap layer. To have the smooth transition, only two terms are enough but taking more leads to a best fitting with the practice.

4.3.3 Inner zone (rough wall)

The wall are never exactly smooth in rality. We already discussed that to have exact similarity between the model and the prototype we must have the same relative rougness (λ/L). Let's see the effect on velocity profile. In practice it is impossible to do an exhaustive study because of the number of parameters. There is also various forms of roughness:

- **uniform sand roughness:** when you blew sand particles on a paper for example
- **non uniform sand roughness:** when particles have different scale and form
- **periodic roughness:** this can be obtained for example if we put wires or square ribs at regular interval.

We will consider the first one. In the overlap layer, we expect that (4.37) remains correct but B will be function of k^+ which is k, the hight of the grains, in wall units (divided by the inner zone length scale)

$$u^+ = \frac{1}{\kappa}\ln y^+ + B_1(k^+) \qquad \textit{with} \qquad k^+ = k\frac{u_\tau}{\nu} \tag{4.40}$$

We expect that roughness will slow down the fluid meaning that $B_1(k^+) < B$. We can rewrite our equation as

$$u^+ = \frac{1}{\kappa}\ln y^+ + B - \Delta u^+(k^+) \qquad \textit{where} \qquad \Delta u^+(k^+) = B - B_1(k^+) \tag{4.41}$$

where Δu^+ is the velocity deficit, expected to be > 0.

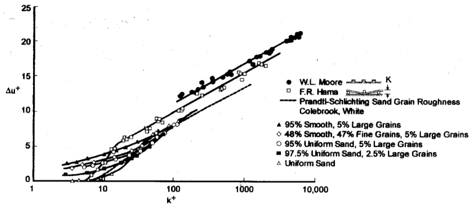

Figure 4.9

In experiments we see indeed that Δu^+ is a function of k^+ and depends on the type of roughness. In which we concerns, we have to look to the triangles. The first observation is that $\Delta u^+ = 0$ for $k^+ \leq 5$, so if the roughness is such that the hight doesn't exceed 5 in wall units, the fluid behaves exactly as the wall was smooth : **hydraulically smooth regime**. Let's remind that 5 is the upper limit of the viscous sublayer. The conclusion is that, as long as the grains remains barried within the viscous sublayer, roughness does not influence the average velocity profile. For non uniform roughness, there are bigger grains that goes throw this limit. The second observation is that for higher k^+ (≥ 80) on the logarithmic axis Δu^+ respect a line of constant slope

$$\Delta u^+ = \frac{1}{\kappa}\ln k^+ + B_3 \qquad k^+ \geq 80 \tag{4.42}$$

This means that the velocity profile in this zone is

$$u^+ = \frac{1}{\kappa}\ln y^+ + B - \left(\frac{1}{\kappa}\ln k^+ + B_3\right) = \frac{1}{\kappa}\ln\frac{y^+}{k^+} + B - B_3 = \frac{1}{\kappa}\ln\frac{y}{k} + B - B_3 \tag{4.43}$$

We see that the appropriate length scale changes from ν/u_τ to beeing k itself and this is called the **fully rough regime** where we can make another manipulation for u^+

$$u^+ = \tfrac{1}{\kappa}\ln y^+ + B - \Delta u^+(k^+) = \tfrac{1}{\kappa}\ln\tfrac{y^+}{k^+} + \underbrace{B + \frac{1}{\kappa}\ln k^+ - \Delta u^+(k^+)}_{B_2(k^+)}$$

$$\textit{where} \tag{4.44}$$

$$k^+ \leq 5 \quad B_2(k^+) \approx B + \tfrac{1}{\kappa}\ln k^+ \qquad k \geq 80 \quad B_2(k^+) \approx B - B_3$$

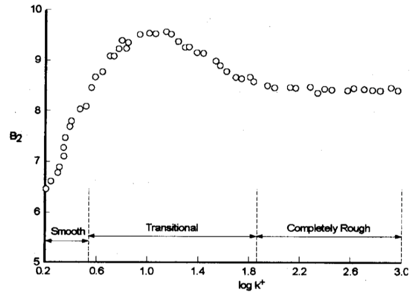

Figure 4.10

This is shown on this diagram where we have a linear curve for $k \leq 5$ giving $B = 5.5$ (y axis) and then the constant area ($k \leq 80$). Notice that we have an overshoot in the transitional zone between the two, this is important to see. Let's see what happens in the irregular roughness case. If we come back to Figure 4.9, we can see that in the fully rough regime, the irregular curve corresopnds to a simple shift of the regular one. In order to approach this second case, let's compute B_3 for uniform sand roughness by extrapolating $B - B_2 = 5.5 - 8.5 = -3$. The equivalent sand roughness will be defined in such a way that we have the same Δu^+ in the fully rough regime where B_S is the universal value of -3

$$\Delta u^+ = \frac{1}{\kappa}\ln k^+ + B_3 = \frac{1}{\kappa}\ln k_S^+ + B_{3S} \qquad \Leftrightarrow \frac{1}{\kappa}\ln\frac{k_S^+}{k^+} = \frac{1}{\kappa}\ln\frac{k_S}{k} = B_3 - B_{3S}. \tag{4.45}$$

4.3.4 Outer zone

Let's remind that we assumed

$$\langle u\rangle = f(y, u_\tau, u_c, h) \qquad \text{or for a boundary layer: } u_c \text{ outer flow velocity and } h \to \delta \tag{4.46}$$

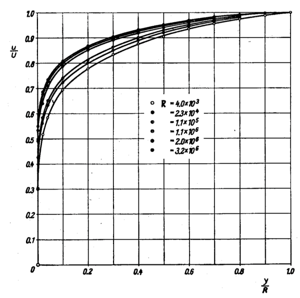

Figure 4.11

To determine what this function should be, let's go to the velocity profile in the pipe where we started from to be guided from the curves. These are more or less flat depending on the Reynolds number whith $u/u_c \to 1$ (u_c velocity at the center of the pipe). Let's imagine that we plot now $1 - \frac{\langle u\rangle}{u_c} = \Delta u/u_c$ in function of y/R, 1 will become 0 and $0 \to 1$, the graph will be reversed (Figure 4.12 left). The curves seems to be similar, so maybe I take the velocity deficit of the half hight $\Delta u(0.5)/u_c$ which depends on Reynolds number and plot that for $\frac{u_c - \langle u\rangle}{u_c}\frac{u_c}{\Delta u}$ (Figure 4.12 center). When this $= 1$, $y/r = 0.5$ and we see that it fall on the same curve as Figure 4.12 left but we get a single curve. This means that

$$\frac{u - \langle u\rangle}{\Delta u} = f\left(\frac{y}{R}\right) \tag{4.47}$$

where the only scale that doesn't appear based on (4.46) is u_τ. So if we plot $\frac{u_e - \langle u\rangle}{u_\tau} = f\left(\frac{y}{R}\right)$ (u_e max velocity at the center) we will obtain a single curve shown on Figure 4.12 right.

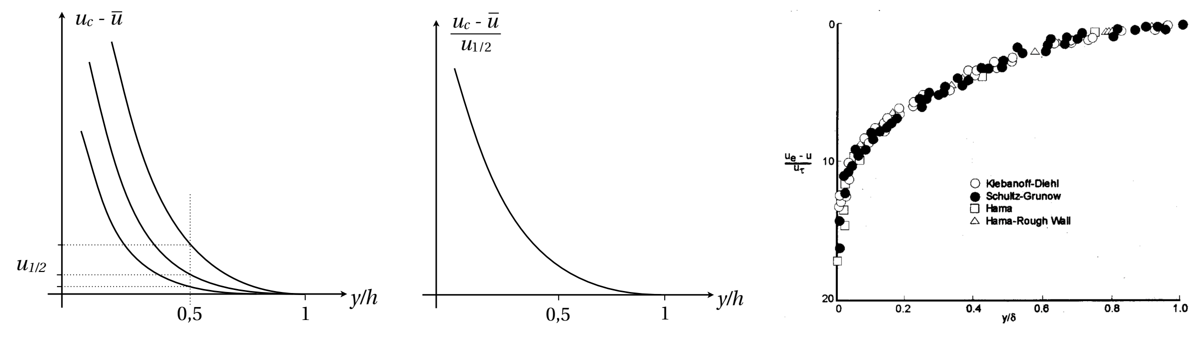

Figure 4.12

Interestingly enough, the logarithmic low is compatible with the outer scaling

$$\frac{\langle u \rangle}{u_\tau} = u^+ = \frac{1}{k} \ln y^+ + B \tag{4.48}$$

But what's u_e/u_τ? We know that

$$u_\tau = \sqrt{\frac{\tau_{wall}}{\rho}} \qquad \Rightarrow \frac{u_\tau}{u_e} = \sqrt{\frac{\tau_{wall}}{\rho u_e^2}} = \sqrt{\frac{C_f}{2}} \qquad \Rightarrow \frac{u_e}{u_\tau} = \sqrt{\frac{2}{C_f}} \tag{4.49}$$

