Round 5 Precalculus

Trigonometric Analysis and Complex Numbers in Trigonometric form

GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 – DECEMBER 1998

ROUND 5 – Trig. analysis and Complex Numbers, Trig Form

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Put $\dfrac{\left(-2+2i\sqrt{3}\right)^5}{\left(\sqrt{2}+i\sqrt{2}\right)^7}$ in the form $r\ cis\ \theta$ where $r > 0$ and $0° \le \theta < 360°$.

2. Given $\cos A = \dfrac{\sqrt{2}}{3}$ and $\tan A < 0$, find the value for $\csc\left(180° - A\right)\tan\left(90° + A\right)$.

3. Solve the following equation over the complex numbers: $z^3 i\sqrt{2} = 125 - 125i$ and put all values for z in the form $r\ cis\ \theta$ where $r > 0$ and $0° \le \theta < 360°$.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 – DECEMBER 1999

ROUND 5 – Trig. analysis and Complex Numbers, Trig Form

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. If $\sin\theta = \frac{3}{4}$ and $\cos\theta < 0$, compute $\tan(90° + \theta)$.

2. Find the **<u>smallest two positive degree measures</u>** for θ satisfying the equation:
$\cos\theta = 2\sin 19.5° \cos 19.5°$

3. Given z is a second quadrant point in the complex plane, z is a solution to the equation, $z^3 = 13.5 - 13.5i\sqrt{3}$, and $zw = -6\sqrt{2} - 6i\sqrt{2}$, solve for w in the polar form $r \operatorname{cis}\theta$, where $r > 0$ and $0° \le \theta < 360°$.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 – DECEMBER 2000

ROUND 5 – Trig. Analysis and Complex Numbers, Trig Form

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Find all complex solutions to the equation $z^3 = -4 + 4i\sqrt{3}$ in the polar form $r \operatorname{cis} \theta$, where $r > 0$ and $0° \le \theta < 360°$.

2. Find all values of x such that $0° \le x < 360°$ and

$$\left(\sqrt{2}\operatorname{cis}315°\right)^6 = (4\operatorname{cis}855°)^2 \cos(270° + x)$$

3. Find all solutions to the equation $\sin(x + 40°) + \sin(x - 40°) = \sin 50° \cdot \tan x$ where $0° \le x < 360°$.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 – DECEMBER 2001

ROUND 5 – Trig. Analysis and Complex Numbers, Trig Form

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Find $\left(\frac{1}{2} cis\, 20°\right)^5 (2 cis\, 26°)^{10}$ in rectangular form.

2. Given $\tan\alpha = \dfrac{cis\, 270°}{2cis\, 90°}$ and $\cos\alpha < 0$, find the value for $\sin(2\alpha)$.

3. Given $0 \le x \le 180$, $0 \le y \le 180$, $\cos x° \cos y° - \sin x° \sin y° = -0.5$, and $\sin x° \cos y° - \cos x° \sin y° = 1$, find all possible ordered pairs (x, y).

GREATER BOSTON MATHEMATICS LEAGUE
MEET 3 – DECEMBER 2005

ROUND 5 – Trig Analysis and Complex Numbers - Trig Form

1. ____________________

2. ________ + ________ **i**

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND.

1. Given $\tan\theta = 2$ and $\sin\theta < 0$.
   Find the exact numerical value of the following expression in simplified form.

$$\frac{1+\sin(\theta)}{1-\sin(\theta)} - 4\sec(\theta)$$

2. Let $z_1 = 4\text{cis}(30°)$ and $z_2 = \sqrt{3}\,\text{cis}(150°)$.
   Determine $z_1(z_2)^2$ in $a + bi$ form.

3. Find all values of x $(0 \le x < 360°)$ that make the following statement true.

$$\cos(x + 60°) - \sin(x + 30°) = (\sec 120°)(\cos 150°)\cos x$$

GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 – DECEMBER 2006

ROUND 5 – Trig Analysis and Complex Numbers in Trig Form

1. ______________________________

2. ______________________________

3. ______________________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND.

1. If $\cot\theta = -7$ and $\sin\theta < 0$, determine the exact value, in simplified form, of $\sec\theta - \cos\theta$.

2. Write all solutions to the system $\begin{cases} r = 2\csc\theta \\ r = 8\sin\theta \end{cases}$ in the form $(r,\ \theta)$, where $r > 0$ and $0 \le \theta < 2\pi$.

3. Find all values of x, $0° \le x < 360°$ for which $2\sin(2x) + \tan(2x) = 0$.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 – DECEMBER 2007

ROUND 5 – Trig Analysis and Complex Numbers in Trig Form

1. ______________________

2. ______________________

3. ______________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND.

