

In-Place Parallel-Partition Algorithms using Exclusive-Read-and-Write Memory

William Kuszmaul*

Massachusetts Institute of Technology
kuszmaul@mit.edu

Alek Westover†

alek.westover@gmail.com

Abstract

The parallel-partition problem, which is the first step of Quicksort and appears in many other algorithms, is given an array A of length n , and must partition the array based on some pivot property. We present two algorithms for the parallel-partition problem. Both algorithms are fully in place, have linear work and polylogarithmic span, and use only exclusive read/write shared-memory variables (i.e., are EREW). This means that they can be implemented using only parallel for loops, and do not require the use of any concurrency mechanisms such as locks or atomic variables. Despite the parallel-partition problem being heavily studied, no algorithms have achieved these theoretical guarantees before.

Our first algorithm is an in-place EREW algorithm with span $O(\log n \cdot \log \log n)$, which is within a factor of $O(\log \log n)$ of optimal. Although the algorithm achieves practical performance improvements over the standard out-of-place parallel partition due to improved cache behavior, the algorithm remains bottlenecked by memory bandwidth in practice.

Our second algorithm is an in-place EREW algorithm with polylogarithmic span and *provably optimal cache behavior*, up to small-order factors. The resulting algorithm achieves near-ideal scaling in practice by avoiding the memory-bandwidth bottleneck. The algorithm’s performance is comparable to that the Blocked Strided Algorithm of Francis, Pannan, Frias, and Petit, which is the previous state-of-the-art for parallel EREW sorting algorithms, but which lacks theoretical guarantees on its span and cache behavior.

1 Introduction

A *parallel partition* operation rearranges the elements in an array so that the elements satisfying a particular *pivot property* appear first. In addition to playing a central role in parallel quicksort, the parallel partition operation is used as a primitive throughout parallel algorithms.¹

A parallel algorithm can be measured by its *work*, the time needed to execute in serial, and its *span*, the time to execute on infinitely many processors. There is a well-known algorithm for parallel partition on arrays of size n with work $O(n)$ and span $O(\log n)$ [1, 6]. Moreover, the algorithm uses only exclusive read/write shared memory variables (i.e., it is an *EREW* algorithm). This eliminates the need for concurrency

mechanisms such as locks and atomic variables, and ensures good behavior even if the time to access a location is a function of the number of threads trying to access it (or its cache line) concurrently. EREW algorithms also have the advantage that their behavior is internally deterministic, meaning that the behavior of the algorithm will not differ from run to run, which makes test coverage, debugging, and reasoning about performance substantially easier [7].

The parallel-partition algorithm suffers from using a large amount of auxiliary memory, however. Whereas the serial algorithm is typically implemented in place, the parallel algorithm relies on the use of two auxiliary arrays of size n . To the best of our knowledge, the only known linear-work and $\text{polylog}(n)$ -span algorithms for parallel partition that are in-place require the use of atomic operations (e.g. fetch-and-add) [5, 20, 28].

An algorithm’s memory efficiency can be critical on large inputs. The memory consumption of an algorithm determines the largest problem size that can be executed in memory. Many external memory algorithms (i.e., algorithms for problems too large to fit in memory) perform large subproblems in memory; the size of these subproblems is again bottlenecked by the algorithm’s memory-overhead [29]. In multi-user systems, processes with larger memory-footprints can hog the cache and the memory bandwidth, slowing down other processes.

For sorting algorithms, in particular, special attention to memory efficiency is often given. This is because (a) a user calling the sort function may already be using almost all of the memory in the system; and (b) sorting algorithms, and especially parallel sorting algorithms, are often bottlenecked by memory bandwidth. The latter property, in particular, means that any parallel sorting algorithm that wishes to achieve state-of-the-art performance on a large multi-processor machine must be (at least close to) in place.

Currently the only practical in-place parallel sorting algorithms either rely heavily on atomic operations or other concurrency mechanisms [5, 20, 28], or eschew theoretical guarantees [13]. Parallel merge sort [17] was made in-place by Katajainen [21], but has proven too sophisticated for practical applications. Bitonic sort [8] is naturally in-place, and can be practical in certain applications on super computers, but suffers in general from requiring work $\Theta(n \log^2 n)$ rather than $O(n \log n)$. Parallel quicksort, on the other hand, despite the many efforts to optimize it [5, 13, 14, 20, 28], has eluded any

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¹In several well-known textbooks and surveys on parallel algorithms [1, 6], for example, parallel partitions are implicitly used extensively to perform what are referred to as *filter* operations.

in-place EREW (or CREW) algorithms due to its reliance on parallel partition.²

Results. We show that parallel partition can be implemented in place, and without the use of concurrency mechanisms. All of the algorithms considered in this paper use only exclusive read/write shared variables, and can be implemented using parallel-for-loops without any additional concurrency considerations.

Our first result is a set of techniques that allows us to adapt the standard parallel partition algorithm to be fully in place. The new algorithm has work $O(n)$ and span $O(\log n \cdot \log \log n)$. As an immediate consequence, we also get an in-place quick-sort algorithm with work $O(n \log n)$ and span $O(\log^2 n \log \log n)$.

Using our algorithmic techniques, we implement a space-efficient parallel partition. Because the in-place algorithm eliminates the use of large auxiliary arrays, the algorithm is able to achieve a reduction in cache misses over its out-of-place counterpart, resulting in performance improvements over the standard parallel partition algorithm.

The in-place algorithm remains bottlenecked by memory bandwidth, however, due to the fact that multiple passes over the input array are required. The memory-bandwidth bottleneck has led past researchers [13, 14] to introduce the so-called **Strided Algorithm**, which has near optimal cache behavior in practice, but which exhibits theoretical guarantees only on certain random input arrays.

The main result of this paper is an algorithm that we call the **Smoothed Striding Algorithm**, which achieves theoretical guarantees not only on span, work, and memory usage, but also on cache behavior. By randomly perturbing the internal structure of the Strided Algorithm, and adding a recursion step that was previously not possible, we arrive at a new algorithm with provably good span and cache behavior. The Smoothed Striding Algorithm is in-place, has polylogarithmic span, and exhibits provably optimal cache behavior up to small-order factors.

Using the Smoothed Striding Algorithm, we implement and optimize a fully in-place parallel partition. The implementation of the Smoothed Striding Algorithm performs within 15% of the Strided Algorithm on a large number of threads, while achieving theoretical guarantees that were previously unattainable.

2 Preliminaries

We begin by describing the the parallelism and memory model used in the paper, and by presenting background on parallel partition.

Workflow Model. We consider a simple language-based model of parallelism in which algorithms achieve parallelism through the use of **parallel-for-loops** (see, e.g., [1, 6, 12]); function calls within the inner loop then allow for more complicated parallel structures (e.g., recursion). Our algorithms

can also be implemented in the less restrictive PRAM model [1, 6].

Formally, a parallel-for-loop is given a range $R \in \mathbb{N}$, a constant number of arguments $\arg_1, \arg_2, \dots, \arg_c$, and a body of code. For each $i \in \{1, \dots, R\}$, the loop launches a thread that is given loop-counter i and local copies of the arguments $\arg_1, \arg_2, \dots, \arg_c$. The threads are then taken up by processors and the iterations of the loop are performed in parallel. Only after every iteration of the loop is complete can control flow continue past the loop.

A parallel algorithm may be run on an arbitrary number p of processors. The algorithm itself is oblivious to p , however, leaving the assignment of threads to processors up to a scheduler.

The **work** T_1 of an algorithm is the time that the algorithm would require to execute on a single processor. The **span** T_∞ of an algorithm is the time to execute on infinitely many processors. The scheduler is assumed to contribute no overhead to the span. In particular, if each iteration of a parallel-for-loop has span s , then the full parallel loop has span $s + O(1)$ [1, 6].

The work T_1 and span T_∞ can be used to quantify the time T_p that an algorithm requires to execute on p processors using a greedy online scheduler. If the scheduler is assumed to contribute no overhead, then Brent's Theorem [11] states that for any p ,

$$T_1/p \leq T_p \leq T_1/p + T_\infty.$$

The work-stealing algorithms used in the Cilk extension of C/C++ realize the guarantee offered by Brent's Theorem within a constant factor [9, 10], with the added caveat that parallel-for-loops typically induce an additional additive overhead of $O(\log R)$.

Memory Model. Memory is **exclusive-read** and **exclusive-write**. That is, no two threads are ever permitted to attempt to read or write to the same variable concurrently. The exclusive-read exclusive-write memory model is sometime referred to as the **EREW model** (see, e.g., [17]).

Note that threads are not in lockstep (i.e., they may progress at arbitrary different speeds), and thus the EREW model requires algorithms to be data-race free in order to avoid the possibility of non-exclusive data accesses.

In an **in-place** algorithm, each thread is given $O(\text{polylog} n)$ memory upon creation that is deallocated when the thread dies. This memory can be shared with the thread's children. However, the depth of the parent-child tree is not permitted to exceed $O(\text{polylog} n)$.

