Round 4
Sequences and Complex Numbers

GREATER BOSTON MATHEMATICS LEAGUE

MEET 4 – JANUARY 1999

ROUND 4 – Sequences and Complex Numbers

1. ____________________

2. ____________________

3. (______ , ______ , ______)

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Given the arithmetic sequence, $5 + 4i$, $7 + i$, $9 - 2i$, …, find the sum of its first 20 terms.

2. Given the geometric sequence where $a_1 = 2$ and $r = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$, find a_{1999}.

3. Given the following sequence of **positive numbers**, 4, x, y, z, 100, where the first three numbers form a geometric sequence, the middle three numbers form an arithmetic sequence, and the last three numbers form a geometric sequence, find the ordered triple, (x, y, z).

GREATER BOSTON MATHEMATICS LEAGUE

MEET 4 – JANUARY 2000

ROUND 4 – Sequences and Complex Numbers

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Given a 100 term arithmetic series whose sum is 1800, and whose last term is 3 times its first term, find its last term.

2. Given a geometric sequence whose third term is $-4 + 8i$ and whose fourth term is $-16 - 8i$, find its first term.

3. Find the following sum of complex numbers:

$$\sum_{k=1}^{22}\left(5i^{k} + 2i^{3k}\right)$$

GREATER BOSTON MATHEMATICS LEAGUE

MEET 4 – JANUARY 2001

ROUND 4 – Sequences and Complex Numbers

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Given the arithmetic sequence of complex numbers whose first term is $3+i$ and whose tenth term is $-15+28i$, find the sum of the first 20 terms of this sequence. Note $i=\sqrt{-1}$.

2. Find the following sum: $\sum_{k=1}^{165} \log_{10}\left(\frac{3k+2}{3k+5}\right)$

3. The sum of all the terms of an infinite geometric sequence is 512 and the second term of this sequence is 96, find all possible values for its first term.

GREATER BOSTON MATHEMATICS LEAGUE
MEET 4 – JANUARY 2002

ROUND 4 – Sequences and Complex Numbers

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Given a geometric sequence in which $a_1 = \frac{1}{8}$, a_3 is a real number, and $a_4 = -i$, find a_{10}. Note $i = \sqrt{-1}$.

2. Given the series, $-47 - 39 - 31 - 23 - 15 - \ldots$, what is the least number of terms necessary for the sum to be greater than 200?

3. The three terms, $x, 5x+1$, and y, form an arithmetic sequence. If 2 is added to the first term, 3 is subtracted from the second term, and 4 is subtracted from the third term, the sequence is now geometric. Find all values for x which will make this true.

GREATER BOSTON MATHEMATICS LEAGUE
MEET 4 – JANUARY 2006

ROUND 4 – Sequences and Complex Numbers

1. (________ , ________)

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND.

1. $i^{11} - 25 \cdot \dfrac{(1-i)^2}{3-4i} + (2-i)^3$ may be expressed in the form $a + bi$. Determine the ordered pair (a, b).

2. If $\underline{x}$ is added to each of the numbers $7i$, $11i$ and $13i$, the new numbers in the same order are the first three terms in an infinite geometric progression. Determine the sum S_∞.

3. The 7 terms a, $5\text{-}b$, c, $\dfrac{3c-a}{2}$, $4b\text{-}17$, $4\text{-}2a$, $5b\text{–}6$, in this order, form an arithmetic progression. Determine the product abc.

GREATER BOSTON MATHEMATICS LEAGUE
MEET 4 – JANUARY 2007

ROUND 4 – Sequences and Complex Numbers

1. ______________________

2. ______________________

3. ______________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND.

1. Simplify the following expression completely: $(6)^{-\frac{1}{2}} \cdot \left(\dfrac{\sqrt{-6}}{1+i} + \dfrac{\sqrt{3}}{\sqrt{-2}} \right)$

2. The third term of an arithmetic progression is $(8+7i)$ and the sixth term is $(17+19i)$. Find the sum of the first 5 terms of this sequence. Express your answer in $a + bi$ form.

3. The second term of the geometric progression is $(13x-1)$, the fourth term is $(6x-6)$, and the sixth term is $2x+14$. The first term is a positive integer k. Find k.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 4 – JANUARY 2008

ROUND 4 – Sequences and Complex Numbers

If you would like to receive email announcements regarding upcoming competitions, please print your email on the reverse side of this paper when you have finished answering the problems.

1. ______________________

2. (__________) + (__________)i

3. A: (____, ____)   G: (____, ____)

CALCULATORS ARE NOT ALLOWED ON THIS ROUND.

