MODULE 5

Linear Algebra

You will learn about the topic, ‘Linear Algebra’ in this module.

Module Learning Objectives

At the end of this module, you will be able to:

- Understand the introduction, history and the applications of linear algebra.
- Understand and apply the important notations in linear algebra.
- Describe the important concepts of linear algebra:
  - Scalars
  - Vectors
  - Vector spaces
  - Basis
  - Matrices
- Understand the important definitions on linear system of equations.

Module Topics

The following topics that will be covered in the module:

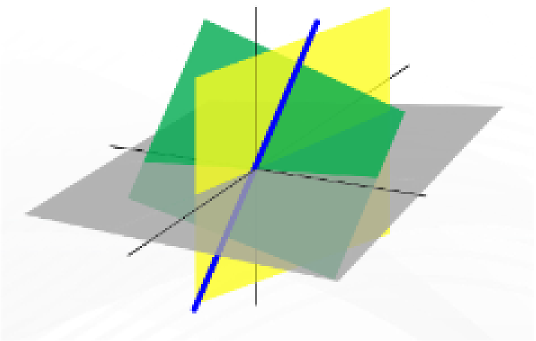
1. Introduction to Linear Algebra
2. Notations in Linear Algebra
3. Important Concepts of Linear Algebra
4. Definitions of Linear Algebra

1 Introduction to Linear Algebra

1.1 Introduction to Linear Algebra

Linear Algebra:

- The branch of mathematics that involves the study of vector spaces and the linear transformations between them.
- It involves the study of linear equations and linear functions and their representation through matrices and vector spaces.
- To be precise, Linear algebra is the study of vectors and linear structures.
- Initial use cases were solving systems of linear equations.



Linear algebra, a branch of mathematics, is widely used in the field of science and engineering. Probability theory, linear algebra and matrices are the three important topics to be understood clearly for understanding and working with data processing and machine learning algorithms.

Linear algebra is very similar to the Algebra taught in the High school curriculum. The only difference is that linear algebra deals with vectors in place of ordinary single numbers, i.e., scalars. Many of the operations that you perform on scalars, such as addition, subtraction and multiplication, can be generalized to be performed on vectors.

Thus Linear algebra is the study of vectors and the linear transformations between them.

1.2 History of Linear Algebra

The following timelines illustrate the history of Linear Algebra:

4000 years ago

4000 years ago - **Babylonians** - Solving simple 2X2 system of linear equations with two unknowns

200 BC

200 BC - **Chinese** - Solution for the 3X3 system of equations

1693

1693 - Determinants by Leibniz (who discovered calculus)

1750

1750 - Cramer rule by Cramer - Introduction to the analysis of algebraic curves

1776

1776 - **Euler** - Euler's theorem

1811

1811 - **Gauss** - Method of least squares; Gaussian elimination

1848

1848 - **JJ Sylvester** - Introduced the term 'matrix' for an array of numbers

1855

1855 - **Arthur Cayley** - Matrix multiplication - Identity matrix and Inverse of a square matrix

The foundations of linear algebra were laid long back by Babylonians 4000 years ago, as they knew how to solve a simple 2X2 system of linear equations with two unknowns. Around 200 BC, the Chinese published that "Nine Chapters of the Mathematical Art," they displayed the ability to solve a 3X3 system of equations. The power and progress in linear algebra did not come to fruition until the late 17th century.

The modern study of systems of linear equations can be said to have originated with Leibniz, who in 1693 invented the notion of a determinant for this purpose. But his investigations remained unknown at the time. In 1750 Cramer published *Introduction to the Analysis of Algebraic Curves*, in this work he published a rule after him, Cramer's rule, for the solution of an nXn system, but he provided no proofs.

It was in the 1770s that Euler was perhaps the first to observe that a system of n equations in n unknowns does not necessarily have a unique solution, noting that to obtain uniqueness it is necessary to add conditions. He had in mind the idea of the dependence of one equation on the others, although he did not give precise conditions. In the eighteenth century the study of linear equations was usually subsumed under that of determinants, so no consideration was given to systems in which the number of equations differed from the number of unknowns. In 1776, Euler's theorem was proposed.

