# Round 5 Trigonometry

## Trigonometric Equations

**GREATER BOSTON MATHEMATICS LEAGUE**

**MEET 2 – NOVEMBER 1998**

**ROUND 5** – Trig. Equations

1. ____________________

2. ____________________

3. ____________________

**CALCULATORS ARE NOT ALLOWED ON THIS ROUND**

1. Given $\mathbf{0° \leq \theta < 360°}$, solve the following equation for $\theta$:
$\tan\theta \cdot \sin\theta + \cos\theta = \sec\theta$

2. Given $\cos 2x = \tan^2 x$, find all values for $\mathbf{\cos x}$ in simplified radical form.

3. Given $\mathbf{0° \leq \theta \leq 180°}$, solve the following equation for $\theta$: $\sin\theta + \cos\theta = \frac{\sqrt{6}}{2}$

# GREATER BOSTON MATHEMATICS LEAGUE

## MEET 2 – NOVEMBER 1999

**ROUND 5** – Trig. Equations

1. ____________________

2. ____________________

3. ____________________

**CALCULATORS ARE NOT ALLOWED ON THIS ROUND**

1. Given $0° \leq x < 360°$ and $\sin x + \tan 60° \cos x = 0$, find all solutions for $x$.

2. Given $0° \leq x < 360°$ and $\cos 2x + \sin 2x = \sin 270°$, find all solutions for $x$.

3. Given $0° \leq x < 360°$ and $2 + 2\cos x = \dfrac{\sin x}{1 - \cos x}$, find all solutions for $x$.

## GREATER BOSTON MATHEMATICS LEAGUE

## MEET 2 – NOVEMBER 2000

**ROUND 5** – Trig. Equations

1. ____________________

2. ____________________

3. ____________________

**CALCULATORS ARE NOT ALLOWED ON THIS ROUND**

1. Given $0° \leq x < 360°$ and $\cos^2 x + \cos x \cdot \sin x = 0$, find all solutions for $x$.

2. Given $0° < x < 45°$ and $\sin x + \cos x = \frac{4}{3}$, compute $\cos 2x$ in simplest radical form.

3. Given $0° \leq x < 360°$ and $3\sec^4 x - 3\tan^4 x = 5$, find all solutions for $x$.

# GREATER BOSTON MATHEMATICS LEAGUE

## MEET 2 – NOVEMBER 2001

**ROUND 5** – Trig. Equations

1. ____________________

2. ____________________

3. ____________________

**CALCULATORS ARE NOT ALLOWED ON THIS ROUND**

1. If $\tan x = 5$, find the possible value(s) of $\dfrac{2\sin x + 3\cos x}{5\sin x + \cos x}$.

2. Given $0° \le x < 360°$ and $\tan x + \sec x = \cos x$, find all solutions for $x$.

3. Given $0° \le x < 360°$ and $\tan^3 x + \sec^2 x = 3\tan x + 4$, find the sum of all solutions for $x$.

# GREATER BOSTON MATHEMATICS LEAGUE
# MEET 2 – NOVEMBER 2006

## ROUND 5 – Trig Equations

1. ______________________

2. ______________________°

3. ______________________°

**CALCULATORS ARE NOT ALLOWED ON THIS ROUND.**
**Answers must be expressed in simplified rationalized form.**

1. Given: $\sin(2A) = \cos(A)$ and $0 \le A < 360°$
   Determine all possible numeric values of $\tan(A)$.

2. Find all values of $x$, $0 \le x < 360°$, for which

$$\sec^2(x) + \cos^2(180°) = \cos(0°) - 2\sec(x)$$

3. Find all values of $x$, $0 \le x < 360°$, for which

$$8\sin^2(x) + \csc^2(x) - 6 = 0 \ \ \underline{\textbf{and}} \ \ \sin(x)\cos(x) = \cos^2(x)$$

# GREATER BOSTON MATHEMATICS LEAGUE

## MEET 2 – NOVEMBER 2007

**ROUND 5** – Trig Equations

1. ______________________ °

2. ______________________ °

3. ______________________ °

## CALCULATORS ARE NOT ALLOWED ON THIS ROUND.

1. Determine all values(s) of $x$, $0° \leq x < 360°$ that satisfy $\dfrac{4\sin^2 x - 3}{2\cos x - 1} = 0$.

2. Find all values of $x$ such that $0° \leq x < 360°$ that satisfy: $\cos 2x + \sin 2x = \tan 675°$

3. Find all values of $A$ such that $0° \leq A < 360°$ that satisfy:
$\cos(270° - A) \cdot \sec(-A) = \cos(144°) \cdot \csc(324°)$

