E.O. SMITH

THE NEW ENGLAND ASSOCIATIONS OF MATHEMATICS LEAGUES

NEAML

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41st *ANNUAL MATH COMPETITION*

April 26, 2013

CANTON HIGH SCHOOL

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NEW ENGLAND ASSOCIATION OF MATHEMATICS LEAGUES

PLAYOFFS – 2013

Round 1: Arithmetic and Number Theory

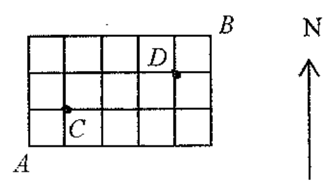
1. ____________________

2. ____________________

3. ____________________

1. A healthy cereal has 5 grams of protein in a 50-gram serving. If the cereal costs \$3.30 for a 16-ounce box, compute the cost per ounce of protein in dollars and cents, rounded to the nearest cent.

2. Going either north or east, how many different routes go from A to B that don't go through points C or D?



3. Given that n is an integer with $1 \le n \le 2013$, for how many values of n does the number $2(n + 3)$ end in 0?

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PLAYOFFS – 2013

Round 2: Algebra 1

1. ___(_____,_____)_____

2. ____________________

3. ____________________

1. If $(x, y) = \left(\frac{2}{3}, \frac{3}{5}\right)$ satisfies the system $\begin{matrix} mx - 5y = -1 \\ 3x + ny = 8 \end{matrix}$, compute the ordered pair (m, n)

2. Compute all real solutions to $\frac{6x^{-1} + 1}{12x^{-1} + 2} = \frac{1}{2}$.

3. If 10101 in base b equals 101 in base $2b$ for $b > 0$, what is the value of b?

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Round 3: Geometry

1. ____________________

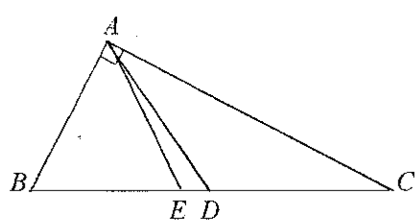
2. ____________________

3. ____________________

1. The measures of two of the angles of a triangle are $x + 40$ and $3x + 10$. All of the angle measures are integers. Determine the smallest possible degree measure of an angle of the triangle.

2. A circle is circumscribed about regular hexagon *ABCDEF*. Rectangle *BCEF* is drawn. Let the area of region I be the total area inside the circle but outside the hexagon. Let the area of region II be the total area outside the rectangle but inside the hexagon. Compute the ratio of the area of region I to the area of region II, given that each side of the hexagon measures 4 units.

3. ΔBAC is a right triangle with $m\angle BAC = 90$, D is the midpoint of $\overline{BC}$, $\overline{AB} \cong \overline{AE}$, $ED = 2$ and $DC = 8$. Compute the value of AB.



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Round 4: Algebra 2

1. ____________________

2. ____________________

3. ___(_____,_____)______

1. For $a \in \{2,3,4,5\}$ and $b \in \{2,3,4,5\}$, determine the number of distinct values of x such that x is a non-zero root of $ax^2 + bx = x$.

2. If $2013^a = 100$ and $0.2013^b = 100$, compute $\frac{1}{a} - \frac{1}{b}$.

3. Computer the coordinates of the ordered pair (x, y) satisfying the following system:

$$x + \left(\frac{7-\sqrt{51}}{2}\right)y = \left(\frac{7-\sqrt{51}}{2}\right)^2$$

$$x + \left(\frac{7+\sqrt{51}}{2}\right)y = \left(\frac{7+\sqrt{51}}{2}\right)^2$$

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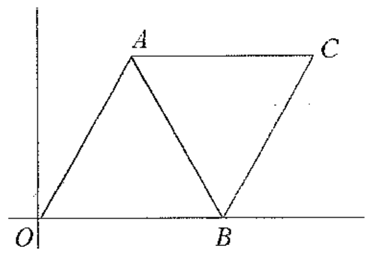
Round 5: Analytic Geometry

1. ____________

2. (___ , ___ , ___)

3. ____________

1. ΔAOB and ΔCAB are equilateral triangles. Compute the slope of $\overline{OC}$.