In 1811, Gauss, who invented the method of least squares, introduced a systematic procedure, now called Gaussian elimination, for the solution of systems of linear equations, though he did not use the matrix notation. Gaussian elimination uses the concepts of combining, swapping, or multiplying rows with each other in order to eliminate variables from certain equations. After variables are determined, the student is then to use back substitution to help find the remaining unknown variables.

In 1848, J.J. Sylvester introduced the term "matrix," the Latin word for womb, as a name for an array of numbers. Matrix multiplication or matrix algebra came from the work of Arthur Cayley in 1855. His work dealing with Matrix multiplication culminated in his theorem, the Cayley-Hamilton Theorem. Simply stated, a square matrix satisfies its characteristic equation. Cayley's efforts were published in two papers, one in 1850 and the other in 1858. His works introduced the idea of the identity matrix as well as the inverse of a square matrix.

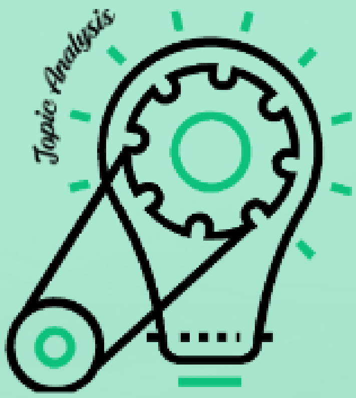
1.3 Applications of Linear Algebra

Some of the disciplines where Linear algebra is applied:	Applications of Linear Algebra:
Mathematics	Oil exploration
Engineering	Electrical networks
Computer Science	Linear Programming
Physics	Aircraft design
Biology	Space flight and control systems
Economics	GPS navigation
Statistics	Multichannel image processing
	Computer graphics

Some of the domains and the applications where linear algebra concepts are applied, are given in the slide. There is a saying, there is no domain where linear algebra concepts are not applied.

Linear algebra forms the centre of mathematics around which most of the mathematical methods revolve around. Linear algebra forms the basis of many science and engineering disciplines, as it is used in deriving the models of many naturally occurring phenomena and performing efficient computing with these models.

What did You Grasp?



1. Who introduced the method of least squares?
   A) Euler
   B) Cramer
   C) Gauss
   D) JJ Sylvester

2 Notations in Linear Algebra

2.1 Linear Algebra Notations

Important notations of linear algebra are listed in the table below.

$\mathbb{R}, \mathbb{R}^+, \mathbb{R}^n$	real numbers, reals greater than 0, ordered n-tuples of reals
$\mathbb{N}$	natural numbers: $\{0, 1, 2, \ldots\}$
$\mathbb{C}$	complex numbers
$\{\ldots \mid \ldots\}$	set of . . . such that . . .
$(a\,..\,b)$, $[a\,..\,b]$	interval (open or closed) of reals between a and b
$\langle\ldots\rangle$	sequence; like a set but order matters
V, W, U	vector spaces
$\vec{v}, \vec{w}$	vectors

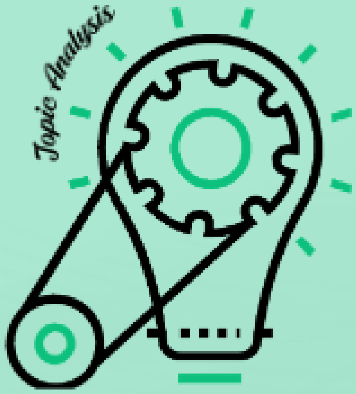
$M \oplus N$	direct sum of subspaces
$V \cong W$	isomorphic spaces
h, g	homomorphisms, linear maps
H, G	matrices
t, s	transformations; maps from a space to itself
T, S	square matrices

$\vec{0}, \vec{0}_V$	zero vector, zero vector of V
B, D	bases
$\mathcal{E}_n = \langle \vec{e}_1, \ldots, \vec{e}_n \rangle$	standard basis for $\mathbb{R}^n$
$\vec{\beta}, \vec{\delta}$	basis vectors
$\mathrm{Rep}_B(\vec{v})$	matrix representing the vector
$\mathcal{P}_n$	set of n-th degree polynomials
$\mathcal{M}_{n \times m}$	set of $n \times m$ matrices
$[S]$	span of the set S

$\mathrm{Rep}_{B,D}(h)$	matrix representing the map h
$h_{i,j}$	matrix entry from row i, column j
$\lvert T \rvert$	determinant of the matrix T
$\mathcal{R}(h), \mathcal{N}(h)$	rangespace and nullspace of the map h
$\mathcal{R}_\infty(h), \mathcal{N}_\infty(h)$	generalized rangespace and nullspace

The important notations and their explanation in Linear algebra are given in the table. The knowledge of these notations is important as you will come across the notations in many of the upcoming sections.

