# MATH 4281 Risk Theory–Ruin and Credibility

Module 3: Credibility Theory (cont.)

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# A General Model

# A useful result

For which pairs $f_{X|\Theta}(x|\theta)$ and $\pi(\theta)$ is $P^{Bayes} = \widetilde{\mu(\Theta)}$ linear?
Equivalently, when is $\widetilde{\mu(\Theta)}$ of the form

$$\widetilde{\mu(\Theta)} = z\bar{X} + (1-z)m\ ?$$

- It is the case for about half a dozen famous examples.
- Jewell (1974) unified these examples
- Gerber (1995) proposed an alternative formulation

# A general model

Suppose

- $$f_{X|\Theta}(x|\theta) = \frac{a(x) \cdot b(\theta)^x}{c(\theta)}, \quad x \in A$$

  where

  $$c(\theta) = \int_A a(x) \cdot b(\theta)^x \mathrm{d}x,$$

  and

- $$\pi(\theta) = \frac{c(\theta)^{-m_0} \cdot b(\theta)^{x_0} \cdot b'(\theta)}{d(m_0, x_0)},$$

  where

  $$d(m_0, x_0) = \int c(\theta)^{-m_0} \cdot b(\theta)^{x_0} \cdot b'(\theta) \mathrm{d}\theta.$$

# A general model

Then

- $\pi_{\mathbf{x}}(\theta)$ is in the same family of $\pi(\theta)$, with the following updated parameter values (for $m_0$ and $x_0$):

$$m_0 + T \quad \text{and} \quad x_0 + \sum_{j=1}^{T} X_j$$

- and finally,

$$P^{Bayes} = \widetilde{\mu(\Theta)} = E[\Theta|X] = \frac{x_0 + S + 1}{m_0 + T} = z\bar{X} + (1 - z)m$$

with

$$z = \frac{T}{m_0 + T}.$$

# Bühlmann model

# Hans Bühlmann-Curriculum Vitae

- PhD ETHZ, 1959
- Swiss Re, 1958-1961 & 1963-1966
- Berkeley, 1961-1963
- ETHZ, full professor since 1966, Emeritus since 1997, President 1987-1990
- 5 times *doctor honoris causa*
- honorary president of ASA and ASTIN
- the father of modern credibility
- Still alive (91 years old!) and coauthored a book in 2016!

# Idea: BLUE vs MVUE

Consider the following. Going back to the Frequentist world:

- Given a sample $\{x_1, x_2, .., x_n\}$ drawn from a uniform distribution $X \sim U[0, \theta]$ is $2\bar{x}$ the "best" or "efficient" (minimum variance, unbiased?) statistic for $\theta$?
- The MLE is $\max(x_1, ..., x_n)$ and adjustiing for bias we get $\tilde{\theta} = \frac{n+1}{2n} \max(x_1, ..., x_n)$.
- The MLE is asymptotically efficient and adjusting for bias we *can*[1] show $\tilde{\theta}$ is the MVUE.

Q Wither $\bar{x}$? What special properties does it have in general?

A It is BLUE!

[1] it's sufficient + Lehmann–Scheffé theorem

## Here's the story.... About a little guy that lives in a blue world ...And all day and all night and everything he sees is just blue...

- BLUE = Best Linear Unbiased Estimator!
- Sample mean (and OLS in general) guaranteed by theorems like Guass-Markov etc...
- What this means is that for any other linear stat $\ell$:

$$E[(\mu(\theta) - \bar{x})^2] \leq E[(\mu(\theta) - \ell)^2]$$

# Back to the Bühlmann model

Bayes premium is the best possible estimator of all. BUT:

- It does not fulfill the requirement of simplicity (closed analytical results are scarce $\longrightarrow$ tedious numerical procedures)
- One has to specify the conditional and a prior distributions.
- May not have nice form of the kind $z\bar{X} + (1 - z)m$.

# Idea: we restrict to linear estimators

- Idea of Bühlmann:

  "*restrict* the class of allowable estimator functions to those which are *linear* in the observations $\mathbf{X}$"

- We are looking for the *best estimator* (in the Bayesian sense) in the class of all *linear estimator functions*.

- The following estimators are then linear Bayes estimators.

