# Team Round

# GREATER BOSTON MATHEMATICS LEAGUE

## MEET 2 – NOVEMBER 1998

### TEAM ROUND

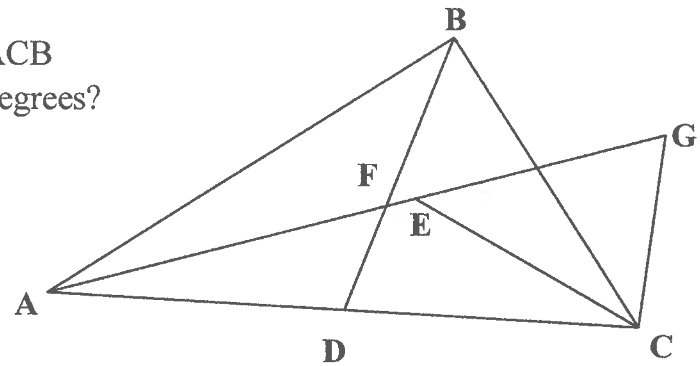
3 pts. 1. ____________________

3 pts. 2. ____________________

4 pts. 3. ____________________

**SAT APPROVED CALCULATORS ALLOWED ON THIS ROUND**

1. Given $AD = DC = BD$, $EC = EG$, $m\angle AFB = 100° + m\angle DCE$, and $\overline{AG}$ and $\overline{CE}$ bisect $\angle BAC$ and $\angle ACB$ respectively, what is $m\angle BCG$ in degrees?




2. The five digit base ten number, $9x3y6$, is divisible by 36. How many different ordered pairs $(x,y)$ satisfy this condition?

3. Given the equation in $x$, $kx^2 + p + 6kx - 3x^2 = 0$, with real numbers $k$ and $p$. Find all ordered pairs $(k, p)$ for which the sum of the roots of the equation will be $2k + 3$ and the roots are real and equal.

# GREATER BOSTON MATHEMATICS LEAGUE

## MEET 2 – NOVEMBER 1999

### TEAM ROUND

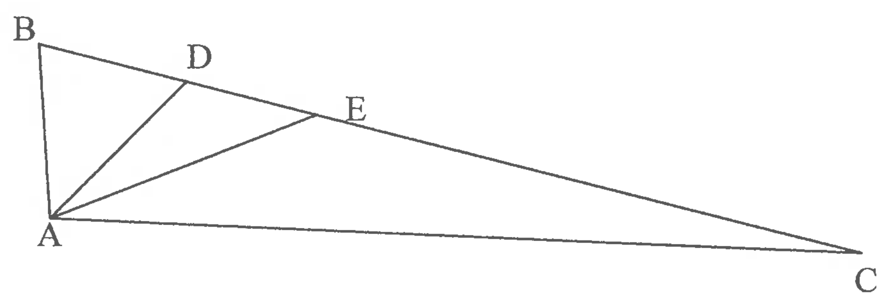
3 pts. 1. ________________

3 pts. 2. ________________

4 pts. 3. ________________

**SAT APPROVED CALCULATORS ALLOWED ON THIS ROUND**

1. Given $\overline{AD}$ bisects $\angle$ BAC, $\overline{AE}$ bisects $\angle$ CAD, and m $\angle$ ABC: m $\angle$ AEB: m $\angle$ ACB = 6:3:1, compute m $\angle$ ADC: m $\angle$ AEC.




2. If the 4 digit base 10 number $9xy1$ is divisible by 11, find the number of ordered pairs $(x, y)$ which satisfy this condition.

3. If a 3-digit number with a non-zero units digit is subtracted from the number formed by reversing its digits, the result is between 200 and 300. The sum of its digits is 14. Find the sum of all possible 3-digit numbers satisfying these conditions.

**GREATER BOSTON MATHEMATICS LEAGUE**

**MEET 2 – NOVEMBER 2000**

**TEAM ROUND**

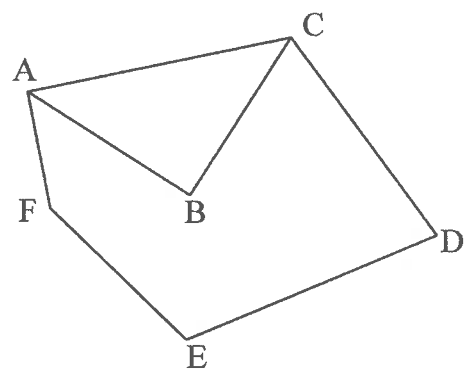
3 pts. 1. ________________

3 pts. 2. ________________

4 pts. 3. ________________

**SAT APPROVED CALCULATORS ARE ALLOWED IN THE TEAM ROUND, EXCEPT FOR CALCULATORS WITH SYMBOLIC MANIPULATION PROGRAMS, (FOR EXAMPLE THE TI-89) WHICH ARE NOT ALLOWED IN THIS ROUND.**

1. Given the figure on the right such that $m\angle E - m\angle ABC = 24°$, $m\angle F - m\angle D = 70°$, and $m\angle BCD - m\angle BAF = 22°$, compute the number of degrees in $m\angle BCD + m\angle D$.




