Université Libre de Bruxelles

Synthèse

Fluid mechanics and transport processes
MECA-H-300

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Appel à contribution

Synthèse OpenSource

Ce document est grandement inspiré de l'excellent cours donné par Allessandro Parente à l'EPB (École Polytechnique de Bruxelles), faculté de l'ULB (Université Libre de Bruxelles). Il est écrit par les auteurs susnommés avec l'aide de tous les autres étudiants et votre aide est la bienvenue ! En effet, il y a toujours moyen de l'améliorer surtout que si le cours change, la synthèse doit être changée en conséquence. On peut retrouver le code source à l'adresse suivante

`https://github.com/nenglebert/Syntheses`

Pour contribuer à cette synthèse, il vous suffira de créer un compte sur *Github.com*. De légères modifications (petites coquilles, orthographe, ...) peuvent directement être faites sur le site ! Vous avez vu une petite faute ? Si oui, la corriger de cette façon ne prendra que quelques secondes, une bonne raison de le faire !

Pour de plus longues modifications, il est intéressant de disposer des fichiers : il vous faudra pour cela installer LaTeX, mais aussi *git*. Si cela pose problème, nous sommes évidemment ouverts à des contributeurs envoyant leur changement par mail ou n'importe quel autre moyen.

Le lien donné ci-dessus contient aussi le `README` contient de plus amples informations, vous êtes invités à le lire si vous voulez faire avancer ce projet !

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Merci !

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Chapitre 1

Introduction

Before beginning the summary, I want to tell you that my English level isn't perfect. Please collaborate and correct the gramatically wrong sentences.

1.1 Reminder

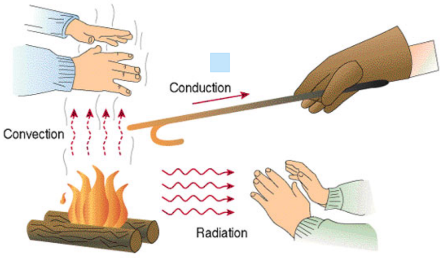
The governing equations in transport processes are the followings :

- Mass conservation :
$$\frac{\partial \rho}{\partial t} + \nabla(\rho v) = 0 \tag{1.1}$$
- Navier-Stokes :
$$\rho\left(\frac{Dv}{Dt} + v\nabla v\right) = -\nabla p + \mu\nabla^2 v \tag{1.2}$$
- Energy equation :
$$\frac{DT}{Dt} = \nabla(\alpha\nabla T) + \frac{\dot{Q}v}{\rho c} \tag{1.3}$$
- Species conservation
$$\frac{\partial \rho_A}{\partial t} + \nabla(\rho_A v_A) = r_A \tag{1.4}$$

Let's precise that there are many applications using these equations like in the aerospace and automotive industry, in safety and fire prevention or in buildings design.

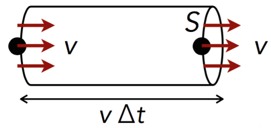
1.2 Convection and diffusion

1.2.1 Definitions



Here is a picture illustrating the principles of **convection**, **conduction** and **radiation**. Imagine that you have a fire and you put your hands above. You will feel a flow of heat transmitted by convection. If someone comes with a stick, there will be conduction in the material transmitting the energy from particles to particles. Finally, if the hands are next to the fire, there is no flow but you feel the heat. The energy is transmitted by radiation.

1.2.2 Convection



Convection is a transfer always associated to **bulk (ensemble) fluid motion**. We consider a fluid with **uniform** velocity and a cylinder of section S and lenght Δt. In that time interval, the fluid in the cylinder will have crossed the section S. We are now able to express the convective flux of momentum (quantité de mouvement), energy and mass knowing that the flux of a physical quantity is given by

$$flux_A = \frac{A}{S\Delta t} \tag{1.5}$$

Mass

We know that the mass is given by density × volume and that the volume of the cylindre is $Sv\Delta t$. Using the new expression of the mass and (1.5), we can find the **flux of mass**

$$M_c = \rho v S\Delta t \qquad and \qquad J_{M_c} = \frac{M_c}{S\Delta t} = \rho v \tag{1.6}$$

Momentum

Similarly to the calcul of the mass

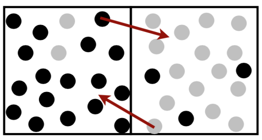
$$Q_c = mv = \rho v^2 S\Delta t \qquad and \qquad J_{Q_c} = \rho v^2 \tag{1.7}$$

