Round 2 Coordinate Geometry of the Straight Line

GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 – DECEMBER 1998

ROUND 2 – Coordinate Geometry of the Straight Line

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Given line l: $3y - 4mx = 6m$ with the sum of its x intercept and y intercept equaling -6, find all values of m which satisfy these conditions.

2. Given the three lines $l_1 : y = (3k+2)x + 1$, $l_2 : y = (6k+1)x + 2$, and $l_3 : y = mx + 3$, where l_2 and l_3 are parallel and l_1 and l_2 are perpendicular, find all possible values for m.

3. Find the area bounded by the y axis and the lines $3x - 2y = 12$ and $3x + 4y = 48$.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 – DECEMBER 1999

ROUND 2 – Coordinate Geometry of the Straight Line

1. (,)

2. ____________

3. ____________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Given line L: $\{(x, y) : 2x - 3y - 24 = 0\}$, find the coordinates of the midpoint of the line segment cut off line L by the coordinate axes.

2. If line λ is the reflection of the line $3x - 2y + 8 = 0$ about the line $x = 4$, find the y-intercept of line λ.

3. If lines L_1: $ax + 3y = 31$, L_2: $5x - 2y = 26$, and L_3: $3x - 4y = 24$ are concurrent, find the area of the triangle formed by lines L_1, L_2 and the x axis.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 – DECEMBER 2000

ROUND 2 – Coordinate Geometry of the Straight Line

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Given line $\ell, \{(x,y) | 2x - 3y = 12\}$, find the area of the triangle formed by line ℓ, the line $x = 3$ and the line $y = 2$.

2. Given points $A(-4,5), B(2,-7)$, and P, on $\overline{AB}$ such that $AP : PB = 1:2$. Line L is drawn through point P perpendicular to $\overline{AB}$. Find the x-intercept of line L.

3. Given line $L_1, \{(x, y) | 3x - 4y = 24\}$ and point $P(9,-8)$, line L_2, with negative slope, is drawn through point P making a 45° angle with the x axis. Find the area of the quadrilateral formed by lines L_1, L_2, and the coordinate axes.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 – DECEMBER 2001

ROUND 2 – Coordinate Geometry of the Straight Line

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. A triangle is formed by the intersection of line $\ell:\{(x,y)|2x-3y-36=0\}$ and the coordinate axes. The line $y=mx$ divides this triangle into two triangles with equal area. Find the value for m.

2. Given points $P(-5,-1)$ and $Q(7,14)$, point R is on $\overline{PQ}$ such that $PR:RQ=1:2$, and S is a point on the x axis such that $\overline{RS}$ is perpendicular to $\overline{PQ}$, find the first coordinate of point S.

3. Given point $P(-2a,a+4)$ lies on line L, $\{(x,y)|3ax+7y=5a\}$, solve for a.

GREATER BOSTON MATHEMATICS LEAGUE
MEET 3 – DECEMBER 2005

ROUND 2 –Coordinate Geometry of the Straight Line

1. a = ______ b = ______

2. ________ : ________

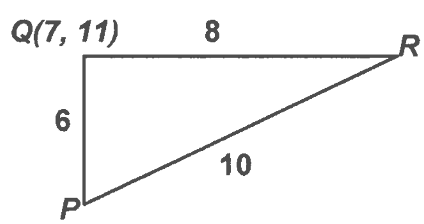
3. (________, ________)

CALCULATORS ARE NOT ALLOWED ON THIS ROUND.

1. Line L_1: $2ax + 3by + 1 = 0$ and Line L_2: $3ax + 2by + 24 = 0$ intersect at point $P(7, 3)$. Find the values of a and b.

2. Line L_1: $4x - 9y - 8 = 0$
   Line L_2: $10x + 6y - 1 = 0$
   Line L_3: $Px + Ty = 0$
   These three lines are concurrent, i.e. intersect at a common point.
   Find the simplified ratio $P : T$.

3. If $\overline{PQ}$ is vertical, determine the coordinates of the foot of the altitude drawn from Q to the side $\overline{PR}$.



GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 – DECEMBER 2006

ROUND 2 – Coordinate Geometry of the Straight Line

1. ______________ : ______________

2. ______________________________

3. ______________________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND.

1. Given: $A(-2, 2)$ and $B(13, 2)$
   The segment $\overline{AB}$ is intersected by the line L_1: $2x + y - 9 = 0$ at point P.
   Find the ratio $AP : PB$.

2. The area of the region bounded by $\begin{cases} x+2y=12 \\ 2x+y=12 \\ y=0 \end{cases}$ is k square units. Find k.

3. Find all ordered pairs (J, K) for which the point $P(-1, J)$ is the point of intersection of

   $$L_1\text{: } 4x + Jy = 5 \text{ and } L_2\text{: } Jx + Ky = 18.$$

GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 – DECEMBER 2007

ROUND 2 – Coordinate Geometry of the Straight Line

1. ____________________ square units

2. __________________________ units

3. ______________________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND.

1. The line containing point $A(-3,0)$ with slope $\frac{1}{2}$ intersects the line containing point $B(5,0)$ with slope $\frac{3}{2}$ at point P. Find the area of ΔABP.

2. A perpendicular line from the point $A(8,2)$ to the line $L_1 : 4x + 3y = 13$ intersects L_1 at M. Determine the exact distance from point M to the origin.