# GREATER BOSTON MATHEMATICS LEAGUE

## MEET 2 – NOVEMBER 2008

**ROUND 5** – Trig Equations

**Numerical answers below are expressed in degrees.**

1. ______________________________

2. ______________________________

3. ______________________________

## CALCULATORS ARE NOT ALLOWED ON THIS ROUND.

1. Solve for $x$ over $0 \leq x < 360°$: $\sin(x+36°) = \cos 48°$

2. Solve for $x$ over $0 \leq x < 360°$: $2\sin x \tan x + \sin x \sec x = \sec x$

3. Solve for $x$ over $0 \leq x < 360°$: $\sin 42° \cos 48° = \sin(2x - 48°) - \cos 138° \sin 228°$

# GREATER BOSTON MATHEMATICS LEAGUE

## MEET 2 – NOVEMBER 2009

**ROUND 5** – Trig Equations

1. ______________________

2. ______________________

3. ______________________

**CALCULATORS ARE NOT ALLOWED ON THIS ROUND.**
The degree symbol is not required in any answer.

1. Find all values of $x$, $0° \le x < 360°$, which satisfy the following system of equations

$$2\sin^2 x - \sin x - 1 = 0 \text{ and } 4\sin^2 x = 1$$

2. How many solutions, $-360° < \theta < 360°$, are possible when $\cos\theta + \tan\theta + \sec\theta = 0$.

3. Find all values of $\theta$, $0° \le \theta < 360°$, such that $4\sin^2\theta - 1 + \cos\theta\csc\theta - 4\sin\theta\cos\theta = 0$.

# GREATER BOSTON MATHEMATICS LEAGUE

## MEET 2 – NOVEMBER 2010

**ROUND 5** – Trig Equations

1. ______________________

2. ______________________

3. ______________________

## CALCULATORS ARE NOT ALLOWED ON THIS ROUND.

1. Given: $\sec^2 x = 4$ and $\sin x = -\dfrac{\sqrt{3}}{2}$, where $0° \le x < 360°$.

   Compute the <u>sum</u> of all values of $x$ (in degrees) for which this is true.

2. Given: $\begin{cases} \sin x = -\dfrac{2}{3}, & \cos x < 0 \\ \tan y = -\dfrac{\sqrt{20}}{4}, & \sin y < 0 \end{cases}$

   If $(\tan x)(\sin y) = \cos z,\ 180° < z < 360°$, compute $\tan z$.

3. Given: $\cos 2x + 5\sin^2 x = 4\sin x + 5$ and $180° < x < 360°$
   Compute $\cos x$.

# MASSACHUSETTS MATHEMATICS LEAGUE
## FEBRUARY 2004
## ROUND 3: TRIG. IDENTITIES OR INVERSES

ANSWERS

A)______________

B)______________

C)______________

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**A)** Simplify $\dfrac{(\cot\theta - \cos\theta)(1 + \sin\theta)}{\cos^3\theta}$ to the form T(θ) where T is one of the six trig functions.

**B)** For $0^0 \le \theta < 360^0$, solve $\dfrac{2\tan\theta}{1+\tan^2\theta} = \dfrac{\sqrt{3}}{2}$.

**C)** Using principle values, express $\cos(\sec^{-1}\dfrac{3}{2} - \cos^{-1}\dfrac{1}{5})$ in simple radical form.

**MASSACHUSETTS MATHEMATICS LEAGUE**
**FEBRUARY 2005**
**ROUND 3 TRIG: IDENTITIES & INVERSE FUNCTIONS**

**ANSWERS**

**A)**______________

**B)**______________

**C)**______________

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**A)** If $\sec^2(x) + \tan(x) = 1$ and $\tan(x) + \csc(x) = y$ find all exact real values for y.

