GREATER BOSTON MATHEMATICS LEAGUE

MEET 5 – MARCH 1999

ROUND 1 – Arithmetic

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. The retail price of an item underwent the following changes from year to year over a four year period: a 25% increase, a 25% decrease, a 20% increase, and finally a 20% decrease. **What percent increase or decrease** was the final retail price over the original retail price of the item? You must write the word **increase** or **decrease** as part of your answer.

2. Thirty children were polled about their likes or dislikes of Dawson's Creek or The Rugrats. The number of children who like Dawson's Creek is twice the number who like The Rugrats. The number who dislike Dawson's Creek equals the number who like The Rugrats. There are four children who like neither. Find the number of children who like Dawson's Creek and do not like The Rugrats.

3. Given the following base 5 addition:

$$\begin{array}{cccc} a & b & c & d_{(5)} \\ + & b & d & c_{(5)} \\ \hline a & a & d & b_{(5)} \end{array}$$

where $a \neq 0$, $b \neq 0$ and a, b, c, and d are different digits, find the four digit number $a\ b\ c\ d_{(5)}$

GREATER BOSTON MATHEMATICS LEAGUE

MEET 5 – MARCH 2000

ROUND 1 – Arithmetic

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Given the 5-digit base ten number, $53{,}7T2$, is divisible by 12, find all possible values for *T*.

2. Given the following "*incomplete*" addition problem of 4-digit numbers in base *r*. Find the **sum** of all possible values for *r*.

$$\begin{array}{cccc} 1 & 0 & 3 & 8_r \\ 2 & 5 & 6 & 7_r \\ 3 & 6 & 1 & 8_r \\ 3 & 5 & 0 & 7_r \\ 7 & 8 & 2 & 8_r \\ \hline \ldots & \ldots & \ldots & 2_r \end{array}$$

3. How many 2-digit natural numbers have exactly 8 factors?

GREATER BOSTON MATHEMATICS LEAGUE

MEET 5 – MARCH 2001

ROUND 1 – Arithmetic
Problem submitted by Newton South.

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. For what value of k is the following equation true?

   $2000^2 + 2000^3 = 2001k$

2. Given the following multiplication in base 8, find the base 10 value of the number $x\,y_8$.

$$\begin{array}{r} x\,y_8 \\ 2 \\ \hline y\,x_8 \end{array}$$

3. The greatest common factor of two whole numbers is 12 and their least common multiple is 2520. Find the smallest possible value for their sum.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 5 – MARCH 2002

ROUND 1 – Arithmetic
Problems submitted by Newton South and Maimonides.

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. A fraction has the value $\frac{12}{13}$. If the sum of the numerator and denominator is 2000, find the fraction's denominator.

2. Find the only whole number that equals 12 times the sum of its digits.

3. Given N is a factor of 2002 and N has exactly 4 factors. How many possible values for N are there?

GREATER BOSTON MATHEMATICS LEAGUE
MEET 5 – MARCH 2006

ROUND 1 – Arithmetic: Open

1. ____________________

2. (________, ________)

3. ____________________ (9)

CALCULATORS ARE NOT ALLOWED ON THIS ROUND.

1. Let A denote the sum of the first 5 primes and B the sum of the next 4 primes. What is the least common multiple of A and B?

2. Let $\sqrt{1\frac{25}{144}} - \sqrt[3]{0.125} - 0.2\bar{3}_{10} = \frac{a}{b}$, where a and b are relatively prime positive integers. Determine the ordered pair (a, b).

3. What is the base 9 equivalent of 0.412_6?

GREATER BOSTON MATHEMATICS LEAGUE
MEET 5 – March 2007

ROUND 1 – Arithmetic: Open

1. ______________$_{(7)}$

2. ______________

3. ______________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND.

Express all answers requested in terms of the <u>exact</u> number of units.

1. Given: $\dfrac{3}{13_{(7)}} + \dfrac{5}{21_{(7)}} = \dfrac{p}{q_{(7)}}$ If $\dfrac{p}{q}$ is a reduced fraction, determine the sum $(p+q)_{(7)}$.

   Note: The subscript $_{(7)}$ denotes the fraction is written in base 7.