2. If the equation $\dfrac{x}{x+1} + \dfrac{x+2}{x-1} = k$ has only one solution for $x$, find exactly all possible values for $k$.

3. An inheritance between \$50,000 and \$51,000 when divided almost evenly among seven heirs, two get an extra dollar, when divided almost evenly among eleven heirs, three get an extra dollar, and when divided among thirteen heirs, they all get the same amount. Find the number of dollars in this inheritance.

**GREATER BOSTON MATHEMATICS LEAGUE**

**MEET 2 – NOVEMBER 2001**

**TEAM ROUND ( 12 MINUTES LONG )**

3 pts. 1. ____________________

3 pts. 2. $y =$ ____________

4 pts. 3. ____________________

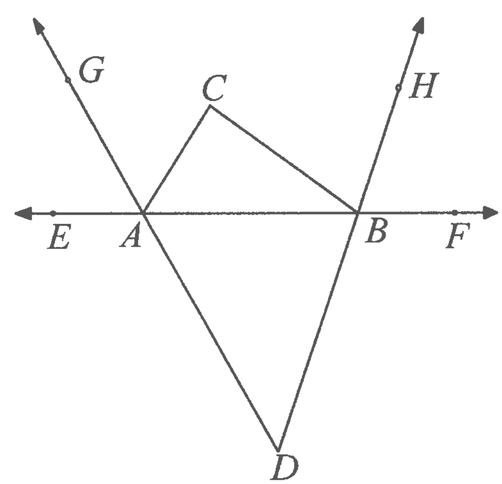
**SAT APPROVED CALCULATORS ARE ALLOWED IN THE TEAM ROUND, EXCEPT FOR CALCULATORS WITH SYMBOLIC MANIPULATION PROGRAMS, (FOR EXAMPLE THE TI-89) WHICH ARE NOT ALLOWED IN THIS ROUND.**

1. Given the following matrix multiplication, find all possible ordered pairs $(a,b)$.

   Write all answers as ordered pairs.

$$\begin{pmatrix} x & y \\ -2 & 4 \end{pmatrix}\begin{pmatrix} -5 & y \\ x & 2 \end{pmatrix}=\begin{pmatrix} 12 & -1 \\ a & b \end{pmatrix}$$

2. Given $\overline{EABF}$, $\overline{DAG}$, $\overline{DBH}$, $\overrightarrow{AG}$ bisects $\angle CAE$ and $\overrightarrow{BH}$ bisects $\angle CBF$. If $m\angle C = (x+y)°$ and $m\angle D = (x-y)°$, find $y$ in terms of $x$.




3. The number 10! (10 factorial) has how many perfect square factors?

# GREATER BOSTON MATHEMATICS LEAGUE
# MEET 2 – NOVEMBER 2006

## TEAM ROUND

3 pts. 1. ______________________

3 pts. 2. ______________________

4 pts. 3. ______________________

**SAT APPROVED CALCULATORS ARE ALLOWED IN THIS ROUND.**

1. Determine the minimum value of $n > 2006$, for which the expression $54^n - 4(27)^n$ evaluates to a multiple of 10.

2. For some positive integers $k$, $\sin\left(k(A-15^\circ)\right) = \pm\frac{1}{2}$, $0 \le A < 360^\circ$, $\tan(A) = 1$, but $A$ is <u>not</u> in quadrant 1. Determine the <u>sum</u> of the four smallest values of $k$.

3. Two natural numbers are simultaneously chosen at random from a box containing the natural numbers 1, 2, 3, 4, …, 29, 30. In how many ways can the two numbers be selected from the box so that their product has exactly four factors? The order in which the two numbers are selected is irrelevant.

# GREATER BOSTON MATHEMATICS LEAGUE

## MEET 2 – NOVEMBER 2007

## TEAM ROUND

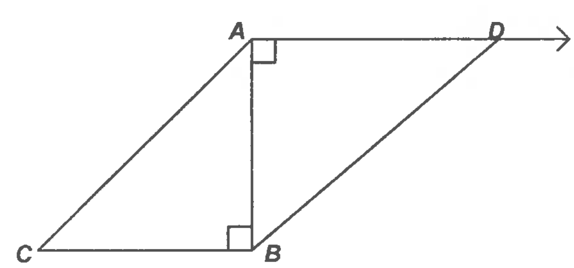
| | | |
|---|---|---|
| 3 pts. | 1. | ______________ |
| 3 pts. | 2. | ______________ |
| 4 pts. | 3. | ______________ |

**SAT APPROVED CALCULATORS ARE ALLOWED IN THIS ROUND.**

1. The value of the units digit of $17^{519} - 23^{118}$ is multiplied by the units digit of the product $\left(17^{519}\right)\left(23^{118}\right)$. The result is raised to the $25^{th}$ power and then divided by 5. The remainder is $w$. Find the value of $w$.

2. $A$, $B$, $C$ and $D$ are 4 consecutive positive multiples of 3, where $A < B < C < D$. The sum $A + B + C + D$ when multiplied by 6 is 27 more than the square of $A$. Find the number of even factors of the product $CD$.

