JANUARY
GBML

Round 1
Volume and Surface Area of Solids

GREATER BOSTON MATHEMATICS LEAGUE

MEET 4 – JANUARY 1999

ROUND 1 – Volume and Surface Area of Solids

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. The total surface area of a cube is 72 square inches. Find the number of inches in the length of one diagonal of the cube. Write the result in simplest radical form.

2. A regular triangular prism has a volume of $12\sqrt{3}$ cm^3. If the height of the prism equals the perimeter of the base, find the number of centimeters in the height of the prism. Write the result in simplest radical form.

3. A sphere with a radius of length six is inscribed in a right circular cone with a height of length fifteen. Find the volume interior to the cone and exterior to the sphere.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 4 – JANUARY 2000

ROUND 1 – Volume and Surface Area of Solids

1. ____________________

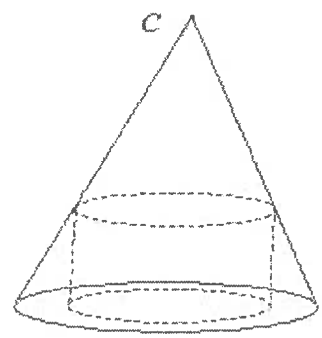
2. ____________________

3. ____________________

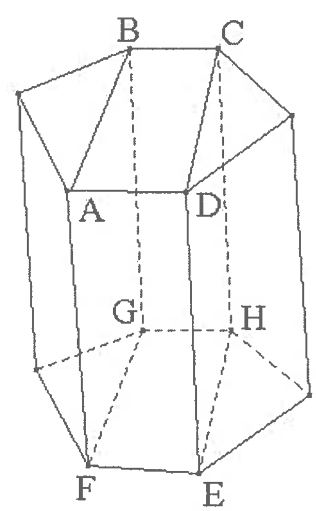
CALCULATORS ARE NOT ALLOWED ON THIS ROUND
DIAGRAMS ARE NOT NECESSARILY DRAWN TO SCALE

1. A rectangular parallelepiped with dimensions 7cm. by 5cm. by 1 cm. has a diagonal the same length as the diagonal of a cube. Find the volume of the cube in cubic centimeters.

2. A right circular cylinder with a radius of 4 inches and a height of 3 inches is inscribed in right circular cone C with a radius of 6 inches. (See the figure.) Find the volume in cubic inches of cone C.



3. A regular hexagonal right prism has a height of $6\sqrt{3}$ cm. (See the figure.) If the volume of <u>rectangular prism</u> ABCDEFGH is 162 cm^3, find the number of square centimeters in the total surface area of the hexagonal prism.



GREATER BOSTON MATHEMATICS LEAGUE

MEET 4 – JANUARY 2001

ROUND 1 – Volume and Surface Area of Solids

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND
DIAGRAMS ARE NOT NECESSARILY DRAWN TO SCALE

1. A face diagonal of cube C is $2\sqrt{3}$ inches long. Find the number of cubic inches in the volume of cube D whose side has the same length as a main diagonal (not the face diagonal) of cube C.

2. A spherical orange is sliced into four congruent pieces. If the total surface area (plane and curved) of one piece of the orange is 32π cm^2, find the number of cubic centimeters in the volume of this one piece.

3. A regular pyramid with a square base has each of its lateral faces making a $60°$ angle with the plane of the square. If the total surface area of this pyramid is 36 m^2, find the number of cubic meters in its volume.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 4 – JANUARY 2002

ROUND 1 – Volume and Surface Area of Solids

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. The volume of a cube is 64 cubic inches. Let the length of a diagonal of this cube divided by the length of its face diagonal equal P. Find the number of square inches in the total surface area of a cube with side of length P inches.

2. Find the number of square centimeters in the lateral area of a regular hexagonal pyramid with a base perimeter of 36 cm and a height of 3 cm.

3. A right circular cylinder has a radius of 6 in. and a height of 30 in. Its total surface area is equal to the total surface area of a hemisphere. If this hemisphere is filled to half of its capacity with water and this water is then poured into the empty cylinder, what would be the number of inches in the depth of the water in the cylinder?

GREATER BOSTON MATHEMATICS LEAGUE
MEET 4 – JANUARY 2006

ROUND 1 – Volumes and Surface Areas of Solids

1. ________________ **cm^3**

2. ________________ **cm^3**

3. ________________ **cm^2**

CALLATORS ARE NOT ALLOWED ON THIS ROUND.

Express all answers requested in terms of the <u>exact</u> number of units.

1. The vertex of a cone coincides with the center of a face of a cube 6 cm on a side. The base of the cone is inscribed in the opposite face. Find the volume of the region inside the cube and outside the cone.

2. Eight spheres are placed in two layers of four in a cube of side 10 cm. Each sphere is externally tangent to three adjacent spheres and is tangent to three sides of the cube. Find the volume of the region external to the spheres and internal to the cube?

3. A right prism with a regular hexagonal base has a height equal to that of a face diagonal of a cube whose volume is 64 cubic centimeters. A longer diagonal of the base of the prism is equal to the length of a diagonal of that same cube. What is the total surface area of this prism?