Whereas the EREW memory model prohibits concurrent accesses to memory, on the other side of the spectrum are CRCW (concurrent-read-concurrent-write) models, which allow for both reads and writes to be performed concurrently (and in some variants even allow for atomic operations) [1, 6, 24]. One approach to designing efficient EREW algorithms is to simulate efficient CRCW algorithms in the EREW model [24]. The

²In a CRCW **CREW** algorithm, reads may be concurrent, but writes may not. CREW stands for *concurrent-read exclusive-write*.

known simulation techniques require substantial space overhead, however, preventing the design of in-place algorithms [24].³

The Parallel Partition Problem. The parallel partition problem takes an input array A of size n , and a **decider function** dec that determines for each element $A[i] \in A$ whether or not $A[i]$ is a **predecessor** or a **successor**. That is, $\text{dec}(A[i]) = 1$ if $A[i]$ is a predecessor, and $\text{dec}(A[i]) = 0$ if $A[i]$ is a successor. The behavior of the parallel partition is to reorder the elements in the array A so that the predecessors appear before the successors.

The (Standard) Linear-Space Parallel Partition. The linear-space implementation of parallel partition consists of two phases [1, 6]:

The Parallel-Prefix Phase: In this phase, the algorithm constructs an array B whose i -th element $B[i] = \sum_{j=1}^i \text{dec}(A[j])$ is the number of predecessors in the first i elements of A . The transformation from A to B is called a **parallel prefix sum** and can be performed with $O(n)$ work and $O(\log n)$ span using a simple recursive algorithm: (1) First construct an array A' of size $n/2$ with $A'[i] = A[2i-1] + A[2i]$; (2) Recursively construct a parallel prefix sum B' of A' ; (3) Build B by setting each $B[i] = B'[\lfloor i/2 \rfloor] + A[i]$ for odd i and $B[i] = A'[i/2]$ for even i .

The Reordering Phase: In this phase, the algorithm constructs an output-array C by placing each predecessor $A[i] \in A$ in position $B[i]$ of C . If there are t predecessors in A , then the first t elements of C will now contain those t predecessors in the same order that they appear in A . The algorithm then places each successor $A[i] \in A$ in position $t + i - B[i]$. Since $i - B[i]$ is the number of successors in the first i elements of A , this places the successors in C in the same order that they appear in A . Finally, the algorithm copies C into A , completing the parallel partition.

Both phases can be implemented with $O(n)$ work and $O(\log n)$ span. Like its serial out-of-place counterpart, the algorithm is stable but not in place. The algorithm uses two auxiliary arrays of size n . Kiu, Knowles, and Davis [23] were able to reduce the extra space consumption to $n + p$ under the assumption that the number of processors p is hard-coded; their algorithm breaks the array A into p parts and assigns one part to each thread. Reducing the space below $o(n)$ has remained open until now, even when the number of threads is fixed.

3 An In-Place Algorithm with Span $O(\log n \log \log n)$

In this section, we present an in-place algorithm for parallel partition with span $O(\log n \log \log n)$. Each thread in the algorithm requires memory at most $O(\log n)$.

We assume without loss of generality that the total number of successors in A exceeds the number of predecessors, since otherwise their roles can simply be swapped in the algorithm.

Further, we assume for simplicity that the elements of A are distinct; this assumption is removed at the end of the section.

Algorithm Outline. We begin by presenting an overview of the key algorithmic ideas needed to construct an in-place algorithm.

Consider how to remove the auxiliary array C from the Reordering Phase. If one attempts to simply swap in parallel each predecessor $A[i]$ with the element in position $j = B[i]$ of A , then the swaps will almost certainly conflict. Indeed, $A[j]$ may also be a predecessor that needs to be swapped with $A[B[j]]$. Continuing like this, there may be an arbitrarily long list of dependencies on the swaps.

To combat this, we begin the algorithm with a Preprocessing Phase in which A is rearranged so that every prefix is **successor-heavy**, meaning that for all t , the first t elements contain at least $\frac{t}{4}$ successors. Then we compute the prefix-sum array B , and begin the Reordering Phase. Using the fact that the prefixes of A are successor-heavy, the reordering can now be performed in place as follows: (1) We begin by recursively reordering the prefix P of A consisting of the first $4/5 \cdot n$ elements, so that the predecessors appear before the successors; (2) Then we simply swap each predecessor $A[i]$ in the final $1/5 \cdot n$ elements with the corresponding element $B[A[i]]$. The fact that the prefix P is successor-heavy ensures that the final $\frac{1}{5} \cdot n$ elements of the reordered P will consist of successors. This implies that for each of the swaps between predecessors $A[i]$ in the final $1/5 \cdots n$ elements and earlier positions $B[A[i]]$, the latter element will be a successor. In other words, the swaps are now conflict free.

Next consider how to remove the array B from the Parallel-Prefix Phase. At face value, this would seem quite difficult since the reordering phase relies heavily on B . Our solution is to *implicitly* store the value of every $O(\log n)$ -th element of B in the ordering of the elements of A . That is, we break A into blocks of size $O(\log n)$, and use the order of the elements in each block to encode an entry of B . (If the elements are not all distinct, then a slightly more sophisticated encoding is necessary.) Moreover, we modify the algorithm for building B to only construct every $O(\log n)$ -th element. The new parallel-prefix sum performs $O(n/\log n)$ arithmetic operations on values that are implicitly encoded in blocks; since each such operation requires $O(\log n)$ work, the total work remains linear.

In the remainder of the section, we present the algorithm in detail. It proceeds in three phases.

A Preprocessing Phase. The goal of the Preprocessing phase is to make every prefix of A successor-heavy. To perform the Preprocessing phase on A , we begin with a parallel-for-loop: For each $i = 1, \dots, \lfloor n/2 \rfloor$, if $A[i]$ is a predecessor and $A[n-i+1]$ is a successor, then we swap their positions in A . To complete the Preprocessing phase on A , we then recursively perform a Preprocessing phase on $A[1], \dots, A[\lfloor n/2 \rfloor]$.

Lemma 3.1. The Preprocessing Phase has work $O(n)$ and span $O(\log n)$. At the end of the Preprocessing Phase, every prefix of A is successor-heavy.

³The known simulation techniques also increase the total work in the original algorithm, although this can be acceptable if only a small number of atomic operations need to be simulated.

Proof. Recall that for each $t \in 1, \dots, n$, we call the t -prefix $A[1], \dots, A[t]$ of A successor-heavy if it contains at least $\frac{t}{4}$ successors.

The first parallel-for-loop ensures that at least half the successors in A reside in the first $\lceil n/2 \rceil$ positions, since for $i = 1, \dots, \lceil n/2 \rceil$, $A[n-i+1]$ will only be a successor if $A[i]$ is also a successor. Because at least half the elements in A are successors, it follows that the first $\lceil n/2 \rceil$ positions contain at least $\lceil n/4 \rceil$ successors, making every t -prefix with $t \geq \lceil n/2 \rceil$ successor-heavy.

After the parallel-for-loop, the first $\lceil n/2 \rceil$ positions of A contain at least as many successors as predecessors (since $\lceil n/4 \rceil \geq \frac{\lceil n/2 \rceil}{2}$). Thus we can recursively apply the argument above in order to conclude that the recursion on $A[1], \dots, A[\lceil n/2 \rceil]$ makes every t -prefix with $t \leq \lceil n/2 \rceil$ successor-heavy. It follows that, after the recursion, every t -prefix of A is successor-heavy.

Each recursive level has constant span and performs work proportional to the size of the subarray being considered. The Preprocessing phase therefore has total work $O(n)$ and span $O(\log n)$. \square

An Implicit Parallel Prefix Sum. Pick a **block-size** $b \in \Theta(\log n)$ satisfying $b \geq 2\lceil \log(n+1) \rceil$. Consider A as a series of $\lfloor n/b \rfloor$ blocks of size b , with the final block of size between b and $2b-1$. Denote the blocks by $X_1, \dots, X_{\lfloor n/b \rfloor}$.

Within each block X_i , we can implicitly store a value in the range $0, \dots, n$ through the ordering of the elements:

Lemma 3.2. Given an array X of $2\lceil \log(n+1) \rceil$ distinct elements, and a value $v \in \{0, \dots, n\}$, one can rearrange the elements of X to encode the bits of v using work $O(\log n)$ and span $O(\log \log n)$; and one can then later decode v from X using work $O(\log n)$ and span $O(\log \log n)$.

Proof. Observe that X can be broken into (at least) $\lceil \log(n+1) \rceil$ disjoint pairs of adjacent elements $(x_1, x_2), (x_3, x_4), \dots$, and by rearranging the order in which a given pair (x_j, x_{j+1}) occurs, the lexicographic comparison of whether $x_j < x_{j+1}$ can be used to encode one bit of information. Values $v \in [0, n]$ can therefore be read and written to X with work $O(b) = O(\log n)$ and span $O(\log b) = O(\log \log n)$ using a simple divide-and-conquer recursive approach to encode and decode the bits of v . \square

To perform the Parallel Prefix Sum phase, our algorithm begins by performing a parallel-for loop through the blocks, and storing in each block X_i a value v_i equal to the number of predecessors in the block. (This can be done in place with work $O(n)$ and span $O(\log \log n)$ by Lemma 3.2.)

The algorithm then performs an in-place parallel-prefix operation on the values $v_1, \dots, v_{\lfloor n/b \rfloor}$ stored in the blocks. This is done by first resetting each even-indexed value v_{2i} to $v_{2i} + v_{2i-1}$; then recursively performing a parallel-prefix sum on the even-indexed values; and then replacing each odd-indexed v_{2i+1} with $v_{2i+1} + v_{2i}$, where v_0 is defined to be zero.