where we define the **friction coefficient** $C_f = \frac{\tau_{wall}}{\rho u_e^2/2}$ as we defined the pressure coefficient $\frac{p-p_\infty}{\rho u_\infty^2/2}$. By combining the two last equation, we have

$$\begin{aligned} \frac{u_e - \langle u \rangle}{u_\tau} &= \sqrt{\frac{2}{C_f}} - \frac{1}{\kappa} \ln\left(\frac{y u_\tau}{\nu} \frac{\delta}{\delta}\right) - B = \sqrt{\frac{2}{C_f}} - \frac{1}{\kappa} \ln \frac{\delta u_\tau}{\nu} - B - \frac{1}{\kappa} \ln \frac{y}{\delta} \\ &= cst - \frac{1}{\kappa} \ln \frac{y}{\delta} \end{aligned} \tag{4.50}$$

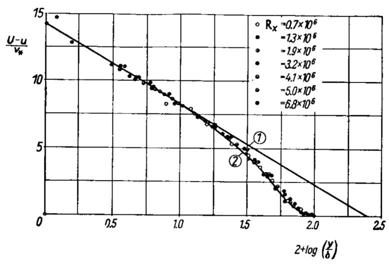

Figure 4.13

that confirms the previously founded relation. Wa can plot this in logarithmic scale that gives Figure 4.13. We see that it obbeys the logarithmic law as in the overlap layer which is compatible with the inner and outer zone scaling. In fact we can rewrite our last equation as

$$\frac{u_e - \langle u \rangle}{u_\tau} = f\left(\frac{y}{\delta}\right) = cst - \frac{1}{\kappa} - g\left(\frac{y}{\delta}\right) \tag{4.51}$$

where $g\left(\frac{y}{\delta}\right)$ is the deviation compared to the logarithmic law. Because of velocity deficit = 0 when $y/\delta = 1$, it turns out that the cst = g(1). As conclusion, we found that the velocity profile is given by

$$u^+ = \frac{1}{\kappa} \ln y^+ + B + g\left(\frac{y}{\delta}\right) \tag{4.52}$$

and this is observed on **??** where we observe the deviation at the very right side of the figure. For the case of zero pressure gradient, Coles observed that the deviation from the logarithmic velocity profile $g\left(\frac{y}{\delta}\right)$ is similar to the velocity profile in a half-jet or wake

$$g\left(\frac{y}{\delta}\right) = \Pi(x) w\left(\frac{y}{\delta}\right) \approx \Pi(x) 2 \sin^2\left(\frac{\pi}{2}\frac{y}{\delta}\right) \tag{4.53}$$

which was subsequently called the **law of the wake**. The coefficient $\Pi(x)$ is the amplitude of the wake, which varies slightly with Reynolds number as shown on Fig. 29, ultimately reaching a constant value equal to about 0.55 for $Re\theta > 5000$.

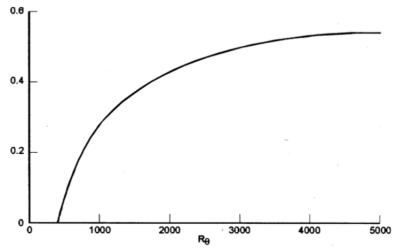

Figure 4.14

4.4 Introduction to turbulent modelling

Let's remind the average Navier-Stokes equation

$$\rho\left[\frac{\partial \bar{u}_i}{\partial t}+\frac{\partial \overline{u_i u_j}}{\partial x_j}\right]=-\frac{\partial \bar{p}}{\partial x_i}+\frac{\partial}{\partial x_j}\Bigg[\underbrace{\mu\left(\frac{\partial \bar{u}_i}{\partial x_j}+\frac{\partial \bar{u}_j}{\partial x_i}\right)}_{\tau_{ij}^v}-\underbrace{\rho\bar{u_i}'\bar{u_j}'}_{\tau_{ij}^t}\Bigg] \tag{4.54}$$

There is still some unknowns, we will try to find equations describing these unknowns. It is possible, but the problem is that we introduce more unknowns than we had before, it doesn't solve the problem. This means that to close the system of equation, one of them has to stop at one stage and model the unkonown terms. One approach is to model directly the Reynold stresses as a function of the average flow quantities, this is called **first order closure approach**. The other strategy is to retain the transport equations for Reynolds stresses and to model the unknows in these equations, **second order closure model**. We will make an introduction with the first approach. The most corresponding method to this is Bousinesq approach or Eddy viscosity approach. The idea is to use a model analogous to viscous stresses. For these we have the **Newton's model** that says

$$\tau_{ji}=\mu\Big(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\qquad\underbrace{-\frac{2}{3}\delta_{ij}\frac{\partial u_k}{\partial x_k}}_{=0\text{ constant density}}\qquad\Big)\Rightarrow-\rho\bar{u}_i'\bar{u}_j'=\mu_t\left(\frac{\partial \bar{u}_i}{\partial x_j}+\frac{\partial \bar{u}_j}{\partial x_i}\right)-\rho\,\underbrace{\overline{u_k' u_k'}}_{2k}\,\frac{\delta_{ij}}{3} \tag{4.55}$$

where μ_t is the **Eddy viscosity coefficient** and where we needed to add the last term in the second expression because if we consider the trace of $-\rho\bar{u_i}'\bar{u}_j'$, so if we contract i and j which gives $\overline{(u_1')^2+(u_2')^2+(u_3')^2}\neq 0$ that can be seen as twice the **fluctuation kinetic energy 2k** while $2\frac{\partial u_i}{\partial x_i}=0$. This is one more unknown but we know that

$$\frac{\partial}{\partial w_j}\left(-\rho\frac{2k}{3}\delta_{ij}\right)=\frac{\partial}{\partial x_i}\left(-\frac{2}{3}\rho k\right)\qquad\Rightarrow-\frac{\partial}{\partial x_i}\left(p+\frac{2}{3}\rho k\right) \tag{4.56}$$

This is add to the pressure gradient because it has the same form and is an effective pressure. The only unknown is now the viscosity to model the 6 unknowns of the Reynolds stresses. Contrary to μ which is only function of the the fluid thermodynamic state, $\mu_t=f$(fluid properties, average flow field quantities, space coordinates) and varies within the flow. The simpliest model to find it is an algebric model

Algebric model

It uses some physical intuition and dimentional analysis. We know that $\mu_t=\rho\nu_t$ and $[\nu_t]=L^2T^{-1}$. So we need a caracteristic fluctuation length scale and time scale. We will describe the **Prandtl's Mixing Length Model** which says, for

$$T^{-1}:\qquad\frac{\partial \bar{u}}{\partial y} \tag{4.57}$$

and for the length scale, we take the average distance travelled by the fluctuating particle, this is clear that this distance is limited by the wall. This leads Prandtl to consider

$$L=\kappa y\qquad\Rightarrow\mu_t=\rho(\kappa y)^2\frac{\partial \bar{u}}{\partial y}. \tag{4.58}$$

We will now see an application of this for the average velocity profile in the overlap layer Figure 4.7. We remind that

$$\tau_{xy}^{tot}(y) = \tau_{xy}^{V}(y) + \tau_{xy}^{R}(y) \approx \tau_{wall} \tag{4.59}$$

we will assume that the total stress is essentialy the stress at the wall. We know that Reynolds stress is dominant in this region which is given by Eddy model (4.55)

$$\begin{aligned} &\tau_{wall} = \tau_{xy}^{R}(y) = \mu_t \frac{\partial \bar{u}}{\partial y} = \rho(\kappa y)^2 \left(\frac{\partial \bar{u}}{\partial y}\right)^2 \qquad \Rightarrow \frac{\tau_{wall}}{\rho} = (\kappa y)^2 \left(\frac{\partial \bar{u}}{\partial y}\right)^2 \\ &\Rightarrow \frac{\partial \bar{u}}{\partial y} = \frac{u_\tau}{\kappa y} \qquad \Rightarrow \frac{\partial u^+}{\partial y^+} = \frac{1}{\kappa y} \qquad \Rightarrow u^+ = \frac{1}{\kappa} \ln y^+ + B \end{aligned} \tag{4.60}$$

and this is consistant with the universal logarithmic profile in the overlap layer. For the buffer layer, μ_t is too large, so it's not accurate. So the mixing length has to be reduced near the wall. A popular model for this is the Van Dreast dampings

$$l(y) = \left(1 - \log\left(-\frac{y^+}{A^+}\right)\right)\kappa y = \left(1 - \exp\left(-\frac{y u_\tau}{26\nu}\right)\right)\kappa y \tag{4.61}$$

where $A^+ = 26$. This is a good approximation for mixing length in the buffer layer.

