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SUMMARY

Structural Analysis & Finite Elements MECA-H421

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Appel à contribution

Synthèse Open Source

Ce document est grandement inspiré de l'excellent cours donné par Pyl Lincy et Peter Berke à l'EPB (École Polytechnique de Bruxelles), faculté de l'ULB (Université Libre de Bruxelles). Il est écrit par les auteurs susnommés avec l'aide de tous les autres étudiants et votre aide est la bienvenue ! En effet, il y a toujours moyen de l'améliorer surtout que si le cours change, la synthèse doit être changée en conséquence. On peut retrouver le code source à l'adresse suivante

`https://github.com/nenglebert/Syntheses`

Pour contribuer à cette synthèse, il vous suffira de créer un compte sur *Github.com*. De légères modifications (petites coquilles, orthographe, ...) peuvent directement être faites sur le site ! Vous avez vu une petite faute ? Si oui, la corriger de cette façon ne prendra que quelques secondes, une bonne raison de le faire !

Pour de plus longues modifications, il est intéressant de disposer des fichiers : il vous faudra pour cela installer LaTeX, mais aussi *git*. Si cela pose problème, nous sommes évidemment ouverts à des contributeurs envoyant leur changement par mail ou n'importe quel autre moyen.

Le lien donné ci-dessus contient aussi un `README` contenant de plus amples informations, vous êtes invités à le lire si vous voulez faire avancer ce projet !

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Merci !

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Models

2.1 Successive modeling phases

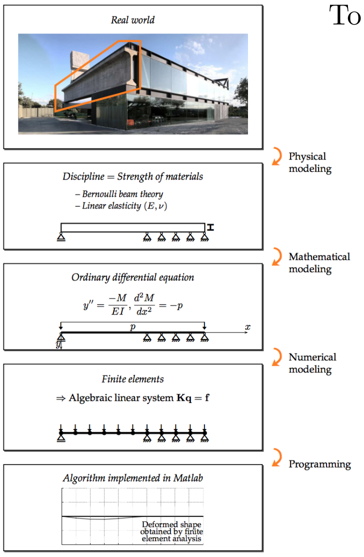

Figure 2.1

To implement a model on the computer we need:

- a **physical model:** step of defining the disciplines involved and (e.g. fluid dynamics) and the hypothesis regarding the material law (e.g. elasticity); In our example we can see a building where there is a huge beam. We want to model this and first this is a volume. We will represent it in 2D and the first question is "do I have shear stress". Yes there is a wall below leading to the small triangles on the figure, assuming that there is no displacement in y axis. Then the second question is "do I care of elastic behavior of concrete or is linear elasticity enough?" $\rightarrow$ assumption.

- a **mathematical model:** translation of the physical principles into mathematical language; In the example we have the relation with y'' and the one concerning the load, which is the roof part. We are assuming here that the weight of the roof is equally distributed.

- a **numerical model:** implementing an algorithm able to solve the previous point equations; What we do is in fact cutting our element in several elements, separated by nodes. The distributed forces will then be applied on that nodes. **THE** equation for finite element is: $Kq = f$ where K is a 6x6 matrix and represents **stiffness**, q **displacement** and f **force** are 6x1 vectors.

- a **computer model:** implementation of an in-house code or a commercial software product, based on the previous point.
  In this course we will be using a displacement based finite element model, the only unknown is the displacement, then we can find the stresses.

Be aware that some steps of the process introduce errors. Indeed, the choice of the physical model, then the mathematical model (choice), the discretization (we solve for the nodes and not the whole model) and the computer-based model (inversion of matrix thousand and thousand times) are not perfect.

Chapter 3

Solid mechanics

3.1 Continuum mechanics

3.1.1 Statics

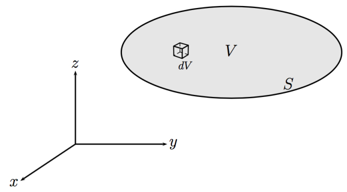

Figure 3.1

We will establish the governing equations in **continuum** mechanics. Consider a volume V delimited by a surface S, defined in a right-handed Cartesian coordinate. Continuum means, despite the microscopic description, that the material is assumed to behave as a continuum whole body. We can define the physical quantity in every point (x,y,z) of V by means of continuous function:

$$\rho = \lim_{dV \to 0} \frac{dm}{dV}. \tag{3.1}$$

In addition, we assume the **differentiability** that allows to write these equations in function of infinitesimal quantities. We also assume that the material is **homogeneous** and **isotropic** (same mechanical properties in all directions). Birth and propagation of cracks are causes of loss of continuity. In this case the continuity approach is not valid anymore. In numerical methods, it requires extensions as X-FEM and others.

