F.O. SMITH CT

THE NEW ENGLAND ASSOCIATIONS OF MATHEMATICS LEAGUES

# NEAML

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## 42nd *ANNUAL MATH COMPETITION*

## May 2, 2014

## *CANTON HIGH SCHOOL*

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NEW ENGLAND ASSOCIATION OF MATHEMATICS LEAGUES

PLAYOFFS – 2014

**Round 1: Arithmetic and Number Theory**

1. ____________________

2. ____________________

3. ____________________

1. Compute: $1+\cfrac{1}{.\overline{1}+\cfrac{.\overline{8}}{.\overline{2}+\cfrac{.\overline{7}}{.\overline{3}+\cfrac{.\overline{6}}{.\overline{4}+.\overline{5}}}}}$

2. A store clerk needs to give a customer \$0.23 in change. The change is to be made with $k$ coins, a collection made up exclusively of pennies, nickels and/or dimes. How many possible values of $k$?

3. A combination lock for a bicycle consists of four cylinders, each of which has all the integers from 1 to 6. Alvin forgot his combination so he tried each integer in order, starting with 1111, then 1112, followed by 1113, and so on until the lock finally opened at 2563. Compute the number of unsuccessful integers that he tried.

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**PLAYOFFS – 2014**

**Round 2: Algebra 1**

**1.** ____________________

**2.** ____________________

**3.** ____________________

1. If $x = 3.000001$, approximate $\dfrac{x^2+5x+2}{x^2-9}$ to the nearest hundred thousand.

2. Compute the largest possible product $xy$ given that $(x, y)$ is a solution of the system:

$x^2 + y = 8$ and $y - x = -4$.

3. Compute all values of $k$ for which there are real $x$ and $y$ such that $\dfrac{k}{x-y} = \dfrac{1}{x} - \dfrac{1}{y}$.

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**Round 3: Geometry**

**1.** ____________________

**2.** ____________________

**3.** ____________________

1. A sphere is inscribed in a cube of edge 6. Determine the exact number of cubic units in the volume of the space interior to the cube but exterior to the sphere.

2. The lengths of three segments are $x^2 + 3x$, $x^2 + x$, and $16$. Find all values of $x$ such that when the three segments are joined at their endpoints, a triangle is formed.

3. The lengths of the sides of a triangle are 3, 9, and 10. By what non-zero factor should the lengths be multiplied so that the numerical value of the perimeter of the triangle equals the numerical value of its area?

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**PLAYOFFS – 2014**

**Round 4: Algebra 2**

**1.** ______________________

**2.** ______________________

**3.** ______________________

1. Let $f^2(x) = f(f(x))$, $f^3(x) = f\left(f(f(x))\right)$, etc.

   For $f(x) = \frac{x-1}{x+1}$, find the value of $f^{2014}(-2)$.

2. If $4x^2 - kx + 15$ is factorable over the integers, compute the number of possible integer values of $k$.

3. Find the least positive integer value of $n$ such that

   $\log_{10}\frac{1}{2} + \log_{10}\frac{1}{3} + \log_{10}\frac{1}{4} + \ldots + \log_{10}\frac{1}{n} < -4$

## NEW ENGLAND ASSOCIATION OF MATHEMATICS LEAGUES

## PLAYOFFS – 2014

**Round 5: Analytic Geometry**

**1.** ____________________

**2.** ____________________

**3.** ____________________

1. A line with negative slope passes through (4, 5). If its $x$-intercept is four times its $y$-intercept, and if point ($x$, 18) lies on the line compute the value of $x$.

2. A diameter of a circle is the transverse axis of a hyperbola whose equation is

$$4(x-5)^2 - 9(y-3)^2 = 36$$

Also, the circle is tangent to the hyperbola at both of its vertices. Compute the length of the vertical chord of the circle which passes through $(4, 0)$.

3. Compute the <u>largest</u> possible value of the $y$-intercept of the tangents to $(x-5)^2 + (y-5)^2 = 25$ that have a slope of $-\dfrac{4}{3}$.

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PLAYOFFS – 2014

**Round 6: Trig and Complex Numbers**

1. ____________________

2. ____________________

3. ____________________

1. Let $\alpha$ and $\beta$ be angles of a triangle. For positive $a$ and $b$, if $\tan\alpha = \frac{a}{b}$ and $\tan\beta = \frac{b+a}{b-a}$, compute $\tan(\alpha - \beta)$.

