FinTech 545 - Week 4

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1. **For Classical Brownian Motion:**

$$P_t = P_{t-1} + r_t$$

Calculating the expected value,

$$\begin{aligned}\mathbb{E}[P_t] &= \mathbb{E}[P_{t-1} + r_t] \\ &= \mathbb{E}[P_{t-1}] + \mathbb{E}[r_t] \\ &= P_{t-1}\end{aligned}$$

To get the standard deviation, we need to calculate the variance first,

$$\begin{aligned}\mathrm{Var}(P_t) &= \mathrm{Var}(P_{t-1} + r_t) \\ &= \mathrm{Var}(r_t) \\ &= \sigma^2\end{aligned}$$

Thus we have the standard deviation,

$$\sigma_t = \sigma$$

For Arithmetic Return System:

$$P_t = P_{t-1}(1 + r_t)$$

Calculating the expected value,

$$\begin{aligned}\mathbb{E}[P_t] &= \mathbb{E}[P_{t-1}(1 + r_t)] \\ &= P_{t-1}\mathbb{E}[1 + r_t] \\ &= P_{t-1}\end{aligned}$$

Similarly,

$$\begin{aligned}\mathrm{Var}(P_t) &= \mathrm{Var}(P_{t-1}(1 + r_t)) \\ &= P_{t-1}^2\mathrm{Var}(1 + r_t) \\ &= P_{t-1}^2\sigma^2\end{aligned}$$

Thus we have the standard deviation,

$$\sigma_t = P_{t-1}\sigma$$

For Geometric Brownian Motion:

$$P_t = P_{t-1}e^{r_t}$$

Calculating the expected value,

$$\begin{aligned}\mathbb{E}[P_t] &= \mathbb{E}[P_{t-1}e^{r_t}] \\ &= P_{t-1}\mathbb{E}[e^{r_t}]\end{aligned}$$

Here, we need to use the moment-generating function of normal distribution -

$$M_X(t) = \mathbb{E}[e^{tX}] = e^{(t\mu + \frac{1}{2}\sigma^2 t^2)}$$

Thus,

$$\begin{aligned}\mathbb{E}[e^{r_t}] &= M_{r_t}(1) \\ &= e^{\frac{1}{2}\sigma^2}\end{aligned}$$

Thus we have the expectation,

$$\begin{aligned}\mathbb{E}[P_t] &= P_{t-1}\mathbb{E}[e^{r_t}] \\ &= P_{t-1}e^{\frac{1}{2}\sigma^2}\end{aligned}$$

Similarly,

$$\begin{aligned}\mathrm{Var}(P_t) &= \mathrm{Var}(P_{t-1}e^{r_t}) \\ &= P_{t-1}^2\mathrm{Var}(e^{r_t}) \\ &= P_{t-1}^2(\mathbb{E}[(e^{r_t})^2] - (\mathbb{E}[e^{r_t}])^2) \\ &= P_{t-1}^2(M_{r_t}(2) - (P_{t-1}e^{\frac{1}{2}\sigma^2})^2) \\ &= P_{t-1}^2(e^{2\sigma^2} - e^{\sigma^2}) \\ &= P_{t-1}^2 e^{\sigma^2}(e^{\sigma^2} - 1)\end{aligned}$$

Thus we have the standard deviation,

$$\sigma_t = P_{t-1}\sqrt{e^{\sigma^2}(e^{\sigma^2} - 1)}$$

WLOG: Set $P_{t-1} = 10$ and $\sigma = 1$, then simulate 1000000 times for the return equation.

For the mean-

	Classical Brownian Motion	Arithmetic Return System	Geometric Brownian Motion
Expectation	10	10	16.4872
Simulation	10.0007	10.0065	16.4833
Difference	-0.0007	-0.0065	0.0039

From the difference between expectation and simulation, we can tell that the simulation for the mean basically is the same as the expectations.

For the standard deviation-

	Classical Brownian Motion	Arithmetic Return System	Geometric Brownian Motion
Expectation	1	10	21.6120
Simulation	0.9995	9.9949	21.5161
Difference	0.0005	0.0051	0.0959

From the difference between expectation and simulation, we can tell that the simulation for the standard deviation basically is the same as the expectations.

2. By implementing a similar function to the "`return_calculate()`", we can calculate the arithmetic returns for all prices. The following picture contains some of the results.