2. An ellipse has one end of its major axis at the y-intercept of $7x - 8y = 32$ and the ends of its minor axis on the ends of the vertical diameter of $(x-3)^2 + (y+4)^2 = 4$. If the equation of the parabola whose vertex is at the lower end of the minor axis of the ellipse and which passes through the ends of the major axis is in the form $y = ax^2 + bx + c$, compute the coordinates of the ordered triple (a, b, c)?

3. The center of circle Q lies in the first quadrant below the graph of $xy = 2$. Circle Q is tangent to the positive x- and y-axes as well as to $xy = 2$. Compute the sum of the coordinates of the center of Q.

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Round 6: Trig and Complex Numbers

1. ____________________

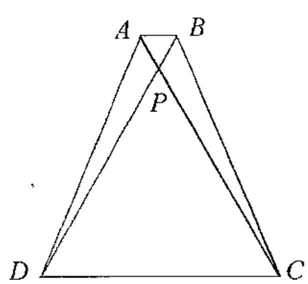
2. ____________________

3. ____________________

1. Determine the rectangular form of $(4 \text{ cis } 45°)^2 \cdot (2 \text{ cis } 150°)$

2. With one end stuck on the ground a telephone pole is raised to the vertical position. If the shadow of the pole loses 20 feet in length as the pole's angle of inclination with the ground increases from 30° to 60°, compute the length of the pole.

3. $ABCD$ is an isosceles trapezoid in which $m\angle CAB = 60°$. If the sum of the areas of ΔAPD and ΔBPC equals 18, compute the value of the product $BP \cdot PC$.



MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

NEW ENGLAND PLAYOFFS – 2013

Team Round

1. ____________________ **4.** ____________________

2. ____________________ **5.** ____________________

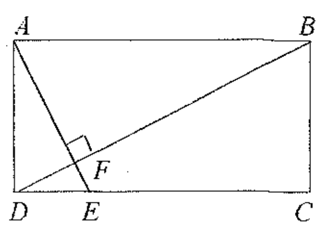
3. ____________________ **6.** ____________________

1. For $a \ge 1$ and $c \ge 1$, if $2^a, 4, 2^c$ form an increasing arithmetic sequence, compute the largest possible value of c.

2. Compute the value of $1003^3 - 3 \cdot 1001^3 + 3 \cdot 999^3 - 997^3$.

3. The equation $I = \dfrac{AM}{ME}$ represents a two-digit number being divided by a two-digit number. The result is a single digit. If the letters I, A, M, and E represent different non-zero digits, what values can I take on?

4. $ABCD$ is a rectangle, $\overline{AE} \perp \overline{DB}$, the area of ΔDFA is 24, the area of ΔAFB is 72, and area of quadrilateral $BCEF$ is 88. Compute the value of the product $(AE)(DB)$.



5. Compute the area of the convex polygon in the complex plane whose vertices are the complex solutions to $\left(z^2+\frac{1}{z^2}\right)^2+\left(z+\frac{1}{z}\right)^2=4$.

6. Point A lies on the positive x-axis, B lies on the positive y-axis, and O is the origin. P and Q are trisection points of $\overline{AB}$. If the slope of $\overline{AB}$ is k, find the product of the slopes of $\overline{OP}$ and $\overline{OQ}$ in terms of k.

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PLAYOFFS – 2013

Answer Sheet

Round 1

1. \$2.06
2. 12
3. 403

Round 2

1. (3, 10)
2. All reals except 0, –6
3. $\sqrt{3}$

Round 3

1. 1
2. $\frac{2\pi\sqrt{3}-9}{3}$
3. $4\sqrt{3}$

Round 4

1. 13
2. 2
3. $\left(\frac{1}{2}, 7\right)$

Round 5

1. $\frac{\sqrt{3}}{3}$
2. $\left(\frac{2}{9}, -\frac{4}{3}, -4\right)$
3. $4\sqrt{2}-4$

Round 6

1. $-16-16i\sqrt{3}$
2. $20\sqrt{3}+20$
3. $12\sqrt{3}$

Team

1. $\log_2 6$
2. 48
3. 2, 3, 4, 7
4. 256
5. $\frac{3\sqrt{3}}{2}$
6. k^2

$2\pi\sqrt{3}-9 \div 3$

MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

NEW ENGLAND PLAYOFFS – 2013 - SOLUTIONS

Round 1 Arithmetic and Number Theory

1. There are 1.6 ounces of protein in the box so $\frac{\$3.30}{16} = \$2.0625$ Thus, an ounce of protein costs $\boxed{\$2.06}$.

