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Université Libre de Bruxelles

Synthèse

Fluid mechanics II
MECA-H-305

Auteur :
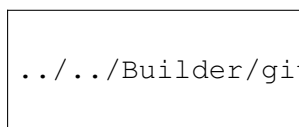Enes Ulusoy

Professeur :
Gérard Degrez

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Appel à contribution

Synthèse Open Source

`../../Builder/git.png` Ce document est grandement inspiré de l'excellent cours donné par Gérard Degrez à l'EPB (École Polytechnique de Bruxelles), faculté de l'ULB (Université Libre de Bruxelles). Il est écrit par les auteurs susnommés avec l'aide de tous les autres étudiants et votre aide est la bienvenue ! En effet, il y a toujours moyen de l'améliorer surtout que si le cours change, la synthèse doit être changée en conséquence. On peut retrouver le code source à l'adresse suivante

https://github.com/nenglebert/Syntheses

Pour contribuer à cette synthèse, il vous suffira de créer un compte sur *Github.com*. De légères modifications (petites coquilles, orthographe, ...) peuvent directement être faites sur le site ! Vous avez vu une petite faute ? Si oui, la corriger de cette façon ne prendra que quelques secondes, une bonne raison de le faire !

Pour de plus longues modifications, il est intéressant de disposer des fichiers : il vous faudra pour cela installer LaTeX, mais aussi *git*. Si cela pose problème, nous sommes évidemment ouverts à des contributeurs envoyant leur changement par mail ou n'importe quel autre moyen.

Le lien donné ci-dessus contient aussi le README contient de plus amples informations, vous êtes invités à le lire si vous voulez faire avancer ce projet !

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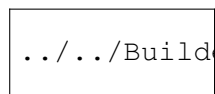
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Merci !

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Chapter 1

Generalities

1.1   Fundamental laws

Reminder

Let's first remind the 3 basic principles of *Fluid mechanics I* :

- **Mass conservation :** *The mass of a closed system remains constant in time.*
  This is much a definition of a closed system than a principle. We have to notice that related to Einstein law of relativity, $E = mc^2$, mass must vary with energy. But if we exclude nuclear reactions, our approximation is valid. Indeed, the square of light velocity has a greater impact on energy than the mass term. If the energy exchange is huge like in nuclear reaction, mass vary, but in smaller energies domain (combustion for example), the mass can be considered as constant.

- **Newton's law :** *the time rate of change of momentum of a closed system is equal to the sum of the forces applied on the system.*

- **First principle of thermodynamics :** *the time rate of change of the total energy of a closed system is equal to the sum of the power of the forces applied on the system and the thermal power provided to the system.*

Useful equations

Let's consider the integral on a moving volume of a function depending on time and position $f(\vec{x}, t)$. Imagine that Figure 1.1 represents the moving volume at initial time containing mass m. An infinitesimal part of that volume contains an infinitesimal mass $dm = \rho dV$, where ρ is mass density. We deduce the expression of the total mass at any time by that of the initial time
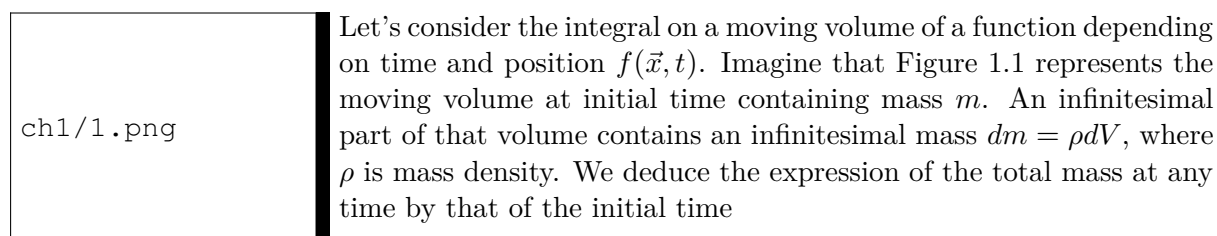
Figure 1.1

$$M(t_0) = \int_{V(t_0)} \rho(\vec{x}, t_0)\, dV \qquad \Rightarrow M(t) = \int_{V(t)} \rho(\vec{x}, t)\, dV \qquad (1.1)$$

By considering $\rho(\vec{x}, t)$ as $f(\vec{x}, t)$, the derivative of the integral is given by

The second equation that will be used in the developement is given by

1.1.1 Mass conservation equation

If $V(t)$ is the moving volume occupied by the closed system as time varies, then by definition of a closed system $\frac{dM(t)}{dt} = 0$. The corresponding equation using Reynolds transport theorem is

$$M(t) = \int_{V(t)} \rho\, dV \qquad \Rightarrow \frac{dM(t)}{dt} = \int_{V(t)} \frac{\partial \rho}{\partial t}\, dV + \oint_{S(t)=\partial V(t)} \rho\,\vec{b}\,\vec{n}\, dS = 0 \qquad (1.4)$$

Figure 1.2

We have to express that this volume is not traversed by material. There is no flux of fluid and the particles in the volume are always the same. By definition, the infinitesimal distance traveled by the surface and the fluid are

$$d\vec{x} = \vec{b}\, dt \qquad and \qquad d\vec{x}' = \vec{u}\, dt \qquad (1.5)$$

where $\vec{u}$ is the fluid velocity. Under wich condition do we know that the fluid has not traversed the boundary? We have to define the relative dispolacement $d\vec{x}' - d\vec{x}$ of the fluid in regard to the fluid. For a closed system

$$
\begin{aligned}
(d\vec{x}' - d\vec{x}) \cdot \vec{n} = 0 \quad &\Leftrightarrow \quad dt(\vec{u} - \vec{b}) \cdot \vec{n} = 0 \quad \Leftrightarrow \quad (\vec{u} - \vec{b}) \cdot \vec{n} = 0 \\
&\Rightarrow \quad \vec{b}\vec{n} = \vec{u}\vec{n}
\end{aligned}
\qquad (1.6)
$$

How to write this equation in a different way? Let's consider now a fixed open system composed of fluid particles in the fixed volume $V_0(t) = V(t_0)$. Similarly to the previous point, the mass variation in this fixed volume is expressed like

$$M_0(t) = \int_{V_0(t)} \rho\, dV \qquad \Rightarrow \int_{V_0(t)} \frac{\partial \rho}{\partial t}\, dV + \oint_{S_0(t)=\partial V_0(t)} \rho\,\vec{b}\,\vec{n}\, dS. \qquad (1.8)$$

The volume integral expresses the variable mass in the fixed volume and the surface integral is nul due to the nul surface velocity (since the volume is fixed). This relation enables us to write the

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Mass conservation equation for fixed open systems (integral form)

$$\frac{dM_0}{dt} + \underbrace{\oint_{S_0(t)=\partial V_0(t)} \rho \, \vec{u} \, \vec{n} \, dS}_{\text{mass flow out of the system}} = 0 \tag{1.9}$$

Let's finally consider an arbitrary open system containing fluid particles in a moving volume $V_*(t)$ such that $V_*(t_0) = V(t_0) = V_0$. Similarly we have using the Reynolds transport theorem

$$M_*(t) = \int_{V_*(t)} \rho \, dV \qquad \Rightarrow \frac{dM_*(t)}{dt} = \int_{V_*(t)} \frac{\partial \rho}{\partial t} \, dV + \oint_{S_*(t)=\partial V_*(t)} \rho \, \vec{b} \, \vec{n} \, dS \tag{1.10}$$

Using the definition of the volume at $t = t_0$, we can equalize the volume integral with that of (1.7) to find

---

Mass conservation equation for arbitrary open systems (integral form)

$$\frac{dM_*(t_0)}{dt} + \oint_{S(t_0)=\partial V(t_0)} \rho(\vec{u} - \vec{b}) \, \vec{n} \, dS = 0 \tag{1.11}$$

Let's now take (1.7) again and apply Gauss theorem

$$\int_{V(t)} \frac{\partial \rho}{\partial t} \, dV + \oint_{S(t)=\partial V(t)} \rho \, \underbrace{\vec{u} \, \vec{n}}_{\vec{a}} \, dS = 0 \qquad with \qquad \oint_{S(t)} \rho \, \underbrace{\vec{u} \, \vec{n}}_{\vec{a}} \, dS = \int_{V(t)} \nabla \rho \vec{u} \, dV$$

$$\Leftrightarrow \int_{V(t)} \left[\frac{\partial \rho}{\partial t} + \nabla \rho \vec{u} \right] dV = 0 \tag{1.12}$$

For this last equation to be true for all systems, the integrated term must be equal to zero

---

Mass conservation equation (differential form (1) - divergent form)

$$\frac{\partial \rho}{\partial t} + \nabla \rho \vec{u} = 0 \tag{1.13}$$

In order to find the second differential form, let's consider 2 points Q and P as described in Figure 1.3. The difference of density between the 2 points is

Figure 1.3

$$\rho_Q(t + dt) - \rho_P(t) = \rho(x_1 + dx_1, x_2 + dx_2, x_3 + dx_3, t + dt) - \rho(x_1, x_2, x_3)$$

$$= d\rho = \frac{\partial \rho}{\partial x_1} dx_1 + \frac{\partial \rho}{\partial x_2} dx_2 + \frac{\partial \rho}{\partial x_3} dx_3 + \frac{\partial \rho}{\partial t} dt \tag{1.14}$$

In general, the fluid particles at $P(t)$ and $Q(t + dt)$ are different. However, if $dx_1 = u_1 dt, dx_2 = u_2 dt, dx_3 = u_3 dt$, then the fluid particles at the 2 points are the same. By making appear these velocities in (1.14),

$$d\rho = \left(\frac{\partial \rho}{\partial x_1} u_1 + \frac{\partial \rho}{\partial x_2} u_2 + \frac{\partial \rho}{\partial x_3} u_3 + \frac{\partial \rho}{\partial t} \right) dt \tag{1.15}$$

Finally, after dividing by dt the 2 members of the equation, we obtain the definition of the time rate of change of density when I follow the fluid $\dot{\rho}$. As (1.13) can be expressed in term of indicial notation like

$$\frac{\partial \rho}{\partial t} + u_i \frac{\partial \rho}{\partial x_i} + \rho \frac{\partial u_i}{\partial x_i} = 0 \tag{1.16}$$

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Replacing the sum of first and second term by $\dot{\rho}$ gives the last form

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Mass conservation equation (differential form (2) - substancial form)

$$\dot{\rho} + \rho\nabla\vec{u} = 0 \tag{1.17}$$

1.1.2   Newton's second law : Momentum equation

Figure 1.4

Momentum in this course is noted $\vec{P}(t)$. For closed systems,

$$\frac{d\vec{P}(t)}{dt} = \sum \vec{F} = \frac{d}{dt}\int_{V(t)} \rho\vec{u}\,dV \tag{1.18}$$

where $\rho\vec{u}$ is the momentum density. We will spell out the expression of the 2 members. First, the derivative, using the Reynolds transport theorem gives

$$\frac{d\vec{P}}{dt} = \int_{V(t)} \frac{\partial \rho\vec{u}}{\partial t}\,dV + \oint_{S(t)=\partial V(t)} \rho\vec{u}(\vec{u}\vec{n})\,dS \tag{1.19}$$

This written in indicial notation

$$\frac{dP_i}{dt} = \int_{V(t)} \frac{\partial \rho u_i}{\partial t}\,dV + \oint_{S(t)=\partial V(t)} \underbrace{\rho u_i u_j}_{tensor:\,\vec{u}\otimes\vec{u}}\ n_j\,dS$$
$$= \int_{V(t)} \frac{\partial \rho u_i}{\partial t}\,dV + \oint_{S(t)=\partial V(t)} \rho(\vec{u}\otimes\vec{u})\vec{n}\,dS \tag{1.20}$$

and by applying Gauss theorem to the surface integral

$$\frac{d\vec{P}}{dt} = \int_{V(t)} \left[\frac{\partial \rho\vec{u}}{\partial t} + \nabla\rho\vec{u}\otimes\vec{u}\right] dV \qquad and \qquad \frac{dP_i}{dt} = \int_{V(t)} \left[\frac{\partial \rho u_i}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_i u_j)\right] dV \tag{1.21}$$