Energy

The energy in the system is given by the specific heat energy of each particles [1]. If T_0 is the reference temperature,

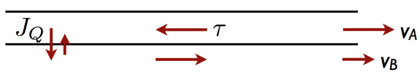
$$E_c = \rho c_v v(T - T_0)S\Delta t \qquad and \qquad J_{E_c} = \rho c_v v(T - T_0) \tag{1.8}$$

1.2.3 Diffusion



Diffusion is a transfer associated to the **particles random walk** and is due to the presence of a gradient of physical quantity (temperature for example). The picture on the left illustrates that, for an infinite time, the process will reequilibrate the gradient, difference between the two boxes.

1.3 Diffusive flux



If we consider two parallel fluid layers to bulk velocity $v_A > v_b$, the gradient of velocity will vanish [2] ($v_A = v_b$). This effect is due to the initial velocity gradient that cause the diffusion of faster particles towards the slower ones, transferring a momentum flux J_Q. In fact, the origin of the flux in the transversal direction to the flow is due to the **friction** between the two layers, parallel to the flow direction.

1. See *Chimie générale* for the expression
2. Disparaître

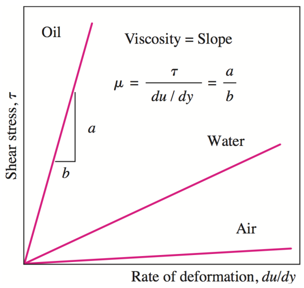
1.3.1 Shear stress

To characterize the friction between to layers, we introduce the **shear stress**[3] that is proportional to the gradient of velocity

$$\tau = \tau\left(\frac{dv}{dy}\right) \tag{1.9}$$

According to the Newton's law, for newtonian fluids like gases and liquids, the equation becomes

$$\tau = \mu\left(\frac{dv}{dy}\right) \tag{1.10}$$

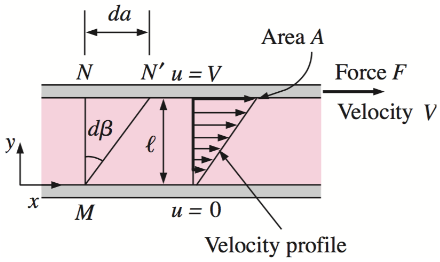


- μ is the **dynamic viscosity**, the **intrinsic resistance** of the fluid to motion $[kg/m.s]$
- dv/dy the gradient of velocity $[s^{-1}]$
- τ a force per unit surface $[N/m^2] = [Pa]$

If we see the shear stress to the **rate of deformation**[4] on a graph, we can see that the **viscosity** is the slope[5]. We can also see that viscosity of air is miner than water that's miner than oil.

1.3.2 Planar Couette flow

Let's consider a fluid layer between two very large plates sparated by a distance l. A constant parallel force F (drag force[6]) is applied to the upper plate and after the initial transients, it moves continuously to a constant velocity V. What are the consequences on the fluid ?



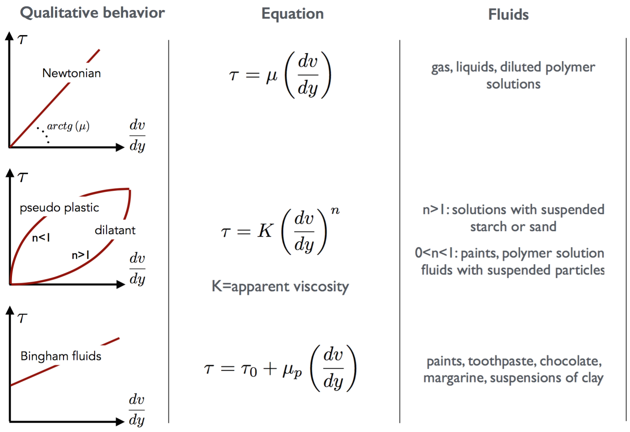
- **Empirical observations :**
  No slip conditions[7] at walls $\rightarrow u(0) = 0$ and $u(l) = V$.

- **The flow is organized in parallel layers :**
  We suppose that the flow is laminar (no turbulence), so the velocity profile is a sole function of y and is **linear** : $u(y) = V\frac{y}{l}$.

