MODULE 7

Mathematical Modeling

You will learn about the 'Mathematical Modeling' in this module.

Module Learning Objectives

At the end of this module, you will be able to:

- Understand mathematical modeling.
- Explain the reason for developing mathematical models along with applications.
- Describe the principles of mathematical modeling.
- Enumerate the different stages in developing a mathematical model.
- Enumerate the different ways to classify mathematical models.
- Describe the methodology for conceptualizing a mathematical model.
- Explain the concept of boundary conditions.

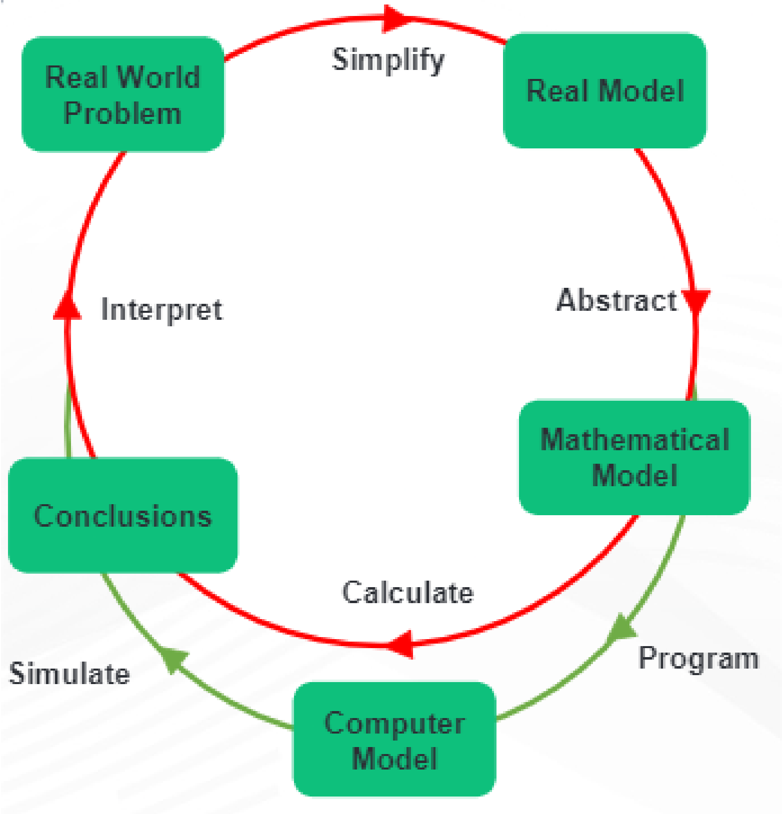
Module Topics

The following topics that will be covered in the module:

1. Introduction to mathematical modeling
2. Applications of mathematical modeling
3. Principles and stages involved in developing a mathematical model
4. Classification of mathematical modeling
5. Conceptualizing a mathematical model
6. Concept of boundary conditions

1. Introduction to Mathematical Modeling

- Mathematical model is a mathematical relation that describes a real-life situation.
- For a problem, a mathematical model is created and the model is used to analyse and derive the solution.
- The solution is then interpreted to the real-life problem being studied.
- Mathematical modeling is thus defined as the "conversion of physical situation into mathematics with some suitable conditions."



Machine Learning theory is a field that intersects statistical, probabilistic, computer science and algorithmic aspects arising from learning iteratively from data and finding hidden insights which can be used to build intelligent applications. Despite the immense possibilities of Machine and Deep Learning, a thorough mathematical understanding of many of these techniques is necessary for a good grasp of the inner workings of the algorithms and getting good results.

In mathematical modeling, we take a real-world problem and write it as an equivalent mathematical problem. We then solve the mathematical problem and interpret its solution in terms of the real-world problem. After this, we see to what extent the solution is valid in the context of the real-world problem.