GREATER BOSTON MATHEMATICS LEAGUE
MEET 4 – JANUARY 2007

ROUND 1 – Volumes and Surface Areas of Solids

1. ________________**cm**

2. __________ : __________

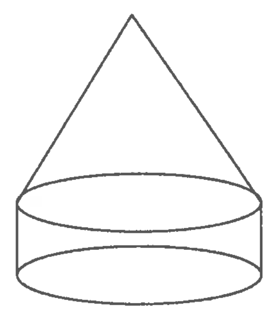
3. ________________**cm^2**

CALCULATORS ARE NOT ALLOWED ON THIS ROUND.

Express all answers requested in terms of the <u>exact</u> number of units.

1. Two spheres are concentric. A plane passing through the larger sphere is tangent to the smaller sphere. The cross section has a circumference of 24π cm^2. What is the radius of the larger sphere if the radius of the smaller sphere is 9 cm?

2. A right circular cone is placed on top of a right circular cylinder such that the base of the cone is a base of the cylinder. as illustrated in the diagram. The total height of the described solid is 20 units. The ratio of the volume of the cone to the volume of the cylinder is 9 : 5, the ratio of the altitude of the cone to the altitude of the cylinder is $A : B$. Express $A : B$ as a simplified ratio of integers.



3. A right pyramid with a square base is carved out of a rectangular solid made of wood. The base of the pyramid coincides with one base of the rectangular solid and the vertex of the pyramid coincides with the center of the other base. If the diagonal of the base is equal to $10\sqrt{2}$ cm and the height of the solid is 12 cm, find the total surface area of the resulting object.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 4 – JANUARY 2008

ROUND 1 – Volume and Surface Area of Solids

If you would like to receive email announcements regarding upcoming competitions, please print your email on the reverse side of this paper when you have finished answering the problems.

1. ______________________

2. ______________________

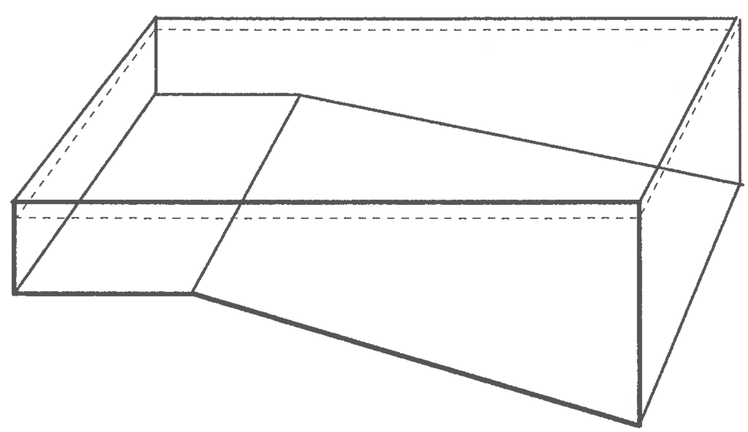
3. ______________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND.

1. A cube is inscribed in a sphere. Find the ratio of the surface area of the cube to the surface area of the sphere.

2. A cylinder has a height of 8 inches and a radius of 6 inches. Two cones with heights in a ratio of 3 : 1 are placed within the cylinder such that their bases coincide with the bases of the cylinder and their vertices also coincide. Find the exact volume of the region interior to the cylinder and exterior to the cones.

3. A swimming pool is 20 yards wide by 36 yards long. The shallow end is 4 feet deep for a length of 8 yards. Then it drops off linearly to a depth of 8 feet at the deep end. The pool is filled with water to within 1 foot of the top. <u>To the nearest integer</u>, what percent of the pool is filled with water?



GREATER BOSTON MATHEMATICS LEAGUE

MEET 4 – JANUARY 2009

ROUND 1 – Volume and Surface Area of Solids

1. ______________________

2. ______________________

3. ______________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND.

1. The length of the shorter diagonal of a regular hexagon (*H*) is $2\sqrt{6}$ cm. A cube (*C*) has a side whose length is equal to the length of a longer diagonal of *H*. The volume of cube *C* is k cm^3. Find the value of k.

2. A right circular cone and a right circular cylinder have the same height, namely $\frac{9}{7}$ inches, but the cone has a radius equal to twice the radius of the cylinder. The sum of their volumes equals the volume of a sphere whose radius is 3 inches. Find the radius of the cone.

3. A triangular right prism has a base that is a right triangle with a perimeter of 40 units. A circle inscribed in a base of the prism has a circumference of 6π units. The ratio of the height of the prism to the length of the shorter leg of the base is 5 : 12. Find the numerical value of the volume of the prism.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 4 – JANUARY 2010

ROUND 1 – Volume and Surface Area of Solids

1. _____________ : _____________

2. _____________ : _____________

3. _________________________ feet

CALCULATORS ARE NOT ALLOWED ON THIS ROUND.

1. The ratio of the lengths of the distinct edges in a rectangular solid are in a ratio of 1 : 3 : 4. The area of the larger faces is 75 square inches. What is the numerical ratio of the volume of this solid to its total surface area?