- **The drag force is imposed :**
  $\tau = \frac{F}{A} = \mu\frac{dv}{dy} = c$

3. Contrainte de cisaillement
4. Gradient of velocity
5. La pente
6. Force de traînée
7. Condition de non-glissement

1.3.3 Fluid rheology



Rheology is the study of the flow that establish the relation between the shear stress and the velocity gradient. Let's have a look to different fluids.
As first observation, we can see that newtonian fluids respect a linear relation for shear stress to the rate of deformation. It's not the case for others like second and third line of the table[8]. For bingham fluids, there is a critical shear stress to reach before the behavior becomes like newtonian fluids

Dynamic viscosity of liquids and gases

When we observe the table of dynamic viscosity for the water, the dynamic viscosity decreases with temperature increase for liquid water. It's not the case for the gases for wich dynamic viscosity increases with temperature. See section 1.5.3 below.

1.4 Constitutive relations

It's an important section because it shows that transport processes have a uniform approach. Let's introduce 3 new variables : **kinematic viscosity** ν, **thermal diffusivity** α **internal energy** u **mass diffusivity** D. Let's precise that $\nu, \alpha, D = [L^2T^{-1}]$

- **Momentum diffusion flux** (Newton's law)

$$J_{Q_d} = -\mu \frac{dv}{dy} \quad \xrightarrow{\nu=\frac{\mu}{\rho}} \quad J_{Q_d} = -\nu \frac{d(\rho v)}{dy} \tag{1.11}$$

- **Energy diffusive flux** (Fourrier's law)

$$J_{E_d} = -k \frac{dT}{dy} \quad \xrightarrow{\alpha=\frac{k}{\rho c_v} \ and \ u=c_v(T-T_0)} \quad J_{E_d} = -\alpha \frac{d(\rho u)}{dy} \tag{1.12}$$

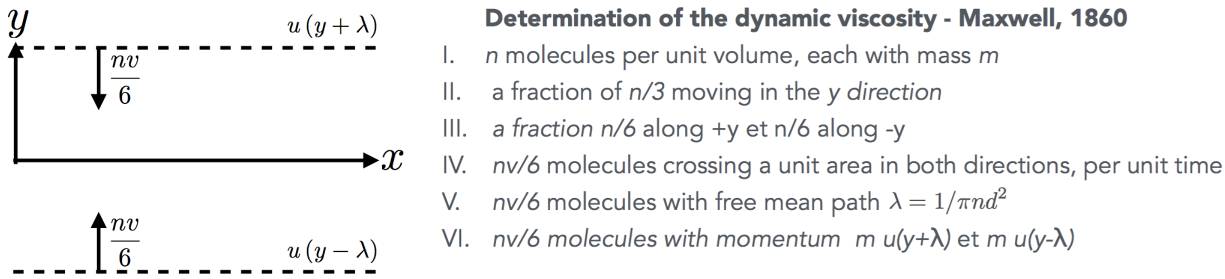
- **Mass diffusion flux** (Fick's law)

$$J_{M_d} = -D \frac{dc}{dy} \qquad (c = \text{concentration}) \tag{1.13}$$

8. Starch = amidon

1.5 Gas viscosity

1.5.1 Another expression of viscosity



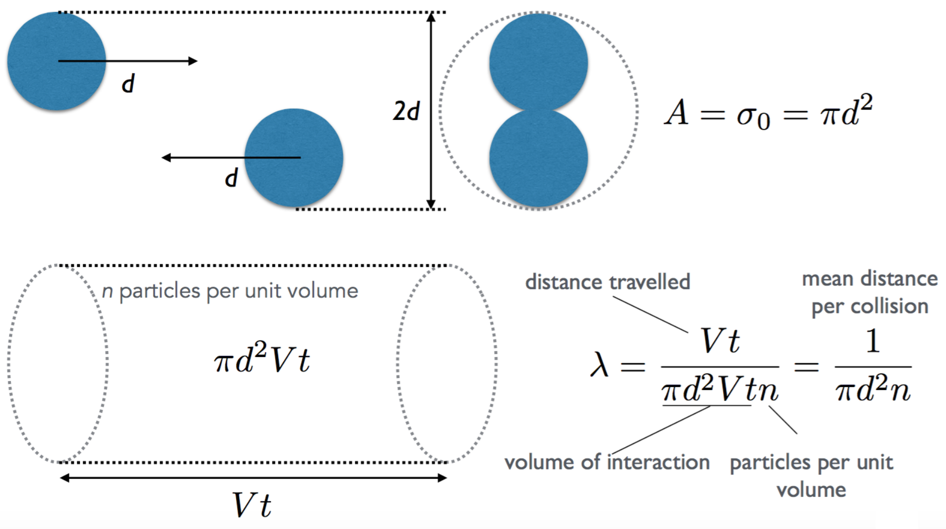
Let's consider a gas moving in the x direction with a velocity $u = u(y)$. The **kinetic theory** gives us the random velocity in y using $v = \sqrt{\frac{3k_bT}{m}}$. The development of Maxwell, above, allows us to calculate the momentum flux crossing the x axis. $\frac{nv}{6}$ gives us the flux of particles for the y direction.