3. The lines $y = kx + 1$ and $y = (2/5)x - k$ intersect at $(5, t)$. Find <u>all</u> possible values of t.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 – DECEMBER 2008

ROUND 2 – Coordinate Geometry of the Straight Line

1. ______________________________

2. ____________________ square units

3. ______________________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND.

1. Find <u>all</u> values of A for which the following is true:
   Line L_1 has equation $Ax - 7y = 2A$ and the point $P(3A, 3) \in L_1$.

2. Given: L_1 with equation $2x + 3y = 12$. Line L_2 is perpendicular to L_1 and passes through the midpoint of line segment connecting its intercepts. A triangle is formed by L_1, L_2 and the x-axis. What is the area of this triangle?

3. Find <u>all</u> values of a so that the line $x = a$, the line $3x - 2y = 9$ and the x-axis form a triangle whose area is $\frac{25}{3}$ square units.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 – DECEMBER 2009

ROUND 2 – Coordinate Geometry of the Straight Line

1. ________________

2. ________________

3. (________ , ________)

1. Find the perimeter of the triangle whose vertices are the origin, and the intercepts of the line $L:\{(x,y)\,|\,3x-4y=20\}$.

2. L_1: $\{(x,y)\,|\,3x-4y=7\}$ and $L_2:\{(x,y)\,|\,7x+5y=45\}$
L_3 is perpendicular to L_1 at the point of intercsection of L_1 and L_2. Line L_4 is parallel to L_2 and contains the point $Q(13,-9)$. Find the coordinates of point P if $P \in L_3 \cap L_4$

3. Given: $M(8,7)$, $A(k,j)$ and $B(-3k,5j)$,
$AM+MB=AB$, $AM:MB=3:1$.
Find the ordered pair $(kj, k-1)$.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 – DECEMBER 2010

ROUND 2 – Coordinate Geometry of the Straight Line

1. ______________________

2. ______________________

3. ______________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND.

1. Given: $A(-4, 6)$ and $B(8, -14)$, line $\mathcal{L}_1$ is perpendicular to $\overline{AB}$ at its midpoint. Find the value of the x-intercept for line $\mathcal{L}_1$.

2. Given: $\mathcal{L}_1 : \{(x, y) \mid 3x + 2y - 5 = 0\}$, $\mathcal{L}_2 : \{(x, y) \mid 4x - 3y - 18 = 0\}$ and $\mathcal{L}_3 : \{(x, y) \mid 2x - 3y - 24 = 0\}$
   Point $P \in \mathcal{L}_1 \cap \mathcal{L}_2$ and point $Q \in \mathcal{L}_2 \cap \mathcal{L}_3$.
   Find the length of segment $\overline{PQ}$.

3. $P(A, B)$, $Q(-5, -9B)$, $R(-14, 3A)$, where point Q lies on $\overline{PR}$.
   Compute the coordinates of point P, if $PQ : QR = 5 : 3$.

MASSACHUSETTS MATHEMATICS LEAGUE
OCTOBER 2005
ROUND 3 ALG 1: LINEAR EQUATIONS

ANSWERS

A)______________

B)____________ mph

C)______________

A) If $[r]$ represents the largest integer which is less than or equal to r and

$$\frac{x+2}{9} - \frac{x-1}{16} = 1 \quad \text{find} \quad [x]$$

B) I took 6 hours to reach my destination. After averaging 60 mph for the first 2.5 hours, bad weather forced me to reduce my speed for the remainder of the trip. If my overall average speed was 39 mph, what was my average speed for the second part of my trip?

C) Suppose for all x, $3Ax - ABx + 15 - 5B = 13(4x + 5)$ with A and B real numbers. Find the value of A+B.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2006
ROUND 3 ALG 1: LINEAR EQUATIONS

ANSWERS

A) ______________________________

B) ______________________________

C) $ ____________________________

A) Farmer Euclid MacDonald states: "I have 360 livestock, horses and chickens. The total number of legs, excluding my own, is 1100." How many chickens does farmer MacDonald have?

B) If (2006, b) and (a, 2006) are two points on the line $y = \frac{1}{4}x + 17$, what is the numeric value of $\frac{2006-a}{2006-b}$?

C) A shopper went into 10 stores and at each spent half the money she had upon entering the store PLUS an additional 25¢. Her purchases at the last store took all the money she had left. How much money did she have after leaving the third store?

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2007
ROUND 3 ALG 1: LINEAR EQUATIONS

ANSWERS

A) ______________________

B) ______________________

C) ______________________

A) Find a simplified expression for the value of x in terms of a and b, given $a + b \neq 0$:
$a(a - 2x) = b(b + 2x)$

B) For how many ordered pairs of <u>positive</u> integers does $x - \frac{11-y}{3} = 18$

C) What is the original cost of a dozen eggs, if buying an additional 4 eggs for 32¢ lowers the cost per dozen by 4¢?

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2010
ROUND 3 ALG 1: LINEAR EQUATIONS

ANSWERS

A) _______________________ lbs.

B) _______________________ units^2

C) _______________________

***** NO CALCULATORS ON THIS ROUND ****

A) 8 lbs of a mixture of grass seed and lime is 72% lime. A second mixture of grass seed and lime is 57% lime. How many lbs of the second mixture when combined with the first mixture will produce a mixture that is 65% lime?