2. A regular cone and a regular cylinder have equal bases. The ratio of the lateral surface area of the cone to the lateral surface area of the cylinder is 5 : 22. What is the ratio of the slant height of the cone to the height of the cylinder?

3. A cylindrical drum whose diameter is 3 feet and whose height is 8 feet contains water to a height of $2\frac{1}{2}$ feet. A sphere of ice whose diameter is 2 feet is placed in the water. If the ice floats so that the ratio of the volume of the ice above the surface of the water to the volume of the ice below the surface is 7 : 9, what would now be the height of the water in the cylinder?

GREATER BOSTON MATHEMATICS LEAGUE
MEET 4 – JANUARY 2011

ROUND 1 – Volume and Surface Area of Solids

1. $k =$ ____________________

2. ____________________

3. (________ , ________ , ________)

CALCULATORS ARE NOT ALLOWED ON THIS ROUND.

1. The center of a cross section of a sphere is 20 units from the center of the sphere. If the cross section were 4 units further from the center of the sphere, its area would decrease by $k\pi$ square units. Compute k.

2. The total surface area of a cube is numerically equal to the volume of a pyramid with a square base. The length of a face diagonal of the cube equals the length of an edge of the base of the pyramid. Compute the height of the pyramid.

3. The radius of the base of a right circular cone is r. Its lateral surface area is twice the area of its base. A cylinder has a base whose diameter equals the radius of the cone and has a volume equal to that of the cone. As a simplified expression in terms of r, the lateral surface area of the cylinder may be written in the form $k\pi r^2$,
where $k = \frac{A}{B}\sqrt{C}$, for integers A, B and C. Determine the ordered triple (A, B, C).

MASSACHUSETTS MATHEMATICS LEAGUE
OCTOBER 2003
ROUND 1: VOLUMES & SURFACES

ANSWERS

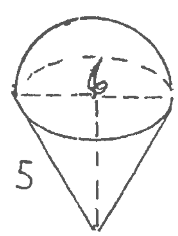
A)______________

B)______________

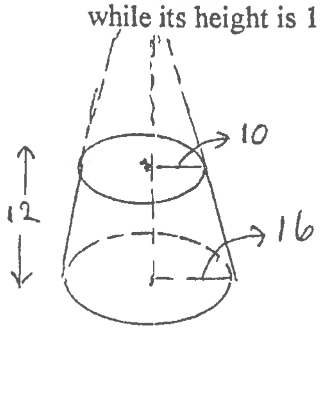
C)______________

A) A right circular cylinder of height 9 cm and diameter 8 cm has a hole of diameter 4 cm drilled out of its center* Find the total surface area of solid remaining leaving the result in terms of pi. * =(along the axis of symmetry)

B) A right circular cone, apex down as shown, has a slant height of 5 cm and a base diameter of 6 cm. A hemisphere is sitting on top of the cone. Find the volume in terms of pi of the solid formed by the cone and the hemisphere.



C) The truncated cone shown was formed by cutting off the top of a right circular cone with a plane parallel to its base. The radii of the bases of the truncated cone are 10 cm and 16 cm while its height is 12 cm. Calculate the volume of truncated cone in terms of pi.



MASSACHUSETTS MATHEMATICS LEAGUE
OCTOBER 2005
ROUND 1 VOLUME & SURFACES

ANSWERS

A)________cubic in.

B)____________cm

C)____________

A) **The base of a right pyramid is a square with perimeter 10"; the pyramid's altitude is 9". Find the exact volume of the pyramid.**

B) **ABCD is a square of side 8 cm in a horizontal plane. $\overline{DK}$ is a 7 cm. line segment perpendicular to the plane. Q is the midpoint of $\overline{AB}$. Find the exact length of the longest edge of pyramid KDAQ.**

C) **A right triangle with legs of 30 and 40 is rotated around each of its three sides. Rotation about the shorter leg produces cone #1; rotation about the longer leg produces cone #2; rotation about the hypotenuse produces two cones sharing a common base. Find the sum of the volumes of the four cones. Leave answer in terms of π.**

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - OCTOBER 2006
ROUND 1 VOLUME & SURFACES

ANSWERS

A) ______________________________

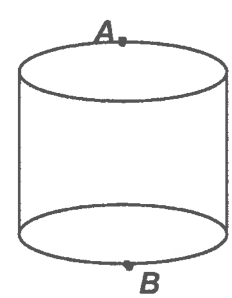
B) ____________________________cm^3

C) ______________________________

A) The volume of a right circular cylinder is 48π and the circumference of the base is 8π. Find the total surface area in terms of π.

B) Six slices, each with a uniform thickness of half a centimeter, are removed from a wooden cube, one slice per face, reducing the volume of the original cube by 169 cm^3. What is the volume of the resulting smaller cube in cm^3?

C) Points A and B are on diametrically opposite "sides" of the cylinder. Find the exact shortest possible distance from A to B along the surface of the cylinder. The diameter and height of the cylinder are 4 units and 1 unit respectively.



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 – OCTOBER 2007
ROUND 1 VOLUME & SURFACES

ANSWERS

A) ______________________

B) ______________________

C) ______________________

A) The radius of a right circular cylinder is 10 inches and its height is 4 inches. Determine the value of x (> 0) which, when added to either the radius or the height, produces two different right circular cylinders of equal volume.