Based on the previous forms, we can generalize this for any arbitrary function ϕ

$$\frac{d}{dt}\int_{V(t)} \rho\phi\,dV = \int_{V(t)} \left[\frac{\partial \rho\phi}{\partial t} + \frac{\partial}{\partial x_j}\rho\phi u_j\right] dt$$
$$= \int_{V(t)} \left[\rho\frac{\partial \phi}{\partial t} + \phi\underbrace{\left(\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_j}{\partial x_j}\right)}_{=0\ (1.13)} + \rho u_j\frac{\partial \phi}{\partial x_j}\right] dV \tag{1.22}$$
$$= \int_{V(t)} \rho\left[\frac{\partial \phi}{\partial t} + u_j\frac{\partial \phi}{\partial x_j}\right] dV$$

Similarly to thermodynamics courses, we can introduce an extensive variable Φ and an intensive ϕ to have

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General relation for any arbitrary function in closed systems

$$\frac{d\Phi}{dt} = \int_{V(t)} \left[\frac{\partial \rho\phi}{\partial t} + \nabla\rho\phi\vec{u}\right] dV = \int_{V(t)} \rho\dot{\phi}\,dV \tag{1.23}$$

For the specific case where $\Phi = \vec{P}$ and $\phi = \vec{u}$, we obtain

$$\frac{d\vec{P}}{dt} = \int_{V(t)} \rho \underbrace{\left[\frac{\partial \vec{u}}{\partial t} + \vec{u}\nabla\vec{u}\right]}_{\dot{u}} dV \tag{1.24}$$

We can now express the forces applied on the system. There are 2 main classes :

- **Distance forces (volume)** $\vec{F}_V$:
  This type of force allows a body to influence another without being in contact with.
  — The most present one is gravity which is applied on each fluid particles ($d\vec{F} = dm\vec{g}$).
  We can imagine that there exists a force density $\vec{f}$ such that

$$\vec{F}_V = \int_{V(t)} \vec{f}\,dV = \int_{V(t)} \rho\vec{a}\,dV \tag{1.25}$$

  where $\vec{a}$ is a force per unit mass, so an acceleration (gravity : $\vec{f} = \rho\vec{g}$).
  — If we have an electric material, we can talk about electromagnetic forces, which can be modelled as

$$\vec{f} = \rho_c(\vec{E} + \vec{u} \times \vec{B}) + \vec{J} \times \vec{B} \tag{1.26}$$

  where ρ_c is the charge density $[C/m^3]$ and the second term is the Lorentz force. Indeed, if we have a lot of particles, we can talk of an average velocity $\vec{v}_k = \vec{u} + \vec{C}_k$, where C_k is a particular velocity due to molecular agitation. The force applied on the system is

$$\vec{F}_k = q_k[\vec{E} + \vec{v}_k \times \vec{B}] \qquad \Leftrightarrow \qquad \underbrace{\frac{\sum \vec{F}_k}{V}}_{} = \underbrace{\frac{\sum q_k}{V}}_{\rho_c}(\vec{E} + \vec{u} \times \vec{B}) + \underbrace{\frac{\sum q_k \vec{C}_k}{V}}_{\vec{J}} \times \vec{B} \tag{1.27}$$

  Mollecules are in general neutral, but containing non-neutral regions. Fluids are essentially neutral, $\vec{F}_V = 0$ in most cases. They are called quasi-neutral fluids. Electric influenced fluids will not be considered in that course but they existence has to be known.
  — They are also entrainment and Coriolis forces in rotating frame of references. These forces due to the rotation of Earth are not considered due to the small rotative velocity, unlike pomps and turbines.

- **Contact forces (surface)** $\vec{F}_S$:
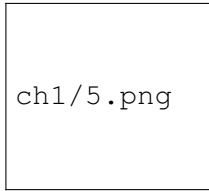
Figure 1.5

These forces results from the contact of an internal and external fluid in regard of a region of surface $dS(t)$. We have

$$d\vec{F}_S = \vec{T}\,dS \qquad \Rightarrow \vec{F}_S = \oint_S \vec{T}\,dS \tag{1.28}$$

$\vec{T}$ is a force per unit area, a continuous function of space depending on location $\vec{x}$ and also linearly on the infinitesimal surface orientation $\vec{n}$. If $\vec{T}$ is the force per unit area for a surface element normal to the unit vector in the j direction e_j, $\vec{T}(\vec{x}) = \vec{T}_j n_j$. (1.28) becomes

$$\vec{F}_S = \oint_S \vec{T}_j n_j\,dS \qquad and \qquad F_{S_i} = \oint_S \underbrace{\tau_{j,i}}_{\sigma_{ji}} n_j\,dS \tag{1.29}$$

where σ_{ji} is the stress tensor.

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We can now take (1.18) and replace the terms using (1.24), (1.25) and (1.29) to obtain the

Momentum equation (integral form)

$$\frac{dP_i}{dt} = \int_{V(t)} \left[\frac{\partial \rho u_i}{\partial t} + \nabla \rho u_i \vec{u} \right] dV = \int_{V(t)} \rho \dot{u} \, dV = \int_{V(t)} \rho a_i \, dV + \oint_{S(t)} \sigma_{ji} n_j \, dS \qquad (1.30)$$

We can see that σ_{ji} and $\rho u_i u_j$ have the same mathematical nature. This is not surprising because in fact these forces result from molleculare agitation in fluids. Let's discuss it. We said that $\vec{v}_k = \vec{u} + \vec{C}_k$. Let's consider a surface element and make the hypothesis that the fluid is in rest, so the average velocity $\vec{u} = 0$. It doesn't mean that the particles are immobile, but that if all particles have the same mass (pure fluid) and if a certain number of particles are going from right to left with velocity $\vec{c}$, there are the same number of particles going from left to right with velocity $-\vec{c}$. There is no global mass flux. So for n particles going in one direction, the mass flux

$$2nm\vec{u} = nm\vec{c} + 2nm(-\vec{c}) = 0 \qquad (1.31)$$

To obtain the momentum in direction x_1, we have to multiply the mass flow in this direction by the velocity in this direction

$$nm(\vec{c} \cdot \vec{e_1})c_1 + nm(-\vec{c_1} \cdot \vec{e_1})(-c_1) = 2nmc_1^2 \qquad (1.32)$$

The global momentum flux traversing the unit surface is so positive going out of the volume. We need so a balance force in the opposite direction to keep the mass in. This explains the presence and nature of σ_{ji} which is a momentum flux.

Let's finally establish the differential form of the momentum equation, applying Gauss theorem to the second right side of (1.30)

$$\int_{V(t)} \rho a_i \, dV + \oint_{S(t)} \sigma_{ji} n_j \, dS = \int_{V(t)} \left[\rho a_i + \frac{\partial \sigma_{ji}}{\partial x_j} \right] dV \qquad (1.33)$$

and by considering the whole equation

$$\int_{V(t)} \left[\frac{\partial \rho u_i}{\partial t} + \frac{\partial \rho u_i u_j}{\partial x_j} - \rho a_i - \frac{\partial \sigma_{ji}}{\partial x_j} \right] dV = 0 = \int_{V(t)} \left[\rho \dot{u}_i - \rho a_i - \frac{\partial \sigma_{ji}}{\partial x_j} \right] dV \qquad (1.34)$$

and for this to be true for all systems we consider, we obtain

Momentum equation (differential form (1) - divergent form)

$$\frac{\partial \rho u_i}{\partial t} + \frac{\partial \rho u_i u_j}{\partial x_j} = \rho a_i + \frac{\partial \sigma_{ji}}{\partial x_j} \qquad and \qquad \frac{\partial \rho \vec{u}}{\partial t} + \nabla \rho \vec{u} \otimes \vec{u} = \rho \vec{a} + \nabla \bar{\bar{\sigma}} \qquad (1.35)$$

Momentum equation (differential form (2) - substancial form)

$$\rho a_i + \frac{\partial \sigma_{ji}}{\partial x_j} = \rho \dot{u}_i \qquad and \qquad \rho \vec{a} + \nabla \bar{\bar{\sigma}} = \rho \dot{\vec{u}} \qquad (1.36)$$

1.1.3 Angular momentum equation

This is a corollary of the momentum equation that states that *the time rate of change of the angular momentum of a closed system is equal to the sum of the torques applied to the system.* There is no additional information except that the stress tensor should be symetric

$$\sigma_{ji} = \sigma_{ij} \qquad (1.37)$$

1.1.4 Energy equation - First principle of thermodynamics

If we note $\mathcal{E}$ the total energy of the system, the first principle tells that

$$\frac{d\mathcal{E}}{dt} = \dot{W} + \dot{Q} \tag{1.38}$$

where $\dot{W}$ is the mechanical power provided by the forces applied on the system and $\dot{Q}$ the thermal power provided to the system. We will proceed like the previous equation expressing first the left side then the right side. If we note E the total energy per unit mass, e the internal energy per unit mass and k the kinetic energy per unit mass (potential energy is not considered in order not to take into account power coming from potential forces)

$$\mathcal{E} = \int_{V(t)} E\, dm = \int_{V(t)} \rho E\, dV = \int_{V(t)} \rho(e+k)\, dV \quad with \quad k = \frac{\vec{u}\vec{u}}{2} \tag{1.39}$$

The time derivative of the energy using the Reynolds transport theorem, then the Gauss theorem is

$$\begin{aligned}
\frac{d\mathcal{E}}{dt} &= \int_{V(t)} \frac{\rho(e+k)}{dt}\, dV + \oint_{S(t)} \rho(e+k)\vec{u}\vec{n}\, dS \\
&= \int_{V(t)} \left[\frac{\rho(e+k)}{dt} + \nabla\rho(e+k)\vec{u}\right] dV = \int_{V(t)} \rho(e\dot{+}k)\, dV
\end{aligned} \tag{1.40}$$

Let's now go on with with the mechanical power expression. We expressed in (1.30) that there are volume and surface forces. These multiplied by the velocity and using Gauss gives

$$\dot{W} = \int_{V(t)} \rho a_i u_i\, dV + \oint_{S(t)} \sigma_{ji} u_i n_j\, dS = \int_{V(t)} \left[\rho a_i u_i + \frac{\partial}{\partial x_j}\sigma_{ji} u_i\right] dV \tag{1.41}$$

For the thermal power expression, we need to introduce a new concept that is the heat flux vector $\vec{q}$ which qualifies a thermal power per unit area leaving the surface. Physically, there is only two heat transport mecanism which are radiation and conduction. Indeed, convection is a specific conduction case where the temperature gradient region becomes thinner and favorises the exchange. The thermal power is

Figure 1.6

$$\dot{Q} = -\oint_{S(t)} \vec{q}\vec{n}\, dS = -\int_{V(t)} \nabla\vec{q}\, dV \tag{1.42}$$

Replacing the terms of (1.38) by (1.40), (1.41) and (1.42) gives

---

Total energy equation (integral form)

$$\int_{V(t)} \frac{\rho(e+k)}{dt}\, dV + \oint_{S(t)} \rho(e+k)\vec{u}\vec{n}\, dS = \int_{V(t)} \rho\vec{a}\vec{u}\, dV + \oint_{S(t)} (\bar{\bar{\sigma}}\vec{n})\vec{u}\, dS - \oint_{S(t)} \vec{q}\vec{n}\, dS \tag{1.43}$$

The differantial form is obtained using Gauss theorem for the two sides and regrouping all the terms into one integral

$$\begin{aligned}
&\int_{V(t)} \left[\frac{\rho(e+k)}{dt} + \nabla\rho(e+k)\vec{u} - \rho\vec{a}\vec{u} - \nabla\bar{\bar{\sigma}}\vec{u} + \nabla\vec{q}\right] dV = 0 \\
\Leftrightarrow \quad &\int_{V(t)} \left[\rho(e\dot{+}k) - \rho\vec{a}\vec{u} - \nabla\bar{\bar{\sigma}}\vec{u} + \nabla\vec{q}\right] dV = 0
\end{aligned} \tag{1.44}$$

