GREATER BOSTON MATHEMATICS LEAGUE

MEET 5 – MARCH 1999

TEAM ROUND

3 pts. 1. ____________________

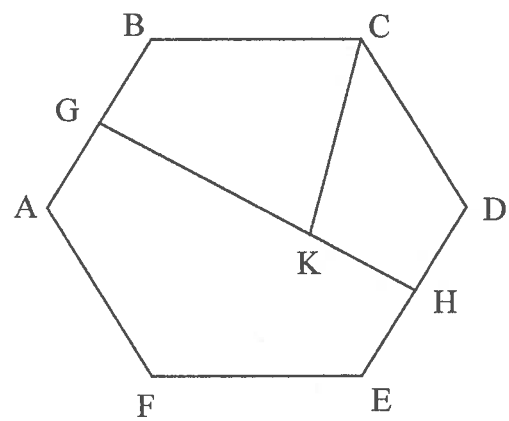
3 pts. 2. ____________________

4 pts. 3. ____________________

SAT APPROVED CALCULATORS ALLOWED ON THIS ROUND
except for the **TI-89 Calculator**, which is not allowed on the Team Round

1. Over the complex numbers, the solutions for x for the cubic equation $x^3 + 6x^2 + 21x + c = 0$ form an arithmetic sequence. Find the value for c.

2. Given regular hexagon, ABCDEF, 12 units on a side, with points G and H, midpoints of sides $\overline{AB}$ and $\overline{DE}$, and GK:KH = 2:1, find the area of quadrilateral BCKG. Write the answer in simplest radical form.



3. Urn A contains 6 red and 4 green marbles. Urn B contains 7 red and 3 green marbles. If 2 marbles are drawn at random from each urn, what is the probability that from those drawn, there will be 2 red and 2 green marbles? Write the answer in the form $\frac{a}{b}$ where a and b are relatively prime whole numbers.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 5 – MARCH 2000

TEAM ROUND

3 pts. 1. ____________________

3 pts. 2. ____________________

4 pts. 3. ____________________

SAT APPROVED CALCULATORS ALLOWED ON THIS ROUND
except for the **TI-89 Calculator or any calculator with symbolic operation capabilities**, which are not allowed on the Team Round

1. The lines $y = 2x + 6$, $y = mx$, and $y = 0$ intersect forming a triangle whose area equals 5. Solve for m.

2. Triangle ABC has vertices A (0, 0), B (6, 0) and C (18, 12). Find the distance PQ where P is the centroid and Q is the circumcenter of Δ ABC. If you estimate this distance, the result should be rounded to four decimal places.

3. Find the probability of drawing at random two cards from a standard deck containing no jokers such that at least one of the cards is a King and at least one of the cards is a diamond. If you estimate this probability, the result should be rounded to four decimal places.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 5 – MARCH 2001

TEAM ROUND

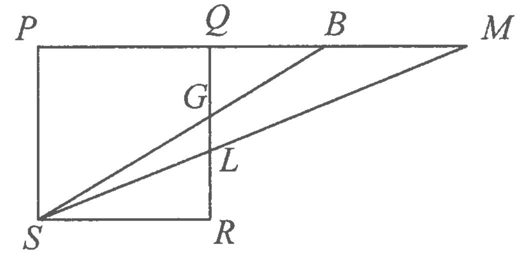
3 pts. 1. ___________________

3 pts. 2. ___________________

4 pts. 3. ___________________

SAT APPROVED CALCULATORS ALLOWED ON THIS ROUND
except for the **TI-89 Calculator or <u>any</u> calculator with symbolic operation capabilities**, which are not allowed on the Team Round

1. *PQRS* is a square. $\overline{PQBM}$, $\overline{QGLR}$, $\overline{SGB}$, $\overline{SLM}$, $QG:GR=2:3$, and $QL:LR=3:2$. Find the ratio of the area of quadrilateral *GBML* to the area of the square *PQRS*.



2. If the cubic equation $x^3-3x^2+2x+k=0$, when solved over the complex numbers, has roots *r*, *s*, and *t*, and $(r+2)(s+2)(t+2)=17$, find the value for *k*.

3. Three cards are picked at random from a standard deck of cards (no jokers). What is the probability that only one of them is a face card and only one of them is a heart? Express the result in the form $\frac{a}{b}$ where *a* and *b* are relative prime whole numbers or, if estimated, round off to 4 decimal places.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 5 – MARCH 2002

TEAM ROUND (<u>12 MINUTES LONG</u>)

Problems submitted by Maimonides.

3 pts. 1. ____________________

3 pts. 2. ____________________

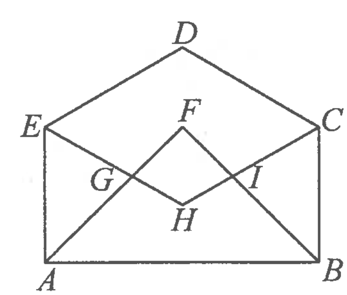
4 pts. 3. ____________________

SAT APPROVED CALCULATORS ALLOWED ON THIS ROUND except for the TI-89 Calculator or <u>any</u> calculator with symbolic operation capabilities, which are not allowed on the Team Round

1. Kaitlin picked a whole number greater than 10, added one, multiplied this result by two, added one again, multiplied this result by three, added one for the last time, and multiplied this result by four. If Kaitlin ended with a perfect cube, find the <u>smallest two</u> whole numbers greater than 10 that she could have picked.

