Assignment 1: Counting/Arrangements
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1. (a) Given any polygon with n sides (where n ≥ 3), I noticed that the triangles formed had a pattern. The triangle $V_1V_2V_3$ was allocated $V_1 = n$ vertex choices, $V_2 = (n-1)$ vertex choices and last $V_3 = (n-3)$ vertex choices. This uses the product rule because there are three steps. $\therefore$ for n vertices, combinations $= n(n-1)(n-2)$, however we must remove three factorial duplicates. This is equivalent to n choose 3. In conclusion, there are

$$C(n,3) = \frac{n!}{(n-3)!3!}$$

triangles formed by the vertices of a polygon with n sides.

(b) To figure out the number of times a polygon can use it's vertices to form a triangle (where no sides of the triangle is on the polygon), I found the number of times one side and two sides where on the polygon:

Two sides: I am looking for a set of consecutive vertices. By example, when n = 4 A, B, C, D, there are $|ABC, BCD, CDA, DAC| = 4$ unique combinations. I know by testing more cases, number of consecutive formations $= n$. $\therefore$ there are n cases where a triangle is on two sides of the polygon.

One side: Again, there are n ways to make a consecutive pair of TWO vertices [n=4 A,B,C,D; AB, BC, CD, DA]. But this time, I need one more vertex. I know I only have $n-2$ choices of vertices left, but I don't want to include the case with three sides $\therefore$ I have $n-4$ (one less in the back of AB and one less in the front) to choose one from. $\therefore$ there are $n \cdot C(n-4,1)$ case where a triangle is with one side of the polygon.

Now I subtract with the total from part A:
$\therefore$ The number of no-side-on-polygon triangles are:

$$\text{Total - One side } \triangle \text{ - Two side } \triangle$$

$$\begin{aligned} C(n,3) - n - (n \cdot C(n-4,1)) &= \frac{n!}{(n-3)!3!} - n - \frac{n \cdot (n-4)!}{(n-4-1)!(1)!} \\ &= \frac{n!}{(n-3)! \cdot 3!} - n - n \cdot (n-4)! \qquad (1) \end{aligned}$$

This equation works for when n ≥ 5. When $n = 5$, there are no possibilities for no-side-on-polygon triangles (evaluating equation (1) - above - at $n = 0$):

$$\frac{5!}{(5-3)! \cdot 3!} - 5 - 5 \cdot (1)! = \frac{20}{2 \cdot 1} - 10 = 0$$

Therefore, for $3 \leq n < 5$, there must be 0 possibilities as well.

2. (a) Each of the first three options is a combination selection without repetition. The final is the sum of a series of combinations of toppings (from none to 4), where order doesn't matter. Each step has k more steps, therefore, I will use the product rule:

First, I will show $C(n,1) = n$

$$C(n,r) = \frac{n!}{(n-r)! \cdot r!} \implies C(n,1) = \frac{n \cdot (n-1) \cdot \ldots \cdot 1}{(n-1) \cdot \ldots \cdot 1 \cdot 1} = n$$

$\therefore C(n,1) = n$
The equation to solve:

$$C(2,1) \cdot C(3,1) \cdot C(5,1) \cdot \sum_{i=0}^{4} C(8,i)$$

$$= 30 * [C(8,0) + C(8,1) + C(8,2) + C(8,3) + C(8,4)]$$
$$= 30 * [1 + 8 + 28 + 56 + 70]$$
$$= 30 * 163 = 4890$$

$\therefore$ there are 4890 different sandwiches I could order.

(b) I need to find the number of combinations for 3 different sandwiches first. I will simply use the combinations formula (4890 choose 3).

$$C(4890,3) = \frac{4890!}{4887! \cdot 3!} = \frac{4890 \cdot 4889 \cdot 4888}{3!} = 19476407080$$

$\therefore$ there are $19,476,407,080$ combinations of 3 different sandwiches I could order. However, I need to also add the orders of 3 similar sandwiches, and 2 similar sandwiches to this.

It must be 4890 total options for 3 similar sandwiches, because there are 4890 combinations to make one sandwich and there is a 1:1 correspondence between a unique sandwich and three of the same sandwiches.

1,1,1; 2,2,2; ...; 4890,4890,4890 = 4890 total choices
$\therefore$ 19476407080 + 4890 = 19476411970 combinations of 3 different sandwiches and 3 similar sandwiches.

It must be 4890*4889 total options for 2 similar sandwiches. The first factor comes for the same reason above (1,1; 2,2; ...; 4890,4890 = 4890 total choices). The second factor, is one less than the first, otherwise I would have three of the same.

19476411970 + 4890*4889 = 19500319180

$\therefore$ there are 19500319180 combinations of 3 sandwiches I could come home with.

