MATH 4281 Risk Theory–Ruin and Credibility

Module 3: Credibility Theory finale and the last lecture!

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Non-Parametric Bühlmann model

Recall from last class

- X_{jt}: claims size of policy j during year t.
- Available data, $1 \leq j \leq J$, $1 \leq t \leq T$:

year t	1	2	3	$\cdots$	T	Risk	Mean
policy $j = 1$	X_{11}	X_{12}	X_{13}	$\cdots$	X_{1T}	θ_1	$\bar{X}_{1\Sigma}$
policy $j = 2$	X_{21}	X_{22}	X_{23}	$\cdots$	X_{2T}	θ_2	$\bar{X}_{2\Sigma}$
policy $j = 3$	X_{31}	X_{32}	X_{33}	$\cdots$	X_{3T}	θ_3	$\bar{X}_{3\Sigma}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$
policy $j = J$	X_{J1}	X_{J2}	X_{J3}	$\cdots$	X_{JT}	θ_J	$\bar{X}_{J\Sigma}$

- $X_{11}, X_{12}, \ldots, X_{JT}$ are *iid* conditional on Θ.
- $\mu(\theta_j) = E[X_{jt}|\Theta = \theta_j]$
- $\sigma^2(\theta_j) = Var(X_{jt}|\Theta = \theta_j)$

$$P^{cred}_{j,T+1} = z\bar{X}_{j\Sigma} + (1-z)m, \quad i = 1, \ldots, J \tag{1}$$

Nonparametric estimation (unbiaised estimators)

Estimation of $E[\mu(\Theta)] = m$:

$$\bar{X}_{\Sigma\Sigma} = \frac{1}{J}\sum_{j=1}^{J}\bar{X}_{j\Sigma} = \frac{\sum_{j=1}^{J}\sum_{t=1}^{T}X_{jt}}{JT} \tag{2}$$

Estimation of $E[\sigma^2(\Theta)] = s^2$:

$$\hat{s}^2 = \frac{1}{J}\sum_{j=1}^{J}\hat{s}_j^2 = \frac{1}{J}\sum_{j=1}^{J}\sum_{t=1}^{T}\frac{(X_{jt}-\bar{X}_{j\Sigma})^2}{T-1} \tag{3}$$

Nonparametric estimation (unbiaised estimators)

Estimation of $Var(\mu(\Theta)) = a$:
(Bühlmann's estimator)

$$\hat{a}_B = Max\left\{\frac{\sum_{j=1}^{J}(\bar{X}_{j\Sigma} - \bar{X}_{\Sigma\Sigma})^2}{J-1} - \frac{1}{T}\hat{s}^2\ ;\ 0\right\} \tag{4}$$

(CAS's estimator)

$$\hat{a}_{CAS} = Max\left\{\frac{\sum_{j=1}^{J}\sum_{t=1}^{T}(X_{jt} - \bar{X}_{\Sigma\Sigma})^2}{JT-1} - \hat{s}^2\ ;\ 0\right\} \tag{5}$$

- If $\hat{a} = 0$ then $z = 0$, which makes sense (all risks have the same parameter)

Example 2

You are given the following past claims data on a portfolio of three classes of policyholders:

	Year		
Class	1	2	3
1	700	800	600
2	625	500	675
3	800	850	750

Estimate the Bühlmann credibility premium to be charged in year 4 for each class of policyholder.

The Bühlmann-Straub model

Adding more realism to the Bühlmann model

- Often we have somewhat coarse data available to us (this is changing but in 1970 when this model was introduced it was even more true).
- Many lines of business have a premium of the type "volume measure" times "premium rate".
- In this case we use an extension of Bühlmann model: Bühlmann-Straub.
- by far the most used and the most important credibility model for insurance practice.

The model

There are $1 \leq j \leq J$ classes of risk (or contracts).
For the j-th class/contract:

- S_{jt} is the aggregate claim amount in year t ($1 \leq t \leq T$)
- w_{jt} is the "volume" associated to S_{jt} in year t
- $X_{jt} = S_{jt}/w_{jt}$ is the claim amount per unit of volume in year t
- One (my favourite) interpretation: average claim costs per year at risk in year t if w_{jt} is the number of years at risk during year t. That is if:

$$S_{jt} = \sum_{k=1}^{w_{jt}} Y_{jt,k}$$

Assumptions

- risk class/contract j is characterized by its specific risk parameter θ_j, which is the realization of a rv Θ_i
- Conditional on Θ_i, the $\{X_{jt} : t = 1, 2, \cdots, T\}$ are iid with $\mu(\theta_j) = E[X_{jt}|\Theta = \theta_j]$ but now:[1]

$$Var(X_{jt}|\Theta = \theta_j) = \frac{\sigma^2(\theta_j)}{w_{jt}}$$

- the pairs $(\Theta_1, \mathbf{X_1}), (\Theta_2, \mathbf{X_2}), \ldots$ are independent
- $\Theta_1, \Theta_2, \ldots$ are iid (from the structural distribution)

[1] In the previous interpretation, $\mu(\theta_j) = E[Y_j]$ and $\sigma^2(\theta_j) = Var(Y_j)$.

