Round 1 – Arithmetic

Number Theory

GREATER BOSTON MATHEMATICS LEAGUE

MEET 1 – OCTOBER 1998

ROUND 1 – Arithmetic-Open

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. When 72% of $\left(\frac{3}{4}+\frac{1}{3}\left(\frac{4}{3}\right)^{-2}\right)$ is put the form $\frac{a}{b}$ where a and b are relatively prime whole numbers, find $a + b$.

2. Find the **sum** of the three smallest non-prime two digit whole numbers each of whose digits is prime.

3. How many natural numbers less than 100 are **not** divisible by 2 or 3?

GREATER BOSTON MATHEMATICS LEAGUE

MEET 1 – SEPTEMBER 1999

ROUND 1 – Arithmetic-Open

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Given the following addition in **base 8**, shown below, compute the base 10 sum, $a + b + c$.

$$\begin{array}{r} a\,5\,7 \\ 1\,6\,c \\ \underline{2\,b\,6} \\ 7\,5\,1 \end{array}$$

2. A stock increased in price 25% after one year and then increased 33⅓% over that price at the end of the second year. After the third year, it was still 10% more than its original price. Compute the percent decrease of the stock after the third year from its price at the end of the second year.

3. How many natural numbers between 267 and 511 are divisible by 4 or 5?

GREATER BOSTON MATHEMATICS LEAGUE

MEET 1 – SEPTEMBER 2000

ROUND 1 – Arithmetic-Open

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND.

1. Find the smallest 4-digit whole number divisible by 12, 15, 18, and 24.

2. A stock underwent the following changes over a three month period: 1st month: 20% drop, 2nd month: up 20 points, 3rd month: 30% gain. If the final value of the stock was 78 points, how many points was the stock worth at the beginning of this three month period?

3. The 3-digit base 6 number, xyz_6 ,$x > y > z$, is divisible by 2, 3, and 5. Find all possible base 6 numbers xyz_6 satisfying these conditions. Note your answer(s) should be left in base 6.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 1 – OCTOBER 2001

ROUND 1 – Arithmetic-Open

1. ______________%

2. ____________________

3. (____,____)

CALCULATORS ARE NOT ALLOWED ON THIS ROUND.

1. If a stock undergoes a 25% decrease, followed by a $46\frac{2}{3}\%$ decrease, followed by a 40% increase, from month to month over a three-month period, find the stock's percent decrease from its original price?

2. Find the sum of all odd 3-digit whole numbers which are divisible by 75.

3. Given $0.12_{(3)} - 0.13_{(4)} = 0.xy_{(12)}$. Find the ordered pair (x, y).

GREATER BOSTON MATHEMATICS LEAGUE

MEET 1 - OCTOBER 2006

ROUND 1 – Arithmetic - Open

1. ______________________________

2. ______________________________

3. ______________________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND.

1. The repeating decimal 1.585858… can be expressed as a rational fraction $\frac{p}{q}$.

   When reduced to lowest terms, what is the square root of the sum of its numerator and denominator?

2. How many two-digit primes have exactly one digit that is a 7 or a 9?

3. When N is written in the positive integer base x, it is represented as 453, and when it is written in base $(x + 3)$ it is represented 252. How would N be written in base $(x - 2)$?

GREATER BOSTON MATHEMATICS LEAGUE

MEET 1 - OCTOBER 2007

ROUND 1 – Arithmetic - Open

1. ______________________

2. ______________________

3. ______________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND.

1. Name the fourth smallest two digit prime number that is one more than a positive multiple of the sum of its digits.

2. Let P = the mean of the non-prime numbers between 43 and 61 exclusive and Q = the mean of the prime numbers between 43 and 61 exclusive.
   Compute $P - Q$.