Chapter 5

Boundary layer

5.1 Derivation of the boundary layer equations

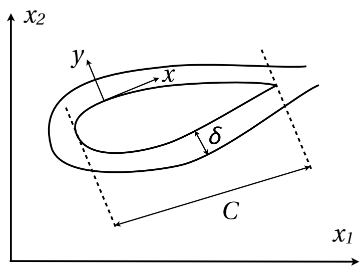

Figure 5.1

Let's remind that in the introduction we analysed the circumstances in whihch viscous forces can be neglected and the conclusion was that it was caracterized by the Reynold's number. When it is large, it means that viscous forces are small compared to convective inertial forces. Despite the small order of magnitude of viscous forces, they can't be neglected everywhere (walls). This leads us to consider 2 region around a body :

- **the distal or outer zone**: wherer the flow is inviscid so viscous forces are negligible (ch3).
- **the thin, proximal or inner zone**: where viscous stresses may not be neglected, leading to the boundary **layer ffi** which is next to a solid wall and a region behind the body called the **wake** (sillage).

We hope that, similarly to the inviscid case where equations simplifies, the case will be here because of the small thickness of the boundary layer. We make a first assumption saying that if C is the caracteristic length of the body in the tengential direction and δ the one in the normal direction

$$when\ Re_C \gg 1 \qquad \Rightarrow \delta \ll C \tag{5.1}$$

The whole chapter is based on constant density flows, so the governing equation in 2D are

$$\begin{aligned}
&\frac{\partial u_1}{\partial x_1} + \frac{Du_2}{\partial x_2} = 0 \\
&\rho\left[u_1\frac{\partial u_1}{\partial x_1} + u_2\frac{\partial u_1}{\partial x_2}\right] = -\frac{\partial p}{\partial x_1} + \mu\left(\frac{\partial^2 u_1}{\partial x_1^2} + \frac{\partial^2 u_1}{\partial x_2^2}\right) \\
&\rho\left[u_1\frac{\partial u_2}{\partial x_1} + u_2\frac{\partial u_2}{\partial x_2}\right] = -\frac{\partial p}{\partial x_2} + \mu\left(\frac{\partial^2 u_2}{\partial x_1^2} + \frac{\partial^2 u_2}{\partial x_2^2}\right)
\end{aligned} \tag{5.2}$$

These equations are for a coordinate system established to study the outer flow. Now as we analyse the flow in a fin layer close to the body it is convinient to rewrite the flow equations in a body fitted curvilinear system where x is the curvilinear coordinate tengent to the body and y the one normal to the body. If we can assume that $\delta \ll R$ which is the body radius of curvature (variable), the transformed curvilinear equations are identical to the original cartesian equations

$$\begin{aligned}
&\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \\
&\rho\left[u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right] = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \\
&\rho\left[u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right] = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)
\end{aligned} \tag{5.3}$$

The condition $\delta \ll R$ is the most likely to be violated where R is the smalest, which corresponds to the front of the body. But let's imagine that the condition is fullfilled. In these equations we didin't use δ so let's rewrite these equations in non-dimensional form by choosing the non-dimensional variables $(\tilde{x}, \tilde{y}, \tilde{u}, \tilde{v}, \tilde{p})$ in such a way that they'll be of order of magnitude 1

$$\begin{aligned}
&\tilde{x} = \frac{x}{C} \qquad && \tilde{u} = \frac{u}{u_\infty} \qquad && \tilde{p} = \frac{C_p}{2} = \frac{p - p_0}{\rho u_\infty^2} \\
&\tilde{y} = \frac{y}{\delta} && \tilde{v} = \frac{v}{v_\delta(?)} &&
\end{aligned} \tag{5.4}$$

where for u we know that for the inviscid case we considered u_∞ as velocity of the flow on the body and v was null on the body wall so we have a "?". The continuity equation in dimensionless variables will help us

$$\frac{u_\infty}{C}\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{v_\delta}{\delta}\frac{\partial \tilde{v}}{\partial \tilde{y}} = 0 \qquad \Rightarrow \underbrace{\frac{\partial \tilde{v}}{\partial \tilde{y}}}_{\theta(1)} = -\frac{u_\infty}{C}\frac{\delta}{v_\delta}\underbrace{\frac{\partial \tilde{u}}{\partial \tilde{x}}}_{\theta(1)} \qquad \Rightarrow \frac{u_\infty}{C}\frac{\delta}{v_\delta} = \theta(1) \Leftrightarrow v_\delta = \theta\left(\frac{\delta u_\infty}{C}\right) \tag{5.5}$$

where $\theta(1)$ means order of magnitude 1. We are going to replace $v_\delta = \frac{\delta u_\infty}{C}$. We are going to do the same operation for momentum equations. Let's begin with the tengential momentum equation

$$\begin{aligned}
&\rho\left[\frac{u_\infty^2}{C}\tilde{u}\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{u_\infty^2 \cancel{\delta}}{C}\frac{\tilde{v}}{\cancel{\delta}}\frac{\partial \tilde{u}}{\partial \tilde{y}}\right] = -\rho\frac{u_\infty^2}{C}\frac{\partial \tilde{p}}{\partial \tilde{x}} + \mu\left(\frac{u_\infty}{C^2}\frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2} + \frac{u_\infty}{\delta^2}\frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2}\right) \\
\Leftrightarrow \quad &\rho\frac{u_\infty^2}{C}\left[\tilde{u}\frac{\partial \tilde{u}}{\partial \tilde{x}} + \tilde{v}\frac{\partial \tilde{u}}{\partial \tilde{y}} + \frac{\partial \tilde{p}}{\partial \tilde{x}}\right] = \mu\frac{u_\infty}{\delta^2}\left(\cancel{\frac{\delta^2}{C^2}\frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2}} + \frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2}\right)
\end{aligned} \tag{5.6}$$

where we make appear $\frac{\delta^2}{C^2}$ which is much smaller than one if $Re_C \gg 1$. Because of all the dimensionless variables are of order of magnitude 1 $\theta(1)$ the two constants must be of the same order of magnitude

$$C\frac{\rho u_\infty^{\cancel{2}}}{C^2}\frac{\delta^2}{\mu \cancel{u_\infty}} = \theta(1) \qquad \Leftrightarrow \left(\frac{\delta}{C}\right)^2 = \theta\left(\frac{\mu}{\rho u_\infty C}\frac{1}{Re_C}\right) = \qquad \Rightarrow \delta = \theta\left(\frac{C}{\sqrt{Re_C}}\right) \ll C \tag{5.7}$$

which confirms the assumption $\delta \ll C$ when $Re_C \gg 1$. From now we will take $\delta = \frac{C}{\sqrt{Re_C}}$. To conclude, it remains the normal momentum equation