$$J_{Q_d} = \left(\frac{nv}{6}\right) mu(y-\lambda) - \left(\frac{nv}{6}\right) mu(y+\lambda) \tag{1.14}$$

Here we use $u(y-\lambda)$ and $u(y+\lambda)$ because the last particles that can cross an axis on y are those who are situated in a distance of maximum λ from there.
The Tylor expansion of the flux $u(y+\lambda) = u(\lambda) + \lambda\frac{du}{dy} + \ldots$ and the constitutive equation (1.11) gives us the final expression of viscosity

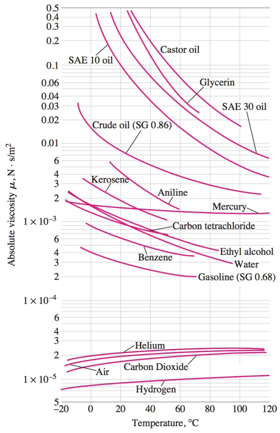
$$\mu = \frac{1}{3}nm\lambda v = \frac{1}{3}\rho\lambda v = \frac{1}{\pi\sqrt{3}}\frac{\sqrt{mk_bT}}{d^2} \tag{1.15}$$

1.5.2 Free mean path



The slide above summarise the way to calculate the free mean path that is the average distance travelled by a particle between 2 collisions. First of all, we have to define a section of interaction. It's given by a circle of radius d. We also have to define a volume within the particles move. It's the cylinder of section A and length Vt. The calculus of λ is given by the distance travelled (without collision) divided by the volume where you can have interactions and by the particles per unit volume.

1.5.3 Viscosity variation with temperature



We have said that gases viscosity incresed with temperature. It's due to the kinetic apprch that gived us the equation (1.15) where $\mu \propto \sqrt{T}$.
For gases, the intermolecular forces are negligible. The increase of temperature makes particles move randomly at higher velocities conducting to more collisions per unit volume per unit time and so more resistance to flow.
For liquids, the increase of temperature decreases the viscosity because particles have higher energies that help them to oppose to the large cohesive intermolecular forces more strongly. They can thus move more freely.

1.6 Viscosity measurements

I HAVE TO ASK TO THE TEACHER IF WE HAVE TO DO THESE 3 NEXT SECTIONS BECAUSE I D'ONT HAVE COMMENTS

1.7 Thermal conductivity

1.8 Mass diffusivity

1.9 Potential flow and boundary layer

1.9.1 Main aspects

- The behavior of a moving fluid element strongly depends on the **interactions** between the **fluid elements** and the **walls** for both internal and external flows.

- Far from the walls, the shear stress and the dissipative forces are negligible. The fluid behaves **ideally**, is **incompressible** and **non-dissipative**. It's called a **potential** flow.

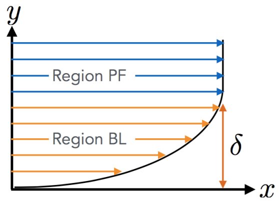
- Potential flow are fully described by **Newtonian mechanics**, using the **conservation laws** for mass and mechanical energy, without any **dissipation** of mechanical **energy into heat**.

- Close to the walls, the fluid velocity is **suddenly modified**, thus introducing **dissipative phenomena** that cannot be described using the potential flow theory.
- The fluid layer thickness [9] affected by the wall is named **boundary layer** [10] (as introduced by Prandtl, 1904).

- Within the boundary layer, the velocity gradients in the flow normal direction as well as the shear stress in the direction parallel to the flow become extremely important.