2. Total number: EEEEENNN gives 56. Through C: 2 times EEEENN for $2 \cdot \frac{6!}{4!2!} = 30$. Through D: EEEENN by 2 gives 30. From A to C to D to B: 2 x 4 x 2 = 16. Thus, $\boxed{56 - (30 + 30 - 16) = 12}$.

3. If n ends in 2 then $n + 3$ ends in 5 so doubling it will result in a 0 at the end. If n ends in 7, then $n + 3$ ends in 0 so doubling it ends in 0. Thus, each set of 10 numbers starting with $[1,10]$ contains two values of n with the desired condition. From 1 to 2010 we have 201 sets of ten numbers making 402 values of n that give a result ending in 0. To that answer we must add 1 for 2012, making a total of $\boxed{403}$.

Round 2 Algebra 1

1. $\frac{2}{3}m - 3 = -1,\ 2 + \frac{3}{5}n = 8 \rightarrow 2m - 9 = -3,\ 10 + 3n = 40 \rightarrow \boxed{m = 3, n = 10}$

2. $\dfrac{6x^{-1}+1}{12x^{-1}+2} = \dfrac{1}{2} \rightarrow \dfrac{\frac{6}{x}+1}{\frac{12}{x}+2} = \dfrac{6+x}{12+2x} = \dfrac{1}{2} \rightarrow 12+2x = 12+2x$. This is true for all x except those that give 0 in the denominator, namely 0 and -6. Answer: all Reals except 0 and -6.

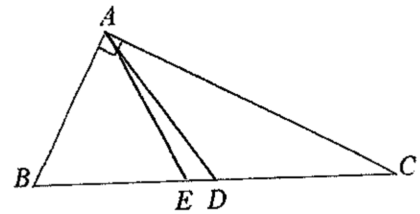
3. $1 \cdot b^4 + 0 \cdot b^3 + 1 \cdot b^2 + 0 \cdot b + 1 = 1 \cdot (2b)^2 + 0 \cdot (2b) + 1 \rightarrow b^4 + b^2 + 1 = 4b^2 + 1$. Simplifying gives $b^4 - 3b^2 = 0 \rightarrow b^2\left(b^2 - 3\right) = 0$. Thus, $\boxed{b = \sqrt{3}}$.

Round 3 – Geometry

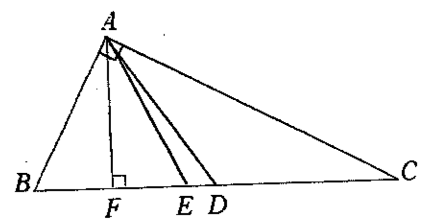
1. Since the angle measures must be integers, the value of x which makes $3x+10$ as small as possible is -3. If $x=-3$, then $3x+10=1$, $\boxed{1°}$ is the smallest angle

2. Since a regular hexagon can be thought of as consisting of six equilateral triangles with a common vertex, the radius of the circle is the same as a side of the hexagon, i.e. 4. Therefore the area of the circle is $\boldsymbol{\pi\cdot 4^2 = 16\pi}$. Similarly the area of the hexagon is $\frac{3}{2}\cdot\boldsymbol{4^2\cdot\sqrt{3}} =$ $\boldsymbol{24\sqrt{3}}$. For the rectangle two of the sides coincide with sides of the hexagon, The other two sides are the bases of isosceles triangles of side 4 and vertex angle 120°. Drawing an altitude to the base creates two 30-60-90 triangles which leads to a base of $\boldsymbol{4\sqrt{3}}$. The area of the rectangle is $\boldsymbol{4\cdot 4\sqrt{4} = 16\sqrt{3}}$. The required ratio is $\boldsymbol{\frac{16\pi-24\sqrt{3}}{8\sqrt{3}} = \frac{2\pi-3\sqrt{3}}{\sqrt{3}} = \boxed{\frac{2\pi\sqrt{3}-9}{3}}}$
Note: The answer is independent of the length of the side of the hexagon..

