On the computational complexity of detecting possibilistic locality

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The proofs of quantum nonlocality due to Greenberger, Horne, and Zeilinger; and due to Hardy are qualitatively different from that of Bell insofar as they rely only on a consideration of whether events are possible or impossible, rather than relying on specific experimental probabilities. We consider the scenario of a bipartite nonlocality experiment, in which two separated experimenters each have access to some measurements they can perform on a system. In a physical theory, some outcomes of this experiment will be labelled possible, others impossible, and an assigment of the values 0 (impossible) and 1 (possible) to these different outcomes forms a *table of possibilities*. Here, we consider the computational task of determining whether or not a given table of possibilities constitutes a departure from possibilistic local realism. By considering the case in which one party has access to measurements with two outcomes and the other three, it is possible to see at exactly which point this task becomes computationally difficult.

I. INTRODUCTION

Since the inception of quantum mechanics, the apparent nonlocality of the theory has been a cornerstone of debate about the foundations of the theory. Einstein, together with Podolsky and Rosen [8], argued that nonlocality, in apparent contravention of the information-propagation bound of the speed of light from special relativity, was evidence of the incompleteness of the quantum-mechanical description of nature. The related concept of nonlocal steering, whereby a choice of measurement made by one party "steers" the partner's system to one of a pair of quantum states, was also the subject of Einstein's scrutiny in a letter to Edwin Schrödinger [7], in which it is noted that a local description of steering necessitates nature to have what we would now call the ψ-epistemic property: that the quantum state is not instantiated as a part of nature's ontology [12, 20].

In this paper, we will consider "Bell experiments" of the classic form. Two experimenters have laboratories in separated locations, performing measurements on two quantum systems; each experimenter has a variety of measurements at their disposal, and once the measurements are done, they compare results. Quantum theory provides a framework in which the probability of the different outcomes, given the measurement settings chosen by each experimenter, can be calculated. In 1966, Bell showed that in some cases, the statistics predicted by quantum mechanics were incompatible with a notion of local realism: namely that each experimenter was performing measurements on a system with a definite state, independent of the measurement choices made at the other location [5]. When such an explanation in terms of local hidden variables is not possible, we would say that the distribution is *nonlocal*, or demonstrates *Bell nonlocality*. This motivates a natural decision problem: we are given a table of empirical data from such a Bell experiment, and asked to decide whether or not it constitutes a demonstration of nonlocality.

In 1991, Avis *et al* [4] demonstrated that determining whether or not a given table of these operational predictions for a Bell experiment had a local hidden variable explanation was **NP**-hard, building on work by Pitowsky [19]. In this paper, we consider a more extreme, possibilistic, manifestation of nonlocality; one in which it is not only true that we cannot account for the specific experimental probabilities with a local hidden variable model, but that we cannot even account for the possibilities. In doing so, we will identify the minimal operational requirements for this problem to remain **NP**-hard, filling in the final gap in a categorisation started by Mansfield and Fritz [14, 15] and continued by Abramsky, Gottlob, and Kolaitis [1]. This nonlocality is a feature only of the table of possibilities at hand, with no dependence on the source of the correlations. Accordingly, the first sections of this paper will focus only on the decision problem, and no knowledge about how such a table might be calculated within the formalism of quantum mechanics is required. In the final section, we turn our focus to those data tables accessible within quantum theory, and demonstrate how we can embed a large set of 3-SAT problems into a quantum formalism.

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II. BACKGROUND

In a Bell experiment, two researchers, canonically referred to as Alice and Bob, are placed at space-like separated locations and perform experiments on a pair of quantum systems that may have interacted in the past. Each of the experimenters can choose between a range of measurement selections: Alice chooses $a \in A$, Bob chooses $b \in B$. Alice's measurement can take different outcomes $\{0, 1, 2, \dots, l-1\}$, and Bob's has possible outcomes $\{0, 1, 2, \dots, m-1\}$. If Alice and Bob have k different measurement choices, and each has a maximum of l different measurement outcomes (that is, $l = m$), this is known as a $(2, k, l)$ scenario [1, 14]: there are two parties, with access to k measurements, each with l different outcomes. In this paper, we will focus on the cases where the measurements available to Alice have a different number of measurement outcomes to those of Bob.

Each combination of measurement settings and measurement outcomes corresponds to an event, and quantum mechanics gives us the ability to calculate the probabilities that a specific pair of measurement outcomes will be observed, given the chosen values of the measurement settings. We can present these in a table, for example in Figure 1. We will follow the notation of Mansfield and Fritz for these tables, in which row and each column, separated by lines, corresponds to a choice of measurement made by one of the two parties respectively. Each row or column contains multiple subrows or subcolumns, which correspond to outcomes of that measurement. The intersection of a row and column is the probability of that outcome occurring, given the choices of measurement. For example, if Alice chooses measurement setting A_2, and Bob chooses measurement setting B_1, then we see from the table that the probability of seeing the outcome $(a_2 = 0, b_2 = 1)$ is $3/8$. We will refer to each of the measurement settings for Alice and Bob as a measurement row, or measurement column respectively. Within a measurement row or column, each measurement outcome is associated with a subrow or subcolumn. We will sometimes refer to the intersection of a measurement row and a measurement column, containing the outcome probabilities or possibilities for that setting, as a "context".