**B)** Given $Sin(Sin^{-1}(2x+1)) = \dfrac{5}{27x+3}$ find exact values for all possible $x$.

**C)** Given $Cos^{-1}(Cos(2x+1)) = \dfrac{5}{x+2}$ find all possible rational values for $x$.

**MASSACHUSETTS MATHEMATICS LEAGUE**
**FEBRUARY 2006**
**ROUND 3 TRIG: IDENTITIES & INVERSE FUNCTIONS**

**ANSWERS**

**A)** $Y =$ ____________

**B)** ____________

**C)** ____________

A) Suppose Arctan( $\sqrt{x}$ ) = $d$, where $0° < d < 90°$. If $d =$ Arcsec( $Y$ ), express $Y$ in terms of $x$.

B) Simplify $\dfrac{\sin\theta}{2(1+\cos\theta)} + \dfrac{1+\cos\theta}{2\sin\theta}$ to obtain a single trigonometric function of $\theta$.

C) If $\sin(4\theta)$ is written in the form $A\sin\theta\cos\theta(B + C\sin^2\theta)$ for integers $A$, $B$ and $C$, find $A^2 + BC$.

**MASSACHUSETTS MATHEMATICS LEAGUE**
**CONTEST 5 – FEBRUARY 2007**
**ROUND 3 TRIG: IDENTITIES AND/OR INVERSE FUNCTIONS**

**ANSWERS**

A) _______________________

B) _______________________

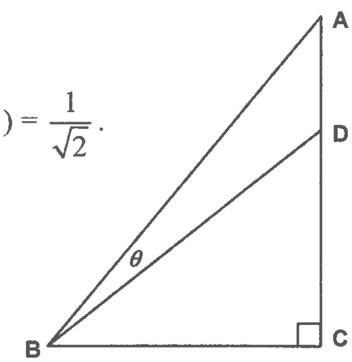
C) _______________________

A) Given $A = Sin^{-1}(\frac{35}{37})$, $B = Cos^{-1}(-\frac{15}{17})$
Find $\sin(A + B)$ as a simplified fraction.

B) In right $\Delta ABC$, $m\angle C = 90°$, $m\angle DBC = 45°$, $AB = 1 + \sqrt{2}$ and $\tan(\theta) = \frac{1}{\sqrt{2}}$.

Determine $BC$ in simplified radical form.

Hint: $BC = AB\cos(\angle ABC)$




C) Let $\theta = Arc\cos(\frac{1}{2x+1})$, where $x > 0$. Express the fraction $\frac{x^2+x}{2x+1}$ as a single simplified fraction in terms of $\cos(\theta)$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 – FEBRUARY 2008
ROUND 3 TRIG: IDENTITIES AND/OR INVERSE FUNCTIONS

ANSWERS

A) ______________________________

B) ______________________________

C) ______________________________

A) Solve for $\theta$, where $0° \leq \theta < 360°$, if $\csc(2\theta) + \cot(2\theta) = 1$

B) Given: $\cos(40°) = k$ and $\sin(x) = 1 - 2k^2$
What are the possible values of $x$ between 0° and 360° exclusive?

C) Determine the positive integer $n$ for which

$$\sin\left( Arc\cos\left(-\frac{n}{11}\right) + Arc\tan\left(-\frac{1}{2\sqrt{6}}\right)\right) = \frac{53}{55}$$

GBML 98

**ROUND 5**

1998

1. $\frac{\sin^2\theta}{\cos\theta} + \frac{\cos\theta}{1} = \frac{1}{\cos\theta}$, $\cos\theta \neq 0 \Rightarrow \frac{\sin^2\theta + \cos^2\theta}{\cos\theta} = \frac{1}{\cos\theta}$, which is an identity $\Rightarrow$
The equation is always true unless $\cos\theta = 0 \Rightarrow$ **$\theta \neq 90°$ and $\theta \neq 270°$**

2. $\cos 2x = \tan^2 x \Rightarrow 2\cos^2 x - 1 = \sec^2 x - 1 \Rightarrow 2\cos^4 x = 1 \Rightarrow \cos x = \pm\frac{1}{\sqrt[4]{2}} = \pm\frac{\sqrt[4]{8}}{2}$

3. $\sin\theta + \cos\theta = \frac{\sqrt{6}}{2} \Rightarrow (\sin\theta + \cos\theta)^2 = \left(\frac{\sqrt{6}}{2}\right)^2 \Rightarrow \sin^2\theta + \cos^2\theta + 2\sin\theta\cdot\cos\theta = \frac{3}{2}$