And considering the fact that this has to be true for all systems, we obtain the two last forms

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Total energy equation (differential form (1) - divergent form)

$$\frac{\rho(e+k)}{dt} + \nabla\rho(e+k)\vec{u} = \rho\vec{a}\vec{u} + \nabla\bar{\bar{\sigma}}\vec{u} - \nabla\vec{q} \qquad (1.45)$$

</div>

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Total energy equation (differential form (2) - substantial form)

$$\rho(\dot{e+k}) = \rho(\dot{e} + \dot{k}) = \rho\vec{a}\vec{u} + \nabla\bar{\bar{\sigma}}\vec{u} - \nabla\vec{q} \qquad (1.46)$$

</div>

Let's finally establish the distribution of the forces in the different energies. If we multiply (1.36) by velocity $\vec{u}$ and if we observe that $\dot{k} = \frac{\dot{u}_i u_i + u_i \dot{u}_i}{2} = u_i \dot{u}_i$, we obtain

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Kinetic - Mechanical energy equation

$$\vec{u}\left(\rho a_i + \frac{\partial\sigma_{ji}}{\partial x_j} = \rho\dot{u}_i\right) \qquad \Leftrightarrow \qquad \rho\underbrace{u_i\dot{u}_i}_{\dot{k}} = \rho u_i a_i + \frac{u_i\partial\sigma_{ji}}{\partial x_j} \qquad (1.47)$$

</div>

The difference between total energy (1.46) and kinetic energy (1.47) gives the internal energy

<div style="border: 1px solid black; padding: 10px;">

Internal energy equation

$$\rho\dot{e} = 0 + \sigma_{ji}\frac{\partial u_i}{\partial x_j} - \nabla\vec{q} \qquad (1.48)$$

</div>

We see that volume forces only contributes to the kinetic energy, heat flux only to the internal energy and the suface forces to both.

1.1.5 Summary - Complementary equation

Let's make the inventory of the 3 substancial equations that we found. How many equations and unknowns do we have?

- In continuity equation (1.17), ρ and u_i are 4 unknowns in 3D.
- In momentum equation (1.36), a_i is an external applied force so is known, σ_{ji} consists in 6 unknowns (symetric matrix).
- In internal energy equation (1.48), e and $\vec{q}$ are 4 most unknowns.

The total unknowns number is 14. The number of disponible equations is 5, 1 thanks to the energy, 1 thanks to the continuity and 3 thanks to the vectorial momentum equation. In this stage, we haven't made any assumption on the nature of the material we're considering. These equations are valid for an elastic solid as a fluid. The main difference is that solids resist to a deformation whereas fluid doesn't. But fluid resists to a rate of deformation. The way that stress tensor σ_{ji} is related to the displacement field is called the constitutuve equations.

Constitutive relations

For a fluid, the stress tensor depends on the fluid rate of deformation (rate of strain). To express σ_{ji}, we have to find a quantity in the field of motion of the fluid that represents the rate of strain. If the velocity field $\vec{u}(\vec{x}, t)$ was uniform, not depending on $\vec{x}$, the fluid will be moving as a bulk and there is no rate of deformation. The rate of strain must be somehow related to the

velocity gradient tensor $\nabla \otimes \vec{u}$. We know that all tensors can be decomposed in an antisymetric and symetric part like

$$\nabla \otimes \vec{u} = \frac{\partial u_j}{\partial x_i} = \Omega_{ji} + S_{ij} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} - \frac{\partial u_i}{\partial x_j}\right) + \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right). \qquad (1.49)$$

For a constant gradient velocity field, the velocity field is linear in the coordinates

$$u_j = \frac{\partial u_j}{\partial x_i}x_i = \Omega_{ji}x_i + S_{ij}x_i \qquad (1.50)$$

Let's look to the mathematical nature of the antisymetric part. If we express using Kronecker δ, we have

$$\Omega_{ji} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} - \frac{\partial u_i}{\partial x_j}\right) = \frac{1}{2}\delta_{kij}\delta_{kqp}\frac{\partial u_p}{\partial x_q} \qquad with \quad \delta_{kij}\delta_{kqp} = \delta_{iq}\delta_{jp} - \delta_{ip}\delta_{jq} \qquad (1.51)$$

Knowing that $(a \times b)_k = \delta_{kpq}a_p b_q$, we can introduce the **curle** (rotationnel) of velocity called vorticity $\vec{\omega}$

$$\delta_{kqp}\frac{\partial u_p}{\partial x_q} = (\nabla \times \vec{u})_k \qquad \Rightarrow \nabla \times \vec{u} = \vec{\omega} \qquad (1.52)$$

Let's look to the way this is linked to (1.50)

$$u_j^{AS} = \Omega_{ji}x_i = \frac{1}{2}\delta_{jki}\omega_k x_i = \frac{1}{2}(\vec{\omega} \times \vec{x})_j \qquad \Leftrightarrow \qquad \vec{u}^{AS} = \frac{1}{2}\vec{\omega} \times \vec{x} \qquad (1.53)$$

In conclusion, we see that the antisymetric part consists in a pure rotation velocity field, a rigid body motion of angular velocity $\frac{1}{2}\vec{\omega}$ without strain. $\vec{\omega}$ is twice the angular velocity of fluid particles around themselves. The quantity representative of the fluid rate of strain can only be the symetric part of the velocity gradient tensor called the rate of strain tensor. For a fluid, $\sigma_{ij} = f(S_{ij})$.

To determine the nature of this relationship, we will assume that σ_{ij} is a linear function of S_{pq}. This is called

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Newton's assumption for stresses

$$\sigma_{ij} = a_{ij} + b_{ijpq}S_{pq}. \qquad (1.54)$$

In this equation, b_{ijpq} is a tensor with four indices, but we know that it's symetric with respect to pq and ij because S is symetric with respect to pq and ij, leading to $6 \times 6 = 36$ coefficients. Symetric tensor a_{ij} counts 6 coefficients, for a total of 42 coefficients.

If we assume that the fluid is **isotropic**, meaning that the fluid react in the same way whatever the sollicitation direction. For example, let's take a case of S_{ij} where all coefficients are null except the S_{11} term. Diagonal terms reprensent a rate of elongation/stretch while the off-diagonal terms represent an angular deformation between two perpendicular direction. The assumption means that if the rate of stress is not in 1 direction but 2, the fluid reaction will be the same. In other words, if we make a rotation of coordinates, the relation in the rotated frame of reference must be the same. In that case, the relation reduces to

$$\sigma_{ij} = a\delta_{ij} + bS_{ij} + c\delta_{ij}S_{kk} \qquad (1.55)$$

where only 3 coefficient must be found. It is natural to think that air and water have no preferential direction unlike certain solid as wood that has a preferential direction related to the

orientation of fibers. Blood or dissolved polymer chains are examples of non isotropic fluids. We will from now consider the fluid to be isotropic.

In (1.55) a is a constant that represents the stress present when the fluid is at rest. The surface force associated to that component is purely normal

$$\sigma_{ij}n_j = a\delta_{ij}n_j = an_i \tag{1.56}$$

This constant corresponds to the pressure exerted by the fluid at rest. Because of its application in the opposite direction to the normal, it's negative. The two other coefficients represents the 2 coefficients of viscosity

$$a = -p \qquad b = 2\mu \qquad c = \lambda \tag{1.57}$$

The stress tensor equation can so be written with a pressure stress and a viscous stress part like

$$\sigma_{ij} = -p\delta_{ij} + \tau_{ij} \qquad with \qquad \tau_{ij} = 2\mu S_{ij} + \lambda\delta_{ij}S_{kk} \tag{1.58}$$

An alternative form to that is the following

$$\tau_{ij} = 2\mu \underbrace{\left(S_{ij} + \frac{1}{3}\delta_{ij}S_{kk} \right)}_{\equiv S_{ij}^S} + \underbrace{\left(\lambda + \frac{2\mu}{3} \right)}_{\equiv \mu_V} \delta_{ij}S_{kk} \tag{1.59}$$

This notation is necessary to make appear the part of the strain tensor which has no trace S_{ij}^S, called the rate of shear. Indeed

$$S_{ii}^S = S_{ii} - \frac{1}{3}\delta_{ii}S_{kk} = 0 \tag{1.60}$$

This means that S_{ij}^S represents the trace less part of the rate of **strain** tensor called the sheer rate tensor. What is now S_{kk} ?

$$S_{kk} = \frac{1}{2}\left(\frac{\partial u_k}{\partial x_k} + \frac{\partial u_k}{\partial x_k} \right) = \frac{\partial u_k}{\partial x_k} = \nabla\vec{u} \tag{1.61}$$

The divergence of the velocity is related to the rate of dilatation of the fluid, the change of volume. We decomposed the rate of strain in a part representing the deformation without change of volume (pure deformation) and another with change of volume (μ_V is the bulk viscosity). Another expression for τ_{ij} with a final net gain of 3 unknowns is

$$\tau_{ij} = 2\mu S_{ij}^S + \mu_V\delta_{ij}\nabla\vec{u} \tag{1.62}$$

At this stage, we have to determine still 6 unknowns from the 9 at the beginning.

Heat flux

We discussed about the fact that heat flux propagates using 2 physical mechanism : conduction and radiation. In the energy equation it's the divergence of the heat fluc that appears. In most application, the radiative effect does not imply heat accumulation or loss. The fluids are so transparent to radiative heat flux $\nabla\vec{q}^{rad} = 0$. We only have conduction and the Fourier law says that

$$\vec{q} \propto \nabla T = d\nabla T = -\kappa\nabla T \qquad \Leftrightarrow \qquad q_i = -\kappa\frac{\partial T}{\partial x_i} \tag{1.63}$$

The negative sign comes from the fact that heat goes from hot to cold (decrease of T). We have a net gain of 1 unknown with this equation.

Thermodynamics

At this stage, we are using 4 thermodynamics intensive variables which are ρ, e, p, T. We know that for a single phase fluid, the variance is 2, meaning that we can use 2 thermodynamics equations of state (EoS) relating them. For example, for a alorically and thermally perfect gas, we will have

$$p = \rho RT \qquad and \qquad e = c_v T \tag{1.64}$$

We have a net gain of 2 unknows, so there remains 2 unknowns.

Transport coefficients

The remaining variables are the shear viscosity μ, the bulk viscosity μ_V and thermal conductivity κ. These are functions of the thermodynamic state. For example for gases we have the relations

$$\mu = f(T) \qquad and \qquad Pr = \frac{\mu c_p}{\kappa} = cst \Leftrightarrow \kappa = \frac{\mu(T) c_p(T)}{Pr} \tag{1.65}$$

The bulk viscosity is more difficult to determine, but it can be shown that for monoatomic gases (no internal degres of freedom) $\mu_V = 0$. For diatomic gases it's much more delcate to measure, but it has been shown that for many flows, the flows is insensitive to the variation of value of bulk viscosity. In fact, for fluids without divergence of velocity, we don't care about μ_V because there is no variation of volume (1.62). We will so make the following assumption

Stokes assumption
$$\mu_V = 0 \quad \text{even for other gases.} \tag{1.66}$$

We're done, we have as many equations as variables. We mentioned the first principle of thermodynamics but not the second. Let's analyse that.