This table of probabilities can be converted into a table of possibilities via a *possibilistic collapse*, explored by Abramsky [2], and by Mansfield and Barbosa [16]. We apply to the event probabilities, the

	$b_1 = 0$	$b_1 = 1$	$b_2 = 0$	$b_2 = 1$
$a_1 = 0$	$1/2$	0	$3/8$	$1/8$
$a_1 = 1$	0	$1/2$	$1/8$	$3/8$
$a_2 = 0$	$3/8$	$1/8$	$1/8$	$3/8$
$a_2 = 1$	$1/8$	$3/8$	$3/8$	$1/8$

	b_1	b_1'	b_2	b_2'
a_1	1	0	1	1
a_1'	0	1	1	1
a_2	1	1	1	1
a_2'	1	1	1	1

FIG. 1. Probability and Possibility tables for a CHSH [6] experiment. Alice has two available measurements: measurement A_1 with outcome a_1 and a_1'; and measurement A_2 with outcomes a_2 and a_2'. Bob has measurements denoted similarly. Typically, however, we will not label outcomes as the properties we are consider are invariant under relabelling.

mapping

$$f(p) = \begin{cases} 1 & p \neq 0 \\ 0 & p = 0, \end{cases} \tag{1}$$

which acts as a homomorphism between the normal probability semiring $\mathbb{R}_+$, and the Boolean semiring on $\{0, 1\}$. The result of such a collapse applied to the CHSH scenario is also presented in Figure 1. The remainder of this article will focus on this possibilistic case. We can consider such a possibility table as a function $T : A \times B \times \{0, \dots, l-1\} \times \{0, \dots, m-1\} \to \{0, 1\}$, taking as its input the measurement settings for the two locations alongside a choice of a particular outcome event, and returning whether or not that event is possible or impossible. This can be thought of as a conditional possibility $P(L, M|A, B)$: the possibility of specific measurement outcomes given the choices of measurement setting.

Given a table of probabilities or possibilities as shown above, there are two questions of particular foundational relevance that we might consider: whether the table provides an example of *signalling*, and if not, whether it still provides an example of nonlocality[18].

The notion of signalling encapsulates the idea that a particular distribution could be used as a resource to superluminally signal information between the two parties. An example of a signalling situation is demonstrated in Figure 2. We will focus on *no-signalling* tables, which cannot be used for

	$b_1 = 0$	$b_1 = 1$	$b_2 = 0$	$b_2 = 1$
$a_1 = 0$	1	1	1	1
$a_1 = 1$	0	0	1	1

FIG. 2. A simple signalling situation; if Alice sees the outcome $a_1 = 1$, she knows Bob must have chosen measurement setting B_2.

$$\begin{array}{cc|cc} 1&0&0&1\\ 0&1&0&1\\ \hline 0&0&0&0\\ 1&1&0&1 \end{array} = \begin{array}{cc|cc} 1&0&0&1\\ 0&0&0&0\\ \hline 0&0&0&0\\ 1&0&0&1 \end{array} + \begin{array}{cc|cc} 0&0&0&0\\ 0&1&0&1\\ \hline 0&0&0&0\\ 0&1&0&1 \end{array}$$

FIG. 3. A table of possibilities that does not exhibit nonlocality; it can be decomposed into a sum of deterministic grids that obey no-signalling.

$$\text{a)}\ \begin{array}{cc|cc} 1&1&0&1\\ 1&1&1&1\\ \hline 0&1&1&1\\ 1&1&1&0 \end{array} \qquad \text{b)}\ \begin{array}{cc|cc} 1&&0&\\ &&&\\ \hline 0&&&\\ &&&0 \end{array}$$

FIG. 4. Tables of possibilities for the Hardy paradoxes: first, a complete table of possibilities; second, a more general presentation in which blank spaces can be replaced with either 1 or 0.

signalling; all quantum-mechanically accessible tables are of this type. In terms of how our tables are constructed, this means that if a sub-row (sub-column) contains a 1, then it must contain a 1 in every column (row); if a measurement outcome is possible, then it is possible no matter which measurement choice is made at the other site.