2. Al and Marty play a game where each of them tosses 4 fair coins and whoever has more coins landing on heads wins. What is the probability that Marty ties the first game and wins the second game? Express the answer in rational form or if estimated round off to exactly <u>5 decimal places</u>.

3. Given pentagon $ABCDE$ with $m\angle BAE = m\angle ABC = 90°$, $m\angle AED = m\angle BCD = 120°$, $AB = 12$, $AE = BC = 6$, $\overline{AF}$, $\overline{BF}$, $\overline{CH}$, and $\overline{EH}$ bisect angles $BAE, ABC, BCD,$ and AED respectively, find the exact area of quadrilateral $FGHI$.



GREATER BOSTON MATHEMATICS LEAGUE
MEET 5 – MARCH 2006

TEAM ROUND: Time Limit – 12 minutes

(3 pts) 1. ____________________

(3 pts) 2. ____________________

(4 pts) 3. ____________________

SAT APPROVED CALCULATORS ARE ALLOWED IN THE TEAM ROUND, EXCEPT FOR CALCULATORS WITH SYMBOLIC MANIPULATION PROGRAMS, (FOR EXAMPLE THE TI-89) WHICH ARE NOT ALLOWED IN THIS ROUND.

1. The following properties are shared by more than one three-digit positive integer in base 10. Six times the hundreds digit increased by the tens digit is four more than five times the units digit. The two-digit integer whose tens and unit digit are the hundreds and tens digit respectively of the three-digit integer equals nine times the units digit of the three-digit integer. Find the sum of the smallest and the largest three-digit integers that satisfy the given information.

2. Find the number of square units in the area of the region bounded by the x-axis and the graph defined by

$$6x^2 - 11xy + 4y^2 + 6x + 7y - 36 = 0$$

3. What is the exact probability that in drawing three cards from a standard deck of 52 cards without replacement that you will obtain a spade, a queen and a diamond in that order?

GREATER BOSTON MATHEMATICS LEAGUE
MEET 5 – March 2007

TEAM ROUND: Time Limit – 12 minutes

(3 pts) 1. ____________________

(3 pts) 2. ____________________

(4 pts) 3. ____________________

SAT APPROVED CALCULATORS ARE ALLOWED IN THE TEAM ROUND, EXCEPT FOR CALCULATORS WITH SYMBOLIC MANIPULATION PROGRAMS, (FOR EXAMPLE THE TI-89) WHICH ARE NOT ALLOWED IN THIS ROUND.

1. A game is played by rolling a single fair sided die. If a 1, 2, or 3 is rolled, then the game is over and the player loses. If a 4 or 5 is rolled, then the player may roll again. If a 6 is rolled, then the player wins automatically. The player continues to roll the die until he/she either wins or loses the game. What is the probability that the player wins the game?

2. Given: ΔABC with medians $\overline{AM}$ and $\overline{BN}$ intersecting in point E.
   (M lies on $\overline{BC}$ and N lies on $\overline{AC}$)
   Draw segment $\overline{MN}$ and shade the interiors of ΔCMN and ΔAEB. Determine the ratio of the area of the shaded regions to the area of the non-shaded regions inside ΔABC.

3. Find all possible three-digit positive base 10 integers with a middle digit of 1 which have the property that when the number is divided by 9, the quotient is the sum of the squares of the original number's digits and the remainder is zero.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 5 – MARCH 2008

TEAM ROUND

3 pts.	1.	____________________
3 pts.	2.	____________________
4 pts.	3.	____________________

SAT APPROVED CALCULATORS ARE ALLOWED IN THIS ROUND.

1. What are the rightmost two digits of $(2007)^{2008} - (2003)^{2008}$ **?**

2. Factor completely over the integers.

$$4^{2x} + 36 - 2^{8x} - 6 \cdot 2^{2x+1}$$

3.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 5 – FEBRUARY 2009

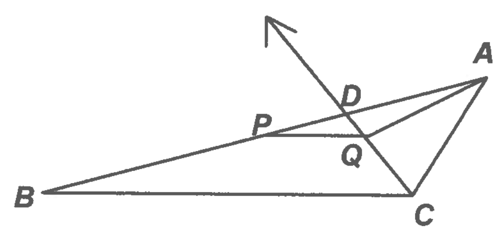
TEAM ROUND

3 pts.	1.	______________________
3 pts.	2.	______________________
4 pts.	3.	______________________

SAT APPROVED CALCULATORS ARE ALLOWED IN THIS ROUND.

1. Find the probability that a 5-digit number written in the form $AB3BA_{(6)}$, where A and B are distinct numerals, will be divisible by 5

2. In ΔABC, $BC > AC$, $\overrightarrow{CD}$ bisects $\angle C$, P is the midpoint of $\overline{AB}$ and $\overline{AQ} \perp \overline{CQ}$
If PQ is the mean proportional between (geometric mean of) BC and AC, determine the exact value of $\dfrac{BC}{AC}$.