Second principle of thermodynamics

We will reuse the internal energy equation (1.48), replace σ_{ij} by it's expression in (1.58) and use the fact that $\frac{\partial u_i}{\partial x_j} = \Omega_{ij} + S_{ij}$

$$\rho \dot{e} = \sigma_{ij} \frac{\partial u_i}{\partial x_j} - \frac{\partial q_i}{\partial x_i} = -p\delta_{ij}\frac{\partial u_i}{\partial x_j} + (2\mu S_{ij}^S + \mu_V \delta_{ij}\nabla\vec{u})(S_{ij} + \cancel{\Omega_{ij}}) - \frac{\partial q_i}{\partial x_i} \tag{1.67}$$

where Ω_{ij} doesn't contribute because the contraction of the symetric tensor by the antisymetric tensor is equal to 0. We have the relation

$$S_{ij}^S = S_{ij} - \frac{1}{3}\delta_{ij}\nabla\vec{u} \qquad \Leftrightarrow \qquad S_{ij} = S_{ij}^S + \frac{1}{3}\delta_{ij}\nabla\vec{u} \tag{1.68}$$

Combined to the fact that $\delta_{ij}\frac{\partial u_i}{\partial x_j} = \frac{\partial u_i}{\partial x_i} = \nabla\vec{u}$, we obtain

$$\rho\dot{e} = -p\nabla\vec{u} + (2\mu S_{ij}^S + \mu_V \delta_{ij}\nabla\vec{u})\left(S_{ij}^S + \frac{1}{3}\delta_{ij}\nabla\vec{u}\right) - \nabla\vec{q} \tag{1.69}$$

The mass conservation equation tells us that we can write the divergence as

$$\dot{\rho} + \rho\nabla\vec{u} = 0 \qquad \Leftrightarrow \qquad \nabla\vec{u} = -\frac{\dot{\rho}}{\rho} = -\rho\left(\frac{\dot{\rho}}{\rho^2}\right) = \rho\left(\frac{\dot{1}}{\rho}\right) = \rho\dot{v} \tag{1.70}$$

If we replace the divergence in the previous equation, we have

$$\rho\dot{e} = -\rho p\dot{v} + (2\mu S_{ij}^S + \mu_V \delta_{ij}\nabla\vec{u})\left(S_{ij}^S + \frac{1}{3}\delta_{ij}\nabla\vec{u}\right) - \nabla\vec{q} \tag{1.71}$$

The first term here looks like the reversible work $-pdv$ in thermodynamics and is so the reversible contribution to the internal energy. Let's make this appear by bringing this to the left side. We make appear $\rho[\dot{e} + p\dot{v}]$, but we have the famous Gibbs relation $de = Tds - pdv \Leftrightarrow \dot{e} = T\dot{s} - p\dot{v}$. We have now

$$\rho\dot{s} = \frac{(2\mu S_{ij}^S + \mu_V \delta_{ij}\nabla\vec{u})\left(S_{ij}^S + \frac{1}{3}\delta_{ij}\nabla\vec{u}\right)}{T} - \frac{\nabla\vec{q}}{T} \tag{1.72}$$

If we remind the relation we demonstrated before for any variable $\dot{\Phi} = \int_V \rho\dot{\phi}dV$, we can make the analogy here to say that when this is integrated over a volume, it gives the time rate of change of the entropy of the closed system that's initially inside this volume. We have to identify the reversible part in this equation. We know that the reversible entropy rate of exchange for a uniform system and its integral over a closed surface is given by

$$\frac{\vec{q}dS}{T} \qquad \Rightarrow \qquad \oint_S \frac{\vec{q}}{T}(-\vec{n})\,dS = -\int_V \nabla\frac{\vec{q}}{T}\,dV. \tag{1.73}$$

We see that we have to make appear a this in the last equation. But we know that

$$\frac{\nabla\vec{q}}{T} = \nabla\frac{\vec{q}}{T} - \vec{q}\nabla\left(\frac{1}{T}\right) = \nabla\frac{\vec{q}}{T} + \vec{q}\frac{\nabla T}{T^2} \tag{1.74}$$

And by introducing this into the relation (1.72), we make appear the reversible entropy rate of exchange

$$\rho\dot{s} = -\nabla\frac{\vec{q}}{T} - \frac{\vec{q}\nabla T}{T^2} + \frac{(2\mu S_{ij}^S + \mu_V \delta_{ij}\nabla\vec{u})\left(S_{ij}^S + \frac{1}{3}\delta_{ij}\nabla\vec{u}\right)}{T} \tag{1.75}$$

We also know that $\vec{q} = -\kappa\nabla T$, making appear $(\nabla T)^2$

$$\rho\dot{s} = -\nabla\frac{\vec{q}}{T} + \frac{\nabla T\nabla T}{T^2} + \frac{(2\mu S_{ij}^S + \mu_V \delta_{ij}\nabla\vec{u})\left(S_{ij}^S + \frac{1}{3}\delta_{ij}\nabla\vec{u}\right)}{T} \tag{1.76}$$

If we imagine a fluid at rest with only a heat exchange operating on it, the third term $= 0$, the first term is reversible so anyway the sign and the second term must be positive. This implies $\kappa \geq 0$ due to the square of the other varaibles (the heat has to go from hot to cold). Let's expand the third term

$$\rho\dot{s} = -\nabla\frac{\vec{q}}{T} + \frac{\nabla T\nabla T}{T^2} + \frac{1}{T}\left[2\mu S_{ij}^S S_{ij}^S + \cancel{\mu_V \nabla\vec{u}\delta_{ij}S_{ij}^S} + \cancel{2\mu S_{ij}^S\frac{\delta_{ij}}{3}\nabla\vec{u}} + \mu_V\frac{\delta_{ij}\delta_{ij}}{3}(\nabla\vec{u})^2\right] \tag{1.77}$$

In this last equation, the second and third terms are nul because $S_{ii}^S = 0$. Let's imagine that we have a fluid with only dilation and no shear S_{ij}^S, the last term must be positive and so μ_V has to be positive (≥ 0). In the other hand, for the first term, we have a quadratic form (sum of squares ≥ 0), so μ has to be positive. To verify the second principle, we have to verify these 3 inequalities. In fluid mechanics, we don't have to worry about the second principle, it's built in the equations as long as the transport coefficient are positive.

1.1.6 Boundary conditions

We have now to establish the boundary conditions which makes the difference between the flow cases. First of all, we have two main categories of flows :

- **External flows** (unbounded domain)
  For example, a flow over a wing, assuming that atmosphere extends to infinity. In that case we have far field boundary conditions, what happens far from the body ($u \to u_\infty, p \to p_\infty, T \to T_\infty$).
- **Internal flows** (bounded domain)
  For example, a flow in a pipe or a fluid in a rotating machine like a pump. In that case we don't have the far field conditions but the inlet and outlet boundary conditions but this problem is not discussed here.

Solid surfaces

In both case we have solid surfaces, we have to make a distinction. We wrote the equation for the general case of a viscous flow, but there is flows where the viscous stresses can be neglected (not = 0 !) leading to what we call the **inviscid flows**. Let's analyse the two cases.

Viscous flows

Viiscosity is associated to the exchange of momentum between neighboring fluid layers due to molecular agitation. If we have a molecule coming from a low velocity region to a high velocity region, it slows down the molecule there and inversely. The same occurs when a fluid particle enter in contact with solid surfaces, it exchange momentum. The result is that velocity and temperature fields must be continuous

$$\vec{u}_{fluid} = \vec{u}_{wall} \qquad and \qquad T_{fluid} = T_{wall} \tag{1.78}$$

In particular, for a surface at rest, the fluid must be at rest on the solid surface as well. This is called the **no-slip condition**.

Inviscid flows

For inviscid flows, this mecanism doesn't exist, the fluid may slip. The boundary condition is that the fluid can't go throw the solid

$$\vec{u}_{fluid}\vec{n} = \vec{u}_{wall}\vec{n} \tag{1.79}$$

This is called the **slip/no penetration condition**. The previous condition is stronger because in fact $\vec{u} = \vec{u}_n\vec{n} + \vec{u}_t$ includes the tangential condition too.

1.2 Special cases

1.2.1 General case

The generale equations are the following :

- Mass conservation equation

$$\dot{\rho} + \rho\nabla\vec{u} = 0 \tag{1.80}$$

- Momentum equation

$$\rho\dot{\vec{u}} = -\nabla p + \nabla\bar{\bar{\tau}} + \rho\vec{F} \tag{1.81}$$

- Energy equation

$$\rho\dot{e} = -p\nabla\vec{u} + \underbrace{\bar{\bar{u}}..\nabla\otimes\vec{u}}_{\epsilon_V} - \nabla\vec{q} \tag{1.82}$$

where $\nabla\otimes\vec{u}$ can be replace by the symetric part, rate of strain tensor $\bar{\bar{S}}$ and ϵ_V is the viscous dissipation.

- Constituve relation

$$\begin{aligned}
\tau_{ij} &= 2\mu\left(S_{ij} - \frac{1}{3}\delta_{ij}\nabla\vec{u}\right) + \mu_V S_{ij}\nabla\vec{u} \\
&= \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\nabla\vec{u}\right) + \mu_V S_{ij}\nabla\vec{u}
\end{aligned} \tag{1.83}$$

- Conductive heat flux

$$\vec{q} = -\kappa\nabla T \tag{1.84}$$

1.2.2 Steady flow

This is caracterized by the fact that $\frac{\partial}{\partial t} = 0$ and implies for example that

$$\dot{\rho} = \frac{\partial\rho}{\partial t} + \vec{u}\nabla\rho = \vec{u}\nabla\rho \tag{1.85}$$

and for the others.

1.2.3 Inviscid flows

They are defined as flows in which vicous stresses and conduction heat flux can be neglected. We are talking about flows and not fluids because there is no fluid with $\mu = 0$ or $\kappa = 0$. This happens for superfluids but we don't care. When we look at the fluid properties tables, in SI units, water and air have very small μ but we can't say that there are negligible because it depends on the system of refference used. If something is negligible it is with respect to something else. Let's start with the viscous stresses. Momentum equation can be written as

$$\rho\dot{\vec{u}} = \frac{\partial\rho\vec{u}}{\partial t} + \nabla\rho\vec{u}\otimes\vec{u} = -\nabla p + \nabla\bar{\bar{\tau}} + \rho\vec{F} \tag{1.86}$$

where the viscous stress tensor is a tensor as the momentum flux tensor. They correspond to the same physical phenomenon but at different scales, the viscous stress tensor is due to the molecular agitation whereas the momentum flux tensor is for the macroscopic scale, the average scale. So it makes sense to compare the order of magnitude of the two ones.

Figure 1.7

Let's consider a fluid flows of far field velocity $\vec{U}$ around a solid body of characteristic lenght L, if we consider the momentum flux tensor, we know that the velocity around the body will vary between 0 and U so the order of magnitude will be $\theta(\rho U^2)$. What about τ? We see that in (1.83) appears the velocity gradient, derivative. What is the order of magnitude of the derivative of a function?

Figure 1.8

Let's consider a function $f(x)$ represented on Figure 1.8. If the function is smooth, so if the function doesn't vary much in the interval, its derivative keeps a constant order of magnitude in the integral. We see it in the figure, the slope varies between a and b, can be twice the slope at the center but keeps the same order of magnitude. So for a smooth function, the order of magnitude of f' remains the same over the interval $f' = \theta\left(\frac{\Delta f}{\Delta x}\right)$. Let's use this to have an approximation for the velocity gradient tensor

$$\nabla \otimes \vec{u} = \theta\left(\frac{U}{L}\right) \qquad \Rightarrow \bar{\bar{\tau}} = \theta\left(\mu\frac{U}{L}\right) \tag{1.87}$$

The relative order of magnitude of viscous stresses with respect to momentum flow is

$$\frac{\mu\frac{U}{L}}{\rho U^2} = \frac{\mu}{\rho U L} = \frac{1}{Re_L} \tag{1.88}$$

We conclude that viscous stresses can be neglected in the case of high Reynolds number.

Now we have to verify the assumption that velocity is a smooth function of the coordinates. Examples of not smooth functions are represented on Figure 1.9 where the green curve is smooth close to the limits but not smooth in a small interval and the yellow one is a periodic function with the characteristic wave length. In these cases, Δx is not the appropriate length scale to determine the order of magnitude of the derivative.