If a probability or possibilitity distribution is nonlocal, it means that it cannot arise from a local hidden variables theory; any model for the process in which each of the separated quantum systems are considered to have their own internal state, independent of choices made in at the other experimental location cannot reproduce the correct statistics. By an extension of Fine's Theorem due to Abramsky and Brandenburger [3, 9], we can, without loss of generality, restrict our attention to deterministic hidden variable theories, where in every measurement context there is a single 1 entry. We say that a table is *local* if it can be decomposed into a mixture of deterministic tables that obey a no-signalling restriction. The mixture of two tables T_1 and T_2 is defined to be the sum $T_1 + T_2$, where addition is performed in the Boolean semiring. Intuitively, an outcome is possible in the mixture of two tables if and only if it is possible in at least one of those tables. Figure 3 demonstrates a simple example of a table that is local, as it permits a decomposition into grids corresponding to deterministic local hidden variables.

In 1993, Hardy [11] demonstrated that the table in Figure 4 was realisable quantum mechanically; there exists a quantum state, and quantum measurements performable on two separated subsystems that yield the pattern of possible and impossible events. The "1" entry in the top left corner cannot be part of a deterministic grid that obeys no-signalling; this can be seen by noting that such a grid would have to include the top-right 1 in the top-right context and the bottom-left 1 in the bottom-left context, and these would together imply that there should be a 1 entry in the bottom right of the bottom right context, where in fact there is a 0. This is the smallest possibility table that can yield such nonlocality [14].

III. THE LOCALITY DECISION PROBLEM AND INCOMPLETENESS OF HARDY'S PARADOX

These considerations motivate a decision problem: does a given a table of possibilities constitute a demonstration of nonlocality?

Definition 1 ((l, m)-PossLoc)**.**
Instance: a possibilistic table of data represented by a function $T : A \times B \times \{1, \ldots, l\} \times \{1, \ldots, m\} \to \{0, 1\}$.

Question: does there exist a set of functions T_i, such that the T_i are no-signalling, deterministic, and $T = \sum_i T_i$?

This represents a two-party nonlocality scenario in which Alice has access to measurements with m outcomes, and Bob has access to measurements with l outcomes. We are given a possibility table and asked whether or not it displays nonlocality. In general, we will take $|A| = |B| = n$ to maintain a single scaling factor; in fact this change is without loss of generality. In the case where $l = m$, we will denote this problem merely l-PossLoc. While it is not essential, we will also require that the data tables obey the possibilistic no-signalling principle; this merely streamlines the wording of the theorems. References [15, 16] call a specific arrangement of 0 and 1 entries in a table an *empirical model*; we shall refer to it as an *instance* of the decision problem.

Theorem 1. *[15] 2-*PossLoc *is in* ***P****.*

Proof. This proof proceeds by showing that the only kind of possibilistic nonlocality that can occur in the two-outcome case is the one originally identified by Hardy [11], as depicted in Figure 4 (b); a table is nonlocal if and only if it contains an example of the Hardy paradox.

Since the appearance of this structure is equivalent to possibilistic nonlocality in this case, an algorithm to determine whether or not a given data table is possibilistically local or nonlocal reduces to checking each possible set of four contexts for the structure. There are $n^2(n-1)^2/4$ contexts, so since checking for the appearance of the Hardy paradox is possible in constant time, our algorithm runs in $O(n^4)$. □

However, this argument is specific to the case in which we have measurements with at most two outcomes on each side. As an illustration that there are quantumly-accessible nonlocal data tables that are not reducible to a fine-graining of a Hardy paradox, we will now demonstrate a novel possibilistic nonlocality scenario in the (2,3) case. This will be a generalisation of Hardy's proof of Bell's theorem.

Figure 5 shows an equatorial slice of the Bloch sphere, and a set of states of Bob's system to which he can be "steered" by Alice. The Bloch sphere is a representation of the state space of a quantum system of dimension 2 as a sphere, in which two orthogonal quantum states are mapped to antipodal points on the sphere; this mapping is a case of the Hopf fibration. Taking an equatorial slice can be thought of as restricting to the case of *rebits*, states that can be represented as vectors over $\mathbb{R}^2$ rather than $\mathbb{C}^2$. For each pair of points on the circle, the line between which goes through ρ, there is a measurement that Alice can perform such that if she gets one outcome, Bob's system is left in the state corresponding to one of the points, and if she gets the other, Bob's system is left in the other. As such, we can associate Alice's measurements with such pairs of points, which are said to be *decompositions* of ρ. The measurements Bob can perform on the state manifest themselves in two ways in the diagram. Any set of points such that the centre of the circle is in their convex hull is a valid POVM measurement for Bob; we will restrict him to measurements with either 2 or 3 outcomes, so his measurements are either a pair of antipodal

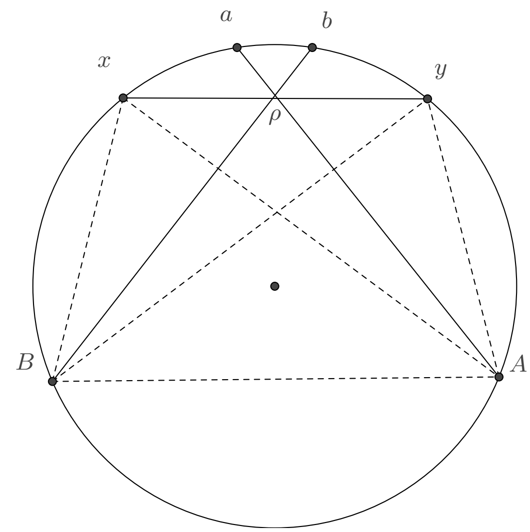

FIG. 5. Points on the Bloch sphere to which Alice's ensemble can be steered, revealing a generalisation of the Hardy paradox.

