MODULE 6

Matrices

You will learn about 'Matrices' in this module.

Module Learning Objectives

At the end of this module, you will be able to:

- Give an Introduction to Matrices along with the basic terminologies.
- Explain Matrix notations and types of matrices.
- Understand Matrix equality.
- Perform operations on matrices, like addition, subtraction and multiplication.
- Describe determinants.
- Understand the singularity of a Matrix.
- Explain orthogonal Matrix.
- Understand elementary transformations and elementary matrices and Matrix. inverse computation using elementary transformations.
- Describe echelon forms and echelon transformations.
- Find Matrix rank and reduce a Matrix to its normal form.
- Explain Vector Spaces and enumerate the axioms.
- Describe linear dependence and independence of vectors.
- Understand the consistency of linear system of equations.
- Describe eigenvalues and eigenvectors.
- State and prove Cayley Hamilton Theorem.

- Explain linear transformation and orthogonal transformation.
- Explain Matrix factorization and enumerate a few types such as LU, QR and SVD decomposition.

Module Topics

The following topics that will be covered in the module:

1. Introduction to Matrices
2. Matrix Notations and Types
3. Matrix Equality
4. Operations on Matrices
5. Determinants
6. Singularity of a Matrix
7. Orthogonal Matrix
8. Elementary Transformations and elementary matrices
9. Echelon forms and echelon transformations
10. Matrix Rank and Normal Form of a matrix
11. Vector Spaces and the axioms
12. Linear Dependence and Independence of vectors
13. Consistency of linear system of equations
14. Eigenvalues and eigenvectors
15. Cayley Hamilton Theorem
16. Linear Transformation and Orthogonal transformation
17. Matrix Factorization and Types

1. Introduction to Matrices

1.1 Introduction to Matrices

Matrix:

- A collection of numbers arranged into a fixed number of rows and columns.
- Usually the numbers are real numbers, but a Matrix can also contain complex numbers.

- A rectangular array of elements, where horizontal arrangements are called rows and vertical arrangements are columns.
- Order or dimension of a Matrix is defined as: Number of rows X Number of columns
- In the example, Matrix A has 2 rows and 3 columns, and the order of the Matrix is 2 X 3.

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{pmatrix}$$

A Matrix is a rectangular array of numbers arranged in rows and columns. The order of a matrix also called the dimension of a Matrix refers to the number of rows and columns in a matrix. While specifying a matrix, the general way is to list the rows first and the columns next. Thus the order of a Matrix is represented as Number of rows X Number of columns.

The numbers present in the rows and columns of a Matrix are called elements of the matrix. In the example above, the element in the first column of the first row is 1; the element in the second column of the first row is 2, etc.

1.2 Matrix Notation

- Matrices are identified by numbers and symbols.
- Matrices are represented by boldface letters.
- Matrix $\mathbf{A}_{ij}$ refers to an i X j Matrix ('i' rows and 'j' columns)
- Matrix elements take the notation $\mathbf{A}_{ij}$, where 'i' is the row number and 'j' is the column number.

$$\mathbf{A} = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \end{bmatrix}$$

- Boldface, capital letters are used to represent matrices, italic capital letters refer to Matrix elements, and subscripts reveal Matrix dimension. For example,
- **A** and **X** refer to matrices A and X, respectively.
- A_{ij} refers to the element in row i and column j of Matrix **A**.
- $\mathbf{A}_{ij}$ refers to an i x j Matrix **A**.

Notations for special matrices are listed below:

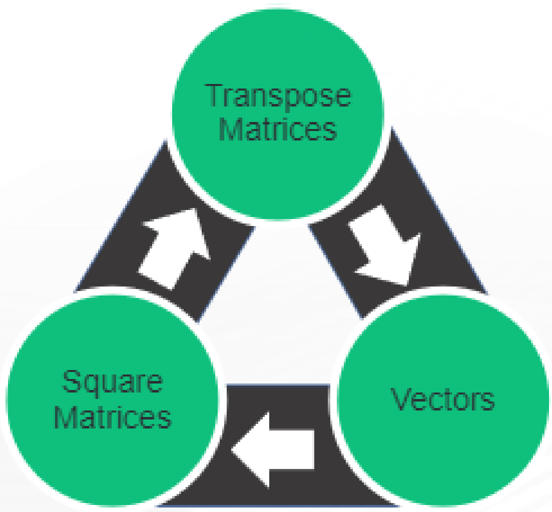
- **A'** refers to the transpose of Matrix **A**.
- **I** refers to an identity matrix.
- $\mathbf{I}_n$ refers to an n x n identity matrix.

- **1** refers to the sum vector, a column vector having all of its elements equal to one.
- $\mathbf{1}_n$ is a 1 x n sum vector.
- |**A**| refers to the determinant of Matrix A.
- **x** refers to a Matrix of deviation scores derived from the raw scores of Matrix X.

2. Matrix Notations and Types

2.1 Types of Matrices

There are three important types of matrices:



There are three important types of matrices.

- Transpose matrices
- Vectors
- Square matrices

We will see each of these in detail in the upcoming sections.

2.2 Transpose Matrix

Transpose of a Matrix:

- The Matrix that is obtained by using the rows of the first Matrix as columns of the second matrix.
- Transpose of Matrix A is represented as A' (A-prime).

Example:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{pmatrix} \qquad A' = \begin{pmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{pmatrix}$$

As given in the slide, transpose of a Matrix is defined as the Matrix that is obtained by using the rows of the first Matrix as columns of the second matrix.

Transpose of Matrix **A** is represented as **A'**.

Row 1 of Matrix **A** becomes column 1 of **A'**; row 2 of A becomes column 2 of **A'**; and so on. The order of a Matrix will be reversed after it is transposed. For example, the transpose of a 2X3 Matrix will be a 3X2 matrix, since rows and columns are interchanged.

2.3 Vectors

Vectors:

- Matrices that have only one column or one row.
- Has two categories:
  - Column vectors
  - Row vectors

Example: $\boldsymbol{a}' = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$

Here a is a column vector and its transpose a' is a row vector.

Vector is a type of Matrix that has only one column or one row. There are two types of vectors: row vectors and column vectors.

Lower-case, boldface letters are used to represent column vectors. The transpose of a column vector is a row vector, we use lower-case, boldface letters plus a prime to represent row vectors. In the example above, vector **a** would be a column vector, and vector **a'** would be a row vector.

2.4 Square Matrices

- A Matrix is said to be a square matrix, if the number of rows and columns are equal.
- The order of a square Matrix in n x n. Example:

$$\boldsymbol{A} = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 5 & 4 \\ 4 & 5 & 8 \end{pmatrix}$$

- Different types of square matrices:

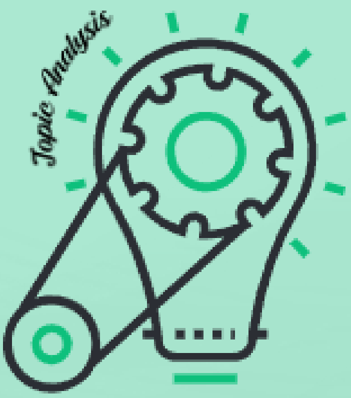
Symmetric matrix	Diagonal matrix	Scalar matrix
$\mathbf{A} = \mathbf{A'} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$	$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$	$\mathbf{A} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

A square Matrix is denoted by n x n, where the number of rows and columns are equal. Some of the types of square matrices are:

1. Symmetric matrix: a symmetric Matrix is one whose transpose is equal to itself.
2. Diagonal matrix: a diagonal Matrix is also a type of symmetric matrix, where the off-diagonal elements are zeroes (diagonal refers to the elements that run from the upper left corner to the lower right corner).
3. Scalar matrix: scalar Matrix is also a diagonal matrix, where the diagonals are equal valued numbers.

Examples of these types are given above.

What did You Grasp?



1. Which of the following options is the transpose of $A = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \end{bmatrix}$?

   A) $A' = \begin{bmatrix} 1 & 5 \\ 2 & 6 \\ 3 & 7 \end{bmatrix}$

   B) $A' = \begin{bmatrix} 1 & 2 \\ 3 & 5 \\ 6 & 7 \end{bmatrix}$

   C) $A' = \begin{bmatrix} 5 & 6 & 7 \\ 1 & 2 & 3 \end{bmatrix}$

   D) $A' = \begin{bmatrix} 1 & 6 \\ 5 & 2 \\ 3 & 7 \end{bmatrix}$

3. Matrix Equality

3.1 Matrix Equality

Two matrices are said to be equal if they satisfy the following conditions:

- Each Matrix has the same number of rows.
- Each Matrix has the same number of columns.

- Corresponding elements within each Matrix are equal.

Consider the three matrices **A**, **B** and **C**.

$$\mathbf{A} = \begin{bmatrix} 111 & x \\ y & 444 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 111 & 222 \\ 333 & 444 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} l & m & n \\ o & p & q \end{bmatrix}$$

The criteria for two matrices being equal is given above.

If **A** = **B**, then x = 222 and y = 333; since corresponding elements of equal matrices are also equal. Matrix **C** is not equal to **A** or **B**, since it has more columns than **A** or **B**.

3.2 Identity Matrices

- The identity Matrix is an n x n diagonal Matrix with 1's in the diagonal and zeros everywhere else.
- The identity Matrix is denoted by I or In. Two identity matrices are given below.