B) Given: $L = \left\{ (x, y) : \frac{x}{A} + \frac{y}{B} = 1 \right\}$

If $\begin{cases} A + B = 127 \\ A - B = 7 \end{cases}$, compute the area of the region bounded by L, the vertical line $x = 0$ and the horizontal line $y = 0$.

C) Given: $\frac{1}{2}y = \frac{2}{3}x + \frac{3}{5}$

Solving for x in terms of y, we get $x = \frac{Ay + B}{C}$, where $A < 0$ and A, B and C are integers.

Compute $\frac{ABC}{AB + AC}$

MASSACHUSETTS MATHEMATICS LEAGUE
DECEMBER 2004
ROUND 3 ANALYTIC GEOM OF LINE

ANSWERS

A)______________

B)______________

C)______________

Definition: A lattice point is one whose coordinates are each integers.

A) A line passes through P(– 8, 3) with slope $\frac{-5}{2}$. Moving from P to the right along the line, what are the coordinates of the next lattice point on the line?

B) A line whose equation is $7x - 3y = c$ passes through the lattice points (9, 16) and (a, b) where $a > 100$. Find the minimum possible value of the sum $a + b$.

C) A triangular region is bounded by the lines $y = 0$, $3x - 2y = 0$, and $3x + 4y = 108$. Find the number of lattice points <u>strictly</u> in the interior of the triangle (that is, do <u>not</u> count points on the boundary.)

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2006
ROUND 3 ANALYTIC GEOMETRY OF THE STRAIGHT LINE

ANSWERS

A) ______________ : ______________

B) ______________________________

C) $x =$ __________ , $y =$ ___________

A) The segment connecting $A(1, 8)$ and $B(6, -2)$ crosses the x-axis at point P. Determine the ratio BP: AP.

B) Given: $A(0, 2006)$ and $B(4250, 0)$
The point $C(p, q)$ is the point on $\overline{AB}$ with integer coordinates that is closest to, but different from, point A.
The point $D(r, s)$ is the point on $\overline{AB}$ with integer coordinates that is closest to, but different from, point B.

Find $p + q + r + s$

C) ΔPQR has vertices at $P(-12, 0)$, $Q(14, 0)$ and $R(2, 42)$. There is a single point $S(x, y)$ in the interior of ΔPQR that is equidistant from points P, Q and R. Find the numerical values of x and y.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2007
ROUND 3 ANALYTIC GEOMETRY OF THE STRAIGHT LINE

ANSWERS

A) ______________________

B) (________ , ________)

C) ______________________

A) L_1 is a line with equation $2x - ay = 7$.
L_2 is a line with equation $ax - 4y - 12 = 0$.
L_1 intersects L_2 at the point $P(8, a)$. Find all possible values of a.

B) A line segment has endpoints at $A(-6, 10)$ and $B(29, -18)$.
Find the coordinates of the point P that is 5/7 of the way from A to B.

C) The line perpendicular to $3x + 2y - 13 = 0$ at $(1, 5)$ passes through the points $P(a, b)$ and $Q(b, a)$.
Compute the distance between P and Q.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2009
ROUND 3 COORDINATE GEOMETRY OF LINES AND CIRCLES

******* NO CALCULATORS IN THIS ROUND *******

ANSWERS

A) (____, ____) (____, ____)

B) ________________________

C) ________________________ = 0

A) The line $4x-3y-11=0$ passes through the center of the circle $(x-2)^2+(y+1)^2=25$
Determine the coordinates of the two points of intersection.

B) Given $A(-2.9, 5.9)$, $B(0.3, k)$ and $AB = 3.2\sqrt{5}$
Determine all possible values of k.

C) Three vertices of parallelogram $PQRS$ are $P(2, 1)$ $Q(6,11)$ and $S(12, 9)$.
Determine the equation of $\overleftrightarrow{PR}$, in $ax + by + c = 0$ form, where a, b and c are integers, $a > 0$ and $\text{GCF}(a, b, c) = 1$.

GBML '98

ROUND 2

1. $l: 3y - 4mx = 6m$: If $x = 0$, then $y = 2m$; if $y = 0$, then $x = -\frac{3}{2} \Rightarrow 2m - \frac{3}{2} = -6 \Rightarrow m = -\frac{9}{4}$

2. $(3k+2)(6k+1) = -1 \Rightarrow 18k^2 + 15k + 3 = 0 \Rightarrow 6k^2 + 5k + 1 = 0 \Rightarrow$
$(3k+1)(2k+1) = 0 \Rightarrow k = -\frac{1}{2}$ or $-\frac{1}{3} \Rightarrow m = 6\left(-\frac{1}{3}\right) + 1$ or $6\left(-\frac{1}{2}\right) + 1 = $ **-1 or -2**

3. The y intercepts of the two lines are -6 and 12. To find the x coordinate of the point of intersection: $6x - 4y = 24$ and $3x + 4y = 48 \Rightarrow 9x = 72 \Rightarrow x = 8$. The area of the triangle $= \frac{1}{2}(12 - (-6))8 = 72$

GBML '99

ROUND 2

1. $x = 0: -3y - 24 = 0 \Rightarrow y = -8$; $y = 0: 2x - 24 = 0 \Rightarrow x = 12$; The midpoint of the segment whose endpoints are (0, -8) and (12, 0) is (6, -4).