9. Epaisseur
10. Couche limite

1.9.2 Representation of the boundary layer



We have to delimit two regions. The one (PF) within we have the **potential flow** that conserve his kinetic and potential energy (no dissipation) and the other (BL) within we have the **boundary layer** of thickness δ, a velocity gradient with shear stress and dissipation of mechanical energy into heat. In the BL region appear two forces, a **drag** force that is due to friction, parallel to the flow and **lift** force that is a resistance perpendicular to the flow.
In order to characterize the flow, we introduce the **Reynolds number**

$$Re = \frac{\text{Convection}}{\text{Diffusion}} = \frac{J_{Q_c}}{J_{Q_d}} = \frac{\rho v^2}{\mu \frac{v}{L}} = \frac{vL}{\nu} \tag{1.16}$$

that compares the convective and diffusive effect of the fluid momentum and the **Peclet number**

$$P_{e_t} = \frac{J_{E_c}}{J_{E_d}} = \frac{\rho c_v v(T - T_0)}{k\frac{T}{L}} = \frac{vL}{\alpha} \qquad \textit{and} \qquad P_{e_m} = \frac{J_{M_c}}{J_{M_d}} = \frac{vL}{D} \tag{1.17}$$

that compares the convective and diffusive effect of the fluid energy and mass. Let's precise that if a uniform fluid flows towards a plate, we will have an **inviscid** flow region away from the plate and a **viscous** flow region next to the plate.

1.9.3 Flow classification and friction force

The classification is based on the Reynolds number. We have two types of fluid :

- **Re $\gg$ 1 $\rightarrow$ tubulent flow**, convection > diffusion.
  In that case, the fluid resistance (drag force) is independent of the viscosity and is proportional to the momentum flux

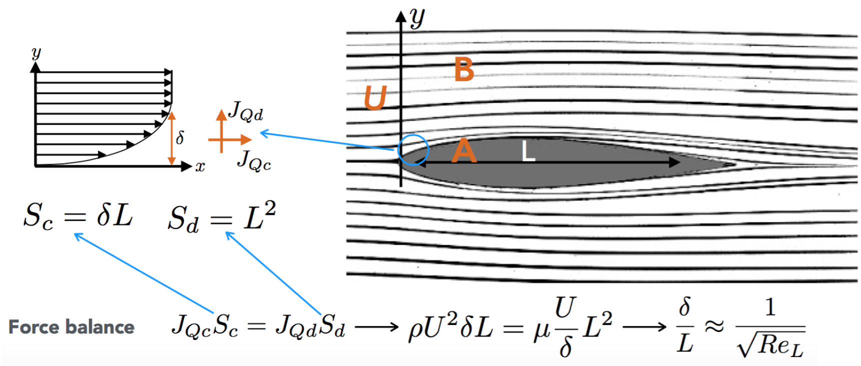
$$F \approx \rho v^2 S \tag{1.18}$$

- **Re $\ll$ 1 $\rightarrow$ laminar flow**, convection < diffusion.
  In that case, the fluid resistance is a function of the viscosity and is proportional to the diffusive momentum flux

$$F \approx \mu L v \tag{1.19}$$

1.9.4 Determination of the boundary layer thickness

The separation between the two regimes above is not possible. Consider a wind flow around a building. The Reynolds number will be high meanig the flow is turbulent. Indeed, the **approaching flow** can be classified as a potential flow, whereas **close to the walls** the dissipative effects will take place with the conversion of mechanical energy into heat.
So, far from the walls the dominant transport mechanism will be convection and close to the walls viscous transport. At a distance equal to the boundary layer thickness, **convection and diffusion will be comparable**. This allows us determining its **order of magnitude**.



We can qualitatively find δ as the distance from the wall at which the convective momentum flux (**parallel** to the flow) equals the diffusive one (**perpendicular** to the flow). For that, let's take a section within convection dominate S_c and another for diffusion S_d. When we egalize the two flux expressed respectively for convection and diffusion, we arrive to the magnitude of δ compared to L.

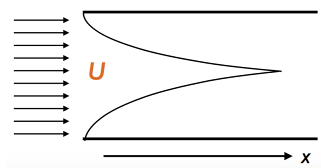
1.9.5 Features of the boundary layer and practical consequences

The relative dimension of the boundary layer is inversely proportional to $Re^{0.5}$ and the absolute dimension is proportional to $L^{0.5}$.