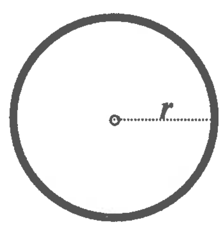
B) **Definition**: A <u>locus of points</u> refers to a set of all points (and only those points) that satisfy a given condition.

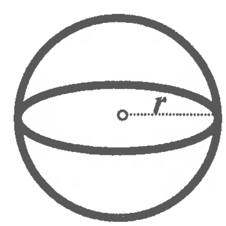
Example: The locus of points equidistant from a given point

on a line: 2 points

in a plane: a circle

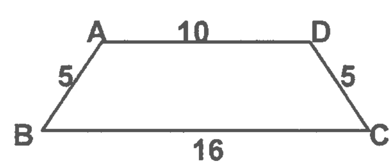
in space: a sphere





<u>Compute</u> the total area of the region(s) bounded by the locus of points <u>in space</u> 4 cm from a given plane <u>and</u> 6 cm from a fixed point in that plane.

C) The following diagram is the cross-section of the frustum of a right circular cone. <u>Compute</u> the volume of the frustum.



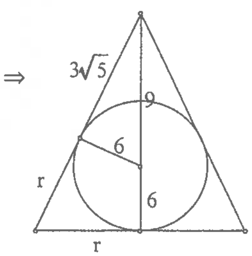
ROUND 1

'99 GBML

1. If x is the length of an edge of the cube, the total surface area = $6x^2 = 72 \Rightarrow$ $x = 2\sqrt{3}$. The length of the diagonal = $x\sqrt{3} = 2\sqrt{3}\cdot\sqrt{3} = 6$ *in.*

2. The base of the prism is an equilateral triangle. Call the length of one side x. $\Rightarrow$ $A = \frac{x^2\sqrt{3}}{4}$ and $h = 3x \Rightarrow V = \frac{x^2\sqrt{3}}{4}\cdot 3x = 12\sqrt{3} \Rightarrow x^3 = 16 \Rightarrow x = 2\sqrt[3]{2} \Rightarrow h = \mathbf{6\sqrt[3]{2}}$

3. $r^2 + 15^2 = \left(r + 3\sqrt{5}\right)^2 \Rightarrow r^2 + 225 = r^2 + 6\sqrt{5}\,r + 45 \Rightarrow$ $r = 6\sqrt{5}$; $V = \frac{1}{3}\left(6\sqrt{5}\right)^2 15\,\pi - \frac{4}{3}\cdot 6^3\,\pi = \mathbf{612\pi}$



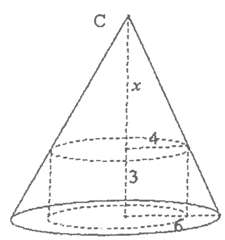
Round 1

'00 GBML

1. diagonal of rectangular parallelepiped = $\sqrt{7^2 + 5^2 + 1^2} = \sqrt{75} = 5\sqrt{3} \Rightarrow$ edge of the cube = 5 cm. $\Rightarrow$ volume of the cube = 125cm.^{3}

2. $\frac{x}{4} = \frac{x+3}{6} \Rightarrow x = 6 \Rightarrow V = \frac{1}{3}\pi\cdot 6^2\cdot 9 = 108\pi\ in.^3$

3. AB:AD = $\sqrt{3}$:1 $\Rightarrow$ If AD = x, volume of ABCDEFGH = $(x)(x\sqrt{3})(6\sqrt{3}) = 162 \Rightarrow x^2 = 9 \Rightarrow x = 3$ cm. $\Rightarrow$
total surface area of the hexagonal prism =
2×area of hexagon + perimeter of hexagon ×height of prism
$= 2\cdot\frac{6\cdot 3^2\sqrt{3}}{4} + 18\cdot 6\sqrt{3} = 27\sqrt{3} + 108\sqrt{3} = 135\sqrt{3}$ cm.^{2}



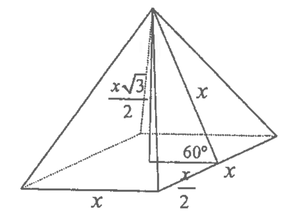
Round 1

'01 GBML

1. The length of the side of cube $C = \frac{2\sqrt{3}}{\sqrt{2}} = \sqrt{6}$ in. The diagonal of cube $C = \sqrt{6}\sqrt{3} = 3\sqrt{2}$ in. The volume of the second cube = $\left(3\sqrt{2}\right)^3 = 27\left(2\sqrt{2}\right) = 54\sqrt{2}$ cubic inches.

