GREATER BOSTON MATHEMATICS LEAGUE

MEET 5 – MARCH 1999

ROUND 2 – Algebra 1

1. $d =$ ____________

2. ____________

3. ____________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. A club with x members raises a total of d dollars to spend equally at an amusement park. When one member cannot go to the amusement park, each member has an extra dollar to spend. Find d in terms of x.

2. Write the following expression in simplest radical form with the smallest possible index for the radical:

$$\frac{\left(\sqrt[3]{12}\right)\left(\sqrt[6]{72}\right)}{\left(\sqrt[4]{\sqrt[3]{108}}\right)\left(\sqrt{\sqrt[3]{3}}\right)}$$

3. Kaitlin is now eight years younger than half her mother's age. In k years (k a positive integer), Kaitlin will be one-third her mother's age then. What is the oldest possible age now for Kaitlin's mother?

GREATER BOSTON MATHEMATICS LEAGUE

MEET 5 – MARCH 2000

ROUND 2 – Algebra 1

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Simplify the following expression:

$$\frac{x+1-\dfrac{2}{x+2}}{x-4-\dfrac{7}{x+2}}$$

2. Find the sum of all two-digit natural numbers whose tens' digit is three less than twice its units' digit.

3. Factor completely: $x^4 - x^3y - 2xy^3 - 4y^4$

GREATER BOSTON MATHEMATICS LEAGUE

MEET 5 – MARCH 2001

ROUND 2 – Algebra 1

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Jill has three times more money than Jack. If Jill gives Jack \$15, Jill will now have twice Jack's new amount. How many dollars did Jill start with?

2. Two numbers differ by 1 and the sum of their reciprocals equals $\frac{15}{4}$. Find all possible values for the smaller of these two numbers.

3. There are only two lines containing the point $P(9,-1)$ that form a triangle with the x and y axes with an area of 6 square units. Find the slopes of these two lines.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 5 – MARCH 2002

ROUND 2 – Algebra 1
Problems submitted by Newton South.

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Find the set of all $x, x \in \Re$, which are the solution for $\left\{ x \,\middle|\, \dfrac{x+2}{x-2} > 5 \right\}$.

2. Given points $A(6,-2)$, $B(t+1,-4)$, and $C(t,4)$ such that $\angle BAC$ is a right angle, find all possible values for t.

3. Ticket prices for the afternoon movie are \$5.50 for adults, \$4.50 for children and \$4.00 for senior citizens. If 100 tickets were sold, the proceeds were \$465, and more senior citizens than children attended, find the most number of children that could have been at the afternoon movie.

GREATER BOSTON MATHEMATICS LEAGUE
MEET 5 – MARCH 2006

ROUND 2 – Algebra 1: Open

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND.

1. Factor completely. $7x(x - 3) + 14y(3 - 2y)$

2. Determine all ordered pairs of positive integers (x, y) for which $x^2 + 220 = y^2$.
   Answers must be listed using correct ordered pair notation.

3. A two-digit natural number is equal to four times the sum of its digits. 15 times the absolute value of the difference of its digits is 12 more than 6 times its unit digit. Find all possible numbers satisfying these conditions.

GREATER BOSTON MATHEMATICS LEAGUE
MEET 5 – March 2007

ROUND 2 – Algebra 1: Open

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND.

1. Find all possible values of k, if the line $kx-4y+3k=0$ passes through the point $P(k,10)$.

2. The sum of the squares of two numbers is 458. The difference of the squares is 120. What is the maximum positive difference between the numbers?

3. A rectangular picture is surrounded by a rectangular mat that has a uniform 2 inch width. The width of the picture is 80% of the left-to-right width of the mat. If the mat itself has an area of 164 square inches, determine the number of inches in the perimeter of the picture?

GREATER BOSTON MATHEMATICS LEAGUE

MEET 5 – MARCH 2008

ROUND 2 – Algebra 1: Open

1. ______________________________

2. ______________________________

3. ______________________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND.