1. Given: $3\sin A + 4 = 2$
   Compute the smallest possible value of $\tan A$.

2. Find all values of x, $0° \leq x < 360°$, that satisfy the following equation:

$$2\sin(180° - x) - \sec(270° + x) - \tan(90° + x) = 0$$

3. If $x^{3/2} = 2 - 2i\sqrt{3}$, find <u>all</u> distinct solutions in $rcis\theta$ form with arguments θ in the interval $[0°, 360°)$ and $\underline{r > 0}$ in simplified radical form.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 – DECEMBER 2008

ROUND 5 – Trig Analysis and Complex Numbers in Trig Form

1. (_____ , _____)

2. ______________________________

3. ______________________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND.

1. Find the value of $\left(\dfrac{8cis240}{2\sqrt{3}-2}\right)^2$. If the answer is expressed in the form $a + bi$, determine the ordered pair (a, b).

2. Given: $\sin(90+A) = -\dfrac{2}{3}$, $0° \le m\measuredangle A < 180°$ and $\tan(180-B) = \dfrac{1}{2}$, $0° \le m\measuredangle B < 180°$

   Find the numerical value of $\dfrac{\sin A}{\sin(270-B)}$.

3. Find all values of x, where $0° \le x < 360°$, for which

$$\tan(x+45°)(\tan 135° + \tan x) = \sec^2 x - 2$$

Note: Unlabelled solutions are understood to be in degrees.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 – DECEMBER 2009

The degree symbol is <u>not</u> required in any answer.

ROUND 5 – Trig Analysis and Complex Numbers in Trig Form

1. ______________________

2. ______________________

3. ______________________

1. Find all values of x, $0° \leq x < 360°$ which satisfy the following equation:

$$2\sin(270° + x)\cos(90° + x) = 1 + \cos 2x$$

2. Find all values of x in $r\text{cis}\theta$ form, $r > 0$, $0° \leq x < 360°$ which make the following statement true:

$$x^{3/2} = 8cis240°$$

Express your answer(s) as ordered pair(s) (r, θ).

3. Given: $\tan x = \dfrac{2}{3}$ and $\sin x < 0$,

$$\cos y = -\frac{6}{7} \text{ and } \tan y < 0$$

The value of $(\sec x)(\sin y) + \cot(270° + x)$ in simplified form equals $\dfrac{M}{N}$.

Find the product MN.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 – DECEMBER 2010

ROUND 5 – Trig Analysis and Complex Numbers in Trig Form

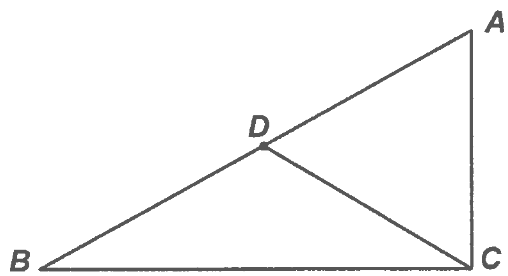
1. ______________________________

2. ______________________________

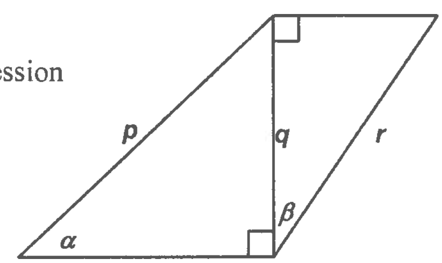
3. (__________) + (__________)i

CALCULATORS ARE NOT ALLOWED ON THIS ROUND.

1. Given: Right ΔABC with right angle at C,
   $AC = 2$, $CD = \sqrt{5}$
   D is the midpoint of $\overline{AB}$
   Compute $\dfrac{DB}{BC}$.



2. Given: $\alpha = 2 \cdot \beta$
   Using the diagram at the right, determine a simplified expression for r in terms of p and β.



3. Given: $P = \left(\sqrt{8}cis55^\circ\right)^4$, $Q = \left(4\sqrt{3} - 4i\right)^{\frac{1}{3}}$
   Compute the product PQ in rectangular form $a + bi$.

MASSACHUSETTS MATHEMATICS LEAGUE
NOVEMBER 2005
ROUND 1 COMPLEX NUMBERS

ANSWERS

A)__________________

B)__________________

C)________________

A) **If $x = a + bi$ for real a and b and if $x^2 = i$ find the product ab.**

B) **Simplify $\left(i^9 - 5i^6 - 3i^8 + 7i^{11}\right)^2$ as much as possible.**

C) Express in simplest form $(\sqrt{-6}-\sqrt{-2})^2 + \dfrac{16i}{1+\sqrt{-3}} - \left(\dfrac{4}{\sqrt{-2}}\right)^2$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 – NOVEMBER 2006
ROUND 1 ALG 2: COMPLEX NUMBERS (No Trig)

ANSWERS

A) ______________________________

B) ______________________________

C) ______________________________

A) Expand and give your answer in $a + bi$ form: $i(2 + 3i)(1 - 4i)$

B) Find $\sqrt{2i}$ in $a + bi$ form, where $b > 0$.