Figure 1.9

Figure 1.10

Now what about the velocity field? Let's assume that velocity is smooth and the fluid inviscid, the boundary conditions to respect are the slip conditions. But we know that μ is not strictly 0, so the velocity must be 0 at the wall and therefore, there must exist a region of another caracteristic scale δ of rapidly changing velocity close to the wall. In this case the study of order of magnitude made is incorrect, U/L must be replaced by U/δ. Due to the smaller scale of δ compared to L, the viscous stress is more important than outside this region and can be of comparable size to the momentum flux tensor. We conclude that the flow can be decomposed into two regions : a region outside of this viscous layer, a distal region where the viscous stresses and heat conduction term can be neglected and a proximal or inner region where viscous stresses may not be neglected. We complete the definition with high Reynolds number by adding "**except close to solid bodies and in their wake**[1]".

1.2.4 Inviscid flows equations

They are the same as the general case, except that the viscous and heat flux terms are neglected.
- Mass conservation equation
$$\dot{\rho} + \rho\nabla\vec{u} = 0 \tag{1.89}$$

- Momentum equation
$$\rho\dot{\vec{u}} = -\nabla p + \nabla\bar{\bar{\tau}} + \rho\vec{F} \tag{1.90}$$

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1. Sillage : refermement de la couche visqueuse à droite de la Figure 1.10.

- Energy equation

$$\rho \dot{e} = -p\nabla \vec{u} + \bar{\bar{u}}\cdot\nabla\otimes\vec{u} - \nabla\vec{q} \qquad \Leftrightarrow \qquad \rho\dot{e} + p\nabla\vec{u} = 0 \tag{1.91}$$

We already analyzed this expression before, with (1.70) we can conclude that

$$\rho(\dot{e} + p\dot{v}) = 0 = \rho T \dot{s} \qquad \Rightarrow \dot{s} = 0 \tag{1.92}$$

Entropy per unit mass is constant along trajectories. Only viscous term and heat flux are responsible of irreversible entropy variations. So all the particles keeps constant entropy and if the incoming fluid particles are uniform it means that the entropy will be constant across the whole flow.

Let's specify the terminology, when we speak about uniform quantity it means that $\nabla q = 0$ and steady q means that it doesn't vary with time $\frac{\partial q}{\partial t} = 0$. Let's now see what happens with the momentum equation

$$\rho\dot{\vec{u}} = -\nabla p + \rho\vec{F} = \rho\left[\frac{\partial u}{\partial t} + (\vec{u}\nabla)\vec{u}\right]$$
$$\rho\dot{u}_i = -\frac{\partial p}{\partial x_i} + \rho F_i = \rho\left[\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial u_j}\right] \tag{1.93}$$

But if we say that, by adding and removing the needed term

$$u_j\frac{\partial u_i}{\partial x_j} = u_j\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) + u_j\frac{\partial u_j}{\partial x_i} \tag{1.94}$$

But what's the last term? We know that the gradient of kinetic energy corresponds to that because

$$u_j\frac{\partial u_j}{\partial x_i} = \frac{\partial \frac{u_j u_j}{2}}{\partial x_i} = \frac{\partial k}{\partial x_i} \qquad \text{additionally} \qquad \frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} = \delta_{kji}\omega_k \tag{1.95}$$

Now if we replace this in (1.93), we find the

Lamb's form of the momentum equation

$$-\frac{\partial p}{\partial x_i} + \rho F_i = \rho\left[\frac{\partial u_i}{\partial t} + \delta_{kji}\omega_k u_j + \frac{\partial k}{\partial x_i}\right]$$
$$\Leftrightarrow -\frac{\partial p}{\partial x_i} + \rho F_i = \rho\left[\frac{\partial u_i}{\partial t} + (\vec{\omega}\times\vec{u})_i + \frac{\partial k}{\partial x_i}\right] \tag{1.96}$$
$$\Leftrightarrow -\nabla p + \rho\vec{F} = \rho\left[\frac{\partial \vec{u}}{\partial t} + \vec{\omega}\times\vec{u} + \nabla k\right]$$

1.2.5   Barotropic flows - Force deriving from a potential

Like the previous one, there are barotropic flows but no barotropic fluids. Let's rewrite the Lamb's equation

$$\frac{\partial \vec{u}}{\partial t} + \vec{\omega}\times\vec{u} + \nabla k = -\frac{\nabla p}{\rho} + \vec{F} \qquad with \vec{F} = \vec{a} \tag{1.97}$$

In general we know that the thermodynamic state of a pure fluid in single phase is determined by 2 thermodynamic variables. This means that in general, p and ρ are independant variables. But when another thermodynamic variable is constant, uniform, then it exists a relation between p

and ρ and hence it exists a certain function $P(p)$ such that $\frac{dP}{dp} = \frac{1}{\rho(p)}$. This implies that the gradient

$$\nabla P = \frac{dP}{dp}\nabla p = \frac{\nabla p}{\rho} \tag{1.98}$$

allowing us to replace this quantity in (1.97). Two examples of constant variables :

- **Constant density flows** : $\rho(p) = \rho = cst$ and so $\frac{dP}{dp} = \frac{1}{\rho} \Rightarrow P = \frac{p}{\rho}$.
- **Isentropic flows** : we have the Gibbs relation $dh = Tds + vdp$ simplifying in $dh = \frac{dp}{\rho}$ and so $P = h$.

If now in addition to the barotropic flow assumption we assume that $\vec{F}$ derives from a potential, we will make other assumptions. This means that the curl $\nabla \times \vec{F} = 0$, allowing to write $\vec{F}$ as the gradient of a certain potential energy per unit mass $\vec{F} = -\nabla\Phi$. Then we can write (1.97) as

$$\frac{\partial \vec{u}}{\partial t} + \vec{\omega} \times \vec{u} = -\nabla(P + k + \Phi) \tag{1.99}$$

Making the additional assumption that we have a steady flow and multiplying by $\vec{u}$ the two members leads to

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Bernouilli's equation (1)

$$\vec{u}(\vec{\omega} \times \vec{u}) = -\vec{u}\nabla(P + k + \Phi) \qquad \Leftrightarrow \qquad P + k + \Phi = e_m \tag{1.100}$$

telling that mecanical energy is constant along streamlines (but can vary between streamlines).

1.2.6 Irrotational flows

This points to flows such that $\vec{\omega} = 0$. In that case we see that

$$-\nabla(P + k + \Phi) = 0 \qquad \Rightarrow P + k + \Phi = cst \tag{1.101}$$

Everywhere in the domain! When the flow is irrotational $\vec{\omega} = \nabla \times \vec{u} = 0$, $\vec{u}$ can be experssed as a velocity potential $\nabla\phi$. Now if we consider an unsteady, barotropic, irrotational inviscid flows with irrotational body forces

$$\frac{\partial \vec{u}}{\partial t} = \frac{\partial \nabla\phi}{\partial t} = \nabla\frac{\partial \phi}{\partial t} = -\nabla(P + k + \Phi) \qquad \Leftrightarrow \qquad \frac{\partial \phi}{\partial t} + P + k + \phi = cst \tag{1.102}$$

Everywhere in the domain.

1.2.7 Incompressible/quasi incompressible flows

Let's consider an inviscid steady flow in absence of body forces (and uniform S due to inlet conditions). Then the momentum equation (1.93) tells that

$$\rho(\vec{u}\nabla)\vec{u} = -\nabla p = -\left.\frac{dp}{d\rho}\right|_S \nabla\rho \tag{1.103}$$

where $\frac{dp}{d\rho}|_S = a^2$, with a the speed of sound.

Figure 1.11

If $\vec{e}_s$ is the unit vector along the following streamline. We have that

$$\vec{u} = u\vec{e}_s \qquad \Rightarrow (\vec{u}\nabla)\vec{u} = u\frac{d}{ds}(u\vec{e}_s) = -\frac{a^2}{\rho}\nabla\rho = -\frac{a^2}{\rho}\left[\frac{d\rho}{ds}\vec{e}_s + \frac{d\rho}{dn}\vec{e}_n\right]$$

$$\Leftrightarrow u\frac{du}{ds}\vec{e}_s + u^2\frac{d\vec{e}_s}{ds} = -\frac{a^2}{\rho}\left[\frac{d\rho}{ds}\vec{e}_s + \frac{d\rho}{dn}\vec{e}_n\right] \tag{1.104}$$

We will not discuss the derivative of the unit vector, so if we look to the streamline's direction component, we have

$$u\frac{du}{ds} = -\frac{a^2}{\rho}\frac{d\rho}{ds} \quad \Leftrightarrow \quad \frac{u^2}{a^2}\frac{du}{u} = -\frac{d\rho}{\rho} \qquad \Rightarrow \frac{d\rho}{\rho} = -M^2\frac{du}{u} \tag{1.105}$$

where $M = \frac{u}{a}$ is the **Mach number** and compares the local velocity to the speed of sound. The conclusion is that when M is small, density variations are small. We can so make the approximation of constant density. Even for compressible fluids (like gases), in the conditions provided by the assumptions, density almost does not vary as long as the Mach number is much smaller than 1 (smaller than 0.3 in practice).

A first counter example is the sound waves because they create unsteady flows and natural convection where the density variation is due to temperature variation (we only considered pressure variation here). In these cases density variation is not negligible even for small Mach number.

1.2.8 Two-dimensional (planar) flows

They are essentially flows over cylindrical geometries. A general cylilnder is a body made of straightlines parallel to each other and wich lie upon a two dimensional curve. When we take a surface and draw infinite straightlines, there is no reason for the solution to vary in the infinite direction, all the derivative are 0. This means, according to Figure 1.12, that $\frac{\partial}{\partial x_3} = 0$. Moreover, in general there is no velocity component in this direction but it isn't necessary $u_3 = 0$.

Figure 1.12

In that case, for **steady** flows, the continuity equation becomes

$$\dot{\rho} + \rho\nabla\vec{u} = \frac{\partial\rho}{\partial t} + \nabla\rho\vec{u} = 0 \qquad \Leftrightarrow \frac{\partial(\rho u_1)}{\partial x_1} + \frac{\partial(\rho u_2)}{\partial x_2} = 0 \tag{1.106}$$

This equation can be made satisfied by introducing an auxiliary funciton ψ (called streamfunction), such that

$$\begin{cases} \rho u_1 = & \rho_0\dfrac{\partial\psi}{\partial x_2} \\ \rho u_2 = & -\rho_0\dfrac{\partial\psi}{\partial x_1} \end{cases} \quad \Rightarrow \quad \underbrace{\frac{\partial\rho u_1}{\partial x_1}}_{\rho_0\frac{\partial^2\psi}{\partial x_1\partial x_2}} + \underbrace{\frac{\partial^2\rho u_2}{\partial x_2}}_{-\rho_0\frac{\partial^2\psi}{\partial x_2\partial x_1}} = 0 \tag{1.107}$$

We see that continuity equation is satisfied for this function. We can replace the two velocity variables by this function and reduce the unknowns from 2 to 1.

Physical meaning of the streamfunction

If we are in 3D with x_3 the direction of homogeneity and a surface composed of straightlines (height h=1) lying upon a curve C in $x_1 x_2$ plan. The mass flow over the surface is $(\vec{b} = 0)$

$$\dot{m} = \int_S \rho \vec{u}\vec{n}\, dS = \int_0^1 dx_3 \int_C \dot{m}\vec{u}\vec{n}\, ds \tag{1.108}$$

Let's look to the curve from top. We will make a little bit geometry on the magnifier of the curve C.