3. $\angle ABC$ and $\angle BAD$ are right angles. If $BC = 5$, $AC = 13$ and $AD$ is an integer, determine all possible values of $BD$.



# GREATER BOSTON MATHEMATICS LEAGUE

## MEET 2 – NOVEMBER 2008

## TEAM ROUND

3 pts. 1. ____________________

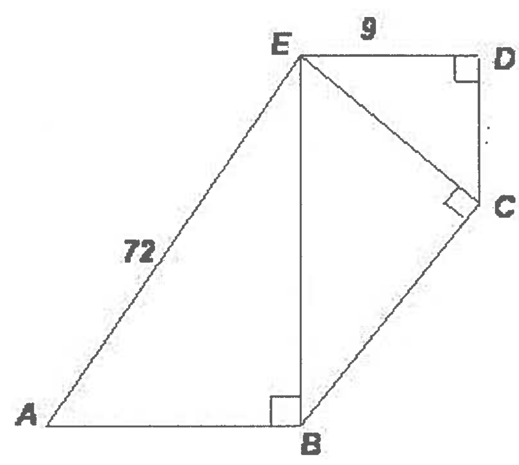
3 pts. 2. ____________________

4 pts. 3. ____________________

**SAT APPROVED CALCULATORS ARE ALLOWED IN THIS ROUND.**

1. $a$ and $b$ are the roots of the quadratic equation $(k+1)x^2-(3k-2)x+6=0$. The sum of the reciprocal of the roots equals $\frac{13}{6}$. Find the numerical value of the expression $\frac{8}{5a}+\frac{8}{5b}$

2. $\measuredangle AED$ in pentagon $ABCDE$ is trisected. $\overline{AB}\perp\overline{BE}$, $\overline{BC}\perp\overline{EC}$, $\overline{ED}\perp\overline{DC}$, $AE=72$, and $ED=9$. Find the exact length of $\overline{DC}$.




3. How many odd integers between 3000 and 6300 have different digits?

# GREATER BOSTON MATHEMATICS LEAGUE

## MEET 2 – NOVEMBER 2009

## TEAM ROUND

| | | |
|---|---|---|
| 3 pts. | 1. | ______________ ° |
| 3 pts. | 2. | ______________ |
| 4 pts. | 3. | ______________ |

**CALCULATORS ARE <u>NOT</u> ALLOWED ON THIS ROUND.**

1. Find the <u>sum</u> of all values of $\theta$, $0° \le \theta < 360°$, which satisfy the following system of equations:

$$\tan\theta - \sqrt{3}\sin\theta + \tan\theta\sin\theta - \sqrt{3} = 0$$
$$2\cos^2\theta + \cos(-\theta) = 0$$

2. Find all equations with integral coefficients whose roots satisfy the following conditions:

   the product of the roots divided by the difference of the roots equals 2, and twice the sum of the roots equals 15.

3. Find all ordered triples $(A, B, C)$ which satisfy the following matrix equation:

$$\begin{pmatrix} 2A & 2 & B \\ 3 & B & C \\ 2B & -A & -4 \end{pmatrix}\begin{pmatrix} -2 \\ 8 \\ 4 \end{pmatrix} = \begin{pmatrix} 12 \\ -14 \\ 3C+4 \end{pmatrix}$$

# GREATER BOSTON MATHEMATICS LEAGUE

## MEET 2 – NOVEMBER 2010

## TEAM ROUND

| | | |
|---|---|---|
| 3 pts. | 1. | ( __________ , __________ ) |
| 3 pts. | 2. | __________________________ |
| 4 pts. | 3. | __________________________ |

**SAT APPROVED CALCULATORS ARE ALLOWED IN THIS ROUND.**

1. If $\begin{pmatrix} 3 & P \\ 1 & Q \end{pmatrix}\begin{pmatrix} -Q & -P \\ 0 & -1 \end{pmatrix} = A$ and $A^{-1} = \begin{pmatrix} 3 & -4 \\ -2 & 6 \end{pmatrix}$, compute $(P, Q)$.

2. Find <u>all</u> values of $x$, where $90° \le x < 270°$ which satisfy the following equation:

$$\tan^2 x - \cot x = \cot^2 x - \tan x$$

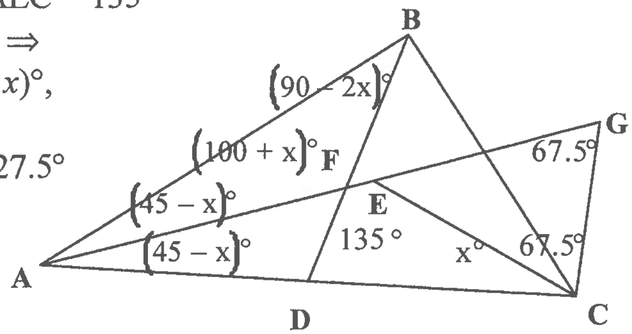
3. Suppose the calendar year is divided into twelve 30-day months with the same names as are currently used and that in a leap year, February has an extra day - day 31.
In the leap year 2016, specify on what day(s) of the week January 1st must fall to maximize the number of Friday the 13ths falling in that calendar year <u>and</u> in what months these Friday the 13ths occur.