C) Evaluate in $a + bi$ form: $\sum_{n=1}^{n=3} (1 - i\sqrt{3})^{(2^n)}$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 – NOVEMBER 2007
ROUND 1 ALG 2: COMPLEX NUMBERS (No Trig)

ANSWERS

A) (__________) + (__________)i

B) ____________________________

C) ____________________________

A) Solve over the complex numbers, expressing your answer in simplified $a + bi$ form.
(Note: $\bar{z} = a - bi$ and denotes the conjugate of z.)

$$z + 6\bar{z} = 7 + 3i$$

B) Find all possible solutions of $z^2 = 75 + 100i$.
Leave your answer(s) in $a + bi$ form.

C) Solve for x.

$$|-3+4i|\,x^2 - |12+16i|\,x = |7-24i|$$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 – NOVEMBER 2009
ROUND 1 COMPLEX NUMBERS (No Trig)

******* NO CALCULATORS IN THIS ROUND *******

ANSWERS

A) ______________________________

B) ______________________________

C) ______________________________

Note: $i = \sqrt{-1}$

A) Simplify completely: $\dfrac{1+2i+3i^2+4i^3}{1-2i+3i^2-4i^3}$

B) Given: $(3+3i)^{40} = r^n$, where r and n are both POSITIVE integers.
Determine the <u>smallest</u> possible value of the sum $r + n$.

C) If $\sqrt{-40-9i} = A + Bi$, compute $\left(\dfrac{A}{B}\right)^2$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 – NOVEMBER 2010
ROUND 1 COMPLEX NUMBERS (No Trig)

ANSWERS

A) ______________________________

B) (____________ , ______________)

C) ______________________ sq. units

****** NO CALCULATORS ON THIS ROUND ******

A) Compute: $\left(\dfrac{1-i}{1+i}\right)^{2010}$

B) Find the ordered pair (x, y) of real numbers that satisfy the equation

$$(x^2 - x - 5) + i(y^2 - 7y + 3) = 1 - 7i$$

and for which $y - x$ is as large as possible.

C) The complex numbers $(1 + i)$, $(-1 + i)$, $(-1 - i)$ and $(1 - i)$ form a square when plotted in the complex plane. If each of these numbers is multiplied by $(1 + i)$, a new figure is formed. Compute the area of the new figure.

GBML '98

ROUND 5

1. $-2+2i\sqrt{3} = 4\ cis\ 120°$ and $\sqrt{2}+i\sqrt{2} = 2\ cis\ 45° \Rightarrow$
$$\frac{\left(-2+2i\sqrt{3}\right)^5}{\left(\sqrt{2}+i\sqrt{2}\right)^7} = \frac{\left(4\ cis\ 120°\right)^5}{\left(2\ cis\ 45°\right)^7} = \frac{2^{10}\ cis\ 600°}{2^7\ cis\ 315°} = \mathbf{8\ cis\ 285°}$$

2. A is a rotation into quadrant IV and $\cos A = \frac{\sqrt{2}}{3} \Rightarrow x = \sqrt{2};\ y = -\sqrt{7};\ r = 3$

   By the reduction formulas, $\csc(180° - A) = \csc A = -\frac{3}{\sqrt{7}}$ and

   $\tan(90° + A) = -\cot A = \frac{\sqrt{2}}{\sqrt{7}} \Rightarrow \csc(180° - A)\tan(90° + A) = -\frac{3\sqrt{2}}{7}$

3. $z^3 i\sqrt{2} = 125 - 125i \Rightarrow z^3\left(\sqrt{2}\ cis\ 90°\right) = 125\sqrt{2}\ cis\ 315° \Rightarrow z^3 = 125\ cis\ 225° \Rightarrow$
   $n = 0,1,2: z = 5\ cis\left(\frac{225°}{3} + 120°n\right) \Rightarrow z = \mathbf{5\ cis\ 75°,\ 5\ cis\ 195°,\ 5\ cis\ 315°}$

GBML '99

ROUND 5

1. $\sin\theta = \frac{3}{4}$ and $\cos\theta < 0 \Rightarrow \cos\theta = -\sqrt{1-\left(\frac{3}{4}\right)^2} = -\frac{\sqrt{7}}{4};$

   $\tan(90° + \theta) = -\cot\theta = -\frac{\cos\theta}{\sin\theta} = -\frac{-\frac{\sqrt{7}}{4}}{\frac{3}{4}} = \frac{\sqrt{7}}{3}$

2. $\cos\theta = 2\sin 19.5°\cos 19.5° \Rightarrow \cos\theta = \sin 39° = \cos 51° = \cos(360° - 51°) \Rightarrow \theta = 51°,\ 309°$