$\Rightarrow 2\sin\theta\cdot\cos\theta = \frac{1}{2} \Rightarrow \sin 2\theta = \frac{1}{2}$; Since $0° \le \theta \le 180°$, then $0° \le 2\theta \le 360° \Rightarrow$

$2\theta = 30°$ or $150° \Rightarrow \theta =$ **15°** or **75°** (which both check into the original equation)

GBML 99

1. $\sin x + \tan 60° \cos x = 0 \Rightarrow \sin x = -\sqrt{3}\cos x \Rightarrow \tan x = -\sqrt{3} \Rightarrow x = 120°, 300°$

1999

2. $\cos 2x + \sin 2x = \sin 270° \Rightarrow 2\cos^2 x - 1 + 2\sin x\cos x = -1 \Rightarrow 2\cos x(\cos x + \sin x) = 0$
$\Rightarrow \cos x = 0$ or $\tan x = -1 \Rightarrow x = 90°, 135°, 270°, 315°$

3. $2 + 2\cos x = \frac{\sin x}{1-\cos x} \Rightarrow 2(1+\cos x)(1-\cos x) = \sin x \Rightarrow 2(1-\cos^2 x) = \sin x \Rightarrow$

$2\sin^2 x - \sin x = 0 \Rightarrow \sin x(2\sin x - 1) = 0 \Rightarrow \sin x = 0$ or $\sin x = \frac{1}{2} \Rightarrow$

$x = 0°, 180°, 30°, 150°$, but $x = 0°$ is extraneous since $1 - \cos 0° = 0 \Rightarrow x = 30°, 150°, 180°$

GBML 00

1. $\cos^2 x + \cos x\cdot\sin x = 0 \rightarrow \cos x(\cos x + \sin x) = 0 \rightarrow \cos x = 0$ or $\sin x = -\cos x \rightarrow$
$\cos x = 0$ or $\tan x = -1 \rightarrow x = 90°, 135°, 270°, 315°$

2000

2. $\sin x + \cos x = \frac{4}{3} \rightarrow (\sin x + \cos x)^2 = \frac{16}{9} \rightarrow \sin^2 x + 2\sin x\cos x + \cos^2 x = \frac{16}{9} \rightarrow$

$1 + \sin 2x = \frac{16}{9} \rightarrow \sin 2x = \frac{7}{9} \rightarrow$ since $0° < 2x < 90°$, $\cos 2x = \sqrt{1 - \frac{49}{81}} = \sqrt{\frac{32}{81}} = \frac{4\sqrt{2}}{9}$

3. $3\sec^4 x - 3\tan^4 x = 5 \rightarrow 3(\sec^4 x - \tan^4 x) = 5 \rightarrow 3(\sec^2 x - \tan^2 x)(\sec^2 x + \tan^2 x) = 5$

$\rightarrow 3(1 + 2\tan^2 x) = 5 \rightarrow 3 + 6\tan^2 x = 5 \rightarrow \tan^2 x = \frac{1}{3} \rightarrow \tan x = \pm\frac{1}{\sqrt{3}} \rightarrow$

$x = 30°, 150°, 210°, 330°$

GBML 01

1. $\frac{2\sin x + 3\cos x}{5\sin x + \cos x} = \frac{2\tan x + 3}{5\tan x + 1} = \frac{2(5)+3}{5(5)+1} = \frac{13}{26} = \frac{1}{2}$

2. $\tan x + \sec x = \cos x \Rightarrow \frac{\sin x}{\cos x} + \frac{1}{\cos x} = \cos x \Rightarrow \sin x + 1 = \cos^2 x$, (and $\cos x \neq 0$) $\Rightarrow$
$\sin x + 1 = 1 - \sin^2 x \Rightarrow \sin^2 x + \sin x = 0 \Rightarrow \sin x(\sin x + 1) = 0 \Rightarrow \sin x = 0, -1 \Rightarrow x = 0°, 180°$
since $\sin x = -1 \Rightarrow x = 270° \Rightarrow \cos x = 0$

3. $\tan^3 x + \sec^2 x = 3\tan x + 4 \Rightarrow \tan^3 x + \tan^2 x + 1 = 3\tan x + 4 \Rightarrow$
$\tan^3 x + \tan^2 x - 3\tan x - 3 = 0 \Rightarrow \tan^2 x(\tan x + 1) - 3(\tan x + 1) = 0 \Rightarrow$
$(\tan^2 x - 3)(\tan x + 1) = 0 \Rightarrow \tan x = \pm\sqrt{3}, -1 \Rightarrow x = 60°, 120°, 240°, 300°, 135°, 315°$;
the sum of all values for $x = 1170°$.

