A Long Ass List of Problems for 33X

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Last Updated: November 2, 2023

The idea for this page came from a YouTube video called "10,000" problems in Analysis, which can be found here!

Of course, the goal is to compile a long list of problems ranging from very difficult to near trivial, all to get a better grasp of what it means to do *real* advanced calculus problems.

Some shorthand on sources:
PMA = Principles of Mathematical Analysis (Rudin)

Question 1

Source: Mathematics Discord (discord message)
By using the Cauchy Criterion for convergence, show that the sequence defined by

$$\{x_n\}_1^\infty = \frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2}.$$

converges.

Question 2

Source: Spring 1981 UC Berkley Mahtematics PhD Prelims, Question 16.
Let $f(x)$ be defined as a real-valued function for all $x \geq 1$, such that $f(1) = 1$ and

$$f'(x) = \frac{1}{x^2 + (f(x))^2}.$$

Prove that

$$\lim_{x\to\infty} f(x)$$

exists and the limit is *less than* $1 + \frac{\pi}{4}$.

Question 3

Source: Real Analysis (Royden) Chapter 6.1 Question 2
Show that there exists a strictly increasing function $f(x)$ over the interval $[0, 1]$, but $f(x)$ is continuous over only the irrationals in $[0, 1]$

Question 4

Source: PMA (Rudin) Chapter 5
Suppose $f'(x)$ is continuous over an interval $[a, b]$ and let $\varepsilon > 0$. Prove that there exists some $\delta > 0$ such that

$$\left|\frac{f(t) - f(s)}{t - s} - f'(x)\right| < \varepsilon$$

whenever $0 < |t - s| < \delta$, $x \in [a, b]$; $y \in [a, b]$.
If this property holds, we say that f is *uniformly differentiable* on $[a, b]$.

Question 5

Source: PMA (Rudin) Chapter 5
If $f(x) = |x|^3$, compute $f'(x)$ and $f''(x)$ for all real x. Then show that $f^{(3)}(0)$ does not exist.

Question 6

Source: Mathematics StackExchange (Question
Determine the points of continuity of $h(x) = \lfloor \sin(x) \rfloor$, where $\lfloor x \rfloor$ is the greatest $m \in \mathbb{Z}$ such that $m \leq x$ (floor function).

Question 7

Source: Sample UC Davids Real Analysis Questions (Has Solution

(a) Suppose $f_n \colon A \to \mathbb{R}$ is *uniformly continuous* on A for every $n \in \mathbb{N}$ and $f_n \to f$ uniformly on A. Prove that f is uniformly continuous on A.

(b) Does the result in (a) remain true if $f_n \to f$ pointwise instead of uniformly?