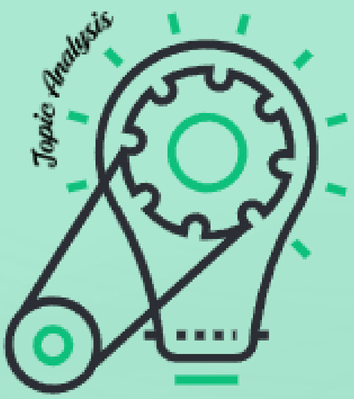
$$\mathbf{I_2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \mathbf{I_3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Any Matrix that is pre or post-multiplied by I remains the same.

i.e , **AI = IA = A**

An identity Matrix has 1's in the diagonals and all other elements are 0's. Examples for identity Matrix is given above.

What did You Grasp?



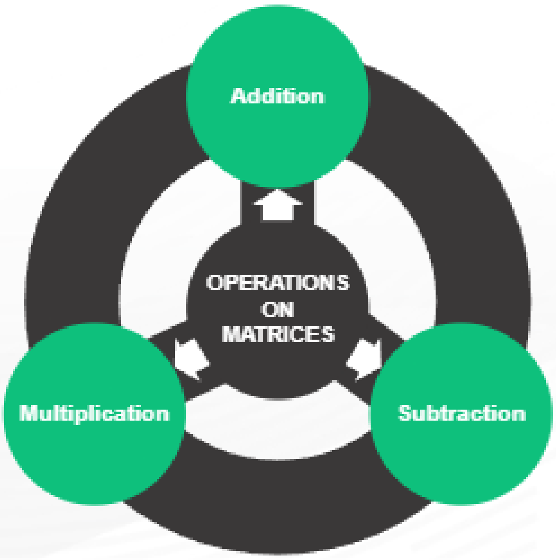
1. What is the order of the following matrix?

$$X = \begin{bmatrix} 1 & 2 & 4 & 4 \\ 3 & 4 & 8 & 0 \end{bmatrix}$$

A) 2x2
B) 2x3
C) 2x4
D) 1x4

3.3 Operations on Matrices

Three major operations can be done on matrices:



Three major operations can be performed on matrices.

- Addition and Subtraction
- Multiplication

We'll see how to perform these operations on matrices, in the upcoming sections.

3.3.1 Matrix Addition and Subtraction

- Matrices can be added or subtracted if they have the same order, i.e., the same number of rows and columns.
- Addition or subtraction can be performed by adding or subtracting the corresponding elements.

For example,

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \end{bmatrix} \qquad \boldsymbol{B} = \begin{bmatrix} 3 & 4 & 5 \\ 7 & 5 & 6 \end{bmatrix}$$

$$\boldsymbol{A} + \boldsymbol{B} = \begin{bmatrix} 1+3 & 2+4 & 3+5 \\ 5+7 & 6+5 & 7+6 \end{bmatrix} \qquad \boldsymbol{A} - \boldsymbol{B} = \begin{bmatrix} 1-3 & 2-4 & 3-5 \\ 5-7 & 6-5 & 7-6 \end{bmatrix}$$

$$\boldsymbol{A} + \boldsymbol{B} = \begin{bmatrix} 4 & 6 & 8 \\ 12 & 11 & 13 \end{bmatrix} \qquad \boldsymbol{A} - \boldsymbol{B} = \begin{bmatrix} -2 & -2 & -2 \\ -2 & 1 & 1 \end{bmatrix}$$

Like any addition or subtraction, matrices can also be added or subtracted. Two matrices can be added or subtracted only if they have the same number of rows and columns, i.e., if they have the same order. Matrices can be added or subtracted by adding or subtracting the corresponding elements in each of the matrices.

An example of Matrix addition and subtraction is given above. From the example we can also understand that the order in which the elements are added is not important, i.e., A + B = B + A. On the other hand, it is also evident that A - B ≠ B - A.

3.3.2 Matrix Multiplication

There are two types of Matrix multiplication:

- Multiplication of a Matrix by a number (scalar)
- Multiplication of a Matrix by another matrix.

The result of both the operations is a Matrix by itself.

Like addition and subtraction, multiplication can also be performed on matrices. There are two types of Matrix multiplication:

- Multiplying a Matrix by a number or scalar.
- Multiplying a Matrix by another matrix.

3.3.2 (a) Multiplication of a Matrix by a Number

- Multiplication of a Matrix by a number involves multiplying each element of the Matrix by that number.
- The result of this multiplication is a matrix, which is called a scalar multiple.

Example:

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \end{bmatrix} \qquad 5\mathbf{A} = 5\begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \end{bmatrix}$$

$$5\mathbf{A} = \begin{bmatrix} 5*1 & 5*2 & 5*3 \\ 5*5 & 5*6 & 5*7 \end{bmatrix}$$

$$5\mathbf{A} = \begin{bmatrix} 5 & 10 & 15 \\ 25 & 30 & 35 \end{bmatrix}$$

As mentioned above, multiplying a Matrix by a number results in a scalar multiple. This multiplication is also referred to as multiplication of a Matrix by a scalar.

In the above example, every element of the Matrix A is multiplied by the scalar 5 to produce the scalar multiple, which is also a matrix.

3.3.2 (b) Multiplication of Matrix by a Matrix

- While multiplying two matrices A and B, the product AB can be defined only if the number of columns in A is equal to the number of rows in B.
- Similarly, the Matrix product BA can be defined only if the number of columns in B is equal to the number of rows in A.
- If Matrix A has order m x n and Matrix B has order p x q,
  - AB exists only if n = p (order of AB will be m x q)
  - BA exists only if q = m (order of BA will be p x n)

Multiplication of Matrix by a Matrix should satisfy the conditions as stated above. The method of computing the Matrix product is explained in the next section.

Multiplication order:

Matrix multiplication is possible in both directions, results may be different. That is the product of two matrices **AB** is not always equal to **BA**. To describe the Matrix product **AB**, we can say **A** is post-multiplied by B; or we can say that B is premultiplied by **A**. Similarly, to describe the Matrix product **BA**, we can say B is post-multiplied by **A**, or we can say that A is premultiplied by B.

3.3.3 Computing the Product of Matrix Multiplication by a Matrix

- Suppose that **A** is an i x j matrix, and **B** is a j x k matrix. Then, the Matrix product **AB** results in a Matrix C, which has i rows and k columns; and each element in C can be computed according to the following formula:

$$C_{ik} = \Sigma_j A_{ij} B_{jk}$$

- *C_{ik} = the element in row i and column k from Matrix C*
- *A_{ij} = the element in row i and column j from Matrix A*
- *B_{jk} = the element in row j and column k from Matrix B*
- *Σ_j = indicates that the A_{ij} B_{jk} terms should be summed over j*

Example:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 6 & 7 \\ 8 & 9 \\ 10 & 11 \end{bmatrix} \quad \mathbf{AB} = \mathbf{C} = \begin{bmatrix} 28 & 31 \\ 100 & 112 \end{bmatrix}$$

Consider the example given above.

Matrix **A** and B are multiplied to get the product C.

Let AB = C. **A** has 2 rows, so C will have two rows; and **B** has 2 columns, so C will have 2 columns.

To compute the value of every element in the 2 x 2 Matrix C, we use the formula $C_{lk} = \Sigma_j A_{lj} B_{jk}$, as shown below.

$C_{11} = \Sigma A_{1j} B_{j1}$ = 0*6 + 1*8 +2*10 = 0 + 8 + 20 = **28**

$C_{12} = \Sigma A_{1j} B_{j2}$ = 0*7 + 1*9 +2*11 = 0 + 9 + 22 = **31**

$C_{21} = \Sigma A_{2j} B_{j1}$ = 3*6 + 4*8 +5*10 = = 18 + 32 + 50 = **100**

$C_{22} = \Sigma A_{2j} B_{j2}$ = 3*7 + 4*9 +5*11 = 21 + 36 +55 = **112**

The result is what you see in the above example, i.e., the Matrix product C.

3.4 Vector Multiplication

- Multiplication of a vector by a vector results two kinds of products:

Vector inner product

$$\mathbf{a} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

$$\mathbf{a'b} = 1*4 + 2*5 + 3*6 = 4 + 10 + 18 = 32$$

Vector outer product

$$\mathbf{a} = \begin{bmatrix} v \\ w \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\mathbf{C} = \mathbf{ab'} = \begin{bmatrix} v*x & v*y & v*z \\ w*x & w*y & w*z \end{bmatrix}$$

Explanation of both these concepts is given below.

Multiplication of a vector by a vector may result in vector inner product or vector outer product.

Vector inner product

Vector inner product is also termed as scalar product or dot product. If **a** and **b** are vectors, each with the same number of elements, then, the inner product of **a** and **b** is s.

a'b = **b'a** = *s*

where

a and **b** are column vectors, each having n elements,

a' is the transpose of **a**, which makes **a'** a row vector,

b' is the transpose of **b**, which makes **b'** a row vector, and

s is a scalar, i.e., a real number and not a matrix.

In the above example, **a'b** = 1*4 + 2*5 + 3*6 = 4 + 10 + 18 = 32, the inner product of **a'b** is equal to **32**.

Vector outer product

Outer product is also called cross product. If **a** and **b** are vectors. Then, the outer product of **a** and **b** is C.