2. In $\Delta ABC$, $A = (0,0)$, $B = (\sin a, \cos a)$, and $C = (\cos b, \sin b)$ where $a$ and $b$ are in radians with $0 < a < b < \frac{\pi}{4}$. If the area of $\Delta ABC$ is 1/10, compute the tangent of $\angle BAC$.

3. Let $a$ and $b$ be unequal positive integers, $0 < b < a < \frac{\pi}{4}$, and let $M(a, 83°)$ and $N(b, 23°)$ be points in the polar coordinate plane. Compute the least possible sum of $a$ and $b$ in which the distance $MN$ is an integer.

**MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES**

**NEW ENGLAND PLAYOFFS – 2014**

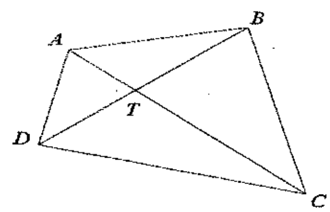
**Team Round - Place all answers on the team round answer sheet.**

1. If $m$ and $n$ are relatively prime positive integers that sum to 143, compute the number of ordered pairs $(m, n)$.

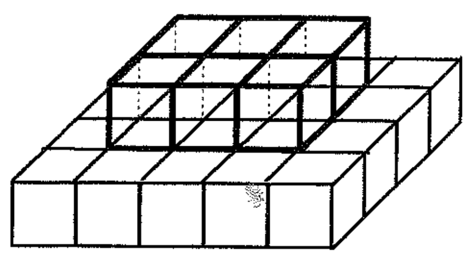
2. Let $\lfloor x \rfloor =$ the greatest integer less than or equal to $x$. For $2 \le a \le 4$, compute the least possible value of $\lfloor \log_a(8a) + \log_{8a}(64a) \rfloor$.

3. Let $S$ be the set of positive 4-digit numbers none of whose digits are 0. A number is chosen at random from $S$. Compute the probability that at least two adjacent digits are the same.

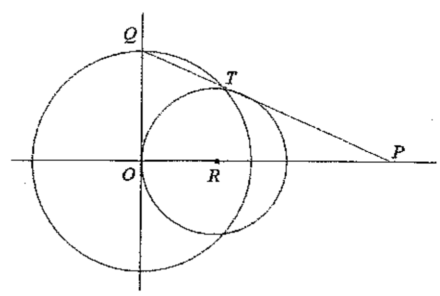
4. The area of quadrilateral $ABCD$ is 120. The diagonals of $ABCD$ meet at $T$. If the area of $ATB$ is 20 and the area of $ATD$ is 28, compute the number of square units in the area of $DTC$.




5. Using sticky toothpicks, a student builds a structure consisting of a base of 20 cubes in a 5 by 4 arrangement and a second layer of 6 cubes in a 2 by 3 arrangement. The top layer has edges in common with the bottom as indicated in the diagram. Whenever edges are in common, only 1 toothpick was used. Compute the number of toothpicks that were used. Note: The diagram doesn't show all the toothpicks.



6. Circle $R$ with center $\left(\frac{25}{2}, 0\right)$ is tangent to the $y$-axis. $Q$ is the positive $y$-intercept of circle $O$ whose center is the origin. $T$ is the intersection of circles $O$ and $R$ while $P$ is the $x$-intercept of $\overleftrightarrow{QT}$. If $7 \le OQ \le 15$, let the smallest possible value of $OP$ be $a$ and the largest be $b$. Compute the ordered pair $(a, b)$.



# NEW ENGLAND ASSOCIATION OF MATHEMATICS LEAGUES

## PLAYOFFS – 2014

### *Answer Sheet*

**Round 1**

1. 2
2. 9
3. 392

**Round 2**

1. 4,300,000
2. 32
3. $k \leq 0$ or $k \geq 4$

**Round 3**

1. $216 - 36\pi$ or $36(6 - \pi)$
2. $(-8, -4) \cup (2,8)$ or
   $-8 < x < -4$ or $2 < x < 8$
3. $\frac{\sqrt{11}}{2}$