2. $3x - 2y + 8 = 0$ when $x = 4 \Rightarrow y = 10$; the slope of the line is $\frac{3}{2} \Rightarrow$ the slope of the reflection of this line $= -\frac{3}{2} \Rightarrow$ the equation of the line is: $y - 10 = -\frac{3}{2}(x - 4)$; the y-intercept of this line is 16.

3. Since $ax + 3y = 31$, L_2: $5x - 2y = 26$, and L_3: $3x - 4y = 24$ are concurrent, find the intersection of L_2 and L_3; $-10x + 4y = -52 \Rightarrow -7x = -28 \Rightarrow x = 4$ *and* $y = -3 \Rightarrow 4a - 9 = 31 \Rightarrow a = 10$; the x-intercept of L_1 is 3.1 and the the x-intercept of L_2 is 5.2 $\Rightarrow$ the area of the triangle $= 0.5(5.2 - 3.1)3 = 3.15$ $\left(\frac{63}{20} \text{ or } 3\frac{3}{20}\right)$

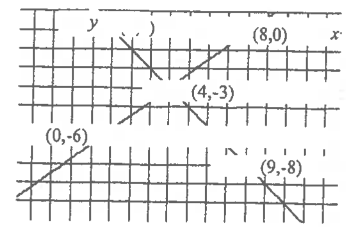
GBML '00

ROUND 2

1. When $x = 3$: $6 - 3y = 12 \rightarrow y = -2$; when $y = 2: 2x - 6 = 12 \rightarrow x = 9 \rightarrow$ vertices of the triangle are $A(3,2)$, $B(3,-2)$, and $C(9,2) \rightarrow$ area of $\Delta ABC = \frac{1}{2} \cdot 4 \cdot 6 = 12$

2. slope of line $L \perp$ to $\overline{AB} = -\left(\frac{-7-5}{2+4}\right)^{-1} = \frac{1}{2}$; $P = \left(\frac{2(-4)+1(2)}{1+2}, \frac{2(5)+1(-7)}{1+2}\right) = (-2,1)$; line L: $y - 1 = \frac{1}{2}(x+2) \rightarrow y = \frac{1}{2}x + 2 \rightarrow (-4,0)$ on L.

3. line $L_2: y + 8 = -1(x - 9) \rightarrow y = -x + 1 \rightarrow (1,0)$ is its x-intercept; $3x - 4(-x+1) = 24 \rightarrow 7x = 28 \rightarrow (4,-3)$ is the point of intersection of L_1 and L_2; area of the quadrilateral $= \frac{1}{2} \cdot 6 \cdot 8 - \frac{1}{2} \cdot 7 \cdot 3 = \frac{27}{2}$



GBML '01

ROUND 2 – Coordinate Geometry of the Straight Line

1. The line ℓ intersects the axes at points P (18,0) and Q (0,−12). For the line $y = mx$ to divide the triangle into two triangles with equal area the line would pass through the midpoint of $\overline{PQ}$ which is $(9,-6) \Rightarrow m = \frac{-6}{9} = -\frac{2}{3}$.

2. To find the coordinates of S: $x = \frac{2(-5)+1(7)}{1+2} = -1$ and $y = \frac{2(-1)+1(14)}{1+2} = 4$. The slope of $\overline{PQ} = \frac{14+1}{7+5} = \frac{15}{12} = \frac{5}{4} \Rightarrow$ slope of $\overline{RS} = -\frac{4}{5} \Rightarrow$ equation of line $\overline{RS}$ is $y - 4 = -\frac{4}{5}(x+1) \Rightarrow$ If $y = 0 \Rightarrow -4 = -\frac{4}{5}(x+1) \Rightarrow 5 = x + 1 \Rightarrow x = 4$.

3. Substituting the coordinates of P into the equation of L: $3a(-2a) + 7(a+4) = 5a \Rightarrow -6a^2 + 7a + 28 = 5a \Rightarrow 6a^2 - 2a - 28 = 0 \Rightarrow 3a^2 - a - 14 = 0 \Rightarrow (3a-7)(a+2) = 0 \Rightarrow a = -2, \frac{7}{3}$

GBML '05

ROUND 2 – Coordinate Geometry of the Straight Line

1. Since $P(7, 3)$ is a point on both lines, the coordinates must satisfy both equations. Substituting $x = 7$ and $y = 3$ we have a system of two equations in two unknowns.

$14a + 9b = -1$ $28a + 18b = -2$
$21a + 6b = -24 \rightarrow -63a - 18b = 72 \rightarrow -35a = 70 \rightarrow$ $\underline{a = -2}$
Substituting, $14(-2) + 9b = -1 \rightarrow 9b = 27 \rightarrow$ $\underline{b = 3}$

Or, using **Cramer's Rule,** $a = \frac{\begin{vmatrix} -1 & 9 \\ -24 & 6 \end{vmatrix}}{\begin{vmatrix} 14 & 9 \\ 21 & 6 \end{vmatrix}}$ and $b = \frac{\begin{vmatrix} 14 & -1 \\ 21 & -24 \end{vmatrix}}{\begin{vmatrix} 14 & 9 \\ 21 & 6 \end{vmatrix}}$

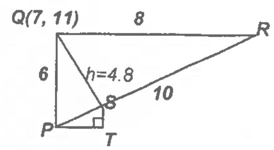
$a = 210/-105 = -2$ and $b = -315/-105 = 3$

Recall:
Notice the denominator (which is the same for both variables) is just the ***determinant*** of the ***matrix of coefficients***. The numerator is the determinant of the same matrix EXCEPT the coefficients of the variable being solved for are replaced by the constants on the right hand side of each equation.

