NEW ENGLAND ASSOCIATION OF MATHEMATICS LEAGUES

PLAYOFFS – 2010

Round 1: Arithmetic and Number Theory

1. ______________

2. ______________

3. ______________

1. How many 2-digit prime numbers have the property that the sum of its digits is divisible by 5?

2. Determine the <u>largest</u> possible prime number that can be expressed by $x^2 + 30x - 175$, where x is an integer.

3. Let x be the two-digit number AB, $A \neq 0$. Let y be the six-digit number $ABABAB$.

   For those values of x for which $\frac{y}{x^2}$ is an integer, determine all prime numbers that could be factors of $\frac{y}{x^2}$.

NEW ENGLAND ASSOCIATION OF MATHEMATICS LEAGUES

PLAYOFFS – 2010

Round 2: Algebra 1

1. ______________________

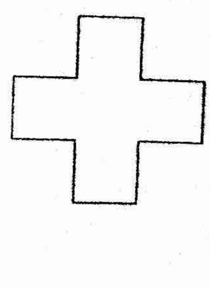
2. ______________________

3. ______________________

1. Determine the <u>smaller</u> value of $\frac{x}{y}$ given $2x^2 + 2xy = 3xy + 3y^2$.

2. Solve for all values of x such that: $x^4 - x^3 - 6x^2 + 6x = x^3 - x^2$.

3. In the figure, all sides are congruent and the angle between each pair of consecutive sides is 90°. If the numerical value of the perimeter subtracted from the numerical value of the area equals K, determine the <u>least</u> possible value of K.



NEW ENGLAND ASSOCIATION OF MATHEMATICS LEAGUES

PLAYOFFS – 2010

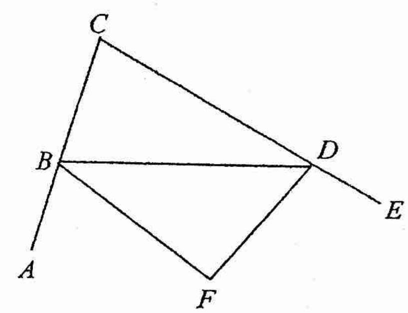
Round 3: Geometry

1. ____________________

2. ____________________

3. ____________________

1. In ΔBCD, $\overline{BF}$ and $\overline{DF}$ are the <u>trisectors</u> of $\angle ABD$ and $\angle EDB$ respectively such that $m\angle DBF = \frac{1}{3}m\angle DBA$ and $m\angle BDF = \frac{1}{3}m\angle BDE$. If $m\angle C = 78°$, determine $m\angle F$.



2. ABC is an equilateral triangle of side 6. Circle P is tangent to $\overline{BC}$ at D and passes through the trisection points of $\overline{AC}$. Find the length of $\overline{CD}$.

3. In ΔABC, $m\angle C = 90°$ and $BC = 7$. Point D lies on $\overline{AB}$ such that $BD = 10$ and $AD = DC$. Find the length of $\overline{AC}$.

NEW ENGLAND ASSOCIATION OF MATHEMATICS LEAGUES

PLAYOFFS – 2010

Round 4: Algebra 2

1. ______________________

2. ______________________

3. ______________________

1. If $\log_3 y = 2\log_5 x$, then $y = x^k$. If k can be written as $\log_a b$, find the ordered pair (a, b), where a and b are integers and $a + b$ has the smallest possible sum.

2. If $4^{1/x} - 8^{1/y} = 0$ and $\log_2 x - \log_4 y = 0$, find all ordered pairs (x, y).

3. Let $f(x) = ax^3 + bx^2 + cx + d$. If $f(1) = 1$, $f(2) = 2$, and $f(3) = 3$, determine the value of $\frac{b}{d}$.

NEW ENGLAND ASSOCIATION OF MATHEMATICS LEAGUES

PLAYOFFS – 2010

Round 5: Analytic Geometry

1. ______________________

2. ______________________

3. ______________________

1. The domain of a relation is $-3 \le x \le 4$ and its range is $-1 \le y \le 8$. If the graph of the relation is rotated through 90° counterclockwise relative to the origin, determine the domain of the rotated graph.