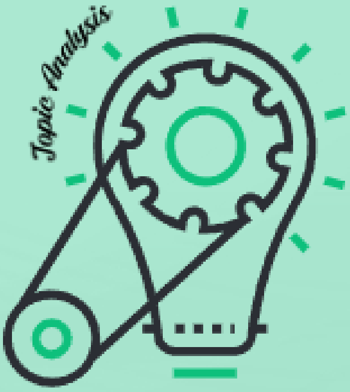
ab’ = C

where

> **a** is a column vector, having *m* elements,
>
> **b** is a column vector, having *n* elements,
>
> **b’** is the transpose of **b**, which makes **b’** a row vector, and
>
> C is a rectangular *m x n* matrix

The outer product of two vectors produces a rectangular matrix, not a scalar, which is illustrated in the example above. The elements of Matrix C consist of the product of elements from vector **a** crossed with elements from vector **b**. Thus, C is a Matrix of cross products from the two vectors.

What did You Grasp?



1. State True or False.

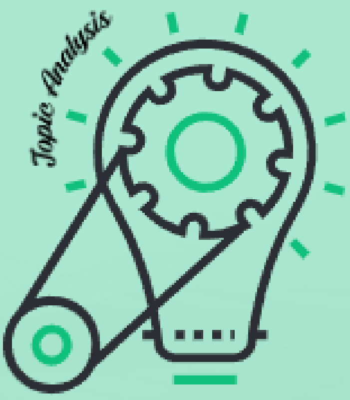
   Outer vector product is also called dot product.

   A) True
   B) False

2. State True or False.

   For two matrices A and B, A - B = B - A.

   A) True
   B) False



3. Find the product AB, given that $A = \begin{bmatrix} 2 & 4 & 6 \\ 1 & 3 & 5 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 1 \\ 3 & 0 \\ 2 & 5 \end{bmatrix}$

   A) $\begin{bmatrix} 10 & 13 \\ 16 & 25 \end{bmatrix}$

   B) $\begin{bmatrix} 24 & 32 \\ 19 & 26 \end{bmatrix}$

   C) $\begin{bmatrix} 24 & 19 \\ 32 & 26 \end{bmatrix}$

   D) $\begin{bmatrix} 12 & 19 \\ 26 & 35 \end{bmatrix}$

4. What is 4A, if $A = \begin{bmatrix} 2 & 3 \\ 6 & 5 \end{bmatrix}$

A) $\begin{bmatrix} 10 & 3 \\ 6 & 25 \end{bmatrix}$

B) $\begin{bmatrix} 8 & 12 \\ 24 & 20 \end{bmatrix}$

C) $\begin{bmatrix} 5 & 3 \\ 6 & 5 \end{bmatrix}$

D) $\begin{bmatrix} 8 & 24 \\ 12 & 20 \end{bmatrix}$

4. Determinants

- The determinant is a value that can be computed from the elements of a square matrix. The determinant of a Matrix A is denoted det(A), det A, or |A|.
- For a 2x2 matrix, the determinant can be defined as:

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$
$$= ad - bc$$

- For a 3x3 Matrix A, the determinant is:

$$|A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} - c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$= aei + bfg + cdh - ceg - bdi - afh$$

Determinant is a unique number that is associated with a square matrix. We have explained how to compute the determinant for 2x2 and 3x3 matrices.

The formula for computing the determinant of any nxn square Matrix is as follows:

$|A| = \Sigma(\pm)\, A_{1q}A_{2r}A_{3s} \ldots A_{nz}$

Points to note in this formula:

- The determinant is the sum of product terms made up of elements from the matrix.
- Each product term consists of *n* elements from the matrix.
- Each product term includes one element from each row and one element from each column.

- The number of product terms is equal to *n*! (where *n*! refers to n factorial).
- By convention, the elements of each product term are arranged in ascending order of the left-hand (or row-designating) subscript.
- To find the sign of each product term, we count the number of inversions needed to put the right-hand (or column-designating) subscripts in numerical order. If the number of inversions is even, the sign is positive; if odd, the sign is negative.

5. Singularity of a Matrix

- If a square Matrix is said to be a singular matrix if its determinant is zero.
- In such a case, the Matrix will have no inverse.

Example:

For a 2x2 matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

ad - bc = 0

For a 3x3 matrix

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \quad a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} - c\begin{vmatrix} d & e \\ g & h \end{vmatrix} = 0$$

$$aei + bfg + cdh - ceg - bdi - afh = 0$$

If the determinant of a Matrix is 0 then the Matrix has no inverse and it is called a Singular Matrix. Examples are given above.

6. Singularity of a Matrix

6.1 Orthogonal Matrix

- If the square Matrix with real elements, $A \in R^{m \times n}$ is the Gram Matrix forms an identity matrix, then the Matrix is said to be an orthogonal matrix.
- Conditions for an orthogonal matrix:

1. $A^T = A^{-1}$
2. $AA^T = A^TA = I$

where, the rows of Matrix **A** are orthogonal.

The definition for orthogonal Matrix is given above. Let's now look at an example to understand the concept.

6.2 Example for Orthogonal Matrix

Example: Identify if the given Matrix is orthogonal.

$$A = \begin{bmatrix} cosx & sinx \\ -sinx & cosx \end{bmatrix}$$

Solution: Check the first condition of orthogonal matrix.

$$A^T = \begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix} \quad ......(1)$$

$$A^{-1} = \frac{adj(A)}{|A|}$$

$$= \frac{\begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix}}{(\cos^2 x + \sin^2 x)}$$

$$= \frac{\begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix}}{(1)}$$

$$A^{-1} = \begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix} \quad ......(2)$$

The first step is to verify that the given Matrix satisfies the first condition of orthogonal matrix.

Check for the second condition.

$$AA^T = \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix} \begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2 + \sin^2 x & -\cos x \sin x + \sin x \cos x \\ -\sin x \cos x + \cos x \sin x & \sin^2 x + \cos^2 x \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= I$$

$$A^T A = \begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix} \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2 + \sin^2 x & \cos x \sin x - \sin x \cos x \\ \sin x \cos x - \cos x \sin x & \sin^2 x + \cos^2 x \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= I$$

The second step proves that the given Matrix satisfies the second condition of orthogonal matrix.

Thus the example Matrix given above satisfies both the conditions of the orthogonal matrix.

What did You Grasp?

1. Select the correct option.

   A = $\begin{bmatrix} 5 & 3 \\ 6 & 5 \end{bmatrix}$ then what is |A|.

   A) 8
   B) 10
   C) 15
   D) 7

7. Orthogonal Matrix

7.1 Elementary transformations of matrices

- Elementary transformations, also called elementary operations, of a Matrix, are:
  - Rearrangement of two rows (or columns).
  - Multiplication of all row (or column) elements of a Matrix to some number, not equal to zero.
  - Multiplication of a row (or column) by a number, not equal to zero, and adding the result to another row.
- Operations performed on rows are called elementary row operations and operations performed on columns are called elementary column operations.

There are three elementary operations that can be performed on matrices. There are certain notations that describe elementary operations. They are listed below:

Row operations:

1. Interchange rows *i* and *j*: R_i <--> R_j

2. Multiply row *i* by *s*, where $s \neq 0$: sR_i --> R_i

3. Add s times row *i* to row *j*: $sR_i + R_j$ --> R_j

Column operations:

1. Interchange columns *i* and *j*: C_i <--> C_j

2. Multiply column *i* by *s*, where $s \neq 0$: sC_i --> C_i

3. Add s times column *i* to column *j*: $sC_i + C_j \rightarrow C_j$

7.2 Elementary Operators

- An Elementary operation may be performed by Matrix multiplication, using square matrices called elementary operators.
- If you want to interchange rows 1 and 2 of Matrix **A**. To accomplish this, you could premultiply **A** by E to produce B.

$$R_1 \leftrightarrow R_2 = \underset{\mathbf{E}}{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}} \quad \underset{\mathbf{A}}{\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}}$$

$$R_1 \leftrightarrow R_2 = \begin{bmatrix} 0+2 & 0+4 & 0+6 \\ 0+1 & 0+3 & 0+5 \end{bmatrix}$$

$$R_1 \leftrightarrow R_2 = \begin{bmatrix} 2 & 4 & 6 \\ 1 & 3 & 5 \end{bmatrix} = \mathbf{B}$$

In the above example, E is an elementary operator. It operates on **A** to produce the desired interchanged rows in B. We'll now see how to find E.

7.3 Elementary Row Operations

- To perform an elementary row operation on a A, an r x c matrix, there are two steps that have to be followed.
  a) To find E, the elementary row operator, apply the operation to an r x r identity matrix.
  b) To carry out the elementary row operation, premultiply A by E.
- Three elementary row operations:
  - Interchange two rows
  - Multiply a row by a number
  - Multiply a row and add it to another row

We learnt about the steps involved in elementary row operations. We'll now learn how to perform the three types of elementary row operations.

7.3.1 Interchange of Two Rows

Step 1:

$$\underset{I_3}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \Rightarrow \underset{E}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}$$

Step 2:

$$R_2 <--> R_3 = \underset{\mathbf{E}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}} \underset{\mathbf{A}}{\begin{bmatrix} 0 & 1 \\ 2 & 3 \\ 4 & 5 \end{bmatrix}}$$

$$R_2 <--> R_3 = \begin{bmatrix} 1*0 + 0*2 + 0*4 & 1*1 + 0*3 + 0*5 \\ 0*0 + 0*2 + 1*4 & 0*1 + 0*3 + 1*5 \\ 0*0 + 1*2 + 0*4 & 0*1 + 1*3 + 0*5 \end{bmatrix}$$

$$R_2 <--> R_3 = \begin{bmatrix} 0 & 1 \\ 4 & 5 \\ 2 & 3 \end{bmatrix}$$

In the above example, we want to interchange the second and third rows of **A**, a 3 x 2 matrix.