What did You Grasp?



1. Which of the following options is a correct notation for a vector space?

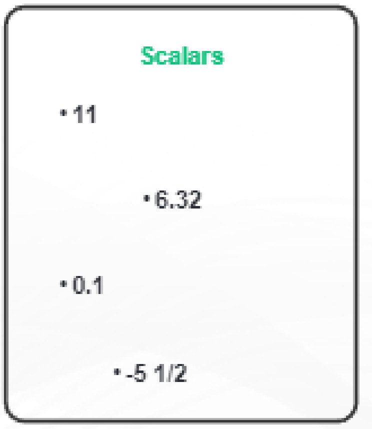
   A) *V*
   B) **v**
   C) *V*
   D) None of the above

3 Important Concepts of Linear Algebra

3.1 Scalars

Scalars:

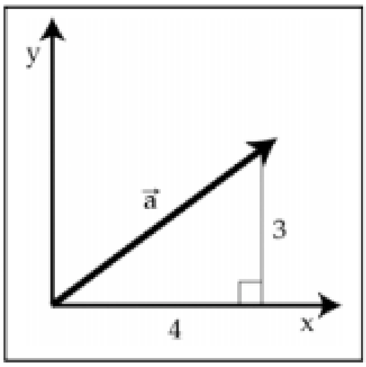
- A single number or an element of a field, which forms the basis of a vector space.
- Refer to real numbers or other elements of a field.
- Have magnitude only, but not direction.
- Are denoted by lowercase italics.
- The kind or number of scalars is specified when scalars are introduced.



To be precise, a scalar is a number. Examples of scalars are temperature, distance, speed, or mass – all quantities that have a magnitude but no “direction”, other than perhaps positive or negative.

3.2 Vectors

Vectors:



- Are arrays of objects, which have magnitude and direction.
- Are not only stacks of numbers, but they also include, but not limited to polynomials, power series, functions, etc.
- Should be added if these are of the same kind.
- Are things that you can add and scalar multiply.

A very special vector can be produced from any vector of any kind by scalar multiplying any vector by the number 0. This is called the zero vector.

We saw that Linear algebra is a study of vectors and linear functions.
We'll now see what 'Vectors' are. Vectors are arrays of objects, but they are not constrained to numbers. Vectors are arrays or list of objects that have both magnitude and direction. Things that can be added and scalar multiplied, can be termed as vectors.

There are (at least) two ways to interpret vectors: One way to think of the vector as being a point in a space. Then vectors are a way of identifying that point in space, where each number represents the vectors component in that dimension. Another way to think of a vector is a magnitude and a direction, e.g. a quantity like 'velocity' ("the fighter jet's velocity is 250 mph north-by-northwest"). In this way, a vector is a directed arrow pointing from the origin to the endpoint given by the list of numbers

An object in R (refer to real numbers in notations section), or in any Rn, comprised of a magnitude and a direction is a vector.

The different kinds of vectors are:

- numbers
- n-vectors
- 2nd order polynomials
- polynomials
- power series
- functions with a certain domain

3.3 Vector Spaces

A vector space is a set that is closed under addition and scalar multiplication and is denoted as V.