Lemma 3.3 analyzes the phase:

Lemma 3.3. The Parallel Prefix Sum phase uses work $O(n)$ and span $O(\log n \log \log n)$. At the end of the phase, each X_i encodes a value v_i counting the number of predecessors in the prefix $X_1 \circ X_2 \circ \dots \circ X_i$; and each prefix $X_1 \circ X_2 \circ \dots \circ X_i$ is successor-heavy.

Proof. If the v_i 's could be read and written in constant time, then the prefix sum would take work $O(n/\log n)$ and span $O(\log n)$, since there are $O(n/\log n)$ v_i 's. Because each v_i actually requires work $O(\log n)$ and span $O(\log \log n)$ to read/write (by Lemma 3.2), the prefix sum takes work $O(n)$ and span $O(\log n \cdot \log \log n)$.

Once of the prefix-sum that has been performed, every block X_i encodes a value v_i counting the number of predecessors in the prefix $X_1 \circ X_2 \circ \dots \circ X_i$. Moreover, because the Parallel Prefix Sum phase only rearranges elements within each X_i , Lemma 3.1 ensures that each prefix of the form $X_1 \circ X_2 \circ \dots \circ X_i$ remains successor-heavy. \square

In-Place Reordering. In the final phase of the algorithm, we reorder A so that the predecessors appear before the successors. Let $P = X_1 \circ X_2 \circ \dots \circ X_t$ be the smallest prefix of blocks that contains at least $4/5$ of the elements in A . We begin by recursively reordering the elements in P so that the predecessors appear before the successors; as a base case, when $|P| \leq 5b = O(\log n)$, we simply perform the reordering in serial.

To complete the reordering of A , we perform a parallel-for-loop through each of the blocks $X_{t+1}, \dots, X_{\lfloor n/b \rfloor}$. For each block X_i , we first extract v_i (with work $O(\log n)$ and span $O(\log \log n)$ using Lemma 3.2). We then create an auxiliary array Y_i of size $|X_i|$, using $O(\log n)$ thread-local memory. Using a parallel-prefix sum (with work $O(\log n)$ and span $O(\log \log n)$), we set each $Y_i[j]$ equal to v_i plus the number of predecessors in $X_i[1], \dots, X_i[j]$. In other words, $Y_i[j]$ equals the number of predecessors in A appearing at or before $X_i[j]$.

After creating Y_i , we then perform a parallel-for-loop through the elements $X_i[j]$ of X_i (note we are still within another parallel loop through the X_i 's), and for each predecessor $X_i[j]$, we swap it with the element in position $Y_i[j]$ of the array A . This completes the algorithm.

Lemma 3.4. The Reordering phase takes work $O(n)$ and span $O(\log n \log \log n)$. At the end of the phase, the array A is fully partitioned.

Proof. After P has been recursively partitioned, it will be of the form $P_1 \circ P_2$ where P_1 contains only predecessors and P_2 contains only successors. Because P is successor-heavy (by Lemma 3.3), we have that $|P_2| \geq |P|/4$, and thus that $|P_2| \geq |X_{t+1} \circ \dots \circ X_{\lfloor n/b \rfloor}|$.

After the recursion, the swaps performed by the algorithm will swap the i -th predecessor in $X_{t+1} \circ \dots \circ X_{\lfloor n/b \rfloor}$ with the i -th element in P_2 , for i from 1 to the number of predecessors in $X_{t+1} \circ \dots \circ X_{\lfloor n/b \rfloor}$. Because $|P_2| \geq |X_{t+1} \circ \dots \circ X_{\lfloor n/b \rfloor}|$ these swaps are guaranteed not to conflict with one-another; and since P_2 consists of successors, the final state of array A will be fully partitioned.

The total work in the reordering phase is $O(n)$ since each X_i appears in a parallel-for-loop at exactly one level of the recursion, and incurs $O(\log n)$ work. The total span of the reordering phase is $O(\log n \cdot \log \log n)$, since there are $O(\log n)$ levels of recursion, and within each level of recursion each X_i in the parallel-for-loop incurs span $O(\log \log n)$. \square

Combining the phases, the full algorithm has work $O(n)$ and span $O(\log \log n)$. Thus we have:

Theorem 3.1. There exists an in-place algorithm using exclusive-read-write variables that performs parallel-partition with work $O(n)$ and span $O(\log n \cdot \log \log n)$.

Allowing for Repeated Elements. In proving Theorem 3.1 we assumed for simplicity that the elements of A are distinct. To remove this assumption, we conclude the section by proving a slightly more complex variant of Lemma 3.2, eliminating the requirement that the elements of the array X be distinct:

Lemma 3.5. Let X be an array of $b = 4\lceil \log(n+1) \rceil + 2$ elements. There is an *encode* function, and a *decode* function such that:

- The encode function modifies the array X (possibly overwriting elements in addition to rearranging them) to store a value $v \in \{0, \dots, n\}$. The first time the encode function is called on X it has work and span $O(\log n)$. Any later times the encode function is called on X , it has work $O(\log n)$ and span $O(\log \log n)$. In addition to being given an argument v , the encode function is given a boolean argument indicating whether the function has been invoked on X before.
- The decode function recovers the value of v from the modified array X , and restores X to again be an array consisting of the same multiset of elements that it began with. The decode function has work $O(\log n)$ and span $O(\log \log n)$.

Proof. Consider the first b letters of X as a sequence of pairs, given by $(x_1, x_2), \dots, (x_{b-1}, x_b)$. If at least half of the pairs (x_i, x_{i+1}) satisfy $x_i \neq x_{i+1}$, then the encode function can reorder those pairs to appear at the front of X , and then use them to encode v as in Lemma 3.2. Note that the reordering of the pairs will only be performed the first time that the encode function is invoked on X . Later calls to the encode function will have work $O(\log n)$ and span $O(\log \log n)$, as in Lemma 3.2.

If, on the other hand, at least half the pairs consist of equal-value elements $x_i = x_{i+1}$, then we can reorder the pairs so that the first $\lceil \log(n+1) \rceil + 1$ of them satisfy this property. (This is only done on the first call to encode.) To encode a value v , we simply explicitly overwrite the second element in each of the pairs $(x_3, x_4), (x_5, x_6), \dots$ with the bits of v , overwriting each element with one bit. The reordering performed by the first call to encode has work and span $O(\log n)$; the writing of v 's bits can then be performed in work $O(\log n)$ and span $O(\log \log n)$ using a simple divide-and-conquer approach.

To perform a decode and read the value v , we check whether $x_1 = x_2$ in order to determine which type of encoding is being

used, and then we can unencode the bits of v using work $O(\log n)$ and span $O(\log \log n)$; if the encoding is the second type (i.e., $x_1 = x_2$), then the decode function also restores the elements x_2, x_4, x_6, \dots of the array X as it extracts the bits of v . Note that checking whether $x_1 = x_2$ is also used by the encode function each time after the first time it is called, in order to determine which type of encoding is being used. \square

The fact that the first call to the encode function on each X_i has span $O(\log n)$ (rather than $O(\log \log n)$) does not affect the total span of our parallel-partition algorithm, since this simply adds a step with $O(\log n)$ -span to the beginning of the Parallel Prefix phase. Lemma 3.5 can therefore be used in place of Lemma 3.2 in order to complete the proof of Theorem 3.1 for arrays A that contain duplicate elements.

4 A Cache Efficient In-Place Parallel Partition

In Section 5, we will see that, although the techniques introduced in Section 3 achieve speedups over the classic parallel-prefix-based partition algorithm, they nonetheless continue to be bottlenecked by cache misses. In this section, we introduce a second algorithm, called the **Smoothed Striding Algorithm**, which exhibits provably optimal cache behavior (up to small-order factors). The Smoothed Striding Algorithm is fully in-place and has polylogarithmic span.

Modeling Cache Misses. We treat memory as consisting of fixed-size cache lines of some size b . Each processor is assumed to have a small cache of $\text{polylog } n$ cache lines. A cache miss occurs on a processor when the line being accessed is not currently in cache, in which case some other line is evicted from cache to make room for the new entry. Each cache is managed with a LRU (Least Recently Used) eviction policy; when child threads are created, they inherit their cache contents from their parent.

We will also assume that the algorithm can choose for certain small arrays to be pinned in cache (i.e. their entries are never evicted from cache). This assumption is without loss of generality in the sense that LRU eviction is competitive (up to resource augmentation) with the optimal off-line eviction strategy OPT (i.e. Furthest in the Future). Formally this is due to the following theorem by Sleator and Tarjan:

Theorem 4.1 (Resource Augmentation Theorem [27]). LRU operating on a cache of size $K \cdot M$ for some $K > 1$ will incur at most $1 + \frac{1}{K-1}$ times the number of times cache misses of OPT operating on a cache of size M , for the same series of memory accesses.

Recall that each processor has a cache of size $\log^c n$ for c a constant of our choice. Up to changes in c LRU incurs no more than a $1 + \frac{1}{\text{polylog } n}$ factor more cache misses than OPT incurs. Thus, up to a $1 + \frac{1}{\text{polylog}(n)}$ multiplicative change in cache misses, and a $\text{polylog}(n)$ change in cache size, we may assume without loss of generality that cache eviction is performed by OPT. Such an assumption will not be necessary for our algorithm analyses, however; instead it will suffice to assume

that certain small arrays are pinned in cache and that other evictions are performed via LRU.