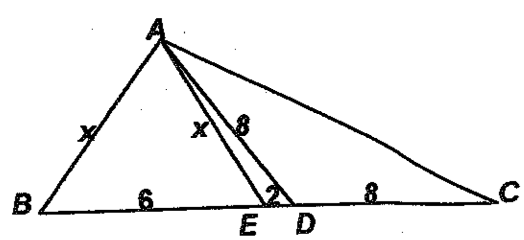
3. Since ABC is a right triangle and D is the midpoint of the hypotenuse, $BD = AD$, so $\angle B \cong \angle BAD$. It is given that $AB = AE$ so $\angle B \cong \angle AEB$. Thus $\Delta ABD \sim \Delta BEA$ giving $\dfrac{AB}{BE} = \dfrac{AD}{AB}$ giving $AB^2 = BE\cdot AD$. Since $DC = 8$ and D is the midpoint of $\overline{BC}$, then $BE = 6$ giving $AB^2 = 6\cdot 8$. Thus $\boxed{AB = 4\sqrt{3}}$.



Alternate solution: Drop the altitude from A. Note that $BF = FE$ since ΔABE is isosceles. By the geometric mean theorem, $AB^2 = BF\cdot BC = \frac{1}{2}BE\cdot 2DC = BE\cdot DC$. Since $DC = 8$ and D is the midpoint of $\overline{BC}$, then $BE = 6$ giving $AB^2 = 6\cdot 8$. Thus $AB = 4\sqrt{3}$.



Alternate Solution 2: Using Stewart's theorem on ΔBAD, we have $x^2\cdot 2 + 8^2\cdot 6 = x^2\cdot 8 + 6\cdot 2\cdot 8$
$\Rightarrow 8^2\cdot 6 - 6\cdot 2\cdot 8 = 6x^2 \Rightarrow 64-16 = x^2 \Rightarrow x = \boxed{4\sqrt{3}}$



Round 4 – Algebra 2

1. Basically, $x = \frac{1-b}{a}$ and checking all possible combinations of a and b gives 13 distinct values for x. Some values of a and b give the same value for x, namely (3, 4), (4, 5), and (2, 3) all give −1, and (2, 2) and (4, 3) give −1/2. So out of the 16 possible combinations of a and b, we reject 3, giving $\boxed{13}$.

2. $\log 2013^a = \log 100 = 2 \to a = \frac{2}{\log 2013} \to \frac{1}{a} = \frac{\log 2013}{2}$. Similarly, $\log .2013^b = 100$ gives $\frac{1}{b} = \frac{\log .2013}{2}$. So. $\frac{1}{a} - \frac{1}{b} = \frac{\log 2013 - \log .2013}{2} = \frac{\log \frac{2013}{.2013}}{2} = \frac{\log 10^4}{2} = \boxed{2}$.

Alternate Solution: The solution is independent to the bases.

$10^b = 100 \to b = 2; [10{,}000 \cdot 10]^a = 100 \to a = \frac{2}{5}. \frac{1}{a} - \frac{1}{b} = \frac{5}{2} - \frac{1}{2} = 2$

3. Solve $x + ry = r^2$ and $x + ty = t^2$ by subtracting to obtain $(r-t)y = r^2 - t^2 \to y = r + t$ and $x + r(r+t) = r^2 \to x = -rt$. With $r = \frac{7-\sqrt{51}}{2}$ and $t = \frac{7+\sqrt{51}}{2}$, we obtain the ordered pair $\boxed{\left(\frac{1}{2}, 7\right)}$.

Round 5 – Analytic Geometry

1. Since O and C are both equidistant from the endpoints of segment $\overline{AB}$, $\overline{OC}$ is the perpendicular bisector of $\overline{AB}$ and $m\angle COB = 30$ so the slope of $\overline{OC}$ = tan 30 = $\boxed{\frac{\sqrt{3}}{3}}$.