2. The total surface of the orange slice is made up of 2 semicircles and 1/4 the surface area of the sphere = $\pi r^2 + \frac{1}{4}\left(4\pi r^2\right) = 2\pi r^2 = 32\pi \rightarrow r = 4$ and the volume of the slice = $\frac{1}{4}\left(\frac{4}{3}\pi\cdot 4^3\right) = \frac{64\pi}{3}$

3. If x is one side of the square $\rightarrow x$ = slant height and $\frac{x\sqrt{3}}{2}$ is the height of the pyramid (see diagram.)
Total area = $x^2 + \frac{1}{2}(4x)x = 3x^2 = 36 \rightarrow x = 2\sqrt{3}$;
Volume = $\frac{1}{3}(12)(3) = 12cm^3$

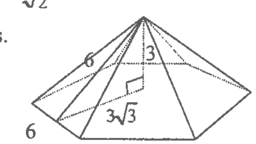


ROUND 1 – Volume and Surface Area of Solids

'02 GBML

1. The volume of the original cube is irrelevant. If s = length of its side, then $s\sqrt{3}$ = its diagonal's length and $s\sqrt{2}$ = length of the diagonal of its face $\Rightarrow \frac{\sqrt{3}}{\sqrt{2}} = P$. The surface area of the 2nd cube = $6P^2 = 6\left(\sqrt{3}/\sqrt{2}\right)^2 = 6\cdot 3/2 = 9$ square inches.

2. The side of the regular hexagon = 6 cm $\Rightarrow$ apothem of the regular hexagon = $3\sqrt{3}$ cm $\Rightarrow$ slant height = 6 cm.
The lateral area = $\frac{1}{2}P\cdot l = \frac{1}{2}\cdot 36\cdot 6 = 108$ cm^2.



3. The total surface area of the cylinder = $2\pi r(r+h) = 12\pi\cdot 36$. If R = radius of the hemisphere $\Rightarrow 3\pi R^2 = 12\pi\cdot 36 \Rightarrow R^2 = 144 \Rightarrow R = 12$. Half the volume of the hemisphere $= \frac{1}{2}\cdot\frac{2}{3}\pi\cdot 12^3 = 12\cdot 12\cdot 4\pi$ in.^{3}. To find the height of the water in the cylinder divide this by 36π in.$^2\Rightarrow$ height of the water $= \frac{12\cdot 12\cdot 4\pi}{36\pi} = 16$in.

'06
GBML

ROUND 1 - Volumes and Surface Areas of Solids

1. The height of the cone is 6 cm and the radius of the base is 3 cm.
   The volume of the region is V(cube) – V(cone) = $6^3 - \frac{1}{3}\pi(3)^2 6 = \underline{\mathbf{216 - 18\pi}}$

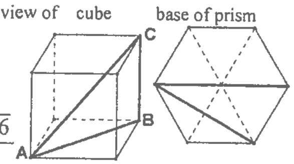
2. The radius of each sphere is 2.5 cm.
   The volume of the region is V(cube) – 8V(sphere) = $10^3 - 8\frac{4}{3}\pi(2.5)^3 =$

   $1000 - 8\frac{4}{3}\pi(\frac{5}{2})^3 = 1000 - \frac{\cancel{8}(4)\pi(125)}{(3)\cancel{8}} = \underline{\frac{3000-500\pi}{3}}$

3. Given: V(cube) = 64
   Since a cube of side s has face diagonals of length $s\sqrt{2}$ and (space) diagonals of length $s\sqrt{3}$, the height of the prism is $4\sqrt{2}$ and a long diagonal of length $4\sqrt{3}$ → length of the side of hexagon is $2\sqrt{3}$.
   Area(regular hexagon) =

   $6\cdot\text{Area(Eq. }\Delta) = 6\cdot\frac{s^2\sqrt{3}}{4} \rightarrow \frac{\cancel{6}^{3}(\cancel{12})\sqrt{3}}{\cancel{4}} = 18\sqrt{3}$

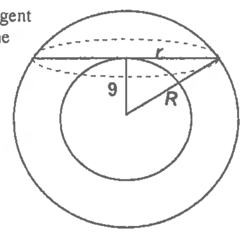
   SA(prism) = 2B+ph = $36\sqrt{3} + 6(2\sqrt{3})(4\sqrt{2}) = \underline{36\sqrt{3}+48\sqrt{6}}$



'07
GBML

ROUND 1 - Volumes and Surface Areas of Solids

1. A cross section of the two concentric spheres and the plane tangent to the smaller sphere is two concentric circles and a chord of the larger circle tangent to the smaller.
   From the diagram, $C = 2\pi r = 24\pi \rightarrow r = 12$ and $R = \underline{\mathbf{15}}$



2. Let the altitudes of the cylinder and cone be h and H respectively.
   Then $\frac{V(\text{cone})}{V(\text{cylinder})} = \frac{\frac{1}{3}\pi r^2 H}{\pi r^2 h} = \frac{9}{5} \rightarrow \frac{H}{3h} = \frac{9}{5} \rightarrow H : h = \underline{\mathbf{27:5}}$
   Note that the fact that "the total height of the solid is 20" was not needed.