3. $z^3 = 13.5 - 13.5i\sqrt{3} \Rightarrow z^3 = 13.5\left(1 - i\sqrt{3}\right) = 13.5(2cis\,300°) = 27cis\,300° \Rightarrow$ since z is in quadrant II, then $z = 3cis\,100°$;

   $-6\sqrt{2} - 6i\sqrt{2} = 6\left(-\sqrt{2} - i\sqrt{2}\right) = 6(2cis\,225°) = 12cis\,225°$

   $w = \frac{12cis\,225°}{3cis\,100°} = 4\ cis\,125°$

ROUND 5

GBML '00

1. $z^3 = -4 + 4i\sqrt{3} = 4\left(-1 + i\sqrt{3}\right) = 4(2\text{cis}120°) = 8\text{cis}120°$

   $\rightarrow z = \sqrt[3]{8}\text{cis}\left(\frac{120°}{3} + \frac{360°}{3}n\right), n = 0,1,2 \rightarrow z = 2\text{cis}40°, 2\text{cis}160°, 2\text{cis}280°$

2. $\left(\sqrt{2}\text{cis}315\right)^6 = (4\text{cis}855°)^2\cos(270° + x) \rightarrow$

   $\rightarrow 8\text{cis}1890° = 16\text{cis}1710°\sin x \rightarrow$

   $\sin x = \frac{1}{2}\text{cis}180° = -\frac{1}{2} \rightarrow x = 210°,\ 330°$

3. $\sin(x + 40°) + \sin(x - 40°) = \sin 50° \cdot \tan x \rightarrow$

   $\sin x\cos 40° + \cos x\sin 40° + \sin x\cos 40° - \cos x\sin 40° = \sin 50° \cdot \tan x \rightarrow$

   $2\sin x\cos 40° = \cos 40° \cdot \frac{\sin x}{\cos x} \rightarrow 2\sin x = \frac{\sin x}{\cos x} \rightarrow 2\sin x\cos x - \sin x = 0$ and $\cos x \neq 0 \rightarrow$

   $\sin x(2\cos x - 1) = 0$ and $\cos x \neq 0 \rightarrow \sin x = 0,\ \cos x = \frac{1}{2}$, and $\cos x \neq 0 \rightarrow$

   $x = 0°,\ 60°,\ 180°,\ 300°$

GBML '01

ROUND 5 – Trig. Analysis and Complex Numbers, Trig Form

1. $\left(\frac{1}{2}cis\,20°\right)^5(2cis\,26°)^{10} = \left(2^{-5}cis\,100°\right)\left(2^{10}cis\,260°\right) = 2^5 cis 360° = 32$ or $32 + 0i$

2. $\tan\alpha = \frac{cis\,270°}{2cis\,90°} = \frac{1}{2}cis\,180° = -\frac{1}{2}$. Since $\cos\alpha < 0 \Rightarrow \sin\alpha > 0 \Rightarrow y = 1$ *and* $x = -2 \Rightarrow$

   $r = \sqrt{5} \Rightarrow \sin\alpha = \frac{1}{\sqrt{5}},\ \cos\alpha = -\frac{2}{\sqrt{5}} \Rightarrow \sin 2\alpha = 2\sin\alpha\cos\alpha = 2\left(\frac{1}{\sqrt{5}}\right)\left(-\frac{2}{\sqrt{5}}\right) = -\frac{4}{5}$.

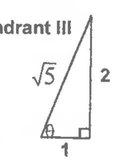
3. Given $0 \le x \le 180,\ 0 \le y \le 180,\ \cos x°\cos y° - \sin x°\sin y° = -0.5$, and $\sin x°\cos y° - \cos x°\sin y° = 1 \Rightarrow \cos(x + y) = -0.5$ and $\sin(x - y) = 1 \Rightarrow$ $x + y = 120$ or 240 and $x - y = 90 \Rightarrow 2x = 210$ or $330 \Rightarrow x = 105$ or $165 \Rightarrow$ $y = 15$ or 75 respectively $\Rightarrow$ ordered pairs are $(105,15), (165,75)$.