There are two types of external forces:

- **Body forces b:** acting throughout the volume V. This depending on position, the resultant:

$$\mathbf{f}^v = \int_V \mathbf{b}(x,y,z)\rho(x,y,z)\,dV. \tag{3.2}$$

  In statics, gravity loads are the main body forces:

$$\mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ -g \end{bmatrix}, \tag{3.3}$$

  z assumed to be the vertical axis oriented upwards.

- **Contact forces t:** present at the contact between two points or surfaces. In practice it is either external forces on S or reactions at attachment points.

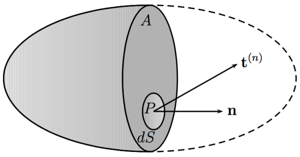

Figure 3.2

To introduce the notion of **Cauchy stresses**, let's cut a section A in the a volume V. The arbitrary surface dS on A is characterized by a resultant force $d\mathbf{f}$ and a resultant moment $d\mathbf{m}$. The **Cauchy stress vector** is defined as:

$$\mathbf{t}^{(n)} = \lim_{dS \to 0} \frac{d\mathbf{f}}{dS}. \tag{3.4}$$

For the rest of the course we will assume $d\mathbf{m}/dS = 0$. Remark that $\mathbf{t}^{(n)}$ is associated to a certain normal, if we make another cut A', we will have another normal $\boldsymbol{n'}$ and a different stress vector. We only have that $\mathbf{t}^{(n)} = -\mathbf{t}^{(-n)}$ (action-reaction).

Since each direction is associated to a stress vector, we define the second-order **stress tensor** $\bar{\bar{\boldsymbol{\tau}}}$. For this, we make the stress vectors defines for the three coordinate planes related to the unit normals $\boldsymbol{e}^{(1)}, \boldsymbol{e}^{(2)}, \boldsymbol{e}^{(3)}$ (see Figure 3.3). From this tensor, we can find any stress vector by projecting the tensor on the normal $\boldsymbol{n}$ associated to the cutting plane:

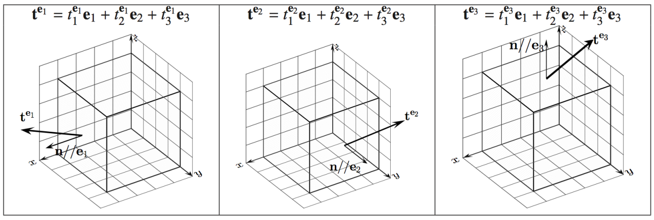

Figure 3.3

$$\bar{\bar{\boldsymbol{\tau}}}.\boldsymbol{n} = \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}. \tag{3.5}$$

As reminder of the notation, $\boldsymbol{\tau}_{xx} = \boldsymbol{\sigma}_x$ (normal stress, $0 <\rightarrow$ tension, $< 0 \rightarrow$ compression) points the normal component in x direction, xy (shear stress) will points to the vector of the plane $\perp x$ oriented to y and xz is the same but oriented to z. Stresses are measured in $N/m^2 =$ Pa.

We have to make the difference between 0-order, 1-order and 2-order tensors. The first means that the same scalar value is associated to each direction of the 3D space (does not depend on the orientation). For a 1-order tensor $\boldsymbol{v}$, assuming a given orientation $\boldsymbol{d}$, a scalar value is associated to $\boldsymbol{d}$ by $v_d = \boldsymbol{d}.\boldsymbol{v}$. This scalar change in function of the considered direction. And for the last, we have a different vector for any direction.