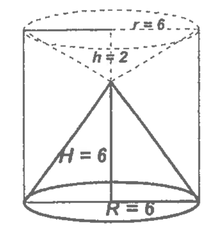
3. The base of the pyramid is 10. The altitude of the pyramid is 12 and, therefore, the slant height is 13. Thus, the total surface area is
   $10^2 + 4\left(\frac{1}{2}\cdot 10\cdot 13\right) = 100 + 260 = \underline{\mathbf{360}}$

'08
GBML

ROUND 1

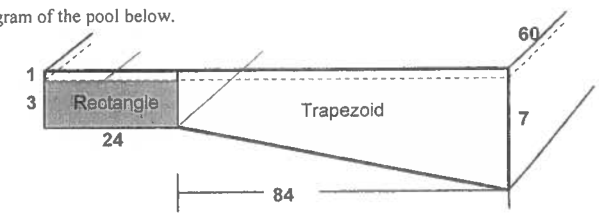
1. Let s, d, r denote the side of the cube, diagonal of the cube and radius of the sphere.
   Since the diameter of the sphere is a diagonal of the inscribed cube, $2r = d = s\sqrt{3}$.
   The surface area of the cube = $6s^2$ and the surface area of the sphere = $4\pi r^2$

   $\frac{6s^2}{4\pi r^2} = \frac{6\left(\frac{2r}{\sqrt{3}}\right)^2}{4\pi r^2} = \underline{\frac{2}{\pi}}$



2. $V_{cyl} = \pi(6)^2(8) = 288\pi$
   $V_{cones} = \frac{1}{3}\pi\cdot 6^2\cdot 2 + \frac{1}{3}\pi\cdot 6^2\cdot 6 = 96\pi$
   Thus, the required volume is $288\pi - 96\pi = \underline{\mathbf{192\pi}}$

3. Consider the diagram of the pool below.



$\frac{V(\text{Water})}{V(\text{Pool})} = \frac{[60]\left(3(24)+\frac{1}{2}(84)(3+7)\right)}{[60]\left(4(24)+\frac{1}{2}(84)(4+8)\right)} = \frac{72+42(10)}{96+42(12)} = \frac{6+7(5)}{8+7(6)} = \frac{41}{50} = \underline{\mathbf{82\%}}$

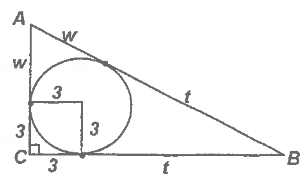
'09
GBML

ROUND 1

1. The diagonals in a regular hexagon with side x have lengths $x\sqrt{3}$ (short) and $2x$ (long).
   $x\sqrt{3} = 2\sqrt{6}$ → side $x = 2\sqrt{2}$ → log diagonal $2x = 4\sqrt{2}$
   Thus, $k = \left(4\sqrt{2}\right)^3 = 64\cdot 2\sqrt{2} = \underline{\mathbf{128\sqrt{2}}}$

2.
   Cone: height - $\frac{9}{7}$ and radius $2r$ Cylinder: height - $\frac{9}{7}$ and radius r
   Sphere: radius 3
   $\frac{1}{3}\pi\cdot 4r^2\cdot\frac{9}{7} + \pi r^2\cdot\frac{9}{7} = \frac{4}{3}\pi\cdot 3^3 \rightarrow 3r^2 = 36 \rightarrow r = 2\sqrt{3} \rightarrow r_{cone} = \underline{4\sqrt{3}}$

3.



$C = 6\pi \rightarrow r = 3$
$(w+3)+(3+t)+(t+w) = 40 \rightarrow 2t+2w = 34$
$\rightarrow t+w = 17$
Using knowledge of special right triangles, the legs must be 8 and 15 and the area of the base is
$\frac{5}{12} = \frac{h}{8} \rightarrow h = \frac{10}{3}$
$B = \frac{1}{2}\cdot 8\cdot 15 = 60$ and the volume of the prism is $V = Bh = 60\cdot\frac{10}{3} = \underline{\mathbf{200}}$
Alternately, to find the legs of the base:
$(w+3)^2 + (t+3)^2 = 17^2$
$(17-t+3)^2 + (t+3)^2 = 17^2$
$(20-t)^2 + (t+3)^2 = (400 - 40t + t^2) + (t^2 + 6t + 9) = 289$
$2t^2 - 34t + 120 = 2(t^2 - 17t + 60) = 2(t-5)(t-12) = 0 \rightarrow t = 5$ to 12 (and $w = 12$ or 5)

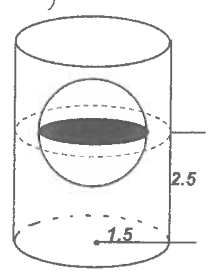
GBML '10

ROUND 1

1. The edges of the largest faces are $3x$ and $4x$. Thus, $12x^2 = 75 \rightarrow x = \frac{5}{2}$ and the dimensions of the rectangular solid are $\frac{5}{2} \times \frac{15}{2} \times 10$.

