Cache-Efficient Parallel Partition Algorithms

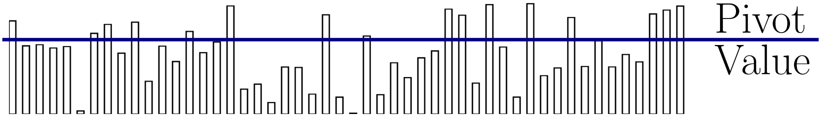
Alek Westover

MIT PRIMES

October 20, 2019

The Partition Problem

An unpartitioned array:



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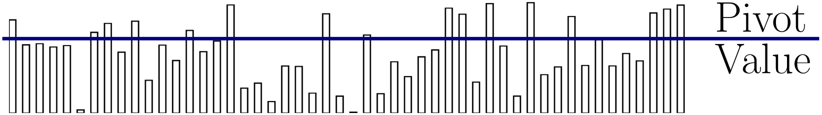
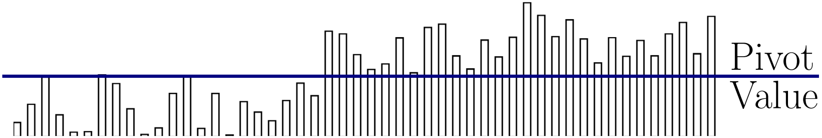


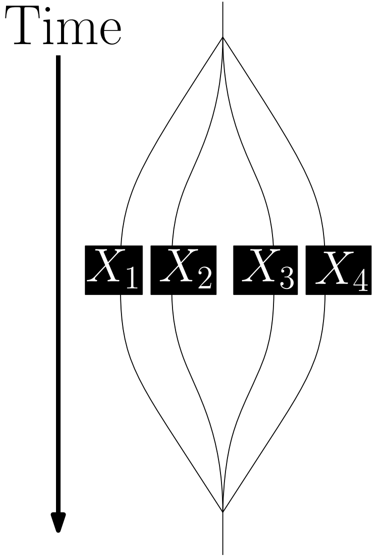
An array partitioned relative to a pivot value:



What is a Parallel Algorithm?

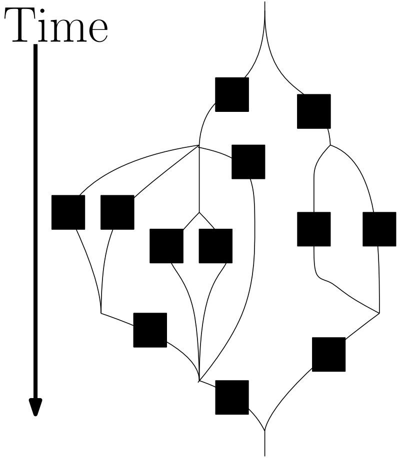
Fundamental primitive:
Parallel for loop

Parallel-For i **from** 1 **to** 4**:**
        Do X_i

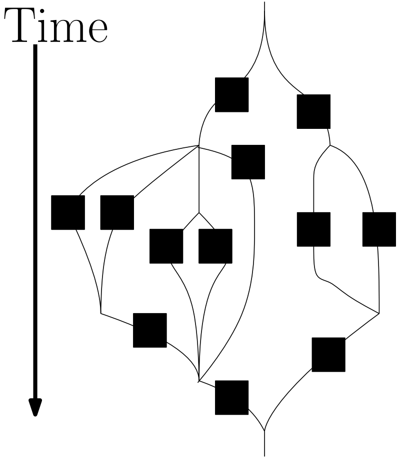


What is a Parallel Algorithm?

More complicated parallel structures can be made by combining parallel for loops and recursion.



T_p: TIME TO RUN ON p PROCESSORS



Important extreme cases:

Work: T_1,

- time to run in serial
- "sum of all work"

Span: T_∞,

- time to run on infinitely many processors,
- "height of the graph"

BOUNDING T_p WITH WORK AND SPAN

Brent's Theorem: [Brent, 74]

$$T_p = \Theta\left(\frac{T_1}{p} + T_\infty\right)$$

Take away: Work T_1 and span T_∞ determine T_p.