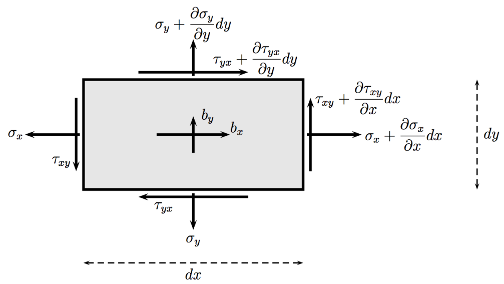

Figure 3.4

It is finally time to derive the equilibrium equations. Instead of compuing this for every point of a body, we will use an infinitesimal element $dx\,dy$ in 2D. By isolating this element, all forces (body and surface) acting on it should be balanced. The balance on the x-axis gives:

$$\begin{aligned} & b_x\,dx\,dy + \left(\frac{D\sigma_x}{\partial x}\,dx\right)dy + \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y}\,dy\right)dx - \sigma_x\,dy - \tau_{yx}\,dx = 0 \\ \Leftrightarrow \quad & b_x\,dx\,dy + \frac{\partial \sigma_x}{\partial x}dx\,dy + \frac{\partial \tau_{yx}}{\partial y}dx\,dy = 0 \\ \Leftrightarrow \quad & b_x + \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} = 0 \end{aligned} \tag{3.6}$$

By using indicial notation, we can generalize this approach in 3D and get a set of three equilibrium equations in translation:

$$b_i + \tau_{ji,j} = 0 \tag{3.7}$$

where i is a free or unrepeated index and j a summed or dummy index. Here are some rules:

- δ_{ij} is the **Kronecker delta** and $= 1$ if $i = j$, $= 0$ otherwise;
- ϵ_{ijk} is the permutation symbol which is $= 1$ if ijk makes a positive permutation, $= -1$ if negative permutation and $= 0$ otherwise (see syllabus if don't remember);
- $u_i v_i$ is a scalar product of $\boldsymbol{u}$ and $\boldsymbol{v}$;
- $\epsilon_{ijk} u_j v_k$ is a cross product;
- $\epsilon_{ijk} \partial_j u_k$ is the curl of $\boldsymbol{u}$ ($\nabla \times \boldsymbol{u}$);
- Gauss theorem in indicial notation:

$$\int_V u_{i,i}\, dV = \oint u_i n_i\, dS. \tag{3.8}$$

Now we have also to verify the rotation equilibrium. If the reference point is the origin of the axes, we denote x_i the current position, then the rotation equilibrium for an arbitrary volume $V' \in V$ delimited by S' is:

$$\int_{V'} \epsilon_{ijk} x_j b_k\, dV' + \oint_{S'} \epsilon_{ijk} x_j t_k^{(n)}\, dS' = 0 \tag{3.9}$$

where we applied the definition of the moment position $\times$ force. We know that $t_k^{(n)} = \tau_q k n_q$. By using this and the Gauss theorem we obtain:

$$\begin{aligned} & \int_V \epsilon_{ijk} \left[x_j b_k + (x_j \tau_{qk}, q) \right] dV' = \int_V \epsilon_{ijk} \left[x_j b_k + x_j \tau_{qk,q} + x_{j,q} \tau_{qk} \right] dV' \\ = \quad & \int_V \epsilon_{ijk} \left[x_j \underbrace{(b_k + \tau_{qk,q})}_{=0} + x_{j,q} \tau_{qk} \right] dV' = \int_V \epsilon_{ijk} x_{j,q} \tau_{qk}\, dV' = 0. \end{aligned} \tag{3.10}$$

Remark that $x_{j,q} = \delta_{jq}$, and since $V' \in V$ is completely arbitrary the integral must vanish:

$$\epsilon_{ijk} \tau_{jk} = 0. \tag{3.11}$$

We can conclude that **in the absence of concentrated body moments, the stress tensor is symmetric.** We revise our equation to

$$\boxed{\tau_k^{(n)} = \tau_{kq} n_q \quad ; \quad b_i + \tau_{ij,j} = 0 \quad ; \quad \tau_{ij} = \tau_{ji} \qquad (3.12)}$$

3.1.2 Kinematics

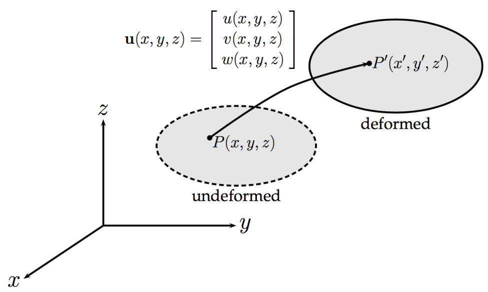

Figure 3.5

Continuum mechanics is also concerned by the way the volume is deformed through displacement and strains. Consider the volume V which is deformed in V' by applying a **displacement** on each point of V. In much applications the displacement can be assumed to be much smaller than the dimensions of the volume, leading to the **infinitesimal strain theory**, also called **small displacement-gradient theory**.