2. Ann bowled a score of 182 in her most recent game, raising her average from 154 to 156. After the next game, her average increased to 158. What was her score for that game?

3. The numbers in the sequence $-13, -2, 9, 20, 31, \ldots$ increase by eleven. The numbers in the sequence $6, 19, 32, 45, 58, \ldots$ increase by 13. The first number that occurs in both sequences is less than 100. Find the sum of the next two numbers that occur in both sequences.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 5 – MARCH 2008

ROUND 1 – Arithmetic: Open

1. (__________, __________)

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND.

1. If $0.33_{(4)} - 0.22_{(3)} = 0.xy_{(12)}$, determine the ordered pair (x, y).

2. Given the following sequence of numbers: (17, 311), (24, 300), (31, 289), (38, 278), …, where (17, 311) is the first term, determine in which term the first coordinate exceeds the second coordinate for the first time.

3. If $a^3 = b^3 + c^3 + d^3$, where a, b, c and d are natural numbers less than 10 and $a > b > c > d$, find all possible quadruples (a, b, c, d).

GREATER BOSTON MATHEMATICS LEAGUE

MEET 5 – FEBRUARY 2009

ROUND 1 – Arithmetic: Open

1. ______________________________

2. ______________________________

3. ______________________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND.

1. Given: $4d5_{(8)} = d003d_{(4)}$ Determine the numerical value of the digit d.

2. If the <u>median</u> of 81, 75, 54, 62, 77, 41, x, 26, and 97 is a prime number, what are the possible values(s) of x?

3. The 20$^{\text{th}}$ term in the following sequence may be expressed in the form $2^A \cdot B \cdot C$, where A is an integer, B and C are primes ($B < C$). Determine the ordered triple (A, B, C).

$$2\cdot5\cdot9,\quad 4\cdot7\cdot10,\quad 8\cdot9\cdot11,\quad 16\cdot11\cdot12,\quad 32\cdot13\cdot13,\quad 64\cdot15\cdot14, \ldots$$

GREATER BOSTON MATHEMATICS LEAGUE

MEET 5 – FEBRUARY 2010

ROUND 1 – Arithmetic: Open

1. ______________________

2. ______________________

3. ______________________ (5)

CALCULATORS ARE NOT ALLOWED ON THIS ROUND.

1. What is the sum of the smallest and largest 3-digit positive palindromes that are divisible by 11?

2. Determine the units value of the following expression: $7^{31} + 18^{43} - 36^{20} - 3^{24}$

3. If $54_{(x-2)} = 45_{(x+1)}$, find x^2 in base 5.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 5 – MARCH 2011

ROUND 1 – Arithmetic: Open

1. ______________________________

2. ______________________________

3. ______________________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND.

1. In a survey of 75 college students, 26 take math, 22 take science and 20 take history or English. Of these 20, no one takes math or science. If 8 students take both math and science, how many do not take any of these subjects?

2. Find all ordered triples (A, B, C), where $A \neq 0$, which satisfy the following statement:

$$ABC_{(7)} = BCA_{(5)}$$

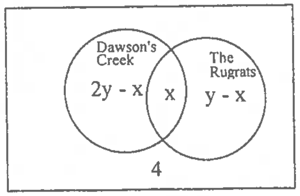
3. What is the sum of the smallest and largest 3-digit positive palindromes that are divisible by 11?

Created with
nitroPDF professional
download the free trial online at nitropdf.com/professional

'99

ROUND 1

1. 25% increase, then a 25% decrease, then a 20% increase, and a 20% decrease ⇒
$\frac{5}{4}\cdot\frac{3}{4}\cdot\frac{6}{5}\cdot\frac{4}{5}=\frac{9}{10}=$ ***10% decrease***



2. Liking The Rugrats = y; Liking Dawson's Creek = $2y$;
Liking both = x; $y-x+4=y \Rightarrow x=4$;
$3y-4=26 \Rightarrow y=10$; Liking Dawson's Creek and not The Rugrats = $2y-x=\mathbf{16}$