3. Find all values of x which satisfy $\dfrac{3}{|x+1|} - \dfrac{2}{|x-2|} \geq 1$

MEET 5 – FEBRUARY 2010

TEAM ROUND

3 pts. 1. ______________

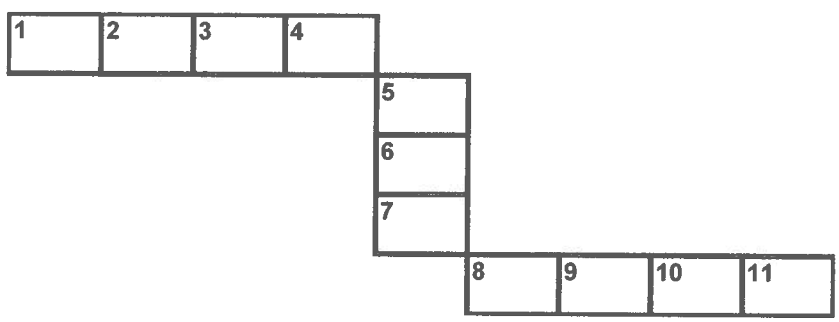
3 pts. 2. ______________

4 pts. 3. ______________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND.

1. Given the digits 1, 2, 4, 5, 7 and 9.
   Using only these digits with no repetition, how many 4-digit natural numbers can be formed that are divisible by 44?

2. If all eleven letters in the word MATHEMATICS are used to fill all the numbered locations in the grid below, what is the probability that the vowels occupy locations containing a prime number?



3. The line, whose equation is $y = x + 2k$, intersects the parabola, whose equation is $y^2 = 4x$, at two points A and B, where $AB = 2\sqrt{3}$. Compute the value of k.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 5 – MARCH 2011

TEAM ROUND

3 pts. 1. $A =$ ______ ________________________

3 pts. 2. (__________ , __________)

4 pts. 3. ___________________________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND.

1. Given: digits 0, 3, 4, 5, 7 and 9.
Compute the probability that a 4-digit natural number whose digits are all distinct will be divisible by 55.

2. Given: the function $\left\{ f \mid f(x) = 6x^3 - 29x^2 - 17x + 60, \text{ where } -\frac{5}{2} \le x \le \frac{13}{2} \right\}$
Compute the probability that $f(x) \geq 0$ for a random choice of x in the stated domain.

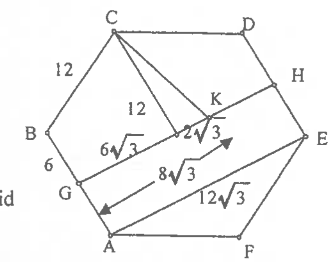
3. Consider the property of having <u>an odd number of factors greater than 3 and no even factors</u>. Determine the <u>two</u> smallest natural number greater than 800 with this property.

'99

TEAM ROUND

1. Call the roots r, $r + d$, $r + 2d$. The sum of the roots is the opposite of the coefficient for $x^2 \Rightarrow 3r + 3d = -6 \Rightarrow r + d = -2 \Rightarrow$ roots are -2, $-2 + d$, and $-2 - d$. The sum of the product of the roots taken 2 at a time is the coefficient of $x \Rightarrow$ $-2(-2+d) + -2(-2-d) + (-2+d)(-2-d) = 21 \Rightarrow 12 - d^2 = 21 \Rightarrow d = 3i \Rightarrow$ roots are -2, $-2+3i$, $-2-3i$; c is the opposite of the product of the roots $\Rightarrow$ $c = 2(-2+3i)(-2-3i) = \mathbf{26}$

2. Draw a perpendicular from C to $\overline{GH}$ dividing the quadrilateral into a trapezoid and a right triangle.
GH = AE = $12\sqrt{3}$
The perpendicular is half the longest diagonal = 12
GK = $\frac{2}{3} \cdot 12\sqrt{3} = 8\sqrt{3}$
Area of triangle $= \frac{1}{2}(2\sqrt{3})(12) = 12\sqrt{3}$; Area of trapezoid $= \frac{1}{2}(6\sqrt{3})(18) = 54\sqrt{3} \Rightarrow$ Area of quadrilateral = $\mathbf{66\sqrt{3}}$



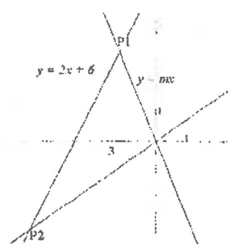
3. There are 3 possibilities: (i) Urn A, 1 red and 1 green, and from Urn B, the same; (ii) Urn A 2 red, Urn B 2 green; (iii) Urn A 2 green, Urn B 2 red; $\Rightarrow$ Probability =

$$\frac{\binom{6}{1}\binom{4}{1}}{\binom{10}{2}} \times \frac{\binom{7}{1}\binom{3}{1}}{\binom{10}{2}} + \frac{\binom{6}{2}}{\binom{10}{2}} \times \frac{\binom{3}{2}}{\binom{10}{2}} + \frac{\binom{4}{2}}{\binom{10}{2}} \times \frac{\binom{7}{2}}{\binom{10}{2}} = \mathbf{\frac{1}{3}}$$