Some of the common definitions of mathematical modeling are:

"A mathematical model is a representation, in mathematical terms, of certain aspects of a non-mathematical system." - Aris, 1999

"A mathematical model is a set of mathematical equations that are intended to capture the effect of certain system variables on certain other system variables." - G. C. Goodwin, et al., 2001

2. Why Mathematical Modeling

Mathematical Modeling:

- An essential tool for understanding the world and the processes around us.
- Provide a better representation of the real-life scenarios, which help in the development of better solutions.
- Can generate accurate mathematical models with the help of powerful computers and computing methods.
- Has countless applications in a number of industries.
- Some examples include:
    - Satellite launch
    - Monsoon prediction
    - Pollution control

Recollect that mathematical modeling is the process of describing a real-life situation mathematically and finding a solution for it so that it can be interpreted to the situation being studied.

Some of the examples where mathematical modeling is of great importance are:

- Finding the width and depth of a river at an unreachable place.
- Estimating the mass of the Earth and other planets.
- Estimating the distance between Earth and any other planet.
- Predicting the arrival of the monsoon in a country.
- Predicting the trend of the stock market.
- Estimating the volume of blood inside the body of a person.
- Predicting the population of a city after 10 years.
- Estimating the number of leaves in a tree.
- Estimating the ppm of different pollutants in the atmosphere of a city.
- Estimating the effect of pollutants on the environment.
- Estimating the temperature on the Sun's surface.

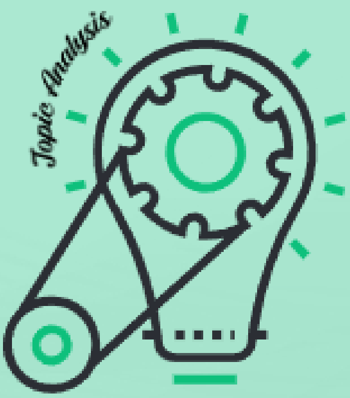
3. Principles of Mathematical Modeling

Following are the principles of Mathematical Modeling:

1. Identify the need for the model. (for what we are looking for)
2. List the parameters/variables which are required for the model.
3. Identify the available relevant data. (what is given?)
4. Identify the circumstances that can be applied. (assumptions)
5. Identify the governing physical principles.
6. Identify:
   a. the equations that will be used.
   b. the calculations that will be made.
   c. the solution which will follow.
7. Identify tests that can check the
   a. consistency of the model.
   b. utility of the model.
8. Identify the parameter values that can improve the model.

Mathematical modeling has a few principles behind it, which are listed above. Based on these principles, there are five stages in mathematical modeling, which we will see in the next section.

What did You Grasp?



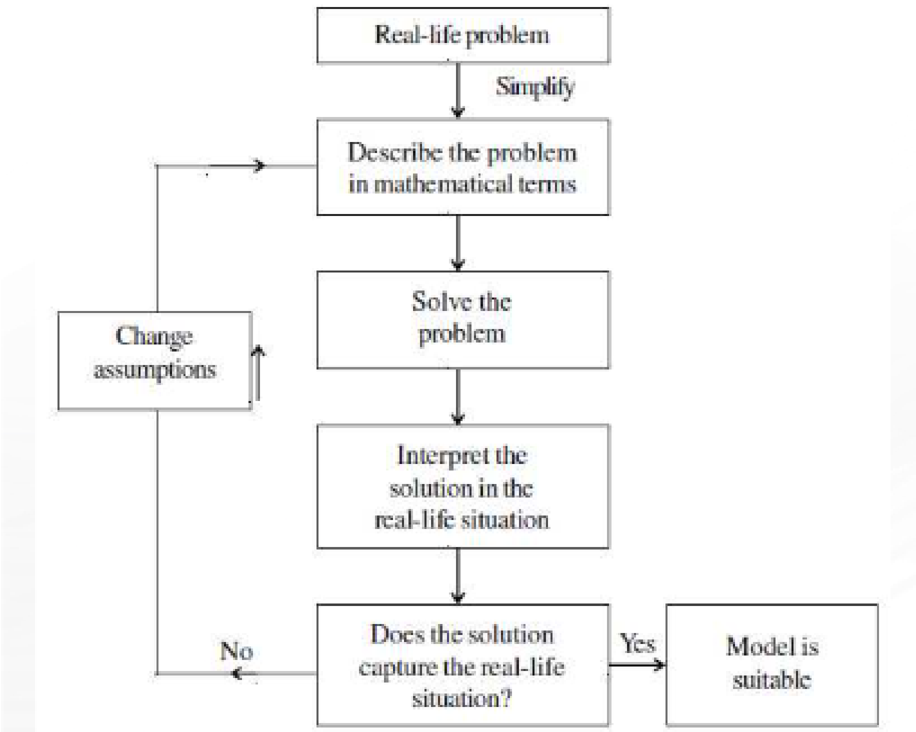
1. Fill in the blanks.