3. A purchase in a hardware store amounts to \$14.20. The customer gives the clerk \$15.00. In how many ways can change be made without using pennies, if there is only one dime in the cash drawer?
   (Allowable coins: nickel, dime, quarter and half dollar)

GREATER BOSTON MATHEMATICS LEAGUE

MEET 1 - OCTOBER 2008

ROUND 1 – Arithmetic - Open

1. ________________________

2. ________________________

3. ________________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND.

1. How many even natural numbers are there between 1 and 10,000 which are divisible by 3, 4, and 14?

2. Using only nickels, dimes and quarters, how many ways can change be made from a $4.30 purchase if a $5.00 bill is handed to the clerk?

3. The sum of $0.1\overline{2}_{(4)} + 0.\overline{24}_{(7)}$ when written in base 10 it is equal to $\frac{a}{b}$ in simplified form.

   Find the sum of $a+b$.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 1 – OCTOBER 2009

ROUND 1 – Arithmetic - Open

1. ______________________________

2. (__________ , __________)

3. ______________________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND.

1. Simplify: $\dfrac{15 \div 5 - 4 + 8 + 2 \times 4}{10 - (4 + 6 \div 2) + 3}$

2. If $14K_9 = 1K4_8$, then $4A_K = A3_{K-1}$. Find (K, A).

3. How many three-digit positive integers will have exactly two of the same digit?

GREATER BOSTON MATHEMATICS LEAGUE

MEET 1 - OCTOBER 2010

ROUND 1 – Arithmetic - Open

1. (____________ , ____________)

2. ________________________

3. ________________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND.

1. Find the ordered pair (x, y) for which $0.33_{(4)} - 0.22_{(3)} = 0.xy_{(12)}$

2. Given: x and y are positive integers and $x! = 120(y!)$
   Compute the sum of the <u>three</u> possible values of x.

3. What are the rightmost two digits of: $2009^{2010} - 2007^{2011} + 2008$?

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MASSACHUSETTS MATHEMATICS LEAGUE
FEBRUARY 2004
ROUND 2: NUMBER THEORY

ANSWERS

A)_______________

B)_______________

C)_______________

A) Given $(ABA)_9 = (BB0)_{11}$ where 0 is zero, and A and B are distinct natural numbers. Determine both possible values of A and B. Write the answers in the form (A, B).

B) Determine the units digit for the sum of $7^{2003} + 9^{2003}$.

C) How many positive even integers are divisors of $(12^3)(18^4)$?

MASSACHUSETTS MATHEMATICS LEAGUE
FEBRUARY 2005
ROUND 2 ELEMENTARY NUMBER THEORY

ANSWERS

A)_______________

B)_______________

C)_{___ ______}_

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positive

A) How many more factors are there for 5292 than for 520 ?

B) How many positive integers less than 200 have exactly three distinct (unrepeated) prime factors?

C) n = 1,111,200,311,112,004,111,1ab is a 22 digit number in base ten whose right-most digits are a and b. If n is divisible by 36 find the set of all possible values of the product of a and b.

MASSACHUSETTS MATHEMATICS LEAGUE
FEBRUARY 2006
ROUND 2 ELEMENTARY NUMBER THEORY

ANSWERS

A)_______________

B)_______________

C) (______, ______)

A) If *abc* is a three digit prime, find the sum of the second largest prime factor and the second smallest prime factor of the six digit number *abcabc*.

B) If A © B is defined as the sum of all composite numbers strictly between A and B, that is, including neither A nor B, evaluate:

(15 © 21) © (28 © 33)

B) If x and y are integers satisfying $2xy - 4x - y - 1 = 0$, which ordered pair (x, y) is furthest from the origin?

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 – FEBRUARY 2007
ROUND 2 ARITHMETIC / NUMBER THEORY

ANSWERS

A) (_____________ , _____________)

B) ____________________________

C) ____________________________

A) Let A be the smallest positive integer value for which $B = \dfrac{7A+1}{13}$ is also an integer.
Find the ordered pair (A, B).

B) The sequences of positive integers generated by $7n + 2$ and $11n + 4$ have exactly one two-digit integer in common. What is the largest three-digit integer that they have in common?