2. Determine the length of the shortest path from the origin to $A(8,6)$ that does not go inside the region determined by the quadrilateral $MNPQ$ given $M(2,4), N(5,4), P(5,1), Q(2,1)$.

3. Points $A(-4, 2)$, $B(4, 2)$, $C(4, -2)$, and $D(-4, -2)$ lie on the graph of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with $a > b$.

   If $\overline{AD}$ and $\overline{BC}$ pass through the focal points of the ellipse, determine the value of $a^2 + b^2$.

NEW ENGLAND ASSOCIATION OF MATHEMATICS LEAGUES

PLAYOFFS – 2010

Round 6: Trig and Complex Numbers

1. ____________

2. ____________

3. ____________

1. Find all real values of a for which $\cos x = -\frac{1}{a}$ and $\cos\frac{x}{2} = \frac{1}{\sqrt{a}}$.

2. For $k > m > 0$, the tangent of the acute angle bounded by $y = kx$ and $y = mx$ is twice the tangent of the first quadrant angle bounded $y = mx$ and the x-axis. If k and m are reciprocals, find k.

3. For $k > 0$ and $i = \sqrt{-1}$, let $a_1 = 1$, $a_n = \frac{i}{k}a_{n-1}$, $b_1 = -1$, and $b_n = -\frac{i}{k}b_{n-1}$. If $\sum_{i=1}^{\infty} a_i - \sum_{i=1}^{\infty} b_i = \frac{16}{9}$, determine the value of k.

MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

NEW ENGLAND PLAYOFFS – 2010

Team Round

1. ____________________ 4. ____________________

2. ____________________ 5. ____________________

3. (____ , ____ , ____ , ____) 6. ____________________

1. For $a, b > 0$, if $\sin\left(\tan^{-1}\frac{a}{b}\right) = \frac{b}{a}$, determine the value of $\frac{a^2}{b^2}$.

2. Let f be a function defined for all real numbers and let f have the property that $f(3-x) = f(x+5)$ for all x. If f has seven distinct roots, determine the average of the roots.

3. Let $P(x)$ be a cubic polynomial whose coefficients are all positive integers. $P(1) = 11$ and $P(P(1)) = 5701$. If $P(x) = ax^3 + bx^2 + cx + d$, determine the ordered quadruple (a, b, c, d).

4. Given weights of 1, 2, 3, 4, 5, and 6 pounds, two are selected at random and placed on one side of a scale. From the remaining four, two are selected at random and placed on the other side of the scale. What is the probability that the scale balances?

5. In ΔABC, $m\angle C = 84°$, $m\angle B = 54°$, $BC = a$ and $AC = b$, where a and b are lengths appropriate for a triangle with those angles.
Determine the length of $\overline{AB}$ solely in terms of a and b.

6. Points A and B lie on circle O of radius 12 such that $m\angle AOB = 120°$. Point P lies on minor arc(AB) such that $m\angle POB = 40°$. Perpendiculars from P intersect $\overline{OA}$ and $\overline{OB}$ at C and D respectively. Determine the length of $\overline{CD}$.

MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

NEW ENGLAND PLAYOFFS – 2010 - SOLUTIONS

Round 1 Arithmetic and Number Theory

1. 5, the numbers are 19, 23, 37, 41, 73

2. $x^2 + 30x - 175 = (x-5)(x+35)$. For the number to be prime, one of the two factors must be 1 and the other a prime, or one must be -1 and the other the negative of a prime. If $x = 6$, the product equals 41. If $x = -36$, the product also equals 41. Answer: $\boxed{41}$.

3. Since $y = 10101x$, $\frac{y}{x^2} = \frac{10101x}{x^2} = \frac{10101}{x} = \frac{3 \cdot 7 \cdot 13 \cdot 37}{x}$. Thus, the primes that could be factors of $\frac{y}{x^2}$ are $\boxed{3, 7, 13, 37}$.

Round 2 Algebra 1

1. $2x^2 + 2xy - 3xy - 3y^2 = 0 \to 2x(x+y) - 3y(x+y) = 0 \to (2x-3y)(x+y) = 0$. Thus, $2x = 3y \to \frac{x}{y} = \frac{3}{2}$ or $x = -y \to \frac{x}{y} = -1$. Ans: $\boxed{-1}$.

