A Long List of Problems for 33X

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The idea for this page came from a YouTube video called "10,000" problems in Analysis, which can be found here!

Of course, the goal is to compile a long list of problems ranging from very difficult to near trivial, all to get a better grasp of what it means to do *real* advanced calculus problems.

Some shorthand on sources:

PMA = Principles of Mathematical Analysis (Rudin)
MSE = Mathematics StackExchange

Question 1

Source: Mathematics Discord (discord message)
By using the Cauchy Criterion for convergence, show that the sequence defined by

$$\{x_n\}_1^\infty = \frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2}.$$

converges.

Question 2

Source: Spring 1981 UC Berkley Mahtematics PhD Prelims, Question 16.
Let $f(x)$ be defined as a real-valued function for all $x \geq 1$, such that $f(1) = 1$ and

$$f'(x) = \frac{1}{x^2 + (f(x))^2}.$$

Prove that

$$\lim_{x\to\infty} f(x)$$

exists and the limit is *less than* $1 + \frac{\pi}{4}$.

Question 3

Source: Real Analysis (Royden) Chapter 6.1 Question 2
Show that there exists a strictly increasing function $f(x)$ over the interval $[0, 1]$, but $f(x)$ is continuous over only the irrationals in $[0, 1]$

Question 4

Source: PMA (Rudin) Chapter 5
Suppose $f'(x)$ is continuous over an interval $[a, b]$ and let $\varepsilon > 0$. Prove that there exists some $\delta > 0$ such that

$$\left|\frac{f(t) - f(x)}{t - x} - f'(x)\right| < \varepsilon$$

whenever $0 < |t - s| < \delta$, $x \in [a, b]$; $y \in [a, b]$.
If this property holds, we say that f is *uniformly differentiable* on $[a, b]$.

Question 5

Source: PMA (Rudin) Chapter 5
If $f(x) = |x|^3$, compute $f'(x)$ and $f''(x)$ for all real x. Then show that $f^{(3)}(0)$ does not exist.

Question 6

Source: MSE (Question)
Determine the points of continuity of $h(x) = \lfloor \sin(x) \rfloor$, where $\lfloor x \rfloor$ is the greatest $m \in \mathbb{Z}$ such that $m \leq x$ (floor function).

Question 7

Source: Sample UC Davids Real Analysis Questions (Has Solution)

(a) Suppose $f_n : A \to \mathbb{R}$ is *uniformly continuous* on A for every $n \in \mathbb{N}$ and $f_n \to f$ uniformly on A. Prove that f is uniformly continuous on A.

(b) Does the result in (a) remain true if $f_n \to f$ pointwise instead of uniformly?

Question 8

Source: Problems in Real Analysis (Rădlescu et. al) Chapter 5.2 No. 3
Let $f(x)$ be a polynomial and a be a real number such that $f(a) \neq 0$. Show that there exists a polynomial with real coefficients $g(x)$ such that $p(a) = 1$, $p'(a) = 0$, and $p''(a) = 0$, where $p(x) = f(x)g(x)$.

Question 9

Source: Problems in Real Analysis (Rălescu et. al) Chapter 5.2 No 8
Find all integers a and b such that $0 < a < b$ and $a^b = b^a$.

Question 10

Source: Problems in Mathematical Analysis (Kazcor and Nowak) Exercise 2.2.29
Show that the only functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$\frac{f(x+h) - f(x)}{h} = f'(x + \frac{h}{2})$$

are polynomials of degree two.

Question 11

Source: Problems in Mathematical Analysis (Kazcor and Nowak) Exercise 2.2.31
Prove that if f is differentiable on an interval $\mathbf{I}$, then the Intermediate Value Theorem can be used on $\mathbf{I}$.
(This is referred to as *Darboux's Theorem*.)

Question 12

Source: Kazcor and Nowak Exercise 2.3.9.
Let f be an $n+1$ times differentiable function. Prove that for every $x \in \mathbb{R}$, there exists some $\theta \in (0,1)$ such that

(a)

$$f(x) = f(0) + xf'(x) - \frac{x^2}{2}f''(x) + \cdots + (-1)^{n+1}\frac{x^n}{n!}f^{(n)}(x) + (-1)^{n+2}\frac{x^{n+1}}{(n+1)!}f^{(n+1)}(\theta x).$$

(b)

$$f\left(\frac{x}{1+x}\right) = f(x) - \frac{x^2}{1+x}f'(x) + \cdots + (-1)^n \frac{x^{2n}}{(1+x)^n}\frac{f^{(n)}(x)}{n!} + (-1)^{n+1}\frac{x^{2n+2}}{(1+x)^{n+1}}\frac{f^{(n+1)}(\frac{x+\theta x^2}{1+x})}{(n+1)!} \qquad x \neq -1.$$

Question 13

Source: MSE (Source)
Suppose we have a function $f\colon \mathbb{R}^2 \to \mathbb{R}$ such that f is smooth (C^∞). Furthermore, suppose that f has two global minima. Prove or disprove that f has a third critical point.

- What happens if we add the constraint that $f(x) \to \infty$ as $\|x\| \to \infty$?

Question 14

Let $f, g\colon [0,1] \to \mathbb{R}$ be continuous functions such that for any continuous $\varphi\colon [0,1] \to \mathbb{R}$ where $\varphi(1) = \varphi(0) = 0$, we have

$$\int_0^1 (f(x)\varphi'(x) + g(x)\varphi(x))\,\mathrm{d}x = 0$$

Prove that f is continuous over it's domain and $f'(x) = g(x)$.
This theorem is called the *du-Bois Reymond Lemma*.