1. The third term of an arithmetic sequence equals the sum of all the terms of a geometric sequence whose first term is $\frac{5}{4}$ and whose second term is $\frac{15}{16}$. If the seventh term of the arithmetic sequence is 19, find the value of the fifth term of the arithmetic sequence.

2. Find the sum of the following series, where $i = \sqrt{-1}$.
   Express your answer in $a + bi$ form.

$$i + 4i^4 + 7i^7 + 10i^{10} + 13i^{13} + 16i^{16} + \ldots + 73i^{73}$$

3. For the sequence 16, x, y, 250, there is an ordered pair (x, y) for which these four terms form an arithmetic sequence (A) and an ordered pair (x, y) for which they form a geometric sequence (G). Find these ordered pairs.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 4 – JANUARY 2009

ROUND 4 – Sequences and Complex Numbers

1. (_________ , _________)

2. ________________________________

3. ________________________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND.

1. For real numbers x and y, $\left(\dfrac{6i^2+2i^3}{(1-i)^2}\right)^2 = x + yi$. Determine the ordered pair (x, y).

2. The first three terms of an arithmetic sequence are $\sqrt{a}$, $\sqrt{a+6}$ and $\sqrt{a+14}$. Find the numerical value of the tenth term.

3. The following six terms form a geometric sequence of positive integers, namely
$A,\ 7x-1,\ 11y-5,\ 15x+3,\ z^2-7$ and $34x+5$.
Find the numerical value of the sum of the first three terms.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 4 – JANUARY 2010

ROUND 4 – Sequences and Complex Numbers

1. (__________, __________)

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND.

1. $\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i\right)^{2010}=a+bi$ Compute the ordered pair (a, b).

2. The 2nd, 4th and 9th terms in an increasing arithmetic sequence form a geometric sequence. If the 15th term of the arithmetic sequence is 86, compute the sum of the three terms that generated the geometric sequence.

3. Given: $a_{n+2}=a_{n+1}-2a_n$
If the 5th term in the sequence $a_5=-24$ and the 2nd term $a_2=10$, compute the sum of the first term a_1 and the seventh term a_7.

GREATER BOSTON MATHEMATICS LEAGUE
MEET 4 – JANUARY 2011

ROUND 4 – Sequences and Complex Numbers

1. (__________ , __________)

2. ______________________________

3. ______________________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND.

1. $\left(3-\sqrt{-4}\right)\left(2+3i\right)+\left(1-4i\right)^2 = A(B+i)$ Compute (A, B).

2. $-2-2i,\ M,\ 6+6i$ is an arithmetic progression.
   $-2-2i,\ J,\ 6+6i$ is a geometric progression.
   Compute the value of $\dfrac{M^2}{J^2}$.

3. Given: the sequence $T, 6, x, y, 27, K$
   $T, 6, x, y$ form an arithmetic progression.
   $x, y, 27, K$ form a geometric progression.
   Compute <u>all</u> possible ordered pairs (T, K).

MASSACHUSETTS MATHEMATICS LEAGUE
FEBRUARY 2004
ROUND 6: SEQUENCES & SERIES

ANSWERS

A)______________

B)______________

C)______________

A) In an arithmetic sequence of ten terms, the tenth term is 14, and their sum is 5. Find the second term.

B) The second term of a geometric sequence is 12, and the sixth term is 1024/27. Find the first term.

C) The six terms $2x - 3$, t, $7 - 12y$, $x + 3$, $3y - 4$, $x + 12$ are in arithmetic sequence. Find the ordered triple (x, y, t).

MASSACHUSETTS MATHEMATICS LEAGUE
FEBRUARY 2005
ROUND 6 ALGEBRA 2: SEQUENCES & SERIES

ANSWERS

A)______________

B)______________

C)______________

A) Find the 2005th term of an arithmetic sequence whose third term is -2000 and whose fifth term is -1996.

B) For an arithmetic sequence a and a geometric sequence g, $a_9 = g_1$ while $a_{81} = g_3$. If $a_0 = 0$ and $g_1 = 6$ find all possible values for $a_2 + g_2$ as improper fractions.

C) At the beginning of each year Shauna adds $100 to her bank account; at the end of each year the bank adds 8% interest to the account. At the beginning of every month Will adds $20 to a shoe box in his closet. If each began with no money when they made their first deposits Jan 1 1990, who had the greater amount after interest was paid to Shauna at the end of 2004- and how much more did they have rounded to the nearest dollar?

MASSACHUSETTS MATHEMATICS LEAGUE
FEBRUARY 2006
ROUND 6 ALGEBRA 2: SEQUENCES & SERIES

ANSWERS

A)_________minutes

B)_______________

C)_______________

A) Sal gets 4 hours of homework every school night. On day 1 he is exceptionally motivated and does all his homework. However, on each successive school night he does only half as much homework as he did on the previous school night. At the end of the school year (180 days) to the nearest minute, how much total homework will Sal have done?