3. $$\begin{array}{r} a\ \ b\ \ c\ \ d_{(5)} \\ +\ \ b\ \ d\ \ c_{(5)} \\ \hline a\ \ a\ \ d\ \ b_{(5)} \end{array}$$
Since the last 2 columns are the same, but the sums are different ⇒
$d+c>4$ and since the 3rd column adds to $d \Rightarrow c=4$; there is no carry from the 2nd column to the 1st column ⇒ $b=1$ or 2; If $b=2$, then the 2nd column would add to 0 because of the carry from the 3rd column ⇒ $b=1 \Rightarrow a=3 \Rightarrow d=2$ since $d+4>4$
$a\,b\,c\,d_{(5)}=\mathbf{3142}_{(5)}$

'00

ROUND 1

1. Since $53{,}7T2$ is divisible by both 3 and 4, 4 must divide evenly into the last 2 digits ⇒
T could be 1, 3, 5, 7, or 9. The sum of the digits must be a multiple of 3 ⇒
$17+T$ is a multiple of 3 ⇒ $T=1$ or 7.

2. $r>8$; the units' column adds to 38 ⇒ $kr+2=38$, for whole number k; $kr=36$ ⇒
r is a factor of 36 greater than 8 ⇒ $r=9,12,18,36$ whose sum equals 75.

3. If a number has exactly 8 factors, it must be of the form:
p^7, or p^3q, or pqr, where p, q, and r are prime numbers. Since $2^7>99$, there are none of the first type; $2^3\cdot3$, $2^3\cdot5$, $2^3\cdot7$, $2^3\cdot11$, $3^3\cdot2$ are the only ones of the second type less than 100; $2\cdot3\cdot5$, $2\cdot3\cdot7$, $2\cdot3\cdot11$, $2\cdot3\cdot13$, $2\cdot5\cdot7$ are the only ones of the third type less than 100; therefore there are 10 possibilities altogether.

'01

ROUND 1

1. $2000^2+2000^3=2001k \rightarrow 2000^2(1+2000)=2001k \rightarrow k=2000^2=4000000$

2. $2(8x+y)=8y+x \rightarrow 16x+2y=8y+x \rightarrow 15x=6y \rightarrow 5x=2y \rightarrow$
$x=2,\ y=5 \rightarrow 8x+y=21$

3. Since their GCF = 12, the numbers are of the form $12x$ and $12y$. Since their LCM =2520, $(12x)(12y)=12\cdot2520 \rightarrow xy=210$. The smallest value for $x+y$ is when the factors of 210 have the smallest difference $\rightarrow x=14,\ y=15 \rightarrow 12x+12y=12\cdot29=348$.

'02

ROUND 1 – Arithmetic

1. Let the numerator = $12x$ ⇒ the denominator = $13x$ ⇒
$12x+13x=2000 \Rightarrow 25x=2000 \Rightarrow x=80 \Rightarrow 13x=1040$.

2. If the number is 2-digit ⇒ $10t+u=12(t+u) \Rightarrow 2t+11u=0$, which is clearly impossible.
If the number is 3-digit ⇒ $100h+10t+u=12(h+t+u) \Rightarrow 88h=2t+11u$ which is true only if $u=8$ and $t=0 \Rightarrow h=1$ ⇒ the number is 108.

3. $2002=2\times7\times11\times13$; any whole number with 4 factors is of the form $p_1\cdot p_2$ or p_1^3 where p_1 and p_2 are primes; since 2002 is the product of 4 distinct primes, there are ${}_4C_2=6$ different pairs of primes ⇒ there are 6 values for n.

'06

ROUND 1 – Arithmetic: Open

1. $A=2+3+5+7+11=28=2^27^1$. $B=13+17+19+23=72=2^33^2$.
The least common multiple is the product of all prime factors, each raised to the highest occurring exponent. Thus, the LCM = $2^37^13^2=$ **504**

2. $\sqrt{1\frac{25}{144}}=\sqrt{\frac{169}{144}}=\sqrt{\frac{13^2}{12^2}}=\frac{13}{12}$. $\sqrt[3]{0.125}=\sqrt[3]{\frac{1}{8}}=\sqrt[3]{\frac{1^3}{2^3}}=\frac{1}{2}$

Let $n=0.2\bar{3}$. Then $100n=23.333\ldots$
$10n=2.333\ldots$ Subtracting, $90n=21 \rightarrow n=\frac{21}{90}=\frac{7}{30}$

$\frac{13}{12}-\frac{1}{2}-\frac{7}{30}=\frac{65-30-14}{60}=\frac{21}{60}=\frac{7}{20} \rightarrow$ **(7, 20)**

3. Method #1: $0.412_6=\frac{4}{6}+\frac{1}{36}+\frac{2}{216}=\frac{152}{216}=\frac{19}{27}=\frac{a}{9}+\frac{b}{81}+\frac{c}{729}+\ldots$

Find the largest value of a for which $\frac{19}{27}\ge\frac{a}{9}\rightarrow\frac{19}{3}\ge\frac{a}{1}\rightarrow a=6$.