Step 1: To create the elementary row operator E, we interchange the second and third rows of the identity Matrix I_3.

Step 2: To interchange the second and third rows of **A**, we premultiply **A** by E.

7.3.2 Multiply a Row by a Number

Step 1:

$$\underset{I_2}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} \Rightarrow \underset{E}{\begin{bmatrix} 1 & 0 \\ 0 & 7 \end{bmatrix}}$$

Step 2:

$$7R_2 --> R_2 = \underset{\mathbf{E}}{\begin{bmatrix} 1 & 0 \\ 0 & 7 \end{bmatrix}} \underset{\mathbf{A}}{\begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \end{bmatrix}}$$

$$7R_2 --> R_2 = \begin{bmatrix} 1*0 + 0*3 & 1*1 + 0*4 & 1*2 + 0*5 \\ 0*0 + 7*3 & 0*1 + 7*4 & 0*2 + 7*5 \end{bmatrix}$$

$$7R_2 --> R_2 = \begin{bmatrix} 0 & 1 & 2 \\ 21 & 28 & 35 \end{bmatrix}$$

Suppose we want to multiply each element in the second row of Matrix A by 7. Assume A is a 2 x 3 matrix.

- **Step 1:** To create the elementary row operator E, we multiply each element in the second row of the identity Matrix I2 by 7.
- **Step 2:** To multiply each element in the second row of A by 7, we premultiply A by E.

7.3.3 Multiply a Row and Add it to Another Row

Step 1:

$$\underset{I_2}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} \Rightarrow \begin{bmatrix} 1 & 0 \\ 0 + 3*1 & 1 + 3*0 \end{bmatrix} \Rightarrow \underset{E}{\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}}$$

Step 2:

$$3R_1 + R_2 \text{-->} R_2 = \underset{E}{\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}} \underset{A}{\begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}}$$

$$3R_1 + R_2 \text{-->} R_2 = \begin{bmatrix} 1*0 + 0*2 & 1*1 + 0*3 \\ 3*0 + 1*2 & 3*1 + 1*3 \end{bmatrix}$$

$$3R_1 + R_2 \text{-->} R_2 = \begin{bmatrix} 0 & 1 \\ 2 & 6 \end{bmatrix}$$

For the above example, assume **A** is a 2 x 2 matrix. Suppose we want to multiply each element in the first row of **A** by 3, and we want to add that result to the second row of **A**.

Step 1: For this operation, creating the elementary row operator is a two-step process. First, we multiply each element in the first row of the identity Matrix I_2 by 3. Next, we add the result of that multiplication to the second row of I_2 to produce E.

Step 2: To multiply each element in the first row of A by 3 and add that result to the second row, we premultiply **A** by E.

7.4 Elementary Column Operations

- To perform an elementary column operation on **A**, an r x c matrix, follow the steps.
  a) To find E, the elementary column operator, apply the operation to an c x c identity matrix.
  b) To carry out the elementary column operation, postmultiply **A** by E.
- The process for performing an elementary column operation on an *r* x *c* Matrix is very similar to the process for performing an elementary row operation. The main differences are:
  1. To operate on the *r* x *c* Matrix **A**, the row operator E is created from an *r* x *r* identity matrix; whereas the column operator, E is created from an *c* x *c* identity matrix.
  2. To perform a row operation, **A** is premultiplied by E; whereas to perform a column operation, **A** is *post-multiplied* by E.

The steps involved in performing an elementary column operation are given above. It is also important to note that the performing an elementary column operation on an *r* x *c* Matrix is very similar to the process for performing an elementary row operation, though there exist a few differences.

7.4.1 Interchange of Two Columns

Step 1:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$\mathbf{I_2}$ **E**

Step 2:

$$C_1 \text{<-->} C_2 = \begin{bmatrix} 0 & 1 \\ 2 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

A **E**

$$C_1 \text{<-->} C_2 = \begin{bmatrix} 0*0 + 1*1 & 0*1 + 1*0 \\ 2*0 + 3*1 & 2*1 + 3*0 \\ 4*0 + 5*1 & 4*1 + 5*0 \end{bmatrix}$$

$$C_1 \text{<-->} C_2 = \begin{bmatrix} 1 & 0 \\ 3 & 2 \\ 5 & 4 \end{bmatrix}$$

In the above example, suppose we want to interchange the first and second columns of **A**, a 3 x 2 matrix.

Step 1: To create the elementary column operator E, we interchange the first and second columns of the identity Matrix $\mathbf{I}_2$.

Step 2: To interchange the first and second columns of **A**, we postmultiply **A** by E.

7.5 Elementary Matrices

A square Matrix E of order n is called an elementary Matrix if it is obtained by applying exactly one elementary row operation to the identity matrix, $\mathbf{I}_n$.

Three types of elementary matrices:

$$(E_{ij})_{(k,\ell)} = \begin{cases} 1 & \text{if } k = \ell \text{ and } \ell \neq i, j \\ 1 & \text{if } (k,\ell) = (i,j) \text{ or } (k,\ell) = (j,i) \\ 0 & \text{otherwise} \end{cases}$$

$$(E_k(c))_{(i,j)} = \begin{cases} 1 & \text{if } i = j \text{ and } i \neq k \\ c & \text{if } i = j = k \\ 0 & \text{otherwise} \end{cases}$$

$$(E_{ij})_{(k,\ell)} \begin{cases} 1 & \text{if } k = \ell \\ c & \text{if } (k,\ell) = (i,j) \\ 0 & \text{otherwise} \end{cases}$$

As mentioned above, there are three types of elementary matrices:

1. E_{ij}, which is obtained by the application of the elementary row operation R_{ij} to the identity matrix, I_n. The (k, l)th entry of E_{ij} is given above in type 1.
2. $E_k(c)$, which is obtained by the application of the elementary row operation $R_k(c)$ to the identity matrix, In. The (i,j)th entry of $E_k(c)$ is given above in type 2.
3. $E_{ij}(c)$, which is obtained by the application of the elementary row operation $R_{ij}(c)$ to the identity matrix, I_n. The (k,l)th entry of $E_{ij}(c)$ is given above in type 3.

7.6 Inverse of a Matrix Using Elementary Transformations

- If **A** is an n x n matrix, the inverse of **A** is another n x n matrix, denoted $\mathbf{A}^{-1}$, such that

  $\mathbf{AA^{-1} = A^{-1}A = I_n}$

  where $\mathbf{I_n}$ is the identity matrix.

- Three types of transformations can be used to find the inverse of a matrix:
  1. Multiplying a row by a constant
  2. Adding a multiple of another row
  3. Swapping two rows

While using the elementary transformation method to find the inverse of a matrix, our goal is to convert the given Matrix into an identity matrix, using elementary transformations.

There are two ways to determine if inverse of a Matrix exists:

1. **Determine the rank of the matrix:** The rank of a Matrix is a unique number associated with a square matrix. If the rank of an n x n Matrix is less than n, the Matrix does not have an inverse.
2. **Compute the determinant of the matrix:** The determinant, a unique number associated with a square matrix. When the determinant for a square Matrix is equal to zero, the inverse for that Matrix does not exist.

A square Matrix that has an inverse is called nonsingular or invertible; a square Matrix that does not have an inverse is called singular.

7.6.1 Example for Finding Matrix Inverse

Consider the Matrix A. Find A^{-1}.

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & -1 \\ 2 & 3 & -1 \end{bmatrix}$$

Step 1:

$$\begin{bmatrix} 1 & 2 & 0 & 1 & 0 & 0 \\ 2 & 0 & -1 & 0 & 1 & 0 \\ 2 & 3 & -1 & 0 & 0 & 1 \end{bmatrix} \begin{array}{c} a \\ b \\ c \end{array}$$

Solution

$$A^{-1} = \begin{bmatrix} 1 & 2/3 & -2/3 \\ 0 & -1/3 & 1/3 \\ 2 & 1/3 & -4/3 \end{bmatrix}$$

Step 2:

$$\begin{bmatrix} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 & -1 & 1 \\ 2 & 3 & -1 & 0 & 0 & 1 \end{bmatrix} \begin{array}{c} a \to a \\ -b+c \to b \\ c \to c \end{array}$$

$$\begin{bmatrix} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1/3 & 1/3 \\ 0 & 1 & 1 & 2 & 0 & -1 \end{bmatrix} \begin{array}{c} a \to a \\ b/3 \to b \\ -c+2a \to c \end{array}$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 2/3 & -2/3 \\ 0 & 1 & 0 & 0 & -1/3 & 1/3 \\ 0 & 0 & 1 & 2 & 1/3 & -4/3 \end{bmatrix} \begin{array}{c} a-2b \to a \\ b \to b \\ c-b \to c \end{array}$$

Step 1 is to write the identity Matrix on the right side of A.

Step 2 is to apply elementary transformations to the whole matrix. The point at which you get an identity Matrix on the left side, you will get the inverse of A on the right side.

8. Elementary Transformations and elementary matrices

8.1 Echelon Matrices

Echelon matrices come in two forms:

Row echelon form

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

A_{ref} B_{ref} C_{ref}

Reduced row echelon form

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

A_{rref} B_{rref} C_{rref}

Row Echelon Form: A Matrix is said to be in row echelon form (ref) when it satisfies the following conditions

- The first non-zero element in each row, called the leading entry, is generally 1, but according to some references, this need not be met.
- Each leading entry is in a column to the right of the leading entry in the previous row.
- Rows with all zero elements, if any, are below rows having a non-zero element.

Example for a row echelon form Matrix is given above.