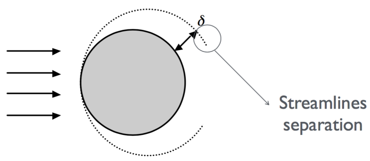
Consequences for internal flows

The boundary layer thickness increases with $x^{0.5}$ and far from the inlet [11], the boundary layer will extend to the full diameter pipe.

$$\frac{\delta}{x} \approx \frac{1}{\sqrt{Re_x}} \qquad \textit{and} \qquad \delta \approx \sqrt{\frac{xv}{U}} \tag{1.20}$$



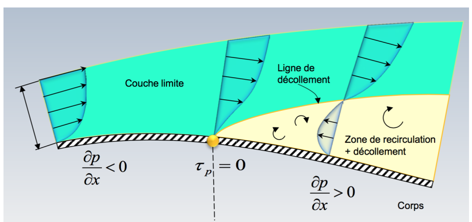
Consequences for external flows



The boundary layer cannot infinitely grow. It will be thinner in front the body and thicker behind it. Behind the body, the flow streamlines detach. If the boundary layer thickness is known, the force exerted by the flow can be directly evaluated, being the velocity gradient of the order $\frac{U}{\delta}$, which gives a shear stress of $\tau = \mu\frac{U}{\delta}$ on the walls. The drag force will be then higher in front of the body (where the boundary layer is thinner) and lower behind it (where the boundary layer is thicker). From symmetry considerations, the drag force only acts in the flow direction.
More commonly, a perpendicular component also exists called **lift** but here it doesn't appear.

$$F = \int_S \mu \frac{dv}{dy} \, dS \tag{1.21}$$

Here's an illustration of the flow separation. There is a **recirculation** region which provides a force opposed to the flow. In order to maximise the body speed, it is necessary to reduce the drag force due to the flow separation.



11. La prise - l'entrée

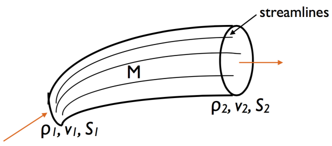
Chapitre 2

Macroscopic and microscopic balances

2.1 Conservation principles

The basic conservation principles in fluid mechanics are the conservation of **mass, energy and momentum.** These conservations are the basis of **continuity, Bernouilli and Navier-Stokes equations** and can be written in **integral** (finite volume of a mouving fluid) or **local** (balances in differential form) form.

2.1.1 Macroscopic mass balance and continuity equation



In a steady [1] flow, the mass balance simply states that the mass entering is equal to the one leaving a volume. But we have to consider the variation of mass for transient [2] problems. We define for that the **mass flow rate** (débit) that gives the mass per time and is easy to measure.

$$\dot{m} = \rho v S \qquad [\dot{m}] = \frac{[M]}{[T]} \tag{2.1}$$

General balance

Let's write what we explained, the mass variation is given by the mass entering at S_1 minus mass leaving at S_2

$$\frac{dM}{dt} = \left(\rho \int_S v dS\right)_1 - \left(\rho \int_S v dS\right)_2 \tag{2.2}$$

To simplify, we introduce the **average velocity** $\bar{v} = \frac{1}{S}\int_S v dS$ giving the final expression

$$\frac{dM}{dt} = (\rho \bar{v} S)_1 - (\rho \bar{v} S)_2 \tag{2.3}$$

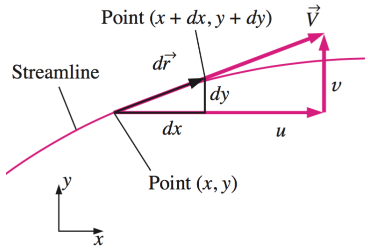
Steady balance

When the fluid is steady, the velocity and the surface are constant in both 2 sections conducting to a constant mass

$$\dot{m} = \rho \bar{v} S = c \tag{2.4}$$

1. Continu, constant.
2. Transitoire.

Streamlines



Let's first define what's a streamline. It's a curve that is everywhere tangent to the instantaneous local velocity vector and so an indicator of the fluid direction. Mathematically, if our velocity vector is

$$\vec{V} = u\vec{i} + v\vec{j} + w\vec{k} \tag{2.5}$$

we can take an infinetisimal arc length along a streamline

$$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k} \tag{2.6}$$

Due to the 2 similar triangles, we have the relations (2.7) that gives the 2D equation of a streamline.

$$\frac{dr}{V} = \frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \qquad \Rightarrow \qquad \left(\frac{dy}{dx}\right)_{\text{along a streamline}} = \frac{v}{u} \tag{2.7}$$

2.1.2 Macroscopic momentum balance

Unlike the mass, momentum can be created and destroyed due to applied forces. The difference between the moementum entering in the control volume at S_1 and leaving at S_2 is equal to the sum of the forces applied to the fluid element.
The momentum flow through a section per unit time is given by

$$\dot{Q} = \dot{m}v = \rho\bar{v}^2 S \tag{2.8}$$

and the forces acting on the fluid element (positive if exerted by the fluid) are :

- **pressure** that is positive at the entry (the fluid push to enter) and negative at the exit (the fluid is pushed).
- **forces on the walls** (normal and tangential). Negative because it's the reaction of the walls.
- **Gravity and other volume forces**. Positive because the forces are due to the presence of a fluid volume.