GBML 06

1. $\sin(2A) = 2\sin A\cos A = \cos A \rightarrow \cos A(2\sin A - 1) = 0 \rightarrow A = 90°, 270°, 30°, 150°$
$\tan(30°) = +\boxed{\frac{\sqrt{3}}{3}}$ and $\tan(150°) = -\boxed{\frac{\sqrt{3}}{3}}$ (At 90° and 270° $\tan(A)$ is undefined.)

2. $\sec^2(x) + 1 = 1 - 2\sec(x) \rightarrow \sec^2(x) + 2\sec(x) = \sec(x)(\sec(x) + 2) = 0$
Since the value of $\sec(x)$ is never zero, the only roots occur when
$\sec(x) = -2$ [ or $\cos(x) = -1/2$ ] $\rightarrow$ 60° family in quadrants 2 and 3 $\rightarrow$ **<u>120°, 240°</u>**

3. Noting that $\csc^2(x)$ is equivalent to $\frac{1}{\sin^2 x}$, in the 1st equation, multiplying by $\sin^2 x$
$\rightarrow 8\sin^4 x - 6\sin^2 x + 1 = (4\sin^2 x - 1)(2\sin^2 x - 1) = 0$
$\rightarrow \sin x = \pm\frac{1}{2}$ (30, 150, 210, 330) or $\pm\frac{\sqrt{2}}{2}$ (45, 135, 225, 315)
The 2nd equation $\sin(x)\cos(x) = \cos^2(x) \rightarrow \cos x(\cos x - \sin x) = 0 \rightarrow$ 90, 270, 45, 225
Taking the intersection (overlap), we have **<u>45°, 225°</u>**.

GBML 07

**ROUND 5**

1. The numerator must be zero (and the denominator must not!).
$4\sin^2(x) = 3 \rightarrow \sin(x) = \pm\frac{\sqrt{3}}{2} \rightarrow 60°$ in all 4 quadrants
However, $\cos(x) \neq \frac{1}{2} \rightarrow x \neq 60°, 300°$
Thus, $x =$ **<u>120°, 240°</u>** only.

2. Square both sides and beware of introducing extraneous answers!
$\cos^2(2x) + 2\cos(2x)\sin(2x) + \sin^2(2x) = \tan^2(675 - 360) \rightarrow 1 + 2\cos(2x)\sin(2x) = \tan^2(315)$
$\rightarrow 1 + \sin(4x) = (-1)^2 = 1 \rightarrow \sin(4x) = 0 \rightarrow 4x = 0 + 180n$ or $x = 0 + 45n$
which would produce 8 answers over the interval $0° \le x < 360°$
However, half of these answers are extraneous, since the left side of the equation evaluates to +1.
Specifically, $x = 0°, 45°, 180°, 225°$ are extraneous, leaving **<u>90°</u>**, **<u>135°</u>**, **<u>270°</u>** and **<u>315°</u>**.

3. $\cos(270° - A)\cdot\sec(-A) = \cos(144°)\cdot\csc(324°)$
$\leftrightarrow -\sin A\cdot\frac{1}{\cos A} = \frac{\cos 144°}{\sin 324°} = \frac{-\cos 36°}{-\sin 36°} = \cot 36° = \tan 54°$
$\leftrightarrow -\tan A = \tan(-A) = \tan(54) \rightarrow -A = 54 + 180n \rightarrow A = -54, -234$
Adding 360° to produce results in the specified range $\rightarrow A =$ **126°, 306°**

GBML 08

1. $\cos 48^\circ = \sin 42^\circ$ or $\sin 138^\circ \rightarrow x + 36 = 42$ or $138 \rightarrow x = \underline{6}$ or $\underline{102}$

3. $\dfrac{2\sin^2 x}{\cos x} + \dfrac{\sin x}{\cos x} = \dfrac{1}{\cos x} \rightarrow 2\sin^2 x + \sin x - 1 = 0 \ (x \neq 90,\ 270) \rightarrow (2\sin x - 1)(\sin x + 1) = 0$