The Strided Algorithm [13]. The Smoothed Striding Algorithm borrows several structural ideas from a previous algorithm of Francis and Pannan [13], which we call the Strided Algorithm. The Strided Algorithm is designed to behave well on random arrays A , achieving span $\tilde{O}(n^{2/3})$ and exhibiting only $n/b + \tilde{O}(n^{2/3}/b)$ cache misses on such inputs. On worst-case inputs, however, the Strided Algorithm has span $\Omega(n)$ and incurs $n/b + \Omega(n/b)$ cache misses. Our algorithm, the Smoothed Striding Algorithm, will build on the Strided Algorithm by randomly perturbing the internal structure of the original algorithm; in doing so, we are able to provide provable guarantees on arbitrary inputs, and to add a recursion step that was previously impossible.

The original **Strided Algorithm** consists of two steps:

- **The Partial Partition Step.** Let $g \in \mathbb{N}$ be a parameter, and assume for simplicity that $gb \mid n$. Partition the array A into $\frac{n}{gb}$ chunks $C_1, \dots, C_{n/gb}$, each consisting of g cache lines of size b . For $i \in \{1, 2, \dots, g\}$, define P_i to consist of the i -th cache line from each of the chunks $C_1, \dots, C_{n/gb}$. One can think of the P_i 's as forming a strided partition of array A , since consecutive cache lines in P_i are always separated by a fixed stride of $g-1$ other cache lines. The first step of the algorithm is to perform an in-place serial partition on each of the P_i s, rearranging the elements within the P_i so that the predecessors come first. This step requires work $\Theta(n)$ and span $\Theta(n/g)$.
- **The Serial Cleanup Step.** For each P_i , define the *splitting position* v_i to be the position in A of the final predecessor in (the already partitioned) P_i . Define $v_{\min} = \min\{v_1, \dots, v_g\}$ and define $v_{\max} = \max\{v_1, \dots, v_g\}$. Then the second step of the algorithm is to perform a serial partition on the sub-array $A[v_{\min}], \dots, A[v_{\max} - 1]$. This completes the full partition.

Note that the Cleanup Step of the Strided Algorithm has no parallelism, and thus has span $\Theta(v_{\max} - v_{\min})$. In general, this results in an algorithm with linear-span (i.e., no parallelism guarantee). When the number of predecessors in each of the P_i 's is close to equal, however, the quantity $v_{\max} - v_{\min}$ can be much smaller than $O(n)$. For example, if $b = 1$, and if each element of A is selected independently from some distribution, then one can use Chernoff bounds to prove that with high probability in n , $v_{\max} - v_{\min} \leq O(\sqrt{n \cdot g \cdot \log n})$. The full span of the algorithm is then $\tilde{O}(n/g + \sqrt{n \cdot g})$, which optimizes at $g = n^{1/3}$ to $\tilde{O}(n^{2/3})$. Since the Partial Partition Step incurs only n/b cache misses, the full algorithm incurs $n + \tilde{O}(n^{2/3})$ cache misses on a random array A .

Using Hoeffding's Inequality in place of Chernoff bounds, one can obtain analogous bounds for larger values of b ; in particular for $b \in \text{polylog}(n)$, the optimal span remains $\tilde{O}(n^{2/3})$

and the number of cache misses becomes $n/b + \tilde{O}(n^{2/3}/b)$ on an array A consisting of randomly sampled elements.⁴

The Smoothed Striding Algorithm. To obtain an algorithm with provable guarantees for all inputs A , we randomly perturb the internal structure of each of the P_i 's. Define U_1, \dots, U_g (which play a role analogous to P_1, \dots, P_g in the Strided Algorithm) so that each U_i contains one randomly selected cache line from each of $C_1, \dots, C_{n/gb}$ (rather than containing the i -th cache line of each C_j). This ensures that the number of predecessors in each U_i is a sum of independent random variables with values in $\{0, 1, \dots, b\}$.

By Hoeffding's Inequality, with high probability in n , the number of predecessors in each U_i is tightly concentrated around $\frac{\mu n}{g}$, where μ is the fraction of elements in A that are predecessors. It follows that, if we perform in-place partitions of each U_i in parallel, and then define v_i to be the position in A of the final predecessor in (the already partitioned) U_i , then the difference between $v_{\min} = \min_i v_i$ and $v_{\max} = \max_i v_i$ will be small (even if the input array A is worst-case!).

Rather than partitioning $A[v_{\min}], \dots, A[v_{\max} - 1]$ in serial, the Smoothed Striding Algorithm simply recurses on the subarray. Such a recursion would not have been productive for the original Strided Algorithm because the strided partition P'_1, \dots, P'_g used in the recursive subproblem would satisfy $P'_1 \subseteq P_1, \dots, P'_g \subseteq P_g$ and thus each P'_i is already partitioned. That is, in the original Strided Algorithm, the problem that we would recurse on is a worst-case input for the algorithm in the sense that the partial partition step makes no progress.

The main challenge in designing the Smoothed Striding Algorithm becomes the construction of U_1, \dots, U_g without violating the in-place nature of the algorithm. A natural approach might be to store for each U_i and each C_j the index of the cache line in C_j that U_i contains. This would require the storage of $\Theta(n/b)$ numbers as metadata, however, preventing the algorithm from being in-place. To save space, the key insight is to select a random offset $X_j \in \{1, 2, \dots, g\}$ within each C_j , and then to assign the $(X_j + i \pmod{g}) + 1$ -th cache line of C_j to U_i for $i \in \{1, 2, \dots, g\}$. This allows for us to construct the U_i 's using only $O\left(\frac{n}{gb}\right)$ machine words storing the metadata $X_1, \dots, X_{n/gb}$. By setting g to be relatively large, so that $\frac{n}{gb} \leq \text{polylog}(n)$, we can obtain an in-place algorithm that incurs $n(1 + o(1))$ cache misses.

The recursive structure of the Smoothed Striding Algorithm allows for the algorithm to achieve polylogarithmic span. As an alternative to recursing, one can also use the in-place algorithm from Section 3 in order to partition $A[v_{\min}], \dots, A[v_{\max} - 1]$. This results in an improved span (since the algorithm from Section 3 has span only $O(\log \log \log n)$), while still incurring only $n(1 + o(1))$ cache misses (since the cache-inefficient algorithm from Section 3 is only used on a small subarray of A). We analyze both the recursive version

⁴The original algorithm of Francis and Pannan [13] does not consider the cache-line size b . Frías and Petit later introduced the parameter b [14], and showed that by setting b appropriately, one obtains an algorithm whose empirical performance is close to the state-of-the-art.

of the Smoothed Striding Algorithm, and the version which uses as a final step the algorithm from Section 3; one significant advantage of the recursive version is that it is simple to implement in practice.

Formal Algorithm Description. Let $b < n$ be the size of a cache line, let A be an input array of size n , and let g be a parameter. (One should think of g as being relatively large, satisfying $\frac{n}{bg} \leq \text{polylog}(n)$.) We assume for simplicity that that n is divisible by gb , and we define $s = \frac{n}{gb}$.⁵

The **Partial Partition Step** if the algorithm partitions the cache lines of A into g sets U_1, \dots, U_g of size $s = \frac{n}{gb}$ and then performs a serial partition on each of those sets U_i in parallel. To determine the sets U_1, \dots, U_g , the algorithm uses as metadata, an array $X = X[1], \dots, X[s]$, where each $X[i] \in \{1, \dots, g\}$.

Formally, the Partial Partition Step performs the following procedure:

- Set each of $X[1], \dots, X[s]$ to be uniformly random and independently selected elements of $\{1, 2, \dots, g\}$. For $i \in \{1, 2, \dots, g\}$, and for each $j \in \{1, 2, \dots, s\}$, define

$$G_i(j) = (X[j] + i \pmod{g}) + (j-1)g + 1.$$

Using this terminology, we define each U_i for $i \in \{1, \dots, g\}$ to contain the $G_i(j)$ -th cache line of A for each $j \in \{1, 2, \dots, s\}$. That is, $G_i(j)$ denotes the index of the j -th cache line from array A to be contained in U_i .

Note that, to compute the index of the j -th cache line in U_i , one needs only the value of $X[j]$. Thus the only metadata needed by the algorithm to determine the U_1, \dots, U_g is the array X . If $|X| = s = \frac{n}{gb} \leq \text{polylog}(n)$, then the algorithm is in place.

- The algorithm performs an in-place (serial) partition on each U_i (and performs these partitions in parallel with one another). In doing so, the algorithm, also collects $v_{\min} = \min_i v_i$, $v_{\max} = \max_i v_i$, where each v_i with $i \in \{1, \dots, g\}$ is defined to be the index of the final predecessor in A (or 0 if no such predecessor exists).⁶ The array A is now partially partitioned, i.e. $A[i]$ is a predecessor for all $i \leq v_{\min}$, and $A[i]$ is a successor for all $i > v_{\max}$.

The second step of the Smoothed Striding Algorithm is to complete the partitioning of $A[v_{\min} + 1], \dots, A[v_{\max}]$. This can be done in one of two ways: The **Recursive Smoothed Striding Algorithm** partitions $A[v_{\min} + 1], \dots, A[v_{\max}]$ recursively using the same algorithm (and resorts to a serial base case when the subproblem is small enough that $g \leq$

⁵This assumption can be made without loss of generality by treating A as an array of size $n' = n + (gb - n \pmod{gb})$, and then treating the final $gb - n \pmod{gb}$ elements of the array as being successors (which consequently the algorithm needs not explicitly access).