"New Quantities"

Risk j:

- individual risk premium
  $\mu(\theta_j) = E[X_{jt}|\Theta = \theta_j]$
- variance within individ. risk
  $\sigma^2(\theta_j) = w_{jt} Var(X_{jt}|\Theta = \theta_j)$
- aggregate volume
  $w_{j\Sigma} = \sum_{t=1}^{T} w_{jt}$
- weighted mean of outcomes
  $\bar{X}_{j\Sigma} = \sum_{t=1}^{T} \frac{w_{jt}}{w_{j\Sigma}} X_{jt}$

Collective:

- collective premium
  $m = E[\mu(\Theta)]$
- variance between individual risk premiums
  $a = Var(\mu(\Theta))$
- average variance within individual risks
  $s^2 = E[\sigma^2(\Theta)]$
- aggregate volume
  $w_{\Sigma\Sigma} = \sum_{j=1}^{J} w_{j\Sigma}$
- weighted mean of outcomes
  $\bar{X}_{\Sigma\Sigma} = \sum_{j=1}^{J} \frac{w_{j\Sigma}}{w_{\Sigma\Sigma}} \bar{X}_{j\Sigma}$

If m, s^2 and a are known

The credibility estimator in the Bühlmann-Straub model is given by

$$P_{j,T+1}^{cred} = z_j \bar{X}_{j\Sigma} + (1 - z_j)m = m + z_j(\bar{X}_{j\Sigma} - m),$$

where

$$\begin{aligned} z_j &= \frac{w_{j\Sigma}}{w_{j\Sigma} + K} \\ K &= \frac{E[\sigma^2(\Theta)]}{Var(\mu(\Theta))} \end{aligned}$$

Remarks:

- the credibility factor z_j now depends on j
- if $w_{jt} = 1$, then $w_{j\Sigma} = T$ and z_j is equivalent to the z of the simple Bühlmann model

If s^2 and a are known but m has to be estimated

$$P_{j,T+1}^{cred} = z_j \bar{X}_{j\Sigma} + (1 - z_j)\widehat{m} = \widehat{m} + z_j(\bar{X}_{j\Sigma} - \widehat{m}),$$

where

$$\widehat{m} = \sum_{j=1}^{J} \frac{z_j}{z_\Sigma} \bar{X}_{j\Sigma}, \quad z_\Sigma = \sum_{j=1}^{J} z_j$$

Remarks:

- it can be shown that $\widehat{m}$ is a better estimator of m than $\bar{X}_{\Sigma\Sigma}$
- Quadratic loss:

$$E[(\widehat{m} + z_j(\bar{X}_{j\Sigma} - \widehat{m}) - \mu(\theta_j))^2] = a(1 - z_j)\left(1 + \frac{1 - z_j}{z_\Sigma}\right)$$

- This makes sense- in an non-i.i.d sample, the weighted average where the weights are inversely proportional to the variances is BLUE.

If m, s^2 and a have to be estimated

$$P_{j,T+1}^{cred} = \widehat{z}_j \bar{X}_{j\Sigma} + (1-\widehat{z}_j)\widehat{m} = \widehat{m} + \widehat{z}_j(\bar{X}_{j\Sigma} - \widehat{m}),$$

Where we use the following unbiased (weighted) sample statistics:

$$\begin{aligned}
\widehat{z}_j &= \frac{w_{j\Sigma}}{w_{j\Sigma} + \frac{\hat{s}^2}{\hat{a}}} \\
\hat{s}^2 &= \frac{1}{J}\sum_{j=1}^{J} \hat{s}_j^2 = \frac{1}{J}\sum_{j=1}^{J}\left(\frac{1}{T-1}\sum_{t=1}^{T} w_{jt}(X_{jt} - \bar{X}_{j\Sigma})^2\right) \\
\hat{a} &= \frac{w_{\Sigma\Sigma}}{w_{\Sigma\Sigma}^2 - \sum_{j=1}^{J} w_{j\Sigma}^2}\left\{\sum_{j=1}^{J} w_{j\Sigma}(\bar{X}_{j\Sigma} - \bar{X}_{\Sigma\Sigma})^2 - (J-1)\hat{s}^2\right\}
\end{aligned}$$

Numerical Example

Past claims data on a portfolio of two groups of policyholders are given below:

	Group	Year 1	2	3	4
Total Claim Amount	1	8000	11,000	15,000	–
Number in Group		40	50	70	75
Total Claim Amount	2	20,000	24,000	19,000	–
Number in Group		100	120	115	95

Estimate the Bühlmann-Straub credibility premium to be charged in year 4 for each group of policyholder.