$\sin x = \dfrac{1}{2}$, $\sin x = -1 \rightarrow x = 30, 150, 270$, but 270 is rejected. Answer: $\underline{30^\circ, 150^\circ}$

4. $\sin 42^\circ \cos 48^\circ + \cos 138^\circ \sin 228^\circ = \sin(2x - 48^\circ) \rightarrow$

$\sin 42^\circ \cos 48^\circ + (-\cos 42^\circ)(-\sin 48^\circ) = \sin(2x - 48^\circ) \rightarrow$

$\sin 42^\circ \cos 48^\circ + \cos 42^\circ \sin 48^\circ = \sin(2x - 48^\circ) \rightarrow$

$\sin(42^\circ + 48^\circ) = \sin 90^\circ = 1 = \sin(2x - 48^\circ) \rightarrow 90^\circ, 450^\circ$

$\rightarrow x = \underline{69}, \underline{249}$

GBML 09

1. $2\sin^2 x - \sin x - 1 = 0 \rightarrow (2\sin x + 1)(\sin x - 1) = 0 \rightarrow \sin x = -\dfrac{1}{2}, +1$

$4\sin^2 x = 1 \rightarrow \sin x = \pm\dfrac{1}{2}$ Thus, the only common solution is $\sin x = -\dfrac{1}{2} \rightarrow \underline{\mathbf{210,\ 330}}$

2. $\cos\theta + \tan\theta + \sec\theta = 0 \rightarrow \cos\theta + \dfrac{\sin\theta}{\cos\theta} + \dfrac{1}{\cos\theta} = 0 \rightarrow \dfrac{\cos\theta}{1} = \dfrac{-(\sin\theta + 1)}{\cos\theta}$

Cross multiplying, $-\sin\theta - 1 = \cos^2\theta = 1 - \sin^2\theta$

$\rightarrow \sin^2\theta - \sin\theta - 2 = (\sin\theta - 2)(\sin\theta + 1) = 0 \rightarrow \sin\theta = -1 \rightarrow \theta = 270$

But $\theta = 270 \rightarrow \cos\theta = 0 \rightarrow \sec\theta$ is undefined

Therefore, there are $\underline{0}$ solutions.

3. $4\sin^2\theta - 1 + \cos\theta\csc\theta - 4\sin\theta\cos\theta = 0 \rightarrow$

$(4\sin^2\theta - 1) + \cos\theta\left(\dfrac{1}{\sin\theta} - 4\sin\theta\right) = (4\sin^2\theta - 1) + \cos\theta\left(\dfrac{1 - 4\sin^2\theta}{\sin\theta}\right)$

$= (4\sin^2\theta - 1) - \dfrac{\cos\theta}{\sin\theta}(4\sin^2\theta - 1) = (4\sin^2\theta - 1)(1 - \cot\theta) = 0$

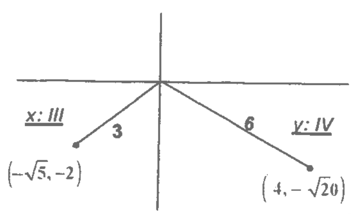
$\rightarrow \sin\theta = \pm\dfrac{1}{2}, \cot\theta = 1 \rightarrow \underline{\mathbf{30,\ 45,\ 150,\ 210,\ 225,\ 330}}$

GBML 10

1. Over $0^\circ \le x < 360^\circ$, $\sec x = \pm 2 \rightarrow x = 60^\circ, 120^\circ, \underline{240^\circ}, \underline{300^\circ}$ and

$\sin x = -\dfrac{\sqrt{3}}{2} \rightarrow x = \underline{240^\circ}, \underline{300^\circ}$. Summing, the total is $\underline{540^\circ}$.

2. $(\tan x)(\sin y) = \left(+\dfrac{2}{\sqrt{5}}\right)\left(-\dfrac{\sqrt{5}}{3}\right) = -\dfrac{2}{3}$




If $\cos z = -\dfrac{2}{3}$ and $180^\circ < z < 360^\circ$,

then $z$ lies in quadrant 3,

$(x, y, r) = (-2, -\sqrt{5}, 3)$ and $\tan z = +\underline{\dfrac{\sqrt{5}}{2}}$.