3. (a) There are 5 ways to arrange b - the last space is left for an 'a': $C(5,1) = 5$, then there are 3 unique ways to arrange an a with b $C(3,1) = 3 \therefore$ by the product rule:

$$C(5,1) \cdot C(3,1) = 5 * 3 = 15$$

There are four spots left for a, a, n, n. I choose 2 spots for the a (from four) and then four spots for the n (from 2). I use the product rule because there are 2 steps that take place:

$$C(4,2) \cdot C(2,2) = \frac{4!}{2! \cdot 2!} = 6$$

Finally, I use the product rule on the last two steps which are: # of ways to make "ba" · # of ways to place the remaining characters:

$$15 \cdot 6 = 90$$

$\therefore$ there are 90 ways to make the pattern "ba" in the arrangements of b, a, n, a, n, a

(b) In this case, I try to make the pattern bnn occur. I select 1 of 4 spots, leaving 2 spots for 2 n's:

$$C(4,1) \cdot C(2,2) \cdot C(3,3) = 4 \cdot 1 \cdot 1 = 4$$

Now I subtract 4 from the total possible arrangements. This is just an arrangement with repetition:

$$P(6;1,2,3) = \frac{6!}{1! \cdot 2! \cdot 3!} = \frac{6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 1 \cdot} = \frac{120}{2} = 60$$
$$60 - 4 = 56$$

$\therefore$ there are 56 ways to make the pattern bnn never occur.

(c) I begin by placing the a's in any spot but one. Then I place the n's in the remaining spots (minus one). Then I put the b in the last place.

$$C(5,3) \cdot C(2,2) \cdot C(1,1) = \frac{5!}{(5-3)! \cdot 3!} \cdot 1 \cdot 1 = \frac{5 \cdot 4}{2 \cdot 1} = \frac{20}{2} = 10$$

$\therefore$ there are 10 ways for b to occur before any a.

4. I begin by calculating the total possibilities for arranging six 3's and four 2's, using $P(n; r_1, r_2 ... r_k)$ (arrangement with repeats).

$$P(10;6,4) = \frac{10!}{6! \cdot 4!} = \frac{10 \cdot 9 \cdot 8 \cdot 7}{4 \cdot 3 \cdot 2 \cdot 1} = \frac{5040}{24} = 210$$

$\therefore$ there are 210 total arrangements of six 3's and four 2's.

Now I figure out the number of times you can possibly have two's together. There are four distinct ways: 2222, 222;2, 22;22 and 2;2;22.

The case with 2222 and 3 6's:

$$P(7;6,1) = \frac{7!}{6! \cdot 1!} = 7$$

The case with 222, 2 and 3 6's:

$$P(8;6,1,1) = \frac{8!}{6! \cdot 1! \cdot 1!} = 8 * 7 = 56$$

The case with 22, 22 and 3 6's:

$$P(9;6,2) = \frac{8!}{6! \cdot 2!} = \frac{56}{2} = 28$$

The case with 22, 2, 2 and 3 6's:

$$P(9;6,(2+1)) = \frac{9!}{6! \cdot (2+1)!} = \frac{504}{6} = 84$$

Now I subtract the initial possible arrangements with the sum of these cases:

$$210 - [7 + 56 + 28 + 84] = 210 - 175 = 35$$

$\therefore$ there are 35 ways for two's to not be together.

5. (a) I simply need to choose 3 people from a group of 12 to make one committee consisting of 3 people:

$$C(12,3) = \frac{12!}{(12-9)! \cdot 3!} = \frac{12 * 11 * 10}{3 * 2 * 1} = 220$$

$\therefore$ there are 220 ways to make a committee of three.

(b) I have to choose 1 person from 12 to be a president, then 1 person from 11 to be a secretary, finally, 1 person from 10 to be a treasurer. In this case, order matters because order determines which position an individual is in. Thus I will use P(12,3):

$$P(12,3) = \frac{12!}{(12-3)!} = \frac{12!}{9!} = 12 * 11 * 10 = 1320$$

$\therefore$ there are 1320 ways to appoint a president, secretary and treasurer.

6. For my first method, I decided to choose 3 people from 10 for decorating, then 2 for sales from remaining 7, and finally, 5 from the remaining five. I used the product rule because there were three steps.

$$C(10,3) \cdot C(7,2) \cdot C(5,5) = \frac{10!}{(10-3)! \cdot 3!} \cdot \frac{7!}{(7-2)! \cdot 2!} \cdot \frac{5!}{(5-5)! \cdot 5!}$$

$$= \frac{10!}{7! \cdot 3!} \cdot \frac{7!}{5! \cdot 2!} \cdot \frac{5!}{0! \cdot 5!} = \frac{10!}{3! \cdot 2! \cdot 5!} = \frac{10 * 9 * 8 * 7 * 6}{(3 * 2 * 1)(2 * 1)} = \frac{5040}{12} = 420$$

$\therefore$ there are 420 ways to arrange the execs in this criteria. Now I will confirm this by a different method.

I also know that this is a case with arrangement with repetitions of various types. In this case, I have $n = 10$ people, $k = 3$ types and $r_1 = 3$ for the decorations, $r_2 = 2$ for the sales and $r_3 = 5$ for the clean up.

$$P(10;3,2,5) = \frac{10!}{3! \cdot 2! \cdot 5!} = \frac{10 * 9 * 8 * 7 * 6}{(3 * 2 * 1)(2 * 1)} = \frac{5040}{12} = 420$$

$\therefore$ there are 420 ways to arrange the execs with this criteria.