B) For an arithmetic sequence a, we find a_{2006} is twice a_{2004} and a_{2006} is 500 more than three times a_{2000}. Find a_{2005} .

C) The sum of the first three terms of a geometric series is 296, while the infinite sum is 80 less than twice that amount. Find the fifth term of the series.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 – FEBRUARY 2007
ROUND 6 ALG 2: SEQUENCES AND SERIES

ANSWERS

A) ______________________

B) ______________________

C) ______________________

A) The symbol Σ in mathematics represents a summation, the addition of several terms of a specific type. For example, $\sum_{k=1}^{k=4} k^2 = 1^2 + 2^2 + 3^2 + 4^2 = 30$.

Evaluate: $\sum_{k=2}^{k=6} (3k-2)$

B) Let $t_1 = (2^{-1} + 2^{-2} + 2^{-3} + \ldots)$, $t_2 = (3^{-1} + 3^{-2} + 3^{-3} + \ldots)$, … $t_n = ((n+1)^{-1} + (n+1)^{-2} + (n+1)^{-3} + \ldots)$
Determine the minimum number of terms that must be added so that the sum exceeds 2.

C) The harmonic mean of <u>nonzero</u> numbers a and b is defined as $\frac{2ab}{a+b}$.
Given a sequence 1, x, y, 2 such that x is the harmonic mean between 1 and y and y is the harmonic mean between x and 2. What is the sum of x and y?

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 – FEBRUARY 2008
ROUND 6 ALG 2: SEQUENCES AND SERIES

ANSWERS

A) ______________________

B) ______________________

C) ______________________

A) 3^2, 5^2, 7^2 are the 2nd, 6th and 12th terms in an arithmetic sequence. What is the 14th term?

B) Find the first term in the arithmetic sequence -2, 5, 12, 19, … that is larger than the 10th term in the geometric sequence -0.75, 1.5, -3, 6, …

C) Given: $\begin{cases} A_{N+2} = 2A_{N+1} + 3A_N \quad \text{for } N \geq 1 \\ A_2 = 4 \\ A_5 = 17 \end{cases}$ <u>Compute</u> $A_1 + A_6$.

'99 GBML

ROUND 4

1. For the arithmetic sequence, $5+4i, 7+i, 9-2i, \ldots, \Rightarrow d = 2-3i$. Now use the formula for arithmetic series, $S_n = \frac{n}{2}\left(2a_1 + (n-1)d\right) \Rightarrow S_{20} = 10\left(2(5+4i) + 19(2-3i)\right) = \mathbf{480 - 490i}$

2. $a_1 = 2$ and $r = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$, $r^2 = \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)^2 = 2\left(\frac{\sqrt{2}}{2}\right)^2 i = i \Rightarrow r^4 = -1 \Rightarrow r^8 = 1$;
$a_{1999} = a_1 \cdot r^{1998}$ and $1998 = 6 \bmod 8 \Rightarrow a_{1999} = 2 \cdot r^6 = 2 \cdot r^4 \cdot r^2 = 2(-1)i = \mathbf{-2i}$

3. Since 4, x, y form a geom. seq. $\Rightarrow x^2 = 4y$; since x, y, z form an arith. seq. $\Rightarrow x + z = 2y$; since $y, z, 100$ form a geom. seq. $\Rightarrow z^2 = 100y$; $x^2 = 4y \Rightarrow$
$25x^2 = 100y \Rightarrow z^2 = 25x^2 \Rightarrow z = 5x$ (both positive); $x + z = 2y \Rightarrow$
$x + 5x = \frac{x^2}{2} \Rightarrow x^2 = 12x \Rightarrow x = 12\,(x \neq 0) \Rightarrow z = 60 \Rightarrow y = 36 \Rightarrow$ **(12, 36, 60)** is the answer

'00 GBML

Round 4

1. $S_n = \frac{n}{2}(a_1 + a_n) \Rightarrow 1800 = \frac{100}{2}(x + 3x) \Rightarrow 200x = 1800 \Rightarrow x = 9 \Rightarrow 3x = 27$

2. $r = \frac{a_4}{a_3} = \frac{-16-8i}{-4+8i} = \frac{-4-2i}{-1+2i} = \frac{(-4-2i)(-1-2i)}{(-1+2i)(-1-2i)} = \frac{4+10i-4}{1+4} = \frac{10i}{5} = 2i$