⁶One can calculate v_{\min} and v_{\max} without explicitly storing each of v_1, \dots, v_g as follows. Rather than using a standard g -way parallel for-loop to partition each of U_1, \dots, U_g , one can manually implement the parallel for-loop using a recursive divide-and-conquer approach. Each recursive call in the divide-and-conquer can then simply collect the maximum and minimum v_i for the U_i 's that are partitioned within that recursive call. This adds $O(\log n)$ to the total span of the Partial Partition Step, which does not affect the overall span asymptotically.

$O(1)$); the **Hybrid Smoothed Striding Algorithm** partitions $A[v_{\min} + 1], \dots, A[v_{\max}]$ using the in-place algorithm given in Theorem 3.1 with span $O(\log n \log \log n)$. In general, the Hybrid algorithm yields better theoretical guarantees on span than the recursive version; on the other hand, the recursive version has the advantage that is simple to implement as fully in place, and still achieves polylogarithmic span. We analyze both algorithms in this section.

Algorithm Analysis. Our first proposition analyzes the Partial Partition Step.

Proposition 4.1. Let $\epsilon \in (0, 1/2)$ and $\delta \in (0, 1/2)$ such that $\epsilon \geq \frac{1}{\text{poly}(n)}$ and $\delta \geq \frac{1}{\text{polylog}(n)}$. Suppose $s > \frac{\ln(n/\epsilon)}{\delta^2}$. Finally, suppose that each processor has a cache of size at least $s + c$ for a sufficiently large constant c .

Then the Partial-Partition Algorithm achieves work $O(n)$; achieves span $O(b \cdot s)$; incurs $\frac{s+n}{b} + O(1)$ cache misses; and guarantees with probability $1 - \epsilon$ that

$$v_{\max} - v_{\min} < 4n\delta.$$

Proof. Since $\sum_i |U_i| = n$, and since the partitioning of each U_i takes time $O(|U_i|)$, the total work performed by the algorithm is $O(n)$.

Assuming that array X is pinned in cache (note, in particular, that $|X| = s \leq \text{polylog}(n)$, and so we are permitted to pin X in cache), algorithm's cache misses consist of: n/b misses from accessing each cache line of A ; s/b for instantiating the array X ; and $O(1)$ for other instantiating costs. This sums to

$$\frac{n+s}{b} + O(1).$$

Note, in particular, that when each cache line in A is accessed, that line continues to be among the $O(1)$ most recently accessed cache lines until the final time that it is accessed, and thus does not get evicted from cache.

The span of the algorithm is $O(n/g + s) = O(b \cdot s)$, since the each U_i is of size $O(n/g)$, and because the initialization of array X can be performed in time $O(|X|) = O(s)$.

It remains to show that with probability $1 - \epsilon$, $v_{\max} - v_{\min} < 4n\delta$. Let μ denote the fraction of elements in A that are predecessors. For $i \in \{1, 2, \dots, g\}$, let μ_i denote the fraction of elements in U_i that are predecessors. Note that each μ_i is the average of s independent random variables $Y_i(1), \dots, Y_i(s) \in [0, 1]$, where $Y_i(j)$ is the fraction of elements in the $G_i(j)$ -th cache line of A that are predecessors. By construction, $G_i(j)$ has the same probability distribution for all i , since $(X[j] + i) \pmod{g}$ is uniformly random in \mathbb{Z}_g for all i . It follows that $Y_i(j)$ has the same distribution for all i , and thus that $\mathbb{E}[\mu_i]$ is independent of i . Since the average of the μ_i s is μ , it follows that $\mathbb{E}[\mu_i] = \mu$ for all $i \in \{1, 2, \dots, g\}$.

Since each μ_i is the average of s independent $[0, 1]$ -random variables, we can apply Hoeffding's inequality (i.e. a Chernoff Bound for a random variable on $[0, 1]$ rather than on $\{0, 1\}$) to each μ_i to show that it is tightly concentrated around its expected value μ , i.e.,

$$\Pr[|\mu_i - \mu| \geq \delta] < 2\exp(-2s\delta^2).$$

Since $s > \frac{\ln(n/\epsilon)}{\delta^2} \geq \frac{\ln(2n/(b\epsilon))}{2\delta^2}$, we find that for all $i \in \{1, \dots, g\}$,

$$\Pr[|\mu_i - \mu| \geq \delta] < 2\exp\left(-2 \frac{\ln(2n/(b\epsilon))}{2\delta^2} \delta^2\right) = \frac{\epsilon}{n/b} < \frac{\epsilon}{g}.$$

By the union bound, it follows that with probability at least $1 - \epsilon$, all of μ_1, \dots, μ_g are within δ of μ .

To complete the proof we will show that the occurrence of the event that all y simultaneously satisfy $|\mu - \mu_y| < \delta$ implies that $v_{\max} - v_{\min} \leq 4n\delta$.

Recall that $G_i(j)$ denotes the index within A of the j th cache-line contained in U_i . By the definition of $G_i(j)$,

$$(j-1)g+1 \leq G_i(j) \leq jg.$$

Note that $A[v_i]$ will occur in the $\lceil s\mu_i \rceil$ -th cache-line of U_i because U_i is composed of s cache lines. Hence

$$(\lceil s\mu_i \rceil - 1)gb + 1 \leq v_i \leq \lceil s\mu_i \rceil gb,$$

which means that

$$s\mu_i gb - gb - 1 \leq v_i \leq s\mu_i gb + gb.$$

Since $sgb = n$, it follows that $|v_i - n\mu_i| \leq gb$. Therefore,

$$|v_i - n\mu| < gb + n\delta.$$

This implies that the maximum of $|v_i - v_j|$ for any i and j is at most, $2bg + 2n\delta$. Thus,

$$\begin{aligned} v_{\max} - v_{\min} &\leq 2n\left(\delta + \frac{n}{bg}\right) = 2n(\delta + s) \\ &\leq 2n\left(\delta + \frac{2\delta^2}{\ln(2n/(b\epsilon))}\right) < 4n \cdot \delta. \end{aligned}$$

□

We will use Proposition 4.1 as a tool to analyze the Recursive and the Hybrid Smoothed Striding Algorithms.

Rather than parameterizing the Partial Partition step in each algorithm by s , Proposition 4.1 suggests that it is more natural to parameterize by ϵ and δ , which then determine s .

We will assume that both the hybrid and the recursive algorithms use $\epsilon = 1/n^c$ for c of our choice (i.e. with high probability in n). Moreover, the Recursive Smoothed Striding Algorithm continues to use the same value of ϵ within recursive subproblems (i.e., the ϵ is chosen based on the size of the first subproblem in the recursion), that way the entire algorithm succeeds with high probability in n .

For both algorithms, the choice of δ results in a tradeoff between cache misses and span. For the Recursive algorithm, we allow for δ to be chosen arbitrarily at the top level of recursion, and then fix $\delta = \Theta(1)$ to be a sufficiently small constant at all levels of recursion after the first; this guarantees that we at least halve the size of the problem size between recursive iterations⁷. Optimizing δ further (after the first level of recursion) would only affect the number of undesired cache misses by a constant factor.

Next we analyze the Hybrid Smoothed Striding Algorithm.

⁷In general, setting $\delta = 1/8$ will result in the problem size being halved. However, this relies on the assumption that $gb \mid n$, which is only without loss of generality by allowing for the size of subproblems to be sometimes artificially increased by a small amount (i.e., a factor of $1 + gb/n = 1 + 1/s$). One can handle this issue by decreasing δ to, say, $1/16$.

Theorem 4.2. The Hybrid Smoothed Striding Algorithm algorithm using parameter $\delta \in (0, 1/2)$ satisfying $\delta \geq 1/\text{polylog}(n)$: has work $O(n)$; achieves span

$$O\left(\log n \log \log n + \frac{b \log n}{\delta^2}\right),$$

with high probability in n ; and incurs fewer than

$$(n + O(n\delta))/b$$

cache misses with high probability in n .

An interesting corollary of the above theorem concerns what happens when b is small (e.g., constant) and we choose δ to optimize span.

Corollary 4.2 (Corollary of Theorem 4.2). Suppose $b \leq o(\log \log n)$. Then the Cache-Efficient Full-Partition Algorithm algorithm using $\delta = \Theta(\sqrt{b/\log \log n})$, achieves work $O(n)$, and with high probability in n , achieves span $O(\log n \log \log n)$ and incurs fewer than $(n + o(n))/b$ cache misses.

Proof of Theorem 4.2. We analyze the Partial Partition Step using Proposition 4.1. Note that by our choice of ϵ , $s = O\left(\frac{\log n}{\delta^2}\right)$.

The Partial Partition Step therefore has work $O(n)$, span $O\left(\frac{b \log n}{\delta^2}\right)$, and incurs fewer than

$$\frac{n}{b} + O\left(\frac{\log n}{b\delta^2}\right) + O(1)$$

cache misses.

By Theorem 3.1, the subproblem of partitioning of $A[v_{\min} + 1], \dots, A[v_{\max}]$ takes work $O(n)$. With high probability in n , the subproblem has size less than $4n\delta$, which means that the subproblem achieves span

$$O(\log n \delta \log \log n \delta) = O(\log n \log \log n),$$

and incurs at most $O(n\delta/b)$ cache misses.

The total number of cache misses is therefore,

$$\frac{n}{b} + O\left(\frac{\log n}{b\delta^2} + \frac{n\delta}{b}\right) + O(1),$$

which since $\delta \geq 1/\text{polylog}(n)$, is at most $(n + O(n\delta))/b + O(1) \leq (n + O(n\delta))/b$, as desired. □

Proof of Corollary 4.2. We use $\delta = \sqrt{b/\log \log n}$ in the result proved in Theorem 4.2.