The determinant of any 2 x 2 matrix $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ is $ad - bc$.

2. Using either of the methods above, the coordinates of the point of intersection of L_1 and L_2 are (1/2, -2/3). (If you use Cramer's Rule, be sure the equations are in the form $Ax + By = C$, i.e. the constant term is on the right hand side of the equation.) Substituting these coordinates in the equation for L_3 we have $P/2 - 2T/3 = 0 \rightarrow 3P - 4T = 0$ $\rightarrow 3P = 4T \rightarrow P : T =$ **4 : 3**

3. Area = ½(6)(8) = ½(10)(h) → $h = 4.8$



$P(7, 5)$

$\Delta PST \sim \Delta QPS \sim \Delta PRQ \sim$ (3-4-5) by AA.

In right triangle QPS, (____, 4.8, 6) = 1.2(____, 4, 5) → $PS = 1.2(3) = 3.6$

Since corresponding sides of similar triangles are in the same ratio, $\frac{3.6}{5} = \frac{PT}{3} = \frac{ST}{4}$

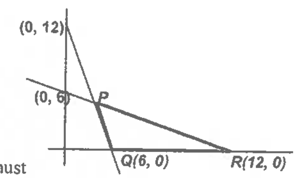
→ $PT = 2.16$ and $ST = 2.88$ → $S(7 + 2.88, 5 + 2.16) =$ **(9.88, 7.16)**

GBML '06

ROUND 2

1. $A(-2, 2), B(13, 2) \rightarrow \overline{AB} : y = 2$ (horizontal) → P, the point of intersection with L_1: $2x + y - 9 = 0$ is (3.5, 2). Thus, $AP = 3.5 - (-2) = 5.5$ and $PB = 13 - 3.5 = 9.5$
5.5 : 9.5 = **11 : 19**

2. The point of intersection of $\begin{cases} x+2y=12 \\ 2x+y=12 \end{cases}$ is $P(4, 4)$.



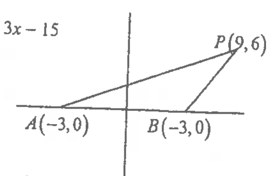
Thus, the bounded region is the interior of ΔPQR.
A(ΔPQR) = ½(4)(6) = **12**.

3. As the point of intersection, the coordinates of $P(-1, J)$ must satisfy both equations.
L_1: $4x + y = 5 \rightarrow -4 + J^2 = 5 \rightarrow J = \pm 3$
L_2: $Jx + Ky = 18 \rightarrow -J + JK = 18 \rightarrow -3 + 3K = 18$ or $3 - 3K = 18 \rightarrow K = 7$ or $K = -5$
Thus $(J, K) =$ **(3, 7)** or **(-3, -5)**.

GBML '07

ROUND 2

1. $y = \frac{1}{2}(x+3)$, $y = \frac{3}{2}(x-5) \rightarrow \frac{1}{2}(x+3) = \frac{3}{2}(x-5) \rightarrow x + 3 = 3x - 15$
→ $(x, y) = (9, 6)$ → Area $= \frac{1}{2} \cdot 8 \cdot 6 =$ **24**



2. $4x + 3y = 13 \rightarrow m = -\frac{4}{3}$ and $m_\perp = \frac{3}{4}$

$y - 2 = \frac{3}{4}(x-8) \rightarrow 4y - 8 = 3x - 24 \rightarrow 16 = 3x - 4y \rightarrow$
$$\begin{array}{r} 4\cdot(4x+3y=13) \\ +3\cdot(3x-4y=16) \\ \hline 25x \quad =100 \end{array}$$

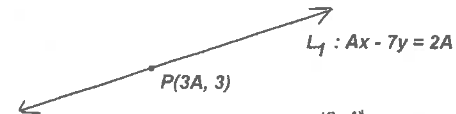
$x = 4, y = -1 \rightarrow \sqrt{(0-4)^2 + (0-(-1))^2} =$ **$\sqrt{17}$**

3. If the point of intersection is $(5, t)$, then $t = 5k + 1 = (2/5)(5) - k \rightarrow 6k = 1 \rightarrow k = 1/6 \rightarrow t =$ **11/6**

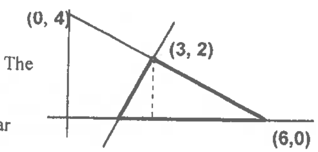
GBML '08

ROUND 2

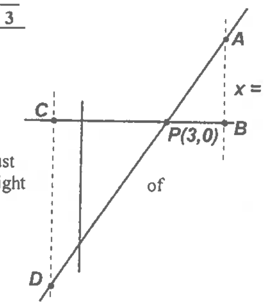
1. The diagram below shows a line with a positive slope, but this is irrelevant. The important fact is: Since point P is on line L_1, the coordinates of point P must satisfy the equation of the line L_1. Thus, substituting for x and y, we have
$A\cdot(3A) - 7(3) = 2A \rightarrow 3A^2 - 2A - 21 = (A-3)(3A+7) = 0$
$\rightarrow A =$ **$3, -\frac{7}{3}$**