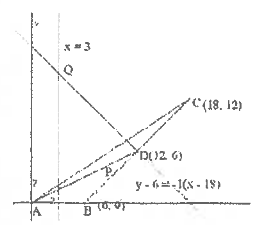
'00

TEAM ROUND

1. Find y coordinate of point P:
$2x + 6 = mx \Rightarrow x = \dfrac{6}{m-2} \Rightarrow$
$y = \dfrac{6m}{m-2} \Rightarrow$ area of the triangle $= \dfrac{1}{2} \cdot 3 \cdot \dfrac{6m}{m-2} = \dfrac{9m}{m-2}$
$= \pm 5 \Rightarrow 9m = 5m - 10 \Rightarrow m = -\dfrac{5}{2}$ (using point P1)
or $9m = -5m + 10 \Rightarrow m = \dfrac{5}{7}$ (using point P2)



2. To find centroid P: 2/3 the distance from A to D (12, 6), the midpoint of BC. $\Rightarrow$
P has coordinates (8, 4); to find circumcenter Q
$\therefore x = 3$ is $\perp$ bis. of $\overline{AB}$; slope of $\overline{BC} = 1 \Rightarrow$
$y - 6 = -1(x - 12)$ is $\perp$ bis. of $\overline{BC}$; $\Rightarrow$
$y - 6 = -1(3 - 12) \Rightarrow y = 15 \Rightarrow$ Q has coordinates
(3, 15); PQ $= \sqrt{5^2 + (-11)^2} = \sqrt{146} = 12.0830$



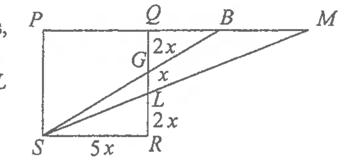
3. two cases: (i) draw a King, which is not a diamond and then a diamond (ii) draw the diamond King and then any card except a second King $\therefore$ the probability equals

$$\frac{\binom{3}{1}\binom{13}{1} + \binom{1}{1}\binom{48}{1}}{\binom{52}{2}} = \frac{29}{442} = 0.0656$$

'01

TEAM ROUND

1. Let side of square = $5x \rightarrow$ because of the ratios, $QG = 2x$, $GL = x$, $LR = 2x$; area of $GBML$ = area of ΔQML – area of ΔQBG; $\Delta QML \sim \Delta RSL$ with ratio of similitude 3:2; area of $\Delta\ RSL = 5x^2$
$\rightarrow$ area of $\Delta\ QML = \dfrac{9}{4} \cdot 5x^2 = \dfrac{45}{4}x^2$;



$\Delta QBG \sim \Delta RSG$ with ratio of similitude 2:3; area of $\Delta\ RSG = \dfrac{15x^2}{2} \rightarrow$ area of ΔQBG =

$\dfrac{4}{9} \cdot \dfrac{15}{2}x^2 = \dfrac{10}{3}x^2$; ratio of area of quad $GBML$: square $= \dfrac{\frac{45}{4} - \frac{10}{3}}{25} = \dfrac{19}{60}$

2. $x^3 - 3x^2 + 2x + k = 0 \rightarrow r + s + t = 3, rs + rt + st = 2$, and $rst = -k$;
$(r+2)(s+2)(t+2) = 17 \rightarrow rst + 2(rs + st + rt) + 4(r + s + t) + 8 = 17 \rightarrow$
$-k + 2(2) + 4(3) + 8 = 17 \rightarrow k = 7$

3. There are two types of successful events: (i) 1 card is a non-heart face card, 1 card is a non-face card heart, and 1 card is a non-heart, non-face card; (ii) 1 card is a heart and a face card and 2 cards are non-hearts, non-face cards; there are 9 non-heart face cards, 10 non-face card hearts, 30 non-hearts, non-face cards, and 3 hearts and face cards;
therefore the probability $= \dfrac{{}_9C_1 \cdot {}_{10}C_1 \cdot {}_{30}C_1 + {}_3C_1 \cdot {}_{30}C_2}{{}_{52}C_3} = \dfrac{801}{4420} \approx 0.1812$.

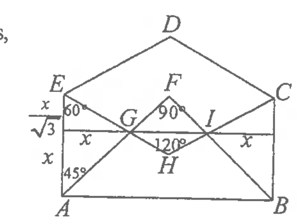
'02

TEAM ROUND

1. Let x = number picked $\Rightarrow 4(3(2(x+1)+1)+1) = 4(6x+10) = 8(3x+5)$ = perfect cube
$\Rightarrow 3x+5$ is a perfect cube, call it y^3. $y^3 = 3x + 5 \equiv 2\bmod 3 \Rightarrow$ since $(2\bmod 3)^3 = 8\bmod 3$
$\equiv 2\bmod 3 \Rightarrow y = 2\bmod 3$. The 2 values for y are 5, 8 $\Rightarrow 3x + 5 = 5^3$ or $8^3 \Rightarrow x = 40, 169$.