GBML '05

ROUND 5 – Trig Analysis and Complex Numbers (Trigonometric Form)

1. $\text{Sin}(\theta) = \frac{-2}{\sqrt{5}}$ and $\text{sec}(\theta) = -\sqrt{5}$ Simplifying,

   θ in quadrant III



$$\frac{1 + \frac{-2}{\sqrt{5}}}{1 - \frac{-2}{\sqrt{5}}} = \frac{\sqrt{5} - 2}{\sqrt{5} + 2} \cdot \frac{\sqrt{5} - 2}{\sqrt{5} - 2} = \frac{9 - 4\sqrt{5}}{5 - 4} = 9 - 4\sqrt{5}$$

   Thus, $9 - 4\sqrt{5} + 4\sqrt{5} = \underline{9}$

2. Recall: $\text{cis}(\theta) = \cos(\theta) + \sin(\theta)i = a + bi$ and
   According to DeMoivre's theorem, if $z_1 = r_1\text{cis}(\alpha)$ and $z_2 = r_2\text{cis}(\beta)$, then

   $z_1 \cdot z_2 = r_1 r_2 cis(\alpha + \beta)$ – multiply the amplitudes, add the angles

   $\frac{z_1}{z_2} = r_1 r_2 cis(\alpha - \beta)$ - divide the amplitudes, subtract the angles

   $z_1^n = r_1^n cis(n\alpha)$ - raise the amplitude to the power, multiply the angle

   $\boxed{6\sqrt{3} - 6i}$

   $z_1 = 4\text{cis}(30°)$ and $z_2 = \sqrt{3}\ \text{cis}(150°) \rightarrow z_1(z_2)^2 = 12(\text{cis}(330°) = 12(\frac{\sqrt{3}}{2} + \frac{-1}{2}i)$

3. $\cos(x+60°) - \sin(x+30°)$
$= \sin(90-(x+60)) - \sin(x+30) = \sin(30-x)-\sin(x+30)$
$= \sin(30)\cos(x) - \cos(30)\sin(x) - \sin(x)\cos(30) - \sin(30)\cos(x) = -2\cos(30)\sin(x)$
$= -\sqrt{3}\sin(x)$

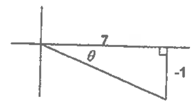
$\sec(120°)\cos(150°)\cos(x) = (-2)(-\frac{\sqrt{3}}{2})\cos(x) = \sqrt{3}\cos(x)$

$\therefore \sin(x) = -\cos(x) \rightarrow \tan(x) = -1 \rightarrow x =$ **<u>135°, 315°</u>**

GBML '06

ROUND 5

1. If $\cot\theta = -7$ and $\sin\theta < 0$, θ must be located in quadrant 4.



$$\sec\theta - \cos\theta = \frac{1-\cos^2\theta}{\cos\theta} = \frac{1-\left(\frac{7}{5\sqrt{2}}\right)^2}{\left(\frac{7}{5\sqrt{2}}\right)} = \left(1-\frac{49}{50}\right)\left(\frac{5\sqrt{2}}{7}\right) = \frac{5\sqrt{2}}{50(7)}$$

$$= \underline{\frac{\sqrt{2}}{70}}$$

2. $r = \frac{2}{\sin\theta} = 8\sin\theta \rightarrow \sin^2\theta = ¼ \rightarrow \sin\theta = ½ \rightarrow \theta = 30, 150, 210, 330$
$r > 0 \rightarrow \theta = 30, 150$ only. Converting to radians, <u>(4, π/6), (4, 5π/6)</u>.

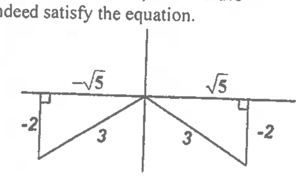
3. $2\sin(2x) + \tan(2x) = 2\sin(2x) + \frac{\sin(2x)}{\cos(2x)} = \frac{\sin(2x)(2\cos(2x)+1)}{\cos(2x)} = 0$
$\sin(2x) = 0 \rightarrow 2x = 0 + 180n \rightarrow x = 90n \rightarrow 0, 90, 180, 270$ or
$\cos(2x) = -½ \rightarrow 2x = \begin{cases}120+360n\\240+360n\end{cases} \rightarrow x = \begin{cases}60+180n\\120+180n\end{cases} \rightarrow x = 60, 120, 240, 300$
Since $\cos(2x) \neq 0$ for any of these eight values, none are extraneous.
$x =$ **<u>0, 90, 180, 270, 60, 120, 240, 300</u>**

Alternate method (using the double angle formula $\sin(2x) = \frac{2\tan x}{1+\tan^2 x}$) – oops!
$\frac{4\tan(x)}{1+\tan^2 x} + \frac{2\tan x}{1-\tan^2 x} = 0 \rightarrow 4\tan x - 4\tan^3 x + 2\tan x + 2\tan^3 x = 6\tan x - 2\tan^3 x$
$= 2\tan x(3 - \tan^2 x) \rightarrow 0, 180, 60, 120, 240, 300$
Notice that two solutions have been lost. Why did this happen?
The formula used was only valid for $x \neq 90 + 180n$.
Since the original equation (in terms of $2x$) does not 'blow up' for these values, these values must be checked directly in the original equation and they indeed satisfy the equation.