The Standard Parallel Partition Algorithm

Step	*Span*
Create filtered array	$O(1)$
Compute prefix sums of filtered array	$O(\log n)$
Use prefix sums to partition array	$O(1)$

Total span: $O(\log n)$

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- Lock-based and atomic-variable based algorithms

  [Michael Axtmann, Sascha Witt, Daniel Ferizovic, and Peter Sanders, 2017; Philip Heidelberger, Alan Norton, and John T. Robinson, 1990; Philippas Tsigas and Yi Zhang, 2003]

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- The Strided Algorithm

  [Francis and Pannan, 92; Frias and Petit, 08]

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Our Question: Can we create an algorithm with theoretical guarantees that is fast in practice?

Our Result: The Smoothed-Striding Algorithm

The Smoothed-Striding algorithm:

- has linear work and polylogarithmic span
  (like the Standard Algorithm)
- is fast in practice
  (like the Strided Algorithm)
- has theoretically optimal cache behavior
  (unlike any past algorithm)

Strided Versus Smoothed Strided Algorithm

Strided Algorithm

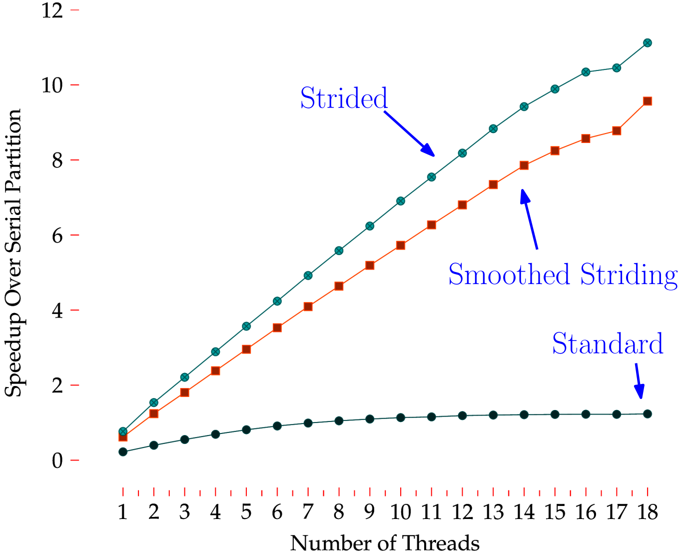
[Francis and Pannan, 92; Frias and Petit, 08]

- Good cache behavior in practice
- Worst case span is $T_\infty \approx n$
- On random inputs span is $T_\infty = \tilde{O}(n^{2/3})$

Smoothed-Striding Algorithm

- Provably optimal cache behavior
- Span is $T_\infty = O(\log n \log \log n)$ with high probability in n
- Uses randomization *inside* the algorithm

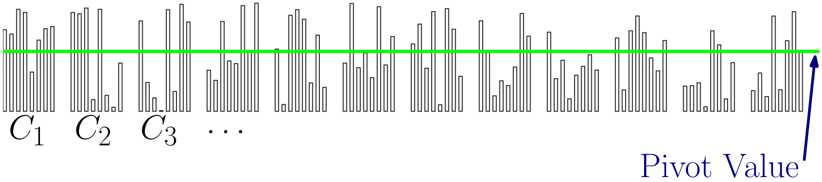
Speedup over Serial Partition: Smoothed-Striding Algorithm's performance



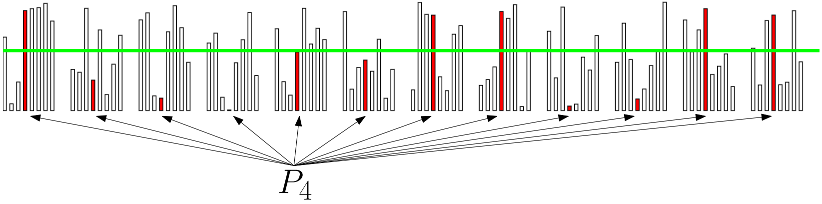
The Strided Algorithm

[Francis and Pannan, 92; Frias and Petit, 08]

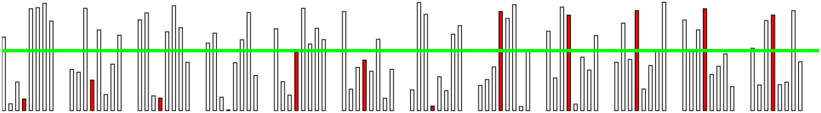
Logically partition the array into chunks of adjacent elements:



Form groups P_i that contain the i-th element from each chunk:

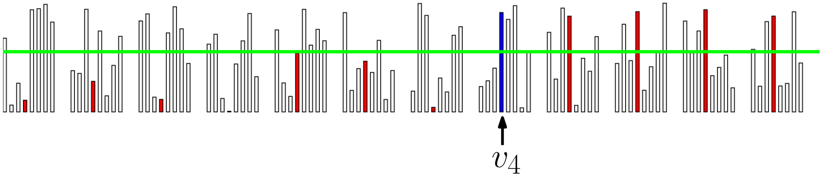


Perform serial partitions on each P_i in parallel over the P_i's:

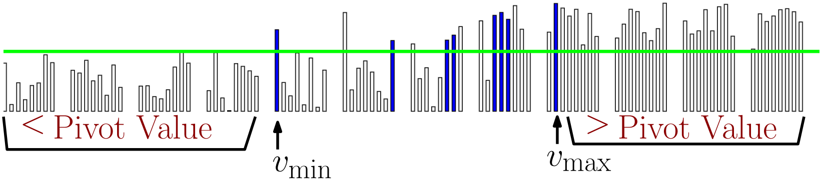


This step is highly parallel.

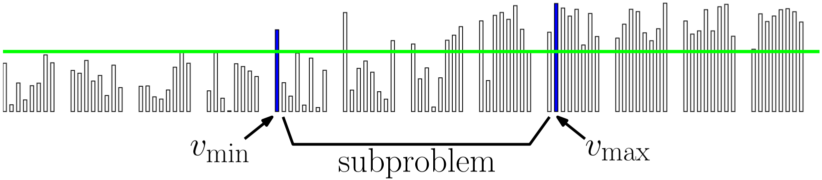
Define v_i = index of first element greater than the pivot in P_i.



Identify leftmost and rightmost v_i.



Final step: Recursively partition the subarray



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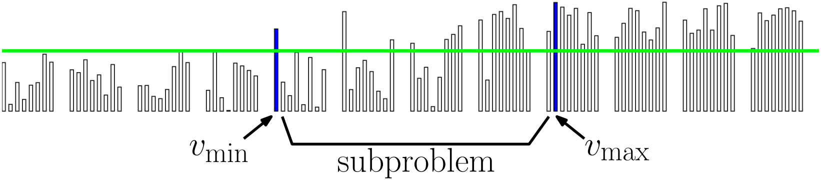

- Recursion is impossible!
- **Final Step:** Partition the subarray *in serial*.

Subproblem Span $T_\infty \approx v_{\max} - v_{\min}$

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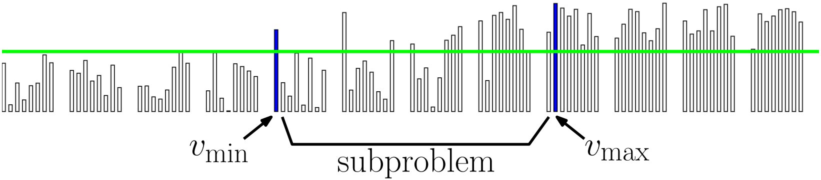
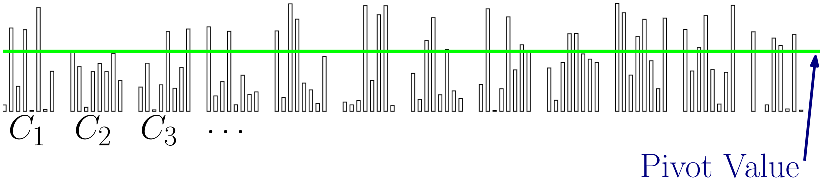


- Recursion is impossible!
- **Final Step:** Partition the subarray *in serial*.

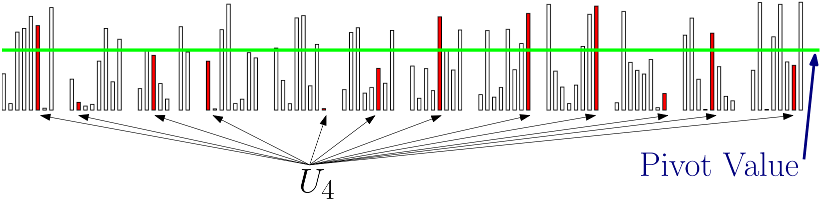
Subproblem Span $T_\infty \approx v_{\max} - v_{\min} \longleftarrow$ n in worst case.