3. $(1 - 2\sin^2 x) + 5\sin^2 x = 4\sin x + 5$

$\rightarrow 3\sin 2x - 4\sin x - 4 = (3\sin x + 2)(\sin x - 2) = 0$

$\rightarrow \sin x = -\dfrac{2}{3}$ and $180^\circ < x < 360^\circ \rightarrow (x, y, r) = (\pm\sqrt{5}, -2, 3) \rightarrow \cos x = \pm\underline{\dfrac{\sqrt{5}}{3}}$

**MASSACHUSETTS MATHEMATICS LEAGUE**
**FEBRUARY 2004**
**ROUND 3: TRIG. IDENTITIES OR INVERSES**
**ANSWERS**

**A)** $\csc\theta$

**B)** $30^\circ, 60^\circ, 210^\circ, 240^\circ$

**C)** $(2 + 2\sqrt{30})/15$

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A) Simplify $\dfrac{(\cot\theta - \cos\theta)(1 + \sin\theta)}{\cos^3\theta}$ to the form T(θ) where T is one of the six trig functions.

$\dfrac{\left(\dfrac{\cos\theta}{\sin\theta} - \cos\theta\right)(1 + \sin\theta)}{\cos^3\theta} = \dfrac{(\cos\theta - \sin\theta\cos\theta)(1 + \sin\theta)}{\sin\theta\cos^3\theta}$

$= \dfrac{\cos\theta(1 - \sin^2\theta)}{\sin\theta\cos^3\theta} = \dfrac{1}{\sin\theta} = \csc\theta$

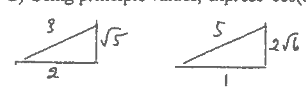
B) For $0^\circ \le \theta < 360^\circ$, solve $\dfrac{2\tan\theta}{1 + \tan^2\theta} = \dfrac{\sqrt{3}}{2}$.

$\dfrac{\dfrac{2\sin\theta}{\cos\theta}}{\sec^2\theta} = \dfrac{2\sin\theta}{\cos\theta} \cdot \dfrac{\cos^2\theta}{1} = 2\sin\theta\cos\theta = \sin 2\theta = \dfrac{\sqrt{3}}{2}$

$2\theta = 60^\circ, 120^\circ, 420^\circ, 480^\circ$

$\theta = 30^\circ, 60^\circ, 210^\circ, 240^\circ$

C) Using principle values, express $\cos(\sec^{-1}\frac{3}{2} - \cos^{-1}\frac{1}{5})$ in simple radical form.




$\cos A \cos B + \sin A \sin B =$

$\dfrac{2}{3} \cdot \dfrac{1}{5} + \dfrac{\sqrt{5}}{3} \cdot \dfrac{2\sqrt{6}}{5} = \dfrac{2 + 2\sqrt{30}}{15}$

MML 2005

**Round Three:**

A. $(\tan^2(x) + 1) + \tan(x) = 1 \Rightarrow \tan(x)=0$ (but then csc(x) undefined) OR tan(x)= –1 so $\csc(x) = \pm\sqrt{2}$

B. $2x+1 = \dfrac{5}{27x+3}$ so $54x^2 + 33x - 2 = 0$ so x = -2/3 or 1/18. Since 2x + 1 must be in the domain of the $\text{Sin}^{-1}$ function, only –2/3 is valid.

C. $\dfrac{5}{x+2} = \pm(2x+1) + 2n\pi$ but x must be rational so n=0. $\dfrac{5}{x+2} = -(2x+1)$ yields no real solutions. $\dfrac{5}{x+2} = 2x+1$ gives $5 = 2x^2+5x+2$ so x = ½ or –3 making $\dfrac{5}{x+2} = 2$ or $-5$. Only the first is in the range of $\text{Cos}^{-1}$ so the only solution is ½

MML 2006

**Round Three:**

A. Right triangle has opposite side $\sqrt{x}$, adjacent 1, hypotenuse $\sqrt{1+x}$

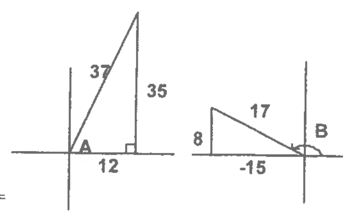
B. Common denominator gets
$$\frac{\sin^2\theta + (1+\cos\theta)^2}{2\sin\theta(1+\cos\theta)} = \frac{\sin^2\theta + \cos^2\theta + 1 + 2\cos\theta}{2\sin\theta(1+\cos\theta)} = \frac{2(1+\cos\theta)}{2\sin\theta(1+\cos\theta)} = \frac{1}{\sin\theta}.$$

C. $\sin(2\cdot 2\theta) = 2\sin(2\theta)\cos(2\theta) = 2(2\sin\theta\cos\theta)(1-2\sin^2\theta) = 4\sin\theta\cos\theta(1+-2\sin^2\theta)$, so $A = 4$, $B = 1$, $C = -2$.