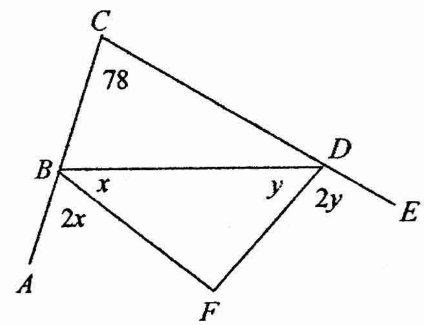
   Alternate Solution Divide by y^2 and factor $\left(2\frac{x}{y} - 3\right)\left(\frac{x}{y} + 1\right) = 0$

2. Rewriting the equation as $x^3(x-1) - 6x(x-1) = x^2(x-1) \to (x-1)(x^3 - x^2 - 6x) = 0$ $\to x(x-1)(x-3)(x+2) = 0$ $\boxed{x = 0,\ 1,\ 3,\ -2}$.

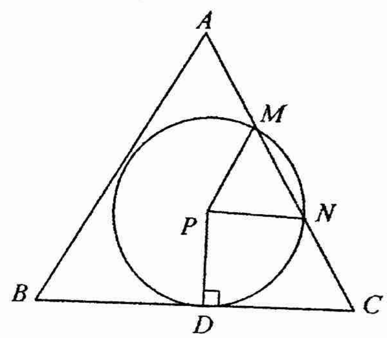
3. Let x = the length of a side. Then $5x^2 - 12x = K \to 5x^2 - 12x - K = 0$. Since $x = \frac{12 + \sqrt{12^2 + 20K}}{10}$, the minimum of the function occurs when $144 + 20K = 0$, making $K = -\frac{36}{5}$ and this occurs when $x = 6/5$. Or the minimum of $y = 5x^2 - 12x - K$ occurs when $x = -\frac{-12}{2 \cdot 5} = \frac{6}{5}$ which makes the area equal to $5 \cdot \left(\frac{6}{5}\right)^2 = \frac{36}{5}$ and the perimeter equal to $12 \cdot \frac{6}{5} = \frac{72}{5}$. The difference is $\boxed{-\frac{36}{5}}$.

Round 3 – Geometry

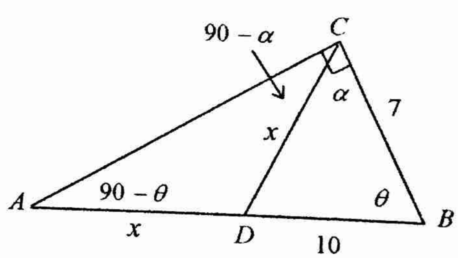
1. By the Exterior Angle Theorem, $3x = 78 + (180 - 3y)$ giving $x + y = 86$. Thus, $\boxed{m\angle F = 94}$.



2. $NC = MN = 2$ and by the power of the point theorem, $CD^2 = CN \cdot CM$, so $CD^2 = 2(2+2) = 8 \rightarrow \boxed{CD = 2\sqrt{2}}$.



3. Let $m\angle B = \theta$, then $m\angle A = 90 - \theta$. Let $m\angle DAC = \alpha$, then $m\angle DCA = 90 - \alpha$. Since $AD = DC$, then $90 - \theta = 90 - \alpha$, making $\theta = \alpha$. Thus $DC = DB$ so $AD = 10$. $AC^2 = 20^2 - 7^2 = 351$. $\boxed{AC = 3\sqrt{39}}$.



Round 4 – Algebra 2

1. $\dfrac{\ln y}{\ln 3} = \dfrac{2\ln x}{\ln 5} \rightarrow \ln y = \left(\dfrac{2\ln 3}{\ln 5}\right)\ln x \rightarrow \ln y = \ln x^{(2\ln 3)/\ln 5}$. Thus, $y = x^{(2\ln 3)/\ln 5}$.

Thus, k equals $\dfrac{2\ln 3}{\ln 5} = \dfrac{\ln 9}{\ln 5} = \log_5 9$, making $(a, b) = \boxed{(5, 9)}$.