	b	$b^\perp$	$A^\perp$	$B^\perp$	$x^\perp$	$A^\perp$	$B^\perp$	$y^\perp$
a	1	1	1	1	1	1	1	1
A	1	1	0	1	1	0	1	1
b	1	0	1	1	1	1	1	1
B	1	1	1	0	1	1	0	1
x	1	1	1	1	0	1	1	1
y	1	1	1	1	1	1	1	0

FIG. 6. A generalised Hardy paradox available for a two qubit measurement, with a two-outcome projective measurement at one site and a three-outcome POVM measurement at the other.

points, or a triple with the centre in their convex hull. Impossible events occur when Bob's system is steered to a state ψ, and a measurement is performed that contains the element $\psi^\perp$, the point antipodal to ψ on the circle. For clarity, points antipodal to those that Alice can steer Bob to are not shown in Figure 5. Constructing the possibility table for this steering scenario reveals the structure presented in Figure 6. The measurement rows denote pairs of states to which Alice can steer Bob; the measurement columns represent measurements that Bob is able to perform. We now present this paradox minimally, demonstrating that it is neither a Hardy paradox, nor a fine-graining of a Hardy paradox. Therefore, the following statement holds:

Proposition 1. The Hardy paradox is not universal even for quantumly-accessible possibilistic nonlocality as long as one party has access to three-outcome measurements.

1	0			0		
0						
		0			0	
			0			
						0

We see that in some sense this can be thought of as a generalisation of Hardy's paradox. The third measurement row acts to convert the two right-hand measurement columns into a single effective column identical to that of the standard Hardy paradox, since we see that in any deterministic, no-signalling grid containing the above specified "1" entry, we cannot have that both measurement columns 2 and 3 have their 1 entries in the rightmost subcolumn. With this rightmost column effectively eliminated, we are left with the form of the Hardy paradox.

IV. (2,3)-POSSLOC IS NP-COMPLETE

We can see that (l, m)-PossLoc is in **NP**; to demonstrate that a data table is possibilistically local, we can specify a possibilistically local hidden variable model that accounts for each of the occurrences of the possible events (1 entries) within the problem instance. Such a witness is only polynomially-sized, since there are only polynomially many 1 entries and each data table is polynomially sized; that they sum to the table in question can also be checked in polynomial time. The rest of this section, then, will be a proof of **NP**-hardness. We note that since Mansfield and Fritz [14] showed that 2-PossLoc was in **P** and Abramsky, Gottlob, and Kolaitis showed that 3-PossLoc and the three-party generalisation of 2-PossLoc were **NP**-complete, this leaves $(2, 3)$-PossLoc as the only remaining case to have its complexity analysed.

We will demonstrate that $(2, 3)$-PossLoc is **NP**-complete. First, we will introduce, following Abramsky, Gottlob and Kolaitis, the concept of a robust satisfiability problem; we will then demonstrate the **NP**-completeness of a specific family of robust satisfaction problems, and show a polynomial-time embedding of such problems into $(2, 3)$-PossLoc.

Definition 2 (r-Robust satisfiability problems)**.** For a satisfiability decision problem S, the problem r-Robust S to be the decision problem that, given an instance of S, asks whether or not every assignment of r variables in the problem that do not directly cause one of the clauses to be false, can be extended to a satisfying assignment of the problem.

Theorem 2. *The* (2,3)-PossLoc *decision problem is* ***NP****-complete.*

Proof. We will now introduce the notion of 0- and 1-valid 3-SAT. This restriction is not strictly necessary for this proof, but we will see later that it has a bearing on the computational hardness of deciding locality for quantum-mechanically accessible possibility tables.

Definition 3. 0-valid 3-SAT is the set of 3-SAT decision problems that are satisfied by assigning a value of 0 to all variables.

We will define 1-valid 3-SAT analogously. Now, we will demonstrate that 2-Robust 0-valid 1-valid 3-SAT, the set of 2-Robust 3-SAT instances which are both 0-valid and 1-valid, is **NP**-complete. We will then demonstrate how such a problem can be embedded into $(2, 3)$-PossLoc.