First note that the assumptions of Theorem 4.2 are satisfied because $O(\sqrt{b/\log \log n}) > 1/\text{polylog}(n)$. The algorithm achieves work $O(n)$. With high probability in n the algorithm achieves span

$$O\left(\log n \log \log n + \frac{b \log n}{\delta^2}\right) = O(\log n \log \log n).$$

With high probability in n the algorithm incurs fewer than

$$(n + O(n\delta))/b = (n + O(n\sqrt{b/\log \log n}))/b$$

cache misses. By assumption $\sqrt{b/\log \log n} = o(1)$, so this reduces to $(n + o(n))/b$ cache misses, as desired. □

The next theorem analyzes the span of the Recursive Smoothed Striding Algorithm.

Theorem 4.3. With high probability in n , the Recursive Smoothed Striding algorithm using parameter $\delta \in (0, 1/2)$ satisfying $\delta \geq 1/\text{polylog}(n)$: achieves work $O(n)$, attains span

$$O\left(b\left(\log^2 n + \frac{\log n}{\delta^2}\right)\right),$$

and incurs $(n + O(n\delta))/b$ cache misses.

A particularly natural parameter setting for the Recursive algorithm occurs at $\delta = 1/\sqrt{\log n}$.

Corollary 4.3 (Corollary of Theorem 4.3). With high probability in n , the Recursive Smoothed Striding Algorithm using parameter $\delta = 1/\sqrt{\log n}$: achieves work $O(n)$, attains span $O(b\log^2 n)$, and incurs $n/b \cdot (1 + O(1/\sqrt{\log n}))$ cache misses.

Proof of Theorem 4.3. To avoid confusion, we use δ' , rather than δ , to denote the constant value of δ used at levels of recursion after the first.

By Theorem 4.1, the top level of the algorithm has work $O(n)$, span $O\left(b\frac{\log n}{\delta^2}\right)$, and incurs $\frac{s+n}{b} + O(1)$ cache misses. The recursion reduces the problem size by at least a factor of 4δ , with high probability in n .

At lower layers of recursion, with high probability in n , the algorithm reduces the problem size by a factor of at least $1/2$ (since δ is set to be a sufficiently small constant). For each $i > 1$, it follows that the size of the problem at the i -th level of recursion is at most $O(n\delta/2^i)$.

The sum of the sizes of the problems after the first level of recursion is therefore a geometric series summing to at most $O(n\delta)$. This means that the total work of the algorithm is at most $O(n\delta) + O(n) = O(n)$.

Recall that each level $i > 1$ uses $s = \frac{\ln(2^{-i}n\delta'/b)}{\delta'^2}$, where $\delta' = \Theta(1)$. It follows that level i uses $s \leq O(\log n)$. Thus, by Proposition 4.1, level i contributes $O(b \cdot s) = O(b\log n)$ to the span. Since there are at most $O(\log n)$ levels of recursion, the total span in the lower levels of recursion is at most $O(b\log^2 n)$, and the total span for the algorithm is at most,

$$O\left(b\left(\log^2 n + \frac{\log n}{\delta^2}\right)\right).$$

To compute the total number of cache misses of the algorithm, we add together $(n+s)/b + O(1)$ for the top level, and then, by Proposition 4.1, at most

$$\sum_{0 \leq i < O(\log n)} \frac{1}{b} O(2^{2-i}n\delta + \log n) \leq O\left(\frac{1}{b}(n\delta + \log^2 n)\right).$$

for lower levels. Thus the total number of cache misses for the algorithm is,

$$\frac{1}{b} \left(n + \frac{\log n}{\delta^2} \right) + O(n\delta + \log^2 n)/b = (n + O(n\delta))/b.$$

□

Proof of Corollary 4.3. By Theorem 4.3, with high probability in n , the algorithm has work $O(n)$, the algorithm has span

$$O\left(b\left(\log^2 n + \frac{\log n}{\delta^2}\right)\right) = O(\log^2 n),$$

and the algorithm incurs

$$(n + O(n\delta))/b = (n + O(n/\sqrt{\log n}))/b = (n + o(n))/b$$

cache misses. □

5 Performance Comparisons

In this section, we implement the techniques from Sections 3 and 4 to build space-efficient and in-place parallel-partition functions.

Each implementation considers an array of n 64-bit integers, and partitions them based on a pivot. The integers in the array are initially generated so that each is randomly either larger or smaller than the pivot.

In Subsection 5.1, we evaluate the techniques in Section 3 for transforming the standard parallel-prefix-based partition algorithm into an in-place algorithm. We compare the performance of three parallel-partition implementations: (1) The **high-space** implementation which follows the standard parallel-partition algorithm exactly; (2) a **medium-space** implementation which reduces the space used for the Parallel-Prefix phase; and (3) a **low-space** implementation which further eliminates the auxiliary space used in the Reordering phase of the algorithm. The low-space implementation still uses a small amount of auxiliary memory for the parallel-prefix, storing every $O(\log n)$ -th element of the parallel-prefix array explicitly rather than using the implicit-storage approach in Section 3. Nonetheless the space consumption is several orders of magnitude smaller than the original algorithm.

In addition to achieving a space-reduction, the better cache-behavior of the low-space implementation allows for it to achieve a speed advantage over its peers, in some cases completing roughly twice as fast as the medium-space implementation and four times as fast as the low-space implementation. We show that all three implementations are bottlenecked by memory throughput, however, suggesting that the cache-optimal Smoothed Striding Algorithm can do better.

In Subsection 5.2, we evaluate the performance of the Recursive Smoothed Striding Algorithm and the Strided Algorithms. The cache efficiency of the two algorithms allows for them to achieve substantially better scaling than their parallel-prefix-based counterparts. The Strided Algorithm tends to slightly outperform the Smoothed Striding Algorithm, though on 18 threads their performance is within 15% of one-another. We conclude that the Smoothed Striding Algorithm allows for one to obtain empirical performance comparable to that of the Strided Algorithm, while simultaneously achieving the provable guarantees on span and cache-efficiency missing from the original Strided Algorithm.

Machine Details. Our experiments are performed on a two-socket machine with eighteen 2.9 GHz Intel Xeon E5-2666 v3 processors. To maximize the memory bandwidth of the machine, we use a NUMA memory-placement policy in which memory allocation is spread out evenly across the nodes of the machine; this is achieved using the *interleave=all* option in the Linux *numactl* tool [22]. Worker threads in our experiments are each given their own core, with hyperthreading disabled.

Our algorithms are implemented using the CilkPlus task parallelism library in C++. The implementations avoid the use of concurrency mechanisms and atomic operations, but do

allow for concurrent reads to be performed on shared values such as n and the pointer to the input array. Our code is compiled using g++ 7.3.0, with `march=native` and at optimization level three.

Our implementations are available at github.com/awestover/Parallel-Partition.

5.1 Comparing Parallel-Prefix-Based Algorithms

In this section, we compare four partition implementations, incorporating the techniques from Section 3 in order to achieve space efficiency. We find that the performances of each of the implementations are closely modeled by cache misses, which allows for the more space-efficient algorithms to achieve practical speedups. The discussion of cache misses and memory-bandwidth bottlenecks in this section also sets the groundwork for understanding the performance of the Smoothed Striding Algorithm in Section 5.2.

The partition implementations considered in this section are:

- *A Serial Baseline:* This uses the serial in-place partition implementation from GNU Libc quicksort, with minor adaptations to optimize it for the case of sorting 64-bit integers (i.e., inlining the comparison function, etc.).
- *The High-Space Parallel Implementation:* This uses the standard parallel partition algorithm [1, 6], as described in Section 2. The space overhead is roughly $2n$ eight-byte words.
- *The Medium-Space Parallel Implementation:* Starting with the high-space implementation, we reduce the space used by the Parallel-Prefix phase by only constructing every $O(\log n)$ -th element of the prefix-sum array B , as in Section 3. (Here $O(\log n)$ is hard-coded as 64.) The array B is initialized to be of size $n/64$, with each component equal to a sum of 64 elements, and then a parallel prefix sum is computed on the array B . Rather than implicitly encoding the elements of B in A , we use an auxiliary array of size $n/64$ to explicitly store the prefix sums. The algorithm has a space overhead of $\frac{n}{32} + n$ eight-byte words.⁸
- *The Low-Space Parallel Implementation:* Starting with the medium-space implementation, we make the reordering phase completely in-place using the preprocessing technique in Section 3.⁹ The only space overhead in this implementation is the $\frac{n}{32}$ additional 8-byte words used in the prefix sum.

We remark that the ample parallelism of the low-space algorithm makes it so that for large inputs the value 64 can easily be increased substantially without negatively effecting algorithm performance. For example, on an input of size 2^{28} ,

⁸In addition to the auxiliary array of size $n/64$, we use a series of smaller arrays of sizes $n/128, n/256, \dots$ in the recursive computation of the prefix sum. The alternative of performing the parallel-prefix sum in place, as in Section 3, tends to be less cache-friendly in practice.

⁹Depending on whether the majority of elements are predecessors are successors, the algorithm goes down separate (but symmetric) code paths. In our timed experiments we test only with inputs containing more predecessors than successors, since this the slower of the two cases (by a very slight amount) for our implementation.

increasing it to 4096 has essentially no effect on the empirical runtime while bringing the auxiliary space-consumption down to a $\frac{1}{2048}$ -fraction of the input size. (In fact, the increase from 64 to 4096 results in roughly a 5% speedup.)