$a_1 = \frac{a_3}{r^2} = \frac{-4+8i}{-4} = 1-2i$

3. Since the powers of i repeat every 4 and $i + i^2 + i^3 + i^4 = 0$, $\sum_{k=1}^{22} 5i^k = 5\left(i^{21} + i^{22}\right) = 5(i-1)$
$\sum_{k=1}^{22} 2i^{3k} = 2\sum_{k=1}^{22}\left(\left(i^3\right)^k\right) = 2\sum_{k=1}^{22}\left((-i)^k\right)$ and since the powers of $-i$ repeat every 4 and
$(-i)^1 + (-i)^2 + (-i)^3 + (-i)^4 = 0$, this second sum =
$2\left((-i)^{21} + (-i)^{22}\right) = 2(-i-1)$; $5(i-1) + 2(-i-1) = -7 + 3i$

'01 GBML

Round 4

1. $-15 + 28i = 3 + i + 9d \rightarrow d = -2 + 3i$;
$S_{20} = \frac{20}{2}\left(2(3+i) + 19(-2+3i)\right) = 10(6 + 2i - 38 + 57i) = -320 + 590i$

2. $\sum_{k=1}^{165} \log_{10}\left(\frac{3k+2}{3k+5}\right) = \sum_{k=1}^{165}\left(\log_{10}(3k+2) - \log_{10}(3k+5)\right)$; since $3(k+1)+2 = 3k+5$, the second term being subtracted = 3rd term being added, and so on. This means all the terms add to 0 except the first and last terms. The sum =
$\log_{10}(3(1)+2) - \log_{10}(3(165)+5) = \log_{10}5 - \log_{10}500 = \log_{10}\left(\frac{5}{500}\right) = \log_{10}\left(\frac{1}{100}\right) = -2$

3. $\frac{a}{1-r} = 512$ and $ar = 96 \rightarrow \frac{96}{r} = 512(1-r) \rightarrow 3 = 16r(1-r) \rightarrow 16r^2 - 16r + 3 = 0$
$\rightarrow (4r-1)(4r-3) = 0 \rightarrow r = \frac{1}{4}, \frac{3}{4} \rightarrow a = 96\left(\frac{4}{3}\right), 96\left(\frac{4}{1}\right) = 128, 384$

'02 GBML

ROUND 4 – Sequences and Complex Numbers

1. $a_4 = a_1 \cdot r^3 \Rightarrow -i = \frac{1}{8}r^3 \Rightarrow r^3 = -8i$. Since a_3 is real the only possible value for $r = 2i \Rightarrow$
$a_{10} = a_4 \cdot r^6 = -i(-2i)^6 = -i(-64) = 64i$.

2. This is an arithmetic series with $d = 8$. The sum of the first n terms =
$\frac{n}{2}\left(2(-47) + (n-1)(8)\right) = \frac{n}{2}(8n - 102) = n(4n - 51)$. You want the smallest value of n such that $n(4n-51) > 200$. You could use the quadratic formula to find what value of n makes the sides equal, but trial and error is quicker and less complicated.
If $n = 15 \Rightarrow n(4n-51) = 15 \cdot 9 < 200$. If $n = 16 \Rightarrow n(4n-51) = 16 \cdot 13 > 200$. Therefore the answer is 16.

3. Since the original terms are arithmetic, then $y + x = 10x + 2 \Rightarrow y = 9x + 2$. The new terms, $x+2$, $5x-2$, and $y-4$ are geometric, therefore $(5x-2)^2 = (x+2)(y-4)$
$\Rightarrow 25x^2 - 20x + 4 = (x+2)(9x-2) = 9x^2 + 16x - 4 \Rightarrow 16x^2 - 36x + 8 = 0 \Rightarrow$
$4x^2 - 9x + 2 = 0 \Rightarrow (4x-1)(x-2) = 0 \Rightarrow x = \frac{1}{4}, 2.$

'06
GBML

ROUND 4 – Sequences and Complex Numbers ($i = \sqrt{-1}$)