2. The intercepts of $L_1 : 2x + 3y = 12$ are (0, 4) and (6, 0). The midpoint of the line segment connecting the intercepts is $(3, 2)$. Since the slope of L_1 is $-\frac{2}{3}$, the perpendicular line L_2 has a slope of $\frac{3}{2}$ and its equation is $y - 2 = \frac{3}{2}(x-3)$, which crosses the x-axis at $\left(\frac{5}{3}, 0\right)$. Thus, the base of triangle has length $6 - \left(\frac{5}{3}\right) = \frac{13}{3}$ and the height of triangle is 2. Therefore, the area of the triangle is $\frac{1}{2} \cdot \frac{13}{3} \cdot 2 =$ **$\frac{13}{3}$**



3. The line $3x - 2y = 9$ intersects the x-axis at (3,0) and the line $x = a$ at $\left(a, \frac{3a-9}{2}\right)$, i.e. points A or D.



Since the value of a could be positive, negative or zero, we must use absolute value to insure that the lengths of the base and height the triangle are positive.

Therefore, the area is given by $\frac{1}{2}|a-3|\left|\frac{3a-9}{2}\right| = \frac{3}{4}\left|(a-3)^2\right| =$

$\frac{3}{4}(a-3)^2 = \frac{25}{3}$

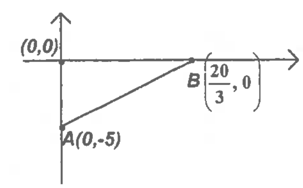
$\rightarrow (a-3)^2 = \frac{100}{9} \rightarrow a - 3 = \pm\frac{10}{3} \rightarrow a =$ **$\frac{19}{3}, -\frac{1}{3}$**

Detailed Solutions for GBML Meet 3 – DECEMBER 2009

GBML '09

ROUND 2

1. $3x - 4y = 20 \rightarrow x$-intercept $\left(\frac{20}{3}, 0\right)$ and y-intercept $(0, -5)$



Rather than using the Pythagorean theorem to determine the length of the hypotenuse, we take out a common factor:
$\left(\frac{20}{3}, 5, ?\right) = \left(\frac{20}{3}, \frac{15}{3}, ?\right) = \frac{5}{3}\left(4, 3, \boxed{5}\right) \rightarrow AB = \frac{25}{3}$

Perimeter $= \frac{15 + 20 + 25}{3} = \frac{60}{3} =$ **20**

$$\begin{aligned} &L_1 \quad 7(3x-4y=7) \\ &L_2 \ \underline{-3(7x+5y=45)} \\ &\qquad -43y=-86 \\ &\qquad\quad y=2, x=5 \end{aligned} \rightarrow L_1 \cap L_2 \text{ at } (5,2).$$

2. Since L_1 has slope $\frac{3}{4}$, L_3 has slope $-\frac{4}{3}$. $L_3: y-2=-\frac{4}{3}(x-5)$ → $4x+3y=20+6=26$

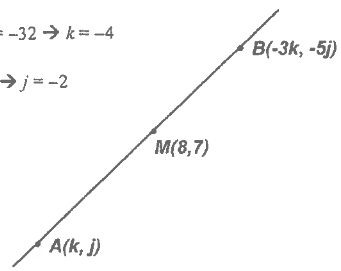
Since L_2 has slope $-\frac{7}{5}$, L_4 has slope $-\frac{7}{5}$. $L_4: y+9=-\frac{7}{5}(x-13)$ → $7x+5y=91-45=46$

$$\begin{aligned} &L_3 \quad 5(4x+3y=26) \\ &L_4 \ \underline{-3(7x+5y=46)} \\ &\qquad -x=130-138 \\ &\qquad\quad x=8, y=-2 \end{aligned} \rightarrow \underline{\mathbf{(8,-2)}}$$

3. $\frac{AM}{MB}=\frac{3}{1}=\frac{8-k}{-3k-8}$ → $8-k=-9k-24$ → $8k=-32$ → $k=-4$

$\frac{7-j}{-5j-7}=\frac{3}{1}$ → $7-j=-15j-21$ → $14j=-28$ → $j=-2$

Thus, $(kj, k-1)=\underline{\mathbf{(8,-5)}}$



GBML '10

ROUND 2

1. The slope of $\overline{AB}$ is $\frac{6-{}^-14}{-4-8}=\frac{20}{-12}=-\frac{5}{3}$ and the midpoint of $\overline{AB}$ is $M\left(\frac{-4+8}{2},\frac{6+{}^-14}{2}\right)=(2,-4)$.

The equation of the perpendicular to $\overline{AB}$ at M is $(y+4)=\frac{3}{5}(x-2)$ → $3x-5y=26$.

