$\cdot\ f_{X|\theta}(x|\theta) = \theta(1-\theta)^x \quad \{x \in \mathbb{N}\} \quad G_{10}(\theta)$

$\cdot\ \pi(\theta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \theta^{\alpha-1}(1-\theta)^{\beta-1} \quad \{0<\theta<1\} \quad Beta(\alpha,\beta)$

$$f(x) = \int_0^1 f_{X|\theta}(x|\theta)\,\pi(\theta)\,d\theta$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 \theta(1-\theta)^x\,\theta^{\alpha-1}(1-\theta)^{\beta-1}\,d\theta$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 \theta^{\alpha}(1-\theta)^{x+\beta-1}\,d\theta$$

We want to use;

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 \theta^{(\alpha+1)-1}(1-\theta)^{(x+\beta)-1}\,d\theta$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(\alpha+1)\Gamma(x+\beta)}{\Gamma(\alpha+1+x+\beta)}$$

(as shown)

This distribution is known as a "Geometric-Beta" distribution. While well known for some appplications it is not as straightforward as the other examples I gave.

Instead just calculate m like so:

$$E[X] = E[E[X|\theta]]$$

$$= E\left[\frac{1-\theta}{\theta}\right]$$

$$= \int_0^1 \frac{1-\theta}{\theta} \cdot \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \cdot \theta^{\alpha-1}(1-\theta)^{\beta-1}\, d\theta$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \underbrace{\int_0^1 \theta^{(\alpha-1)-1}(1-\theta)^{(\beta+1)-1}\, d\theta}_{Beta(\alpha-1,\ \beta+1)}$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(\alpha-1)\Gamma(\beta+1)}{\Gamma(\alpha-1+\beta+1)}$$

$$= \underbrace{\frac{\Gamma(\alpha-1)}{\Gamma(\alpha)}}_{=\frac{\Gamma(\alpha-1)}{\Gamma((\alpha-1)+1)}} \cdot \underbrace{\frac{\Gamma(\beta+1)}{\Gamma(\beta)}}_{=\beta} \cdot \underbrace{\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta}}_{=1}$$

$$= \alpha - 1$$

$$= \frac{\beta}{\alpha-1} \quad \text{as desired}$$

The posterior:

$$\pi(\theta|\vec{X}) \propto \left(\prod_{i=1}^{T} f(X_i|\theta)\right)\pi(\theta)$$

$$= \prod_{i=1}^{T} \theta(1-\theta)^{X_i} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \theta^{\alpha-1}(1-\theta)^{\beta-1}$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \theta^{\alpha+T-1} (1-\theta)^{\beta+\sum X_i-1}$$

$$\propto \theta^{\alpha+T-1}(1-\theta)^{\beta+\sum X_i-1}$$

$\Rightarrow$ Proportional to a Beta dist. w/

$$\tilde{\alpha} = \alpha + T \qquad \tilde{\beta} = \beta + \sum X_i$$

The rest is trivial