1. Determine all possible values of x for which $\left|\left|\left|x-2\right|-3\right|-4\right|=5$.

2. Factor completely over the integer. $81\left(4y^2-1\right)-5x(5x-18)$

3. Today, in the Devaney household, the sum of the ages of mother and father is 10 times the age of their daughter Casandra. When Casandra was born, the sum of the ages of the parents was the square of Casandra's age today. If Mr. Devaney is two years younger than his wife, how old is Mrs. Devaney today?

GREATER BOSTON MATHEMATICS LEAGUE

MEET 5 – FEBRUARY 2009

ROUND 2 – Algebra 1: Open

1. ______________________________

2. ______________________________

3. ______________________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND.

1. Find the largest real number which does <u>not</u> satisfy the inequality $\dfrac{7-2x}{3} < \dfrac{2-5x}{13}$.

2. The reciprocal of a number added to the reciprocal of 2 less than the number gives a result of $\dfrac{11}{60}$. Determine <u>all</u> possible values of the original number.

3. Determine the three-digit integer N for which the following statements are true:

   - the units digit is twice the tens digit
   - the units digit is one-half of one less than the sum of the tens and hundreds digits
   - the sum of the three digits is the sum of the two largest single digit perfect squares

GREATER BOSTON MATHEMATICS LEAGUE

MEET 5 – FEBRUARY 2010

ROUND 2 – Algebra 1: Open

1. ______________________

2. ______________________

3. ______________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND.

1. Determine the average of all integer values of z such that $-8 < \frac{6-5z}{8} < -4$.

2. K gallons of a 60% alcohol solution are mixed with 40 gallons of a $K\%$ alcohol solution resulting in a $33\frac{1}{3}\%$ alcohol solution. Compute the value of K.

3. The sum of 4 consecutive odd numbers is twice the sum of 3 consecutive even integers. The smallest odd number is 1 more than the largest even number. Find the sum of the largest odd and the smallest even number.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 5 – MARCH 2011

ROUND 2 – Algebra 1: Open

1. ________________ AM PM

2. (________ , ________)

3. $\{x \mid$ ____________________ $\}$

CALCULATORS ARE NOT ALLOWED ON THIS ROUND.

1. An empty tank can be filled by pipe A is 4 hours. When full, this same tank can be drained by pipe B in 6 hours. Starting with an empty tank, Pipe A is opened at 8 AM and pipe B is opened at 9 AM on the same morning. When the tank has become completely full, both pipes are immediately closed down to avoid any spillage. At exactly what time must this shutdown occur? Be sure to circle AM or PM.

2. Determine the ordered pair of integers (x, y) for which $20^{2x+y}\left(\frac{1}{5}\right)^{y} = 50^{-2}(500)^{x-y}$

3. Given: $\{x : |3x-2| + 2x \leq |5-x|\}$
Solve for x over the reals.

'99

ROUND 2

1. $\frac{d}{x-1}=\frac{d}{x}+1 \Rightarrow dx=dx-d+x(x-1) \Rightarrow d=x(x-1)$ or x^2-x

2. $\frac{\left(\sqrt[3]{12}\right)\left(\sqrt[6]{72}\right)}{\left(\sqrt[4]{\sqrt[3]{108}}\right)\left(\sqrt{\sqrt[3]{3}}\right)}=\frac{\left(2^2\cdot 3\right)^{1/3}\left(2^3\cdot 3^2\right)^{1/6}}{\left(2^2\cdot 3^3\right)^{1/12}(3)^{1/6}}=\frac{\left(2^{2/3}\cdot 3^{1/3}\right)\left(2^{1/2}\cdot 3^{1/3}\right)}{\left(2^{1/6}\cdot 3^{1/4}\right)(3)^{1/6}}=2^{2/3+1/2-1/6}\cdot 3^{2/3-1/4-1/6}=$
$2\cdot 3^{1/4}=\mathbf{2\sqrt[4]{3}}$

3. Kaitlin's mother's age now $=x$; Kaitlin's age now $=0.5x-8$;
Kaitlin's mother's age in k years $=x+k$; Kaitlin's age in k years $=0.5x-8+k$;
$3(0.5x-8+k)=x+k \Rightarrow 3x-48+6k=2x+2k \Rightarrow x=48-4k \Rightarrow x=\mathbf{44}$ $(k=1)$
is the oldest Kaitlin's mother can be.