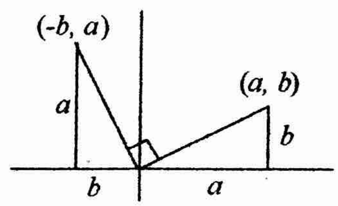
2. $4^{1/x} = 8^{1/y} \to 2^{2/x} = 2^{3/y} \to \frac{2}{x} = \frac{3}{y} \to y = \frac{3}{2}x$. $\log_2 x = \log_4 y \to$
$\log_2 x = \frac{1}{2}\log_2 y \to x = \sqrt{y}$. Thus, $x = \sqrt{\frac{3x}{2}} \to x^2 = \frac{3x}{2}$. We reject $x = 0$ since it lies outside the domain and accept $x = \frac{3}{2}$. Answer: $\boxed{\left(\frac{3}{2}, \frac{9}{4}\right)}$.

3. Substituting we have $f(1) = a + b + c + d = 1$, $f(2) = 8a + 4b + 2c + d = 2$, and $f(3) = 27a + 9b + 3c + d = 3$. Subtracting the first from each of the other two yields two equations: $7a + 3b + c = 1$ and $19a + 5b + c = 1$. Subtracting the first from the second yields $12a + 2b = 0$. Thus, $b = -6a$. Substituting into $7a + 3b + c = 1$ gives $c = 11a + 1$. Substituting into $a + b + c + d = 1$ gives $d = -6a$. Thus, $\boxed{\frac{b}{d} = 1}$.

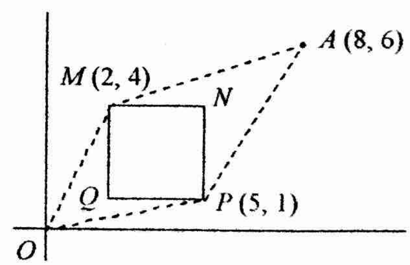
Alternate Solution: Randomly, let $a = 1$, this leads to 3 equations in three unknowns, the solution of which is $b = -6, c = 12, d = -6$

Round 5 – Analytic Geometry

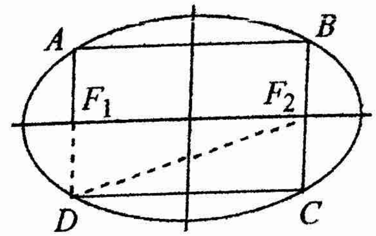
1. From the diagram, note that rotating the point (a, b) through 90° results in the point $(-b, a)$ since the product of the slopes of perpendiculars is -1. Not only are the domain and range interchanged but the new domain is the negative of the old range. Thus, the new domain is $\boxed{-8 \le x \le 1}$.



2. By the triangle inequality it is clear that the shortest path north of the square passes through M and no other point on the square. Similarly, the shortest path south must go through P. $OP + PA = \sqrt{26} + \sqrt{34}$ and $OM + MA = \sqrt{20} + \sqrt{40}$. To compare them, square both sums, obtaining $26 + 2\sqrt{26 \cdot 34} + 34$ and $20 + 2\sqrt{20 \cdot 40} + 40$. The non-radical parts are equal so ignore them. Either calculate $26 \cdot 34 = 884$ and $20 \cdot 40 = 800$ to determine the shortest path or use the GM-AM result that says that the smaller product of pairs of numbers with the same sum occurs with the pair that has the greatest difference. In either way the shortest distance is $OM + MA = \boxed{2\sqrt{5} + 2\sqrt{10}}$.



3. Since $F_1D + DF_2 = 2a$, then $2a = 2 + \sqrt{2^2 + 8^2} = 2 + 2\sqrt{17}$, making $a = 1 + \sqrt{17}$. Thus, $a^2 = 18 + 2\sqrt{17}$. Since $F_2 = (4,0)$, then $c = 4$, and from $b^2 + c^2 = a^2$, we obtain $b^2 = 2 + 2\sqrt{17}$, making $a^2 + b^2 = \boxed{20 + 4\sqrt{17}}$.

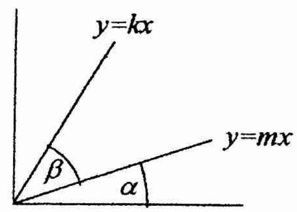


Round 6 – Trig and Complex Numbers

1. Since $\cos\frac{x}{2} = \sqrt{\frac{1+\cos x}{2}}$ we have $\frac{1}{\sqrt{a}} = \sqrt{\frac{1-\frac{1}{a}}{2}} = \sqrt{\frac{a-1}{2a}}$. Thus, $\sqrt{a-1} = \sqrt{2}$ so $\boxed{a = 3}$.