A *vector space* (over $\mathbb{R}$) consists of a set V along with two operations '+' and '·' subject to the conditions that for all vectors $\vec{v}, \vec{w}, \vec{u} \in V$ and all *scalars* $r, s \in \mathbb{R}$:

(1) the set V is closed under vector addition, that is, $\vec{v} + \vec{w} \in V$
(2) vector addition is commutative, $\vec{v} + \vec{w} = \vec{w} + \vec{v}$
(3) vector addition is associative, $(\vec{v} + \vec{w}) + \vec{u} = \vec{v} + (\vec{w} + \vec{u})$
(4) there is a *zero vector* $\vec{0} \in V$ such that $\vec{v} + \vec{0} = \vec{v}$ for all $\vec{v} \in V$
(5) each $\vec{v} \in V$ has an *additive inverse* $\vec{w} \in V$ such that $\vec{w} + \vec{v} = \vec{0}$
(6) the set V is closed under scalar multiplication, that is, $r \cdot \vec{v} \in V$
(7) addition of scalars distributes over scalar multiplication, $(r+s) \cdot \vec{v} = r \cdot \vec{v} + s \cdot \vec{v}$
(8) scalar multiplication distributes over vector addition, $r \cdot (\vec{v} + \vec{w}) = r \cdot \vec{v} + r \cdot \vec{w}$
(9) ordinary multiplication of scalars associates with scalar multiplication, $(rs) \cdot \vec{v} = r \cdot (s \cdot \vec{v})$
(10) multiplication by the scalar 1 is the identity operation, $1 \cdot \vec{v} = \vec{v}$.

A vector space is one that has a set of vectors, a set of scalars and a scalar multiplication operation that takes a scalar k and a vector **v** to another vector *k***v**.

The definition above involves two kinds of addition and two kinds of multiplication. In condition 7, the '+' on the left is addition of two real numbers while the '+' on the right is addition of two vectors in ***V***. r and s are real numbers so 'r + s' can only mean real number addition. In the same way, in condition 9, the left side 'rs' is ordinary real number multiplication, while its right side 's v' is the scalar multiplication defined for this vector space.

For any vector space, a ***subspace*** is a subset that is itself a vector space, under the inherited operations. The ***span*** (or linear closure) of a nonempty subset S of a vector space is the set of all linear combinations of vectors from S. The span of the empty subset of a vector space is its trivial subspace.

Some important definitions are:

Let **v***1*, **v***2*, · · · **v***k* be vectors in a vector space ***V*** , then the span of these vectors is the set of all possible linear combinations of **v***1*, **v***2*, · · · **v***k*. This set is denoted by Span{**v***1*, **v***2*, · · · **v***k*}.

Let *S* be a subspace of a vector space ***V***. We say that **v***1*, **v**2, · · · vk form a spanning set for *S* if *S* = Span{**v***1*, **v**2, · · · **v***k*}.

A set of vectors {**v***1*, **v***2*, · · · **v***k*} in a vector space ***V*** is said to be linearly dependent if there exist scalars, *c1*, *c2*, ….. *ck* not all zero, such that *c1***v1** + *c2***v2** + · · · + *ck***vk** = **0**.

A set of vectors {**v***1*, **v***2*, · · · **v***k*} in a vector space ***V*** is said to be linearly independent if *c1***v1** + *c2***v2** + · · · + *ck***vk** = **0** implies that all the coefficients *c1*, *c2*, ….. *ck* are zero.

3.4 Basis

- A basis for a vector space is a sequence of vectors that is linearly independent and that spans the space.
- For any

$$\mathcal{E}_n = \langle \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \rangle$$

is the standard (or natural) basis, and we denote these vectors, $\vec{e}_1, \ldots, \vec{e}_n$

- In any vector space, a subset is a basis, if and only if each vector in the space can be expressed as a linear combination of elements of the subset in one and only one way.

A set of vectors {**v***1*, **v***2*, · · · **v***k*} in a vector space ***V*** is said to be a basis for a subspace *S* if it is a linearly independent spanning set for *S*.

A basis is a sequence, meaning that bases are different if they contain the same elements but in different orders, we denote it with angle brackets. A sequence is linearly independent if the multiset consisting of the elements of the sequence is independent. Similarly, a sequence spans the space if the set of elements of the sequence spans the space.

The dimension of a vector space *S* is equal to the number of vectors in a basis for *S*.

