MATH 4281 Risk Theory–Ruin and Credibility

Module 2: Ruin Theory (cont.)

Feb 2, 2021

1. Stochastic Processes (cont)
    - The Cramér-Lundberg process

2. Ruin Theory
   (with a preamble on decision theory)
    - Talking about decision theory
    - The probability of ruin

Stochastic Processes (cont)

Motivation

Model for the surplus of a non-life insurer at time t:

$$U(t) = \underbrace{u_0 + ct}_{\text{Revenue}} - \underbrace{\sum_{i=1}^{N(t)} X_i}_{\text{Losses}}$$

where

- u_0 initial surplus
- c premium rate:
- $\sum_{i=1}^{N(t)} X_i$ aggregate loss up to time t

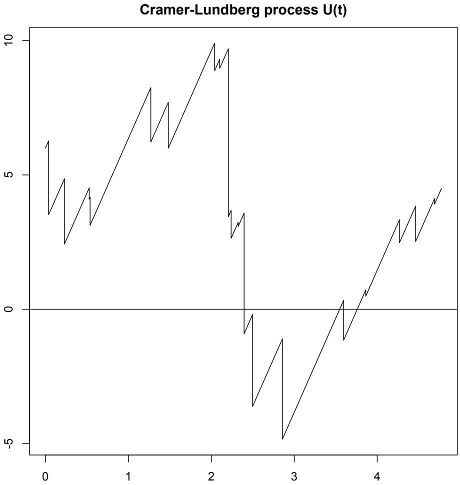
The Cramér-Lundberg process

Furthermore if:

- the premium rate is $c = (1+\theta)\lambda E[X]$
- where θ is called the relative security loading.
- and, $\sum_{i=1}^{N(t)} X_i$ is a Compound Poisson (X_i independent of N Poisson)

$\Longrightarrow \{U(t), t \geq 0\}$ is called the Cramér-Lundberg process.

A path (realization) of Cramér-Lundberg process



Properties

Does the Cramér-Lundberg process have independent increments?

Does the Cramér-Lundberg process have stationary increments?

Ruin Theory
(with a preamble on decision theory)

How to optimize the Cramér-Lundberg process?

The survival of the insurance company will depend on certain variables:

- initial surplus (u_0)
- loading of premiums (θ)
- reinsurance (e.g. α or d–see later)

What is the "best" way to choose/monitor these variables?

How to optimize the Cramér-Lundberg process?

Depends on what criterion the assessment is based:

- probability of ruin—goes back to Lundberg (1909) and Cramér (1930, 1955)
- utility—goes back to von Neumann and Morgenstern (1944)
- present value of dividends—goes back to de Finetti (1957)
- . . . ?

Quick "Review" : Expected Utility Theory

- For most if its history economics was considered more a branch of "worldly philosophy" or "political economy".
- Finance was considered by economists to be a somewhat more applied pursuit or was looked down upon as an area of study outright.
- During the inter-war period previously disparate analytic tools and models were synthesized into a more coherent theoretical framework (the neoclassical synthesis).
- Notable among these was the theory of **utility**.

St. Petersburg paradox

Consider a lottery/game with a fair coin where the initial stake starts at 2 dollars and is doubled every time heads appears. The first time tails appears, the game ends and the player wins whatever is in the pot.

Probability of n tosses with no tails is 2^{-n}.

Winnings on nth toss is 2^n.

St. Petersburg (cont.)

The expected winnings from the lottery is:

$$\begin{aligned} E &= \frac{1}{2} \cdot 2 + \frac{1}{4} \cdot 4 + \frac{1}{8} \cdot 8 + \frac{1}{16} \cdot 16 + \cdots \\ &= 1 + 1 + 1 + 1 + \cdots \\ &= 1 + 1 + 1 + 1 + \cdots \\ &= \infty \end{aligned}$$

St. Petersburg *possible* resolution

- Say I start with wealth w_o and pay c for the lottery.
- Further I measure my *utility* rather than my net wealth as the relevant variable.
- Assume utility is given by the natural logrithm of wealth. i.e . $u = \ln(w)$.
- I say my net expected utility is:

$$\sum_{k=1}^{\infty} \frac{1}{2^k} \left[\ln(w_o + 2^k - c) - \ln(w_o) \right] < \infty$$

- Was and still is a controversial solution, as is the use of utilities generally. However there are some theories justifying this (see Von Neumann–Morgenstern).