**Round 4**

1. $\frac{1}{2}$
2. 12
3. 8

**Round 5**

1. $-48$
2. $4\sqrt{2}$
3. 20

**Round 6**

1. $-1$
2. $\frac{\sqrt{6}}{12}$
3. 11

**Team**

1. 120
2. 4
3. $\frac{217}{729}$
4. 42
5. 157
6. (45,49)

**MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES**

**NEW ENGLAND PLAYOFFS – 2014 - SOLUTIONS**

## Round 1 Arithmetic and Number Theory

1. $$1+\cfrac{1}{.\overline{1}+\cfrac{.\overline{8}}{.\overline{2}+\cfrac{.\overline{7}}{.\overline{3}+\cfrac{.\overline{6}}{.\overline{4}+.\overline{5}}}}} = 1+\cfrac{1}{.\overline{1}+\cfrac{.\overline{8}}{.\overline{2}+\cfrac{.\overline{7}}{.\overline{3}+.\overline{6}}}} = 1+\cfrac{1}{.\overline{1}+\cfrac{.\overline{8}}{.\overline{2}+.\overline{7}}} = 1+\cfrac{1}{.\overline{1}+.\overline{8}} = \boxed{2}.$$

2.

| Dimes | Nickels | Pennies | |
|---|---|---|---|
| 2 | 0 | 3 | There are $\boxed{9}$ ways to make \$0.23 in change |
| 1 | 2 | 3 | |
| 0 | 4 | 3 | |
| 1 | 1 | 8 | |
| 1 | 0 | 13 | |
| 0 | 3 | 8 | |
| 0 | 2 | 13 | |
| 0 | 1 | 18 | |
| 0 | 0 | 23 | |

3. 1111 to 2110 has $1000_6$ numbers for a total of 216. 2111 to 2510 has $400_6$ for a total of 144. 2511 to 2562 has a total of $52_6 = 32$ numbers. The total is 216 + 144 + 32 = $\boxed{392}$.

## Round 2 Algebra 1

1. Express $\dfrac{x^2+5x+2}{x^2-9}$ as $\dfrac{x^2+5x+2}{(x-3)(x+3)} \approx \dfrac{9+15+2}{(.000001)6} = \dfrac{26}{(.000001)6} = 1{,}000{,}000 \cdot 4\frac{1}{3} =$ $4{,}333{,}333$ so the answer would be $\boxed{4{,}300{,}000}$.

2. Rewrite the second equation as $x-y=4$ and add to the first equation, obtaining $x^2+x-12=0$. From $(x+4)(x-3)=0$ we obtain $x=-4$ or $3$, giving $(x,y)=(-4,-8),(3,-1)$. The largest possible product is $\boxed{32}$.

3. $\dfrac{k}{x-y} = \dfrac{1}{x} - \dfrac{1}{y} = \dfrac{y-x}{xy} \rightarrow kxy = -(x-y)^2 \rightarrow x^2 + (k-2)xy + y^2 = 0$. Treat this equation as a quadratic in $x$. If the discriminant is positive then there are solutions for $x$ and $y$. Then $x = \dfrac{(2-k)y \pm \sqrt{(k-2)^2 y^2 - 4y^2}}{2} = \dfrac{(2-k)y \pm |y|\sqrt{(k-2)^2 - 4}}{2}$. There will be real values for $x$ and therefore for $y$ if $(k-2)^2 - 4 \geq 0$. From $k^2 - 4k \geq 0$ we obtain $\boxed{k \leq 0 \text{ or } k \geq 4}$.

**Round 3 – Geometry**

1. $6^3 - \frac{4}{3}\pi \cdot 3^3 = 216 - 36\pi$

2. Since the sides must be positive we have $x^2 + 3x > 0 \rightarrow x < -3$ or $x > 0$. Similarly, $x^2 + x > 0 \rightarrow x < -1$ or $x > 0$. From the triangle inequality we obtain

(1) $\left(x^2 + 3x\right) + \left(x^2 + x\right) > 16 \rightarrow x^2 + 2x - 8 > 0 \rightarrow x < -4$ or $x > 2$.

(2) $\left(x^2 + x\right) + 16 > x^2 + 3x \rightarrow x < 8$.

(3) $\left(x^2 + 3x\right) + 16 > x^2 + x \rightarrow x > -8$.

The intersection of the four solution sets is $\boxed{-8 < x < -4 \text{ or } 2 < x < 8}$. This could be written as $\boxed{(-8,-4) \cup (2,8)}$.