'00

ROUND 2

1. $\frac{x+1-\frac{2}{x+2}}{x-4-\frac{7}{x+2}}=\frac{(x+1)(x+2)-2}{(x-4)(x+2)-7}=\frac{x^2+3x}{x^2-2x-15}=\frac{x(x+3)}{(x+3)(x-5)}=\frac{x}{x-5}$

2. $t=2u-3 \Rightarrow$ If $u=2$, $t=1$, if $u=3$, $t=3$, if $u=4$, $t=5$,
if $u=5$, $t=7$, if $u=6$, $t=9$; $12+33+54+75+96=270$

3. $x^4-x^3y-2xy^3-4y^4=x^4-4y^4-x^3y-2xy^3=$
$(x^2-2y^2)(x^2+2y^2)-xy(x^2+2y^2)=(x^2-xy-2y^2)(x^2+2y^2)=$
$(x+y)(x-2y)(x^2+2y^2)$

'01

ROUND 2

1. Let x = Jack's original amount $\rightarrow 3x$ = Jill's $\rightarrow$
$3x-15=2(x+15)\rightarrow 3x-15=2x+30\rightarrow x=45\rightarrow 3x=135$

2. $\frac{1}{x}+\frac{1}{x+1}=\frac{15}{4}\rightarrow 4x+4+4x=15x^2+15x\rightarrow 15x^2+7x-4=0\rightarrow$
$(5x+4)(3x-1)=0\rightarrow x=-\frac{4}{5},\frac{1}{3}$

3. $y+1=m(x-9)\rightarrow$ if $x=0\rightarrow y=-9m-1$ or if $y=0\rightarrow x=\frac{1}{m}+9$;
the area of the triangle $=\frac{1}{2}(-9m-1)\left(\frac{1}{m}+9\right)=6\rightarrow -9-81m-\frac{1}{m}-9=12\rightarrow$
$81m+\frac{1}{m}+30=0\rightarrow 81m^2+30m+1=0\rightarrow(27m+1)(3m+1)=0\rightarrow m=-\frac{1}{3},-\frac{1}{27}$

'02

ROUND 2 – Algebra 1

1. $\frac{x+2}{x-2}>5\Rightarrow\frac{x+2}{x-2}-\frac{5}{1}>0\Rightarrow\frac{x+2-5x+10}{x-2}>0\Rightarrow\frac{12-4x}{x-2}>0$; key values for the inequality are 2 and 3 (the numbers that make the numerator and denominator 0). Now section off the number line using 2 and 3 and check each interval. Therefore the solution set is $\{x\mid 2<x<3\}$.

(+)/(-) (+)/(+) (-)/(+)
– 2 + 3 –

2. Since the points $A(6,-2)$, $B(t+1,-4)$, and $C(t,4)$ form a right angle at A, the slope of $\overline{AB}$ times the slope of $\overline{AC}$ equals $-1\Rightarrow\left(\frac{-4+2}{t+1-6}\right)\left(\frac{4+2}{t-6}\right)=-1\Rightarrow\left(\frac{-2}{t-5}\right)\left(\frac{6}{t-6}\right)=-1\Rightarrow$
$t^2-11t+30=12\Rightarrow t^2-11t+18=0\Rightarrow(t-2)(t-9)=0\Rightarrow t=2,9$.

3. Let x = number of adults, y = number of children, and z = number of senior citizens $\Rightarrow$
$x+y+z=100$ *and* $5.5x+4.5y+4z=465$ *and* $z>y\Rightarrow$
$5.5x+5.5y+5.5z=550$ *and* $5.5x+4.5y+4z=465\Rightarrow y+1.5z=85$. If $z=y$, then
$2.5y=85\Rightarrow y=z=34$. Since $z>y$, then the smallest value for z would be $36\Rightarrow$
$y+54=85\Rightarrow y=31$ is the largest possible value.