1. The powers of $\underline{i}$ have a cycle of 4. i^n cycles through the values $\{1, i, -1, -i\}$ as $\underline{n}$ takes on values that are of the form $4k$, $4k+1$, $4k+2$ and $4k+3$ in this order. Divide the exponent by 4, retain only the remainder and you can easily evaluate any power of $\underline{i}$.
$i^{11} = i^3 = -i$, $(1-i)^2 = -2i$, $(2-i)^3 = 2^3 + 3(2^2)(-i) + 3(2)(-i)^2 + (-i)^3 = 8 - 12i - 6 + i = 2 - 11i$
Rationalizing the denominator by multiplying both numerator and denominator by $3 + 4i$ reduces the middle term to $(-2i)(3+4i) = -6i + 8$
$-i - (-6i + 8) + (2 - 11i) = -6 - 6i \rightarrow (a, b) = \underline{\mathbf{(-6, -6)}}$
2. The new numbers are: $7i + x$, $11i + x$ and $13i + x$. A geometric progression requires that the ratio of successive terms must be equal. This implies that the product of the first and third terms must equal the square of the middle term.
Thus, $(11i + x)^2 = (7i + x)(13i + x) \rightarrow -121 + 22ix + x^2 = -91 + 20ix + x^2 \rightarrow 2ix = 30$
$\rightarrow x = 30/(2i) = -15i$.
The first three terms of the geometric progression are: $-8i$, $-4i$ and $-2i$
The common ratio is ½. The sum of the terms in the infinite progression is determined by the formula $\frac{a}{1-r}$ (where $\underline{a}$ denotes the first term). $S_\infty = -8i/(1-.5) = \underline{\mathbf{-16i}}$
4. An arithmetic progression requires that the difference between successive terms must be equal. Let $(t_1, t_2, t_3, t_4, t_5, t_6, t_7) = (a, 5-b, c, (3c-a)/2, 4b-17, 4-2a, 5b-6)$ and d denote the common difference.
$d = t_2 - t_1 = t_3 - t_2 \rightarrow a + 2b + c = 10$ (#1)
$d = t_4 - t_3 = t_5 - t_4 \rightarrow a + 4b - 2c = 17$ (#2)
$d = t_6 - t_5 = t_7 - t_6 \rightarrow 4a + 9b = 31$ (#3)
2(#1) + #2 $\rightarrow 3a + 8b = 37$ (#4)
-3(#3)+4(#4) $\rightarrow 5b = 55 \rightarrow b = 11$
Substituting, $a = -17$, $c = 5 \rightarrow abc = \underline{\mathbf{-935}}$

'07
GBML

ROUND 4 – Sequences and Complex Numbers ($i = \sqrt{-1}$)

1. $\frac{1}{\sqrt{6}}\left(\frac{i\sqrt{6}}{1+i} + \frac{\sqrt{3}}{i\sqrt{2}}\right) = \frac{i}{1+i} + \frac{1}{2i} = \frac{i(1-i)}{2} + \frac{i}{-2} = \frac{i+1-i}{2} = \underline{\mathbf{\frac{1}{2}}}$

2. Let $t_1 = a$ Then $t_3 = a + 3d = 8 + 7i$ and $t_6 = a + 5d = 17 + 19i$
Subtracting, $3d = 9 + 12i \rightarrow d = 3 + 4i$
Substituting, $a = 2 - i$
$S_n = \frac{n}{2}(2a + (n-1)d) \rightarrow S_5 = \frac{5}{2}(4 - 2i + 4(3+4i)) = \frac{5}{2}(16 + 14i) = \underline{\mathbf{40 + 35i}}$

3. Let r denote the common multiplier of the geometric progression.
$t_4/t_2 = t_6/t_4 = r^2 \rightarrow \frac{6x-6}{13x-1} = \frac{2x+14}{6x-6} \rightarrow 6^2(x-1)^2 = (13x-1)(2x+14) = 26x^2 + 180x - 14$
$\rightarrow 10x^2 - 252x + 50 = (5x-1)(x-25) = 0$
$\rightarrow x = 1/5 \rightarrow r^2 = -3$ (rejected)
or $x = 25 \rightarrow r^2 = 144/324 = 12^2/18^2 \rightarrow r = \pm 2/3$
Only $r = 2/3$ produces a <u>positive</u> first term, $k(2/3) = 324 \rightarrow k = \underline{\mathbf{486}}$

GBML
'08

ROUND 4

1. $t_3 = a + 2d = S_\infty = \frac{A}{1-R} = \frac{5/4}{1-3/4} = 5$ and $t_7 = a + 6d = 19 \rightarrow (a, d) = (-2, 7/2)$
Thus, $t_5 = a + 4d = -2 + 4(7/2) = \underline{\mathbf{12}}$

2. The coefficient and exponent of the nth term in the sum are given by $3n - 2$.
Therefore, $3n - 2 = 73 \rightarrow n = 25$ and the sum consists of 25 terms.
The sum of the first block of 4 terms is $(i + 4 - 7i - 10) = -6 - 6i$.
The sum of the second block of 4 terms is $(13i + 16 - 19i - 22) = -6 - 6i$.
Thus, the sum of the 25 terms is $6(-6 - 6i) + (73i^{73}) = -36 - 36i + 73i = \underline{\mathbf{-36 + 37i}}$