C) The product of the first 2007 positive prime numbers is divisible by several 3-digit positive integers of the form $AAA_{(10)}$. Find the sum of all 3-digit positive integers of this form.
Note: 1 is not considered a prime number.

MASSACHUSETTS MATHEMATICS LEAGUE
DECEMBER 2003
ROUND 2: NUMBER THEORY

ANSWERS

A)_______________

B)_______________

C)_______________

A) During rush hour on December fourth, 20% of the cars on Route 314 took exit 1, 25% of those remaining took exit 2, and 10% remaining after that took exit 3. If 162 cars continued on Route 314, how many cars traveled the route during rush hour that day?

B) A palindrome reads the same from right to left or vice versa. For example, 37673 is a palindrome. How many palindromes are there between 10,000 and 20,000?

C) The length of each side of a triangle is a prime, and its perimeter is also a prime. What is the smallest possible perimeter that the triangle could have?

MASSACHUSETTS MATHEMATICS LEAGUE
DECEMBER 2004
ROUND 2 ELEMENTARY NUMBER THEORY

ANSWERS

A)______________

B)______________

C)______________

A) **Three men who are no longer teenagers find the product of their current ages is 26,390. Find the sum of their current ages.**

B) **How many positive integers less than 500 each have exactly three different positive integer divisors?**

C) **Find all primes of the form $8n^3 - 2197$**

MASSACHUSETTS MATHEMATICS LEAGUE
DECEMBER 2005
ROUND 2 ELEMENTARY NUMBER THEORY

ANSWERS

A)_______________

B)_______________

C)_______________

A) How many different positive integers are factors of 160?

B) The product of 123_4 with 567_8 is $1ABC0_9$ a five-digit base nine number. Determine the ordered triplet of digits (A, B, C).

C) The digits of a two-digit base ten positive integer are reversed, resulting in a 108% increase in value. What was the original number?

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2006
ROUND 2 ARITHMETIC/ ELEMENTARY NUMBER THEORY

ANSWERS

A) ______________________________

B) ______________________________

C) ______________________________

A) W is the units digit of a base 10 integer N. When N is raised to a positive integer power, the units digit may equal exactly two distinct values. Find all values of W for which this is true.

B) A positive integer has exactly 8 positive factors. Two of them are 77 and 119. Find this integer.

C) The 4-digit base 10 positive integer $ABBA$ (where $A > 0$) is divisible by 12. A and B are distinct digits. Find the sum of all integers satisfying this condition.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2007
ROUND 2 ARITHMETIC/ ELEMENTARY NUMBER THEORY

ANSWERS

A) ______________________________

B) ______________________________

C) ______________________________

A) A magic integer is defined to be a positive integer that is both a perfect square and a perfect cube. Determine the sum of all magic integers less than 100,000.

B) Given the following pattern:

$$\begin{array}{ccccccccccc} & & 1 & & 4 & & 4 & & 1 & & \\ & 1 & & 5 & & 8 & & 5 & & 1 & \\ 1 & & 6 & & 13 & & 13 & & 6 & & 1 \end{array}$$

The sum of the entries in row 1 is 10.
Each row has one more entry than the previous row.
Each row begins and ends with 1 and the in-between entries are the sum of the entries immediately to the right and left in the previous row.
What is the sum of the entries in the 16^{th} row?