106

ROUND 2 – Algebra 1: Open

1. $7x(x-3) + 14y(3-2y) = 7(x^2 - 3x + 6y - 4y^2) = 7(x^2 - 4y^2 - 3x + 6y)$
   $= 7[(x+2y)(x-2y) - 3(x-2y)] = \underline{\mathbf{7(x-2y)(x+2y-3)}}$

2. $y^2 - x^2 = 220 = 2^2 5^1 11^1$ (12 factors → 6 possible pairs of factors)
   $(y-x)(y+x) = 1(22), 2(110), 4(55), 5(44), 10(22)$ or $11(20)$
   Setting up a system of equations, only factorizations in terms of two even integers will result in integer values for x and y. Thus, only two systems need be solved.
   $2(110) \rightarrow (x, y) = \underline{\mathbf{(54, 56)}}$
   $10(22) \rightarrow (x, y) = \underline{\mathbf{(6, 16)}}$

3. $N = 10t + u = 4(t+u) \rightarrow u = 2t$
   $15|t-u| = 15(u-t) = 12 + 6u \rightarrow 3u = 5t + 4$
   Substituting, $3(2t) = 5t + 4 \rightarrow t = 4, u = 8 \rightarrow N = \underline{\mathbf{48}}$

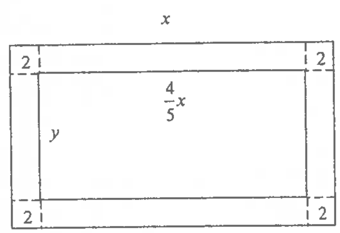
107

ROUND 2 – Algebra 1: Open

1. If point $P(k, 10)$ is on the line $kx - 4y + 3k = 0$. then the coordinates must satisfy the equation.
   $k^2 - 40 + 3k = k^2 + 3k - 40 = (k+8)(k-5) = 0 \rightarrow k = \underline{\mathbf{-8 \text{ or } +5}}$

2. If $\begin{cases} x^2 + y^2 = 458 \\ x^2 - y^2 = 120 \end{cases}$, then $x = \pm 17$ and $y = \pm 13$. Thus, the maximum positive difference is
   $17 - (-13) = \underline{\mathbf{30}}$.

3. Let the width of the frame (left to right) be x inches.
   Let the height of the picture (top to bottom) be y inches.
   Thus, $\frac{4}{5}x + 4 = x \rightarrow x = 20$
   The area of the frame $= 20(y+4) - 16y = 164$
   $\rightarrow 4y = 84 \rightarrow y = 21$
   Finally, the perimeter of the picture $= 2(21 + 16) = \underline{\mathbf{74}}$



108

ROUND 2

1. $\left|\left|\left|x-2\right|-3\right|-4\right| = 5 \rightarrow \left|\left|x-2\right|-3\right|-4 = \pm 5 \rightarrow \left|\left|x-2\right|-3\right| = 9$ (-1 is rejected)
   $\left|x-2\right|-3 = \pm 9 \rightarrow |x-2| = 12$ (-6 is rejected)
   $x - 2 = \pm 12 \rightarrow x = 2 \pm 12 = \underline{\mathbf{-10, 14}}$

2. $81(4y^2-1) - 5x(5x-18) = 324y^2 - 25x^2 + 90x - 81 = 18y^2 - (25x^2 - 90x + 81)$
   $= (18y)^2 - (5x-9)^2 = \underline{\mathbf{(18y + 5x - 9)(18y - 5x + 9)}}$

3. Let M, F and C denote the ages of mother, father and Casandra respectively. Then:
   $\begin{cases} (1)\ M + F = 10C \\ (2)\ (M-C) + (F-C) = C^2 \\ (3)\ M = F + 2 \end{cases}$ The second equation simplifies to $M + F = C^2 + 2C$
   Thus, $C^2 + 2C = 10C \rightarrow C^2 - 8C = C(C-8) = 0 \rightarrow C = 8 \rightarrow M + F = 80$
   Using equation (3) and substituting for M, $2F + 2 = 80 \rightarrow F = 39$ and $M = \underline{\mathbf{41}}$.