GBML '07

ROUND 5

1. $\sin A = -2/3 \rightarrow A$ lies in quadrant 3 or 4.
Since we require the smallest possible value of $\tan A$, A must lie in quadrant 4, where $\tan A$ is negative.
$\rightarrow \tan A = \underline{-\frac{2\sqrt{5}}{5}}$



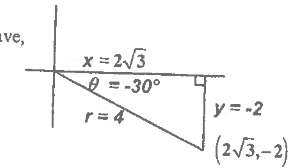
2. $2\sin(x) - \csc(x) + \cot(x) = 0 \rightarrow 2\sin(x) - \frac{1}{\sin(x)} + \frac{\cos(x)}{\sin(x)} \rightarrow 2\sin^2(x) - 1 + \cos(x) = 0$
$\rightarrow 2 - 2\cos^2(x) - \cos(x) - 1 = 0 \rightarrow 2\cos^2(x) - \cos(x) - 1 = (2\cos(x)+1)(\cos(x)-1) = 0$
$\rightarrow x =$ **<u>120°, 240°</u>** (0° is extraneous)

3. $x^{3/2} = 2 - 2i\sqrt{3} \rightarrow x = \left(2-2i\sqrt{3}\right)^{2/3} = \left(\left(2-2i\sqrt{3}\right)^2\right)^{1/3} = \left(-8-8i\sqrt{3}\right)^{1/3} = (-8)^{1/3}\left(1+i\sqrt{3}\right)^{1/3}$
Converting to trigonometric form, $= -2\left(2cis60°\right)^{1/3}$
$\theta = (60° + 360k)/3 \rightarrow 20° + 120k \rightarrow \begin{cases}-2\sqrt[3]{2}cis20°\\-2\sqrt[3]{2}cis140°\\-2\sqrt[3]{2}cis260°\end{cases} \rightarrow \begin{cases}\mathbf{2\sqrt[3]{2}cis200°}\\\mathbf{2\sqrt[3]{2}cis320°}\\\mathbf{2\sqrt[3]{2}cis80°}\end{cases}$

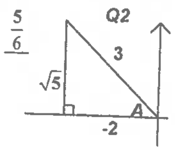
GBML '08

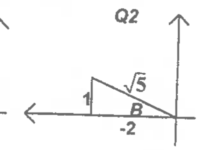
ROUND 5

1. Converting the denominator to trigonometric form we have,
$\left(\frac{8cis240}{2\sqrt{3}-2}\right)^2 = \left(\frac{8cis(240°)}{4cis(-30°)}\right)^2 = (2cis(270))^2 =$
$(2(\cos 270° + i\sin 270°))^2 = 4(0-1i)^2 = -4 \rightarrow$ <u>(-4, 0)</u>



2. $\sin(90+A) = -\frac{2}{3}$ and $0° \le m\measuredangle A < 180° \rightarrow \cos A = -\frac{2}{3}$ and $90° < A < 180°$ (quadrant 2)
$\tan(180-B) = \frac{1}{2}$ and $0° \le m\measuredangle B < 180° \rightarrow \tan B = -\frac{1}{2}$ and $90° < B < 180°$ (quadrant 2)
$\frac{\sin A}{\sin(270-B)} = \frac{\sin A}{-\cos B} = \frac{\sqrt{5}/3}{2/\sqrt{5}} = \frac{\sqrt{5}}{3}\cdot\frac{\sqrt{5}}{2} = \underline{\frac{5}{6}}$





3. $\tan(x+45)(\tan 135 + \tan x) = \sec^2 x - 2 \rightarrow \frac{\tan x + 1}{1-\tan x}\cdot(-1+\tan x) = (\tan^2 x + 1) - 2$
$\rightarrow \frac{\tan x + 1}{1-\tan x}\cdot(-1+\tan x) = \frac{\tan x + 1}{-1} = -\tan x - 1$ (provided $x \neq 45°, 225°$)
Thus, we have $-\tan x - 1 = \tan^2 x - 1 \rightarrow \tan^2 x + \tan x = \tan x(\tan x + 1) = 0$
$\rightarrow x =$ **<u>0°,135°,180°,315°</u>**

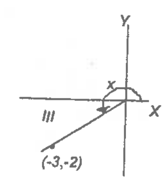
GBML '09

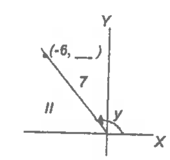
ROUND 5

1. $2\sin(270°+x)\cos(90°+x) = 1 + \cos 2x \rightarrow 2(-\cos x)(-\sin x) = \left(2\cos^2 x - 1\right) + 1$
$\rightarrow 2\cos x\sin x = 2\cos^2 x \rightarrow \cos x(\cos x - \sin x) = 0 \rightarrow \cos x = 0$ or $\sin x = \cos x$
$\rightarrow x =$ **<u>45°, 90°, 225°, 270°</u>**