   Alternate Solution Use $\cos x = 2\cos^2\left(\frac{x}{2}\right) - 1$, then $-\frac{1}{a} = \frac{2}{a} - 1$

2. Since $\tan\alpha = m$ and $\tan\beta = \frac{k-m}{1+km} = \frac{k-m}{2} = 2m$, then $k = 5m$. Since $km = 1$, then $k = \frac{5}{k} \rightarrow \boxed{k = \sqrt{5}}$.



3. $\sum_{i=1}^{\infty} a_i = 1 + \frac{i}{k} - \frac{1}{k^2} - \frac{i}{k^3} + \cdots = \frac{1}{1-\frac{i}{k}} = \frac{k}{k-i} = \frac{k^2+ki}{k^2+i}$. In similar fashion

   $\sum_{i=1}^{\infty} b_i = \frac{-k^2+ki}{k^2+1}$. Then $\sum_{i=1}^{\infty} a_i - \sum_{i=1}^{\infty} b_i = \frac{2k^2}{k^2+1} = \frac{16}{9} \rightarrow k^2 = 8$. Thus, $\boxed{k = 2\sqrt{2}}$.

Team Round

1. Let $\theta = \tan^{-1}\frac{a}{b} \to \tan\theta = \frac{a}{b} \to \sin\theta = \frac{a}{\sqrt{a^2+b^2}}$. Thus $\frac{a}{\sqrt{a^2+b^2}} = \frac{b}{a} \to$

$a^2 = b\sqrt{a^2+b^2} \to a^4 - a^2b^2 - b^4 = 0$. Divide by b^4 to obtain $\left(\frac{a}{b}\right)^4 - \left(\frac{a}{b}\right)^2 - 1 = 0$.

Solving for $\frac{a^2}{b^2}$ gives $\boxed{\frac{1+\sqrt{5}}{2}}$.

Alternate Solution, Let $b = 1$, then proceed as above

2. If $3 - x_1$ is a root, then $x_1 + 5$ is also a root. Likewise for $3 - x_2, x_2 + 5, 3 - x_3$, and $x_3 + 5$ The sum of these six roots is 24. For there to be seven roots $3 - x_7$ must equal $x_7 + 5 \to$ $x_7 = -1$, making the seventh root equal to 4. Thus the average is $\frac{24+4}{7} = \boxed{4}$.

Alternate solution
$3 - x = x + 5 \rightarrow x = -1$
Thus, the function is symmetric about $x = 4$. Three roots are less than 4 and three roots are greater than 4 (mirror images of the first three). The average of these 6 roots is clearly 4. The 7th root must be 4 (or it would have a mirror image and there would be 8 distinct roots). Therefore, the average is $\boxed{4}$.

3. Let $P(x) = ax^3 + bx^2 + cx + d$, then $P(11) = a \cdot 11^3 + b \cdot 11^2 + c \cdot 11 + d = 5701$. Thus, if we write 5701 in base 11 we'll have a, b, c, and d. Since 5701 = $4 \cdot 1331 + 3 \cdot 121 + 1 \cdot 11 + 3$, then $\boxed{P(x) = 4x^3 + 3x^2 + x + 3}$.

Alternate solution
$5701 = 11^3a + 11^2b + 11c + d$. Dividing by 11 leaves a quotient of 518 and a remainder of 11 on the left side; on the right side, the quotient is $121a + 11b + c$ and the remainder is d.
Thus, $d = 3$ and $121a + 11b + c = 518$
$P(1) = 11 \rightarrow a + b + c + d = 11 \rightarrow a + b + c = 8$
Subtracting, $510 = 120a + 10b \rightarrow b = 3(17 - 4a)$
Since a, b, c and d are positive integers, the only possible a values are 1, 2, 3 and 4.
The only one that allows c to be positive is $a = 4$.
Thus, $(a, b, c, d) = \boxed{(4, 3, 1, 3)}$.