   In your answer, use the abbreviations
   SUN MON TUE WED THU FRI SAT for the days of the week (DOW) and
   JAN FEB MAR APR MAY JUN JUL AUG SEP OCT NOV DEC for the names of the months.
   Express your answer(s) in the form DOW: months.

**TEAM ROUND**

98

1. Since AD = BD = CD, Δ ABC is right and because of the angle bisectors, m ∠AEC = 135° and m ∠ECG = 67.5°. Call m ACE = $x°$ ⇒ m ∠AFB = $(100 + x)°$, m ∠BAF = $(45 - x)°$, and m ∠ABD = $(90 - 2x)°$; Equation: $100 + x + 45 - x + 90 - 2x = 180 \Rightarrow x = 27.5°$ ⇒ m ∠BCG = 67.5° – 27.5° = **40°**

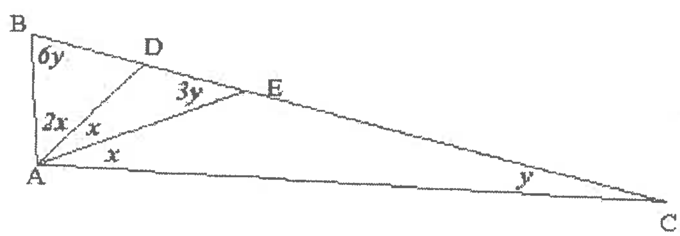



2. In order for 9x3y6 to be divisible by 36, it must be divisible by 4 and 9. Divisibility by 4 ⇒ $y$ = 1, 3, 5, 7, or 9. (4 divides evenly into the last 2 digits.) Now consider each case: $y = 1 \Rightarrow x = 8$; (The sum of the digits is divisible by 9.) $y = 3 \Rightarrow x = 6$; $y = 5 \Rightarrow x = 4$; $y = 7 \Rightarrow x = 2$; $y = 9 \Rightarrow x = 0$ or 9; ⇒ There are **6** possible ordered pairs.

3. $kx^2 + p + 6kx - 3x^2 = 0 \Rightarrow (k-3)x^2 + 6kx + p = 0 \Rightarrow$ sum of the roots $= \dfrac{-6k}{k-3} = 2k + 3$
$2k^2 + 3k - 9 = 0 \Rightarrow (2k-3)(k+3) = 0 \Rightarrow k = -3,\ 1.5$; Two equal roots ⇒
$(6k)^2 - 4(k-3)p = 0$; If $k = -3$, then $(-18)^2 - 4(-6)p = 0 \Rightarrow p = -13.5$
If $k = 1.5$, then $(9)^2 - 4(-1.5)p = 0 \Rightarrow p = -13.5 \Rightarrow$ **(-3, -13.5) and (1.5, -13.5)** are the ordered pairs.

99

1. $3y = y + x \Rightarrow x = 2y$;
$4x + 7y = 180° \Rightarrow$
$15y = 180° \Rightarrow y = 12°$;
m ∠ ADC =
$2x + 6y = 10y = 120°$;
m ∠ AEC = 180° – $3y$ = 144°; m ∠ ADC: m ∠ AEC = 120:144= 5:6

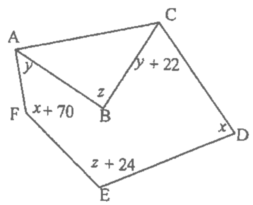



2. $9xy1$ is divisible by 11 $\Rightarrow (9 + y) - (x + 1) = 0, 11, 22 \ldots \Rightarrow y = x - 8$ or $y = x + 3$
Note 22 is too large to produce any ordered pairs.
If $y = x - 8$ results in ordered pairs (8, 0) and (9,1)
If $y = x + 3$ results in ordered pairs (0, 3),(1, 4) … (6, 9).
Therefore there are 9 possibilities

3. Call the original number, $100h + 10t + u$. The two results are $h + t + u = 14$ and $200 < (100u + 10t + u) - (100h + 10t + u) < 300 \Rightarrow 200 < 99u - 99h < 300 \Rightarrow$
$200 < 99(u - h) < 300 \Rightarrow u - h = 3$; adding the equations: $2u + t = 17$ or $t = 17 - 2u$;
now list all the possibilities: $u = 4 \Rightarrow t = 9 \Rightarrow h = 1$ (194); $u = 5 \Rightarrow t = 7 \Rightarrow h = 2$ (275); $u = 6 \Rightarrow t = 5 \Rightarrow h = 3$ (356); $u = 7 \Rightarrow t = 3 \Rightarrow h = 4$ (437); $u = 8 \Rightarrow t = 1 \Rightarrow h = 5$ (518);
Finally adding the five possibilities: $518 + 437 + 356 + 275 + 194 = 1780$