Lemma 1. 2-Robust 0-valid 1-valid 3-SAT *is* ***NP****-complete.*

Proof of Lemma. Consider a general 3-SAT instance $C = \bigwedge_i c_i$, where c_i are clauses consisting of the disjunction of three literals or negated literals. We will introduce two new variables x and y, and apply the mapping f to each clause c_i defined by:

$$f(l_1 \cup l_2 \vee \circ l_3) = (l_1 \vee l_2 \vee \circ l_3 \vee \neg x), \qquad (2)$$

$$f(\neg l_1 \vee \neg l_2 \vee \circ l_3) = (\neg l_1 \cup \neg l_2 \vee \circ l_3 \vee y), \qquad (3)$$

in which the $\circ$ symbol is being used to denote the presence or absence of a $\neg$ symbol, and should be taken as constant across instances of the same variable within the equations. By the pigeonhole principle, each clause must contain either at least two positive literals, or at least two negated literals, and so f is total. We note that now, every clause of $C' = \bigwedge_i f(c_i)$ contains at least one positive literal and at least one negative literal, and so C' is both 0-valid and 1-valid.

We note that it is possible now to convert this 4-SAT instance back to a 3-SAT instance by the following procedure, mapping each clause to a pair of equisatisfiable clauses (which we will refer to as an effective clause):

$$(\circ l_1 \cup \circ l_2 \vee \circ l_3 \vee \neg x) \to (\circ l_1 \cup \circ l_2 \vee z) \wedge (\neg z \vee \circ l_3 \vee \neg x) \tag{4}$$

$$(\circ l_1 \cup \circ l_2 \vee \circ l_3 \vee y) \to (\circ l_1 \cup \circ l_2 \vee z) \wedge (\neg z \vee \circ l_3 \vee y) \tag{5}$$

We note also that this adjustment maintains 0-validity and 1-validity, since we can choose an order on the $\{\circ l_i\}$ such that each of the two clauses in the effective clause contain a positive and a negative literal. We note that since each effective clause contains either $\neg x$ or y, setting x to 0 and y to 1 leads to a satisfying assignment for any assignment choices of the other variables.

We can now verify that this transformed instance is 2-Robustly satisfiable if and only if our original 3-SAT instance was satisfiable. We will consider the possibilities on a case-by-case basis, depending on whether or not the variables x or y have been assigned valuations of 1 and 0 respectively:

1. If neither x is fixed to 1 nor y is fixed to 0, then we set x to 0 and y to 1. As mentioned above, this yields a satisfying assignment. If one or more z has been assigned a value, we may need to assign some l_i to make the affected clauses valid. Since this affects at most two clauses it can be seen that it is always possible to satisfy such a modified pair of clauses.

2. If exactly one of x is fixed to 1 or y is fixed to 0, then, removing all instances of the $\neg x$ or y literals respectively still leaves the instance either 0-valid or 1-valid, and so the instance is satisfiable by definition.

3. If we fix both x to 1 and y to 0, then removing the $\neg x$ and y literals from C' transform it back into C, and therefore this restriction has a satisfying assignment if and only if C did originally.

Therefore, since 3-SAT is **NP**-hard, so is 2-Robust 0-valid 1-valid 3-SAT. It is in **NP** since there are only $O(n^2)$ different pairs of variables to set, and so the witness size to demonstrate a set of assignments that display robustness is still only polynomially sized. Hence, 2-Robust 0-valid 1-valid 3-SAT is **NP**-complete.

□

We note that a stronger version of this result was independently derived by Ham [10].

Corollary 1. 2-Robust 3-SAT *is* ***NP****-complete.*

We note now that we will in fact only need the fact that 2-Robust 3-SAT is **NP**-complete for the rest of this proof; the motivation for having proven that the 0-valid 1-valid is also **NP**-complete will become apparent when considering which of these possibility tables are realisable within quantum mechanics in Section V.

Lemma 2. *There is a polynomial-time embedding of* 2-Robust 3-SAT *into* (2,3)-PossLoc.

Proof of Lemma. The reduction algorithm is as follows:

- For each variable in the 2-Robust 3-SAT instance, we add a measurement to the party with two-outcome measurements available to them, to which we will assign the measurement rows. We pick any ordering of the variables to do this.

- For each clause in the 2-Robust 3-SAT instance, we add a measurement to the party with three-outcome measurements available to them; these are our measurement columns. This is a departure from the strategy of Abramsky, Gottlob, and Kolaitis, whose constructions have a direct symmetry between the rows and columns.

- For each intersection of a variable row and clause column, such that the variable is represented positively in the clause, we have the measurement possibilities given by:

  $$\boxed{\begin{matrix}0&1&1\\1&1&1\end{matrix}}, \boxed{\begin{matrix}1&0&1\\1&1&1\end{matrix}}, \text{or} \boxed{\begin{matrix}1&1&0\\1&1&1\end{matrix}},$$

  depending on whether the variable is the first, second, or third variable in the clause with respect to our variable ordering.

- For each intersection of a variable row and clause column, such that the variable is represented negatively in the clause, we have the measurement possibilities given by:

  $$\boxed{\begin{matrix}1&1&1\\0&1&1\end{matrix}}, \boxed{\begin{matrix}1&1&1\\1&0&1\end{matrix}}, \text{or} \boxed{\begin{matrix}1&1&1\\1&1&0\end{matrix}},$$

  depending on whether the variable is the first, second, or third variable in the clause with respect to our variable ordering.