The remainder is $\frac{19}{27}-\frac{6}{9}=\frac{1}{27}$.

Find the largest b. $\frac{1}{27}\ge\frac{b}{81}\rightarrow\frac{1}{3}\ge\frac{b}{9}\rightarrow b=3$.

The remainder is 0, the process stops. Thus, $0.412_6=0.63_9$

Method #2: Write the new base (9) as a number in the old base (6) → $9=13_6$
Multiply the given fraction by this value and keep track of the integer and fractional parts separately. The process terminates when the fractional part becomes zero or when a repeating pattern is observed. Multiplying in base 6 may seem a bit strange. Remember:
multiples of 6 are carried to the next column.
(Ex (in base 6): 2 x 3 = 10, 2 x 4 = 12, 4 x 5 = 32)

```
 0.412
x   13
  2040
  4120
 6.200
x   13
  1000
  2000
 3.000
```

Method #3: $0.412_6=19/27=0.\overline{703}$
$9(.\overline{703})=6.\bar{3}$, $9(\bar{3})=3.\bar{0}$
Short, but computation is difficult to perform without a calculator.

'07

ROUND 1 – Arithmetic: Open

1. $\frac{3}{13_{(7)}} + \frac{5}{21_{(7)}} = \frac{3}{10_{(10)}} + \frac{1}{3_{(10)}} = \frac{19}{30_{(10)}} \rightarrow (p+q)_{(10)} = 49_{(10)} = \underline{\mathbf{100}}_{(7)}$

2. Let n denote the number of games bowled to establish an average of 154.
Thus, $154n$ points must increase to $156(n+1)$ after bowling 182. $2n + 156 = 182 \rightarrow n = 13$
To increase the average 2 points above the previous average of 156 in the next game, i.e. the 15^{th} game, we require a score $15(2) = 30$ points higher $\rightarrow$ **186**

3. 1st sequence continues 42, 53, 64, 75, 97,...
2nd sequence continues 58, 71, 84, 97, ...
1st term = 97. Subsequent terms are determined by adding 143, the LCM of 11 and 13.
Thus, 240 + 383 = **623**

'08

ROUND 1

1. $0.33_{(4)} - 0.22_{(3)} = \left(\frac{3}{4}+\frac{3}{16}\right) - \left(\frac{2}{3}+\frac{2}{9}\right) = \frac{15}{16} - \frac{8}{9} = \frac{135-128}{144} = \frac{7}{144} = 0.07_{(12)}$
Thus $(x, y) =$ **(0, 7)**.

2. Since the first coordinates are increasing by 7 and the second coordinates are decreasing by 11,
$17 + 7k > 311 - 11k \rightarrow 18k > 294 \rightarrow k > 16\frac{1}{3} \rightarrow k = 17 \rightarrow$ **18**th term

3. Starting with a list of the perfect cubes: 1, 8, 27, 64, 125, 216, 343, 512, 729
Clearly, $a > 5$, since $8 + 27 + 64 = 99 < 125$
For $a = 6$, we have $27 + 64 + 125 = 216 \rightarrow$ **(6,5,4,3)** as the first solution.
For $a = 7$, b must be 6 forcing $c^3 + d^3 = 343 - 216 = 127$ and this is not possible.
For $a = 8$, b must be 7 forcing $c^3 + d^3 = 512 - 343 = 169$ and this is not possible.
For a = 9, $c^3 + d^3 = 729 - 512 = 217$ $(= 261 + 1) \rightarrow$ **(9, 8, 6, 1)**