3. Given: the sequence 16, x, y, 250
For an arithmetic sequence, $x - 16 = y - x = 250 - y \rightarrow \begin{cases} 2x = y + 16 \\ 2y = 250 + x \end{cases}$
$\rightarrow 2(2x - 16) = 250 + x \rightarrow 3x = 282 \rightarrow (x, y) = \underline{\mathbf{(94, 172)}}$
For the geometric sequence, $\frac{x}{16} = \frac{y}{x} = \frac{250}{y} \rightarrow \begin{cases} x^2 = 16y \\ y^2 = 250x \end{cases} \rightarrow x^4 = 16^2(250x)$
$\rightarrow x^3 = 2^8(250) \rightarrow 2^9 5^3 \rightarrow x = 8(5) = 40 \rightarrow (x, y) = \underline{\mathbf{(40, 100)}}$

GBML
'09

ROUND 4

1.
$\left(\frac{6i^2 + 2i^3}{(1-i)^2}\right)^2 = \left(\frac{-6-2i}{-2i}\right)^2 = \left(\frac{3+i}{i}\right)^2 = \frac{9 + 6i - 1}{-1} = -8 - 6i$
Thus, $(x, y) = \underline{\mathbf{(-8, -6)}}$.

2.
Since $\varnothing$ are the first three terms of an AP, $2\left(\sqrt{a+6}\right) = \sqrt{a} + \sqrt{a+14}$.
$\rightarrow 4(a+6) = a + 2\sqrt{a(a+14)} + a + 14$
$\rightarrow 4a + 24 = 2a + 14 + 2\sqrt{a(a+14)}$
$\rightarrow 2a + 10 = 2\sqrt{a(a+14)}$
$\rightarrow a + 5 = \sqrt{a(a+14)}$
$\rightarrow a^2 + 10a + 25 = a^2 + 14a$
$\rightarrow 4a = 25 \rightarrow a = \frac{25}{4} \rightarrow$ first three terms: $\frac{5}{2}, \frac{7}{2}, \frac{9}{2} \rightarrow$ common difference $d = 1$
Thus, $t_{10} = \frac{5}{2} + 9(1) = \underline{\mathbf{\frac{23}{2}}}$

3.
Since the terms A, $\underline{7x-1}$, $11y-5$, $\underline{15x+3}$, z^2-7, $\underline{34x+5}$ form a geometric sequence, we have $(15x+3)^2 = (7x-1)(34x+5)$. Therefore,
$225x^2 + 90x + 9 = 238x^2 + x - 5 \rightarrow 0 = 13x^2 - 89x - 14 \rightarrow 0 = (13x+2)(x-7)$
$x = \frac{13}{2}$ is rejected since all six terms must be positive integers. Thus, $x = 7$ only.
The sequence is now A, 48, $11y - 5$, 108, $z^2 - 7$, 243 and
$(11y-5)^2 = (48)(108) = 4 \cdot 12 \cdot 9 \cdot 12 = 6^2 12^2 = (72)^2$
Therefore, $11y - 5 = 72$ and $y = 7$ also. Since the first four terms of this geometric sequence are A, 48, 72, 108, the common multiplier is 3/2 and $A = \frac{2}{3}(48) = 32$
The sum of the first three terms is $32 + 48 + 72 = \underline{\mathbf{152}}$

GBML '10

ROUND 4

1. 1st approach: $\left[\frac{\sqrt{2}}{2}(1+i)\right]^{2010} = \left(\frac{\sqrt{2}}{2}\right)^{2010}(1+i)^{2010} =$

$\left(\frac{1}{\sqrt{2}}\right)^{2010}\left((1+i)^2\right)^{1005} = \frac{1}{2^{1005}}(2i)^{1005} = i^{1005} = i^{1004}i = \left(i^4\right)^{251} i = (1)^{251} i = i$

$\rightarrow (a, b) = \underline{(0, 1)}$

2nd approach:

$\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i\right)^{2010} = (1cis45°)^{2010} = \left((\cos 45° + i\sin 45°)^4\right)^{502}(\cos 45° + i\sin 45°)^2$

$= (cis180°)^{502}(cis90°) = (-1+0i)^{502}\cdot(0+i) = 1(i) = i \rightarrow (a, b) = \underline{(0, 1)}$

2. $(t_2, t_4, t_9) = (a+d, a+3d, a+8d)$

Since these terms generate a geometric sequence, $(a+3d)^2 = (a+d)(a+8d)$

$\rightarrow a^2 + 6ad + 9d^2 = a^2 + 9ad + 8d^2 \rightarrow d^2 = 3ad \rightarrow d = 3a, \cancel{d=0}$

$(d = 0 \rightarrow a = 86 \rightarrow$ non-increasing arithmetic sequence 86, 86, 86, …)

$t_{15} = a + 14d = a + 42a = 86 \rightarrow a = 2$

$d = 3a \rightarrow (t_2, t_4, t_9) = (4a, 10a, 25a) = (8, 20, 50) \rightarrow$ sum = $\underline{78}$