4. The sums of weights that could be balanced on the other side are 9, 8, 7, 6, and 5. We can obtain 11 with (6, 5), but not with a second combination. We can obtain 10 with (6, 4) but not with a second combination. The combinations for 9, 8, 7, 6, and 5 are shown below:

9: (6, 3), (5, 4)   8: (6, 2). (5, 3)   7: (6, 1), (5, 2), (4, 3)
6: (5, 1), (4, 2)   5: (4, 1), (3, 2)

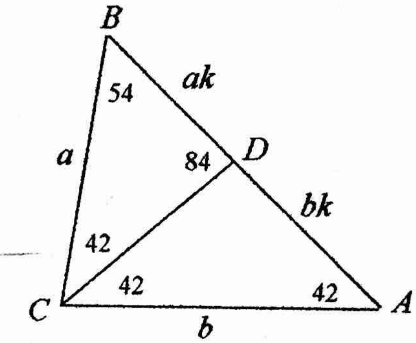
The probability of obtaining a balancing combination of weights is:

$$\frac{{}_2C_1 + {}_2C_1 + 3\cdot 2 + {}_2C_1 + {}_2C_1}{{}_6C_2 \cdot {}_4C_2} = \frac{14}{15\cdot 6} = \boxed{\frac{7}{45}}.$$

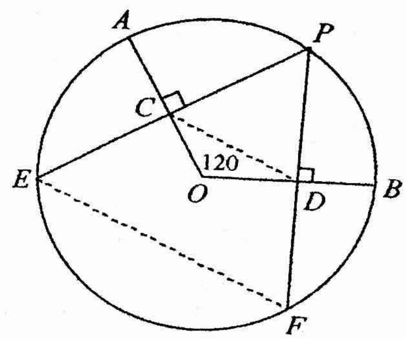
5. Draw the bisector of $\angle BCA$. Then $\Delta BDC \sim \Delta BCA$ giving $\frac{BD}{BC} = \frac{BC}{BA}$. Since $\overline{CD}$ bisects $\angle BCA$, then $BD = ak$ and $DA = bk$ for some value k. Substituting gives $\frac{ak}{a} = \frac{a}{ak + bk}$.

Thus, $k^2 = \frac{a}{a+b}$ giving $k = \sqrt{\frac{a}{a+b}}$. Then

$$AB = (a+b)\sqrt{\frac{a}{a+b}} = \boxed{\sqrt{a(a+b)}}.$$



6. Extend $\overline{PC}$ and $\overline{PD}$ to meet O at E and F respectively. Quadrilateral $PCOD$ has two 90° angles and one 120° angle so $m\angle P = 60°$. Since $\overline{OA}$ and $\overline{OB}$ are radii perpendicular to chords, they bisect the chords. Thus, $\overline{CD}$ is a midline of ΔPEF and equals $\frac{1}{2}EF$. $\overline{EF}$ is a chord cutting off a 120° arc in a circle of radius 12, so $EF = 12\sqrt{3}$, making $\boxed{CD = 6\sqrt{3}}$. Note that this is just the length of the altitude from A to $\overline{OB}$.



Alternate solution: Let $OC = x$ and $OD = y$. Then $\cos 80 = \dfrac{x}{12}$, $\cos 40 = \dfrac{y}{12}$.

$CD^2 = x^2 + y^2 - 2xy\cos 120 = x^2 + y^2 + xy =$
$12^2\cos^2 80 + 12^2\cos^2 40 + 12^2\cos 40\cos 80$. From
$\dfrac{CD^2}{12^2} = (\cos(60+20))^2 + (\cos(60-20))^2 + \cos(60+20)\cos(60-20)$ we obtain

$$\frac{CD^2}{144} = (\cos 60\cos 20 - \sin 60\sin 20)^2 + (\cos 60\cos 20 + \sin 60\sin 20)^2 +$$
$$(\cos 60\cos 20 - \sin 60\sin 20)(\cos 60\cos 20 + \sin 60\sin 20)$$

This reduces to $3\cos^2 60\cos^2 20 + \sin^2 60\sin^2 20 = \dfrac{3}{4}\cos^2 20 + \dfrac{3}{4}\sin^2 20 = \dfrac{3}{4}$. Thus,

$CD^2 = 144\cdot\dfrac{3}{4} = 108$, making $CD = \sqrt{108} = 6\sqrt{3}$.