The Smoothed-Striding Algorithm

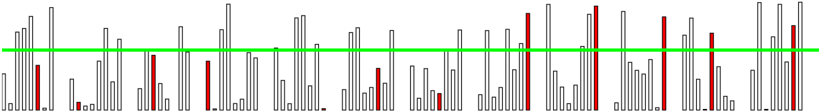
Logically partition the array into chunks of adjacent elements:



Key difference: Form groups U_i that contain a random element from each chunk (rather than containing the i-th element from each each chunk every time as the Strided Algorithm's P_i's do)

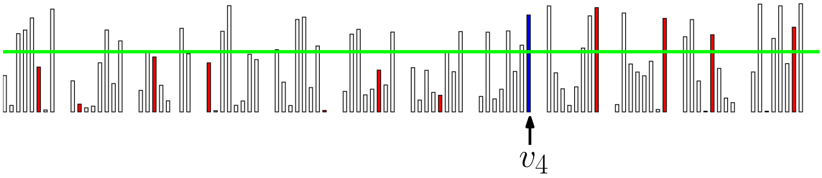


Perform serial partitions on each U_i in parallel over the U_i's:

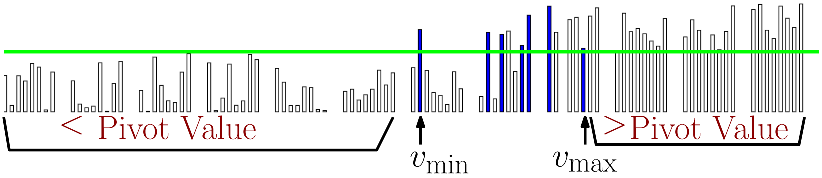


This step is highly parallel.

Define v_i = index of first element greater than the pivot in U_i.



Identify leftmost and rightmost v_i.



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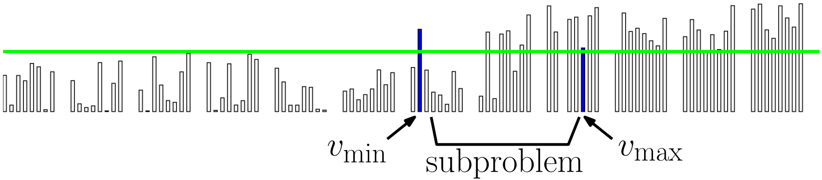

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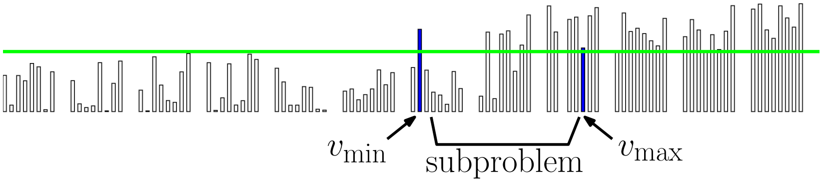


- Recursion is now possible!

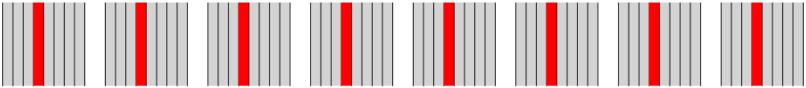
Subproblem can be solved with an In-Place algorithm that has span $O(\log n \log \log n)$.

A Key Challenge

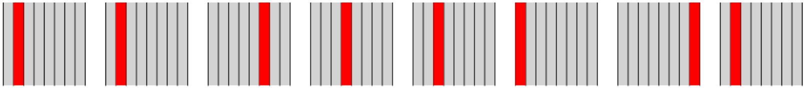
How do we store the U_i's if they are all random?

Storing which elements make up each U_i takes too much space!

Strided Algorithm P_i.



Smoothed-Striding Algorithm U_i.



An Open Question

Our algorithm: span $T_\infty = O(\log n \log \log n)$

Standard Algorithm: span $T_\infty = O(\log n)$.

Can we get optimal cache behavior and span $O(\log n)$?

Acknowledgments

I would like to thank

- MIT PRIMES
- William Kuszmaul, my PRIMES mentor
- My parents