'09

ROUND 1

1. $4d5_{(8)} = 4(8)^2 + d(8)^1 + 5(8)^0 = 256 + 8d + 5 = 261 + 8d$
$d003d_{(4)} = d(4)^4 + 3(4)^1 + d(4)^0 = 256d + 12 + d = 257d + 12$
Equating, $249d = 249 \rightarrow d =$ **1**

2. Arranging the 8 known numbers into increasing order: 26, 41, 54, 62, 75, 77, 81, 97
The median of 9 numbers that have been arranged in increasing order will be located in the 5^{th} position.
If $x \le 62$, then the median would be 62 (which is not prime).
If $62 < x < 75$, then x would be the median. $\rightarrow$ **67, 71, 73**
If $x \ge 75$, then the median would be 75 (which is not prime).

3. $t_N = (2^N)(2N+3)(N+8) \rightarrow 2^{20}(43)(28) = 222(7)(43) \rightarrow$ **(A, B, C) = (22, 7, 43)**

'10

ROUND 1

1. Applying the divisibility rule for 11 to a 3-digit numeral ABA, $2A - B = 0$ or 11
$\rightarrow B = 2A$ or $B = 2A - 11$
The first condition produces a smallest of 121 and a largest of 484.
The second condition produces a smallest of 616 and a largest of 979.
Thus, the sum of the maximum and minimum = 121 + 979 = **1100**.

Alternately, brute force works very well.
We require that $11n \ge 100$. Start with $n = 10$.

2. Consider the units digits of powers of 7: 7, 49, __3, __1, __7, ...
The cyclical pattern (7, 9, 3, 1) indicates that 7^4 ends in 1 and $7^{31} = (7^4)^7 \cdot 7^3$ ends in 3.
The units digits of powers of 18 form another 4-cycle (8, 4, 2, 6).
Since the units digit of any integer ending in 6 also ends in 6, $18^{43} = (18^4)^{10} \cdot 18^3$ ends in 2.
36^{20} ends in 6.
The units digits of powers of 3 form a 4-cycle (2, 9, 7, 1) and $3^{24} = (3^4)^6$ ends in 1.
Thus, $3 + 2 - 6 - 1 = -2$ and the units digit is $10 - 2 =$ **8**.

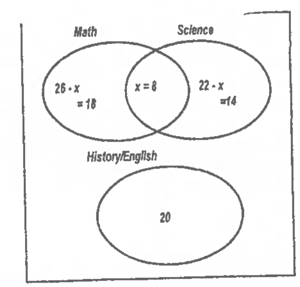
3. $5(x-2) + 4 = 4(x+1) + 5 \rightarrow 5x - 6 = 4x + 9 \rightarrow x = 15$

$$\begin{array}{ccc} & 5^3\ 5^2\ 5^1\ 5^0 & \\ 15^2 = 225_{(10)} \rightarrow & 125 + 100 = 225 & \rightarrow \underline{\mathbf{1400}}_{(5)} \\ & 1\ \ 4\ \ 0\ \ 0 & \end{array}$$

'11

ROUND 1

1. The Venn diagram at the right summarizes the given information. The count of 'other' students is
$75 - (18 + 8 + 14 + 20) =$ **15**



2. Since A, B and C must be legal digits in bases 5 and 7, the literals A, B and C may only assume values of 0, 1, 2, 3 and 4 and both A and B must be nonzero.
$\rightarrow 49A + 7B + C = 25B + 5C + A \rightarrow 48A - 18B - 4C = 0$
$\rightarrow C = \frac{3(8A - 3B)}{2} \rightarrow B$ must be even.
$B = 2 \rightarrow C = 3(4A - 3) \rightarrow (A, C) = (1, 3)$, $\cancel{(2, 15)}$
$B = 4 \rightarrow C = 3(4A - 6) \rightarrow (A, C) = \cancel{(2, 6)}$
Thus, the only solution is **(1, 2, 3)**.

3. Applying the divisibility rule for 11 to a 3-digit numeral ABA, $2A - B = 0$ or 11
$\rightarrow B = 2A$ or $B = 2A - 11$
The first condition produces a smallest of 121 and a largest of 484.
The second condition produces a smallest of 616 and a largest of 979.
Thus, the sum of the maximum and minimum = 121 + 979 = **1100**.