The required ratio $\frac{V}{TSA} = \frac{\frac{5}{2}\cdot\frac{15}{2}\cdot 10}{2\left(\frac{5}{2}\cdot\frac{15}{2}+\frac{5}{2}\cdot 10+\frac{15}{2}\cdot 10\right)} = \frac{\frac{25\cdot 15}{2}}{2\left(\frac{75}{4}+25+75\right)} = \frac{25\cdot 15}{75+400} = \underline{\mathbf{\frac{15}{19}}}$

2. $\frac{LSA(\text{cone})}{LSA(\text{cylinder})} = \frac{\pi r l}{2\pi r h} = \frac{5}{22} \rightarrow \frac{l}{h} = \underline{\mathbf{\frac{5}{11}}}$



3. $\text{Vol(water)} = \pi r^2 h = \pi\left(\frac{3}{2}\right)^2\cdot\frac{5}{2} = \frac{45\pi}{8}$

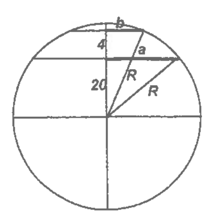
$\text{Vol(sphere)} = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi(1)^3 = \frac{4\pi}{3}$

$\frac{9}{16}$ is below the surface $\rightarrow \frac{3\pi}{4}$ is submerged

Therefore, $\pi\left(\frac{3}{2}\right)^2\cdot h = \frac{45\pi}{8}+\frac{3\pi}{4} = \frac{51\pi}{8} \rightarrow \frac{9h}{4} = \frac{51}{8} \rightarrow h = \underline{\frac{17}{6}}$ or $\underline{2\frac{5}{6}}$

GBML '11

ROUND 1



1. Let the radius of the sphere be R and the radii of the circular cross sections be a and b. To visualize the sphere and its two circular cross sections, imagine the diagram rotated about the vertical axis. The difference in area between the cross sections is $k\pi = \pi a^2 - \pi b^2$ $\rightarrow k = a^2 - b^2$.
From the diagram at the right, using the Pythagorean Theorem,
$\begin{cases} 20^2 + a^2 = R^2 \\ 24^2 + b^2 = R^2 \end{cases} \rightarrow a^2 - b^2 = 24^2 - 20^2 = 576 - 400 = \underline{\mathbf{176}}$

2. Let the side of the cube be s, the face diagonal be d and the side of the base of the pyramid be x.
Then: $6s^2 = \frac{1}{3}Bh = \frac{1}{3}x^2 h$ and $d^2 = 2s^2 = x^2 \rightarrow 18s^2 = 2s^2 h \rightarrow h = \underline{\mathbf{9}}$.

3. Dimensions of the cone: base radius r, height h and slant height l
Dimensions of the cylinder: base diameter D, height H
Note: According to the given, $D = r$.
$LSA_{cone} = 2(\text{area of base}) \rightarrow \pi r l = 2\pi r^2 \rightarrow l = 2r$
Pythagorean Theorem on cone $\rightarrow h^2 + r^2 = l^2 = (2r)^2 \rightarrow h = r\sqrt{3}$

$Vol_{cone} = Vol_{cyl} \rightarrow \frac{1}{3}\pi r^2 h = \pi\left(\frac{D}{2}\right)^2 H \rightarrow H = \frac{4}{3}h$

$LSA_{cyl} = 2\pi\left(\frac{D}{2}\right)H = \pi D H = \pi r\cdot\frac{4}{3}h = \frac{4}{3}\pi r\left(r\sqrt{3}\right) = \frac{4}{3}\sqrt{3}\pi r^2 \rightarrow (A, B, C) = \underline{(4, 3, 3)}$.

MML '03

MASSACHUSETTS MATHEMATICS LEAGUE
OCTOBER 2003
ROUND 1 VOLUMES & SURFACES

ANSWERS

A) 132π

B) [illegible]

C) 2064π

A) [illegible] drilled out of its center. Find the total surface area of solid remaining leaving the result in terms of pi

$R = 4 \quad r = 2 \quad h$ [illegible]

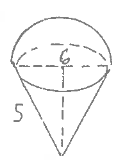
[illegible] $= 72\pi$ [illegible] Total surface [illegible]

[illegible] $= 36\pi$

[illegible] 2π [illegible] $16 - 2\pi$ [illegible]

Ans $108\pi + 24\pi = 132\pi$

B) A right circular cone apex down as shown has a slant height of 5 cm and a base diameter of 6 cm. A hemisphere is sitting on top of the cone. Find the volume in terms of pi of the solid formed by the cone and the hemisphere

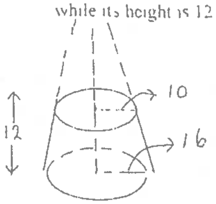


$V_s = \frac{4}{3}\pi r^3 \quad V_c = \frac{1}{3}\pi r^2 h \qquad V_c = \frac{1}{3}\pi \cdot 3^2 \cdot 4 = 12\pi$

$r = 3,\ h = \sqrt{5^2 - 3^2} = 4$

$V_s = \frac{\frac{4}{3}\pi\ 3^3}{2} = \frac{2}{3}\ 27 \cdot \pi = 18\pi \qquad$ ANS 30π

C) The truncated cone shown was formed by cutting off the top of a right circular cone with a plane parallel to its base. The radii of the bases of the truncated cone are 10 cm and 16 cm while its height is 12 cm. Calculate the volume of truncated cone in terms of pi



$V = \frac{1}{3}\pi h (r_1^2 + r_1 r_2 + r_2^2)$

$= \frac{1}{3} 12\pi (100 + 160 + 256) = 4\pi \cdot 516$

$= 2064\pi$

MML '05

Round One:

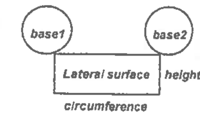
A. Base has side 2.5, area 6.25 Vol = 9(6.25)/3 = 18.75

B. Hypotenuse KA = $\sqrt{98}$ while DQ and AQ are each $\sqrt{8^2 + 4^2} = \sqrt{80}$ Since $\overline{KD} \perp \overline{DQ}$ hypotenuse KQ = $\sqrt{80 + 49} = \sqrt{129}$.

