Cache-Efficient Parallel Partition Algorithms

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Partition

Definition (Partitioned Array)

$A[i]$ *predecessor*, $A[j]$ *successor* $\implies i < j$

Definition (Array partitioned relative to pivot value p)

$A[i] \leq p, A[j] > p \implies i < j$

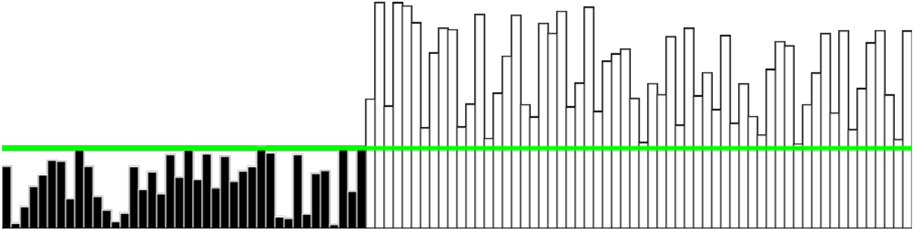
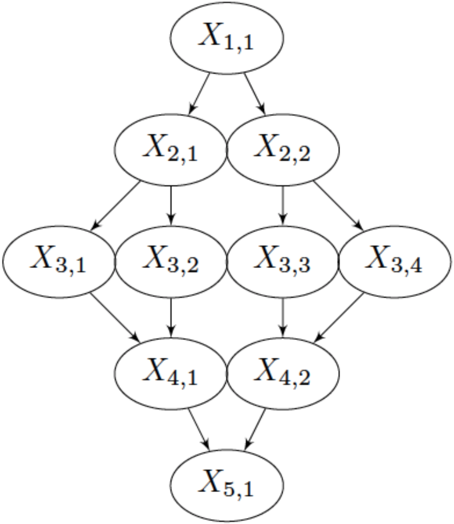


Figure: Black: Predecessor, White: Successor, Green: Pivot Value

Parallel Algorithm



Work and Span

Definition (T_p)

Running time on p processors. Note: $T_p \geq T_\infty$, $T_p \geq \frac{T_1}{p}$.

Definition (Work)

Running time on a single processor. $T_1 = \sum_i W_i$.

Definition (Span)

Running time on infinitely many processors. $T_\infty = \sum_i 1$.

Theorem (Brent's Theorem)

$$T_p = \sum_i \left\lceil \frac{W_i}{p} \right\rceil \leq \sum_i \left(\frac{W_i}{p} + 1 \right) = \frac{T_1}{p} + T_\infty.$$

Serial Partition

while low $<$ high **do**
    while $A[\text{low}] \leq$ pivotValue **do**
        low $\leftarrow$ low $+ 1$
    end while
    while $A[\text{high}] >$ pivotValue **do**
        high $\leftarrow$ high $- 1$
    end while
    Swap $A[\text{low}]$ with $A[\text{high}]$
end while
if $A[\text{low}] \leq$ pivotValue **then**
    low $\leftarrow$ low $+ 1$
end if

Randomized Algorithms

Definition (With high probability in n)

Probability of success is

$$1 - \frac{1}{n^c}$$

for c of our choice. i.e. the probability can be made arbitrarily close to 1.

Memory Bandwidth Bound

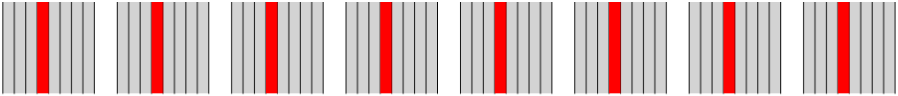
Definition (Cache miss)

A cache miss occurs when the algorithm must load a cache-line not stored in cache into cache.

- In-place $\implies$ spatial-locality
- Kuszmaul developed in-place parallel partition algorithm
  - Outperforms standard out-of-place algorithm
  - Outperformed by more cache-efficient higher span algorithm
- Experiments show Memory Bandwidth Bound (incurring too many cache misses) is the problem
- Want Temporal and Spatial locality

Strided Algorithm [1, 2] Description

Partition A into chunks $C_1, C_2, \ldots C_n/gb$ each consisting of g cache lines of size b. Let P_i be the union of the i-th cache-line from each chunk C_j.