$$
\begin{aligned}
& \rho\left[\frac{u_\infty^2\delta}{C^2}\tilde{u}\frac{\partial\tilde{v}}{\partial\tilde{x}}+\frac{u_\infty^2\cancel{\delta^2}}{C^2}\frac{\tilde{v}}{\cancel{\delta}}\frac{\partial\tilde{v}}{\partial\tilde{y}}\right]=-\rho\frac{u_\infty^2}{\delta}\frac{\partial\tilde{p}}{\partial\tilde{y}}+\mu\left(\frac{u_\infty\delta}{C^3}\frac{\partial^2\tilde{v}}{\partial\tilde{x}^2}+\frac{u_\infty\delta}{C\delta^2}\frac{\partial^2\tilde{v}}{\partial\tilde{y}^2}\right)\\
\Leftrightarrow\quad & \rho\frac{u_\infty^2\delta}{C^2}\left[\tilde{u}\frac{\partial\tilde{v}}{\partial\tilde{x}}+\tilde{v}\frac{\partial\tilde{v}}{\partial\tilde{y}}\right]=-\rho\frac{u_\infty^2}{\delta}\frac{\partial\tilde{p}}{\partial\tilde{y}}+\mu\frac{u_\infty}{C\delta}\left(\cancel{\frac{\delta^2}{C^2}\frac{\partial^2\tilde{v}}{\partial\tilde{x}^2}}+\frac{\partial^2\tilde{v}}{\partial\tilde{y}^2}\right)\\
\Leftrightarrow\quad & \rho u_\infty^2\left[\tilde{u}\frac{\partial\tilde{v}}{\partial\tilde{x}}+\tilde{v}\frac{\partial\tilde{v}}{\partial\tilde{y}}\right]=-\rho u_\infty^2\left(\frac{C}{\delta}\right)^2\frac{\partial\tilde{p}}{\partial\tilde{y}}+\mu u_\infty\underbrace{\left(\frac{C}{\delta}\right)^2}_{Re_C=\rho\frac{Cu_\infty}{\mu}}\frac{1}{C}\frac{\partial^2\tilde{v}}{\partial\tilde{y}^2}\\
\Leftrightarrow\quad & \rho u_\infty^2\left[\tilde{u}\frac{\partial\tilde{v}}{\partial\tilde{x}}+\tilde{v}\frac{\partial\tilde{v}}{\partial\tilde{y}}\right]=-\rho u_\infty^2\left(\frac{C}{\delta}\right)^2\frac{\partial\tilde{p}}{\partial\tilde{y}}+\frac{\cancel{\mu}u_\infty}{\cancel{C}}\rho\frac{\cancel{C}u_\infty}{\cancel{\mu}}\frac{\partial^2\tilde{v}}{\partial\tilde{y}^2}\\
\Leftrightarrow\quad & \underbrace{\tilde{u}\frac{\partial\tilde{v}}{\partial\tilde{x}}+\tilde{v}\frac{\partial\tilde{v}}{\partial\tilde{y}}-\frac{\partial^2\tilde{v}}{\partial\tilde{y}^2}}_{\theta(1)}=-\left(\frac{C}{\delta}\right)^2\frac{\partial\tilde{p}}{\partial\tilde{y}}\qquad\Rightarrow\frac{\partial\tilde{p}}{\partial\tilde{y}}=\theta\left(\frac{\delta}{C}\right)^2
\end{aligned}
\tag{5.8}
$$

where we see that the pressure gradient accross the boundary layer normal to the wall cannot be of order of magnitude 1 but of that of $\left(\frac{\delta}{C}\right)^2$ which is **negligible**

$$
\tilde{p}(\tilde{x},\tilde{y})=\tilde{p}_e(\tilde{x})\qquad\Rightarrow p(x,y)=p_e(x)\tag{5.9}
$$

where $p_e(x)$ is the outer inviscid flow pressure distribution. The pressure variation inside the boundary layer beeing null, the pressure inside is equal to the outer pressure distribution computed on the wall. The pression is no longer an unknown. The final form of the equations in the boundary layer are

$$
\left.\begin{array}{l}
\frac{\partial\tilde{u}}{\partial\tilde{x}}+\frac{\partial\tilde{v}}{\partial\tilde{y}}=0\\
\tilde{u}\frac{\partial\tilde{u}}{\partial\tilde{x}}+\tilde{v}\frac{\partial\tilde{u}}{\partial\tilde{y}}=-\frac{d\tilde{p}_e}{d\tilde{x}}+\frac{\partial^2\tilde{u}^2}{\partial\tilde{y}^2}\\
\tilde{p}(\tilde{x},\tilde{y})=\tilde{p}_e(\tilde{x})
\end{array}\right|
\begin{array}{l}
\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0\\
\rho\left[u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right]=-\frac{dp_e}{dx}+\mu\frac{\partial^2 u}{\partial y^2}\\
p(x,y)=p_e(x)
\end{array}
\tag{5.10}
$$

So it means that if we replace the third equation in the second, we end up with a system of two equations and two unknowns. We also see that the geometry of the body does not appear at all in the equations since we assumed that $\delta\ll R$. The boundary layer is only sensitive the pressure distribution.

5.2 Zero-pressure gradient (flat plate) boundary layer

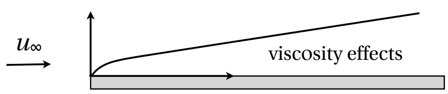

Figure 5.2

We want to solve the simpliest problem using this when there is no pressure gradient. This correspond to a uniform flow over a flat plate of 0 thickness. The coordinate system is the cartesian one and the outer flow is the uniform flow because the flat plate does not perturb the flow. So for the inviscid flow

$$
u=u_\infty\qquad v=0\qquad p=p_\infty\qquad\Rightarrow p_e(x)=p_\infty\Rightarrow\frac{dp_e}{dx}=0\tag{5.11}
$$

The simplified equations and the initial condition IC and boundary conditions BC are

$$\begin{aligned}&\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 && IC: u(0,y)=u_\infty\\&u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2} && BC: u(x,0)=v(x,0)=0 \text{ (non-slip)}\end{aligned}\tag{5.12}$$

Then we have also the matching boundary conditions that says to the edge of the boundary layer, the velocity should tend to its inviscid value

$$\lim_{y\to\infty} u(x,y)=u_e(x)=u^{inv}(x,0)=u_\infty \tag{5.13}$$

the far field limit of the boundary flow should be equal to the inner limit of the outer inviscid flow. So the solution at ∞ of the boundary flow should be equal to the solution at 0 of the inviscid flow. This is the matching condition for the tengential velocity and not the normal velocity. We are looking solutions $u(x,y)$ and $v(x,y)$. We'll try to represent the sollution to have an idea of how to find it, for $u(x,y)$ in a 3 dimentional coordinate system x,y,u. Following IC, when $x=0$, $u=u_\infty$ and $y=0, u=0$. We see that there is a jump, a discontinuity when $x=0=y$ (Figure 5.2).

Another way to represent is to plot y in function of u for various axis (positions x). We already know that $u=u_\infty$ in the far field and it start from 0 for all axis. If we look at our boundary profile, we know that δ is increasing with x, meaning that velocity will vary slower along y for increasing x leading to Figure 5.2. These last curves have the same shape, such that we can maybe contract them on a same curve.

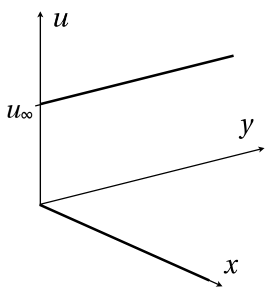

Figure 5.3

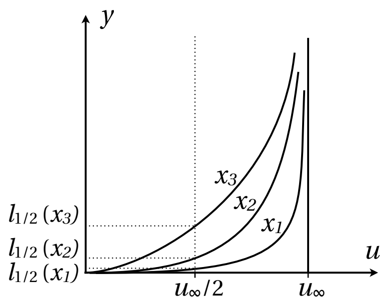

Figure 5.4

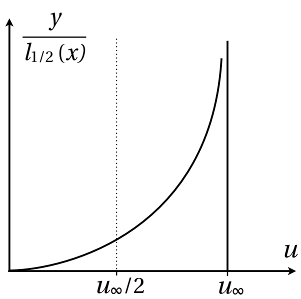

Figure 5.5

Let's take a value $u_\infty/2$ with the half velocity thickness $l_{1/2}(x)$. If we repllot u/u_∞ in function of $y/l_{1/2}(x)$, we know that at $u_\infty/2$ we will have 1 and 0 at 0. If we assume that all the curves have the same shape, they all pass from these 2 points as represented on Figure 5.2. This is called the **self similarity assumption** and we'll have to check it.

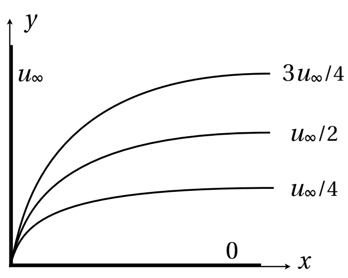

Figure 5.6

In order to verify the assumption, let's first represents Figure 5.2 as Figure 5.6, this is equivalent to plot the **contour lines**. The contour $u=u_\infty$ is vertical and $u=0$ horizontale, $u=u_\infty/2$ will have the equation $y=l_{1/2}(x)$ by definition. If Figure 5.2 is valid, we will have for example for $\frac{u}{u_\infty}=0.1$, $\frac{y}{l_{1/2}(x)}=c_{0.1}$. So the contour line $u=0.1u_\infty$ has as equation $y=c_{0.1}l_{1/2}(x)$. We see that in fact the self similarity assumption implies that the contour lines are stretched expression of the same function.