3. Let $k$ be the scaling factor giving a triangle with sides $3k$, $9k$, and $10k$. The perimeter is $22k$. By Heron's formula the area is $\sqrt{(11k)(11k-3k)(11k-9k)(11k-10k)} = 4k^2\sqrt{11}$. From $4k^2\sqrt{11} = 22k$ we obtain $\boxed{k = \dfrac{\sqrt{11}}{2}}$.

**Round 4 – Algebra 2**

1. $f(-2) = \frac{-3}{-1} = 3$, $(3) = \frac{2}{4} = \frac{1}{2}$. $f\left(\frac{1}{2}\right) = \dfrac{\frac{1}{2}-1}{\frac{1}{2}+1} = -\frac{1}{3}$, $f\left(-\frac{1}{3}\right) = \dfrac{-\frac{1}{3}-1}{-\frac{1}{3}+1} = \dfrac{-\frac{4}{3}}{\frac{2}{3}} = -2$
$\therefore f^4(-2) = -2$. For $n$ divisible by 4, $f^n(-2) = -2$. $\therefore f^{2012}(-2) = -2$. $f^{2014}(-2) = f^2(-2) = f(f(-2)) = \boxed{\frac{1}{2}}$

2. Solution 1: The number of possible pairs of factors of 60 and $-60$ will give the result. Hence 60,1; 30,2; 20,3; 4,15; 5,12; 6,10. Hence $\boxed{12}$ possible values of $k$.

Solution 2: Try all the factor combinations for $ab = 4$ and $cd = 15$ in $(ax - c)(bx - d)$.

Solution 3:
Suppose for integers $a$, $b$, $c$, and $d$, $4x^2 - kx + 15 = (ax - b)(cx - d) = acx^2 - (ad + bc)x + bd$. We must consider all possible quadruples $(a,b,c,d)$ for which $(ac)(bd) = 4(15) = 60$. There are exactly three possible ordered pairs $(a,c)$, namely (1, 4), (2, 2) and (4, 1). There are only two possible ordered pairs $(b,d)$, namely (1, 15) and (3, 5) that need be considered. Thus, there are $3 \cdot 2 = 6$ ordered quadruples of positive integers and an equal number consisting of negative integers, for a total of $\boxed{12}$. Since we have included reversals of the ordered pairs $(a,c)$, considering reversals of the ordered pairs $(b,d)$ would result in duplicates. Thus, there are <u>at most</u> 12 different $k$-values $(ac + bd)$ .If we show that no two distinct ordered quadruples $(a,b,c,d)$ produce the same $k$-value, then we have exactly 12 different $k$-values. For $abcd = 60$, a list of the possibilities suffices.

3. $\log_{10}\frac{1}{2} + \log_{10}\frac{1}{3} + \ldots + \log_{10}\frac{1}{n} = \log_{10}1 - \log_{10}2 + \log_{10}1 - \log_{10}3 + \ldots + \log_{10}1 - \log_{10}n$

Since $\log_{10}1 = 0$, we have $-(\log_{10}2 + \log_{10}3 + \ldots + \log_{10}n) = -\log_{10}(n!) = \log_{10}\frac{1}{n!}$. Since $\log_{10}10^{-4} = -4$, we want $\frac{1}{n!} < \frac{1}{10,000}$. Since 7! = 5040 and 8! = 40,320, then $\boxed{n = 8}$.

## Round 5 – Analytic Geometry

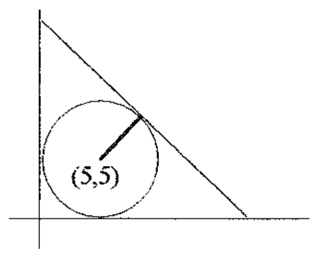
1. Let the $y$-intercept equal $b$. Using the intercept form for the equation of a line we have $\frac{x}{4b} + \frac{y}{b} = 1$. If (4, 5) lies on the line, we have $\frac{4}{4b} + \frac{5}{b} = 1 \rightarrow \frac{1}{b} + \frac{5}{b} = 1$ so $b = 6$. The equation of the line is $\frac{x}{24} + \frac{y}{6} = 1$ and when $y = 18$, $\frac{x}{24} = -2$, so $\boxed{x = -48}$.

