

A Long Ass List of Problems for 33X

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The idea for this page came from a YouTube video called “10,000” problems in Analysis, which can be found [here](#)!

Of course, the goal is to compile a long list of problems ranging from very difficult to near trivial, all to get a better grasp of what it means to do *real* advanced calculus problems.

Some shorthand on sources:

PMA = Principles of Mathematical Analysis (Rudin)

Question 1

Source: Mathematics Discord (discord message)

By using the Cauchy Criterion for convergence, show that the sequence defined by

$$\{x_n\}_1^\infty = \frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2}.$$

converges.

Question 2

Source: Spring 1981 UC Berkley Mathematics PhD Prelims, Question 16.

Let $f(x)$ be defined as a real-valued function for all $x \geq 1$, such that $f(1) = 1$ and

$$f'(x) = \frac{1}{x^2 + (f(x))^2}.$$

Prove that

$$\lim_{x \rightarrow \infty} f(x)$$

exists and the limit is *less than* $1 + \frac{\pi}{4}$.

Question 3

Source: Real Analysis (Royden) Chapter 6.1 Question 2

Show that there exists a strictly increasing function $f(x)$ over the interval $[0, 1]$, but $f(x)$ is continuous over only the irrationals in $[0, 1]$

Question 4

Source: PMA (Rudin) Chapter 5

Suppose $f'(x)$ is continuous over an interval $[a, b]$ and let $\varepsilon > 0$. Prove that there exists some $\delta > 0$ such that

$$\left| \frac{f(t) - f(x)}{t - x} - f'(x) \right| < \varepsilon$$

whenever $0 < |t - x| < \delta$, $x \in [a, b]$; $y \in [a, b]$.

If this property holds, we say that f is *uniformly differentiable* on $[a, b]$.

Question 5

Source: PMA (Rudin) Chapter 5

If $f(x) = |x|^3$, compute $f'(x)$ and $f''(x)$ for all real x . Then show that $f^{(3)}(0)$ does not exist.

Question 6

Source: Mathematics StackExchange (Question)

Determine the points of continuity of $h(x) = \lfloor \sin(x) \rfloor$, where $\lfloor x \rfloor$ is the greatest $m \in \mathbb{Z}$ such that $m \leq x$ (floor function).

Question 7

Source: Sample UC Davids Real Analysis Questions (Has Solution)

- (a) Suppose $f_n: A \rightarrow \mathbb{R}$ is *uniformly continuous* on A for every $n \in \mathbb{N}$ and $f_n \rightarrow f$ uniformly on A . Prove that f is uniformly continuous on A .
- (b) Does the result in (a) remain true if $f_n \rightarrow f$ pointwise instead of uniformly?

Question 8

Source: Problems in Real Analysis (Rădulescu et. al) Chapter 5.2 No. 3

Let $f(x)$ be a polynomial and a be a real number such that $f(a) \neq 0$. Show that there exists a polynomial with real coefficients $g(x)$ such that $p(a) = 1$, $p'(a) = 0$, and $p''(a) = 0$, where $p(x) = f(x)g(x)$.

Question 9

Source: Problems in Real Analysis (Rădulescu et. al) Chapter 5.2 No 8

Find all integers a and b such that $0 < a < b$ and $a^b = b^a$.