2. $x^{3/2} = 8cis240°$
$\rightarrow x = (8cis240°)^{2/3} = 8^{2/3}cis\left(\frac{2}{3}(240° + 360k)\right) = 4cis(160° + 240k),\ k = 0, 1, 2$
$\rightarrow x = 4cis40°, 4cis160°, 4cis280° \rightarrow$ **<u>(4, 40°), (4, 160°), (4, 280°)</u>**

3. $(\sec x)(\sin y) + \cot(270° + x) = \frac{\sin y}{\cos x} + (-\tan x) =$
$= \left(-\frac{\sqrt{13}}{3}\right)\left(\frac{\sqrt{13}}{7}\right) + \left(-\frac{2}{3}\right) = -\frac{13}{21} - \frac{2}{3} = \frac{-13-14}{21} = \frac{-27}{21} = \frac{-9}{7}$
$\rightarrow$ <u>-63</u>

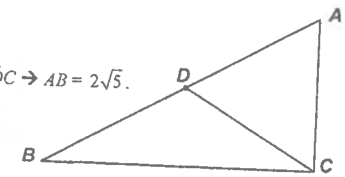




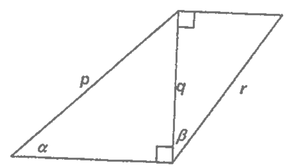
GBML 4/10

ROUND 5

1. Since $\overline{CD}$ is the median to the hypotenuse $\overline{AB}$, $AB = 2DC \rightarrow AB = 2\sqrt{5}$.
   $(2\sqrt{5})^2 = 2^2 + BC^2 \rightarrow BC = \sqrt{20-4} = 4 \rightarrow \frac{DB}{BC} = \underline{\frac{\sqrt{5}}{4}}$



2. $\sin\alpha = \frac{q}{p}, \cos\beta = \frac{q}{r} \rightarrow p\sin\alpha = r\cos\beta$
   $\rightarrow p\sin 2\beta = p(2\sin\beta\cos\beta) = r\cos\beta$
   $\rightarrow r = \underline{2p\sin}$



3. $P = \left(\sqrt{8}cis55°\right)^4 \rightarrow P = \left(\sqrt{8}\right)^4 \cdot cis(4\cdot 55°) = 64cis(220°)$
   $Q = \left(4\sqrt{2}-4i\right)^{\frac{1}{3}} = \left(8cis330°\right)^{\frac{1}{3}} = 2cis110°$
   $PQ = 128cis(330°) = 128\left(\frac{\sqrt{3}}{2} + \left(-\frac{1}{2}\right)i\right) = \underline{64\sqrt{3} + (-64)i}$

MML 11/05

Round One:

A. $(a+bi)^2 = a^2 - b^2 + 2abi = 0 + 1i$ so 2ab=1.
B. $(i + 5 - 3 - 7i)^2 = 4 - 24i - 36 = -32 - 24i$
C. $-6 - 2\sqrt{12} - 2 + \frac{16i(1-i\sqrt{3})}{1-(-3)} - (-8) = -4\sqrt{3} + \frac{16i + 16\sqrt{3}}{4} = 4i$

MML 11/06

Round 1

A) $i(2+3i)(1-4i) = i(2 - 8i + 3i - 12i^2) = i(2 - 5i + 12) = \underline{\mathbf{5 + 14i}}$

B) Let $\sqrt{2i} = \sqrt{0+2i} = \sqrt{(a+bi)^2} = \sqrt{(a^2-b^2)+(2ab)i} \rightarrow a^2 - b^2 = 0$ and $ab = 1$
   Thus, $b > 0 \rightarrow b = 1$ and $a = 1 \rightarrow \underline{\mathbf{1+i}}$

C) $(1-i\sqrt{3})^2 = -2 - 2i\sqrt{3} = -2(1+i\sqrt{3})$
   $(1-i\sqrt{3})^4 = [-2(1+i\sqrt{3})]^2 = 4(-2+2i\sqrt{3}) = -8(1-i\sqrt{3})$
   $(1-i\sqrt{3})^8 = [-8(1-i\sqrt{3})]^2 = 64(-2-2i\sqrt{3}) = -128(1+i\sqrt{3})$
   Thus, the sum is $(-2-8-128) + (-2+8-128)i\sqrt{3}$ = $\underline{\mathbf{-138 - (122\sqrt{3})i}}$

MML 11/07

Round 1

A) $\bar{z} = a - bi$ Thus, $z + \bar{z} = 7a - 5bi = 7 + 3i \rightarrow 7a = 7$ and $-5b = 3 \rightarrow (a, b) = \left(1, -\frac{3}{5}\right)$
   $\rightarrow \underline{\mathbf{1 + \left(-\frac{3}{5}\right)i}}$ or $\underline{\mathbf{1 + (-0.6)i}}$