Coordinate transformation

Let's imagine that we make a change of variables

$$\xi = x \qquad \eta = \frac{y}{l_{1/2}(x)} = \frac{y}{l(x)}. \tag{5.14}$$

By the way, we use the value at 1/2 but we could take what we want, the only change is the constant. Now if we plot η in function of ξ, the self similarity assumption in the case of contour plot means that if all velocity profile have the same value of x so the same ξ. It means that the contour lines will be horizontal lines in the transformed variables. So if the transformation induces no variation along ξ

$$u(x,y) \to u(\cancel{\xi}, \eta) \qquad \Rightarrow \frac{u}{u_\infty} = g(\eta). \tag{5.15}$$

The solution only depends on η. This is an assumption and we have to check. For that we have to make the change of variables in the equations using the relation

$$\begin{gathered}\forall \varphi(x,y) = \hat{\varphi}(\xi(x,y), \eta(x,y)) \\ \frac{\partial \varphi}{\partial x} = \frac{\partial \hat{\varphi}}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial \hat{\varphi}}{\partial \eta}\frac{\partial \eta}{\partial x} \qquad and \qquad \frac{\partial \varphi}{\partial y} = \frac{\partial \hat{\varphi}}{\partial \xi}\cancel{\frac{\partial \xi}{\partial y}} + \frac{\partial \hat{\varphi}}{\partial \eta}\frac{\partial \eta}{\partial y}.\end{gathered} \tag{5.16}$$

We can now compute the derivative of the velocities

$$\begin{aligned}\frac{\partial u}{\partial x} &= \cancel{\frac{\partial (u_\infty g)}{\partial xi}}\frac{\partial xi}{\partial x} + \underbrace{\frac{\partial (u_\infty g)}{\partial \eta}}_{u_\infty g'} \underbrace{\frac{\partial \eta}{\partial x}}_{-\frac{y}{l^2(x)}\frac{dl}{dx}} = -u_\infty g'(\eta)\frac{y}{l^2(x)}\frac{dl(x)}{dx} = -u_\infty g'(\eta)\eta\frac{1}{l(\xi)}\frac{dl(\xi)}{d\xi} \\ \frac{\partial u}{\partial y} &= \frac{\partial u_\infty g}{\partial \eta}\frac{1}{l(\xi)} = u_\infty \frac{g'}{l(\xi)} \qquad and \qquad \frac{\partial^2 u}{\partial y} = u_\infty \frac{g''}{l^2(\xi)}\end{aligned} \tag{5.17}$$

Continuity equation

Let's integrate and replace by what we expressed ($\zeta = z/l(\xi), z \equiv y, \zeta \equiv \eta$)

$$\begin{aligned}\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} = u_\infty g'(\eta)\frac{\eta}{l(\xi)}\frac{dl(\xi)}{d\xi} \qquad &\Leftrightarrow v(x,y) - v(x,0) = u_\infty \int_0^y g'(\zeta)\frac{\zeta}{l(\xi)}\frac{dl(\xi)}{d\xi}\, dz \\ &\Leftrightarrow v(x,y) = u_\infty \frac{dl(\xi)}{d\xi}\underbrace{\int_0^\eta g'(\zeta)\zeta\, d\zeta}_{F(\eta)}\end{aligned} \tag{5.18}$$

where if we integrate by part we find $F(\eta) = \eta g - \int g\, d\zeta = \eta f' - f(\eta)$. This implies that (5.17) becomes

$$\frac{\partial u}{\partial x} = -u_\infty f''\eta\frac{1}{l(\xi)}\frac{dl}{d\xi} \qquad \frac{\partial u}{\partial y} = u_\infty \frac{f''}{l} \qquad \frac{\partial^2 u}{\partial y} = u_\infty \frac{f'''}{l^2}. \tag{5.19}$$

A first conclusion is that the same similarity of the tengential velocity profile implies a same similarity of the normal velocity profile in the form

$$\boxed{v = u_\infty \frac{dl}{d\xi}(\eta f' - f).} \tag{5.20}$$

Momentum equation

We know that $u = u_\infty g = u_\infty f'$, so the continuity equation becomes

$$
\begin{aligned}
&\underbrace{u_\infty f' \left(-u_\infty f'' \eta \frac{1}{l(\xi)} \frac{dl}{d\xi} \right)} + u_\infty \frac{dl}{d\xi} (\underbrace{\eta f'} - f) u_\infty \frac{f''}{l} = \nu u_\infty \frac{f'''}{l^2} \\
&\Leftrightarrow - u_\infty^2 \frac{dl}{l d\xi} f f''' = \frac{\nu u_\infty f''}{l^2} \qquad \Leftrightarrow - u_\infty^{\cancel{2}} \frac{dl}{\cancel{l} d\xi} f f'' \frac{l^{\cancel{2}}}{\nu \cancel{u_\infty}} = f''' \\
&\Leftrightarrow - \frac{u_\infty l}{\nu} \frac{dl}{d\xi} f f'' = f'''
\end{aligned}
\tag{5.21}
$$

At this stage, we can already conclude something about the validity of the self similarity assumption. Indeed, l is function of ξ, f a function of θ, if we bring all f to the right member, we have an equality between a function of ξ and a function of η. The only way for these to be equal, and so for the assumption to hold, is for the expression to be a **constant**

$$
\boxed{\frac{u_\infty l}{\nu} \frac{dl}{d\xi} = cst \qquad f''' + f f'' = 0} \tag{5.22}
$$

The l is chosen arbitrary so we can choose the constant as wish. We take $cst = 1$. Notice that we can write the last equation in terms of Re as

$$
Re_l \frac{dl \frac{u_\infty}{\nu}}{d\xi \frac{u_\infty}{\nu}} = Re_l \frac{dRe_l}{dRe_\xi} = 1 \qquad \Leftrightarrow Re_l^2 = 2 Re_\xi + \cancel{Re_{l_0}^2}. \tag{5.23}
$$

We can easely solve (5.22)

$$
\frac{u_\infty}{\nu} \frac{\frac{dl^2}{2}}{d\xi} = 1 \qquad \Leftrightarrow l^2 = \frac{2\nu}{u_\infty} \xi + \cancel{l_0^2} \qquad \Leftrightarrow l = \frac{\sqrt{2}x}{\sqrt{Re_x}} \tag{5.24}
$$

The caracteristic length scale l_0 appearing in the equations is the one when $\xi = x = 0$ at the leading edge where $l = 0$. The condition for the self similarity assumption to hold is that the caracteristic length scale is of this form. We have checked the compatibility with the governing equations, we now have to check the IC and BC.

Compatibility with IC/BC

We have for the initial condition that

$$
IC: \quad u(0, y) = u \inf \qquad \Leftrightarrow \frac{u(0, y)}{u_\infty} = g \left(\eta = \frac{y}{l(0)} \right) = 1 \tag{5.25}
$$

If $l(0)$ was a bounded number, since y can take all values, η has an infinite set of value which is impossible. In order to get one value, $l(0) = 0$ or $l(0) = \infty$. The only possible value is $l(0) = l_0 = 0$ which is matching with our result before and we get the condition

$$
\lim_{\eta \to \infty} g(\eta) = 1. \tag{5.26}
$$

Now for the boundary condition we have

$$
BC: \quad
\begin{aligned}
&u(x, 0) = 0 \qquad \Rightarrow \frac{u(x, 0)}{u_\infty} = g \left(\eta = \frac{0}{l(x)} \right) = 0 \qquad \Rightarrow f'(0) = 0 \\
&v(x, 0) = 0 \qquad \Rightarrow f(0) = 0 \quad \text{using (5.20)}
\end{aligned}
\tag{5.27}
$$

It stays only the matching condition that says

$$\lim_{y\to\infty} u(x,y) = u_\infty \qquad \Leftrightarrow \lim_{y\to\infty} \frac{u(x,y)}{u_\infty} = \lim_{\eta\to\infty} g\left(\eta = \frac{y}{l(x)}\right) = 1 \tag{5.28}$$

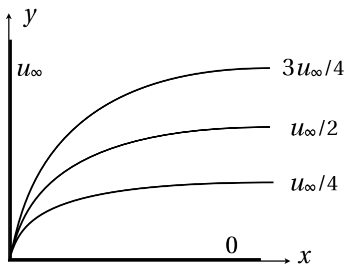

Figure 5.7

We fall on the same condition as the IC. This is fortunate because (5.22) is a third order equation so it needs only 3 conditions. It remains to find f by solving this equations with these conditions and that's what Blasius did. It looks simple but there isn't analytical solutions, he used expansion in series. The solutions are shown at the top on Figure 5.7 and at the bottom we find the experimental data giving the velocity profile for various value of x. We see that the data nicely collapse with the same curve. But why we determined the boundary layer? For friction! So let's exploite the results.