C) Determine a simplified factored expression, in terms of the positive integer x, for the number of even factors of the following expression:

$$\left(12^{x+1}\right)\cdot\left(18^{x-1}\right)\cdot\left(75^{3}\right)$$

GBML 1998

ROUND 1

1. 72% of $\left(\frac{3}{4}+\frac{1}{3}\left(\frac{4}{3}\right)^{-2}\right)=\frac{18}{25}\left(\frac{3}{4}+\frac{3}{16}\right)=\frac{18}{25}\cdot\frac{15}{16}=\frac{27}{40}\Rightarrow a+b=\mathbf{67}$

2. The three smallest two digit non-prime whole numbers each of whose digits is prime are 22, 25, and 27. Their sum equals **74**

3. Method 1: There are 99 numbers altogether; $2\cdot1, 2\cdot2, \dots 2\cdot49$ are divisible by 2; $3\cdot1, 3\cdot2, \dots 3\cdot33$ are divisible by 3; $6\cdot1, 6\cdot2, \dots 6\cdot16$ are divisible by 6, which are numbers in the multiples of 2 list and the multiples of 3 list $\Rightarrow 99-49-33+16=33$
Method 2: In mod 6, 1,5 are not divisible by 2 or 3. The 99 numbers are 16 groups of 6 and 1 group of 3 $\Rightarrow 16\cdot2+1=33$ numbers not divisible by 2 or 3.

GBML 1999

ROUND 1

1. $\begin{array}{r} a\,5\,7 \\ 1\,6\,c \\ 2\,b\,6 \\ \hline 7\,5\,1 \end{array}$. Since the right column adds to 1, and $17=1 \bmod 8 \Rightarrow c=4$. There is a 2 carry to the middle column, and since the middle column adds to $5=13 \bmod 8 \Rightarrow b=0$. There is a 1 carry to the left column which adds to $7 \Rightarrow a=3 \Rightarrow a+b+c=7$

2. $\frac{5}{4}\cdot\frac{4}{3}x=\frac{11}{10}\Rightarrow x=\frac{33}{50}=66\%\Rightarrow$ **34%** decrease

3. $267\div4=66$ R3; $511\div4=127$ R3; $267\div5=53$ R2; $511\div5=102$ R1; $267\div20=13$ R7; $511\div20=25$ R11; $\Rightarrow$ there are $127-67+1=61$ multiples of 4, there are $102-54+1=49$ multiples of 5, and there are $25-14+1=12$ multiples of $20\Rightarrow$ there are $61+49-12=\mathbf{98}$ numbers divisible by either 4 or 5.

GBML 2000

ROUND 1

1. The LCM of 12, 15, 18, and 24 = 8×9×5 = 360.
The smallest 4-digit multiple of 360 = 360×3 = 1080

2. Let x = original value of the stock:
$1.3(.8x+20)=78\rightarrow .8x+20=60\rightarrow .8x=40\rightarrow x=50$

3. Since xyz_6 is divisible by 2, 3, and 5, it must be divisible by $6\rightarrow z=0\rightarrow$ the base 10 value of the number $=36x+6y=6(6x+y)$; this is divisible by $30\rightarrow 6x+y$ is divisible by 5; since $x>y>z=0$, the only possibilities are $x=3, y=2$ or $x=4, y=1\rightarrow$ answers are 320_6 and 410_6.

GBML 2001

ROUND 1

1. $\frac{3}{4}\cdot\frac{8}{15}\cdot\frac{7}{5}=\frac{14}{25}=56\%\Rightarrow 44\%$ decrease.

2. Case I: last 2 digits 25: First : 2, 5, 8
Case II: last 2 digits 75: First: 3, 6, 9 $\Rightarrow 225+525+825+375+675+975=3600$.

3. $\left(\frac{1}{3}+\frac{2}{9}\right)-\left(\frac{1}{4}+\frac{3}{16}\right)=\frac{17}{144}=\frac{1}{12}+\frac{5}{144}\Rightarrow(x,y)=(1,5)$.

GBML 2006

ROUND 1

1.
Let $n = 1.585858\ldots$ Then $100n = 158.585858\ldots$ Subtracting $157n = 99$ or $n = 99/157$
Since this fraction is already reduced, the sum of the numerator and denominator is 256, giving a square root of **16**.