	SPY	AAPL	MSFT	AMZN	NVDA	GOOGL	TSLA	GOOG	BRK-B	META	...
1	-0.010544	-0.013611	-0.016667	-0.002425	-0.020808	-0.017223	-0.025076	-0.016915	-0.016854	-0.030479	...
2	-0.003773	-0.008215	-0.010974	-0.010980	-0.013336	-0.009643	0.015581	-0.011042	-0.003890	-0.011103	...
3	0.017965	0.009254	0.019111	0.026723	0.018795	0.024717	0.033817	0.027912	0.016089	0.011669	...
4	0.006536	-0.009618	0.001666	0.002626	0.020126	-0.009776	0.019598	-0.009595	0.008184	0.010412	...
5	0.015535	0.018840	0.022977	0.026575	0.028377	0.020945	0.036023	0.021568	0.008576	0.043749	...
...	...	...	...	...	...	...	...	...	...	...	...
261	0.000586	0.016913	-0.003513	-0.002920	0.001503	0.005895	-0.033201	0.004772	0.006986	0.007459	...
262	-0.002074	0.006181	-0.001246	-0.016788	-0.010144	-0.001230	0.004599	-0.000936	0.000135	0.008329	...
263	-0.009193	-0.019992	-0.023977	-0.017002	-0.029435	-0.031150	-0.014672	-0.030541	-0.009879	-0.017701	...
264	-0.016528	-0.008889	-0.003866	-0.044053	-0.028931	-0.024675	-0.026239	-0.023999	-0.009651	-0.013148	...
265	-0.002249	0.004945	-0.007887	-0.001624	0.014457	-0.001457	-0.042315	-0.000837	-0.008588	0.011328	...

Calculating the VaR with percentage and dollar basis respectively.

Percentage Basis -

Normal Distribution	0.0543
Normal Distribution with EW Variance	0.0300
MLE Fitted T Distribution	0.0431
AR(1)	0.0542
Historic Simulation	0.0395

Dollar Basis - Calculated with the most recent price

Normal Distribution	16.2361
Normal Distribution with EW Variance	8.9670
MLE Fitted T Distribution	12.9007
AR(1)	16.2189
Historic Simulation	11.8160

For the dollar basis VaR, it is simply percentage basis with a scale. WLOG, we will only compare percentage basis VaR for here. The maximum of VaR is 0.0543 for normal distribution, and it is close to the VaR for the AR(1) model. The minimum of VaR is 0.0300 for normal distribution with EW Variance, which contrasts with simply normal distribution. This implied the EW Variance is much smaller than that of simply normal distribution. It is probably the effect of the weight balancing the present and the forecast.

The MLE fitted T Distribution' is the median VaR, which is close to the Historic Simulation. Still, there is a big difference between these two. For the Historic Simulation, it does not have a really close VaR with other models, indicating the four models do not depict the real distribution of the price.

3. To compute the VaR for the portfolio, we need to know about the standard deviation for the portfolio. To get it, we have several steps to take.

   We need to know about the weight of each stock first, which we can acquire by dividing the individual value by the total value.

   After doing that, we need to calculate the covariance matrix for the return of the stock, where we can use our method of `return_calculate()` to get the return from the price we have. If we get the result of the return, we can then calculate the exponentially weight covariance matrix for the portfolio.

   By applying vector and matrix multiplication, we can acquire the desired standard deviation. Also, by setting the significance level at 5%, we can calculate the dollar basis VaR.

Here are the results of using the Arithmetic Return System:

Portfolio A	15206.3910
Portfolio B	7741.2510
Portfolio C	17877.7331
Total Portfolio	37972.2945

We can see that the VaR for portfolio C is the biggest, then for A, and for B. We need to take more care of managing the risk for portfolio C. Noted that the VaR for total portfolio is smaller than the sum of A, B and C, indicating the diversification of the portfolio and reduction of the risk.

Aside from the arithmetic return system, here I choose the Geometric Brownian Motion to depict the return.

I chose this model for several reasons. GBM is the fundamentals of lots of financial theories. For it depicts the randomness of the asset movement, it would make much more sense for financial modeling.

Also, calculating the log return is much easier than calculating the percentage change. For a small amount of change, it could be approximated as the percentage change.

Log return also lowers the scale for the data, making it more stationary.

Here are the results of using the Geometric Brownian Motion:

Portfolio A	15242.0889
Portfolio B	7775.0860
Portfolio C	17836.0019
Total Portfolio	38039.2767

Not much has changed if we switch from Arithmetic Return System to Geometric Brownian Motion. The individual VaR for portfolio C has dropped slightly, but the VaR for portfolio A, B and the total portfolio has increased slightly.