Quiz #4 (a & b)

a) $\pi(\theta \mid \vec{X})$

$$\propto \prod_{i=1}^{T} f(x_i \mid \theta)\, \pi(\theta)$$

$$\prod_{i=1}^{T} \theta^{x_i} (1-\theta)^{1-x_i} \left(\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\right) \theta^{\alpha-1} (1-\theta)^{\beta-1}$$

$$\propto \theta^{\alpha+\sum x_i - 1} (1-\theta)^{T-\sum x_i+\beta-1}$$

Proportional to a Beta dist w/

$$\tilde{\alpha} = \alpha + \sum X_i, \quad \tilde{\beta} = \beta + T - \sum X_i$$

$$\Rightarrow E[\theta \mid \vec{X}] = \frac{\tilde{\alpha}}{\tilde{\alpha}+\tilde{\beta}} = \frac{\alpha+\sum x_i}{\alpha+\beta+T}$$

as desired

b) $f(x) = \int_0^1 f(x|\theta)\, \pi(\theta)\, d\theta$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 \theta^x (1-\theta)^{1-x}\, \theta^{\alpha-1} (1-\theta)^{\beta-1}\, d\theta$$

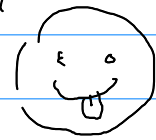
$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \underbrace{\int_0^1 \theta^{(x+\alpha)-1} (1-\theta)^{(\beta-x+1)-1}}_{= \mathrm{Beta}(x+\alpha,\ \beta-x+1)}$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(x+\alpha)\Gamma(\beta-x+1)}{\Gamma(\cancel{x}+\alpha+\beta-\cancel{x}+1)}$$

$$= \frac{1}{\alpha+\beta} \frac{\Gamma(x+\alpha)}{\Gamma(\alpha)} \frac{\Gamma(\beta-x+1)}{\Gamma(\beta)}$$

$$= \begin{cases} \dfrac{\alpha}{\alpha+\beta} & \text{if } x = 1 \\ \dfrac{\beta}{\alpha+\beta} & \text{if } x = 0 \end{cases}$$

Hence Bernoulli$\left(\frac{\alpha}{\alpha+\beta}\right)$

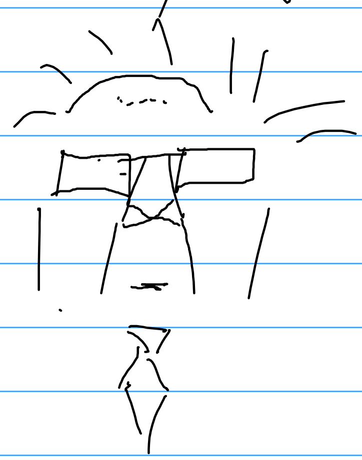
(To check if anyone bothers to read this.... )

Now, what if instead of Berноulli....

.... it was BERNoulli.

And if was Bernie Sanders as a probability distribution?

"professional"



$\sim p = 1/2$

"Mittens"

$\sim p = 1/2$