Definition (Partially Partitioned Array)

$\exists u, l$ *such that*

$$i < u \implies A[i] \text{ is predecessor } , i \geq l \implies A[i] \text{ is successor}$$

- Perform serial partitions on all P_i in parallel.
- Let v_i be the position in A of the first successor in P_i. Perform a serial partition on $A[\min_i v_i], \ldots, A[\max_i v_i - 1]$.

Strided Algorithm Analysis

- Partial partiton step: work $O(n)$, span $\Theta(n/g)$.
- Serial cleanup step: span $\Theta(v_{\max} - v_{\min})$, which is $O(n)$ in general.
- If the number of predecessors in each P_i is similar, $v_{\max} - v_{\min}$ can be small.
- In particular, if $b \in \text{polylog}(n)$, and the array values are selected independently at random from some distribution, and g is chosen to optimize span ($g = n^{1/3}$), then with high probability in n,

  $$v_{\max} - v_{\min} < \tilde{O}(n^{2/3}),$$

  the span is

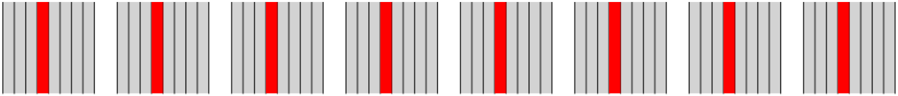
  $$\tilde{O}(n^{2/3}),$$

  and the number of cache misses is fewer than

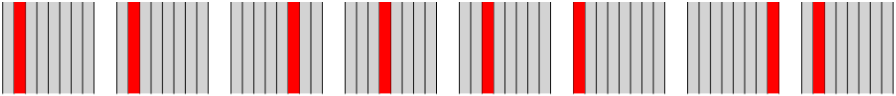
  $$\frac{n}{b} + \frac{\tilde{O}(n^{2/3})}{b}.$$

Blocked Strided Algorithm to Smoothed Striding Algorithm

Blocked Strided Algorithm P_i.

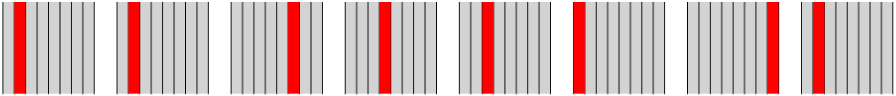


Smoothed-Striding Algorithm U_i.



Smoothed Striding Algorithm Description

Let $X[1], \ldots, X[s]$ be chosen uniformly at random from $\{1, \ldots, g\}$. Let U_i be the union of the $(X[j] + i) \mod g$-th cache-line from each chunk C_j.



- Perform serial partitions on all U_i in parallel.
- The array is partially now partitioned with $A[i]$ a predecessor for all $i < v_{\min}$ and $A[i]$ a successor for all $i \geq v_{\max}$.

Note that we will make $s = \frac{n}{gb} < \text{polylog}(n)$ so the algorithm remains in-place.

Partial Partition Step Analysis

Proposition

Let $\epsilon \in (0, 1/2)$ *and* $\delta \in (0, 1/2)$ *such that* $\epsilon \geq \frac{1}{\mathrm{poly}(n)}$ *and* $\delta \geq \frac{1}{\mathrm{polylog}(n)}$. *Suppose* $s > \frac{\ln(n/\epsilon)}{\delta^2}$. *Finally, suppose that each processor has a cache of size at least* $s + c$ *for a sufficiently large constant* c.
Then the Partial-Partition Algorithm achieves work $O(n)$*; achieves span* $O(b \cdot s)$*; incurs* $\frac{s+n}{b} + O(1)$ *cache misses; and guarantees with probability* $1 - \epsilon$ *that*

$$v_{max} - v_{min} < 4n\delta.$$

From Partial Partition to Full Partition

Partial Partition Step:

- Use $\epsilon = 1/n^c$ for c of our choice (i.e. with high probability).
- Choice of δ results in tradeoff between cache misses and span.

Recursive strategies:

- **Hybrid Smoothed Striding Algorithm**: Use algorithm with span $O(\log n \log\log n)$. Note: recursive algorithm's cache behavior doesn't affect overall cache behavior because subarray is small. This algorithm can be tuned to give optimal span and cache misses.
- **Recursive Smoothed Striding Algorithm**: Use the Partial Partition step recursively to solve subproblems. Recursive applications of the Partial Partition step use the same ϵ the top-level (to guarantee success with high probability in n), and use $\delta \in \Theta(1)$ such that the problem size is reduced by half at each step. This algorithm has slightly worse span, but is very simple to implement.