We can see that the choice of subrow in a measurement row correspond to an assignment of the associated variable, and that the clause structure effectively bans the assignment disallowed by that specific clause, for example the clause $(x_1 \vee x_2 \vee x_3)$ corresponds to the possibility table

$x_1 = 0$	0	1	1
$x_1 = 1$	1	1	1
$x_2 = 0$	1	0	1
$x_2 = 1$	1	1	1
$x_3 = 0$	1	1	0
$x_3 = 1$	1	1	1

We see we cannot simultaneously assign a 1 to each of the top subrows as part of a deterministic grid. It now only remains to be shown that choosing a specific "1" instance, which needs to be extended to a deterministic grid, is the equivalent of fixing at most two variable assignments in the embedded 3-SAT instance. We note that choosing any "1" entry fixes the valuation of the variable associated with its measurement row; the "1" is in some measurement subrow and corresponds to a value assignment for that variable. Additionally, each measurement subcolumn has exactly one "0" entry by construction: if this "0" instance is not in the same measurement row as the chosen "1" entry, this also fixes a variable assignment for the measurement row containing the "0" entry. In the example directly above, if the bottom-left "1" is chosen, then we see from the table that this is equivalent to asking that $x_3 = 1$, $x_1 = 1$. Therefore this construction constitutes a polynomial-time reduction from 2-Robust 3-SAT to $(2,3)$-PossLoc.

□

Therefore, since 2-Robust 3-SAT is **NP**-hard; any instance of 2-Robust 3-SAT can be reduced to an instance of (2,3)-PossLoc in polynomial time; and (2,3)-PossLoc is in **NP**, (2,3)-PossLoc is **NP**-complete. □

V. QUANTUM REALISATION

Thus far in the paper we have cared only about whether or not a given table was achievable within the set of no-signalling theories, rather than whether or not it could arise from a particular quantum mechanical scenario. When we restrict ourselves to quantumly accessible distributions, all these problems become open.

Theorem 3. *All the quantumly accessible instances of the above* (2,3)-PossLoc *construction in which the party with three-outcome measurements has access only to a 2-dimensional quantum system are 0-valid and 1-valid under a variable renaming that can be calculated in linear time.*

Proof. Each measurement row corresponds to a two-outcome measurement. Without loss of generality, we can take these to be projective measurements since any two-outcome POVM is a convex mixture of such projectors. Since the possibility table would then be a statistical mixture of two or more other tables, noting that the statistical mixture of two local tables is always local, if there exists a quantum mechanically accessible table using POVMs for measurement choices for Alice, there must exist one using PVMs. If our entangled state is pure, then this means that each outcome steers Bob's system to one of two pure states that convexly mix to Bob's reduced state. Pure states, and measurement directions, of Bob's system are represented by rays in a 2-dimensional Hilbert space, which can be represented as a collection of points a sphere $\mathcal{S}^2$, a representation known as the Bloch sphere.

We note that each measurement column corresponds to a three-element POVM, and as such the convex hull of each triple contains the origin. Additionally, these POVM elements have an additional geometrical constraint: the POVM element with a 0 entry when $x_i = 0$, say, must be orthogonal to the state to which Bob is steered when he gets the outcome $x_i = 0$. This restricts each of these elements to be proportional to the projector onto the unique element orthogonal to the steered states.

We note that each variable is associated with two POVM elements, one representing positive occurrences of the variable, the other representing the negative occurrences of the variable; since the union of the supports of such a pair is the whole Hilbert space, at least one of these must have support in our chosen hypersphere and we choose one such POVM element to represent the positive occurrences, where by the support of a matrix we mean the set of vectors that are not mapped to the zero vector via the action of the matrix. Now, if we chose our hemisphere such that no POVM elements lie on its equator, which is always possible since there are only finitely many POVM elements, then we see that each triple must have at least one POVM element from that half of the Hilbert space. Each clause therefore contains at least one positively-represented variable and so the all-true assignment of the variables is a satisfying assignment. Additionally, the opposite holds and so the all-false assignment of the variables is satisfied also after this transformation has taken place.

