MATH 4281 Risk Theory–Ruin and Credibility

Module 1 (cont.)

January 21, 2021

Insurance in Practice

- We now have enough "tools in our belt" to look at some more applied insurance problems involving the IRM.
- Today we will look at policy transforms and Reinsurance.
- Next week we will dive into the CRM again.

Policy Transforms

The Story So Far...

- So far we have looked at how to model insurance losses.
- Often this is *exogenous* i.e. outside of our control.
- How can we control the cost/variably of claim losses?

Deductible and Policy Limit

One way to control the cost (and variability) of individual claim losses is to introduce deductibles and policy limits.

- Deductible d: the insurer starts paying when claim amounts exceed the deductible d
- Limit L: the insurer pays up to the limit L.

If we denote the damage random variable by D, then if a claim occurs the insurer is liable for

$$B = \min\left[\max\left(D - d, 0\right), L\right].$$

Numerical Example 1

Consider an automobile insurance where:

- the policyholder has a probability 0.10 of getting in an accident
- If an accident occurs, the damage to the vehicle is uniformly distributed between 0 and 2 500.
- The policy has a deductible amount of 500 and a policy limit of 1500.

Find:

1. The CDF and the probability density/mass function of X, the claim amount the insurer is liable for.
2. The expected value of X.

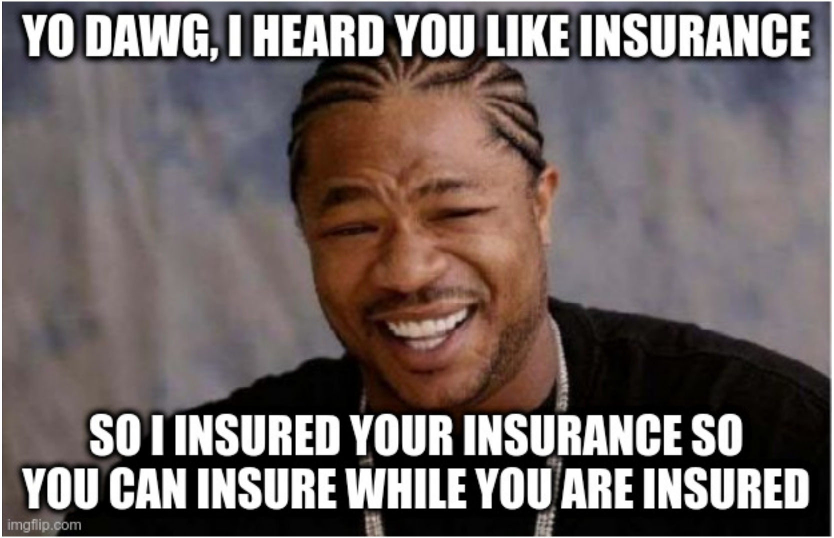
Numerical Example

Numerical Example

Numerical Example

Reinsurance

Insurance for Insurance Companies?



Insurance for Insurance Companies?

Reinsurance is a contract wherin one company (the reinsurer) takes on the catastrohic risks of another insurer e.g.:

- Earthquakes and other disasters
- Industrial accidents
- Terrorism/war/riots
- ....Aliens? (some losses are **very** difficult to anticipate)

It is risk transfer from an insurer (the direct writer) to a reinsurer: swap of deterministic against random. The risk that the insurer keeps is called the retention.

In general there are two broad categories of Reinsurance "Treaties":

- Random walk type (e.g. proportional, excess of loss, stop-loss).
- Extreme value type (e.g. Largest claim, Excédent du coût moyen relatif or ECOMOR).

It is also worth noting the existence of things like *Catastrophe Bonds*.

Reinsurance

- Proportional
  - quota share: the proportion is the same for all risks
  - surplus: the proportion can vary from risk to risk
- Non-Proportional
  - (individual) excess of loss: on each individual loss (X_i)
  - stop loss: on the aggregate loss (S)
- Cheap (reinsurance premium is the expected value), or non cheap (reinsurance premium is loaded)

Proportional reinsurance

The retained proportion α defines who pays what:

- the insurer pays $Y = \alpha X$
- the reinsurer pays $Z = (1-\alpha)X$

This is nothing else but a change of scale and we have

$$\mu_Y = \alpha\mu_X, \quad \sigma_Y^2 = \alpha^2\sigma_X^2, \quad \gamma_Y = \gamma_X.$$

Proportional reinsurance

In some cases it suffices to adapt the scale parameter. For example if X is exponential with parameter β:

$$\Pr[Y \leq y] = \Pr[\alpha X \leq y] = \Pr[X \leq y/\alpha] = 1 - e^{-\beta y/\alpha}$$

...and thus Y is exponential with parameter β/α.