An Additional Optimization for The High-Space Implementation. The optimization of reducing the prefix-sum by a factor of $O(\log n)$ at the top level of recursion, rather than simply by a factor of two, can also be applied to the standard parallel-prefix algorithm when constructing a prefix-sum array of size n . Even without the space reduction, this reduces the (constant) overhead in the parallel prefix sum, while keeping the overall span of the parallel-prefix operation at $O(\log n)$. We perform this optimization in the high-space implementation.

Performance Comparison. Figure 1 graphs the speedup of each of the parallel algorithms over the serial algorithm, using varying numbers of worker threads on an 18-core machine with a fixed input size of $n = 2^{30}$. Both space optimizations result in performance improvements, with the low-space implementation performing almost twice as well as the medium-space implementation on eighteen threads, and almost four times as well as the high-space implementation.

Figure 2 compares the performances of the implementations in serial. Parallel-for-loops are replaced with serial for-loops to eliminate scheduler overhead. As the input-size varies, the ratios of the runtimes vary only slightly. The low-space implementation performs within a factor of roughly 1.9 of the serial implementation. As in Figure 1, both space optimizations result in performance improvements.

The Source of the Speedup. If we compare the number of instructions performed by the three parallel implementations, then the medium-space algorithm would seem to be the clear winner. Using Cachegrind to profile the number of instructions performed in a (serial) execution on an input of size 2^{28} ,¹⁰ the high-space, medium-space, and low-space implementations perform 4.4 billion, 2.9 billion, and 4.6 billion instructions, respectively.

Cache misses tell a different story, however. Using Cachegrind to profile the number of top-level cache misses in a (serial) execution on an input of size 2^{28} , the high-space, medium-space, and low-space implementations incur 305 million, 171 million, and 124 million cache misses, respectively.

To a first approximation, the number of cache misses by each algorithm is proportional to the number of times that the algorithm scans through a large array. By eliminating the use of large auxiliary arrays, the low-space implementation has the opportunity to achieve a reduction in the number of such scans. Additionally, the low-space algorithm allows for steps from adjacent phases of the algorithm to sometimes be performed in the same pass. For example, the enumeration of the number of predecessors and the top level of the Preprocessing phase can be performed together in a single pass on

¹⁰This smaller problem size is used to compensate for the fact that Cachegrind can be somewhat slow.

the input array. Similarly, the later levels of the Preprocessing phase (which focus on only one half of the input array) can be combined with the construction of (one half of) the auxiliary array used in the Parallel Prefix Sum phase, saving another half of a pass.

The Memory-Bandwidth Limitation. The comparison of cache misses suggests that performance is bottlenecked by memory bandwidth. To evaluate whether this is the case, we measure for each $t \in \{1, \dots, 18\}$ the memory throughput of t threads attempting to scan through disjoint portions of a large array in parallel. We measure two types of bandwidth, the **read-bandwidth**, in which the threads are simply trying to read from the array, and the **read/write bandwidth**, in which the threads are attempting to immediately overwrite entries to the array after reading them. Given read-bandwidth r bytes/second and read/write bandwidth w bytes/second, the time needed for the low-space algorithm to perform its memory operations on an input of m bytes will be roughly $3.5m/w + .5m/r$ seconds. We call this the **bandwidth constraint**. No matter how optimized the implementation of the low-space algorithm is, the bandwidth constraint serves as a hard lower bound for the running time.¹¹

Figure 3 compares the time taken by the low-space algorithm to the bandwidth constraint as the number of threads t varies from 1 to 18. As the number of threads grows, the algorithm becomes bandwidth limited, achieving its best possible parallel performance on the machine. The algorithm scales particularly well on the first socket of the machine, achieving a speedup on nine cores of roughly six times better than its performance on a single core, and then scales more poorly on the second socket as it becomes bottlenecked by memory bandwidth.

Implementation Details. In each implementation, the parallelism is achieved through simple parallel-for-loops, with one exception at the beginning of the low-space implementation, when the number of predecessors in the input array is computed. Although CilkPlus Reducers (or OpenMP Reducers) could be used to perform this parallel summation within a parallel-for-loop [15], we found a slightly more ad-hoc approach to be faster: Using a simple recursive structure, we manually implemented a parallel-for-loop with Cilk Spawns and Syncs, allowing for the summation to be performed within the recursion.

5.2 Comparing the Smoothed Striding and Strided Algorithms

In this section we consider the performance of the Strided Algorithm and the Recursive Smoothed Striding Algorithm. Past work [14] found that, on large numbers of threads, the Strided Algorithm has performance close to that of other non-EREW state-of-the-art partition algorithms (i.e., within 20% of the

best atomic-operation based algorithms). The Strided Algorithm does not offer provable guarantees on span and cache-efficiency, however; and indeed, the reason that the algorithm cannot recurse on the subarray $A[v_{\min} + 1], \dots, A[v_{\max}]$ is that the subarray has been implicitly constructed to be worst-case for the algorithm. In this subsection, we show that, with only a small loss in performance, the Smoothed Striding Algorithm can be used to achieve provable guarantees on arbitrary inputs. We remark that we do not make any attempt to generate worst-case inputs for the Strided Algorithm (in fact the random inputs that we use are among the only inputs for which the Strided Algorithm does exhibit provable guarantees!).

Figures 2 and 1 evaluate the performance of the Smoothed Striding and Strided Algorithms in serial and in parallel. On a single thread, the Smoothed Striding and Strided Algorithms perform approximately 1.5 times slower than the Libc-based serial implementation baseline. When executed on multiple threads, the performances of the Smoothed Striding and Strided Algorithms scale close to linearly in the number of threads. On 18 threads, the Smoothed Striding Algorithm achieves a 9.6 \times speedup over the Libc-based Serial Baseline, and the Strided Algorithm achieves an 11.1 \times speedup over the same baseline.

The nearly-ideal scaling of the two algorithms can be explained by their cache behavior. Whereas the parallel-prefix-based algorithms were bottlenecked by memory bandwidth, Figure 3 shows that the same is no longer true for the Smoothed Striding Algorithm. The figure compares the performance of the Smoothed Striding Algorithm to the minimum time needed simply to read and overwrite each entry of the input array using 18 concurrent threads without any other computation (i.e., the memory bandwidth constraint). On 18 threads, the time required by the memory bandwidth constraint constitutes 58% of the algorithm's total running time.

NUMA Effects. We remark that the use of the Linux *numactl* tool [22] to spread memory allocation evenly across the nodes of the machine is necessary to prevent the Smoothed Striding Algorithm and the Strided Algorithm from being bandwidth limited. For example, if we replicate the 18-thread column of Figure 3 without using *numactl*, then the speedup of the Smoothed Striding Algorithm is 8.2, whereas the memory-bandwidth bound for maximum possible speedup is only slightly larger at 10.2.

Implementation Details. Both algorithms use $b = 512$. The Smoothed Striding Algorithm uses slightly tuned ϵ, δ parameters similar to those outlined in Corollary 4.3. Although v_{\min} and v_{\max} could be computed using CilkPlus Reducers [15], we found it advantageous to instead manually implement the parallel-for-loop in the Partial Partition step with Cilk Spawns and Syncs, and to compute v_{\min} and v_{\max} within the recursion.

Example Application: A Full Quicksort. In Figure 4, we graph the performance of a parallel quicksort implementation using the low-space parallel-prefix-based algorithm, the Smoothed Striding Algorithm, and the Strided Algorithm. We compare the algorithm performances with varying numbers

¹¹Empirically, on an array of size $n = 2^{28}$, the total number of cache misses is within 8% of what this assumption would predict, suggesting that the bandwidth constraint is within a small amount of the true bandwidth-limited runtime.

of worker threads and input sizes to GNU Libc quicksort; the input array is initially in a random permutation.

Our parallel quicksort uses the parallel-partition algorithm at the top levels of recursion, and then swaps to the serial-partitioning algorithm once the input size has been reduced by at least a factor of $8p$, where p is the number of worker threads. By using the serial-partitioning algorithm on the small recursive subproblems we avoid the overhead of the parallel algorithm, while still achieving parallelism between subproblems. Small recursive problems also exhibit better cache behavior than larger ones, reducing the effects of memory-bandwidth limitations on the performance of the parallel quicksort, and further improving the scaling.

6 Conclusion and Open Questions

Parallel partition is a fundamental primitive in parallel algorithms [1, 6]. Achieving faster and more space-efficient implementations, even by constant factors, is therefore of high practical importance. Until now, the only space-efficient algorithms for parallel partition have relied extensively on concurrency mechanisms or atomic operations, or lacked provable performance guarantees. If a parallel function is going to be invoked within a large variety of applications, then provable guarantees are highly desirable. Moreover, algorithms that avoid the use of concurrency mechanisms tend to scale more reliably (and with less dependency on the particulars of the underlying hardware).