2000

1. If you draw $\overline{BE}$ you see that the following is true:
$m\angle BAF + (360 - m\angle ABC) + m\angle BCD +$
$m\angle D + m\angle E + m\angle F = 720$
$\rightarrow y + 360 - z + y + 22 + x + z + 24 + x + 70 = 720 \rightarrow$
$2x + 2y = 244 \rightarrow x + y + 22 = 144$



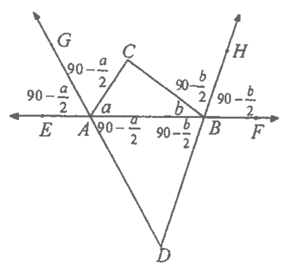

2. $\dfrac{x}{x+1} + \dfrac{x+2}{x-1} = k \rightarrow$
$x^2 - x + x^2 + 3x + 2 = k\left(x^2 - 1\right) \rightarrow 2x^2 + 2x + 2 = kx^2 - k \rightarrow$
$(k-2)x^2 - 2x + (-k-2) = 0$; 1 solution $\rightarrow$ discriminant $= 0 \rightarrow$
$(-2)^2 - 4(k-2)(-k-2) = 0 \rightarrow 4 + 4(k-2)(k+2) = 0 \rightarrow$
$1 + k^2 - 4 = 0 \rightarrow k^2 = 3 \rightarrow k = \pm\sqrt{3}$ or if $k = 2$, then the equation becomes linear and thus has one solution. Therefore the values for $k$ are $2, \pm\sqrt{3}$.

3. The number $= 7x + 2 = 11y + 3 = 13z$; first find the smallest number such that $x$ and $y$ satisfies the first equation: $y = \dfrac{7x-1}{11} \rightarrow x = 8$ and $y = 5 \rightarrow$ number is $58 \equiv 6 \bmod 13$; adding $77n$ to 58 produced numbers with the same property; $77 \equiv -1 \bmod 13 \rightarrow$
$58 + 6\cdot 77 = 520 \equiv 6 + 6(-1) \bmod 13 \equiv 0 \bmod 13$; $7\cdot 11\cdot 13 = 1001$; adding $1001n$ to 520 produced numbers with the same property; $1001\cdot 50 + 520 = 50570$

2001

1. $\begin{pmatrix} x & y \\ -2 & 4 \end{pmatrix}\begin{pmatrix} -5 & y \\ x & 2 \end{pmatrix} = \begin{pmatrix} 12 & -1 \\ a & b \end{pmatrix} \Rightarrow \begin{cases} -5x + xy = 12 \\ xy + 2y = -1 \end{cases} \Rightarrow \begin{cases} y = \dfrac{12+5x}{x} \\ y = \dfrac{-1}{x+2} \end{cases} \Rightarrow \dfrac{-1}{x+2} = \dfrac{12+5x}{x} \Rightarrow$
$-x = 5x^2 + 22x + 24 \Rightarrow 5x^2 + 23x + 24 = 0 \Rightarrow (5x+8)(x+3) = 0 \Rightarrow$
$x = -1.6 \Rightarrow y = -2.5,\ x = -3 \Rightarrow y = 1$; $\begin{pmatrix} -1.6 & -2.5 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} -5 & -2.5 \\ -1.6 & 2 \end{pmatrix} = \begin{pmatrix} 12 & -1 \\ 3.6 & 13 \end{pmatrix}$
$\begin{pmatrix} -3 & 1 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} -5 & 1 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} 12 & -1 \\ -2 & 6 \end{pmatrix} \Rightarrow (a,b) = (3.6, 13), (-2, 6)$

2. Let $m\angle CAB = a°$ and $m\angle CBA = b° \Rightarrow$
$m\angle C = \left(180 - (a+b)\right)°$;
$m\angle D = \left(180 - \left(90 - \dfrac{a}{2}\right) - \left(90 - \dfrac{b}{2}\right)\right)° = \left(\dfrac{a+b}{2}\right)°$;
$x + y = 180 - (a+b)$ and $2x - 2y = a + b \Rightarrow 3x - y = 180$
$\Rightarrow y = 3x - 180$.




3. $10! = 2^8\cdot 3^4\cdot 5^2\cdot 7$; any perfect square factor contains
0 or 2 or 4 or 6 or 8 factors of 2 (5 possibilities);
any perfect square factor contains 0 or 2 or 4 factors of 3 (3 possibilities);
any perfect square factor contains 0 or 2 factors of 5 (2 possibilities);
by the basic counting principle the number of perfect square factors of 10! =
$5\cdot 3\cdot 2 = 30$ possibilities.

2006

**TEAM ROUND**

1. Let $P = 54^n$ and $Q = 4(27)^n$. Consider the following table.

| n | Units' digit of P | Units' digit of Q |
|---|---|---|
| 1 | 4 | 8 |
| 2 | 6 | 6 |
| 3 | 4 | 2 |
| 4 | 6 | 4 |
| 5 | 4 | 8 |

Clearly, for $n = 2$ (and $n = 6, 10, \ldots$), the difference ends in 0 (and is a multiple of 10).