Figure 1.13

In order to determine the normal to the circuit, we have to close it following an anticlockwise fashion (A to B). If we write the normal following x_1 and x_2 direction, according to Figure 1.14 we have $(dx_1 < 0)$

Figure 1.14

$$\left.\begin{array}{l} n_1 = \cos\theta = \dfrac{dx_2}{ds} \\[2mm] n_2 = \sin\theta = -\dfrac{dx_1}{ds} \end{array}\right\} \Rightarrow \begin{array}{l} \rho\vec{u}\vec{n}\, ds = \rho(u_1 n_1 + u_2 n_2)\, ds \\[2mm] = \rho\left[u_1 \dfrac{dx_2}{ds} - u_2 \dfrac{dx_1}{ds}\right] ds \\[2mm] = \rho(u_1 dx_2 - u_2 dx_1) \end{array} \tag{1.109}$$

And now if we remplace in (1.108) and use the definition of the streamline function, we have

$$\dot{m} = \int_C \rho(u_1 dx_2 - u_2 dx_1) = \int_C \rho_0 \underbrace{\left[\frac{\partial\psi}{\partial x_2}dx_2 - \left(-\frac{\partial\psi}{\partial x_1}\right)dx_1\right]}_{d\psi} = \rho_0(\psi_B - \psi_A) \tag{1.110}$$

So, the physical meaning is that ψ on a certain point is the mass flow between this point and a reference point where $\psi = 0$. And because of a steady flow, it's the same mass flow over the surface from A to B whatever the curve used for. If 2 points A and B are on the same streamline then $\psi_A = \psi_B$, so lines with $\psi = cst$ are streamlines.

Streamfunction equation (constant-density flow)

We are interested in finding an equation the streamfunction has to satisfied. The assumption of constant density allow us to consider $\rho = \rho_0$ as it's not a function of space in (1.107)

$$\begin{cases} u_1 = \dfrac{\partial\psi}{\partial x_2} \\[4mm] u_2 = -\dfrac{\partial\psi}{\partial x_1} \end{cases} \Rightarrow \nabla\vec{u} = 0 \text{ (continuity equation)} \tag{1.111}$$

If we compute the vorticity vector

$$\vec{\omega} = \nabla \times \vec{u} = \begin{vmatrix} \vec{e_1} & \vec{e_2} & \vec{e_3} \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ u_1 & u_2 & u_3 \end{vmatrix} = \vec{e_3}\underbrace{\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right)}_{\omega_3} \tag{1.112}$$

Now if we replace the velocity components by the streamfunction equivalence, we found the

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Streamfunction equation

$$\omega_3 = -\frac{\partial^2\psi}{\partial x_1^2} - \frac{\partial^2\psi}{\partial x_2^2} \qquad\Leftrightarrow\qquad -\omega_3(x_1, x_2) = \nabla^2\psi \tag{1.113}$$

An equation for vorticity can be obtained by taking the curl of the momentum equation. This last equation compatible with our assumptions is

$$\rho\dot{\vec{u}} = \rho\left[\frac{\partial\vec{u}}{\partial t} + (\vec{u}\nabla)\vec{u}\right] = -\nabla p + p\vec{F} + \nabla\bar{\bar{\tau}} \tag{1.114}$$

Combined with (1.111), we have a set of 4 equations and 4 unknowns in 3D ($\vec{u}$ and p). So the continuity and momentum equations can be solved independently of the energy equation, the thermal problem is dissociated from the hydrodynamic problem. Let's finally point that for irrotational flows we have

$$\nabla^2\psi = 0 \qquad \text{(Laplace's equation)} \tag{1.115}$$

Chapter 2

Similarity and dimensional analysis

This was hystorically first developped by engineers. We derived the governing equations for fluids flows but in practice we can not everytime find an analytical solution. The main weapon of engineers is testing. But it's not easy to do tests on full scale prototypes. We will do tests on a scaled model. But there rises the question of the representativity of the tests.

2.1 Experimental testing - Similarity

Under wich conditions are experiments carried out over a scaled model (m) representative of the flow over the actual prototype (p) ? The response is that the flow has to verify 3 different similarity conditions.

Geometrical similarity

The first and easy one. There exists a unique constant C_x such that space coordinates at analogous points in the model and the prototype are related by a proportionality relation

$$x_{i,m} = C_x x_{i,p} \tag{2.1}$$

This implies that the model and the prototype are **homothetic** $L_m = C_x L_p$. It seems easy but in reality when we have to manufacture something, there are irregularities in molecular level. All machined surfaces are characterized by a certain roughness ϵ. So, the roughness should also be proportional. We have in terms of relative roughness

$$\epsilon_m = C_x \epsilon_p \qquad \Leftrightarrow \qquad \left(\frac{\epsilon}{L}\right)_m = \left(\frac{\epsilon}{L}\right)_p \tag{2.2}$$

We see here that for a model x time smaller than the prototype, the roughness must be x time smaller in order to have the same relative roughness, which is very difficult.

Kinematic similarity

The relation that relates the analogous points of the model and the prototype is (2.1). Similarly, for analogous points in time or analogous time, we have

$$t_m = C_t t_p \tag{2.3}$$

Time is not necesseraly the same. For example, for the humming bird testing, we need to slow down the movement to do measurements leading to more time to accomplish the same

movement as the prototype. The kinematic similarity says that velocities at analogous points and times are related by a unique proportionality constant

$$u_{j,m}(x_{i,m}t_m) = C_u u_{j,p}(x_{i,p}, t_{i,p}) = C_u u_{j,p}\left(\frac{x_{i,m}}{C_x}, \frac{t_m}{C_t}\right) \tag{2.4}$$

Dynamic similarity

Forces per unit volue of area at analogous points and times should also be related by a unique proportionality constant. For example if we consider the pressure, we must have

$$p_m = C_p p_p \left(\frac{x_{i,m}}{C_x}, \frac{t_m}{C_t}\right) \tag{2.5}$$

If the dimensional similarity is easy to check, it's not the case for the txo others for which we must study the equaitions of motion.

2.2 Non-dimensional form of the governing equations - Similarity conditions

We deduces from the previous similarities that at analogous points

$$\left.\begin{array}{r}x_{i,m} = C_x x_{i,p}\\L_m = C_x L_p\end{array}\right\} \qquad \Rightarrow \frac{x_{i,m}}{L_m} = \frac{x_{i,p}}{L_p} \tag{2.6}$$

At this stage we can define non-dimensional coordinates and times like

$$\tilde{x}_i = \frac{x_i}{L} \quad \Rightarrow \tilde{x}_{i,m} = \tilde{x}_{i,p} \qquad and \qquad \tilde{t} = \frac{t}{T} \quad \Rightarrow \tilde{t}_m = \tilde{t}_p \tag{2.7}$$

Analogous points are caracterized by the fact that they have the same non-dimensional coordinates and time value. We have to define a non-dimensional velocity like

$$\tilde{u}_j = \frac{u_j}{U} \qquad \Rightarrow \tilde{u}_{j,m} = \tilde{u}_{j,p} \tag{2.8}$$

For a kinematically similar flows, the non-dimensional velocity fields should be the same for the model and the prototype. How can we verify these idenity? This can only be achieved if the non-dimensional equations are the same for the model and the prototype.

2.2.1 Continuity equation

The dimensional and index form of the continuity equation were

$$\frac{\partial \rho}{\partial t} + \nabla \rho u = 0 = \frac{\partial \rho}{\partial t} + \frac{\partial \rho u_j}{\partial x_j} \tag{2.9}$$

We have to introduce the following independent non-dimensional variables

$$\tilde{t} = \frac{t}{T} \qquad \tilde{x}_j = \frac{x_j}{L} \qquad \tilde{u}_j = \frac{u_j}{U} \qquad \tilde{\rho} = \frac{\rho}{\rho_0} \tag{2.10}$$

And by replacing the variables in general equation and dividing by convective term coefficient, we have

$$\frac{\rho_0}{T}\frac{\partial \tilde{\rho}}{\partial \tilde{t}} + \frac{\rho_0 U}{L}\frac{\partial \tilde{\rho}\tilde{u}_j}{\tilde{x}_j} = 0 \qquad \Leftrightarrow \qquad \frac{L}{UT}\frac{\partial \tilde{\rho}}{\partial \tilde{t}} + \frac{\partial \tilde{\rho}\tilde{u}_j}{\partial \tilde{x}_j} = 0 \tag{2.11}$$

There appears a non-dimensional number that we define as **Strouhal number** $St = \frac{L}{UT}$. L is the caracteristic lenght of the body and UT the lenght travelled in a caracteristic time scale. So St is the ratio of the two length.

Non-dimensional continuity equation

$$St\frac{\partial \tilde{\rho}}{\partial \tilde{t}} + \frac{\partial \tilde{\rho}\tilde{u}_j}{\partial \tilde{x}_j} = 0 \qquad (2.12)$$

This equation is the same for the model and the prototype. For the solution to be the same, Strouhal numbers must be the same $St_m = St_p$. It is not a very strict condition to verify because the caracteristic time can be chosen the way to verify the identity.

2.2.2 Momentum equation

The procedure is exactly the same. The divergence form was (considering first the gravitation force)

$$\frac{\partial \rho u_i}{\partial t} + \frac{\partial \rho u_i u_j}{\partial x_j} = \rho g \alpha_i - \frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ji}}{\partial x_j} \qquad (2.13)$$

where α_i is the orientation of the gravity vector. Here we have to express the viscous stress tensor using **Stokes hypothesis** ($\mu_V = 0$)

$$\tau_{ji} = 2\mu S_{ij}^S = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\frac{\partial u_k}{\partial x_k} \right) \qquad (2.14)$$

There we only have to introduce a non-dimensional viscosity $\tilde{\mu} = \frac{\mu}{\mu_0}$. We have so

$$\tau_{ji} = \frac{\mu_0 U}{L}\tilde{\mu} \left(\frac{\partial \tilde{u}_i}{\partial \tilde{x}_j} + \frac{\partial \tilde{u}_j}{\partial \tilde{x}_i} - \frac{2}{3}\delta_{ij}\frac{\partial \tilde{u}_k}{\partial \tilde{x}_k} \right) = \frac{\mu_0 U}{L}\tilde{\tau}_{ji} \qquad (2.15)$$

We have also to introduce a non-dimensional pressure but if we rewrite left side of (2.13) like

$$\rho \left[\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j} \right] = \rho g \alpha_i - \frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ji}}{\partial x_j} \qquad (2.16)$$

where p only appears in differentiated form while ρ and u appear also in non-differentiated form. This implies that we define a relative non-dimensional pressure

$$\tilde{p} = \frac{p - p_0}{\Delta p} \qquad (2.17)$$

where p_0 is a reference pressure and Δp a caracteristic pressure variation scale. So we're done, we can replace the variables

$$\frac{\rho_0 U}{T}\frac{\partial \tilde{\rho}\tilde{u}_i}{\partial \tilde{t}} + \frac{\rho_0 U^2}{L}\frac{\partial \tilde{\rho}\tilde{u}_i\tilde{u}_j}{\partial \tilde{x}_j} = \rho_0 \tilde{\rho} g \alpha_i - \frac{\Delta p}{L}\frac{\partial \tilde{p}}{\partial \tilde{x}_i} + \frac{\mu_0 U}{L^2}\frac{\partial \tilde{\tau}_{ji}}{\partial \tilde{x}_j} \qquad (2.18)$$

Dividing by the convective term coefficient $\frac{\rho_0 U^2}{L}$, we have

$$\underbrace{\frac{L}{UT}}_{St}\frac{\partial \tilde{\rho}\tilde{u}_i}{\partial \tilde{t}} + \frac{\partial \tilde{\rho}\tilde{u}_i\tilde{u}_j}{\partial \tilde{x}_j} = \underbrace{\frac{gL}{U^2}}_{\frac{1}{Fr^2}}\tilde{\rho}\alpha_i - \underbrace{\frac{\Delta p}{\rho_0 U^2}}_{Eu}\frac{\partial \tilde{p}}{\partial \tilde{x}_i} + \underbrace{\frac{\mu_0}{\rho_0 LU}}_{\frac{1}{Re_L}}\frac{\partial \tilde{\tau}_{ji}}{\partial \tilde{x}_j} \qquad (2.19)$$

where $Fr = \frac{U}{\sqrt{gL}}$ is the **Froude number**, the ratio of the caracteristic velocity with another wich is the velocity of propagation of waves on shallow water (in a pond for example). There is also the **Euler number** $Eu = \frac{\Delta p}{\rho_0 U^2}$.