It allows us to write the equation, considering that the surface is a vector of module equal to the scalar surface and of same direction of the normal to the surface.

$$\sum F = p_1 S_1 - p_2 S_2 - F_w + F_g \tag{2.9}$$

Steady momentum balance

It's simply the use of equations (2.8) and (2.9)

$$\dot{Q}_2 - \dot{Q}_1 = \sum F \qquad \Leftrightarrow \qquad \dot{m}_2 v_2 - \dot{m}_1 v_1 = p_1 S_1 - p_2 S_2 - F_w + F_g \tag{2.10}$$

We can use equation (2.4) and have a compact final relation

$$\Delta_1^2(\rho\bar{v}S + p) = -F_w + F_g \tag{2.11}$$

General momentum balance (transient)

To consider the case within the velocity is not constant into the 2 sections, we have to remind the mechanical relation for the resultant and do an infinitesimal deomposition in volume V

$$\frac{dR}{dt} = \sum F \qquad \Leftrightarrow \qquad \begin{aligned} \frac{d}{dt}\int_V \rho v dV &= (\rho\bar{v}^2 + p)_1 S_1 - (\rho\bar{v}^2 + p)_2 S_2 - F_w + F_g \\ &= \Delta_1^2(\rho\bar{v}^2 + p)S - F_w + F_g \end{aligned} \tag{2.12}$$

You can find an example of exercice on slide 8 and 9 where we study the forces on an elbow [3]. Keep in mind that we often **consider a steady-state flow** to simplify $d/dt = 0$.

2.1.3 Local form of the conservation principles

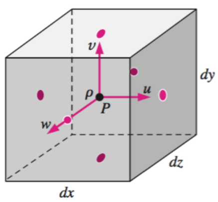
Macroscopic balances

They are used to have a general view an a volume, when we are interested in the **overall features** [4].

Microscopic balances

They consist on differantial and local equations and are valid on every point of the fluid to give the velocity, density or pressure.

2.2 Continuity equation



The continuity equation caracterise the net rate of change of mass in a control volume. We start from the fact that this rate is equal to the net rate at wich mass flows through the volume.

Démonstration. The variation of mass in function of time in an infinitesimal volume is given by

$$\partial M = \frac{\partial \rho}{\partial t}\partial x \partial y \partial z \tag{2.13}$$

If we do the difference between the enter and the exit of the flow in the volume following axis x, we have $(\rho v S)$

$$\rho u \partial y \partial z - \left(\rho u + \frac{\partial(\rho u)}{\partial x}\partial x\right)\partial y \partial z = -\frac{\partial(\rho u)}{\partial x}\partial x \partial y \partial z \tag{2.14}$$

and using the same way for the other axis

$$-\frac{\partial(\rho v)}{\partial y}\partial x \partial y \partial z \qquad and \qquad -\frac{\partial(\rho w)}{\partial z}\partial x \partial y \partial z \tag{2.15}$$

We take care of the contributions in all direction and simplify the infinitesimal volume in each term

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0 \qquad \Rightarrow \qquad \frac{\partial \rho}{\partial t} + \nabla(\rho v) = 0 \tag{2.16}$$

3. Coude
4. Caractéristiques générales

Where the final v is global and not specific to an axis. Let's remind that the material derivative is defined as $\frac{D\rho}{Dt} = \frac{\partial\rho}{\partial t} + v\nabla\rho$, we have

$$\frac{D\rho}{Dt} + \rho\nabla v = 0 \tag{2.17}$$

$\square$

2.3 Momentum equation

2.3.1 Cauchy equation

To solve this rapidly we use the Langrangian approach. We suppose that the control volume is moving with the fluid and contain a fixed mass. The momentum rate of change is is equal to the net force acting on the control volume (volumetric and superficial)

$$\frac{D}{Dt}\int_{V_m(t)} \rho v\, dV = \int_{V_m(t)} \rho g\, dV + \int_{S_m(t)} f\, dS \tag{2.18}$$