3. $a_{n+2} = a_{n+1} - 2a_n \rightarrow \begin{cases} a_5 = a_4 - 2a_3 \\ a_4 = a_3 - 2a_2 \end{cases} \rightarrow -24 = a_3 - 2a_2 - 2a_3 = -a_3 - 2a_2 = -a_3 - 20 \rightarrow a_3 = 4$

$\rightarrow a_4 = 4 - 20 = -16,\ a_5 = -24,\ a_6 = -24 - 2(-16) = 8,\ a_7 = 8 - 2(-24) = 56,$

$a_4 = a_3 - 2a_2 \rightarrow -16 = 4 - 2a_2 \rightarrow a_2 = 10$

$a_3 = a_2 - 2a_1 \rightarrow 4 = 10 - 2a_1 \rightarrow a_1 = 3$

Therefore, $a_1 + a_7 = 3 + 56 = \underline{59}$

GBML '11

ROUND 4

1. $(3-\sqrt{-4})(2+3i) + (1-4i)^2 = (12+5i) + (-15-8i) = -3-3i = -3(1+i) \rightarrow (A, B) = \underline{(-3, 1)}$.

2. AP: $-2-2i, M, 6+6i \rightarrow M = \frac{(-2-2i)+(6+6i)}{2} = 2+2i = 2(1+i) \rightarrow M^2 = 8i$

GP: $-2-2i, J, 6+6i \rightarrow J^2 = (-2-2i)(6+6i) = -24i$

Therefore, $\frac{M^2}{J^2} = \underline{-\frac{1}{3}}$

3. AP: $T, 6, x, y \rightarrow x - 6 = y - x \rightarrow y = 2x - 6$

GP: $x, y, 27, K \rightarrow y^2 = 27x$

Therefore, $(2x-6)^2 = 27x \rightarrow 4x^2 - 61x + 36 = (x-12)(4x-3) = 0$

$\rightarrow (x, y) = (12, 18), \left(\frac{3}{4}, -\frac{9}{2}\right)$

$6 - T = x - 6 \rightarrow T = 12 - x$

$\frac{27}{y} = \frac{K}{27} \rightarrow \frac{27^2}{y}$

For (12, 18), $(T, K) = \underline{\left(0, \frac{81}{2}\right)}$ For $\left(\frac{3}{4}, -\frac{9}{2}\right)$, $(T, K) = \left(\frac{45}{4}, -162\right)$

MML '04

MASSACHUSETTS MATHEMATICS LEAGUE
FEBRUARY 2004
ROUND 6: SEQUENCES & SERIES

ANSWERS

A) -10

B) ± 9

C) $\left(-\frac{15}{2}, \frac{4}{3}, -\frac{27}{2}\right)$

A) In an arithmetic sequence of ten terms, the tenth term is 14, and their sum is 5. Find the second term.

$a_1 + 9d = 14,\quad 5(2a_1 + 9d) = 5$ so $a_1 + 9d = 14$

$\underline{2a_1 + 9d = 1}\quad a_1 = -13$

$-13 + 9d = 14,\ 9d = 27,\ d = 3.$

$a_2 = a_1 + d = -13 + 3 = -10$

B) The second term of a geometric sequence is 12, and the sixth term is 1024/27. Find the first term.

$12r^4 = \frac{1024}{27},\quad r^4 = \frac{1024}{12\cdot 27} = \frac{256}{81},\quad r = \pm\frac{4}{3}$

$a_1 = \frac{a_2}{r} = \frac{12}{\pm 4/3} = \pm\frac{12}{1}\cdot\frac{3}{4} = \pm 9$

C) The six terms 2x – 3, t, 7 – 12y, x + 3, 3y – 4, x + 12 are in arithmetic sequence. Find the ordered triple (x, y, t).

$(x+12) - (x+3) = 9 = 2d,\ d = \frac{9}{2}.\ (3y-4) - (7-12y) = 9,$

$15y - 11 = 9,\ 15y = 20,\ y = \frac{20}{15} = \frac{4}{3},\ a_3 = 7 - 12y = 7 - \frac{12}{1}\cdot\frac{4}{3} =$

$7 - 16 = -9,\ t = a_2 = a_3 - d = -9 - \frac{9}{2} = -\frac{27}{2},\ a_4 = x + 3 = a_3 + d =$

$-9 + \frac{9}{2} = -\frac{9}{2}$ so $x + 3 = -\frac{9}{2}$ and $x = -3 - \frac{9}{2} = -\frac{6-9}{2} = -\frac{15}{2}.$

