MATH 4281 Risk Theory–Ruin and Credibility

Module 2: Ruin Theory (cont.)

Feb 23, 2021

Post Reading Week/test #1 Review

Recall the following

Since it's been since the 4th we will review what we covered of Ruin Theory so far. Recall we covered:

- Stochastic processes and their properties (independent \stationary increments, etc...) .
- Counting processes, specifically Poisson processes.
- Compound Poisson processes.
- Leading to the Cramér-Lundberg process:

$$U(t) = \underbrace{u_0 + ct}_{\text{Revenue}} - \underbrace{\sum_{i=1}^{N(t)} X_i}_{\text{Losses}}$$

The probability of ruin

- Recall the Cramér-Lundberg model:

$$U(t) = u_0 + ct - \sum_{i=1}^{N(t)} X_i$$

- The time to ruin T is defined as

$$T = \inf\{t \geq 0 | U(t) < 0\}.$$

- The probability that the company would be ruined by time t is denoted by

$$\psi(u_0, t) = \Pr[T < t].$$

Avoiding Ultimate Ruin

- Finally, the probability of ultimate ruin is

$$\psi(u_0) = \Pr(T < \infty) = \lim_{t \to \infty} \psi(u_0, t) \geq \psi(u, t).$$

- The Net Profit Condition (NPC):

$$c \leq \lambda \mathbb{E}[X_i] \Rightarrow \psi(u_0) = 1$$

- To ensure the NPC holds we add our "safety loading" :

$$c = (1 + \theta)\lambda \mathbb{E}[X]$$

Recall we introduced an approximation

We can approximate ψ easy via The Lundberg Inequality:

$$\psi(u) \leq e^{-Ru}$$

Where R (the adjustment coefficient) solves the equation[1]

$$e^{rpt} = \mathbb{E}[e^{rS_t}]$$

Today we will discuss this in more detail.

[1] Where $S_t = \sum_{i=1}^{N(t)} X_i$

The Lundberg Inequality

Avoiding Ultimate Ruin

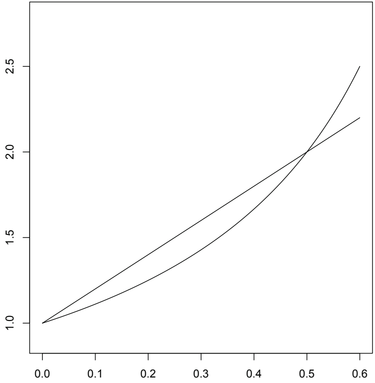
In the Cramér-Lundberg model, consider the excess of losses over premiums over the interval $[0, t]$: $S(t) - ct$. We define the adjustment coefficient R as the first positive solution of the following equation in r:

$$M_{S(t)-ct}(r) = E\left[e^{r(S(t)-ct)}\right] = e^{-rct}e^{\lambda t[M_X(r)-1]} = 1,$$

Recall $c = (1+\theta)\lambda E[X]$. So, the adjustment coefficient R is the first positive of the following equation:

$$1 + (1+\theta)rE[X] = M_X(r)$$

Does such an R exist?



- Recall that Jesen's inequalty gives $1+(1+\theta)rE[X] = M_X(r) \geq e^{rE[X]}$
- How else could this fail?

The Theorem

1. Let $R > 0$ be the adjustment coefficient. If $\{U(t)\}$ is a Cramér-Lundberg process with $\theta > 0$, then for $u \geq 0$

$$\psi(u) = \frac{e^{-Ru}}{E\left[e^{-RU(T)} \middle| T < \infty\right]}.$$

2. Since $U(T) < 0$, we have then (Lundberg's exponential upper bound)

$$\psi(u) < e^{-Ru}.$$

An Example

Assume $X \sim \exp(\beta)$ (the mean is $1/\beta$). Find R and $\psi(u)$.

An Example

Proving the Cramér-Lundberg Inequality

Proving our theorems

To start show that $\left\{e^{-RU(t)}\right\}$ is a martingale[2]

[2] i.e. $E\left[e^{-RU(t)}|e^{-RU(s)}\right] = e^{-RU(s)}$ for $s < t$ or $E\left[e^{-RU(t)}\right] = e^{-Ru}$ for all t

A *very very* useful theorem

(Given we don't have all the machinery we need at this point- we will define a *stopping time* as a random time dependent on another stochastic process exhibiting some behaviour)

Theorem (Optimal Stopping Theorem)

Given a bounded stopping time T, i.e. $T \leq t_0 < \infty$ for a martingale[a] *M_t them:*

$$M_0 = E[M_T]$$

[a] For those who know we must also impose right continuity

An example: Gambler's Ruin

A gambler enters a casino with n dollars and plays a game with a win probability p. He gains \$1 for every win and losses \$1 for every loss. He leaves when he wins N or looses everything. What is the probability he leaves ruined?

An example: Gambler's Ruin

Proof of the Cramér-Lundberg Inequality

Proof of the Cramér-Lundberg Inequality