An Introduction to "Dimensionality"
Some Warping Required

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0 Introduction

This paper is about the ideas of dimensionality, both in the relatively physical sense of Linear Algebra as well as into the non-standard way of thinking for fractals.

1 A Brief Summary of Linear Algebra

First we begin by reviewing some general axioms and definitions from linear algebra.

Definition 1. *(Vector Space Axioms) We say that a space X with scalars in a field $\mathbb{F}$ is said to be a vector space if the following conditions are satisfied:*

- *(Associativity) For any u, v, and w in the space X, we have that*

$$u + (v + w) = (u + v) + w.$$

- *(Commutativity) For any $u, v \in X$ we have that*

$$u + v = v + u.$$

- *(Identity Element) There exists a vector $e \in X$ such that for any $v \in X$, $ev = ve = v$.*
- *(Zero Vector) There exists a vector $0 \in X$ such that for any $v \in X$, $0 + v = v + 0 = v$.*
- *(Inverse Element) For any $v \in X$, there exists another vector – denoted as the element $-v$ – such that*

$$v + (-v) = 0.$$

- *(Associativity of Scalars)For any two scalars $a, b \in \mathbb{F}$ and $v \in X$,*

$$a(bv) = b(av).$$

- *(Identity scalar) There exists a $1 \in \mathbb{F}$ such that for any $v \in X$,*

$$1v = v.$$

Remark: If we have a subset, Y, of a vector space, X, that satisfies the vector space axioms, we call Y a *subspace* of X.

When we consider any vector space, we need some way to refer to an arbitrary point in space. In $\mathbb{R}^2$, we would say that a point in the space is described as (x, y) where x and y are real numbers, denoting how far they are from the origin with respect to the x and y axes. Now within that description, there was a fundamental choice between the "building blocks" of the vector space. Those blocks being $(1, 0)$ and $(0, 1)$ respectively. How can we determine what we can choose as those blocks?

Definition 2. *(Linear Combination) For any two vectors, u and v, we call the linear combination of the two the collection of vectors such that we have vectors of the form*

$$au + bv,$$

where a and b are scalars.

Definition 3. *(Span) We say that a collection of vectors $\{v_1, v_2, \ldots, v_n\}$ span a vector space V when:*

- *Every vector in space can be represented as a linear combination of our basis vectors.*

Remark: We call the collection of vectors that span the vector space a *basis* for the space V.

1.1 Dimension of Vector Spaces

We say that a vector space has a finite dimension if there are a finite number of vectors in the basis. Otherwise we say that the vector space is *infinite dimensional*.

So, if a vector space $\mathcal{V}$ has 3 vectors in its basis, then we say that $\mathcal{V}$ has a dimension of 3.

2 Manifolds

2.1 What is a manifold?

Example to strive towards: An example of a manifold would be the Earth, as when we "zoom" in, the surface of the object appears to look like $\mathbb{R}^2$.

2.2 Atlases and Charts

A **homeomorphism** is a concept in topology that describes a strong form of equivalence between topological spaces, meaning that they are structurally the same space.

Definition 4. *Given two topological spaces X and Y, we say a function $f : X \to Y$ is called a **homeomorphism** if the following are true:*

- *Bijective: f is a one-to-one (injective) and onto (surjective) function.*
- *Continuous: The function f is continuous, meaning that the preimage of any open set in Y is open in X.*
- *Continuous inverse: The inverse function $f^{-1} : Y \to X$ is also continuous.*

*If such a function f exists, then the spaces X and Y are said to be **homeomorphic**.*

Said in another way, if these two spaces are homeomorphic, we can treat the spaces like Play-Doh, where we can stretch and warp X into a space that looks like Y.

In topology, an atlas on a manifold M is a collection of charts that together describe the structure of the manifold.

Definition 5. *An atlas for a topological space M is a collection of pairs*

$$\{(U_i, \phi_i)\}_{i \in I},$$

where:

- *$\{U_i\}$ is an open cover of M, meaning that $M = \bigcup_{i \in I} U_i$ and each U_i is an open subset of M.*
- *Each $\phi_i : U_i \to V_i \subset \mathbb{R}^n$ is a **homeomorphism** from U_i onto an open subset V_i of $\mathbb{R}^n$.*

The maps ϕ_i are known as *coordinate charts*. An atlas thus provides a way of covering the manifold with coordinate systems that describe its local geometry. When the transition maps between overlapping charts ϕ_i and ϕ_j are differentiable (or satisfy other specified conditions), the atlas is said to be differentiable, giving rise to a *differentiable manifold* or *smooth manifold*.

2.2.1 Open Mapping Condition

3 Box Covering Dimension

3.1 Looking at the Coast of Britain and Fractals

4 Hausdorff Covering Dimension

4.1 Infinite "Area"; Zero Area