2. The center of the circle is at $(3, -4)$ with radius 2. The lower end of the vertical diameter and hence the vertex of the parabola is $(3, -6)$. One end of the major axis is $(0, -4)$, the y-intercept of the graph of the linear equation. By symmetry, the other

end is $(6,-4)$. Substituting these points in to $y = ax^2 + bx + c$ gives three equations $-6 = 9a + 3b + c$, $-4 = 0a + 0b + c$, and $-4 = 36a + 6b + c$. The second equation gives $c = -4$. Substituting this into the other two equations and solving them as a linear system of two equations in two variables gives $a = \frac{2}{9}$ and $b = -\frac{4}{3}$. The ordered triple is $\boxed{\left(\frac{2}{9}, -\frac{4}{3}, -4\right)}$

Alternate Solution: The vertex of the parabola is (3, –6) and it contains (0, –4), so the equation is $y + 6 = a(x-3)^2$. Substituting $(0,-4)$ for x and y gives $a = \frac{2}{9}$.

3. By the symmetry of the situation the center of the circle would be $Q(a,a)$, the radius would have length a, and the tangent point of intersection with $xy = 2$ would be $T\left(\sqrt{2},\sqrt{2}\right)$. Then $\left(a-\sqrt{2}\right)^2 + \left(a-\sqrt{2}\right)^2 = a^2 \to a^2 - 4a\sqrt{2} + 4 = 0$. Solving gives $a = 2\sqrt{2} \pm 2$. The larger value lies above the graph so we choose $a = 2\sqrt{2} - 2$. The sum of the coordinates of the center is $\boxed{4\sqrt{2} - 4}$.

Alternate Solution: With $Q = (a,a)$ and $T = (\sqrt{2},\sqrt{2})$, we have $TO = a + a\sqrt{2} = 2 \to$ $a = \frac{2}{\sqrt{2}+1} = 2\sqrt{2} - 2 \to 2a = 4\sqrt{2} - 4$

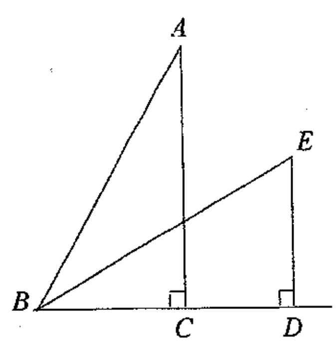
Round 6 – Trig and Complex Numbers

1. $\left[4\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)\right]^2 \cdot 2\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = 16\left(\frac{1}{2} + i - \frac{1}{2}\right)\left(-\sqrt{3} + i\right) = 16i\left(-\sqrt{3} + i\right) =$ $\boxed{-16 - 16i\sqrt{3}}$

Alternate Solution:

$[(4cis45)^2][2cis150] = [16cis90][2\ cis\ 150] = 32cis240 = -16 - 16i\sqrt{3}$.

2. Let the length of the pole be x. Then $ED = \frac{x}{2}$ and $BD = \frac{x}{2}\sqrt{3}$. Also, $BC = \frac{x}{2}$. Since $CD = 20$, then $BD - BC = 20 \to \frac{x}{2}\sqrt{3} - \frac{x}{2} = 20$. Then $x = \frac{40}{\sqrt{3}-1} = \boxed{20\sqrt{3} + 20}$.



3. Since $\Delta APD \cong \Delta BPC$, the area of ΔBPC is 9. Since triangles PAB and PCD are equilateral, then $m\angle BPC = 120°$, giving $\frac{1}{2} \cdot BP \cdot PC \cdot \sin 120 = 9$. Thus, $BP \cdot PC = 18 \cdot \frac{2}{\sqrt{3}} = \boxed{12\sqrt{3}}$.

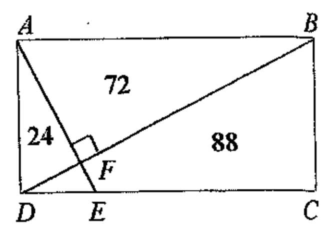
Team Round

1. From $4 - 2^a = 2^c - 4$ we obtain $8 = 2^a + 2^c$. The value of c is as large as possible when 2^a is as small as possible, so let $a = 1$, making $2^c = 6$ giving $\boxed{c = \log_2 6}$.