Reduced Row Echelon Form

A Matrix is in reduced row echelon form (rref) when it satisfies the following conditions.

- The Matrix satisfies conditions for a row echelon form.
- The leading entry in each row is the only non-zero entry in its column.

Example for reduced row echelon form is given above.

8.2 Echelon Transformation

- Any Matrix can be transformed into its echelon form, using elementary operations.
- Steps are as follows:
  a) Find the pivot, the first non-zero entry in the first column of the matrix.
  b) To get the Matrix in row echelon form, repeat the pivot
  c) To get the Matrix in reduced row echelon form, process non-zero entries above each pivot.

$$\underset{\mathbf{A}}{\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 7 & 8 \end{bmatrix}} \Rightarrow \underset{\mathbf{A_1}}{\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 2 & 7 & 8 \end{bmatrix}} \Rightarrow \underset{\mathbf{A_2}}{\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 3 & 6 \end{bmatrix}} \Rightarrow \underset{\mathbf{A_{ref}}}{\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}} \Rightarrow \underset{\mathbf{A_{rref}}}{\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}}$$

Elementary operations are used to transform a Matrix to its echelon forms. The steps are as follows:

1. Pivot the matrix
   - Find the pivot, the first non-zero entry in the first column of the matrix.
   - Interchange rows, moving the pivot row to the first row.
   - Multiply each element in the pivot row by the inverse of the pivot, so the pivot equals 1.
   - Add multiples of the pivot row to each of the lower rows, so every element in the pivot column of the lower rows equals 0.
2. To get the Matrix in row echelon form, repeat the pivot
   - Repeat the procedure from Step 1 above, ignoring previous pivot rows.
   - Continue until there are no more pivots to be processed.

3. To get the Matrix in reduced row echelon form, process non-zero entries above each pivot.
   - Identify the last row having a pivot equal to 1, and let this be the pivot row.

- Add multiples of the pivot row to each of the upper rows, until every element above the pivot equals 0.
- Moving up the matrix, repeat this process for each row.

In the example given above, the following steps were followed to transform Matrix **A** into its echelon forms using a series of elementary row operations.

1. The first non-zero entry in the first column of the Matrix in row 2; so rows 1 and 2 were interchanged to get Matrix $\mathbf{A}_1$.
2. Each element of row 1 in Matrix $\mathbf{A}_1$ was multiplied by -2 and the result was added with row 3. This resulted in $\mathbf{A}_2$.
3. Each element of row 2 in Matrix $\mathbf{A}_2$ by -3 and the result was added to row 3 to produce $\mathbf{A}_{ref}$. Now, Aref is in row echelon form, based on the conditions specified in the previous slide.
4. The second row of $\mathbf{A}_{ref}$ was multiplied by -2 and the result was added to the first row to produce $\mathbf{A}_{rref}$.

9. Echelon forms and echelon transformations

9.1 Rank of a Matrix

- Imagine an r x c Matrix as a set of r row vectors, each having c elements; it can also be thought of a set of c column vectors, each having r elements.
- The rank of a Matrix is defined as:
    a) the maximum number of linearly independent column vectors in the matrix, or
    b) the maximum number of linearly independent row vectors in the matrix. Both definitions are equivalent.

The definition for rank of a Matrix is given above.

For an r x c matrix,

- If r is less than c, then the maximum rank of the Matrix is r.
- If r is greater than c, then the maximum rank of the Matrix is c.

The rank of a Matrix would be zero, only if the Matrix had no elements. If a Matrix had even one element, its minimum rank would be one.

9.2 Finding the rank of a matrix

- The maximum number of linearly independent vectors in a Matrix is equal to the number of non-zero rows in its row echelon matrix.
- To find the rank of a matrix, we need to transform the Matrix to its row echelon form and count the number of non-zero rows.
- Consider Matrix **A** and its row echelon matrix, $\mathbf{A}_{ref}$.

$$\underset{\mathbf{A}}{\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 7 & 8 \end{bmatrix}} \Rightarrow \underset{\mathbf{A}_{ref}}{\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}}$$

In the above example, the row echelon form $\mathbf{A}_{ref}$ has two non-zero rows, and Matrix **A** has two independent row vectors; and the rank of Matrix **A** is 2.

Row 1 and Row 2 of Matrix **A** are linearly independent. However, Row 3 is a linear combination of Rows 1 and 2. Specifically, Row 3 = 3*(Row 1) + 2*(Row 2). Therefore, Matrix *A* has only two independent row vectors.

9.3 Full Rank Matrices

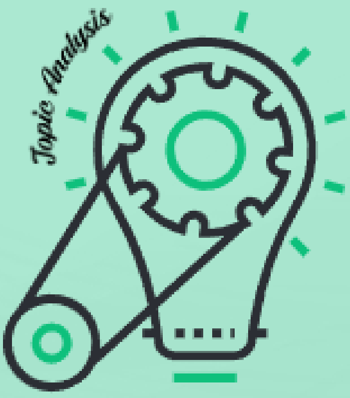
- When all of the vectors in a Matrix are linearly independent, the Matrix is said to be full rank. Consider the matrices **A** and **B** below:

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$$

- Row 2 of Matrix **A** is a scalar multiple of row 1; i.e., row 2 is equal to twice row 1. Therefore, rows 1 and 2 are linearly dependent. Matrix **A** has only one linearly independent row, so its rank is 1. Hence, Matrix **A** is not full rank.
- Now, if we take Matrix **B**, all of its rows are linearly independent, so the rank of Matrix **B** is 3. Matrix **B** is full rank.

The definition of a full rank Matrix and examples are given above.

What did You Grasp?



1. What is the rank of the Matrix X?

$$X = \begin{bmatrix} 1 & 2 & 4 & 4 \\ 3 & 4 & 8 & 0 \end{bmatrix}$$

A) 1
B) 2
C) 3
D) 4

10. Matrix Rank and Normal Form of a matrix

10.1 Normal Form of a Matrix

Normal form of a Matrix is a Matrix satisfying following conditions:

- consist of only ones and zeros.
- every row has a maximum of single one and rest are all zeros (there can be rows with all zeros).

We can produce the normal form of a Matrix by using elementary operations

Rank of a Matrix can be found from its normal form by counting the number of rows with non zero elements. But using echelon forms is an easy alternative.

We saw about normal form of a matrix. We'll learn how to reduce a Matrix to its normal form, using an example.

10.2 Reducing a Matrix to its Normal Form

Reduce the matrix:

$$A = \begin{bmatrix} 5 & 3 & 8 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

Solution: We can reduce the Matrix to its normal form using elementary operations.

Step 1:

Applying $R_1 \leftrightarrow R_3$, we have

$$A \sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 5 & 3 & 8 \end{bmatrix}$$

Step 2:

Applying $R_3 \to R_3 - 5R_1$, we have

$$A \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 8 & 8 \end{bmatrix}$$

Steps involved in reducing a Matrix to its normal form is explained above.

Step 3: Applying elementary row operations $R_1 \to R_1 + R_2$ and $R_3 \to R_3 - 8R_2$, we have

$$A \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Step 4: Now, we apply elementary column operation $C_3 \to C_3 - C_2$, to get

$$A \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Step 5: Again, applying $C_3 \to C_3 - C_1$, we have

$$A \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Steps involved in reducing a Matrix to its normal form is explained above.

We'll now move on to learn about vectors, vector spaces and associated techniques.

11. Vector Spaces and the axioms

11.1 Vector Spaces

Vector Space:

- A space in mathematics is a set in which the list of elements is defined by a collection of guidelines or axioms for how each element relates to another within the set.
- A space in which the elements are sets of numbers themselves. Each element in a vector space is a list of objects that have a specific length, which we call vectors.
- Often we refer to the elements of a vector space as n-tuples, with n as the specific length of each of the elements in the set.

To define a Vector Space, first we need a few basic definitions.

- A set is a collection of distinct objects called elements. The elements are usually real or complex numbers when we use them in mathematics, but the elements of a set can also be a list of things.
- We notate a set by encasing the elements within curly braces. Note that to be distinct, an element cannot be repeated within the same set.
- {0, 1, 2, 3, 4, 5, 6, 7, 8, 9} is the set of single digit numbers that we use in mathematics.
- {a, b, c, d, ..., y, z} is the set of letters in the alphabet.

The definition of a vector space is given above. Each element of a vector space of length n can be represented as a matrix, or a collection of numbers within parentheses.

Field:

- We refer to any vector space, as a vector space defined over a given field F.
- A field is a space of individual numbers, usually real or complex numbers.
- The specific axioms to define a field are similar to those of a vector space.
- We'll define a field as a vector space whose elements are single numbers that adhere to the same set of axioms as listed in the next section.

Understand fields from the context of vector spaces, as given in the slide above.

1. For any $\vec{v}, \vec{w} \in V : \vec{v} + \vec{w} \in V$.
2. For any $\vec{v}, \vec{w} \in V : \vec{v} + \vec{w} = \vec{w} + \vec{v}$.
3. For any $\vec{u}, \vec{v}, \vec{w} \in V : (\vec{v} + \vec{w}) + \vec{u} = \vec{v} + (\vec{w} + \vec{u})$.
4. There is a **zero vector** $\vec{0} \in V$ such that $\vec{v} + \vec{0} = \vec{v}$ for all $\vec{v} \in V$.
5. Each $\vec{v} \in V$ has an **additive inverse** $\vec{w} \in V$ such that $\vec{w} + \vec{v} = \vec{0}$.
6. If r is a **scalar**, that is, a member of $\mathbb{R}$ and $\vec{v} \in V$ then the **scalar multiple** $r \cdot \vec{v}$ is in V .
7. If $r, s \in \mathbb{R}$ and $\vec{v} \in V$ then $(r+s) \cdot \vec{v} = r \cdot \vec{v} + s \cdot \vec{v}$.
8. If $r \in \mathbb{R}$ and $\vec{v}, \vec{w} \in V$, then $r \cdot (\vec{v} + \vec{w}) = r \cdot \vec{v} + r \cdot \vec{w}$.
9. If $r, s \in \mathbb{R}$ and $\vec{v} \in V$, then $(rs) \cdot \vec{v} = r \cdot (s \cdot \vec{v})$
10. For any $\vec{v} \in V$, $1 \cdot \vec{v} = \vec{v}$.