   While developing a mathematical model, tests are needed to check the ___ and ___ of the model.

   A) Performance, Utility
   B) Consistency, Utility
   C) Efficiency, Robustness
   D) Parameters, Consistency

4. Stages in Mathematical Modeling

The stages in mathematical modeling are depicted in the illustration.



There are five important stages in developing a mathematical model as given in the above illustration.

1. **Understanding the problem:** The primary step is to understand and define the real problem. If you're working in a team, discuss the problem and the points that need to be understood clearly.
2. **Mathematical description and formulation:** To formulate, it is necessary to describe the different aspects of the problem mathematically. The following ways are used to describe the problem mathematically:
    - Define variables
    - Write equations or inequalities
    - Gather data and organise into tables
    - Make graphs
    - Calculate probabilities
3. **Solving the mathematical problem:** By means of mathematical description, the problem in hand is simplified. It is then solved using various mathematical techniques.
4. **Interpreting the solution:** The solution obtained in the previous stage is now interpreted to the real-life situation that is being studied.

5. **Validating the model:** The final and the most important step is model validation. Here we go back to the problem being studied and check if the developed solution works and if it makes sense. If the model works, the model will be used until new information becomes available or the assumptions change.

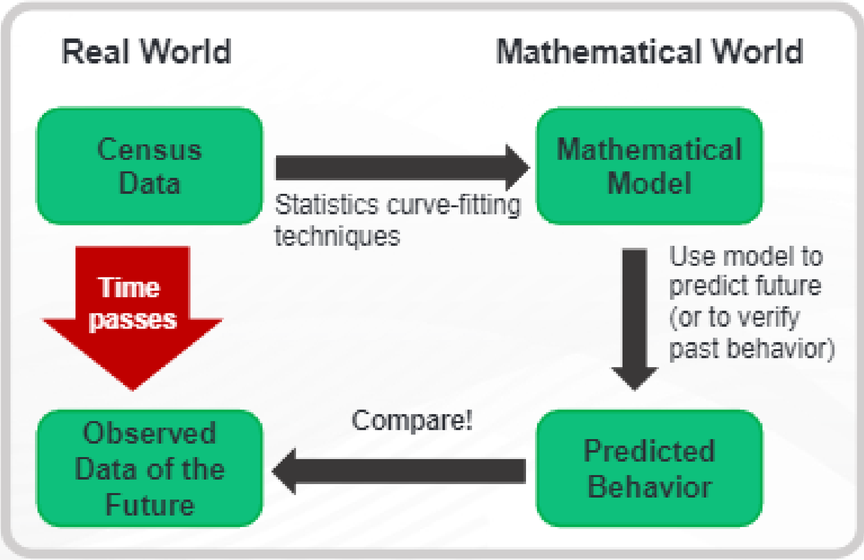
In some cases, because of the simplified assumptions, essential aspects of the real problem may be lost, while giving its mathematical description. In such cases, the solution could very often be off the mark, and may not make sense in the real situation. If this happens, assumptions made in stage 1 need to be reconsidered and revised to be more realistic, possibly by including some factors which were not considered earlier.