f is a function of the position r on the surface defined by the unit vector n perpendicular to the surface and pointing outward. f can be expressed as the scalar product between the tensor T and normal n

$$f(n, r) = nT(r) \tag{2.19}$$

Démonstration. Let's consider a tetrahedron on the axis x_1, x_2, x_3 and the forces exerted by the surrounding fluid. The force balance can be expressed

$$f(n)S_{ABC} = f(e_1)S_{AOB} + f(e_2)S_{AOC} + f(e_3)S_{BOC} = f(e_1)S_{ABC}ne_1 + f(e_2)S_{ABC}ne_2 + f(e_3)S_{ABC}ne_3 \tag{2.20}$$

if we do a factorisation and we define the tensor T as

$$T = e_1 f(e_1) + e_2 f(e_2) + e_3 f(e_3) \tag{2.21}$$

we find the equation (2.19). We can so write the equation (2.18) with this expression and use the **divergence theorem**

$$\int_S nT\, dS = \int_V \nabla T\, dV \tag{2.22}$$

we obtain the equation

$$\int_V \left[\frac{\partial(\rho v)}{\partial t} + \nabla(\rho v v) - \nabla T - \rho g\right] dV = 0 \tag{2.23}$$

Considering that it must be valid for any and every point of the volume, we get the **Cauchy equation** [5]

$$\rho\frac{Dv}{Dt} = \nabla T + \rho g \tag{2.24}$$

$\square$

5. We can get ρ out of the material derivative because we consider a constant ρ

2.3.2 Constitutive equations

Static conditions

Imagine that we have a static fluid. The only force f will be due to the pressure p

$$f = -pn \Leftrightarrow T = -pI = \begin{matrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{matrix} \tag{2.25}$$

In that case, the use of equation (2.24) give us the information

$$\rho g = \nabla p \tag{2.26}$$

Dynamic conditions

Let's now take care of the viscosity effects on a Newtonian fluid. In that case we have to add to the pressure the forces function of the velocity gradient

$$T = -pI + f(\nabla v) \tag{2.27}$$

where (∇v) has a symetrical (pure deformations) and an antisymetrical (pure rotation) part given by

$$S = \frac{1}{2}(\nabla v + \nabla v^T) \qquad \textit{and} \qquad A = \frac{1}{2}(\nabla v - \nabla v^T) \tag{2.28}$$

Final form of the stress tensor

Given the upper discussion, the final expression is

$$T = \left[-p + \lambda(\nabla v)\right] I + 2\mu S \tag{2.29}$$

where S is the symetric part of the gradient. Let's precise that for **incompressible flows**, we have

$$T = -pI + 2\mu S \tag{2.30}$$

Navier-Stokes equation

Théorème : Navier-Stokes equation
For incompressible, Newtonian and isothermal flows

$$\rho \frac{Dv}{Dt} = \rho \left[\frac{\partial v}{\partial t} + v\nabla v\right] = -\nabla p + \mu \nabla^2 v + \rho g \tag{2.31}$$

Where we used the Cauchy equation (2.24) and equation (2.30) for T.

2.4 Bernouilli equation

To access the Bernouilli equation, let's take the Navier-Stokes equation (2.31) and disregard the viscous term (with μ). The scalar multiplication of that equation provides the **kinetic energy balance**

$$\rho v \frac{Dv}{Dt} = \rho \frac{D}{Dt}\frac{v^2}{2} = -v\nabla(p + \rho\Phi) \qquad \textit{with} \qquad \Phi = g \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix} \tag{2.32}$$

For incompressible and steady fluids, $\frac{d\rho}{dt} = 0 = \frac{dp}{dt}$, so

$$\frac{D}{Dt}\left(\rho\frac{v^2}{2} + p + \rho g z\right) = 0 \tag{2.33}$$

leading us to the Bernouilli equation after integration

Théorème : Bernouilli equation
The sum of the kinetic, potential and flow energies of a fluid particle is **constant** along a streamline during **steady** flow when the **compressibility** and **frictional effects** are negligible.

$$\rho\frac{v^2}{2} + p + \rho g z = c = p_{tot} \tag{2.34}$$

- $\frac{v^2}{2}$ is the **dynamic pressure** : pressure rise when the fluid in motion is brought to a stop isentropically.
- p is the static pressure : thermodynamic pressure of the fluid.
- $\rho g z$ is the **hydrostatic pressure** : for the elevation effects.

Finally, we can say that the sum of these 3 type of pressure is the **total pressure** and that it's constant along a srteamline.