# Bühlmann model-Assumptions

- 1 insured
- $\Theta$ risk parameter (random variable)
- $\Pi(\theta) = \Pr[\Theta \leq \theta]$, structural function
- $X_1, X_2, \ldots, X_T | \Theta$ conditionally iid (given $\Theta$)
- $\mathbf{x} = (x_1, x_2, \ldots, x_T)'$ observations
- $\mu(\theta) = E[X_1 | \Theta = \theta]$ and $\sigma^2(\theta) = Var(X_1 | \Theta = \theta)$

# Bühlmann model-Formulas

We want to find the best estimator that is linear in the observations:

$$P_{T+1}^{cred} = \sum_{j=1}^{T} \hat{a}_j X_j + \hat{a}_0$$

such that

$$E[(\mu(\Theta) - \sum_{j=1}^{T} \hat{a}_j X_j - \hat{a}_0)^2] = \min_{a_0, a_1, \cdots, a_T \in \mathbb{R}} E[(\mu(\Theta) - \sum_{j=1}^{T} a_j X_j - a_0)^2]$$

# Bühlmann model-Formulas

As the distribution of $X_1, \cdots, X_T$ is invariant under permutations of $X_j$, the estimator has the form

$$P_{T+1}^{cred} = z\bar{X} + b,$$

where $z$ and $b$ are the solution of the minimizing problem

$$E[(\mu(\Theta) - z\bar{X} - b)^2] = \min_{a_1, a_0 \in \mathbb{R}} E[(\mu(\Theta) - a_1\bar{X} - a_0)^2]$$

# Bühlmann model-Formulas

- Taking partial derivatives with respect to $a_0$, and $a_1$ respectively:

$$E[-2(\mu(\Theta) - a_1\bar{X} - a_0)] = 0 \ \& \ E[-2\bar{X}(\mu(\Theta) - a_1\bar{X} - a_0)] = 0$$

- Solving[2] the equations, we get

$$a_1 = \frac{Var(\mu(\Theta))}{\frac{E[\sigma^2(\Theta)]}{T} + Var(\mu(\Theta))} \ \& \ a_0 = (1 - a_1)E[\mu(\Theta)].$$

- Can see $a_1 = z$ (the Bühlmann credibility factor) $a_0 = (1 - z)m$ as we expected.

[2]requires a great deal of manipulation etc...not important (pg. 416 of Loss models if you are curious)

# The credibility premium

- The **credibility premium (credibility estimator)** is then:

$$P^{cred}_{T+1} = z\bar{X} + (1-z)m \quad (1)$$

where

$$z = \frac{T}{T+K}, \quad K = \frac{E[\sigma^2(\Theta)]}{Var(\mu(\Theta))}$$

and $m = E[\mu(\Theta)]$.

- $P^{cred}$ is a weighted mean of $P^{coll} = m$ and the individual observed average $\bar{X}$
- $K = \frac{E[\sigma^2(\Theta)]}{Var(\mu(\Theta))}$ is called the **credibility coefficient**
- When the Bayes estimator is linear in the observations, the Bayes estimator is equal to the credibility estimator. We refer to such cases as **exact credibility**.

# Example

The amount of a claim has the exponential distribution with mean $\frac{1}{\Theta}$. Among the class of insureds and potential insureds, the parameter $\Theta$ varies according to the gamma distribution with $\alpha$ and scale parameter $\frac{1}{\beta}$. Suppose that a person had claims of $X_1$, $X_2$, $\cdots$, and $X_n$. Determine the credibility premium of the $n+1$th claim.

# Quadratic losses

$m$: best estimator based on the a priori knowledge alone. Quadratic loss:

$$E[(m-\mu(\Theta))^2] = Var(\mu(\Theta)) = a.$$

$\bar{X}$: best linear and individually (conditionally) unbiased (given $\Theta$) estimator, based on the observation vector $\mathbf{X}$. Quadratic loss:

$$E[(\bar{X}-\mu(\Theta))^2] = E[\sigma^2(\Theta)/T] = s^2/T.$$

$P^{cred}$: weighted mean of $\bar{X}$ and $m$, where weights are *proportional to their inverse quadratic loss (precision)*

$$z = \frac{T/s^2}{T/s^2+1/a} \;\&\; (1-z) = \frac{1/a}{T/s^2+1/a}$$

# Study of $K$

$$K = \frac{E[\sigma^2(\Theta)]}{Var(\mu(\Theta))} \ \& \ z = \frac{T}{T+K}$$

The credibility factor $z$ increases as

- the number of observations $T$ increases
- the heterogeneity of the portfolio increases.
- the within risk variability decreases.