Exploitation of the results

Let's remind the expression for friction

$$\tau_w = \mu\left(\frac{\partial u}{\partial y}\right)_w = \frac{\mu}{l} f''(0)u_\infty = \sqrt{\frac{u_\infty}{2\nu x}}\mu f''(0)u_\infty \tag{5.29}$$

In fluid mecanics we love dimensionless numbers so we use the **friction coefficient**

$$C_f = \frac{\tau_w}{\rho u_\infty^2/2} = \frac{2u_\infty}{\rho u_\infty^2}\sqrt{\frac{u_\infty}{2\nu x}}\mu f''(0) = \sqrt{\frac{2\nu}{u_\infty x}} f''(0) = \frac{\sqrt{2}f''(0)}{\sqrt{Re_x}} = \frac{2f''(0)}{Re_l} \tag{5.30}$$

where the last equivalence comes from (5.23). Looking at the table, we find the last result

$$C_f = \frac{0.664}{\sqrt{Re_x}} \tag{5.31}$$

What's the physical interpretation of l. We said it was some percentage velocity thickness. When $\eta = 1$, $f'(\eta) = 0.46$, so l is the 46% velocity $\Rightarrow u = 0.46u_\infty$. Now we want to construct more physically based boundary layer thicknesses.

Other characteristic thicknesses

Conventional thickness There is one called the **conventional thickness** δ or $\delta_{.99}$, beeing the thickness where velocity reaches 99% of the outer velocity. So

$$\eta_\delta = \frac{\delta_{.99}}{l(x)} \text{ is such that } f'(\eta_\delta) = 0.99. \tag{5.32}$$

Using the tables we find that it corresponds to

$$\eta_\delta = 3.5 \qquad \Rightarrow \delta_{.99} = 3.5l(x) = 3.5\sqrt{\frac{2\nu x}{u_\infty}} \approx 5\sqrt{\frac{\nu x}{u_\infty}} \qquad \Leftrightarrow \frac{\delta_{.99}}{x} = \frac{5}{\sqrt{Re_x}} \tag{5.33}$$
$$\Rightarrow Re_\delta = 5Re_x$$

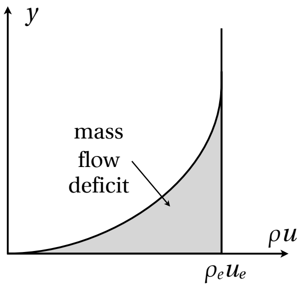

Figure 5.8

This is necessary because if we take the table and try to identify $\delta_{.99}$, the angle is not accurate. This is no more physical than the 46% thickness.

Mass flow defect thickness Something more physical is found when we replot ρu in function of y and compare it to the inviscid vertical profile $u = u_\infty$. When we integrate the mass flow near the wall, there is a mass flow defecit and a thickness based on which are

$$\begin{gathered}\text{Mass flow deficit} = \rho \int_0^\infty (u_\infty - u)\, dy \\ \text{Mass flow defect thickness :} \\ \delta^* = \frac{\rho \int_0^\infty (u_\infty - u) dy}{\rho u_\infty} = \int_0^\infty \left(1 - \frac{u}{u_\infty}\right) dy\end{gathered} \tag{5.34}$$

The value for the flate plate/zero pressure gradient is

$$\delta^* = \int_0^\infty (1 - f') l \, dy = l \underbrace{\int_0^\infty (1 - f')\, dy}_{\eta^*} \qquad \Rightarrow Re_{\delta^*} = Re_l \eta^* = \sqrt{2Re_x}\eta^*. \tag{5.35}$$

Let's now give a physical meaning to that mass flow deficit. Let's consider the infinitely thick flat plate and a streamline in the outer region flow which is deviated because of the presence of the boundary layer $y(x) \neq y_0$. Mass conservation tells us that

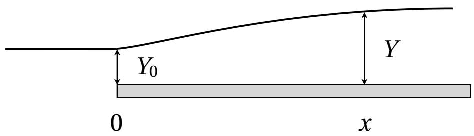

Figure 5.9

$$\begin{gathered}\rho \int_0^{y_0} u_\infty \, dy = \rho \int_0^y u(x,y)\, dy \qquad \Leftrightarrow \rho \int_0^y u_\infty \, dy - \rho \int_{y_0}^y u_\infty \, dy = \rho \int_0^y u(x,y) \infty \, dy \\ \Leftrightarrow \rho \int_0^y (u_\infty - u)\, dy = \rho \int_{y_0}^y u_\infty \, dy \qquad \Leftrightarrow y - y_0 = \int_0^y \left(1 - \frac{u}{u_\infty}\right) dy = \delta^*\end{gathered} \tag{5.36}$$

We see that the deviation of the streamline is exactly δ^* called the **displacement thickness**. One last thing to draw attention, we saw that $f' \to 1 \Rightarrow f = \eta + C$. This constant is

$$C = \lim_{\eta \to \infty} (f - \eta) = \lim_{\eta \to \infty} \int_0^\eta (f' - 1)\, d\eta = \int_0^\infty (f' - 1)\, d\eta = -\eta^*. \tag{5.37}$$

We can see this on the table graph where if we extrapolate to the x axis we find η^*. We have imposed the matching condition to the tengential velocity profile. What's the asymptotic value of the normal velocity? It is the limiting value of $F = \eta f' - f$

$$\lim_{y \to \infty} v = u_\infty \frac{dl}{dx} \eta^* \neq 0. \tag{5.38}$$

There we have normal velocity mismatch. This is due to the consideration we made at the beginning saying that streamlines were straight but now we conclude that they are not straight. We do not take into account the perturbation of the outer inviscid flow induced by the presence of the viscous layer.

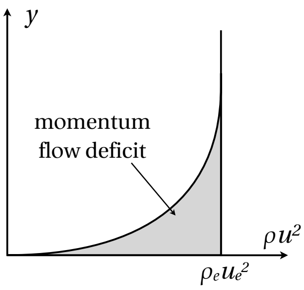

Figure 5.10

Momentum flow defect thickness Similarly to the mass flow defect, we can define the momentum flow defect as beeing

$$\text{MoFD} = \int_0^\infty \rho \left(u_\infty^2 - u^2\right) \, dy \tag{5.39}$$

If we draw attention to to the math definition of the previous δ^*, we notice that it is the hight in the plot of the rectangle that have the same area as the one formed by the curve. We can define a momentum induced by the mass flow by multiplying its expression by u_∞ and by defining the **extra momentum flow defect** and the corresponding **momentum flow defect thickness**

$$\begin{aligned} \text{XMoFD} &= \int_0^\infty \rho \left(u_\infty^2 - u^2\right) \, dy - u_\infty \int_0^\infty \rho \left(u_\infty - u\right) \, dy = \int_0^\infty \rho u \left(u_\infty - u\right) \, dy \\ \text{MoFDT} &= \frac{\text{XMoFD}}{\rho u_\infty^2} = \theta = \int_0^\infty \frac{u}{u_\infty} \left(1 - \frac{u}{u_\infty}\right) \, dy \end{aligned} \tag{5.40}$$

For the case of the flat plate, we know $\frac{u}{u_\infty} = f'$, so

$$\theta = l \underbrace{\int_0^\infty f'(1-f') \, d\eta}_{\theta^*} = l\theta^* = \frac{\sqrt{2}x}{\sqrt{Re_x}}\theta^*. \tag{5.41}$$

To find the value of θ^*, we can integrate by part we find

$$\begin{aligned} \theta^* &= \int_0^\infty (1-f') \underbrace{f' \, d\eta}_{df} = \underbrace{\left[f(1-f')\right]_0^\infty}_{=0} + \int_0^\infty f f'' \, d\eta \\ (5.22) \Rightarrow &= \int_0^\infty -f''' \, dy = f''(0) = 0.664. \end{aligned} \tag{5.42}$$

This conclues this section but keep in mind that we have a normal velocity mismatch. However, in (5.20) appears $\frac{fl}{dx}$ which is equal by (5.22) to

$$\frac{dl}{dx} = \frac{1}{Re_l} \qquad = 0 \; for \; Re_l \to \infty \tag{5.43}$$

So we see that the mismatch disappear for Re number going to ∞. This means that classical boundary theory is valid only in the case of infinite Re number, for a finite Re there is a finite mismatch.