Nonproportional reinsurance

(Individual) Excess of Loss reinsurance (EoL):

- For each individual loss X, the reinsurer pays the excess over a retention (excess point) d
  - the insurer pays $Y = \min(X, d)$
  - the reinsurer pays $Z = (X - d)_+$
- in EoL, the reinsurer may limit his payments to an amount L. In that case
  - the insurer pays $Y = \min(X, d) + (X - L - d)_+$
  - the reinsurer pays $Z = \min\{(X - d)_+, L\}$

Nonproportional reinsurance

Stop loss reinsurance: For aggregate loss S, the reinsurer pays the excess over a retention (excess point) d

- the insurer pays $Y = \min(S, d)$
- the reinsurer pays $Z = (S - d)_+$

A useful identity

Note that

$$\min(X, c) = X - (X - c)_+$$

and thus

$$E[\min(X, c)] = E[X] - E[(X - c)_+].$$

The amount $E[(X - c)_+]$

- is commonly called "stop loss premium" with retention c.
- is identical to the expected payoff of a call with strike price c, and thus results from financial mathematics can sometimes be directly used (and vice versa).

Reinsurance premium

- Cheap reinsurance: Reinsurance premium equals to the expected value of random losses covered by the reinsurer

- Non-cheap reinsurance: Reinsurance premium equals to the expected value of random losses covered by the reinsurer plus a loading

Example A life insurance company covers 16000 lives for 1-year term life insurance in amounts shown below

Benefit Amt (in 10,000s)	No. of lives covered
1	8000
2	3500
3	2500
5	1500
10	500

The probability of a claim q for each of the 16000 lives is 0.02. The excess of loss reinsurance with retention limit 30000 is available at a cost of 0.025 per dollar of coverage. Use the Normal approximation method to calculate the probability that the total cost will exceed 8250000.

Example

The portfolio of retained business is given by:

Benefit Amt (in 10,000s)	No. of lives covered
1	8000
2	3500
3	4500

Example

Example

Stop loss reinsurance on the aggregate loss

- For a stop-loss contract with deductible d, the amount paid by the reinsurer to the ceding insurer is
$$I_d = (S-d)_+ = \begin{cases} 0 & S \le d \\ S-d & S > d \end{cases}$$
where S is the aggregate claims.
- Then we have
  - If S is continuous positive RV with pdf $f_S(x)$, then
$$E[I_d] = \int_d^\infty (x-d) f_S(x)dx = \int_d^\infty [1 - F_S(x)]dx$$
  - If S is discrete positive RV with possible values x_k with PMF $f_S(x_k)$ for $k = 0, 1, \cdots$, then
$$E[I_d] = \sum_{k: x_k \ge d} (x_k - d) f_S(x_k)$$

Aside: The Darth Vader Rule

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Example

Calculate $E[I_d]$ if S is Exponential with mean $1/\beta$.

Stop loss reinsurance - recursive formulas

If the possible values of S are non-negative integers, then
First moment:

- if d is an integer

$$E[I_{d+1}] = E[I_d] - [1 - F_S(d)] \text{ with } E[I_0] = E[S]$$

- if d is not an integer

$$E[I_d] = E[I_{\lfloor d \rfloor}] - (d - \lfloor d \rfloor)[1 - F_S(\lfloor d \rfloor)],$$

  where $\lfloor d \rfloor$ is the integer part of d.

Second moment $E[I_d^2] = E[(S-d)_+^2]$:

$$E[I_d^2] = E[I_{d-1}^2] - 2E[I_{d-1}] + [1 - F_S(d-1)] \text{ with } E[I_0^2] = E[S^2].$$

This seems strange...what's going on?

Solution to Convolution ex on Jan14

x	$f_1(x)$	$f_2(x)$	$f_{1+2}(x)$	$f_3(x)$	$f_{1+2+3}(x)$	$F_{1+2+3}(x)$
0	$1/4$	$1/2$	$1/8$	$1/4$	$1/32$	$1/32$
1	$1/2$	0	$2/8$	0	$2/32$	$3/32$
2	$1/4$	$1/2$	$2/8$	$1/2$	$4/32$	$7/32$
3	0	0	$2/8$	0	$6/32$	$13/32$
4	0	0	$1/8$	$1/4$	$6/32$	$19/32$
5	0	0	0	0	$6/32$	$25/32$
6	0	0	0	0	$4/32$	$29/32$
7	0	0	0	0	$2/32$	$31/32$
8	0	0	0	0	$1/32$	$32/32$

Worked Example

For the distribution F_{1+2+3} derived earlier in the lecture (refer to Page 11) we have $E[S] = 4 = 128/32$ and $E[S^2] = 19.5 = 624/32$ and thus

d	$f_{1+2+3}(d)$	$F_{1+2+3}(d)$	$E[I_d]$	$E[I_d^2]$	$Var((X-d)_+)$
0	1/32	1/32	128/32	624/32	3.500
1	2/32	3/32	97/32	399/32	3.280
2	4/32	7/32	68/32	234/32	2.797
3	6/32	13/32	43/32	123/32	2.038
4	6/32	19/32	24/32	56/32	1.118
5	6/32	25/32	11/32	21/32	0.538
6	4/32	29/32	4/32	6/32	0.172
7	2/32	31/32	1/32	1/32	0.030
8	1/32	32/32	0	0	0.000

$$E[I_{2.6}] = E[I_2] - (2.6-2)\cdot(1-F_{1+2+3}(2)) = \frac{68}{32} - 0.6\times(1-\frac{7}{32}) = \frac{53}{32}.$$