Utility Theory

Popular choice is the *hyperbolic absolute risk aversion* (HARA):

$$u(x) = \frac{1-\gamma}{\gamma}\left(\frac{ax}{1-\gamma} + b\right)^{\gamma}$$

This includes:

- Logarithmic (see above) as $a = 1$ and $\gamma \to 0$.
- Exponential $u(x) = (1 - e^{ax})/a$ (often they drop the $1/a$ term)
- Linear, quadratic, etc...

Ruin Theory

- Around the same time the theoretical foundations of actuarial science were being laid *in parallel*!
- Filip Lundberg and Harald Cramér where developing *Ruin Theory*!
- More specific to Insurance companies but...

[W]ith insurance you have a cash flow. And they understand the problem very well, since Cramer [...] They looked at some process that compensate the risk you are taking because you are making some money to accumulate in some reservoir that's going to be depleted, but not 100%. So the idea is to calibrate the risk taking to what you are getting into the reservoir.

-Nassim Taleb

The probability of ruin

- Recall the Cramér-Lundberg model:

$$U(t) = u_0 + ct - \sum_{i=1}^{N(t)} X_i$$

- The time to ruin T is defined as

$$T = \inf\{t \geq 0 | U(t) < 0\}.$$

- The probability that the company would be ruined by time t is denoted by

$$\psi(u, t) = \Pr[T < t].$$

- Finally, the probability of ultimate ruin is

$$\psi(u) = \Pr(T < \infty) = \lim_{t \to \infty} \psi(u, t) \geq \psi(u, t).$$

- The risk based capital / solvency problem is similar to looking for

$$\inf\{u | \psi(u, t) \leq \alpha_t\} \quad \text{for given } t.$$

How to calculate the probability of ruin

There are different ways of calculating ruin probabilities ψ:

- analytically
  - $\psi(u)$ is very hard to calculate, but possible for exponential and mixtures of exponential losses
  - $\psi(u, t)$ is even more difficult to determine
- using Panjer's recursion via a special trick
- Monte-Carlo methods (simulations)

In what follows, we assume we are using the Cramér-Lundberg model.

But first...

We have a result that tells us when we fact ruin with certainty!

- The Net Profit Condition (NPC):

$$p \leq \lambda \mathbb{E}[X_i] \Rightarrow \psi(u_o) = 1$$

- To ensure the NPC holds we add our "safety loading" :

$$p = (1 + \theta)\lambda \mathbb{E}[X]$$

- Often for large U_t the the so called "net premium" $p = \lambda \mathbb{E}[X]$ is effective. Interest on U_t alone is a safety loading.

The adjustment coefficient

We will return to this in more detail on the 16th of Feb (after your test). But for now consider the following without proof. We can approximate ψ easy via The Lundberg Inequality:

$$\psi(u) \leq e^{-Ru}$$

Where R (the adjustment coefficient) solves the equation[1]

$$e^{rpt} = \mathbb{E}[e^{rS_t}]$$

[1] Where $S_t = \sum_{i=1}^{N(t)} X_i$

A familiar premium calculation

- Setting $\psi(u) \leq e^{-Ru} = \varepsilon$ yields $R = \frac{|\ln(\varepsilon)|}{u}$ and gives:

$$pt = \frac{1}{R}\ln(\mathbb{E}[e^{RS_t}])$$

- This is also called the "Entropic Risk Measure":

$$\pi[L] = \frac{1}{\alpha}\ln(M_L(\alpha))$$

- Or the Exponential Risk Principle. If $u(x) = -\alpha e^{-\alpha}$ then:

$$\mathbb{E}[u(L - \pi[L])] = 0$$

a.k.a the the certainty equivalent under an exponential utility!

- That is all for now till we return to Module 2 on the 16th. We will prove the Lundberg Inequality.

- Your Quiz this week will feature a simple stochastic processes question.

- On Tuesday the 9th there will be a Module 1 review and your test will take place on the 11th.