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Non-dimensional momentum equation

$$St\frac{\partial \tilde{\rho}\tilde{u}_i}{\partial \tilde{t}} + \frac{\partial \tilde{\rho}\tilde{u}_i\tilde{u}_j}{\partial \tilde{x}_j} = \frac{1}{Fr^2}\tilde{\rho}\alpha_i - Eu\frac{\partial \tilde{p}}{\partial \tilde{x}_i} + \frac{1}{Re_L}\frac{\partial \tilde{\tau}_{ji}}{\partial \tilde{x}_j}$$

(2.20)

Therefor, for the similarity we need to have additionally

$$Fr_m = Fr_m \qquad Eu_m = Eu_p \qquad Re_m = Re_p \tag{2.21}$$

2.2.3 Energy equation

We will go from the total energy equation wich was

$$\frac{\partial \rho E}{\partial t} + \frac{\partial \rho E u_j}{\partial x_j} = \rho g \alpha_i u_i - \frac{\partial p u_j}{\partial x_j} + \frac{\partial \tau_{ji}u_i}{\partial x_j} - \frac{\partial q_i}{\partial x_i}$$

$$\Leftrightarrow \quad \frac{\partial \rho E}{\partial t} + \frac{\partial (\rho E + p)u_j}{\partial x_j} = \rho g \alpha_i u_i + \frac{\partial \tau_{ji}u_i}{\partial x_j} - \frac{\partial q_i}{\partial x_i}$$

(2.22)

We remember that

$$\rho E + p = \rho(e + k) + p = \rho(e + pv) + \rho k = \rho(h + k) = \rho H \tag{2.23}$$

where H is defined as the total enthalpy per unit mass. If we replace $E = H - p$ and the others

$$\frac{\partial \rho H}{\partial t} + \frac{\partial \rho H u_j}{\partial x_j} - \frac{\partial p}{\partial t} = \rho g \alpha_i u_i + \frac{\partial \tau_{ji}u_i}{\partial x_j} - \frac{\partial q_i}{\partial x_i} \tag{2.24}$$

We have to introduce 2 non-dimensional variables, one for h and the other for q. Again, the enthalpy appears only in derivated form, so we introduce

$$\tilde{h} = \frac{h - h_0}{\Delta h} \quad and \quad k = \frac{u_i u_j}{2} = U^2 \underbrace{\frac{\tilde{u}_i\tilde{u}_j}{2}}_{\tilde{k}} \qquad \Rightarrow \begin{cases} \partial H = \Delta h\partial \tilde{h} + U^2\partial \tilde{k} \\ \\ = \Delta h\left(\partial \tilde{h} + \frac{U^2}{\Delta h}\partial \tilde{k}\right) \end{cases} \tag{2.25}$$

Where the coefficient $Ec = \frac{U^2}{\Delta h}$ is the **Eckert number**. Let's attack the heat flux and remind that $Pr = \frac{\mu c_p}{\kappa}$

$$q_i = -\kappa\frac{\partial T}{\partial x_i} = -\frac{\kappa}{c_p}\frac{c_p\partial T}{\partial x_i} = -\frac{\mu}{Pr}\frac{\partial h}{\partial x_i} = \frac{\mu_0\Delta h}{PrL}\underbrace{-\left(\tilde{\mu}\frac{\partial \tilde{h}}{\partial \tilde{x}_i}\right)}_{\tilde{q}_i} \tag{2.26}$$

We are ready to write the non dimensional form of (2.24)

$$\frac{\rho_0\Delta h}{T}\tilde{\rho}\left[\frac{\partial \tilde{h} + Ec\partial \tilde{k}}{\partial \tilde{t}} + \frac{UT}{L}\tilde{u}_j\frac{\partial \tilde{h} + Ec\partial \tilde{k}}{\partial \tilde{x}_j}\right] - \frac{\Delta p}{T}\frac{\partial \tilde{p}}{\partial \tilde{t}} = \rho_0 g U\alpha_i\tilde{\rho}\tilde{u}_i + \frac{\mu_0 U^2}{L^2}\frac{\partial \tilde{\tau}_{ji}\tilde{u}_i}{\partial \tilde{x}_j} - \frac{\mu_0\Delta h}{PrL^2}\frac{\partial \tilde{q}_i}{\partial \tilde{x}_i}$$

(2.27)

Again, if we divide by the coefficient of the convective term $\frac{\rho_0\Delta h}{T}\frac{UT}{L}$

$$\underbrace{\frac{L}{UT}}_{St}\tilde{\rho}\frac{\partial \tilde{h} + Ec\partial \tilde{k}}{\partial \tilde{t}} + \tilde{\rho}\tilde{u}_j\frac{\partial \tilde{h} + Ec\partial \tilde{k}}{\partial \tilde{x}_j} - \underbrace{\frac{L}{UT}\frac{\Delta p}{\rho_0 U^2}\frac{U^2}{\Delta h}}_{StEuEc}\frac{\partial \tilde{p}}{\partial \tilde{t}}$$

$$= \underbrace{\frac{glU^2}{U^2\Delta h}}_{\frac{Ec}{Fr^2}}\alpha_i\tilde{\rho}\tilde{u}_i + \underbrace{\frac{\mu_0 U^2}{L\rho_0\Delta hU}}_{\frac{Ec}{Re_L}}\frac{\partial \tilde{\tau}_{ji}\tilde{u}_i}{\partial \tilde{x}_j} - \underbrace{\frac{\mu_0}{PrL\rho_0 U}}_{\frac{1}{PrRe_L}}\frac{\partial \tilde{q}_i}{\partial \tilde{x}_i}$$

(2.28)

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Non-dimensional form of the energy equation

$$St\tilde{\rho}\frac{\partial \tilde{h} + Ec\partial \tilde{k}}{\partial \tilde{t}} + \tilde{\rho}\tilde{u}_j\frac{\partial \tilde{h} + Ec\partial \tilde{k}}{\partial \tilde{x}_j} - StEuEc\frac{\partial \tilde{p}}{\partial \tilde{t}} = \frac{Ec}{Fr^2}\alpha_i\tilde{\rho}\tilde{u}_i + \frac{Ec}{Re_L}\frac{\partial \tilde{\tau}_{ji}\tilde{u}_i}{\partial \tilde{x}_j} - \frac{1}{PrRe_L}\frac{\partial \tilde{q}_i}{\partial \tilde{x}_i}$$

$$(2.29)$$

At this stage what we have a last condition which is $Ec_m = Ec_p$. For the tests over the scaled model to be representative of the actual flow over the prototype, all these dimensionless parameters should be the same for the experiment and for the true configuration. This is called complete similarity. This is very difficult, impossible to achieve. It is why we have to study the relax we can give to the parameters. What is the interpretation we can give to the dimensionless numbers? For those who appear in the momentum equation, ρg is a force per unit volume, so all terms in the dimentional momentum equation are force density. And these dimensionless numbers represents the ratio of the divers force densities. For example, St number represents the relative magnitude of the inertial force density to the convective force density. We can also give other interpretations like previously with the definition of the numbers.

2.2.4 Partial similarity

We will not consider Prandtl number because is relatively constant for most fluids.

Strouhal number

For flows where there are intrinsic time scales, for example the flapping bird has a certain period of flapping. This is imposed by the problem. For flows there is no imposed period, for example steady flows. For those we can choose

$$T = \frac{L}{U} \qquad \Rightarrow St = 1 \tag{2.30}$$

for both model and prototype. The fact that there is no caracteristic time scale doesn't mean that the flow is steady. The example of that is the flow at low speed around a cylinder of diameter D. When $Re > 40$, the flow becomes unsteady naturally. Even though the cylinder is fiwed, even the velocity of the fluid constant in time, the flow develops natural unsteadiness by its own. In fact, we have a shedding of vortices alternatively on the upper and lower side (Karman vortex street - Aelion times). The oscillation takes place at a very specific frequency and the non-dimensional period is

$$\tilde{T}_{osc} = \frac{T_{osc}}{T} = \frac{T_{osc}U}{L} \tag{2.31}$$

and is the same for both model and prototype. We have also $St_{osc} = \frac{f_{osc}L}{U} \Rightarrow St_{osc} = f(Re_L)$ that is function of the other parameters of the oscillation like the Re number.

Euler number

We know that the governing equations must be supplemented by some additional equations. We never spoke about the thermodynamic equations of states that must be the same for the model and prototype. We will assume a flow of thermally and calorically perfect gas

$$\left.\begin{array}{l} p = \rho RT \\ h = c_p T = \dfrac{\gamma}{\gamma - 1}RT \end{array}\right\} \quad \Leftrightarrow \quad h = \frac{\gamma}{\gamma - 1}\frac{p}{\rho} \quad \Leftrightarrow \quad \rho = \frac{\gamma}{\gamma - 1}\frac{p}{h} \tag{2.32}$$

By introducing non-dimentional variables

$$\rho_0 \tilde{\rho} = \frac{\gamma}{\gamma - 1} \frac{p_0 + \Delta p \tilde{p}}{h_0 + \Delta h \tilde{h}} \quad \Leftrightarrow \quad \rho_0 \tilde{\rho} = \frac{\gamma}{\gamma - 1} \frac{p_0 \left(1 + \frac{\Delta p \tilde{p}}{p_0}\right)}{h_0 \left(1 + \frac{\Delta h \tilde{h}}{h_0}\right)} \quad \Leftrightarrow \quad \tilde{\rho} = \frac{1 + \frac{\Delta p \tilde{p}}{p_0}}{1 + \frac{\Delta h \tilde{h}}{h_0}} \tag{2.33}$$

In the other hand, if we remind that $a^2 = \gamma R T = \gamma \frac{p}{\rho}$, we have

$$\frac{\Delta p}{p_0} = \frac{\Delta p}{\rho_0 U^2} \frac{\rho_0 U^2}{p_0} = \underbrace{\frac{\Delta p}{\rho_0 U^2}}_{Eu} \gamma \underbrace{\frac{U^2}{a^2}}_{Ma^2} \tag{2.34}$$

And if we replace in previous relation

$$\tilde{\rho} = \frac{1 + \gamma Eu Ma^2 \tilde{p}}{1 + \frac{\Delta h \tilde{h}}{h_0}} \tag{2.35}$$

There is no imposed scale for pressure, so if we choose $\Delta p = \rho_0 U^2 \Rightarrow Eu = 1$, we don't care about. In that case

$$\tilde{\rho} = \frac{1 + \gamma Ma^2 \tilde{p}}{1 + \frac{\Delta h \tilde{h}}{h_0}} \tag{2.36}$$

So here we have to satisfy the Mach number similarity, we didn't gain anything, we can replace Euler number similarity by Mach number similarity. We see that if Ma is negligible, the term is small compared to 1 and so the Ma number disappears. So we will not have to take into account that similarity in the last case

$$\tilde{\rho} = \frac{1}{1 + \frac{\Delta h \tilde{h}}{h_0}} \tag{2.37}$$

Froude number

We're going to speak about the governing equation. In fact, for flows in which there is no free surface (for example a pipe where the fluid is contained), we can integrate the gravity term with the pressure term (hydrostatic pressure) in momentum equation. The reason this is not possible in free surface cases is because the pressure does not vary on the surface. Consider a flow of a liquid of a gas without free surface. The corresponding terms for pressure gradient and gravity are respectively

$$-\frac{\partial p}{\partial x_i} + \rho g \alpha_i \quad \Rightarrow \quad -\frac{\partial}{\partial x_i}(p + \rho g \alpha_i x_i) \tag{2.38}$$

We will define $\delta p = p - (p_0 + \rho_0 g \alpha_i x_i)$ where appears the hydrostatic pressure field, due to the fact that hydrostatic pressure force in (2.38) can be expressed by $-\rho \frac{\partial}{\partial x_i}(g \alpha_i x_i)$, derivative of the potential energy. It comes that

$$\frac{\partial \delta p}{\partial x_i} = \frac{\partial p}{\partial x_i} - \frac{\partial}{\partial x_i}(\rho_0 g \alpha_i x_i) \tag{2.39}$$

Making δp appear in (2.38), we have

$$-\frac{\partial}{\partial x_i}\underbrace{(p - p_0 - \rho_0 g \alpha_i x_i)}_{\delta p} + (\rho - \rho_0)\frac{\partial}{\partial x_i} g \alpha_i x_i = -\frac{\partial \delta p}{\partial x_i} + \underbrace{(\rho - \rho_0)}_{\delta \rho} g \alpha_i \tag{2.40}$$

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The corresponding terms in non-dimensional equation are

$$- Eu\frac{\partial \delta\tilde{p}}{\partial \tilde{x}_i} + \frac{1}{Fr^2} \underbrace{(\tilde{\rho} - 1)}_{\delta\tilde{\rho}} \alpha_i \tag{2.41}$$

Let's consider a low Mach number in order not to worry about the Mach number similarity ($p = \rho_0 U^2 \leftarrow Eu = 1$). Let's look to the contribution of gravity with Froude number. Using (2.37), we know that

$$\tilde{\rho} - 1 = \delta\tilde{\rho} = \frac{1}{1 + \frac{\Delta h \tilde{h}}{h_0}} - 1 \approx -\frac{\Delta h \tilde{h}}{h_0} \tag{2.42}$$

and we see that for small temperature variations in the flow, it means that the density variations are small and the term with Fr number in (2.41) can be neglected. The gravity has no influence on the flow. And so we don't have to worry about Froude number when temperature variations are small. Liquids have a thermal expansion coefficient making them sensible to the temperature variation (natural convection can take place).