4.1 An Example for Mathematical Modeling

Construction of a mathematical model to predict the population of a country after 10 years.

Steps:

1. Formulation
2. Solution
3. Interpretation and Validation



Imagine a population control board of a certain country wants to predict the population of that country after 10 years. Let's look at the steps involved in constructing this model.

Step 1: Formulation

We know that population changes with time, and it increases with births and decreases with deaths. Suppose we want to find the population at a particular time 't', which is time in years. t takes the value of 1, 2, 3, … n. If t = 0 denotes the current year, t = 1, will be the next year. For any time t, let p(t) denote the population in that particular year.

If we want to find the population of the country in a particular year, say, $t_0 = 2018$. Add the number of births and subtract the number of deaths in that year. If B(t) represents the number of births in one year between t and t + 1 and D(t) represents the number of deaths between t and t + 1. Then, we get the relation,

$P(t + 1) = P(t) + B(t) - D(t)$ (Equation 1)

Let's make some assumptions and definitions now.

1. For the time interval t to t + 1, birth rate would be $B(t)/P(t)$.

2. For the time interval t to t + 1, death rate would be $D(t)/P(t)$.

Assumptions:

1. The birth rate and the death rates are the same for all intervals. So, there are two constants now, b and d, the birth and death rates, respectively, for all t >/= 0.

2. There is no migration into or out of the population, so the sources of change in population could be birth or death only. Based on assumptions 1 and 2, we deduce that, for t >/= 0,

$P(t + 1) = P(t) + B(t) - D(t)$

$= P(t) + bP(t) - dP(t)$

$= (1 + b - d) P(t)$ (Equation 2)

Setting t = 0 in equation 2 gives,

$P(1) = (1 + b - d) P(0)$ (Equation 3)

Setting t = 1 in equation 2 gives,

$P(2) = (1 + b - d) P(1)$

$= (1 + b - d) (1 + b - d) P(0)$ (Using equation 3)

$= (1 + b - d)^2 P(0)$

Continuing this way, we get

$P(t) = (1 + b - d)^t P(0)$, for t = 0, 1, 2, …. (Equation 4)

The constant 1 + b - d is often abbreviated by r and called the growth rate, which is also known as Malthusian parameter. In terms of r, the equation 4 becomes

$P(t) = P(0) r^t$, for t = 0, 1, 2…. (Equation 5).

P(t) is an example of an exponential function. Any function of the form cr^t, where c and r are constants, is an exponential function. Equation 5 gives the mathematical formulation of the problem.

Step 2: Solution

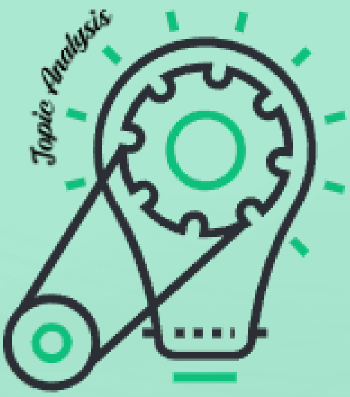
Suppose the current population is 250,000,000 and the rates are b = 0.02 and d = 0.01. What will the population be in 10 years? Using the formula, we calculate P(10).

$$P(10) = (1.01)^{10}\,(250{,}000{,}000)$$
$$= (1.104622125)\,(250{,}000{,}000)$$
$$= 276{,}155{,}531.25$$

Step 3: Interpretation and Validation

The result we arrived in cannot be valid, as there cannot be 0.25 person. So we round it off and conclude that population would be 276,155,531 (approx). Here, we are not getting the exact answer, because of the assumptions that we have in our mathematical model. The above example shows how modeling is done to predict things that can happen a few years later, in a quantitative way.

What did You Grasp?