We've seen about these conditions in the previous module. This is a recap of the vector spaces section in the previous module. We'll see the verification for these conditions in the upcoming sections.

Verification of Conditions 1 to 3

ere are five conditions in item 1. For 1, closure of addition, note that for any $v_1, v_2, w_1, w_2 \in \mathbb{R}$ the result of the sum

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \end{pmatrix}$$

a column array with two real entries, and so is in $\mathbb{R}^2$. For 2, that addition of vectors commutes, take all entries to be real numbers and compute

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \end{pmatrix} = \begin{pmatrix} w_1 + v_1 \\ w_2 + v_2 \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

e second equality follows from the fact that the components of the vectors are real numbers, and the addition of real numbers is commutative).
ndition 3, associativity of vector addition, is similar.

$$\begin{aligned} \left(\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \right) + \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} &= \begin{pmatrix} (v_1 + w_1) + u_1 \\ (v_2 + w_2) + u_2 \end{pmatrix} \\ &= \begin{pmatrix} v_1 + (w_1 + u_1) \\ v_2 + (w_2 + u_2) \end{pmatrix} \\ &= \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \left(\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} + \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \right) \end{aligned}$$

Verification of Conditions 4 to 7

For the fourth condition we must produce a zero element — the vector of zeroes is it.

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

For 5, to produce an additive inverse, note that for any $v_1, v_2 \in \mathbb{R}$ we have

$$\begin{pmatrix} -v_1 \\ -v_2 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

so the first vector is the desired additive inverse of the second.

The checks for the five conditions having to do with scalar multiplication are just as routine. For 6, closure under scalar multiplication, where $r, v_1, v_2 \in \mathbb{R}$,

$$r \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} rv_1 \\ rv_2 \end{pmatrix}$$

is a column array with two real entries, and so is in $\mathbb{R}^2$. Next, this checks 7.

$$(r+s) \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} (r+s)v_1 \\ (r+s)v_2 \end{pmatrix} = \begin{pmatrix} rv_1 + sv_1 \\ rv_2 + sv_2 \end{pmatrix} = r \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + s \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

Verification of Conditions 8 to 10

For 8, that scalar multiplication distributes from the left over vector addition, we have this.

$$r \cdot \left(\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \right) = \begin{pmatrix} r(v_1 + w_1) \\ r(v_2 + w_2) \end{pmatrix} = \begin{pmatrix} rv_1 + rw_1 \\ rv_2 + rw_2 \end{pmatrix} = r \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + r \cdot \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

The ninth

$$(rs) \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} (rs)v_1 \\ (rs)v_2 \end{pmatrix} = \begin{pmatrix} r(sv_1) \\ r(sv_2) \end{pmatrix} = r \cdot \left(s \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right)$$

and tenth conditions are also straightforward.

$$1 \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1v_1 \\ 1v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

11.2 Axioms for Vector Spaces

There are ten axioms that define a vector space. Let x, y, and z be elements of the vector space V. Let a and b be elements of the field F:

1. Closed under addition: For each element x and y in V, x + y is also in V.

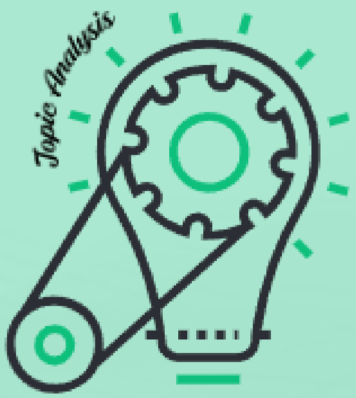
2. Closed under scalar multiplication: For each element x in V and scalar a in F, ax is in V.

3. Commutativity of addition: For each element x and y in V, x + y = y + x.

4. Associativity of addition: For each element x, y, and z in V, (x + y) + z = x + (y + z).
5. Existence of the additive identity: There exists an element in V denoted as 0 such that x + 0 = x, for all x in V.
6. Existence of the additive inverse: For each element x in V, there exists another element in V that we will call -x such that x + (-x) = 0.
7. Existence of the multiplicative identity: There exists an element in F notated as 1 such that for all x in V, 1x = x.
8. Associativity of scalar multiplication: For each element x in V, and for each pair of elements a and b in F, (ab)x = a(bx).
9. Distribution of elements to scalars: For each element a in F and each pair of elements x and y in V, a(x + y) = ax + ay.
10. Distribution of scalars to elements: For each element x in V, and each pair of elements a and b in F, (a + b)x = ax + bx.

The ten axioms of vector spaces are listed above.

What did You Grasp?



1. For any vector in the vector space, addition follows associativity.
   A) True
   B) False

12. Linear Dependence and Independence of vectors

12.1.1 Linear Dependence and Independence of Vectors

Definition 1:

- Let $A = \{ \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r \}$ be a collection of vectors from R^n .
- If r > 2 and at least one of the vectors in A can be written as a linear combination of the others, then **A** is said to be **linearly dependent**.

- If none of the vectors can be expressed as a linear combination of the others, then the vectors are independent.

The motivation for this description is simple. At least one of the vectors depends (linearly) on the others. On the other hand, if no vector in A can be expressed as a linear combination of the other two, then the vectors are independent. It is also quite common to say that “the vectors are linearly dependent (or independent)” rather than “the set containing these vectors is linearly dependent (or independent).”

12.1.1 (a) Example 1

1. Are the vectors $\mathbf{v}_1 = (2, 5, 3)$, $\mathbf{v}_2= (1, 1, 1)$, and $\mathbf{v}_3 = (4, −2, 0)$ linearly independent?

   From $k_1\mathbf{v}_1 + k_2\mathbf{v}_2= \mathbf{v}_3$,

   $k_1(2,5,3) + k_2(1,1,1) = (4,-2,0)$

   which is $2k_1 + k_2 = 4$

   $5k_1 + k_2 = -2$

   $3k_1 + k_2 = 0$

From this, the vectors can be said to be linearly independent.

If none of these vectors can be expressed as a linear combination of the other two, then the vectors are independent; otherwise, they are dependent. If, for example, $\mathbf{v}_3$ were a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$, then there would exist scalars $k_1$and k_2 such that $k_1\mathbf{v}_1 + k_2\mathbf{v}_2= \mathbf{v}_3$.

The example given above is an inconsistent system. For instance, subtracting the first equation from the third yields k 1 = −4, and substituting this value into either the first or third equation gives $k_2 = 12$. However, $(k_1, k_2) = (−4, 12)$ does not satisfy the second equation.

The conclusion is that $\mathbf{v}_3$ is not a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$. A similar argument would show that $\mathbf{v}_1$ is not a linear combination of $\mathbf{v}_2$ and $\mathbf{v}_3$ and that $\mathbf{v}_2$ is not a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_3$. Thus, these three vectors are indeed linearly independent.

12.1.2 Linear Dependence and Independence of Vectors

Definition 2:

- An alternative, but entirely equivalent and often simpler definition of linear independence reads as follows. A collection of vectors v_1, v_2, …, v_r from R^n is linearly independent if the only scalars that satisfy are
- $k_1 = k_2 = ... = k_r = 0$. This is called the trivial linear combination.
- If, on the other hand, there exists a *nontrivial* linear combination that gives the zero vector, then the vectors are dependent.

Read out and understand the alternative definition for linear dependence and independence of vectors.

12.1.2 (a) Example 2

Using the second definition, find out if the vectors $\mathbf{v}_1 = (2, 5, 3)$, $\mathbf{v}_2 = (1, 1, 1)$, and $\mathbf{v}_3 = (4, -2, 0)$ linearly independent?

These vectors are linearly independent if the only scalars that satisfy are $k_1 = k_2 = k_3 = 0$. But (*) is equivalent to the homogeneous system. $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 = \mathbf{0} \quad (*)$

$$\begin{bmatrix} | & | & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \\ | & | & | \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \mathbf{0} \qquad (**)$$

Row reduction of the coefficient Matrix yields:

$$\begin{bmatrix} | & | & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} 2 & 1 & 4 \\ 5 & 1 & -2 \\ 3 & 1 & 0 \end{bmatrix} \xrightarrow{-2r_1 \text{ added to } r_2} \begin{bmatrix} 2 & 1 & 4 \\ 1 & -1 & -10 \\ 3 & 1 & 0 \end{bmatrix}$$

$$\xrightarrow{r_1 \leftrightarrow r_2} \begin{bmatrix} 1 & -1 & -10 \\ 2 & 1 & 4 \\ 3 & 1 & 0 \end{bmatrix}$$

$$\xrightarrow[-3r_1 \text{ added to } r_3]{-2r_1 \text{ added to } r_2} \begin{bmatrix} 1 & -1 & -10 \\ 0 & 3 & 24 \\ 0 & 4 & 30 \end{bmatrix}$$

$$\xrightarrow{(-4/3)r_2 \text{ added to } r_3} \begin{bmatrix} 1 & -1 & -10 \\ 0 & 3 & 24 \\ 0 & 0 & -2 \end{bmatrix}$$

This echelon form of the Matrix makes it easy to see that $k_3 = 0$, from which follow $k_2 = 0$ and $k_1 = 0$. Thus, equation (**), and therefore (*), is satisfied only by $k_1 = k_2 = k_3 = 0$, which proves that the given vectors are linearly independent.