$y=0$ → x-intercepts: $3x=26$ → $\underline{\mathbf{\frac{26}{3}}}$

2. To find P: $\begin{cases}3(3x+2y=5)\\2(4x-3y=18)\end{cases} \rightarrow \begin{cases}9x+6y=15\\8x-6y=36\end{cases} \rightarrow 17x=51 \rightarrow (x,y)=(3,-2)$

To find Q: $\begin{cases}4x-3y=18\\2x-3y=24\end{cases} \rightarrow 2x=-6 \rightarrow (x,y)=(-3,-10)$

$PQ=\sqrt{(-2-{}^-10)^2+(3-{}^-3)^2}=\sqrt{64+36}=\underline{\mathbf{10}}$

3. $P(A,B)$, $Q(-5,-9B)$, $R(-14,3A)$ and $\frac{PQ}{QR}=\frac{5}{3}$ → $\frac{-5-A}{-14+5}=\frac{5}{3}$

→ $-15-3A=-70+25$ → $A=10$

$\frac{-9B-B}{3A+9B}=\frac{5}{3}$ → $-30B=15A+45B=150+45B$ → $B=-2$ → $(A,B)=\underline{\mathbf{(10,-2)}}$.

MML 10/05

Round Three:

A. $16(x+2)-9(x-1)=144$; $7x+33=144$; $7x=103$; $x=14\frac{5}{7}$

B. x = second leg avg. speed. Total distance was 60(2.5) + x(3.5) = 39(6) . Solve x= 24

C. If x=0, 15 – 5B = 65 so B = –10. if x=1, 3A + 10A + 65 = 117, A = 4. Sum is –6

MML 10/06

Round 3

A) Let H and C denote the number of horses and chicks respectively.
Then
$H+C=360$ → $H=360-C$
$4(360-C)+2C=1100$ → $2C=1440-1100=340$ → $C=\underline{\mathbf{170}}$

B) Since the slope of the line is $\frac{1}{4}$, $\frac{b-2006}{2006-a}=\frac{1}{4}$. Inverting both sides and multiplying through by -1, it follows that $\frac{2006-a}{2006-b}=\underline{\mathbf{-4}}$.

C) If she has \$$x$ entering a store, she has \$$\frac{x}{2}-\frac{1}{4}$ when she leaves.
Thus, we half her money and then subtract 0.25. To make life easier, we'll work <u>backwards</u>, by starting with \$0 and first adding 0.25, then doubling!

Upon leaving store #10	9	8	7	6	5	4	3
0 →	0.5	→ 1.5	→ 3.5	→ 7.5	→ 15.5 →	31.5 →	**<u>63.5</u>**

MML 10/07

Round 3

A) $a(a-2x)=b(b+2x)$ → $a^2-b^2=2bx+2ax$ → $(a+b)(a-b)=2x(a+b)$ → $x=\underline{\mathbf{\frac{a-b}{2}}}$

B) Solving for y in terms of x → $y=65-3x$. Clearly, for $x=1\ldots21$, y will be a positive integer. Thus, there are <u>**21**</u> solutions.

C) Assume 12 eggs cost x¢ and 16 cost $(x+32)$¢ or $\frac{3}{4}(x+32)$ ¢/dozen

Then $\frac{3}{4}(x+32)=x-4$ → $3x+96=4x-16$ → $x=$ <u>**112 or \$1.12**</u>

IL o/10

Round 3 Linear Equations

A) 7 lbs B) 2010 C) 180

MML 12/04

Round Three:

A. – 8 + 2 = 6; 3 – 5 = – 2.

B. Substitute (9, 16) to get c = 15 thus y = 7/3 x – 5. To get lattice pt x is a multiple of 3 so a = 102, b = 7/3 (102) – 5 = 233 sum is 335

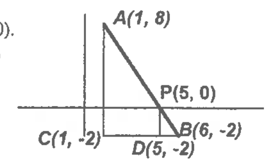
C. Vertices are (0,0) (12, 18) and (36, 0) Consider x= 1 to 12; interior lattice point counts along vertical lines are 1,2, 4, 5, 7, 8, 10,11, 13, 14, 15, 17 sum 108. Counting from x= 35 back to x = 13 gets 0, 1, 2, 2, 3, 4, 5, 5, 6, 7, 8, 8, 9, 10, 11, 11, 12, 13, 14 14, 15 16, 17 sum 193 total is 301. OR Picks Thm: Area = (#Interor pts) + (#Boundary Pts)/2 –1 so 324 = I + (48/2) – 1 so I = 301

MML
12/06

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2006 SOLUTION KEY

Round 3

A) Method 1:
Since the equation of $\overline{AB}$ is $y = -2x + 10$, the x-intercept is at (5, 0).
From the diagram, it is clear that $\overline{PD} \parallel \overline{AC}$ and the required ratio is the same as $BD : CD = \underline{1:4}$.



Method 2:
Alternately, after finding the x-intercept P, using the distance formula, you could compute the distance between P and B ($\sqrt{5}$) and the distance between A and P ($\sqrt{80} = 4\sqrt{5}$) → $\underline{1:4}$.

Method 3 (**without** finding the equation or x-intercept of $\overline{AB}$):
Let X denote the x-intercept of the vertical line $\overline{AC}$. Clearly, the coordinates of X are (1, 0) and $CX : AX = 1 : 4$ → $BP : AP = \underline{1:4}$ (since $\Delta BPD \sim \Delta BAC$).