MML 2007

**Round 3**

A) $\sin(A+B) = \sin A\cos B + \cos A\sin B = \dfrac{35}{37}\cdot\dfrac{-15}{17} + \dfrac{12}{37}\cdot\dfrac{8}{17}$
$= \dfrac{-525+96}{629} = -\boxed{\dfrac{429}{629}}$

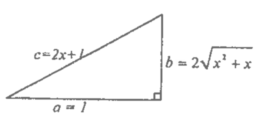



B) $\tan(\theta) = \dfrac{1}{\sqrt{2}}$ for an acute angle $\theta \rightarrow (\cos(\theta), \sin(\theta)) =$
$\left(\dfrac{\sqrt{2}}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}\right)$

$\cos(\theta + 45°) = \cos(\theta)\cos(45°) - \sin(\theta)\sin(45°) = \dfrac{\sqrt{2}}{\sqrt{3}}\cdot\dfrac{1}{\sqrt{2}} - \dfrac{1}{\sqrt{3}}\cdot\dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}-1}{\sqrt{6}}$

$BC = AB\cos(\theta + 45°) = \dfrac{(\sqrt{2}+1)(\sqrt{2}-1)}{\sqrt{6}} = \dfrac{1}{\sqrt{6}} = \boxed{\dfrac{\sqrt{6}}{6}}$

C)




Then $\dfrac{x^2+x}{2x+1} = \dfrac{b^2/4}{c} = \dfrac{1}{4}\cdot\dfrac{b}{1}\cdot\dfrac{b}{c} = \dfrac{1}{4}\tan\theta\sin\theta = \dfrac{1}{4}\dfrac{\sin^2\theta}{\cos\theta} = \boxed{\dfrac{1-\cos^2\theta}{4\cos\theta}}$

MML 2008

**Round 3**

A) $\dfrac{1}{\sin 2\theta} + \dfrac{\cos 2\theta}{\sin 2\theta} = \dfrac{1+\cos 2\theta}{\sin 2\theta} = \dfrac{1+2\cos^2\theta - 1}{2\sin\theta\cos\theta} = \dfrac{\cos\theta}{\sin\theta} = \cot(\theta) = 1$
$\rightarrow \theta =$ **<u>45°, 225°</u>**

B) Using the double angle formula, $\sin(x) = 1 - 2\cos^2 40° = -(2\cos^2 40° - 1) = -\cos(80°) = -\sin(10°)$
The related values of 10° in quadrants II, III and IV are 170°, 190° and 350°.
Since $\sin(x)$ is negative only in quadrants III and IV, $x =$ **<u>190° or 350°</u>**.

C) Let $A = Arc\cos(-n/11)$ and $B = Arc\tan\left(-1/\left(2\sqrt{6}\right)\right)$.
Then $\pi/2 < A < \pi$ (quadrant 2) and $-\pi/2 < B < 0$ (quadrant 4)
and $\sin(A+B) = \sin A\cos B + \sin B\cos A = \dfrac{\sqrt{121-n^2}}{11}\cdot\dfrac{2\sqrt{6}}{5} + \dfrac{-1}{5}\cdot\dfrac{-n}{11} = \dfrac{2\sqrt{6}\sqrt{121-n^2}+n}{55}$

Thus, $2\sqrt{6}\sqrt{121-n^2} + n = 53$ and the radicand $121 - n^2$ must be 6 times a perfect square
Additionally, since $n/11$ is a cosine value, the only possible integer values of $n$ are 1 ... 11.
Only $n = \underline{5}$ satisfies both conditions ($2\sqrt{6}\sqrt{96} + 5 = 2\cdot 6\cdot 4 + 5 = 53$).