5.3 Other pressure gradient

The equations and initial conditions are the same except for the pressure term

$$\begin{aligned} &\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \\ &u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dp_e}{dx} + \nu\frac{\partial^2 u}{\partial y^2} \end{aligned} \tag{5.44}$$

Because of the variation of the tangential pressure, there is a variation of the tangential velocity. Indeed, if we take the momentum equation for the inviscid flow we have

$$u\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{dp_e}{dx} \tag{5.45}$$

which means that the outer velocity is not constant if the outer pressure is not constant. In other words, for a position x_1 we will have a u_{e_1} different of the velocity u_{e_2} of a position x_2. We also see that a positive pressure gradient corresponds to a decelarating flow due to the minus sign and vice-versa. We can now wonder if there is also a self-similar solution.

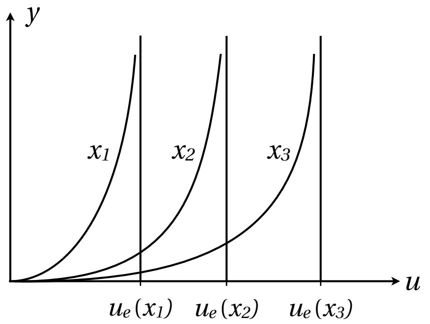

Figure 5.11

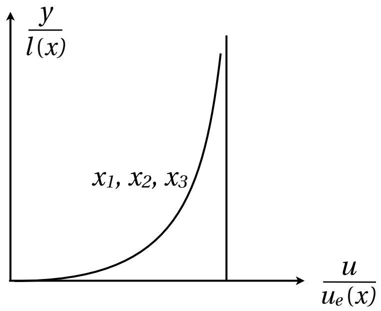

Figure 5.12

If we replot y in funtion of u, we will have a different plot compared to the flat plate because of the changing value of u_{e_x} as shown on Figure 5.3. So to make these velocity profiles collapse in a single one we need to scale the velocity and y and not only y as previously. So if we plot $\frac{y}{l(x)}$ in function of $\frac{u}{u_e(x)}$, we must have a single velocity profile with an asymptote at 1, whatever the value of x as shown on Figure 5.3. Again we don't know we will have to check. There is a difference between the two plot axis, $l(x)$ is unknown but $u_e(x)$ is known by the calculation on the outer flow. For the same coordinate transformation (5.14)

$$\frac{u}{u_e(x)} = g(\cancel{\xi}, \eta) \qquad \Leftrightarrow u = u_e(x) \tag{5.46}$$

That's the **self-similarity assumption**. (5.17) is therefore valid to the condition of replacing u_∞ by $u_e(x)$

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{du_e}{dx} g(\eta) + u_e \underbrace{\frac{\partial g}{\partial \eta}}_{u_e(x) g'} \underbrace{\frac{\partial \eta}{\partial x}}_{-\frac{y}{l^2(x)}\frac{dl}{dx}} = \frac{du_e}{dx} g(\eta) - u_e(x) g'(\eta) \eta \frac{1}{l(\xi)} \frac{dl(\xi)}{d\xi} \\ \frac{\partial u}{\partial y} &= u_e(x) \frac{\partial g}{\partial \eta} \frac{1}{l(\xi)} = u_e(x) \frac{g'}{l(\xi)} \qquad and \qquad \frac{\partial^2 u}{\partial y} = u_e(x) \frac{g''}{l^2(\xi)} \end{aligned} \tag{5.47}$$

Continuity equation

The process is exactly the same as last time except that we will have the additional term due to $\frac{\partial u}{\partial x}$

$$\begin{aligned} \frac{\partial v}{\partial y} &= -\frac{\partial u}{\partial x} = -\frac{du_e}{dx} g(\eta) + u_e g' \frac{\eta}{l} \frac{dl}{d\xi} \\ v(x,y) &= -\frac{du_e}{dx} l \int_0^\eta \underbrace{g(\zeta)\, d\zeta}_{f} + u_e \frac{dl}{d\xi} \underbrace{\int_0^\eta \zeta g'(\zeta)\, d\zeta}_{F = \eta f' - f} \\ v &= -l \frac{du_e}{dx} f + u_e \frac{dl}{dx} (\eta f' - f) \end{aligned} \tag{5.48}$$

which is the same expression with an additional term.

Momentum equation

Knowing that $u = u_e f'$, the momentum equation becomes

$$u_e f'\left[\frac{du_e}{dx}f' \cancel{-u_e f''\frac{\eta}{l}\frac{dl}{dx}}\right] + \left[-l\frac{du_e}{dx}f + u_e\frac{dl}{dx}(\cancel{\eta f'} - f)\right]u_e\frac{f''}{l} = u_e\frac{du_e}{dx} + \nu u_e\frac{f'''}{l^2}$$

$$\Leftrightarrow \qquad \underbrace{\frac{\nu u_e}{l^2}}_{F_1(x)} f''' + \underbrace{u_e\frac{du_e}{dx}}_{F_2(x)}\left[1 - f'^2 + ff''\right] + \underbrace{\frac{u_e^2}{l}\frac{dl}{dx}}_{F_3(x)} ff'' = 0 \tag{5.49}$$

$$\Leftrightarrow \qquad \frac{F_1(x)}{F_3(x)}f''' + \frac{F_2(x)}{F_3(x)}(1 - f'^2 + ff'') + ff'' = 0$$

For this last equation to admit a solution, it must be reductible to a simple function of η. This implies that **the self similarity conditions** are

$$\frac{F_1(x)}{F_3(x)} = cst = \frac{F_3(x)}{F_1(x)} \qquad and \qquad \frac{F_2(x)}{F_3(x)} = cst = \frac{F_3(x)}{F_2(x)} \tag{5.50}$$

Let's write completely $\frac{F_2(x)}{F_3(x)}$ and $\frac{F_3(x)}{F_1(x)}$

$$\frac{F_2(x)}{F_3(x)} = \frac{\frac{du_e}{u_e dx}}{\frac{dl}{l dx}} = \frac{\frac{d\ln u_e}{dx}}{\frac{d\ln l}{dx}} = k \qquad \Leftrightarrow \ln u_e = k\ln l + c \qquad \Leftrightarrow u_e = Kl^k$$
$$\frac{F_3(x)}{F_1(x)} = u_e\frac{dl}{dx}\frac{l}{\nu} = \frac{K}{\nu}l^{k+1}\frac{dl}{dx} = \frac{K}{\nu(k+2)}\frac{dl^{k+2}}{dx} = cst \qquad \Leftrightarrow \begin{cases} l \propto x^{\frac{1}{k+2}} \\ u_e \propto x^{\frac{k}{k+2}} \end{cases} \tag{5.51}$$

The conclusion is that we will have a solution if the velocity distribution is a power of x

$$\boxed{u_e = ax^m \qquad and \qquad \frac{du_e}{dx} = amx^{m-1} = m\frac{u_e}{x}} \tag{5.52}$$

Now we can look at $\frac{F_2(x)}{F_1(x)}$ to see what happens

$$\frac{F_2(x)}{F_1(x)} = \frac{u_e\frac{du_e}{dx}}{\frac{\nu u_e}{l^2}} = cst \qquad \Leftrightarrow l^2 \propto \frac{1}{\frac{du_e}{dx}} \propto \frac{x}{u_e} \qquad \Leftrightarrow l \propto \sqrt{\frac{bx}{u_e}} \tag{5.53}$$

where b is an arbitrary constant to specify that l is defined up to an arbitrary constant. Let's now compute the coefficients F_1, F_2, F_3

$$\frac{\nu u_e}{l^2} = \frac{\nu u_e^2}{bx} \qquad u_e\frac{du_e}{dx} = m\frac{u_e^2}{x} \qquad \frac{u_e^2}{l}\frac{dl}{dx} = u_e^2\frac{d\ln l}{dx} = \frac{u^2}{2}\left(\frac{1}{x} - \frac{m}{x}\right) \tag{5.54}$$

After simplifying $\frac{u_e^2}{x}$ and replacing in (5.49), we have

$$\frac{\nu}{b}f''' + m\left[1 - f'^2 + ff''\right] + \frac{1}{2}(1-m)ff'' = 0 = f''' + ff''\underbrace{\frac{1+m}{2}\frac{b}{\nu}} - f'^2 + \frac{mb}{\nu}(1 - f'^2) \tag{5.55}$$

b being an arbitrary constant, we will choose it such that we obtain the equation in the case of the flat plate with $\frac{1+m}{2}\frac{b}{\nu} = 1$ which is the

Falkner-skan equation

$$b = \frac{2\nu}{1+m} \qquad \Rightarrow f''' + ff'' + \underbrace{\frac{2m}{1+m}}_{\beta}(1 - f'^2) = 0 \tag{5.56}$$

At this stage, let's notice that when:

- $m > 0$: accelerating flow $\Rightarrow \beta > 0$.
- $m < 0$: decelerating flow $\Rightarrow \beta < 0$.

where β can be seen as an acceleration parameter. We can see it easely by seeing that

$$\frac{du_e}{dx} = m\frac{u_e}{x} \qquad \Rightarrow l^2\frac{du_e}{dx} = m\frac{bx}{u_e}\frac{u_e}{x} = \frac{2m\nu}{1+m} = \beta\nu \qquad \Rightarrow \beta = \frac{l^2}{\nu}\frac{du_e}{dx} \tag{5.57}$$

where β is really related to the velocity gradient. So when the velocity profile is a power of x, the self similar solution exist and its shape depends on β.