1. In which of the following stages of mathematical modeling, graphs are generated?
   A) Problem definition
   B) Mathematical description
   C) Solution interpretation
   D) Model validation

5. Classification of Mathematical Modeling

Different categories of classification:

1. Linear and Nonlinear
2. Static and Dynamic
3. Explicit and Implicit
4. Discrete and Continuous
5. Deterministic and Probabilistic (Stochastic)
6. Deductive, Inductive and Floating

There are different ways to classify mathematical models:

1. **Linear and Nonlinear:** The model is said to be linear, if all the operators in it exhibit linearity, otherwise the model is said to be nonlinear. The definition of linearity and nonlinearity is dependent on context, and there is a possibility of linear models containing nonlinear expressions in them. For example, in a statistical linear model, it is assumed that a relationship is linear in the parameters, but it may be nonlinear in the predictor variables. Similarly, a differential equation is said to be linear if it can be written with linear differential operators, but it can still have nonlinear expressions in it. In a mathematical programming model, if the objective functions and constraints are represented entirely by linear equations, then the model is regarded as a linear model. If one or more of the objective functions or constraints are represented with a nonlinear equation, then the model is known as a nonlinear model.

   Nonlinearity is often associated with phenomena such as chaos and irreversibility. Although there are exceptions, nonlinear systems and models tend to be more difficult to study than linear ones. A common approach to nonlinear problems is linearization, but this can be problematic if one is trying to study aspects such as irreversibility, which is strongly tied to nonlinearity.

2. **Static and Dynamic:** A dynamic model accounts for time-dependent changes in the state of the system, while a static (or steady-state) model calculates the system in equilibrium, and thus is time-invariant. Dynamic models typically are represented by differential equations or difference equations.

3. **Explicit and Implicit:** If all of the input parameters of the overall model are known, and the output parameters can be calculated by a finite series of computations, the model is said to be explicit. But sometimes it is the output parameters which are known, and the corresponding inputs must be solved for by an iterative procedure, such as Newton's method (if the model is linear) or Broyden's method (if non-linear). In such a case the model is said to be implicit. For example, a jet engine's physical properties such as turbine and nozzle throat areas can be explicitly calculated given a design thermodynamic cycle (air and fuel flow rates, pressures, and temperatures) at a specific flight condition and power setting, but the engine's operating cycles at other flight conditions and power settings cannot be explicitly calculated from the constant physical properties.

4. **Discrete and continuous:** A discrete model treats objects as discrete, such as the particles in a molecular model or the states in a statistical model; while a continuous model represents the objects in a continuous manner, such as the velocity field of fluid in pipe flows, temperatures and stresses in a solid, and electric field that applies continuously over the entire model due to a point charge.

5. **Deterministic and Probabilistic (stochastic):** A deterministic model is one in which every set of variable states is uniquely determined by parameters in the model and by sets of previous states of these variables. Therefore, a deterministic model always performs the same way for a given set of initial conditions. On the other hand, in a stochastic model randomness is present, and variable states are not described by unique values, but rather by probability distributions.

6. **Deductive, Inductive, and Floating:** A deductive model is a logical structure based on a theory. An inductive model arises from empirical findings and generalization from them. The floating model rests on neither theory nor observation but is merely the invocation of expected structure.

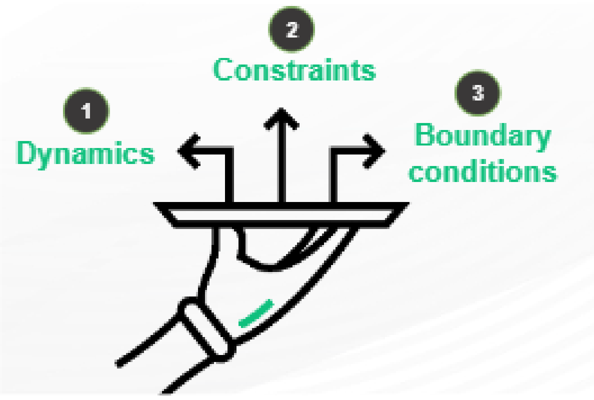
Application of mathematics in social sciences outside of economics has been criticized for unfounded models. Application of catastrophe theory in science has been characterized as a floating model.

