Recursive Strategies

The smoothed striding algorithm allows for the subproblems to be solved with recursion. There are two algorithms that can be used in recursion:

- In the *Recursive Smoothed Striding Algorithm* we recurse with the smoothed striding algorithm. This gets span $O(\log^2 n)$ which is not great, but this algorithm is very simple to implement, while maintaining optimal cache behavior.
- We can also recurse with a Cache-Inneficient In-Place Parallel-Partition algorithm, that was developed concurrently to our algorithm. Doing so we achieve span $O(\log n \log \log n)$ and optimal cache behavior.

Analysis Overview

Let μ be the faction of elements of the array that are less than the pivot, and μ_i be the fraction of elements of U_i that are less than the pivot.

- Each U_i has a random element from each chunk of the array, so each element of each U_i is randomly either greater than or less than the pivot, with probabilities $1-\mu, \mu$.
- $|U_i| = \text{polylog}\, n$, so a Chernoff Bound guarantees that all U_i's will have μ_i's similar to $\mathbb{E}[\mu_i] = \mu$ with high probability in n.
- The concentration of μ_i's induces a concentration of v_i's.
- This guarantees that $v_{\max} - v_{\min}$ is small.

Pseudocode

Figure: The Smoothed Striding Algorithm

Recall:
A is the array to be partitioned, of length n.
We break A into chunks, each consisting of g cache lines of size b.
We create g groups $U_1, \ldots, U_g$ that each contain a single cache line from each chunk,
U_i's j-th cache line is the $(X[j] + i \bmod g + 1)$-th cache line in the j-th chunk of A.

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procedure GET BLOCK START INDEX(X, g, b, i, j)      ▷ This procedure returns the index in A of the start of U_i's j-th block.
    return b · ((X[j] + i mod g) + (j − 1) · g) + 1
end procedure

procedure PARALLELPARTITION(A, n, g, b)
    if g < 2 then
        serial partition A
    else
        for j ∈ {1,2,...,n/(gb)} do
            X[j] ← a random integer from [1,g]
        end for
        for all i ∈ {1,2,...,g} in parallel do      ▷ We perform a serial partition on all U_i's in parallel
            low ← GetBlockStartIndex(X,g,b,i,1)      ▷ low ← index of the first element in U_i
            high ← GetBlockStartIndex(X,g,b,i,n/(gb)) + b − 1      ▷ high ← index of the last element in U_i
            while low < high do
                while A[low] ≤ pivotValue do
                    low ← low+1
                    if low mod b ≡ 0 then      ▷ Perform a block increment once low reaches the end of a block
                        k ← number of block increments so far (including this one)
                        low ← GetBlockStartIndex(X,g,b,i,k)      ▷ Increase low to start of block k of G_i
                    end if
                end while
                while A[high] > pivotValue do
                    high ← high−1
                    if high mod b ≡ 1 then      ▷ Perform a block decrement once high reaches the beginning of a block
                        k ← number of block decrements so far (including this one)
                        k' ← n/(gb) − k
                        high ← GetBlockStartIndex(X, g,b,i,k') +b − 1      ▷ Decrease high to end of block k' of G_i
                    end if
                end while
                Swap A[low] and A[high]
            end while
        end for
        Recurse on A[v_min],...,A[v_max − 1]
    end if
end procedure
```