2. Let $x = 1000$, giving $(x+3)^3 - 3(x+1)^3 + 3(x-1)^3 - (x-3)^3$. The first and last terms sum to $(x^3 + 9x^2 + 27x + 27) - (x^3 - 9x^2 + 27x - 27) = 18x^2 + 54$. The second and third terms sum to $3(x^3 - 3x^2 + 3x - 1) - 3(x^3 + 3x^2 + 3x + 1) = -18x^2 - 6$. Adding this to $18x^2 + 54$ gives $\boxed{48}$. The sum is invariant and does not depend on x.

3. M can't be greater than or equal to 5. If $M = 1$ we have $3 = \frac{51}{17}$ or $7 = \frac{91}{13}$, if $M = 2$, $2 = \frac{42}{21}$ and $2 = \frac{52}{26}$ but both fail since $M = I$. However, $3 = \frac{72}{24}$ and $4 = \frac{92}{23}$ both work. If $M = 3$ we have $3 = \frac{93}{31}$ which fails since $M = I$. If $M = 4$, $2 = \frac{84}{42}$ fails since $E = I$, but $2 = \frac{94}{47}$ works. Thus, the solutions are $2 = \frac{94}{47}$, $3 = \frac{51}{17}$, $7 = \frac{91}{13}$, $3 = \frac{72}{24}$, $4 = \frac{92}{23}$. So I takes on the values of $\boxed{2, 3, 4, \text{and } 7}$.

4. The area of $\Delta EDF = (24 + 72) - 88 = 8$. Since ΔADF and ΔEDF have the same height, the ratio of their areas equals the ratio of their bases so $\frac{AF}{EF} = 3$. Let $AF = 3x$ and $EF = x$. Similarly, let $DF = y$ and $BF = 3y$. Then



$AE \cdot DB = (4x)(4y)$. Since the area of $\Delta DFE = \frac{1}{2} \cdot x \cdot y = \frac{xy}{2} = 8$, then $xy = 16$. This makes $AE \cdot DB = (4 \cdot 4)xy = 16 \cdot 16 = \boxed{256}$. One can solve for the lengths and obtain $FE = \frac{4}{\sqrt[4]{3}}$, $FA = \frac{12}{\sqrt[4]{3}}$, $DF = 4\sqrt[4]{3}$, and $BF = 12\sqrt[4]{3}$,

5. Expanding $\left(z^2+\frac{1}{z^2}\right)^2+\left(z+\frac{1}{z}\right)^2=4$ we obtain $z^4+2+\frac{1}{z^4}+z^2+2+\frac{1}{z^2}=4$. Subtracting 4 and multiplying by z^4 gives $z^8+z^6+z^2+1=0$. This factors as $z^6\left(z^2+1\right)+\left(z^2+1\right)=0 \to \left(z^6+1\right)\left(z^2+1\right)=0$. The six solutions to $z^6=-1$ form a hexagon of radius 1 centered at the origin. The two solutions to $z^2=-1$ are also solutions to the first equation since $\left(z^2\right)^3=z^6=(-1)^3=-1$ so they don't add any vertices. Thus the area is the area of the hexagon which is $6\cdot\frac{1^2\sqrt{3}}{4}=\boxed{\frac{3\sqrt{3}}{2}}$.

6. Let $OA=3a$ and $OB=3b$, making $AB=3\sqrt{a^2+b^2}$. Since $PA=\sqrt{a^2+b^2}$, then $PT=b$, $AT=a$, $OT=2a$, making the slope of $\overline{OP}=\frac{b}{2a}$. Since $QA=2\sqrt{a^2+b^2}$, $QR=2b$, $AR=2a$, $OR=a$, making the slope of $\overline{OQ}=\frac{2b}{a}$. The product of the slopes is $\frac{b^2}{a^2}$. The slope of $\overline{AB}=-\frac{3b}{3a}=-\frac{b}{a}=k$. Thus, the product of the slopes equals $\boxed{k^2}$.