C. #1 ht = 30, radius = 40, vol = 16000π. #2 ht = 40, radius = 30, vol = 12000π. Other 2 have radius = 24 (40x30/50) and hts of 18 and 32, vol=9600π.

MML '06

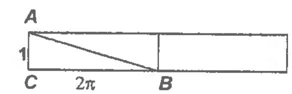
Round 1



A) $C = 2\pi r = 8\pi \rightarrow r = 4$, $V = \pi r^2 h = 48\pi \rightarrow h = 3$,
TSA = $2(\pi r^2) + (2\pi r)h = 32\pi + 24\pi = \underline{56\pi}$

B) Let x denote the edge of the original cube. Then $(x - 1)$ denotes the edge of the smaller cube. The decrease in volume is $x^3 - (x-1)^3 = 169 \rightarrow x^3 - (x^3 - 3x^2 + 3x - 1) = 169$
$\rightarrow 3x^2 - 3x - 168 = 3(x^2 - x - 56) = 3(x-8)(x+7) = 0 \rightarrow x = 8$
Thus, the volume of the smaller cube is $7^3 = \underline{343}$ cm^3.

C) A path from point B along the lateral surface from the bottom base to the top base (perpendicularly) and then through the center of the top base to point A has length $1 + 4 = 5$
Cutting the surface of the cylinder along a line through point A perpendicular to a base and rolling out the lateral surface of the cylinder into a plane, we get a rectangle whose width equals the height of the cylinder and whose length equals the circumference of the base of the cylinder. The shortest path along this surface is the hypotenuse of ΔABC.
Using the Pythagorean Theorem, we have $AB = \sqrt{1 + 4\pi^2}$



Since $5 < \sqrt{1 + 4\pi^2}$, the shortest distance is $\underline{5}$
An interesting question is under what circumstances is one path shorter than the other?

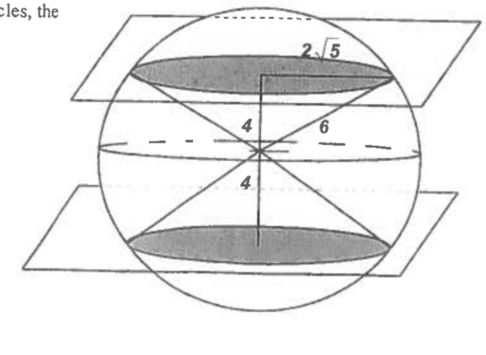
MML '07

Round 1

A) Let x denote the increase in radius or height. Then $\pi(10 + x)^2(4) = \pi(10)^2(4 + x)$
$\rightarrow 4(100 + 20x + x^2) = 400 + 100x \rightarrow 4x^2 - 20x = 4x(x - 5) = 0 \rightarrow x = \underline{5}$

B) The regions consist of two congruent circles, the intersection of a sphere and two parallel planes.

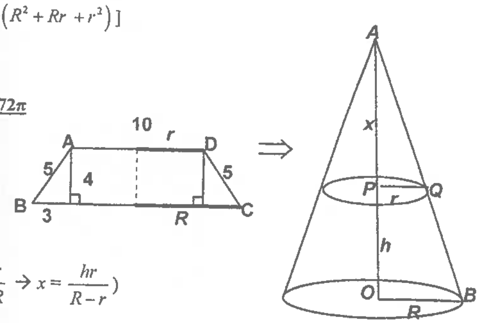
Area = $2\pi(2\sqrt{5})^2 = \underline{40\pi}$



C) Method #1 [using V(frustum) = $\frac{\pi h}{3}\left(R^2 + Rr + r^2\right)$]

$r = 5$, $R = 8$ and $h = 4$

Thus, $\mathbf{V} = \frac{1}{3}\pi(4)[25+40+64] = \underline{172\pi}$



Method #2 [V(cone$_1$) – V(cone$_2$)]

(Note: $\Delta APQ \sim \Delta AOB \rightarrow \frac{x}{x+h} = \frac{r}{R} \rightarrow x = \frac{hr}{R-r}$)

$\frac{x}{x+4} = \frac{5}{8} \rightarrow x = \frac{20}{3}$

V(cone$_1$) = $\frac{1}{3}\pi(8)^2\left(4+\frac{20}{3}\right) = \frac{2048\pi}{9}$ V(cone$_2$) = $\frac{1}{3}\pi(5)^2\left(\frac{20}{3}\right) = \frac{500\pi}{9}$

Therefore, V(frustum) = $\frac{1548\pi}{9} = \underline{172\pi}$