As there is no intrinsic velocity scale for natural convection flows, we will see if we can make the same trick considering a velocity in order to have $Fr = 1$ and not wonder about the similarity. If we take the Fr term in (2.41), what $\delta\tilde{\rho}$ relative density variation? When there are small density variations we can write this and make a first order expansion

$$\frac{\delta\rho}{\rho_0} = -\beta\Delta T \qquad \Leftrightarrow \qquad \rho = \rho_0(1 - \beta\Delta T) \qquad \Leftrightarrow \qquad -\beta\rho_0 = \left.\frac{\partial \rho}{\partial T}\right|_0 \tag{2.43}$$

where β is the thermal expansion parameter $\beta = -\frac{1}{\rho_0}\left.\frac{\partial \rho}{\partial T}\right|_0$. The minus sign describes the decrease of density with temperature. Fr term in (2.41) becomes

$$\frac{1}{Fr^2}\delta\tilde{\rho}\alpha_i = -\frac{gl}{U^2}\beta\Delta T \tilde{\Delta}t\alpha_i \tag{2.44}$$

where $\tilde{\Delta}t$ is the non-dimensional local temperature variation. For natural convection, we choose $U^2 = gL\beta\Delta T$ in order to not worry about Froude number similarity. If we do that, there is an interresant corollary for Reynolds number

$$Re_L = \frac{UL}{\nu_0} = \sqrt{\frac{U^2 L^2}{\nu_0^2}} = \sqrt{\frac{g\beta\Delta T L^3}{\nu_0^2}} \equiv \sqrt{Gr} \tag{2.45}$$

The Grashoff number is nothing else but the square root of Reynolds number in the case where U is chosen like Fr = 1. The conclusion is that the only case where we have to consider the Froude number is the **flows with free surfaces**.

Eckert number

If we look to energy equation for inviscid flows, we are not worrying about Froude number and are not considering free surfaces, making all the right side of the energy equation disappear. It reduces to

$$\frac{\partial H}{t} + \vec{u}\nabla H - \frac{\partial p}{\partial t} = 0 \tag{2.46}$$

We see that for steady flows flows $H = cst = h + \frac{u^2}{2}$ so the inviscide enthalpy variation scale $\Delta h^{inv} \approx \frac{u^2}{2} = H_0 - h_0$. If solid bodies are heated, then there is another thermal enthalpy

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variation scale $\Delta h^{th} = H_0 - h(T_w)$. What is the appropriate scale? The actual Δh will be the maximum of the 2 and so the Eckert number

$$Ec = \frac{U^2}{\Delta h} = \min\left(\underbrace{\frac{U^2}{\Delta h^{inv}}}_{2}, \frac{U^2}{\Delta h^{th}}\right) \tag{2.47}$$

So Ec is always smaller than 2. There are instances in which $Ec \ll 1$ and in which case the energy equation can be significantly simplified. We can look at the thermal part, reminding that $(\gamma - 1)h = a^2$ (2.32)

$$\frac{U^2}{\Delta h^{th}} = \frac{U^2}{h_0}\frac{h_0}{\Delta h^{th}} = \frac{(\gamma - 1)Ma^2}{\frac{\Delta h^{th}}{h_0}} \tag{2.48}$$

We see that if Ma is smaal, Eckert number is small, unless the denominator is small too. So this is when $Ma^2 \ll \frac{\Delta h^{th}}{h_0}$, energy equation can be simplified. The final condition for similarity, in the equation of state we have the ratio $\Delta h/h_0$ that has to be the same for the model and the prototype. This reduces the condition to

$$\frac{\Delta h^{th}}{h_0} = \frac{H_0}{h_0} - \frac{h_w}{h_0} = 1 + \frac{\gamma - 1}{2}Ma^2 - \frac{h_w}{h_0} \tag{2.49}$$

So the last term must be the same, implying that $\frac{T_w}{T_0}$ must be the same.

Reynolds number

Experimentally, it has been observed that, for many configurations, flow quantities became insensitive to Reynolds number beyond a certain critical value. In this range, we don't have to worry about Reynolds number similarity as long as we are in the insensitivity range.

2.3 Dimensional analysis - Vashy Buckingam π theorem

Assuming that there exists a relationship between n physical variables $f(q_1, q_2, \ldots, q_3) = 0$ involving j physical dimensions, then their exists a relationship between $n - j$ non-dimensional groups $\Pi_k : g(\Pi_1, \Pi_2, \ldots, \Pi_{n-j}) = 0$.

Construction of dimensionless groups : methods of repeating variables

1. Among the n physical variables, pick j that involve all the physical dimensions $[q_n, q_{n-1}, \ldots, q_{n-j+1}]$. These are the repeating variables.
2. Construct

$$\Pi_k = \frac{q_k}{q_{n-j+1}^{\alpha_1} q_{n-j+2}^{\alpha_2} \ldots q_n^{\alpha_j}} \tag{2.50}$$

3. Adjust the exponents $\alpha_1, \alpha_2, \ldots, \alpha_j$ so that Π_j is dimensionless.

Let's take the example of the drag of a sphere (called D). We must first assume what are the involving variables in the drag. It depends on $D = f(d, U, \mu, \rho)$. We have 5 physical quantities here and 3 physical dimensions, so there is 2 dimensionless groups. The repeating variables will be d, U, ρ. The first group is

$$\Pi_1 = \frac{D}{d^{\alpha_1} U^{\alpha_2} \rho^{\alpha_1}} = \left[\frac{MLT^{-2}}{L^{\alpha_1}(LT^{-1})^{\alpha_2}(ML^{-3})^{\alpha_3}}\right] \Rightarrow \begin{cases} M : 1 - \alpha_3 = 0 \Rightarrow \alpha_3 = 1 \\ L : 1 - \alpha_1 - \alpha_2 + 3\alpha_3 = 0 \Rightarrow \alpha_1 = 2 \\ T : -2 + \alpha_2 = 0 \Rightarrow \alpha_2 = 2 \end{cases} \tag{2.51}$$

Finally, $\Pi_1 = \frac{D}{\rho U^2 d^2}$. When the same is applied for μ $[ML^{-1}T^{-1}]$, we obtain $\Pi_2 = \frac{\mu}{\rho U d} = \frac{1}{Re_d}$. We have so that $\Pi_1 = f(Re_D)$ but rather than using Π_1, we have a similar non-dimensional number (we only make appear other non-dimensional numbers) which is the drag coefficient $C_D = \frac{D}{\frac{1}{2}\rho U^2 S^2}$. We conclude that $C_D = f(Re_D)$ and make vary this two variables only.

Conclusive example : ship hull resistance (Froude)

In order to have similarity between 2 flows, we must have the same non-dimentional numbers. The example is clearly a free surface flow so the numbers are :
- **Strouhal number** : for a steady flow we take $T = L/U$, which make the similarity satisfied.
- **Euler number** : we are working with liquids so there is no density variation and we don't care about this number. Low speed flow $\Delta p = \rho U^2$.
- **Froude number**
- **Reynolds number** : Re number is hugh because dimensions of the ship are hugh, so in general we are in the range of insensitivity of Re number.
- **Eckert number** : we are not concerned about thermal effects so it doesn't matter.

1. Symilarity analysis

We see that the only similarity of the problem to be respected is Froude number similarity

$$Fr_m = Fr_p \qquad \Leftrightarrow \qquad \frac{U_m}{\sqrt{gL_m}} = \frac{U_p}{\sqrt{gL_p}} \qquad \Leftrightarrow \qquad U_m = \sqrt{\frac{L_m}{L_p}}U_p \qquad (2.52)$$

What about the resistance? We know that since the fluids are the same, the pressure fields relation is

$$p_m = C_p p_p \qquad \Leftrightarrow \qquad \rho U_m^2 = C_p \rho U_p^2 \qquad \Leftrightarrow \qquad C_p = \left(\frac{U_m}{U_p}\right)^2 \qquad (2.53)$$

As the drag force is proportional to the pressure $D_m \propto S_m p_m$, using (2.52) we conclude that

$$D_p = \left(\frac{L_m}{L_p}\right)^2 \left(\frac{U_m}{U_p}\right)^2 D_p = \left(\frac{L_m}{L_p}\right)^3 D_p \qquad (2.54)$$

2. Dimensional analysis

If we don't make a reference to the equations of motion. We say that $D = f(\rho, U, L, g)$ which gives 5 quantities and 3 dimensions. And we found the same conclusions

$$\frac{D}{\rho U^2 L^2} = \varphi\left(\frac{U}{\sqrt{gL}}\right) \qquad \Rightarrow \frac{U_m}{U_p} = \sqrt{\frac{L_m}{L_p}} \qquad \Rightarrow \frac{D_m}{D_p} = \left(\frac{U_m L_m}{U_p L_p}\right)^2 = \left(\frac{L_m}{L_p}\right)^3 \qquad (2.55)$$

Chapter 3

Inviscid incompressible potential flows

3.1 Introduction

3.1.1 Governing equations

Let's remind that we have a set of 4 equations and 4 unknowns. It's inviscid so there are no stresses. This is decoupled from the energy equation, for constant density flows, the hydro-dynamic problem gets decoupled from the thermal problem. So when we speak about compressible fluid we have to couple them.

$$\rho = cst \qquad \nabla \vec{u} = 0 \qquad \rho \left[\frac{\partial \vec{u}}{\partial t} + \vec{u} \nabla \vec{u} \right] = -\nabla p + \rho \vec{F} \tag{3.1}$$

3.1.2 Bernouilli equation

The flow is barotropic, in additoin let's consider that the flow is steady and that the force derives from a potential $\vec{F} = -\nabla \Phi$ (F = 0 for most applications). We have the Bernouilli equation

$$\epsilon_m = \frac{p}{\rho} + k + \cancel{\Phi} = cst = \frac{p}{\rho} + \frac{u^2}{2} \tag{3.2}$$

The mechanical energy is constant on a streamline. We can give an interpretation to the constant by calling it $\frac{p_t}{\rho} = \frac{p^0}{\rho}$. Physically, p^0 is the pressure where u = 0 and is called **stagnation pressure**. The constant differs from constant to constant. Indeed we found that $\vec{\omega} \times \vec{u} = -\nabla \epsilon_m$, so if the fluid is rotational ϵ_m will differ from streamline to another, meaning that the stagnation pressure differs.

3.1.3 Irrotational (potential) flow

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