3.5 Matrices

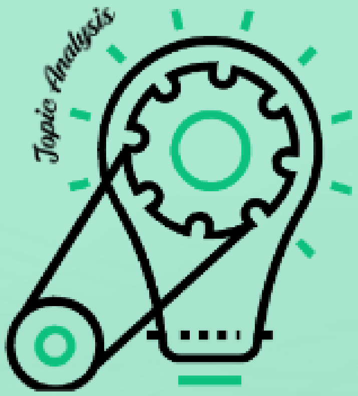
- A matrix, like a vector, is an array of numbers, the difference is that matrix is an organized table of numbers instead of just being a list.
- Suppose that m and n are positive integers. An m x n matrix is a rectangular array of numbers with m rows and n columns. Matrices are usually denoted by capital letters.
- For example:

$$A = \begin{bmatrix} 8 & 10 & -6 & -8 \\ -5 & 7 & 6 & -6 \\ 0 & 5 & -10 & 1 \\ 1 & -3 & -10 & 5 \\ -4 & -8 & -1 & -10 \end{bmatrix}$$

is a 5 x 4 matrix.

Matrices are the result of organizing information related to linear functions. A Matrix is a 2-D array of numbers, so each element is identified by two indices instead of just one. We usually give matrices, upper-case variable names with bold typeface, such as **A**. If a real-valued matrix **A** has a height of m (row) and a width of n (column), then we say that $A \in R^{m \times n}$.

What did You Grasp?



1. State True or False.

   Velocity of wind is a scalar.

   A) True
   B) False

2. State True or False.

   A basis has set of vectors that are linearly dependent.

   B) True
   C) False

4 Definitions of Linear Algebra

4.1 Definitions on Linear Equations

1. A linear equation in the variables x_1, x_2, ... xn is an equation of the form

$$a_1x_1 + a_2x_2 + \ldots\ldots + a_nx_n = b,$$

where a_1, a_2, ... an and b are real or complex numbers (that are often given in advance).

2. If m and n are positive integers, a system of linear equations in the variables x_1, x_2, ... x_n is a set of equations of the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned}$$

where a_{ij} (i = 1, ... m, j = 1, ... n) and bi (i = 1, ... m) are real or complex numbers (that are often given in advance). This system of linear equations is said to consist of m equations with n unknowns.

Example for definition 1:

4x + 4y - 7z + 8w = 8 is a linear equation in the variables x, y, z and w. On the other hand, $3x + 5y^2 = 6$ is not a linear equation (because of the presence of y^2).

Example for definition 2:

$2x_1 + 2x_2 - 5x_3 + 4x_4 - 2x_5 = 4$

$-4x_1 + x_2 - 5x_4 = -1$

$3x_1 - 2x_2 + 4x_4 - 2x_5 = -4$

is a system of three linear equations in five unknowns. (The "unknowns" or "variables" of the system are x_1, x_2, x_3, x_4, x_5).

3. The solution set of a system of linear equations is the set of all solutions of the system.

4. Two linear systems are said to be equivalent to each other if they have the same solution set.

5. A linear system is said to be consistent if it has at least one solution and is otherwise said to be inconsistent.

Definitions 3 and 4 are self-explanatory.

Example for definition 5:

The linear system

$2x_1 + 2x_2 - 5x_3 + 4x_4 - 2x_5 = 4$

$-4x_1 + x_2 - 5x_4 = -1$

$3x_1 - 2x_2 + 4x_4 - 2x_5 = -4$

is consistent because it has at least one solution. (It in fact has infinitely many solutions.)

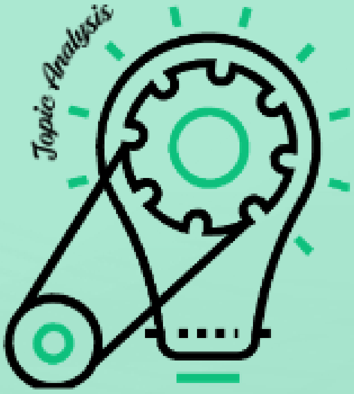
However, the linear system

$x + 2y = 5$

$x + 2y = 9$

is obviously inconsistent.

What did You Grasp?



1. When do two linear systems become equivalent?
   A) When they have unknowns.
   B) If they have the same solution set.
   C) If they have real numbers as solution.
   D) None of the above.

In a nutshell, we learnt:

1. The introduction, history and applications of Linear algebra.
2. The important notations of Linear algebra.
3. Important concepts in Linear algebra:
   - Scalars
   - Vectors
   - Vector spaces
   - Basis
   - Matrices
4. Important definitions on linear system of equations.