Thus, for $n = 4k + 2$ (that is, a number which is 2 more than a multiple of 4), the difference will always be a multiple of 10. $4k + 2 > 2006 \rightarrow k = 502 \rightarrow n = \underline{\mathbf{2010}}$

2. $A$ is located in quadrant 3 and belongs to the 45° family $\rightarrow A = 225°$
Therefore, $210k = \begin{cases} 30 + 180n \\ 150 + 180n \end{cases} \rightarrow 7k = \begin{cases} 1 + 6n \\ 5 + 6n \end{cases}$
$\rightarrow n = 1, 8, 15, \ldots$ with corresponding values of $k = 1, 7, 13\ldots.$
or $n = 5, 12, 19, \ldots$ with corresponding values of $k = 5, 11, 17, \ldots$
$\rightarrow \mathbf{1 + 5 + 7 + 11 = \underline{24}}$

3. Numbers with exactly 4 factors are either the product of two distinct primes or the cube of a prime. The primes in the box are: 2, 3, 5, 7, 11, 13, 17, 19, 23 and 29
Thus, there are ${}_{10}C_2 = 45$ pair products and 3 cubes of a prime ( 8 = 2x4, 27 = 3x9 and 125 = 5x25).
__However__, whenever the product is a number in the box, we must also consider that product as 1 x (the number). So, of the 45 pair products, 2 x 3 = 6, 2 x 5 = 10, 2 x 7 = 14, 2 x 11 = 22, 2 x 13 = 26, 3 x 5 = 15 and 3 x 7 = 21 must be counted twice (1 x 6, 1 x 10 etc.) and likewise for the cubes of a prime. Thus, 38 + 2(7) + 2(2) + 1 = __57__.

2007

**TEAM ROUND**

1. $17^{519} - 23^{118} = 17^{516}17^3 - 23^{116}23^2 = (17^4)^{129}17^3 - (23^4)^{29}23^2$
The units digit of both $17^4$ and 23 is 1. The units digits of $17^3$ and $23^2$ are 3 and 9 respectively. Thus the units digit of the difference $17^{519} - 23^{118}$ is $1(3) - 1(9) = -6 \rightarrow +4$

Recall: For the difference $\begin{array}{r} \_\,{}^13 \\ -\_\,9 \\ \hline \end{array}$ we would have to 'borrow' from the previous digit to perform the subtraction.

The units digit of the product $(17^{519})(23^{118})$ is $(\_3)(\_9) \rightarrow 7$.

Thus, $w$ = the remainder from the quotient $\dfrac{28^{25}}{5}$. Since the units digit of $28^4$ is 6 and, raising a number ending in 6 to any positive integer power produces an integer that still ends in 6, the units digit of the numerator $28^{25} = (28^4)^6 28^1$ is $(\_6)(28) \rightarrow 8$
Dividing by 5 will always leave a remainder of __3__

2. $A + B + C + D = 3k + 3(k+1) + 3(k+2) + 3(k+3) = 12k + 18$
$6(12k + 18) = 27 + (3k)^2 \rightarrow 36(2k + 3) = 9(k^2 + 3) \rightarrow k^2 - 8k - 1 = (k-9)(k+1) = 0$
$\rightarrow k = 9 \rightarrow CD = 33(36) = 2^2 3^3 11^1$
Thus, even factors of $CD$ must be of the form $2^x 3^y 11^z$, where $x$ can be 1 or 2, $y$ can be 0, 1, 2 or 3 and $z$ can be 0 or 1 $\rightarrow$ # positive factors = (2)(4)(2) = __16__

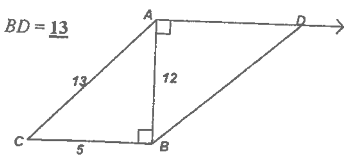
3. $AB = 12$

Case 1: $\Delta BAD \cong \Delta ABC$ (or $ACBD$ is a parallelogram)→ $BD = \underline{13}$

Case 2: $\Delta BAD$ is similar to a 3 – 4 – 5 right triangle

3(3 – 4 – 5) → 9 – 12 – $\underline{15}$

4(3 – 4 – 5) → 12 – 16 – $\underline{20}$




Case 3: a "larger" primitive Pythagorean Triple

All primitive Pythagorean Triples are given by the expressions

$\begin{cases} m^2 - n^2 \\ 2mn \\ m^2 + n^2 \end{cases}$, where $m > n$ and $m$ and $n$ have opposite parity, i.e. if $m$ is even, $n$ is odd and vice versa.