ANS $(x, y, t) = \left(-\frac{15}{2}, \frac{4}{3}, -\frac{27}{2}\right)$

MML '05

Round Six:

A. Constant difference is 4/2=2; $a_3 = a_0 + 3(2)$ so $a_0 = -2006$ and therefore $a_{2005} = -2006 + 2005(2)$.

B. $81d = g_0 r^3$ while $9d = g_0 r$; dividing gives $r = \pm 3$. If r = 3, $g_n = 2 \cdot 3^n$ and since $a_9 = 6$, $a_n = \frac{2}{3}n$ so $a_2 + g_2 = \frac{58}{3}$. If r = –3, $a_n = \frac{-2}{3}n$ and $a_2 + g_2 = \frac{50}{3}$.

C. Will accumulates \$20x12x15=\$3,600. Shauna has the geometric sum 100*1.08 + 100*(1.08)^2 + 100*(1.08)^{15} = 100*1.08*(1.08^{15} – 1)/ 0.08 ≈ \$2,932.43.

MML '06

Round Six:

A. HW done = 4 + 2 + 1 + ... + a geometric progression of 180 terms with r = ½. The difference between the sum of 180 terms and the sum of an infinite sequence is considerably less than 1 minute, so use $a/(1-r) \rightarrow 4/(1-½)$ = 8 hrs = 480 min

B. $a_{2006} = a_{2000} + 6d = 2(a_{2000} + 4d)$, so $a_{2000} = -2d$, while $a_{2000} + 6d = 500 + 3a_{2000}$ so $a_{2000} = 3d - 250$. Thus, $-5d = -250 \rightarrow d = 50$, $a_{2000} = -100$ and $a_{2005} = -100 + 5(50) = 150$.

C. $a + ar + ar^2 = 296$, while $a/(1-r) = 512$. $a = 296/(1 + r + r^2) = 512(1-r)$, so $296/512 = (1 + r + r^2)(1-r) = 1 - r^3 \rightarrow r^3 = 1 - 296/512 = 216/512 \rightarrow$ $r = 3/4$ and $a = 128$.

MML '07

Round 6

A) For $k = 2$ to 6, the expression $3k - 2$ produces the numbers 4, 7, 10, 13 and 16. The sum of these 5 numbers is **50**.

B) Each of these terms is the sum of an infinite geometric sequence. Applying $\frac{a}{1-r}$,

$t_1 = 1$, $t_2 = 1/2$, $t_3 = 1/3$, $t_4 = 1/4$, etc
$1 + 1/2 + 1/3 + 1/4 = 25/12 > 2 \rightarrow n = \underline{4}$

C) By definition, $x = \frac{2y}{1+y}$ and $y = \frac{4x}{x+2}$. Substituting for x in the second equation,

$$y = \frac{4\left(\frac{2y}{1+y}\right)}{\frac{2y}{1+y}+2} = \frac{8y}{4y+2} = \frac{4y}{2y+1} \rightarrow 2y^2 + y = 4y \rightarrow 2y^2 - 3y = y(2y-3) = 0 \rightarrow y = \frac{3}{2}$$

and $x = \frac{3}{5/2} = \frac{6}{5}$. Adding the required sum is $\frac{3}{2} + \frac{6}{5} = \frac{5+12}{10} = \underline{2.7}$

MML '08

Round 6

A) $9 + 4d = 25 \rightarrow d = 4 \rightarrow t_{14} = t_{12} + 2(4) = \underline{57}$

B) The first sequence is an <u>arithmetic</u> sequence with a common difference of 7, i.e. $t_n = 7n - 9$. The second sequence is a <u>geometric</u> sequence with a common ratio of –2, i.e. $t_n = (3/8)(-2)^n$

The 10th term in the geometric sequence is $(3/8)(-2)^{10} = 384$.
$7n - 9 > 384 \rightarrow n > 393/7 = 56+ \rightarrow n = 57 \rightarrow 7(57) - 9 = \underline{390}$

C) Using the recursive part of the definition, $A_{N+2} = 2A_{N+1} + 3A_N$

$N = 3 \rightarrow A_5 = 2A_4 + 3A_3$

$N = 2 \rightarrow A_4 = 2A_3 + 3A_2$

Substituting for A_2 and A_5, $\begin{cases} 17 = 2A_4 + 3A_3 \\ A_4 = 2A_3 + 12 \end{cases} \rightarrow (A_3, A_4) = (-1, 10)$

$A_3 = 2A_2 + 3A_1 \rightarrow -1 = 8 + 3A_1 \rightarrow A_1 = -3$

$A_6 = 2A_5 + 3A_4 = 2(17) + 3(10) = 64$

Thus, $A_1 + A_6 = \underline{61}$