Space Reduction (Spatial Locality) Yields Speedup

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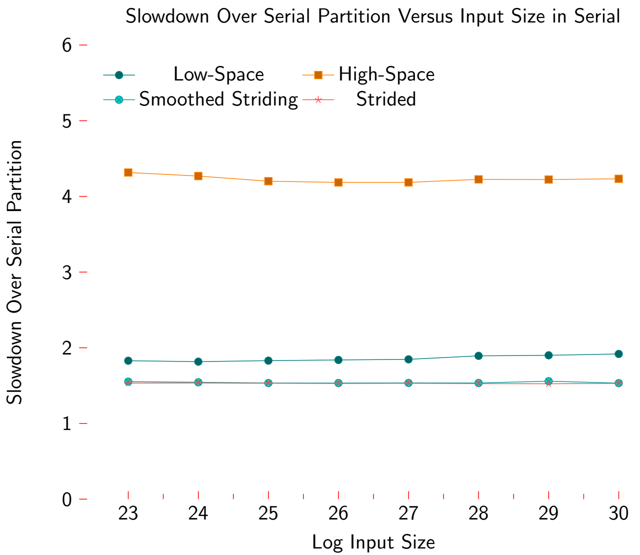
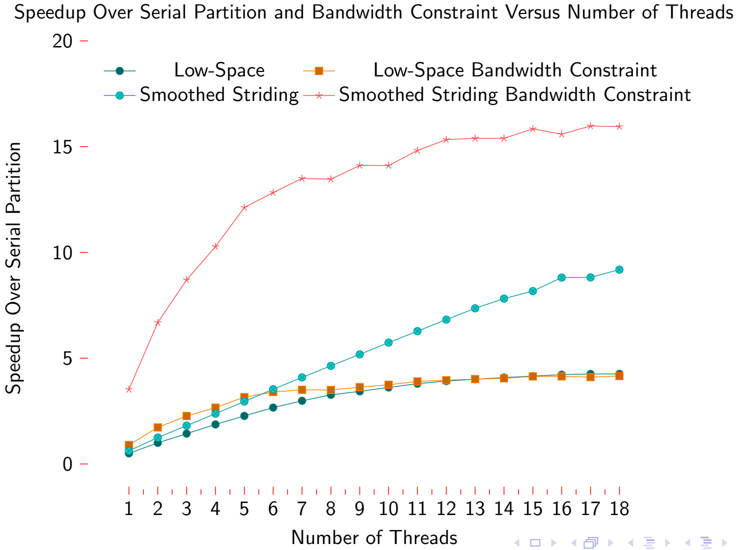
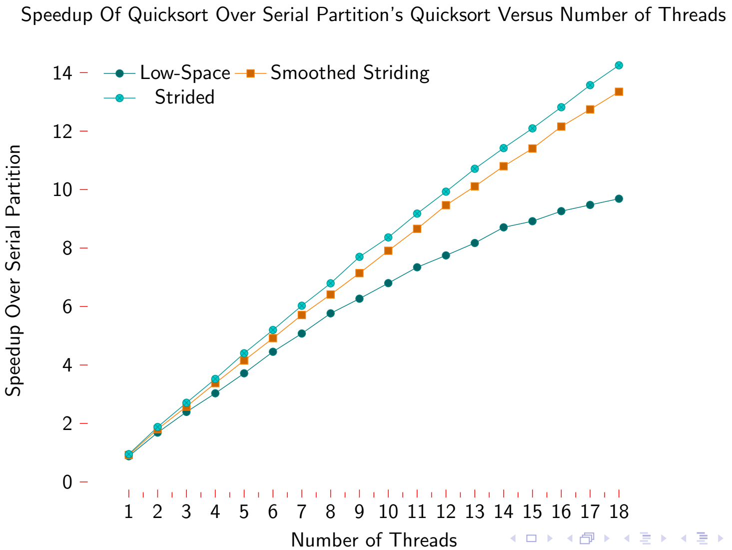


Few passes over input (Temporal locality) reduces memory bandwidth bound



Smoothed-Striding algorithm is comparable to Blocked Strided algorithm



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