2. $P(T) = \left(\dfrac{{}_4C_0}{2^4}\right)^2 + \left(\dfrac{{}_4C_1}{2^4}\right)^2 + \left(\dfrac{{}_4C_2}{2^4}\right)^2 + \left(\dfrac{{}_4C_3}{2^4}\right)^2 + \left(\dfrac{{}_4C_4}{2^4}\right)^2 = \dfrac{35}{128} \Rightarrow$
$P(W) = \left(1 - \dfrac{35}{128}\right) \div 2 = \dfrac{93}{256} \Rightarrow P(TW) = \left(\dfrac{35}{128}\right)\left(\dfrac{93}{256}\right) = \dfrac{3255}{32768} \approx 0.09933$

3. Draw the line through G and I and let x = altitude of ΔAGE from G. Because of the special right triangles,
$AE = 6 = x + \dfrac{x}{\sqrt{3}} \Rightarrow x = \dfrac{6}{1 + \frac{1}{\sqrt{3}}} = \dfrac{3}{2}(6)\left(1 - \dfrac{\sqrt{3}}{3}\right) =$
$9 - 3\sqrt{3}$; $GI = 12 - 2x = 6\sqrt{3} - 6$; area of $FGHI$ = area of ΔGIF + area of ΔGIH =
$\dfrac{1}{4}(6\sqrt{3} - 6)^2 + \dfrac{1}{4}(6\sqrt{3} - 6)^2\left(\dfrac{\sqrt{3}}{3}\right) = 18 - 6\sqrt{3}$.



'06

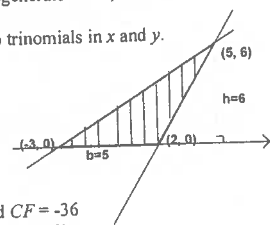
TEAM ROUND

1. $10h + t = 9u$ and $6h + t = 4 + 5u \rightarrow 10h + t - 9u = 0$

$$\begin{array}{r} 10h + t - 9u = 0 \\ \underline{6h + t - 5u = 4} \\ 4h \quad - 4u = -4 \end{array} \rightarrow h = u - 1$$

Substituting in the first equation $\rightarrow t = 10 - u$
Since $h \neq 0$, the smallest allowable value of u is $2 \rightarrow h = 1$ and $t = 8 \rightarrow N_{smallest} = 182$.
The largest possible value of u produces the largest possible value of h.
$u = 9 \rightarrow h = 8$ and $t = 1 \rightarrow N_{largest} = 819$. Sum $= 182 + 819 =$ **1001**.

2. The equation $6x^2 - 11xy + 4y^2 + 6x + 7y - 36 = 0$ represents a degenerate conic, two intersecting lines, $2x - y - 4 = 0$ and $3x - 4y + 9 = 0$.
It remains to show that the original equation factors to these two trinomials in x and y.



Method #1: Using the method of indeterminate coefficients,
The coefficients of the original equation sum to –24.
If the two factors we seek are $(Ax + By + C)$ and $(Dx + Ey + F)$ then the product of the coefficient sums of the factors must also be –24.

$AB = 6$, $AE + BD = -11$, $BE = 4$, $AF + CD = 6$, $BF + CE = 7$ and $CF = -36$
A little experimenting produces the factors $(2x - y - 4)$ and $(3x - 4y + 9)$
Notice $(2 - 1 - 4)(3 - 4 + 9) = (-3)(8) = -24$.
Solving the system of equations yields a point of intersection at (5, 6).
Setting $y = 0$ yields x-intercepts at (-3, 0) and (2, 0). Area $= (1/2)(5)(6) =$ **15**.

Method #2: Using the quadratic formula to find the equations of the lines:
Rewrite the original equation as $6x^2 + (6-11y)x + (4y^2 + 7y - 36) = 0$ and think of this as a quadratic equation in x with coefficients $(A, B, C) = (6, 6-11y, 4y^2+7y-36)$.

$$x = \frac{(11y-6) \pm \sqrt{(6-11y)^2 - 4(6)(4y^2+7y-36)}}{12} = \frac{(11y-6) \pm \sqrt{36 - 132y + 121y^2 - 96y^2 - 168y + 864}}{12}$$

$$= \frac{(11y-6) \pm \sqrt{25(y^2 - 12y + 36)}}{12} = \frac{(11y-6) \pm \sqrt{25(y-6)^2}}{12} = \frac{(11y-6) \pm 5(y-6)}{12} \rightarrow$$

$$x = \frac{16y-36}{12} = \frac{4y-9}{3} \text{ or } x = \frac{6y+24}{12} = \frac{y+4}{2} \rightarrow (3x - 4y + 9)(2x - y - 4) = 0$$

3. What is the second card drawn?

	Card #1 #2 #3
Case 1: Queen of spades	(12)(1)(13) = 156
Case 2: Queen of diamonds	(13)(1)(12) = 156
Case 3: Queen of clubs or hearts	(13)(2)(13) = 338 → total: 650

→ P(spade, queen, diamond) = 650/(52·51·50) = 13/(52·51) = 1/(4·51) = **1/204**

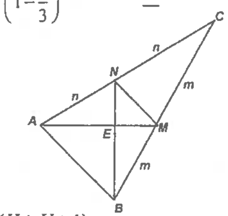
'07

TEAM ROUND

1. P(win) = P(win on roll 1 or roll2 or roll3 or …) $= \frac{1}{6} + \frac{1}{3}\cdot\frac{1}{6} + \frac{1}{3}\cdot\frac{1}{3}\cdot\frac{1}{6} + \ldots = \frac{1}{6}\left(1 + \frac{1}{3} + \frac{1}{3}^2 + \ldots\right)$