109

ROUND 2

1. Cross multiplying, $13(7-2x) < 3(2-5x) \rightarrow 91 - 26x < 6 - 15x \rightarrow 11x > 85 \rightarrow x > \underline{\mathbf{\frac{85}{11}}}$

   $\frac{1}{x} + \frac{1}{x-2} = \frac{11}{60} \rightarrow (x-2) + x = \frac{11}{60}x \cdot (x-2) \rightarrow 60x - 120 + 60x = 11x^2 - 22x$

2. $\rightarrow 11x^2 - 142x + 120 = (11x - 10)(x - 12) = 0$
   $\rightarrow x = \underline{\mathbf{\frac{10}{11}, 12}}$

3. $\begin{cases} (1)\ u = 2t \\ (2)\ u = \frac{1}{2}(h + t - 1) \text{ or } 2u = h + t - 1 \\ (3)\ h + t + u = 4 + 9 = 13 \end{cases}$

Substituting (1) in (2) and (3), we have $\begin{cases} h - 3t = 1 \\ h + 3t = 13 \end{cases} \rightarrow (h, t, u) = (7, 2, 4) \rightarrow N = \underline{\mathbf{724}}$

110

ROUND 2

1. $-8 < (6 - 5z)/8 < -4 \rightarrow -64 < 6 - 5z < -32 \rightarrow 64 > -6 + 5z > 32 \rightarrow 14 > z > 7.6$
   Thus, the integer solutions are 8, 9, 10, 11, 12 and 13. The average is $63/6 = \underline{\mathbf{10.5}}$.

2. $K(0.60) + 40\left(\frac{K}{100}\right) = (K+40)\frac{33\frac{1}{3}}{100} = \frac{K+40}{3}$
   Simplify and cross multiply: $3(0.6K + 0.4K) = K + 40 \rightarrow 2K = 40 \rightarrow K = \underline{\mathbf{20}}$

3. Let N, $(N+2)$, $(N+4)$ and $(N+6)$ denote the four odd integers and M, $(M+2)$ and $(M+4)$ denote the three even integers. Then:
   $4N + 12 = 2(3M + 6) \rightarrow 2N = 3M$
   If, additionally, $N = (M+4) + 1$, we have $2(M+5) = 3M \rightarrow M = 10$ and $N = 15$.
   Thus, the required sum is $21 + 10 = \underline{\mathbf{31}}$.

'||

ROUND 2

1. Let T denote the time (in hours) from 8AM until shutdown. The rates of the two pipes are $(A, B) = \left(\frac{1}{4}, -\frac{1}{6}\right)$, where + rate denotes filling and a – rate denotes draining. Then:

   $\frac{1}{4}T + \left(-\frac{1}{6}\right)(T-1) = 1$ → $3T - 2(T-1) = 12$ → $T = 12 - 2 = 10$

   8 AM + 10 hours → <u>**6 PM**</u>

2. $(2^2 \cdot 5)^{2x+y}(5^{-y}) = (2 \cdot 5^2)(2^2 \cdot 5^3)^{x-y}$ → $2^{4x+2y} \cdot 5^{2x} = 2^{2x-2y-2} \cdot 5^{3x-3y-4}$ →

   $\begin{cases} 4x+2y = 2x-2y-2 \\ 2x = 3x-3y-4 \end{cases}$

   → $\begin{cases} x+2y = -1 \\ x-3y = 4 \end{cases}$ Subtracting, $5y = -5$ → $y = -1$ → <u>**(1, –1)**</u>

3. $|3x-2| + 2x \le |5-x|$ has critical points at: $x = 2/3$ and 5.

   In each case, we look at an equivalent equation <u>valid only within the specified domain.</u>

   Case 1: $x < 2/3$

   → $-3x+2+2x \le 5-x$ → $2 \le 5$ (which is always true) – these values accepted.

   Case 2: $2/3 \le x \le 5$

   → $3x-2+2x \le 5-x$ → $6x \le 7$ → $x \le \frac{7}{6}$ → $\frac{2}{3} \le x \le \frac{7}{6}$

   Case 3: $x > 5$

   → $3x-2+2x \le -5+x$ → $4x < -3$ → $x \le -\frac{3}{4}$ → $\varnothing$ (i.e. no solution, outside domain!)

   Combining the solutions, we have $\underline{\left\{x \mid x \le \frac{7}{6}\right\}}$.