Alternate Solution:

Consider the equation: $mx + y = 4m + 5$ as the equation of a line through $P(4, 5)$. The $x$-intercept is $\frac{4m+5}{m}$ and the y – intercept is $4m + 5$. Solve $\frac{4m+5}{m} = 16m + 20$. This gives $16m^2 + 16m - 5 = 0 \rightarrow (4m - 1)(4m + 5) = 0$. We want $m > 0$ for a negative slope, so $m = \frac{1}{4}$. The equation of the line is $x + 4y = 24$, When $y = 18$, $x = -72 + 24 = -48$.

2. The center of the circle is the same as the center of the hyperbola, namely $(5,3)$. For the hyperbola $a = 3$, so that's the radius of the circle. The equation of the circle is $(x-5)^2 + (y-3)^2 = 9$. At $x = 4$, $1 + (y-3)^2 = 9 \rightarrow (y-3)^2 = 8 \rightarrow$ $y - 3 = \pm\sqrt{8} \rightarrow y = 3 \pm \sqrt{8}$. The length of the chord is $(3+\sqrt{8}) - (3-\sqrt{8}) = \boxed{4\sqrt{2}}$

3. Find the intersection of the radius perpendicular to the tangent with the tangent. The slope of the radius is $\frac{3}{4}$; the equation of the radius is $y - 5 = \frac{3}{4}(x-5) \rightarrow y = \frac{3}{4}x + \frac{5}{4}$. Substituting




into $(x-5)^2 + (y-5)^2 = 25$ gives $(x-5)^2 + \left(\frac{3}{4}(x-5)\right)^2 = 25$. This simplifies to $\frac{25}{16}(x-5)^2 = 25 \rightarrow x - 5 = \pm 4$. Here $x$ must equal 9, making $y = \frac{3}{4}\cdot 9 + \frac{5}{4} = 8$. Thus, the tangent line passes through (9, 8) with slope $-\frac{4}{3}$. Its equation is $y - 8 = -\frac{4}{3}(x-9)$, giving $y = -\frac{4}{3}x + 20$. Answer: $\boxed{20}$.

Alternate Solution: Line $L$: $4x + 3y = 35$ has slope $-\frac{4}{3}$ and passes through the center (5, 5) of the circle, We want a line parallel to $L$ and $r = 5$ units distance from $L$. Solve $\frac{|C-35|}{5} = 5$ to get $C = 60$ as the larger solution. The tangent line is then $4x + 3y = 60$ with y-intercept 20.

## Round 6 – Trig and Complex Numbers

1. $\tan(\alpha-\beta)=\dfrac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}=\dfrac{\frac{a}{b}-\frac{b+a}{b-a}}{1+\frac{a}{b}\cdot\frac{b+a}{b-a}}=\dfrac{ab-a^2-b^2-ab}{b^2-ab+ab+a^2}=\dfrac{-\left(a^2+b^2\right)}{a^2+b^2}=\boxed{-1}$.

Solution #2:
Suppose $a=b$. Then $\tan\beta$ is undefined and we have $\beta=90°$, so the triangle must be an isosceles right triangle $\Rightarrow \alpha=45°$. $\tan(\alpha-\beta)=\tan(45°-90°)=-\tan(45°)=\boxed{-1}$

2. The area of $\Delta ABC$ is $\frac{1}{2}\cdot AB\cdot AC\cdot\sin\angle BAC$. $AB=\sqrt{\sin^2 a+\cos^2 a}=1$. Likewise, $AC=1$.

Thus, $\frac{1}{10}=\frac{1}{2}\cdot 1\cdot 1\cdot\sin\angle BAC\rightarrow\sin\angle BAC=\frac{1}{5}$. Then $\cos\angle BAC=\frac{\sqrt{24}}{5}=\frac{2\sqrt{6}}{5}$, making

$\tan\angle BAC=\dfrac{1}{2\sqrt{6}}=\boxed{\dfrac{\sqrt{6}}{12}}$.

3. Using the Law of Cosines to compute the distance gives

$MN=\sqrt{a^2+b^2-2ab\cos(83-23)}=\sqrt{a^2+b^2-2ab\cos 60}=\sqrt{a^2+b^2-}$

We note that integer values for $(a, b, MN)$ include (3, 8, 7), (5, 8, 7), (7, 15, 13), (8, 15, 13) as indicated below for (3, 8, 7):

$\sqrt{8^2+3^2-8\cdot 3}=\sqrt{73-24}=\sqrt{49}=7$. The minimum sum is $8+3=\boxed{11}$.