B) Let $z = a + bi$. Then $(a+bi)^2 = 25(3+4i) \rightarrow a^2 - b^2 = 3$ and $2ab = 4 \rightarrow (a, b) = (2,1)$ or $(-2,-1)$
   $\rightarrow z = 5(2+i)$ or $5(-2-i) \rightarrow \underline{\mathbf{10+5i, -10-5i}}$

C) $|-3+4i| = \sqrt{(-3)^2+4^2} = 5$, $|12+16i| = \sqrt{12^2+16^2} = 20$, $|7-24i| = \sqrt{7^2+(-24)^2} = 25$
   $5x^2 - 20x - 25 = 5(x^2 - 4x - 5) = 5(x-5)(x+1) = 0 \rightarrow x = \underline{\mathbf{5, -1}}$

MML 11/09

Round 1

A) $\frac{1+2i+3i^2+4i^3}{1-2i+3i^2-4i^3} = \frac{1+2i-3-4i}{1-2i-3+4i} = \frac{-2-2i}{-2+2i} = \frac{1+i}{1-i}\cdot\left(\frac{1+i}{1+i}\right) = \frac{1+i^2+2i}{1-i^2} = \frac{2i}{2} = \underline{\mathbf{i}}$

B) $= 3^{40}(1+i)^{40} = 3^{40}(2i)^{20} = 3^{40}2^{20}i^{20} = 3^{40}2^{20}(1) = 9^{20}2^{20} = 18^{20} \rightarrow r + n = \underline{\mathbf{38}}$
   Note: $18^{20} = \left(18^2\right)^{10} = 324^{10}$. Such equivalent expressions produce larger values of $r + n$.

C) If $z = A + Bi$, then $z^2 = -40 - 9i = A^2 + 2ABi - B^2 = \left(A^2 - B^2\right) + 2ABi$ and $|z| = A^2 + B^2$

$$\left|z^2\right| = |z|^2 = \left(\sqrt{(-40)^2 + (-9)^2}\right)^2 = 41^2 \rightarrow |z| = 41.$$

Equating the real parts, the imaginary parts and the absolute values, $\begin{cases} A^2 - B^2 = -40 \\ 2AB = -9 \\ A^2 + B^2 = 41 \end{cases}$.

The second condition requires A and B have <u>opposite</u> signs.

$$\rightarrow (A, B) = \left(\pm\frac{1}{\sqrt{2}}, \mp\frac{9}{\sqrt{2}}\right) \rightarrow \left(\frac{A}{B}\right)^2 = \frac{\frac{1}{2}}{\frac{81}{2}} = \underline{\frac{1}{81}}$$

Round 1

MML 11/10

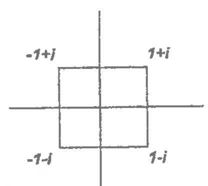
A) $\left(\frac{1-i}{1+i}\right)^{2010} = \left(\frac{1-i}{1+i}\cdot\frac{1-i}{1-i}\right)^{2010} = \left(\frac{(1-i)^2}{1-i^2}\right)^{2010} = \left(\frac{-2i}{2}\right)^{2010} = (-1)^{2010}\cdot i^{2008}\cdot i^2 = 1\cdot 1\cdot -1 = \underline{-1}$

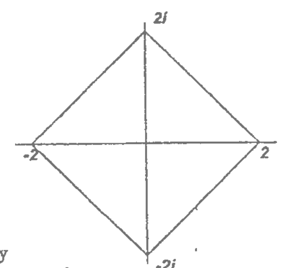
If you know DeMoirve's theorem, you might want to use it to formulate an alternative solution for comparison.

B) Equating the real and imaginary coefficients, $\begin{cases} x^2 - x - 5 = 1 \\ y^2 - 7y + 3 = -7 \end{cases} \rightarrow$

$\begin{cases} x^2 - x - 6 = (x-3)(x+2) = 0 \\ y^2 - 7y + 10 = (y-2)(y-5) = 0 \end{cases} \rightarrow x = 3, -2$ and $y = 2, 5 \rightarrow (3, 5), (-2, 5), (3, 2), (-2, 2) \rightarrow \underline{(-2, 5)}$

C) $(1+i)\cdot\begin{cases}(1+i)\\(-1+i)\\(-1-i)\\(1-i)\end{cases} = \begin{cases}2i\\-2\\-2i\\2\end{cases}$





The new figure is a square with side $2\sqrt{2}$, so the area is <u>8</u>.

Alternate solution: The area of the original square is $2^2 = 4$, and multiplying the vertices by $(1 + i)$ rotates the square 45° and expands each side by a factor of $|1 + i| = \sqrt{2}$. Therefore, the new square will have area $4\left(\sqrt{2}\right)^2 = \underline{8}$.