Since the second factor is an infinite geometric progression, we have $\frac{1}{6}\left(\frac{1}{1-\frac{1}{3}}\right) = \frac{1}{6}\cdot\frac{3}{2} = \mathbf{\frac{1}{4}}$

2. Let $MN = b$. Then $AB = 2b$ Let the altitude from E to $\overline{MN} = h$. Then the altitude from E to $\overline{AB} = 2h$ and the altitude from C to $\overline{MN} = 3h$



Thus, $\frac{Area(\Delta CNM + \Delta EAB)}{Area(\Delta NEA + \Delta MEB + \Delta MEN)} = \frac{\frac{3}{2}bh + 2bh}{bh + bh + \frac{1}{2}bh} = \frac{3.5}{2.5} = \mathbf{\frac{7}{5}}$

3. $\frac{100H + 10 + U}{9} = H^2 + 1^2 + U^2 \rightarrow 9(H^2 + U^2) = 100H + U + 1 = 99H + (H + U + 1)$
Since the left hand side is divisible by 9, $(H + U + 1)$ must be a multiple of 9.
Examining the possible ordered pairs from (H,U) = (0, 8), (1, 7), …, (8,0)
Only (3, 5) works, namely $9(3^2 + 5^2) = 99(3) + (3 + 5 + 1) = 306 \rightarrow$ **315** only

'08

TEAM ROUND

1. The rightmost two digits of $(2007)^x$ for $x = 1,2,3, \ldots$ are: 07, **49, 43 01**, 07
Thus, such exponential expressions exhibit a 4 – cycle, i.e. the rightmost two digits repeat whenever the exponent is increased by 4 or equivalently produce 01 for exponents of the form $4k$, where $k = 1, 2, 3, \ldots$.

The last rightmost two digits of $(2003)^x$ for $x = 1,2,3, \ldots$ are:
03, 09, 27, 81, 43, 29, 87, 61, 83, 49, 47, 41, 23, 69, 07, 21, 63, 89, 67, 01, 03
Thus, such exponential expressions exhibit a 20-cycle, i.e. the rightmost two digits repeat whenever the exponent is increased by 20 or equivalently produce 01 for exponents of the form $20k$, where $k = 1, 2, 3, \ldots$.

Thus, $(2007)^{2008} - (2003)^{2008} = (2007)^{4\cdot502} - (2003)^{20\cdot100+8} =$ ___01 - ___61.
Since the difference is clearly positive, the last two digits are **40**.
(If the first term in the difference were smaller than the second term, the result would have been negative and the rightmost two digits would have been 60.)

2. $4^{2x} + 36 - 2^{8x} - 6\cdot2^{2x+1} = 2^{4x} + 36 - 2^{8x} - 12\cdot2^{x} = \left((2^{2x})^2 - 12\cdot2^{2x} + 36\right) - (2^{4x})^2 =$
$(2^{2x} - 6)^2 - (2^{4x})^2 = (2^{2x} - 6 + 2^{4x})(2^{2x} - 6 - 2^{4x}) = \mathbf{(2^{2x} + 3)(2^{2x} - 2)(2^{2x} - 6 - 2^{4x})}$

3.

109

TEAM ROUND

1.
Testing for divisibility by 5 in base 6 is identical to testing for divisibility by 9 in base 10. Simply determine if the sum of the digits is a multiple of the divisor being tested.

Thus, we need to determine under what conditions, $2A + 2B + 3$ is a multiple of 5?
Since A is the lead digit, we need check out the nonzero digits 1 … 5.

2A+2B+3 is a multiple of 5 if and only if

A	2A+2B+3	B =	
1	2B + 5	0, 5	
2	2B + 7	4	
3	2B + 9	3	rejected (A = B)
4	2B + 11	2	
5	2B + 13	1	
	5 cases		

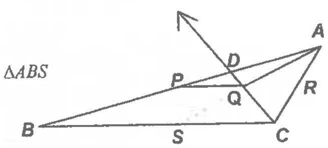
A \ B	0	1	2	3	4	5
0	X	X	X	X	X	X
1		X				
2			X			
3				X		
4					X	
5						X

Of the 36 possible ordered pairs (A, B), 11 are eliminated since A must be nonzero and $A \neq B$.

Thus, the probability of divisibility by 5 is $\frac{5}{36-11} = \underline{\frac{1}{5}}$

Note that if the requirement that the digits A and B be distinct were dropped, the probability would remain the same. $\left[\frac{6}{36-6} = \frac{1}{5}\right]$

2. $\Delta AQC \equiv \Delta SQC$ (by ASA – $\overline{CQ}$ common side)
$\rightarrow AQ = QS,\ SC = AC$
Since P and Q connect midpoints of 2 sides of ΔABS
$\overline{PQ} \parallel \overline{BC}$ and more importantly
$PQ = \frac{1}{2}BS = \frac{1}{2}(BC - SC) = \frac{1}{2}(BC - AC)$
However, we were also given $PQ = \sqrt{BC \cdot AC}$
From the first equation $4PQ^2 = (BC - AC)^2$.
Substituting for PQ, $4BC \cdot AC = (BC - AC)^2 = BC^2 - 2\,BC \cdot AC + AC^2$



$\rightarrow BC^2 - 6\,BC \cdot AC + AC^2 = 0 \rightarrow \left(\frac{BC}{AC}\right)^2 - 6\left(\frac{BC}{AC}\right) + 1 = 0 \rightarrow \frac{BC}{AC} = \frac{6 \pm \sqrt{36-4}}{2} = 3 \pm 2\sqrt{2}$

But, since $BC > AC$, $3 - 2\sqrt{2}$ is extraneous. The required ratio is $\underline{3 + 2\sqrt{2}}$.