We need to show that all sums less than 11 fail.

| $a+b$ | $(a,b)$ | $a^2+b^2-ab$ | $a+b$ | $(a,b)$ | $a^2+b^2-ab$ |
|---|---|---|---|---|---|
| 3 | (2,1) | 5-2=3 | 9 | (8,1) | 65-8=57 |
| 4 | (3,1) | 10-3=7 | | (7,2) | 53-14=39 |
| 5 | (4,1) | 17-4=13 | | (6,3) | 45-18=27 |
| | (3,2) | 13-6=7 | | (5,4) | 41-20=21 |
| 6 | (5,1) | 26=5=21 | 10 | (9,1) | 82-9=73 |
| | (4,2) | 20-8=12 | | (8,2) | 68-16=52 |
| 7 | (6,1) | 37-6=31 | | (7,3) | 58-21=37 |
| | (5,2) | 29-10=19 | | (6,4) | 52-24=28 |
| | (4,3) | 25-12=13 | 11 | (10,1) | 101-10=91 |
| 8 | (7,1) | 50-7=43 | | (9,2) | 85-18=67 |
| | (6,2) | 40-12=28 | | (8,3) | 73-24=**49** |
| | (5,3) | 34-15=19 | | | |

**Team Round**

1. Since $143 = 11 \bullet 13$, neither $m$ nor $n$ can have a factor of 11 or 13. The number of positive integers less than 143 without a factor of 11 or 13 is $142 - 12 - 10 = 120$. So there are 120 values for $m$ and for each value of $m$ there is a value of $n$ equal to $143 - m$, making for $\boxed{120}$ ordered pairs $(m, n)$.

2. If $a = 2$ we have $\log_2 16 + \log_{16} 128 = 5.75$. If $a = 4$, we have $\log_4 32 + \log_{32} 256 = \frac{41}{10}$, so 4 seems to be the answer. Can we obtain a result less than 4? Using the AM-GM we have

$$\frac{\frac{\ln(8a)}{\ln a} + \frac{\ln(64a)}{\ln(8a)}}{2} \geq \sqrt{\frac{\ln(8a)}{\ln a} \cdot \frac{\ln(64a)}{\ln(8a)}} = \sqrt{\frac{\ln 64 + \ln a}{\ln a}} = \sqrt{\frac{\ln 64}{\ln a} + 1}.$$

Clearly, $\sqrt{\frac{\ln 64}{\ln a} + 1}$ is least for the largest value of $a$. Letting $a = 4$ gives $\log_a(8a) + \log_{8a}(64a) \geq 2\sqrt{\frac{\ln 4^3}{\ln 4} + 1}$. Since $2\sqrt{\frac{\ln 4^3}{\ln 4} + 1} = 2\sqrt{3+1} = 4$, $\log_a(8a) + \log_{8a}(64a)$ can't drop below 4, so the least possible value of $\lfloor \log_a(8a) + \log_{8a}(64a) \rfloor$ is $\boxed{4}$.

3. Solution #1: Let the number be $ABCD$. Let's count the number of cases where no two adjacent digits are equal. Since none of digits can be 0, there are 9 choices for $A$. $B$ must be different from $A$ so there are 8 choices for $B$. $C$ must be different from $B$ but it could equal $A$ so there are 8 choices for $C$. $D$ must be different from $C$ but it could equal either $A$ or $B$ so there are 8 choices for $D$. Thus, there are $9 \cdot 8 \cdot 8 \cdot 8$ cases where no two adjacent numbers are the same. There are $9^4$ four digit numbers if 0 is not used so the probability that no two adjacent numbers are equal is $\frac{9 \cdot 8^3}{9^4} = \frac{8^3}{9^3}$. Thus, the probability that at least two adjacent numbers are the same is $1 - \frac{8^3}{9^3} = \boxed{\frac{217}{729}}$.

Solution #2: We'll look at all the possible cases where at least 2 adjacent numbers are equal. There are $9^4$ possible numbers. Several cases:

1) $AABC$, $BAAC$, or $BCAA$. There are $3 \cdot 9 \cdot 8 \cdot 7$ ways to do that.