In this paper, we have shown that, somewhat surprisingly, one can adapt the classic parallel algorithm to completely eliminate the use of auxiliary memory, while still using only exclusive read/write shared variables, and maintaining a polylogarithmic span. Although the superior cache performance of the low-space algorithm results in practical speedups over its out-of-place counterpart, both algorithms remain far from the state-of-the-art due to memory bandwidth bottlenecks. To close this gap, we also presented a second in-place algorithm, the Smoothed Striding Algorithm, which achieves polylogarithmic span while guaranteeing provably optimal cache performance up to low-order factors. The Smoothed Striding Algorithm introduces randomization techniques to the previous (blocked) Striding Algorithm of [13, 14], which was known to perform well in practice but which previously exhibited poor theoretical guarantees. Our implementation of the Smoothed Striding Algorithm is fully in-place, exhibits polylogarithmic span, and has optimal cache performance.

Our work prompts several theoretical questions. Can fast space-efficient algorithms with polylogarithmic span be found for other classic problems such as randomly permuting an array [3, 4, 26], and integer sorting [2, 16, 18, 19, 25]? Such algorithms are of both theoretical and practical interest, and might be able to utilize some of the techniques introduced in this paper.

Another important direction of work is the design of in-place parallel algorithms for sample-sort, the variant of quicksort in which multiple pivots are used simultaneously in each partition. Sample-sort can be implemented to exhibit fewer

cache misses than quicksort, which is especially important when the computation is memory-bandwidth bound. The known in-place parallel algorithms for sample-sort rely heavily on atomic instructions [5] (even requiring 128-bit compare-and-swap instructions). Finding fast algorithms that use only exclusive-read-write memory (or concurrent-read-exclusive-write memory) is an important direction of future work.

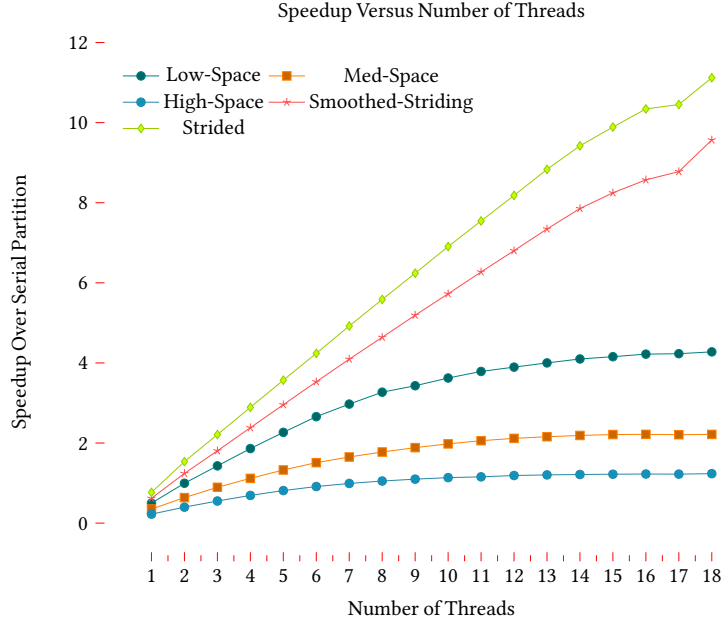


Figure 1. For a fixed table-size $n = 2^{30}$, we compare each implementation's runtime to the Libc serial baseline, which takes 3.9 seconds to complete (averaged over five trials). The x -axis plots the number of worker threads being used, and the y -axis plots the multiplicative speedup over the serial baseline. Each time (including the serial baseline) is averaged over five trials.

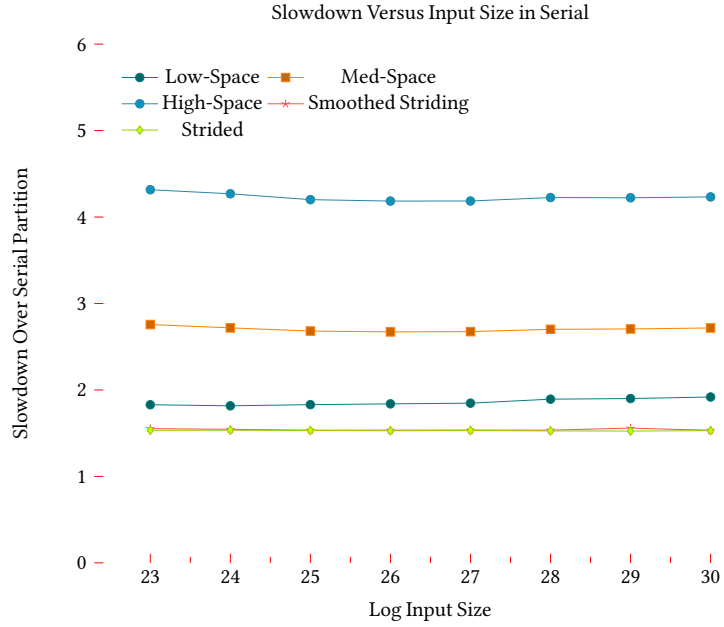


Figure 2. We compare the performance of the implementations in serial, with no scheduling overhead. The x -axis is the log-base-2 of the input size, and the y -axis is the multiplicative slowdown when compared to the Libc serial baseline. Each time (including the baseline) is averaged over five trials.

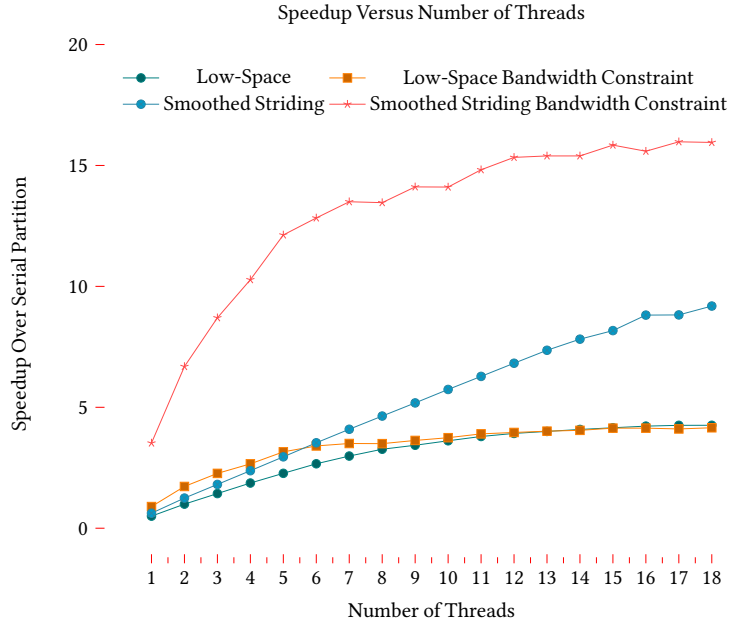


Figure 3. We compare the performances of the low-space and Smoothed Striding parallel-partition algorithms to their ideal performance determined by memory-bandwidth constraints on inputs of size 2^{30} . The x -axis is the number of worker threads, and the y -axis is the multiplicative speedup when compared to the Libc serial baseline (which is computed by an average over five trials). Each data-point is averaged over five trials.

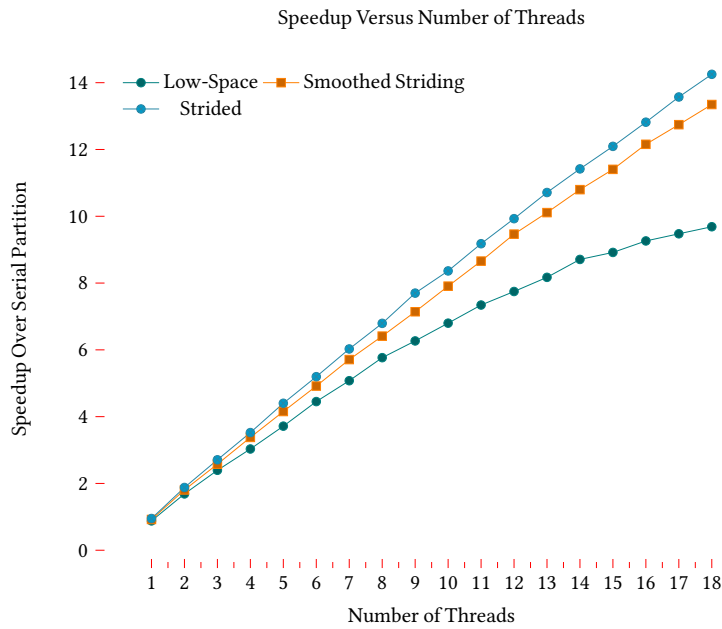


Figure 4. We compare the performance of the low-space and high-span sorting implementations running on varying numbers of threads and input sizes. The x -axis is the number of worker threads and the y -axis is the multiplicative speedup when compared to the Libc serial baseline. Each time (including each serial baseline) is averaged over five trials.

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The work for the project is split as follows:

- **Cache-Efficient In-Place Algorithms:** The main contribution of the project is a cache-efficient in-place EREW parallel-partition algorithm with provable guarantees on span and cache-behavior, and with near-state-of-the-art real-world performance. The theoretical ideas and analysis behind this algorithm, as well as the software implementation and experimentation of the algorithm are completely due to the student, Alek Westover. The writeup of the algorithm and its analysis were also performed by the student, with minor editing help by the mentor.
- **Non-Cache-Efficient In-Place Algorithms:** The paper also presents a non-cache-efficient in-place algorithm for parallel partition, which generalizes the classic parallel-prefix-based algorithm. This contribution is primarily of theoretical interest. Most of this part of the project was performed prior to the student's involvement. The performance bottlenecks of this algorithm were the motivation for the student's work, which produced algorithms with equally compelling theoretical guarantees and far better practical performance.

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