3.

(–1) (+2)

$\frac{3}{-x-1} - \frac{2}{-x+2} \geq 1$

$\frac{-3}{x+1} + \frac{2}{x-2} \geq 1$

$\frac{-3(x-2) + 2(x+1)}{(x+1)(x-2)} - 1 \geq 0$

$\frac{(-x+8) - x^2 + x + 2}{(x+1)(x-2)} \geq 0$

$\frac{x^2 - 10}{(x+1)(x-2)} \leq 0$

Roots: $-\sqrt{10}, -1, +2, \sqrt{10}$
5 intervals: + – + – +
$\underline{\mathbf{-\sqrt{10} \leq x < -1}}$

$\frac{3}{x+1} - \frac{2}{-x+2} \geq 1$

$\frac{3}{x+1} + \frac{2}{x-2} \geq 1$

$\frac{(5x-4) - x^2 + x + 2}{(x+1)(x-2)} \geq 0$

$\frac{x^2 - 6x + 2}{(x+1)(x-2)} \leq 0$

$\frac{(x-3)^2 - 7}{(x+1)(x-2)} \leq 0$

$\mathbf{-1, 3 - \sqrt{7}, 2, 3 + \sqrt{7}}$
5 intervals: + – + – +
$\underline{\mathbf{-1 < x \leq 3 - \sqrt{7}}}$

$\frac{3}{x+1} - \frac{2}{x-2} \geq 1$

$\frac{(x-8) - x^2 + x + 2}{(x+1)(x-2)} \geq 0$

$\frac{x^2 - 2x + 6}{(x+1)(x-2)} \leq 0$

$\frac{(x-1)^2 + 5}{(x+1)(x-2)} \leq 0$

–1, 2
3 intervals: + – +

10

TEAM ROUND

1. Consider 4-digit natural numbers of the form $\underline{A}\,\underline{B}\,\underline{C}\,\underline{D}$, where A, B, C and D ε {1, 2, 4, 5, 7, 9}.
Then $\underline{C}\,\underline{D}$ must be 12, 24, 52, 72 or 92 to guarantee divisibility by 4.
To guarantee divisibility by 11, we examine $(B + D) - (A + C)$ which must be a multiple of 11.
The results are summarized in the following chart:

to insure divisibility by 4 C D	A, B	to insure divisibility by 11 (B+D) - (A+C) simplifies to	B - A		in range multiples of 11	A B
12	{4,5,7,9}	B - A + 1	-5 to +5	-4 to + 6	0	54
24	{1,5,7,9}	B - A + 2	-8 to +8	-6 to +10	0	75, 97
52	{1,4,7,9}	B - A - 3	-8 to +8	-11 to +5	-11, 0	14, 47, 91
72	{1,4,5,9}	B - A - 5	-8 to +8	-13 to +3	-11,0	49
92	{1,4,5,7}	B - A - 7	-6 to +6	-13 to -1	-11	51

Thus, there are $\underline{8}$ possibilities, namely $\underline{5412}$, $\underline{1452}$, $\underline{4752}$, $\underline{7524}$, $\underline{5192}$, $\underline{9152}$, $\underline{4972}$, $\underline{9724}$.

2. Pick the 4 vowel positions and place the vowels in these positions - $\binom{5}{4}\frac{4!}{2!}$

Place the remaining letters in the remaining positions - $\binom{7}{7}\frac{7!}{2!2!}$

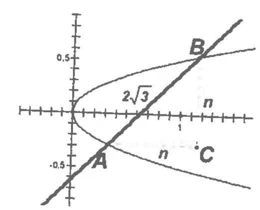
Fill the squares with the letters (no restrictions) - $\frac{11!}{2!2!2!}$

$$P = \frac{\binom{5}{4}\frac{4!}{2!} \cdot \binom{7}{7}\frac{7!}{2!2!}}{\frac{11!}{2!2!2!}} = \frac{5 \cdot 6 \cdot 1 \cdot 7!}{2!} \cdot \frac{2!2!2!}{11!} = \frac{30(4)}{11 \cdot 10 \cdot 9 \cdot 8} = \frac{1}{11 \cdot 3 \cdot 2} = \underline{\frac{1}{66}}$$

3. $\begin{cases} y = x + 2k \\ y^2 = 4x \end{cases} \rightarrow (x + 2k)^2 = 4x$

$\rightarrow x^2 + 4(k-1)x + 4k^2 = 0$

$\rightarrow x = \frac{4(1-k) \pm \sqrt{16(k-1)^2 - 16k^2}}{2} = 2(1-k) \pm 2\sqrt{1-2k}$