This simplifies our live because we assume the deformed volume to remain as the initial one and we can perform the integrals interchangeably on the initial or deformed configuration. This is valid for stiff materials like steel. Other flexible materials are the scope of non linear mechanics. The **linear strain tensor** is derived from the displacement field:

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j+u_{j,i}}) \qquad \bar{\bar{\epsilon}} = \begin{bmatrix} \epsilon_x & \frac{\gamma_{xy}}{2} & \frac{\gamma_{xz}}{2} \\ \frac{\gamma_{yx}}{2} & \epsilon_y & \frac{\gamma_{yz}}{2} \\ \frac{\gamma_{zx}}{2} & \frac{\gamma_{zy}}{2} & \epsilon_z \end{bmatrix} \tag{3.13}$$

where the ϵ_i are the axial strain and the γ_{ij} are the shear strain. **Don't forget the importance of small displacements!**

3.2 Linear elasticity

3.2.1 Material law

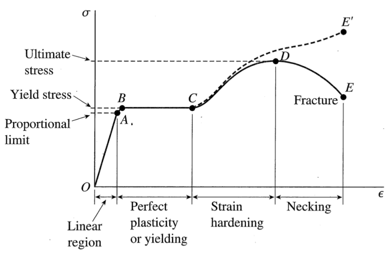

Figure 3.6

The understanding of the mechanical response requires a link between stresses and strains. This is done by the **material law**. It is find experimentally, using a tensile test machine that loads the material in tension at a constant speed until fracture. The **nominal axial stress** and the **average axial strain** are obtained respectively by dividing the force and the displacement by the initial surface and length of the sample:

$$\sigma = \frac{P}{A_0} \qquad \epsilon = \frac{\delta}{L_0}. \tag{3.14}$$

On Figure 3.6, the O state corresponds to no strain no stress, then a straight line to A. The slope of OA is called the **modulus of elasticity** or the **Young's modulus**, noted E $[N/m^2]$. After A, the relation is no longer linear. In AB the strain increases more rapidly than the strain, until a plateau BC where large strains are obtain without increase of load, **perfect plasticity**. The constant stress at this stage is the **yield stress**.

After this, the material **strain harden**, it resists to further deformation. The maximum stress is the **ultimate stress** on D. Then, the section A is shrunk and the bar is necking. The load decreases until the failure. By using the "true" cross-section area A_{true}, we can draw a true stress-strain curve CE'. We will focus on **linear elastic materials**. Elasticity points that σ is a unique function of δ and that the material recovers initial state when unloaded. Linearity points to the proportionality.

Additionally to the axial deformation, in prismatic bar is accompanied by a **lateral contraction** from the very beginning of the load. In the elastic domain, this is the **Poisson effect**:

$$\epsilon_x = \frac{\sigma_x}{E}, \qquad \epsilon_y = \epsilon_z = -\frac{\nu\sigma_x}{E} \tag{3.15}$$

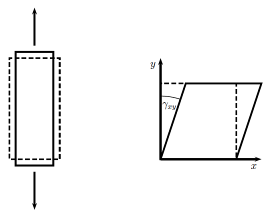

Figure 3.7

where $0 \leq \nu \leq 0.5$ is the Poisson coefficient. Let's remark that similarly to the Hooke's law, a shear version can be:

$$\tau = G\gamma \qquad G = \frac{E}{2(1+\nu)} \tag{3.16}$$

where the shear strain γ_{xy} can be interpreted as the angular deformation in xy. In 3D, as τ_{ij} and ϵ_{ij} are second order tensors, we need a fourth-order tensor C_{ijkl} such that $\tau_{ij} = C_{ijkl}\epsilon_{kl}$. This is simplified for **isentropic elastic materials**:

$$\tau_{ij} = \lambda\epsilon_{kk} + 2\mu\epsilon_{ij} \tag{3.17}$$

where λ, μ are the **Lamé constants**, defined as:

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}, \qquad \mu = \frac{E}{2(1+\nu)} = G. \tag{3.18}$$

Finally, if we replace we get

Stress-strain relationship in isotreopic elasticity

$$\epsilon_{ij} = \frac{1}{E}[(1+\nu)\tau_{ij} - \nu\delta_{ij}\tau_{kk}]. \tag{3.19}$$

In general material have also a non-linear behavior, but as in engineering we design the material to remain below the linear limit, we can make the approx.

3.2.2 Strain energy

When a load is applied, the external work of the force is converted into strain energy. Indeed, the application of σ_x induces an extension $\epsilon_x dx$, physically the energy for a volume is:

$$dW = \frac{1}{2}\sigma_x\epsilon_x\,dx\,dy\,dz. \tag{3.20}$$

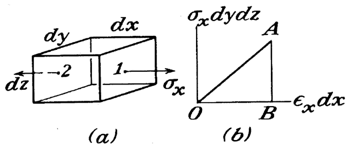

Figure 3.8

In the general case, the conservation of energy for linear elastic materials, the work cannot depend on the order of magnitude of the load. In other word, we have to multiply stress and strain tensors component by component:

$$\begin{aligned} dW = W_V\,dx\,dy\,dz &= \frac{1}{2}\tau_{ij}\epsilon_{ij}\,dx\,dy\,dz. \\ &= \frac{1}{2}(\sigma_x\epsilon_x + \sigma_y\epsilon_y + \sigma_z\epsilon_z + \tau_{xy}\epsilon_{xy} + \tau_{xz}\epsilon_{xz} + \tau_{yz}\epsilon_{yz} + \tau_{yx}\epsilon_{yx} + \tau_{zx}\epsilon_{zx} + \tau_{zy}\epsilon_{zy})\,dx\,dy\,dz \\ &= \frac{1}{2}(\sigma_x\epsilon_x + \sigma_y\epsilon_y + \sigma_z\epsilon_z + \tau_{xy}\gamma_{xy} + \tau_{xz}\gamma_{xz} + \tau_{yz}\gamma_{yz})\,dx\,dy\,dz \end{aligned} \tag{3.21}$$

where W_V is the **strain energy density**: $W_V = \int_{\epsilon_{ij}} \tau_{ij}\,d\epsilon_{ij}$.
We see the utility of defining $\epsilon_{xy} = \frac{1}{2}\gamma_{xy}$ in the equation. Using the Hooke's law:

$$\begin{aligned} W_V &= \frac{1}{2E}(\sigma_x^2 + \sigma_y^2 + \sigma_z^2) - \frac{\nu}{E}(\sigma_x\sigma_y + \sigma_x\sigma_z + \sigma_y\sigma_z) + \frac{1}{2G}(\tau_{xy}^2 + \tau_{xz}^2 + \tau_{yz}^2) \\ &= \frac{1}{2}\lambda(\epsilon_x^2 + \epsilon_y^2 + \epsilon_z^2) + G(\epsilon_x^2 + \epsilon_y^2 + \epsilon_z^2) + \frac{1}{2}G(\gamma_{xy}^2 + \gamma_{xz}^2 + \gamma_{yz}^2). \end{aligned} \tag{3.22}$$

We see that W_V is always **positive**. Let's define the dual $W_V^* = \int_{\tau_{ij}} \epsilon_{ij}\,d\tau_{ij}$, such that:

$$W_V + W_V^* = \tau_{ij}\epsilon_{ij}. \tag{3.23}$$

The strain energy W and the complementary are obtained by $\int_V W_V\,dV$, with $W_V = W_V^* = \frac{1}{2}\tau_{ij}\epsilon_{ij}$.