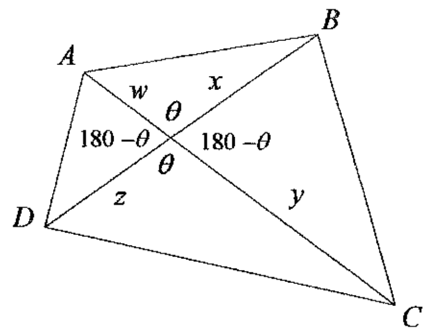
2) $AABB$ or $BBAA$. There are $2 \cdot 9 \cdot 1 \cdot 8 \cdot 1$ such numbers.

3) $AAAB$ or $BAAA$. There are $2\cdot 9\cdot 1\cdot 1\cdot 8$ ways to do that.

4) $AABA$ or $ABAA$ 16 more.

5) $AAAA$ 1 more.

4. Let $T$ be the intersection of diagonals $\overline{AC}$ and $\overline{DB}$. The product of the areas of $\Delta ATB$ and $\Delta DTC$ is $\left(\frac{1}{2}wx\sin\theta\right)\left(\frac{1}{2}yz\sin\theta\right)=\frac{1}{4}xyzw\sin^2\theta$. The product of the areas of $\Delta ATD$ and $\Delta BTC$ is $\left(\frac{1}{2}wz\sin(180-\theta)\right)\left(\frac{1}{2}xy\sin(180-\theta)\right)=\frac{1}{4}xyzw\sin^2\theta$.




Thus, the products of the areas of the opposite triangles are equal: $a(\Delta ATD)\cdot a(\Delta BTC)=a(\Delta ATB)\cdot a(\Delta DTC)$, giving $\frac{a(\Delta ATD)}{a(\Delta ATB)}=\frac{28}{20}=\frac{a(\Delta DTC)}{a(\Delta BTC)}$. Thus, $a(\Delta BTC)=\frac{5}{7}a(\Delta DTC)$. Since $a(\Delta DBC)=120-48=72$, then $a(\Delta DTC)+\frac{5}{7}\left(a(\Delta DTC)\right)=72\to a(DTC)=\boxed{42}$.

Alternate Solution 1:

Since the angles at $T$ aren't given, we can assume any angle. Choose 90°. Then choose lengths for $TA$, $TB$, and $TD$ that wok for the given areas. Let $TA=8, TB=5, TD=7$. Let $TC=x$. Then $\frac{5x}{2}+\frac{7x}{2}=120-48=72, \rightarrow 12x=144 \rightarrow x=12$. Then area $\Delta DTC=\frac{7\cdot 12}{2}=42$.

Alternate Solution 2:

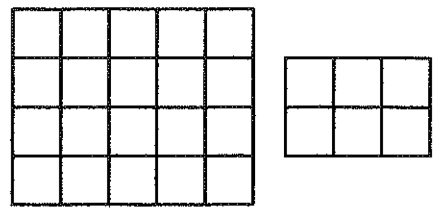
Let the area of $\Delta DTC$ be $X$. Then the area of $\Delta BTC$ is $120-(20+28)-X=72-X$

Since $\Delta DTA$ and $\Delta BTC$ have a common altitude from vertex $A$, their areas are in to ratio of their bases, namely $\frac{DT}{BT}$. The same can be said for $\Delta DTC$ and $\Delta BTC$.

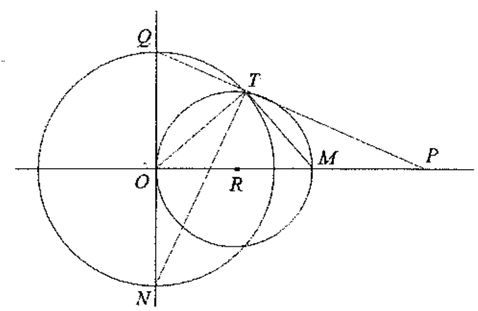
Thus, $\frac{X}{72-X}=\frac{28}{20}=\frac{7}{5}\Rightarrow 12X=7\cdot 72\Rightarrow X=\boxed{42}$

5. Counting the toothpicks on the very bottom gives $5 \cdot 5 + 6 \cdot 4 = 49$ as we count first from front to back and then from side to side. There will be the same number on the top of the first layer. There will be $6 \cdot 5 = 30$ vertical toothpicks in the first row. For the second row, we don't count the toothpicks on the bottom since they have already been counted. We have $3 \cdot 4 = 12$ vertical toothpicks and $3 \cdot 3 + 4 \cdot 2 = 17$ on the top. The total is $49 + 30 + 49 + 12 + 17 = \boxed{157}$.