Thus, even integers must be generated by $2mn$ and $12 = 2mn$ → $mn = 6$ → $(m, n) = (6, 1)$ or $(3, 2)$

(3, 2) produces the 5- 12 – 13 triangle we already have

(6, 1) produces a 12 – 35 – 37 triangle and the last possible length of $\overline{BD}$ is $\underline{37}$

2008

1. $\dfrac{3k-2}{k+1} = \dfrac{13}{6} \to k = 5$, $a + b = \dfrac{13}{6}$, $ab = \dfrac{6}{6}$. $\dfrac{8}{5a} + \dfrac{8}{5b} = \dfrac{8(a+b)}{5ab} = \dfrac{8\left(\frac{13}{6}\right)}{5} = \underline{\dfrac{52}{15}}$

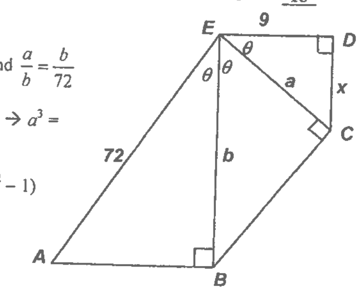
2. $\Delta ABE \sim \Delta BCE \sim \Delta CDE$ → $\dfrac{9}{a} = \dfrac{a}{b}$ and $\dfrac{a}{b} = \dfrac{b}{72}$

→ $\begin{cases} a^2 = 9b \\ b^2 = 72a \end{cases}$ → $a^4 = 81b^2 = 81(72a)$ → $a^3 =$

$81(72) = 2^3 \cdot 3^6$

Thus, $a = 18$ and $x^2 = 18^2 - 9^2 = 9^2(2^2 - 1)$

→ $DC = 9\sqrt{3}$




3. From the integers 3001, …, 3999, there are $1 \cdot 8 \cdot 7 \cdot 4 = \underline{224}$ odd numbers with different digits itemized as follows:

The thousands digit must be a 3. The units digit can be 1, 5, 7, or 9 (4 choices).
This leaves 8 choices for the hundreds digit and 7 choices for the tens digit.

Similar reasoning produces $\underline{224}$ odd numbers with different digits among 5000, 5001, …, 5999.

Of the integers 4000, 4001,…, 4999, there are $1 \cdot 8 \cdot 7 \cdot 5 = \underline{280}$ odd numbers with different digits itemized as follows:

The thousands digit must be a 4. The units digit can be 1,3,5,7, or 9 (5 choices).
This leaves 8 choices for the hundreds digit and 7 choices for the tens digit.

Of the integers 6000, 6001, …, 6299 there are $(1 \cdot 2 \cdot 7 \cdot 5) + (1 \cdot 1 \cdot 7 \cdot 4) = \underline{98}$ odd numbers with different digits itemized as follows: The thousands digit must be a 6.
If the hundreds digit is either a 0 or 2, then there will be 7 choices for the tens digit and 5 choices for the units digit. Thus, 70 choices.
If the hundreds digit is a 1, then there will be 4 choices for the units digit and 7 choices for the tens digit. Thus, 28 choices. Combining totals from these disjoint ranges we have,
$280 + 224 + 224 + 98 = \underline{826}$.

## TEAM ROUND

2009

1. $\begin{matrix} \tan\theta - \sqrt{3}\sin\theta + \tan\theta\sin\theta - \sqrt{3} = 0 \\ 2\cos^2\theta + \cos(-\theta) = 0 \end{matrix}$ → $\begin{cases} \tan\theta(1+\sin\theta) - \sqrt{3}(1+\sin\theta) = 0 \\ \cos\theta(2\cos\theta + 1) = 0 \end{cases}$

→ $\begin{cases} \tan\theta = \sqrt{3} \text{ or } \sin\theta = -1 \\ \cos\theta = 0 \text{ or } \cos\theta = -\dfrac{1}{2} \end{cases}$ → $\begin{cases} \theta = 60°, \underline{240°}, 270° \\ \theta = 90°, \underline{270°}, 120°, \underline{240°} \end{cases}$ → $\underline{\mathbf{510°}}$

2. Let $r_1$ and $r_2$ denote the roots. Then: $\dfrac{r_1 r_2}{r_1 - r_2} = 2$ and $2r_1 + 2r_2 = 15$

$r_1 r_2 = 2r_1 - 2r_2$ → $r_1 r_2 + 2r_2 = r_2(r_1 + 2) = 2r_1$ → $r_2 = \dfrac{2r_1}{r_1 + 2}$

Substituting in the 2[nd] condition, $2r_1 + 2\left(\dfrac{2r_1}{r_1+2}\right) = 15$ → $2r_1^2 + 4r_1 + 4r_1 = 15r_1 + 30$

→ $2r_1^2 + 7r_1 - 30 = (2r_1 + 5)(r_1 - 6) = 0$ → $r_1 = -\dfrac{5}{2}, 6$

→ $(r_1, r_2) = \left(-\dfrac{5}{2}, 10\right)$ or $\left(6, \dfrac{3}{2}\right)$ → $\begin{cases} x^2 - \dfrac{15}{2}x - 25 = 0 \\ x^2 - \dfrac{15}{2}x + 9 = 0 \end{cases}$ → $\underline{\begin{cases} \mathbf{2x^2 - 15x - 50 = 0} \\ \mathbf{2x^2 - 15x + 18 = 0} \end{cases}}$

3. $\begin{pmatrix} 2A & 2 & B \\ 3 & B & C \\ 2B & -A & -4 \end{pmatrix}\begin{pmatrix} -2 \\ 8 \\ 4 \end{pmatrix} = \begin{pmatrix} 12 \\ -14 \\ 3C+4 \end{pmatrix}$ → $\begin{cases} -4A + 16 - 4B = 12 \\ -6 + 8B - 4C = -14 \\ -4B - 8A + 16 = 3C + 4 \end{cases}$ → $\begin{cases} (1)\ A + B = 1 \\ (2)\ 2B - C = -2 \\ (3)\ 8A + 4B + 3C = 12 \end{cases}$