# Nonparametric estimation

# Parametric and Nonparametric estimation

There are three ways of estimating the parameters $m$, $s^2$ and $a$:

- Pure Bayesian procedure: intuitively set using the a priori knowledge of an experienced actuary or underwriter
- Parametric estimation: the distributions $f_{X|\Theta}(x|\theta)$ and $\pi(\theta)$ are known and the parameters are calculated from these distributions.
- Nonparametric estimation: data from a collective of similar risks exist, and values for the parameters are inferred from it. This is an empirical Bayes approach.

The parametric estimation is straightforward and follows from the definition of the parameters.

# Nonparametric estimation

- $X_{jt}$: claims size of policy $j$ during year $t$.
- Available data, $1 \leq j \leq J$, $1 \leq t \leq T$:

| year $t$ | 1 | 2 | 3 | $\cdots$ | $T$ | Risk | Mean |
|---|---|---|---|---|---|---|---|
| policy $j = 1$ | $X_{11}$ | $X_{12}$ | $X_{13}$ | $\cdots$ | $X_{1T}$ | $\theta_1$ | $\bar{X}_{1\Sigma}$ |
| policy $j = 2$ | $X_{21}$ | $X_{22}$ | $X_{23}$ | $\cdots$ | $X_{2T}$ | $\theta_2$ | $\bar{X}_{2\Sigma}$ |
| policy $j = 3$ | $X_{31}$ | $X_{32}$ | $X_{33}$ | $\cdots$ | $X_{3T}$ | $\theta_3$ | $\bar{X}_{3\Sigma}$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| policy $j = J$ | $X_{J1}$ | $X_{J2}$ | $X_{J3}$ | $\cdots$ | $X_{JT}$ | $\theta_J$ | $\bar{X}_{J\Sigma}$ |

- $X_{11}, X_{12}, \ldots, X_{JT}$ are *iid* conditional on $\Theta$.
- $\mu(\theta_j) = E[X_{jt}|\Theta = \theta_j]$
- $\sigma^2(\theta_j) = Var(X_{jt}|\Theta = \theta_j)$

$$P^{cred}_{j,T+1} = z\bar{X}_{j\Sigma} + (1-z)m, \quad i = 1, \ldots, J \qquad (2)$$

# Nonparametric estimation (unbiaised estimators)

Estimation of $E[\mu(\Theta)] = m$:

$$\bar{X}_{\Sigma\Sigma} = \frac{1}{J}\sum_{j=1}^{J}\bar{X}_{j\Sigma} = \frac{\sum_{j=1}^{J}\sum_{t=1}^{T}X_{jt}}{JT} \tag{3}$$

Estimation of $E[\sigma^2(\Theta)] = s^2$:

$$\hat{s}^2 = \frac{1}{J}\sum_{j=1}^{J}\hat{s}_j^2 = \frac{1}{J}\sum_{j=1}^{J}\sum_{t=1}^{T}\frac{(X_{jt} - \bar{X}_{j\Sigma})^2}{T-1} \tag{4}$$

# Nonparametric estimation (unbiaised estimators)

Estimation of $Var(\mu(\Theta)) = a$:
(Bühlmann's estimator)

$$\hat{a}_B = Max\left\{\frac{\sum_{j=1}^{J}(\bar{X}_{j\Sigma} - \bar{X}_{\Sigma\Sigma})^2}{J-1} - \frac{1}{T}\hat{s}^2 \; ; \; 0\right\} \qquad (5)$$

(CAS's estimator)

$$\hat{a}_{CAS} = Max\left\{\frac{\sum_{j=1}^{J}\sum_{t=1}^{T}(X_{jt} - \bar{X}_{\Sigma\Sigma})^2}{JT-1} - \hat{s}^2 \; ; \; 0\right\} \qquad (6)$$

- If $\hat{a} = 0$ then $z = 0$, which makes sense (all risks have the same parameter)

# Example 1

An insurance company sells automobile insurance and has mainly two types of policy: $A$ and $B$. The aggregate claim amounts (in millions of dollars) for the last three years are summarized below:

| Policy Type | Year 1 | Year 2 | Year 3 |
|---|---|---|---|
| $A$ | 5 | 8 | 11 |
| $B$ | 11 | 13 | 12 |

Estimate the Bühlmann's credibility factor and use this to determine the next year's credibility premium for each of the policy type.