This calculation can be done in linear time exactly if we can find a great circle of the sphere that does not go through any of point in linear time. To do this, take any axis of the sphere that does not go through a point, any set of $n+1$ such axes must contain one that does not meet a point, and consider a set of $n+1$ great circles that lie along this axis. Since such great circles share only their intersection points at each end of the axis, at most n can contain one of our chosen points and so one must contain none. □

We note that since our reduction in Lemma 2 was from a robust version of this problem, this proof alone does not demonstrate that the quantum realisation is not **NP**-complete; however it does demonstrate that we have access only to a very restricted set of problems we can embed, and therefore the hardness results of the previous section do not automatically carry through into the quantumly-accessible world. We have seen that when robustness is introduced into the mix, even simple problems can become **NP**-hard; we cannot make any assumptions about the difficulty of 2-robust variants of decision problems based on the difficulty of the underlying problem, but that we can embed *some* 2-Robust 0-valid 1-valid 3-SAT problems into the quantum formalism. We note however that an arbitrary instance cannot be embedded: for instance, one can show that the array shown in Figure 7 is ruled out by the second tier of the NPA-hierarchy [17], which is a heirarchy of semidefinite programs to test whether a given probability distribution is quantum-mechanically realisable: rejection at any tier of the hierarchy constitutes a proof that it is not. The proof is merely by computation and will not be presented here, however the following proposition holds in general:

Proposition 2. The array demonstrateed in Figure 7 is not possible to create in any quantum dimension and any realisation of measurement outcomes.

Having considered some restrictions on which 3-SAT instances can be embedded into (2,3)-PossLoc, we will now present a construction that enables a reasonably large class of 3-SAT instances to be embedded. We will consider the case in which our projective measurements act on a qubit, and are additionally confined to a single plane through the Bloch sphere. It is possible that relaxing either of these assumptions could allow more instances to be embedded, however this is not clear: moving from a two-dimensional to a higher-dimensional quantum system could allow more flexibility with regards to the geometry of the projectors, but since our measurements can only have two outcomes, having more dimensions causes the creation of impossible events to be more difficult; likewise moving from a system in which all the projectors are coplanar makes it harder for impossible events to be created from the projectors since for a qubit at least, we need three coplanar vectors to form a valid POVM. We note that the array in Figure 7 is, in particular, not achievable in this situation, since if projectors $\{P_1, P_2, P_3\}$ in a two-dimensional Hilbert space have the identity projector in their convex hull, then the set $\{P_1, P_2, P_3^{\perp}\}$ does not unless $P_1 = P_2^{\perp}$. In this situation, they cannot form a proper POVM alongside another nonzero element.

0	1	1	0	1	1
1	1	1	1	1	1
1	0	1	1	0	1
1	1	1	1	1	1
1	1	0	1	1	1
1	1	1	1	1	0

FIG. 7. This table of possibilities, which corresponds to the 3-SAT instance $(x_1 \wedge x_2 \wedge x_3) \vee (x_1 \wedge x_2 \wedge \neg x_3)$ can be shown to be unrealisable quantum mechanically, for example by invoking the NPA heirarchy.

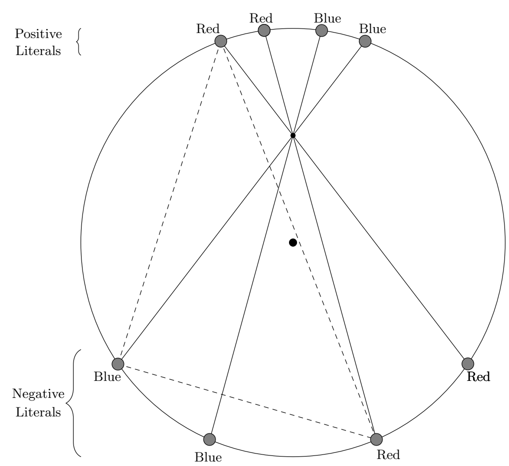

FIG. 8. A red positive literal, a red negative literal and a blue negative literal form a valid POVM.

We will draw inspiration for our projector geometry from the novel possibilistic nonlocality scenario we illustrated in Figure 5 and presented in Figure 6.

For diagrammatic clarity, in this section we will adopt a slight representational change for our diagrams, compared to the previous section. We will consider the scenario in which Alice and Bob share a pure, entangled state and so once again, we will consider an equatorial slice of the Bloch sphere, and the reduced state for Bob is a point, ρ, within the circle. Any pair of points on the edge of the circle, the straight line through which goes through the point ρ constitute a pair of states to which Bob can be steered (we will call this a steering pair); that is, there exists a measurement row such that if the event corresponding to one subcolumn is observed by Alice, Bob's system is left in one of those states, and if the other is observed Bob's system is left in the other. The departure from our previous notation comes with how we will represent POVMs. Each POVM element, which in this section will be proportional to a projector $|\psi\rangle\langle\psi|$, will be shown on the equatorial Bloch plane as being in the location corresponding to the orthogonal projection $\left|\psi^{\perp}\right\rangle\left\langle\psi^{\perp}\right|$. The reason for this is that in this representation of the system, when the same point on the circle is part of a POVM and a steering pair, the possibility at the intersection of the relevant measurement subrow and measurement subcolumn will be 0; Bob's system is left in a state $\left|\psi^{\perp}\right\rangle$ and the POVM element is proportional to $|\psi\rangle$.