Alternate Solution:



Consider the templates for the bottoms of each layer. For the larger template, we would need $5 \cdot 5 + 6 \cdot 4 = 49$ toothpicks. For the smaller templates, we would need $3 \cdot 3 + 2 \cdot 4 = 17$ toothpicks.
We need two of the larger templates (for the top and bottom surfaces of the larger layer of cubes), but only one of the smaller templates.
Erecting a toothpick at each lattice point completes the structure, requiring an additional $6 \cdot 5 + 4 \cdot 3 = 42$ toothpicks. Thus, the total need is $2(49) + 17 + 42 = \boxed{157}$



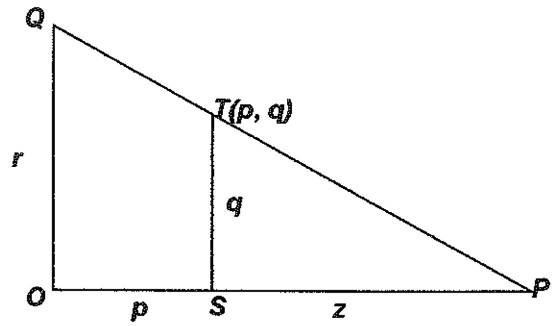

6. Draw lines $\overline{TO}$, $\overline{TN}$, and $\overline{TM}$ forming right triangles $QTN$ and $OTM$. Since $QTN$ and $QOP$ are right triangles sharing acute angle $\angle Q$, then $m\angle QNT = m\angle QPO$. Let $m\angle QNT = \alpha$. Since $ON = OT$, then $m\angle OTN = \alpha$. Since $m\angle QTN = m\angle OTM = 90$, we have $m\angle QTO = m\angle NTM = 90 - \alpha$. This means that $m\angle MTP = \alpha$, making $\Delta TMP$ isosceles with $TM = MP$. Let $OT = x$ and $TM = MP = y$. Since $OM = 25$ we have $x^2 + y^2 = 25^2 \to y = \sqrt{625 - x^2}$. For $x = 15$, $y = 20$, making $OP = 25 + 20 = 45$. For $x = 7$, $y = 24$, making $OP = 25 + 24 = 49$. Thus, $45 \le OP \le 49$, making the answer the ordered pair $\boxed{(45, 49)}$.

Alternate Solution:

Let $OQ = r$. Then the equations of the two circles are: $\begin{cases} x^2 + y^2 = r^2 \\ \left(x - \frac{25}{2}\right)^2 + y^2 = \left(\frac{25}{2}\right)^2 \end{cases}$

Subtracting, we have $-25x = -r^2 \Rightarrow x = \left(\frac{r}{5}\right)^2 \Rightarrow y^2 = r^2 - \left(\frac{r}{5}\right)^4 = \frac{r^2\left(625 - r^2\right)}{625} \Rightarrow y = \pm\frac{r}{25}\sqrt{625 - r^2}$

Thus, the coordinates of point $T$ are $\left(\left(\frac{r}{5}\right)^2, +\frac{r}{25}\sqrt{625 - r^2}\right)$




$$\frac{z}{z+p} = \frac{q}{r} \Rightarrow z = \frac{pq}{r-q} \Rightarrow OP = p + \frac{pq}{r-q} = \frac{pr}{r-q}$$

$r = 15 \Rightarrow T(p,q) = (9,12)$ and we have $OP = \frac{9 \cdot 15}{15 - 12} = 45$

$r = 7 \Rightarrow (p,q) = \left(\frac{49}{25}, \frac{168}{25}\right)$ and we have $OP = \frac{\frac{49}{25} \cdot 7}{7 - \frac{168}{25}} = \frac{49 \cdot 7}{7 \cdot 25 - 168} = \frac{49}{25 - 24} = 49$

Thus, $(a,b) = \boxed{(45,49)}$ .