Substituting $A = 1 - B$ and $C = 2B + 2$ into (3), $8(1 - B) + 4B + 3(2B + 2) = 12$
→ $8 - 8B + 4B + 6B + 6 = 12$ → $2B = -2$ → $(A, B, C) = \underline{(2, -1, 0)}$

2010

1. For any 2 x 2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $A^{-1} = \dfrac{1}{\det(A)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

$A^{-1} = \begin{pmatrix} 3 & -4 \\ -2 & 6 \end{pmatrix}$ → $A = \dfrac{1}{18-8}\begin{pmatrix} 6 & 4 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} \frac{6}{10} & \frac{4}{10} \\ \frac{2}{10} & \frac{3}{10} \end{pmatrix}$ and $\begin{pmatrix} 3 & P \\ 1 & Q \end{pmatrix}\begin{pmatrix} -Q & -P \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -3Q & -4P \\ -Q & -P-Q \end{pmatrix}$

Equating, $-3Q = \dfrac{6}{10}$ → $Q = -\dfrac{1}{5}$ and $-4P = \dfrac{4}{10}$ → $P = -\dfrac{1}{10}$ and these values satisfy the requirements of the remaining positions → $(P, Q) = \underline{\left(-\dfrac{\mathbf{1}}{\mathbf{10}}, -\dfrac{\mathbf{1}}{\mathbf{5}}\right)}$.

Alternate Solution (using the identity matrix):

For any invertible 2 x 2 matrix $A \cdot A^{-1} = A^{-1} \cdot A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

$A = \begin{pmatrix} 3 & P \\ 1 & Q \end{pmatrix}\begin{pmatrix} -Q & -P \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -3Q & -4P \\ -Q & -P-Q \end{pmatrix}$

Therefore, $\begin{pmatrix} -3Q & -4P \\ -Q & -P-Q \end{pmatrix}\begin{pmatrix} 3 & -4 \\ -2 & 6 \end{pmatrix} = \begin{pmatrix} -9Q + 8P & 12Q - 24P \\ 2P - Q & -2Q - 6P \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Equating terms in the second row, $\left.\begin{matrix} 2P - Q = 0 \\ -2Q - 6P = 1 \end{matrix}\right\}$ → $-4P - 6P = 1$ → $P = -\dfrac{1}{10}$ etc.

2. $\tan^2 x - \cot x = \cot^2 x - \tan x$

$\rightarrow \tan^2 x - \cot x - \cot^2 x + \tan x = 0$

$\rightarrow (\tan x + \cot x)(\tan x - \cot x) + (-\cot x + \tan x) = 0$

$\rightarrow (\tan x - \cot x)(\tan x + \cot x + 1) = 0$

$(\tan x - \cot x) = 0$

$\tan x = \cot x$

$90° \le x < 270°$

$x = \cancel{45°}, 135°, 225°, \cancel{315°}$

$(\tan x + \cot x + 1) = 0$

$\tan x = -1 - \cot x = -1 - \dfrac{1}{\tan x}$

$\tan^2 x + \tan x + 1 = 0$

Since $B^2 - 4AC < 0$, there are no more solutions.

Thus, the only solutions are **<u>135°</u>** and **<u>225°</u>**.

3. Note that:
   a) Consecutive Fridays are 7 days apart
   b) Day 74 and day 284 fall on the same day of the week, since day 284 falls exactly 210 days or 30 weeks later in the year.
   c) Let $d$ denote the day of the year ($1 \le d \le 361$). More generally, days $d$, $d + 7$, $d + 14$, … $d + 7n$, where $n$ is an integer, fall on the same day of the week
   d) If two days $d_1$ and $d_2$ leave the same remainder when divided by 7, then they will fall on the same day of the week.

The following table gives the day of the year ($d$) for the 13th of each month and the remainder of that number when divided by 7.

| | JAN | FEB | MAR | APR | MAY | JUN | JUL | AUG | SEP | OCT | NOV | DEC |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $d$ | 13 | 43 | 74 | 104 | 135 | 164 | 194 | 224 | 254 | 284 | 314 | 344 |
| $r$ | 6 | 1 | 4 | 6 | 1 | 3 | 5 | 0 | 2 | 4 | 6 | 1 |

Applying d) above, the 13th of JAN, APR and NOV fall on the same day of the week, as do the 13th of FEB, MAY and DEC. All other remainders occur only once or twice.

Applying c) above, if JAN 13 is a FRI, then so is the JAN 6th and JAN "-1st".
JAN 1st would be two days later, i.e. on a SUN.
In FEB, day 43 = 6(7) + 1. If day 43 is a FRI, then so is day 1 (JAN 1st), which is exactly 6 weeks earlier.

Thus, **<u>SUN → JAN, APR, NOV</u>** and **<u>FRI → APR, MAY, DEC</u>**.