2.
The required prime must be of one of the following 4 forms:

7_ → 71, 73 7_ → 17, 37, 47, 67 _9 → 19, 29, 59, 89 9_ → none Total: **10**

3.
Instead of trying $n \geq 3$ by trial and error, we choose to proceed algebraically.
$N = 4x^2 + 5x + 3 = 2(x+3)^2 + 5(x+3) + 2 = 2x^2 + 17x + 35 \rightarrow 2x^2 - 12x - 32 = 0$
$\rightarrow x^2 - 6x - 16 = (x-8)(x+2) = 0 \rightarrow x = 8$
$453_{(8)} = 4(64) + 5(8) + 3 = 299_{(10)} \rightarrow \underline{1215_{(6)}}$ $6\overline{)299}$ Read the remainders UP from the bottom.

Quotient	Rem
49	5
8	1
1	2
0	1

GBML 2007

ROUND 1

1. $11 = 5(1+1) + 1 \rightarrow 1^{st}$
   $13 = 3(1+3) + 1 \rightarrow 2^{nd}$
   $17 = 2(1+7) + 1 \rightarrow 3^{rd}$
   The following fail: $19 = 1(1+9) + 9$, $23 = 4(2+3) + 3$, $29 = 2(2+9) + 7$
   $31 = 7(3+1) + 3$, $37 = 3(3+7) + 7$

   $\underline{41} = 8(4+1) + 1 \rightarrow 4^{th}$

2. primes: 47, 53 and 59 → sum = 159 and mean = Q = 53
   $44 + 45 + \ldots + 60 = \frac{17}{2}(2 \cdot 44 + 16 \cdot 1) = 884$

   Non-primes → sum = 884 − 159 = 725 and mean = $P = 725/14 = 51\frac{11}{14}$

   Difference $= P - Q = -\frac{17}{14}$ or $-1\frac{3}{14}$

3. construct a table (change due 80¢)
   10 distinct ways

H	D	Bal	N	Q	H	D	Bal	N	Q
1	0	30	1	1	0	0	80	11	1
1	0	30	6	0	0	0	80	16	0
1	1	20	4	0	0	1	70	4	2
0	0	80	1	3	0	1	70	9	1
0	0	80	6	2	0	1	70	14	0

GBML 2008

ROUND 1

1. The least common multiple of 3, 4, and 14 is 84.
   There are **119** multiples of 84 between 1 and 10,000.

2 Let N, D and Q represent the number of nickels, dimes and quarters used. Then:
$5N + 10D + 25Q = 70 \rightarrow N + 2D + 5Q = 14$ Clearly, $Q < 3$.
Examining the 3 possible cases:
$Q = 2 \rightarrow N + 2D = 4 \rightarrow D = (4 - N)/2 \rightarrow N = 0, 2, 4 \rightarrow$ 3 possible triples
$Q = 1 \rightarrow N + 2D = 9 \rightarrow D = (9 - N)/2 \rightarrow N = 1, 3, 5, 7, 9 \rightarrow$ 5 possible triples
$Q = 0 \rightarrow N + 2D = 14 \rightarrow D = (14 - N)/2 \rightarrow$ 8 possible triples
Thus, there are 16 possible triples (N, D, Q).

3. $0.1\overline{2}_{(4)} =$
$100N = 12.\overline{2}$
$10N = 1.\overline{2} \rightarrow 30N = 11 \rightarrow N = \frac{11}{30_{(4)}} = \frac{5}{12_{(10)}}$
Likewise, $0.\overline{24}_{(7)} = \frac{24}{66_{(7)}} = \frac{3}{8_{(10)}}$
$\frac{5}{12} + \frac{3}{8} = \underline{43}$

GBML 2009

ROUND 1

1. $\dfrac{15 \div 5 - 4 + 8 + 2 \times 4}{10 - (4 + 6 \div 2) + 3} = \dfrac{3 - 4 + 8 + 8}{10 - (4+3) + 3} = \dfrac{15}{6} = \underline{\dfrac{5}{2}}$

2. $14K_9 = 1K4_8 \rightarrow 81 + 36 + K = 64 + 8K + 4 \rightarrow 7K = 49 \rightarrow K = 7$

   $4A_K = A3_{K-1} \rightarrow 28 + A = 6A + 3 \rightarrow A = 5$ Thus, $(K, A) = \underline{(7, 5)}$.