13. Consistency of linear system of equations

13.1 Consistency of Linear System of Equations

- Solving a System of Linear Equations in Two Variables:
  - A **consistent system** is a system that has at least one solution.
  - An **inconsistent system** is a system that has no solution.
- The equations of a system are **dependent** if ALL the solutions of one equation are also solutions of the other two equations. In other words, they end up being **the same line**.
- The equations of a system are **independent** if they **do not share ALL solutions**. They can have one point in common (point of intersection), just not all of them.

We saw the definition for consistent and inconsistent systems. There are three possible solutions for a system.

No Solution (No point of Intersection):

If the three planes are parallel to each other, they will never intersect. This means they do not have any points in common. In this situation, you would have no solution.

Unique (one) solution (Only one point of intersection):

If the system in three variables has one solution, it is an ordered triple (*x, y, z*) that is a solution to ALL THREE equations.

Infinite (many) Solutions (Lying on the same line):

If the three planes end up lying on top of each other, then there is an infinite number of solutions. In this situation, they would end up being on the same plane, so any solution that would work in one equation will work in the other.

14. Eigenvalues and eigenvectors

14.1 Eigenvalues

- When you multiply a Matrix (A) by a vector (v) you get a new vector (x).

$$\mathbf{Av = x}$$

- There's also a special case where instead of getting a completely new vector you get a scaled version of the same vector you started with. In other words, a Matrix times a vector equals a scalar (lambda) times that same vector.

$$\mathbf{Av = \lambda v}$$

- When this happens we call the scalar, lambda, an eigenvalue of Matrix A.
- The number eigenvalues a Matrix has will depend on the size of the matrix. An nxn Matrix will have n eigenvalues.

We saw about eigenvalues of a matrix. Let's now learn about the general solution for finding the eigenvalue of a matrix.

We know that $Av = \lambda v$, i.e. $Av - \lambda v = 0$.

The important point to note is that zero is not a scalar but the zero vector. Also, we are searching for a solution to the above equation under the condition that v is not equal to zero. When v equals zero lambda's value becomes trivial because any scalar or Matrix multiplied by a zero vector equals another zero vector.

We now have to multiply λv by an identity Matrix (I). Multiplying by an identity Matrix is like multiplying by one for scalar equations. $Av - \lambda Iv = 0$.

Since both A and lambda*I are multiplied by v we can factor it out. So, $(A - \lambda I)v = 0$

When v is not equal to zero this equation is true only if the Matrix we multiply v by is noninvertible. This means there must not exist a Matrix B such that C*B = B*C = I, where C = A - λI, here. A Matrix is noninvertible only when its determinant equals zero.

$|A - \lambda I| = 0$

When we solve for the determinant we will get a polynomial with eigenvalues as its roots. We call this polynomial characteristic polynomial of the matrix. We can then figure out what the eigenvalues of the Matrix are by solving for the roots of the characteristic polynomial.

14.2 Finding Eigenvalue

Find the eigenvalue of the Matrix **A** = $\begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$

Solution: By the equation, $|A - \lambda I| = 0$.

$$\left| \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0$$

Step 2:

$$\left| \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} 4-\lambda & 2 \\ 1 & 3-\lambda \end{vmatrix} = 0$$

Earlier we stated that an nxn Matrix has n eigenvalues. So a 2x2 Matrix should have 2 eigenvalues.

From the previous section, $|A - \lambda I| = 0$, which is expanded as given above. First we insert our Matrix in for ***A***, and write out the identity matrix. In general an identity Matrix is written as an nxn Matrix with ones on the diagonal starting at the top left and zeroes everywhere else. We'll use a 2x2 identity Matrix here because we want it to be the same size as **A**.

The next step is to simplify everything inside the determinant to get a single matrix, which is given at step 2.

Now, we need to solve for the determinant and find the characteristic polynomial of the matrix.

$(4 - \lambda)(3 - \lambda) - (2)(1) = 0$

$12 - 4\lambda - 3\lambda + \lambda 2 = 0$

$\lambda 2 - 7\lambda + 10 = 0$

If we solve this polynomial, we'll get the solution as **λ = 5**, **λ = 2**

14.3 Properties of Eigenvalues

Properties f Eigenvalues are:

- **Property 1:**

  A square Matrix A and its transpose have the same eigenvalues.

- **Property 2:**

  The eigenvalues of a diagonal or triangular Matrix are its diagonal elements.

- **Property 3:**

  An n x n Matrix is invertible if and only if it doesn't have 0 as an eigenvalue.

The first three properties of eigenvalues are listed above. The proof can be given as follows:

Property 1:

We have that any solution of $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$ is a solution of $\det(\mathbf{A} - \lambda\mathbf{I})^T = 0$ and vice versa. Thus **A** and $\mathbf{A}^T$ have the same eigenvalues.

$\det(\mathbf{A}^T - \lambda\mathbf{I})$

$= \det(\mathbf{A}^T - \lambda\mathbf{I}^T)$

$= \det(\mathbf{A} - \lambda\mathbf{I})^T$

$= \det(\mathbf{A} - \lambda\mathbf{I})$

The matrices **A** and $\mathbf{A}^T$ will usually have different eigen***vectors***.

Property 2:

Suppose the Matrix **A** is diagonal or triangular. If you subtract λ's from its diagonal elements, the result $\mathbf{A} - \lambda\mathbf{I}$ is still diagonal or triangular. Its determinant is the product of its diagonal elements, so it is just the product of factors of the form (diagonal element – λ). The roots of the characteristic equation must then be the diagonal elements.

Property 3:

An n x n Matrix **A** has an eigenvalue 0 if and only if $\det(\mathbf{A} - 0\mathbf{I}) = 0$, i.e. if and only if $\det(\mathbf{A}) = 0$. Since A is invertible if and only if $\det\mathbf{A} \neq 0$, **A** is invertible if and only if 0 is not an eigenvalue of **A**.

- **Property 4:**

  If a Matrix **A** has eigenvalue λ with corresponding eigenvector **x**, then for any k = 1, 2, … , $\mathbf{A}^k$ has eigenvalue λ^k corresponding to the same eigenvector **x**.

- **Property 5:**

  If **A** is an invertible Matrix with eigenvalue λ corresponding to eigenvector **x**, then $\mathbf{A}^{-1}$ has eigenvalue λ^{-1} corresponding to the same eigenvector **x**.

The fourth and fifth properties of eigenvalues are listed above. The proof can be given as follows:

Property 4:

Suppose the Matrix $\mathbf{A}$ has eigenvalue λ with eigenvector $\mathbf{x}$, i.e. suppose that $\mathbf{Ax} = \lambda\mathbf{x}$. Then $\mathbf{A}^2\mathbf{x} = \mathbf{A}(\mathbf{Ax}) = \mathbf{A}(\lambda\mathbf{x}) = \lambda(\mathbf{Ax}) = \lambda(\lambda\mathbf{x}) = \lambda^2\mathbf{x}$. Multiply by more $\mathbf{A}$'s to get $\mathbf{A}^3\mathbf{x} = \lambda^3\mathbf{x}$, $\mathbf{A}^4\mathbf{x} = \lambda^4\mathbf{x}$ and so on.

Property 5:

Multiply the equation $\mathbf{Ax} = \lambda\mathbf{x}$ by $\lambda^{-1}\mathbf{A}^{-1}$:

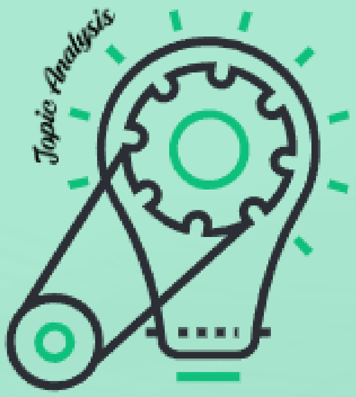
$$\lambda^{-1}\mathbf{A}^{-1}(\mathbf{Ax}) = \lambda^{-1}\mathbf{A}^{-1}\lambda\mathbf{x},$$

i.e.

$$\lambda^{-1}\mathbf{x} = \mathbf{A}^{-1}\mathbf{x}.$$

Thus $\mathbf{A}^{-1}$ has eigenvalue λ^{-1} corresponding to the same eigenvector $\mathbf{x}$.

What did You Grasp?



1. Find the eigenvalues of

$$\mathbf{A} = \begin{pmatrix} 2 & 7 \\ 7 & 2 \end{pmatrix}$$

A) -5, 9
B) 2, 7
C) 3, 8
D) -3, 9

Explanation for previous slide

Find all scalars, such that: λ has a nontrivial solution. That matrix equation has nontrivial solutions only if the matrix is not invertible or equivalently its determinant is zero. This is the characteristic equation.

$$(\mathbf{A} - \lambda\mathbf{I})\cdot\mathbf{x} = \mathbf{0}$$

$$|\mathbf{A} - \lambda\mathbf{I}| = \left|\begin{pmatrix} 2-\lambda & 7 \\ 7 & 2-\lambda \end{pmatrix}\right| = 0$$

$$(\mathbf{A} - \lambda\mathbf{I})\cdot\mathbf{x} = \mathbf{0}$$

$$\mathrm{P}(\lambda) = (2-\lambda)^2 - 49 = 0$$

$$\lambda_{1,2} = -5, 9$$

15. Cayley Hamilton Theorem

15.1 Cayley Hamilton Theorem

- Cayley-Hamilton theorem states that: "A square Matrix satisfies its own characteristic equation."