The x-coordinate of point A, $A_x = 2(1-k) - 2\sqrt{1-2k}$
The x-coordinate of point B, $B_x = 2(1-k) + 2\sqrt{1-2k}$
Since the slope of $\overline{AB}$ is 1, let $AC = BC = n$.
Using the P.T. on ΔABC, $n = \sqrt{6}$.
But we also have $n = B_x - A_x = 4\sqrt{1-2k}$.
Thus, $6 = 16(1 - 2k) \rightarrow 32k = 10 \rightarrow k = \underline{\frac{5}{16}}$

TEAM ROUND

1. An integer divisible by 55 is divisible by both 5 and 11.
Recall: An integer is divisible by 11 if and only if the difference between the sum of the digits in odd positions and the sum of the digits in even positions is a multiple of 11.
Let the 4-digit natural number be denoted $\underline{A}\ \underline{B}\ \underline{C}\ \underline{D}$.
Then: $\begin{cases} D = 0 \text{ or } 5 \\ (B+D)-(A+C) = 11k \end{cases}$
Using the digits given there are $5 \cdot 5 \cdot 4 \cdot 3 = 300$ possible 4-digit positive integers with distinct digits.

Case $D = 0$:
Available digits for A, B and C: $\{3, 4, 5, 7, 9\}$ $(B + D) - (A + C) = B - (A + C)$
We must maximize and minimize the difference between a single digit and the sum of two others. $3 - (7 + 9) = -13 \le B - (A + C) \le 2 = 9 - (4 + 3)$
Thus, the multiples of 11 are either -11 or 0.
-11: $3 - (5 + 9)$ or $5 - (7 + 9)$ → 5390, 9350, 7590 and 9570
0: $9 - (4 + 5)$ or $7 - (3 + 4)$ → 4950, 5940, 3740 and 4730

Case $D = 5$:
Available digits for A, B and C: $\{0, 3, 4, 7, 9\}$
$(B+5)-(A+C) = 11k \rightarrow B-(A+C) = 11k-5$
$-11 \le (B+5)-(A+C) \le +11 \rightarrow -16 \le 11k-5 \le 6$
$k = -1, 0, 1 \rightarrow -16, -5, +6$
-16: $0 - (7 + 9)$ → 7095 and 9075
-5: $4 - (0 + 9)$ or $7 - (3 + 9)$ → 9405, 3795 and 9735
$+6$: $9 - (3 + 0)$ → 3905 only
Thus, 14 out of 300 gives a probability of $\underline{\dfrac{7}{150}}$.

2. The rational root theorem restricts integer zeros of the function to factors of 60. By synthetic substitution, we soon realize that to overcome the negative coefficients of the quadratic and linear terms, we must use a 'larger' positive integer. Starting with 5, instead of 1, 2, 3 or 4, we quickly discover a zero. In fact, $f(x)$ factors as $(x-5)(6x^2+x-12) = (x-5)(3x-4)(2x+3)$ and f has three rational zeros, namely $-\frac{3}{2}, \frac{4}{3}$ and 5.

$$\begin{array}{r|rrrr} & 6 & -29 & -17 & 60 \\ & & 30 & 5 & -60 \\ \hline 5 & \boxed{6} & \boxed{1} & \boxed{-12} & 0 \end{array}$$

For x-values at the extreme left on the number line, all three factors are negative (hence, a negative product) and as each critical point is passed the sign of the product alternates as indicated in the diagram below.

[-] [+] [-] [+]

-5/2 -3/2 4/3 5 13/2

Thus, the probability that $f(x) \ge 0$ over the stated domain requires x be selected from $\left[-\frac{3}{2}, \frac{4}{3}\right]$ or $\left[5, \frac{13}{2}\right]$.

This determines a probability of $\dfrac{\left(\frac{4}{3}-\frac{-3}{2}\right)+\left(\frac{13}{2}-5\right)}{\frac{13}{2}-\frac{-5}{2}} = \dfrac{8+9+39-30}{9 \cdot 6} = \dfrac{26}{54} = \underline{\mathbf{\dfrac{13}{27}}}$

3. Even powers of a prime will have an odd number of factors ($p^2 \rightarrow 1, p, p^2$ $p^4 \rightarrow 1, p, p^2, p^3, p^4, p^5$)
The prime 2 must be excluded since even factors are to be excluded.
In fact, an <u>even power of any product of odd primes</u> will have an odd number of factors, none of which will be even – exactly the property we want our natural numbers to satisfy.
For a single prime, the even power must be at least 4.
$5^4 = 625$ is rejected, but $7^4 = 2401$ has 5 odd factors, but is it one of the two smallest?
Let's systematically consider numbers of the form $(pq)^k$, where p and q are odd primes and k is even.
If $p = 3$, then numbers of the form $(3q)^2 = 3^2 \cdot q^2$ will have 9 factors.
$3^2 \cdot 11^2 = \underline{\mathbf{1089}}$ and $3^2 \cdot 13^2 = 1521$ are the smallest numbers greater than 800.
If $p = 5$, then the smallest possibility is $5^2 \cdot 7^2 = \underline{\mathbf{1225}}$.
A product of more than two odd primes will clearly produce larger numbers, so we have determined the smallest two numbers satisfying the given conditions.