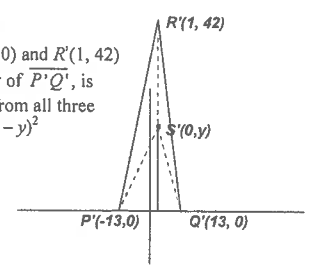
B) The slope of $\overline{AB}$ is $\dfrac{-2006}{4250} = \dfrac{-2(17)(59)}{2(17)(125)} = \dfrac{-59}{125}$
Points with integer coordinates (i.e. lattice points), may be determined by starting at A and increasing the x-coordinate by 125 and decreasing the y-coordinate by 59 or alternately, starting at B and decreasing the x-coordinate by 125 and increasing the y-coordinate by 59.

Both strategies produce: (0, 2006) **(125, 1947)**, (250, 1888) … **(4125, 59)**, (4250, 0)
$125 + 1947 + 4125 + 59 = \underline{\mathbf{6256}}$.

In fact, suppose the slope of $\overline{AB}$ were $\dfrac{-a}{b}$, where a and b are positive integers.
Then $C(b, 2006 - a)$ and $D(4250 - b, a)$ → $p + q + r + s = 4250 + 2006 = 6256$ and it wasn't even necessary to find the slope of $\overline{AB}$. In the worst case scenario, if the slope fraction could not be reduced, point C would coincide with point B and D would coincide with point A.

C) Method 1:
Point S is the intersection of the perpendicular bisectors of the sides of ΔPQR.
The perpendicular bisector of $\overline{PQ}$ is the vertical line $\underline{x = 1}$.
The perpendicular bisector of $\overline{PR}$ is $x + 3y = 58$.
→ $3y = 57$ → $\underline{y = 19}$.

Method 2:
Shifting each vertex of ΔPQR left 1 unit. $P'(-13, 0)$, $Q'(13, 0)$ and $R'(1, 42)$
Clearly, point $S'(0, y)$, a point on the perpendicular bisector of $\overline{P'Q'}$, is equidistant from P' and Q'. To insure that it is equidistant from all three vertices, we require $(S'Q')^2 = (S'R')^2$ → $13^2 + y^2 = 1^2 + (42 - y)^2$
→ $169 + y^2 = 1 + 1764 - 84y + y^2$ → $84y = 1596$ → $y = 19$
→ $S'(0, 19)$ → $S(1, 19)$ → $x = \underline{1}$, $y = \underline{19}$.



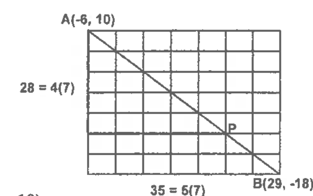
MML
12/07

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2007 SOLUTION KEY

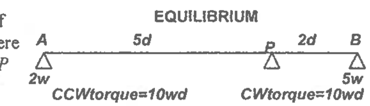
Round 3

A) The coordinates of point P must satisfy both equations.
Thus, both $2 \cdot 8 - a^2 = 7$ and $8a - 4a - 12 = 0$ must be true.
$a^2 = 9$ is satisfied by both ±3, but the second equation is only satisfied by $a = \underline{3}$.

B) Each block in the vertical direction is 4 units and each block in the horizontal direction is 5 units.
$(-6 + 5 \cdot 5, 10 - 5 \cdot 4)$ → $\underline{(19, -10)}$
Alternate solution:
5/7 → $AP : PB = 5 : 2$ Thus, the coordinates of P are determined by 'weighting' the coordinates.
Since P is closer to B than A its 'influence" is greater.
In fact, B will be counted 5 times and A only twice.
$P\left(\dfrac{2(-6)+5(29)}{2+5}, \dfrac{2(10)+5(-18)}{2+5}\right) = \left(\dfrac{145-12}{7}, \dfrac{20-90}{7}\right) = \underline{(19, -10)}$



The diagram at the right illustrates this weighting of x- and y-coordinates in terms of a balancing act where the force producing a clockwise turn around point P equals the force producing a counterclockwise turn around point P; hence, the term equilibrium.



C) Since perpendicular lines have negative reciprocal slopes, the perpendicular to $3x + 2y - 13 = 0$ has the form $2x - 3y + c = 0$.
Since this line must also pass through (1, 5), we can find c by substituting for x and y.
$2(1) - 3(5) + c = 0$ → $c = 13$ and the required line is $2x - 3y + 13 = 0$.

Substituting the coordinates of the points that lie on this line, $\begin{cases} P)\ 2a - 3b + 13 = 0 \\ Q)\ 2b - 3a + 13 = 0 \end{cases}$

Subtracting, $5a - 5b = 0$ → $a = b$ → P and Q are the same point → $PQ = \underline{0}$

Aside #1:
Since the slope of $\overline{PQ}$, given $P(a, b)$ and $Q(b, a)$ is $\dfrac{b-a}{a-b} = -1$ and the slope of the given line $\neq -1$, the only way both P and Q could be on the line is for P and Q to be the same point!

Aside #2:
Suppose both (h, k) and (k, h) lie on a line $Ax + By + C = 0$. Then
$\begin{cases} Ah + Bk + C = 0 \\ Ak + Bh + C = 0 \end{cases}$ Subtracting, $A(h-k) + B(k-h) = 0$ → $A(h-k) = -B(k-h) = B(h-k)$
∴ $A = B$ or $h = k$ → if x- and y- coefficients are unequal, (h, k) and (k, h) must be the same point.

YL
12/09

Round 3 Coordinate Geometry of Lines and Circles

A) (5, 3), (–1, –5) B) –0.5, 12.3 C) $9x - 7y - 11 = 0$ [$R(16, 19)$]