What did You Grasp?

1. State True or False.
   A deterministic model varies with the state of the variables.
   A) True
   B) False
2. Fill in the blank.
   The model in which output parameters are known, but the corresponding input parameters need to be calculated, is called _______.
   A) Deterministic
   B) Explicit
   C) Implicit
   D) Deductive

6. Conceptualizing a Mathematical Model

- Building a mathematical model involves observing and understanding the problem, including the key unknowns.
- The phenomenology includes notes, quotes, questions, little pictures, mind maps, fragments of equations, fragments of pseudo-code, made-up graphs, etc.
- Three types of model building blocks, based on the phenomenology:



To conceptualize a mathematical model it is important to understand the problem deeply, including the unknowns. We can then sort out three types of model building blocks:

- **Dynamics:** Dynamics refers to the patterns in change and the laws that govern those changes. It forms the center of any mathematical thought. It is very easy to get insights with respect to the dynamics and the sudden changes will become easily visible. Dynamics can be better explained using things like a swinging pendulum.

- **Constraints:** At the time of constraint creation, destruction, loosening or tightening, the changes are usually harder to notice, and the effects are often delayed or obscured. In case of a swinging pendulum, if the string of the pendulum is suddenly disturbed, the oscillation changes. But, if we notice only the dynamics, we might miss out the cause of it, that is, the introduction of a constraint. If not properly understood, valuable time will be lost in analysing the dynamics, instead of looking at the constraint.
- **Boundary conditions:** Most of the raw, primitive, numerical data in a mathematical modeling problem lives in the description of boundary conditions. The initial kick you might give a pendulum is an example. The fact that the rim of a vibrating drum skin cannot move is a boundary condition. When boundary conditions change, the effects can be extremely weird, and hard to sort out, if you aren't looking at the right boundaries.

7. Boundary Conditions

Boundary Conditions:

- Are a set of constraints. A boundary value problem comprises of a differential equation along with these additional constraints, termed as boundary conditions.
- If the solution to the differential equation is derived, which satisfies these boundary conditions, it serves as the solution of the boundary value problem as well.
- For every input to the problem, there is a unique solution, and it continuously depends on the input.

A boundary value problem has conditions specified at the extremes ("boundaries") of the independent variable in the equation whereas an initial value problem has all of the conditions specified at the same value of the independent variable (and that value is at the lower boundary of the domain, thus the term "initial" value).

For example, if the independent variable is time over the domain [0,1], a boundary value problem would specify values for y(t) at both t=0 and t=1, whereas an initial value problem would specify a value of y(t) and y'(t) at time t=0.

Finding the temperature at all points of an iron bar with one end kept at absolute zero and the other end at the freezing point of water would be a boundary value problem.

If the problem is dependent on both space and time, one could specify the value of the problem at a given point for all time or at a given time for all space.

Concretely, an example of a boundary value (in one spatial dimension) is the problem y”(x)+y(x)=0 to be solved for the unknown function y(x) with the boundary conditions y(0)=0, y(Π/2)=2.

Without the boundary conditions, the general solution to this equation is y(x)=A sin(x)+B cos(x)

From the boundary condition y(0) = 0 we get 0 = A.0 +B.1, which implies that B = 0. From the boundary condition y(Π/2)=2 we get 2 = A.1 and so A = 2.

We see that imposing boundary conditions allows one to determine a unique solution, which in this case is y(x) = 2 sin(x).

What did You Grasp?

1. State True or False.

   Boundary conditions refer to the constraints specified at the upper extreme of the independent variable.

   A) True
   B) False

In a nutshell, we learnt:

1. Introduction to mathematical modeling.
2. The reason for developing mathematical models along with the applications.
3. The principles and the different stages in developing a mathematical model.
4. Different ways to classify mathematical models.
5. How to conceptualize a mathematical model.
6. The concept of boundary conditions.