Explanation:

Let us consider that **A** be an nxn square Matrix and if its characteristic polynomial is defined as:

$P(\lambda) = |A - \lambda I_n|$

where, I_n is the identity Matrix of same order as **A**.

According to Cayley-Hamilton Theorem:P(A) = O, where, O represents the zero Matrix of same order as A.

We can say that if we replace λ by Matrix A, then the relation would be equal to zero. Hence Matrix A annihilates its own characteristic equation

15.2 Example for Cayley Hamilton Theorem

Prove Cayley Hamilton Theorem for the matrix $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Solution:

Let A = $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Characteristic polynomial of the Matrix is:

$$A - \lambda I = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 2 \\ 3 & 4-\lambda \end{bmatrix}$$

$$P(\lambda) = |A - \lambda I| = (1-\lambda)(4-\lambda) - 6$$

$$= 4 - 5\lambda + \lambda^2 - 6$$

$$= \lambda^2 - 5\lambda - 2$$

For the given Matrix we have derived the characteristic equation. We'll now see how the Matrix satisfies Cayley Hamilton Theorem.

15.3 Proof for Cayley Hamilton Theorem

In order to prove the statement of Cayley Hamilton theorem for A, we need to show that P(A) = O

$$A^2 - 5A - 2 = O$$

$$A^2 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix}$$

$$A^2 - 5A - 2$$

$$= \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix} - 5\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 9 \\ 15 & 22 \end{bmatrix} - \begin{bmatrix} 5 & 10 \\ 15 & 20 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

We have proved that P(A) = O, hence the Cayley Hamilton theorem for the given Matrix is proved.

15.4 Applications of Cayley Hamilton Theorem

- Cayley Hamilton theorem is widely applicable in many fields not only related to mathematics, but in other scientific fields as well.
- This theorem is used all over in linear algebra. One can easily find inverse of a Matrix using Cayley Hamilton theorem.
- It also plays an important role in solving ordinary differential equations. This theorem is quite useful in physics also.
- Cayley Hamilton theorem plays a vital role in computer programming and coding. In a newer subject - Rheology, where behavior of material is studied, this theorem is used to determine the equations that illustrate nature of materials.

Some of the applications of Cayley Hamilton theorem are listed above. In short, there is a vast use of Cayley Hamilton theorem in many areas where **linear equations** and **matrices** are needed to be used.

16. Linear Transformation and Orthogonal transformation

16.1 Linear Transformation

- The main example of a linear transformation is given by Matrix multiplication. Given an nxm Matrix A, define T(v)=Av, where v is written as a column vector (with m coordinates).

For example, consider $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & 2 \\ 1 & 0 \end{bmatrix}$.

- Then, T is a linear transformation from $\mathbf{R}^2$ to $\mathbf{R}^3$ is defined by

$$T(x, y) = (y, -2x + 2y, x).$$

The definition for linear transformation is given above.

If T is a linear transformation, then T0 must be 0. (So if you find T0 ≠ 0, that means your T is not a linear transformation).

16.1 (a) Examples for Linear Transformations

1. $T : \mathbb{R}^5 \to \mathbb{R}^2$ defined by

$$T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 - 5x_3 + 7x_4 + 6x_5 \\ -3x_1 + 4x_2 + 8x_3 - x_4 + x_5 \end{bmatrix},$$

or equivalently,

$$T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 & 2 & -5 & 7 & 6 \\ -3 & 4 & 8 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}.$$

2. Scaling (expansion by factor 5): $T : \mathbb{R}^3 \to \mathbb{R}^3$ with matrix $\begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

A couple of examples for linear transformations are given above.

16.2 Orthogonal Transformation

A linear transformation T from R^n to R^n is called orthogonal if it preserves the length of vectors:

$$\|T(\vec{x})\| = \|\vec{x}\|, \text{ for all } \vec{x} \text{ in } R^n$$

If $T(\vec{x}) = A\vec{x}$ is an orthogonal transformation, we say that A is an orthogonal matrix. For example, the rotation:

$$T(\vec{x}) = \begin{bmatrix} cos\phi & -sin\phi \\ sin\phi & cos\phi \end{bmatrix} \vec{x}$$ is an orthogonal transformation from R^2 to R^2, and

$$A = \begin{bmatrix} cos\phi & -sin\phi \\ sin\phi & cos\phi \end{bmatrix} \vec{x}$$ is an orthogonal Matrix for all angles ϕ.

The definition of orthogonal transformation along with an example is given above. We also need to know that:

- An orthogonal transformation is an isomorphism.
- The inverse of an orthogonal transformation is also orthogonal.
- The composition of orthogonal transformations is orthogonal.

An n x n Matrix is orthogonal if its columns are orthonormal.

17. Matrix Factorization and Types

17.1 Matrix Factorization

Matrix Factorization:

- Matrix decomposition methods, also called Matrix factorization methods.
- Are a foundation of linear algebra in computers, even for basic operations such as solving systems of linear equations, calculating the inverse, and calculating the determinant of a matrix.
- Just as its name suggests, Matrix factorization is to, obviously, factorize a matrix, i.e. to find out two (or more) matrices such that when you multiply them you will get back the original matrix.

Many complex Matrix operations cannot be solved efficiently or with stability using the limited precision of computers. Matrix decompositions are methods that reduce a Matrix into constituent parts that make it easier to calculate more complex Matrix operations.

17.2 LU Decomposition

- A mxn Matrix is said to have a LU-decomposition if there exists matrices L and U with the following properties:
  - L is a mxn lower triangular Matrix with all diagonal entries being 1.
  - U is a mxn Matrix in some echelon form.
  - A = LU
- Let A be a mxn Matrix and E_1, E_2 …. E_k be elementary matrices such that

  U = E_k ….. E_1A is in non-echelon form. If none of the E_i's corresponds to the operation of row interchange, then

  C = E_k ….. E_1 is a lower triangular invertible matrix. Further L = C^{-1} is also a lower triangular Matrix with A = LU .

Suppose we want to solve a mxn system A**X** = b. If we can find a LU-decomposition for A , then to solve A**X** =b, it is enough to solve the systems, LY = b and UX = Y

Thus the system LY = b can be solved by the method of forward substitution and the system UX = Y can be solved by the method of backward substitution.

17.3 QR Decomposition

- Qr Decomposition is applicable to: mxn Matrix **A**.
- Decomposition: A=QR, where Q is a unitary Matrix of size mxm, and R is an upper triangular Matrix of size mxn
- Comment: The QR decomposition provides an alternative way of solving the system of equations Ax=b without inverting the Matrix A.
- The fact that Q is orthogonal means that Q^TQ =I, so that Ax=b is equivalent to Rx=Q^Tb, which is easier to solve since R is triangular.

The QR decomposition (also called the QR factorization) of a Matrix is a decomposition of the Matrix into an orthogonal Matrix and a triangular matrix. A QR decomposition of a real square Matrix A is a decomposition of A as A = QR, where Q is an orthogonal Matrix (i.e. Q^TQ = I) and R is an upper triangular matrix. If A is nonsingular, then this factorization is unique.

In general QR decomposition is not unique, but if A is of full rank, then there exists a single R that has all positive diagonal elements. If A is square, also Q is unique.

17.4 Single Value Decomposition (SVD)

- The singular value decomposition of a Matrix A is the factorization of A into the product of three matrices $A = UDV^T$,
  - *A is an m × n matrix*
  - *U is an m × n orthogonal matrix*
  - *D is an n × n diagonal matrix*
  - *V is an n × n orthogonal matrix*
- For any m x n real Matrix A, the SVD consists of matrices U,D,V which are always real – this is unlike eigenvectors and eigenvalues of A which may be complex even if A is real.

SVD is applicable to: mxn Matrix A.

Decomposition: $A=UDV^T$, where D is a nonnegative diagonal matrix, and U and V are unitary matrices, and V^T is the conjugate transpose of V (or simply the transpose, if V contains real numbers only).

The singular values are always non-negative, even though the eigenvalues may be negative.

The diagonal elements of D are called the singular values of A. The singular value decomposition involves finding basis directions along which Matrix multiplication is equivalent to scalar multiplication, but it has greater generality since the Matrix under consideration need not be square.

Uniqueness: the singular values of A are always uniquely determined. U and V need not to be unique in general.

In a nutshell, we learnt:

1. Introduction to Matrices along with the basic terminologies
2. Matrix notations and types of matrices
3. Matrix equality
4. Operations on matrices like addition, subtraction and multiplication
5. Basics of Determinants
6. Singularity of a matrix
7. Orthogonal Matrix
8. Elementary transformations and elementary matrices and Matrix inverse computation using elementary transformations
9. Echelon forms and echelon transformations

10. How to find Matrix rank and reduce a Matrix to its normal form
11. Vector Spaces and enumerate the axioms
12. Linear dependence and independence of vectors
13. Consistency of linear system of equations
14. Eigenvalues and eigenvectors
15. Cayley Hamilton Theorem and its applications
16. Linear transformation and orthogonal transformation
17. Matrix factorization and enumerate a few types such as LU, QR and SVD decomposition