- The reduced state ρ_B is a point inside or on the edge of a circle.
- A measurement row corresponds to a line going through the point corresponding to ρ_B. The two measurement outcomes are represented by the points at which this line intersects the circle, or, equivalently, the inversion of these points through the centre of the circle. The latter denotation will now be used.
- A measurement column consists of three points around the edge of the circle such that the circle's centre is in their convex hull, each of which is associated with a single measurement subcolumn.
- An impossible event happens when the point corresponding to that measurement subcolumn and the point corresponding to that measurment subrow are the same point.

A specific quantum scenario will now be explored alongside a characterisation of some of the problem instances that can be embedded into it. Given a 3-SAT instance, let us assign each variable a colour: blue or red. Clauses with variables of certain colourings and negation status will be realisable with this construction. For example, as can be seen in Figure 8, a clause consisting of a red positive literal, a red negative literal, and a blue negative literal, *e.g.* $(l_1^R{\wedge}\neg l_2^R{\wedge}\neg l_3^B)$, forms a valid POVM.

If we add projectors for x and y that are close to the eigendecomposition of ρ_B (distinguished only because we want them to have independent possibilities), as shown in Figure 9, we can note that we can also support clauses of forms $(l_1^R{\vee}l_2^B{\vee}\neg x)$ or $(\neg l_1^R{\vee}\neg l_2^B{\vee}y)$.

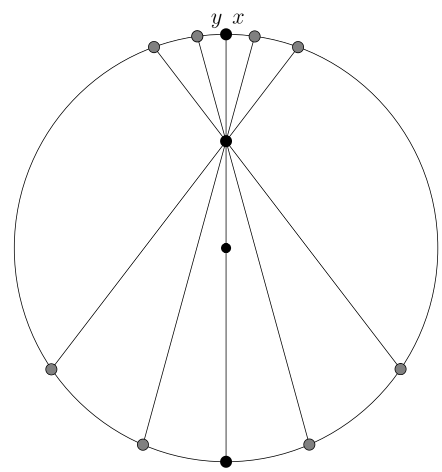

FIG. 9. Adding in x and y projectors sufficiently close to the eigendecomposition of ρ_B, which is signified by the black circles. The positions of x and y are not shown individually due to their closeness to the eigendecomposition.

If we want to perform the transformation procedure in theorem 1, we can see from these constructions that the clauses in the initial 3-SAT instance can have one of the following forms:

- $(l_1^R{\vee}l_2^R{\vee}l_3^B)$, by transforming it to the conjunction of clauses $(l_1^R{\vee}l_3^B{\vee}\neg z) \wedge (z{\vee}l_2^R{\vee}\neg x)$, with z close to x.
- $(\neg l_1^R{\vee}\neg l_2^R{\vee}\neg l_3^B)$, by transforming it to the conjunction of clauses $(\neg l_1^R{\vee}\neg l_3^B{\vee}z) \wedge (\neg z{\vee}\neg l_2^R{\vee}y)$, with all clauses of allowed forms if we choose the colouring z^B.
- $(\neg l_1^R{\vee}\neg l_2^B{\vee}l_3^R)$, by transforming it to $(\neg l_1^R{\vee}\neg l_2^B{\vee}z_2^R){\wedge}(\neg z_2^R{\vee}\neg z_1^B{\vee}y){\wedge}(z_1^B{\vee}l_3^R{\vee}\neg x)$.

- $(l_1^R \vee l_2^B \vee \neg l_3^R)$, since this becomes modified to $(\neg z^R \vee l_2^B \vee \neg l_3^R) \wedge (z^B \vee l_1^R \vee \neg x)$, and by choosing the colouration z^R both the clauses have permitted forms.

We also clearly by symmetry have as permissible clauses the images of the ones above under an exchange of the colours blue and red. The rules seem to be, then, that in the original 3-SAT each variable must be colourable either red or blue such that each clause contains at least one literal of each colour, and that the literal in each clause that is the only one of its colour must not have opposite sense to the other two literals. The author has been unable to produce a proof of **NP**-hardness or membership in **P** of such 3-SAT instances. The scenario in consideration does not fall under the purview of Ham's trichotomy result [10], due to the restrictions being placed on which variables, not just which senses, can appear together in clauses. In any case, the quantum realisation forces a very strong geometrical relationship on clauses of the embedded instance.

VI. CONCLUSION

By demonstrating the **NP**-completeness of $(2,3)$-PossLoc, the computational complexity of all such possibilistic locality experiments have now been characterised. However, the problem of quantum realisation remains open; we have also seen that it is difficult even to rule out a quantum realisation under the restrictive assumption that the party with three-outcome measurements has access only to a two-dimensional quantum system– although this does also imply without loss of generality that the entire state under questioning is an entangled state of two qubits as can be seen by invoking the Schmidt decomposition. A natural extension of this problem would be into the formalism of ontological models; it is possible to prove possibilistic nonlocality results at the level of underlying ontological models that nonetheless rely on operational probabilities rather than possibilities, as is the case in Section IV of reference [13].

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