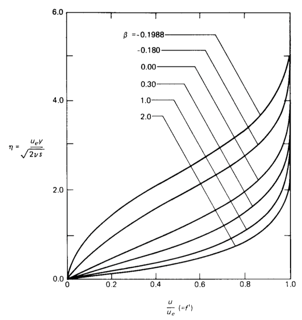

Figure 5.13

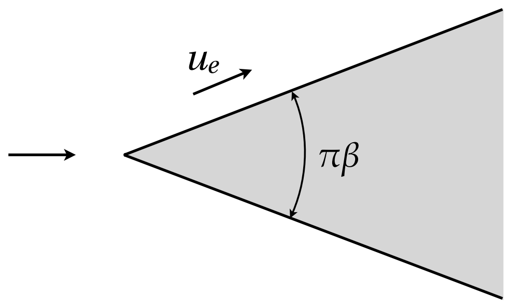

Figure 5.14

Figure 5.3 represents these profiles in function of β. $\beta = 0$ is the zero pressure gradient. For increasing β the boundary layer gets stiller and the profile fuller. When β decelerate, the boundary layer gets thickker and the velocity profile has an inflexion point. For $\beta > -0.1988$, the curve starts with a vertical tengent $\frac{du}{dy} = 0$, no friction. We could imagine this is good but in fact for these values there is separation of the boundary layer (reversed flow). Now we will answer to two questions:

- do this x law velocity profile have a physical meaning?
  It turns out that $u_e = ax^m$ is the velocity disctribution over a wedge where the opening angle is precisly equal to

$$\frac{2m}{1+m}\pi = \pi\beta. \tag{5.58}$$

  When $\beta > 0$ this is an easy practical case represented on Figure 5.3. In particular, $\beta = 1$ corresponds to $m = 1$ and an opening angle π. This is the flow near a stagnation point. Indeed, $m = 1$ means that

$$\frac{du_e}{dx} = a = cst \qquad \Rightarrow l^2 = \frac{\nu}{a} \neq 0 \tag{5.59}$$

  meaning that the boundary layer does not start at 0 thickness at a stagnation point.

- what can we do when this profile is not the case.
  Let's make a qualitative discussion on the influence of pressure on the velocity profile.

5.4 Effect of pressure gradient on velocity procile in a boundary layer - qualitative analysis

y	u	$\frac{\partial u}{\partial y}$	$\frac{\partial^2 u}{\partial y^2}$	$\frac{\partial^3 u}{\partial y^3}$	$\frac{\partial^4 u}{\partial y^4}$
0	0	$\frac{\tau_w}{\mu}$	$\frac{1}{\mu}\frac{\partial p}{\partial x}$	0	$\frac{1}{\nu\mu^2}\tau_w\frac{\partial \tau_w}{\partial x}$
δ	u_e	0	0	0	0

Table 5.1

We want to sketch the velocity profile as a function of pressure gradient. Figure 5.1 lists the value of the velocity and its derivative at the wall and the boundary layer edge. At δ, because of the bigger thickness scale in the outer region, the slope of $u = 0$. For $\frac{du}{dy}$, we use the shear stress at the wall

$$\tau_w = \mu \frac{\partial u}{\partial y}\bigg|_w \tag{5.60}$$

For $\frac{\partial^2 u}{\partial y^2}$, we use the momentum equation computed at the wall

$$\nu \left.\frac{\partial^2 u}{\partial y^2}\right|_w = \frac{1}{\rho}\frac{\partial p}{\partial x} \qquad \Rightarrow \left.\frac{\partial^2 u}{\partial y^2}\right|_w = \frac{1}{\mu}\frac{\partial p}{\partial x} \tag{5.61}$$

For the third order, we will differentiate the momentum equation (5.44) with respect to y. This gives

$$\begin{aligned}
& u\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial u}{\partial y}\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial u}{\partial y} + v\frac{\partial^2 u}{\partial y^2} = 0 + \nu\frac{\partial^3 u}{\partial y^3} \\
\Leftrightarrow \quad & u\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial u}{\partial y}\underbrace{\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)}_{=0 \text{ continuity}} + v\frac{\partial^2 u}{\partial y^2} = 0 + \nu\frac{\partial^3 u}{\partial y^3} \\
\Leftrightarrow \quad & u\frac{\partial^2 u}{\partial x \partial y} + v\frac{\partial^2 u}{\partial y^2} = \nu\frac{\partial^3 u}{\partial y^3} \qquad \Leftrightarrow \nu \left.\frac{\partial^3 u}{\partial y^3}\right|_w = 0
\end{aligned} \tag{5.62}$$

To have the fourth derivation we need to derive one more time

$$\begin{aligned}
& u\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial u}{\partial y}\underbrace{\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)}_{=0 \text{ continuity}} + v\frac{\partial^2 u}{\partial y^2} = 0 + \nu\frac{\partial^3 u}{\partial y^3} \\
\frac{\partial}{\partial y} \Rightarrow \quad & \frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x \partial y} + u\frac{\partial^3 u}{\partial x \partial y^2} + v\frac{\partial^3 u}{\partial y^3} + \frac{\partial v}{\partial y}\frac{\partial^2 u}{\partial y^2} = \nu\frac{\partial^4 u}{\partial y^4} \\
\Leftrightarrow \quad & \frac{\tau_w}{\mu}\frac{\partial}{\partial x}\left(\frac{\tau_w}{\mu}\right) + 0 + 0 + 0 = \nu \left.\frac{\partial^4 u}{\partial y^4}\right|_w \qquad \Rightarrow \left.\frac{\partial^4 u}{\partial y^4}\right|_w = \frac{1}{\nu\mu^2}\tau_w\frac{\partial \tau_w}{\partial x}
\end{aligned} \tag{5.63}$$

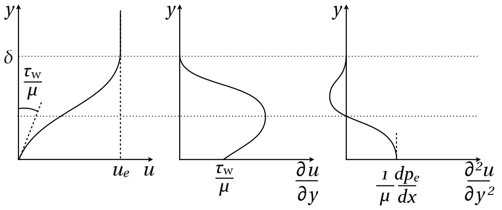

Figure 5.15

Now we will plot the derivative in function of y in the case of decelarating flow so a positive pressure gradient. Let's suppose that $\tau_w > 0$, as the second derivative is positive, it means that the first derivative increases with y. The third derivative beeing equal to 0, the second derivative begins with a slope 0. The first derivative has to reach the value 0, it means that we need a maximum where the derivative is equal to 0 (second derivative vanishes there), meaning that **in the velocity profile we have a inflexion point where we reach a maximum before continuing to increase**. We founded that before but now we conclude that it's a general case for $\beta < 0$. We also see that the second derivative curvature becomes negative, meaning that

$$\frac{\partial^4 u}{\partial y^4} < 0 \qquad \Rightarrow \frac{\partial \tau_w}{\partial x} < 0 \tag{5.64}$$

So we have **a tendancy for decreasing shear stress**. This is confirming what we found in the power law. The presence of the inflexion point makes the boundary layer unstable for disturbancies, it makes it switched to turbulencies more easely (promote transition to turbulence, increased friction). The decresing shear stress can be seen as good but in fact, when shear stress vanishes, separation appears (promote separation). So the conclusion is that we have a lot of bad phenomena for increasing pressure, this is why we call that the **adverse pressure gradient** when positif.

5.5 Approximate solution method for boundary layers (integral method)

Podcast 27