Chapter 4

Formulations in linear elasticity

4.1 Strong formulation

The equilibrium equations omitting concentrated body moments and the material law in linear elasticity for isotropic and homogeneous materials are:

$$b_i + \tau_{ij,j} = 0, \qquad \epsilon_{ij} = \frac{1}{E}[(1+\nu)\tau_{ij} - \nu\delta_{ij}\tau_{kk}]. \tag{4.1}$$

Since there is a linear relation between stresses and strains, strain tensor ϵ_{ij} derives itself from the displacement field, we can rewrite the equilibrium equations in function of the displacements. This leads to the **displacement-based finite element method**, the unknowns are the displacements. All the set of equations are encompassed under the **strong formulation** therminology, because they require to be satisfied **locally**, at each point of the domain V.

4.1.1 Boundary conditions

The main classes of boundary conditions are:

- **essential boundary conditions** (Dirichlet): the displacement is prescribed on a portion of the external surface S, called S_u;

- **natural boundary conditions** (Neumann): the contact force is imposed on another portion of the external surface S, called S_t.

4.1.2 Mathematical properties of the governing equations

First, the **superposition principle** is respected by stresses, strains and displacements because linearity of the static and kinematic equations in linear elasticity (assuming small displacements/strains). Indeed if τ_{ij}^A and τ_{ij}^B are stress tensors associated to load cases A and B, $\tau_{ij}^A + \tau_{ij}^B$ is the solution for the load case A + B.

Secondly, for given surface and body forces, the uniqueness of the solution for the governing equations is guaranteed.

Proof.

As a counterargument, let's assume that there exist two solution $u_i^{(1)}$ and $u_i^{(2)}$ and their difference $u_i' = u_i^{(1)} - u_i^{(2)}$. Let's do the same for strains and stresses:

$$\epsilon'_{ij} = \epsilon^{(1)}_{ij} - \epsilon^{(2)}_{ij}, \qquad \tau'_{ij} = \tau^{(1)}_{ij} - \tau^{(2)}_{ij}. \tag{4.2}$$

Since the body forces are external and identical for the two solutions, we get from the equilibrium equations:

$$\epsilon^{(1)}_{ij,j} - \epsilon^{(2)}_{ij,j} = \epsilon'_{ij,j} = 0. \tag{4.3}$$

On the other hand, the strain energy of the difference is:

$$W' = \frac{1}{2}\int_V \epsilon'_{ij}\tau'_{ij}\,dV = \frac{1}{2}\int_V \frac{1}{2}(u'_{i,j} + u'_{j,i})\tau'_{ij}\,dV = \frac{1}{2}\int_V u'_{i,j}\tau'_{ij}\,dV, \tag{4.4}$$

by symmetry and as i, j are repeated. If we use integration by part and Gauss theorem, we get:

$$\begin{aligned} W' &= \frac{1}{2}\int_V (u'_i\tau'_{ij})_{,j}\,dV - \cancel{\frac{1}{2}\int_V u'_i\tau'_{ij,j}\,dV} \\ &= \frac{1}{2}\oint_S u'_i\tau'_{ij}n_j\,dV = \frac{1}{2}\oint_S u'_i\bar{\tau}'^{(n)}_i\,dV \end{aligned} \tag{4.5}$$

To guarantee the uniqueness of the solution, so to have $W' = 0$, the boundary conditions must satisfy:

1. the displacement field u_i is prescribed on the entire surface, leading directly to $u'_i = 0$
2. the contact forces $t_i^{(n)}$ are imposed over the entire surface while overall equilibrium is satisfied, making $t_i^{\prime(n)} = 0$
3. displacements u_i are prescribed on one part of the surface ($u_i = \bar{u}_i$ on S_u), while contact forces $\tau_i^{(n)}$ are imposed on the rest of the surface ($\tau_i^{(n)} = \bar{\tau}_i^{(n)}$ on S_t), so that either $u'_i = 0$ or $\tau_i^{(n)} = 0$. The $^-$symbol indicates that the values are prescribed.

This explains why it is not allowed to impose both the displacement and the force on the same point and the same direction. Example on Figure 4.1.

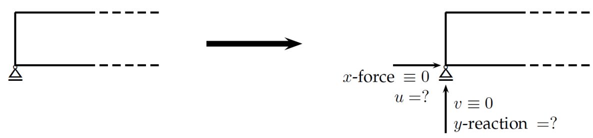

Figure 4.1