3. The three-digit positive integer will be of the form XXY, XYX or YXX, where $X \neq Y$. In each case, there are 9 ways to select the hundred's digit (since zero is not allowed) and 9 ways to select the other different digit (since zero is allowed). Thus, there are $3(9^2) = \underline{243}$ positive integers with exactly two identical digits.

GBML 2010

ROUND 1

1. $0.33_{(4)} = \frac{3}{4} + \frac{3}{16} = \frac{15}{16} = \frac{135}{144}$
   $0.22_{(3)} = \frac{2}{3} + \frac{2}{9} = \frac{8}{9} = \frac{128}{144}$ → difference $= \frac{7}{144} = \frac{0}{12} + \frac{7}{12^2} \rightarrow (x, y) = \underline{(0, 7)}$.

2. $x! = 120y! = 2^3 \cdot 3 \cdot 5 \cdot y! = 2 \cdot 3 \cdot 4 \cdot 5 \cdot y! \rightarrow$ $y = 1, x = 5$; $y = 3, x = 6$; $y = 119, x = 120$ → sum = **131**

3. For $n = 0, 1, 2, \ldots$, the rightmost two digits of $(2009)^n$ cycle through: 09, 81, 29, 61, 49, 41, 69, 21, 89, 01 and the rightmost two digits of $(2007)^n$ cycle through the values: 07, 49, 43, 01
   Thus, 2009^{2010} ends in 01.
   and $(\ldots 01)(\ldots 43)$ ends in $\ldots 43$.

   9
   ~~10~~ 11
   — ~~0~~ 1
   4 3
   5 8 + 8 = 66

   → 58 + 8 → **66**

MML 2/04

A) Given $(ABA)_9 = (BB0)_{11}$ where 0 is zero, and A and B are distinct natural numbers. Determine both possible values of A and B. Write the answers in the form (A, B).

$81A + 9B + A = 121B + 11B$

$82A = 132B - 9B = 123B$

$2A = 3B, \quad A = 3, B = 2$ or $A = 6, B = 4$

$(3,2), (6,4)$

B) Determine the units digit of $7^{2003} + 9^{2003}$.

7^0	1
7^1	7
7^2	9
7^3	3
7^4	1

9^0	1
9^1	9
9^2	1

$4\overline{)2003}$ R=3

$2\overline{)2003}$ R=1

$3 + 9 = 1\,2$

C) How many positive even divisors does $(12^3)(18^4)$ have?

$(2^2 \cdot 3)^3 (2 \cdot 3^2)^4 = 2^6 \cdot 3^3 \cdot 2^4 \cdot 3^8 = 2^{10} \cdot 3^{11}$

divisors $= (10+1)(11+1) = 11(12) = 132$

Ans 132 - odd divisors $= 132 - 12 = \boxed{120}$

MML 2/05

Round Two:

A. $520=2^3 x5x13$ so 4x2x2=16 factors. $5292=2^2 x3^3 x7^2$ so 3x4x3=36 factors, **20** more.

B. An organized list gives 2x3x5, 2x3x7, ... 2x3x31 (9 values); then 2x5x7, ..., 2x5x19 (5 values); then 2x7x11, 2x7x13 (2 values); 3x5x7,..3x5x13(3 values); for **19** values with exactly 3 distinct prime factors. Note that without the distinct requirement we would have additional values such as 2x2x3x5

C. Divisibility by 9 requires 23+a+b be a multiple of 9 or 5+a+b=9 or 18 so a+b=4 or 13. Divisibility by 4 requires ab be a multiple of 4. Only possibilities are 04, 40, and 76 so products are 0 or 42

MML 2/06

Round Two:

A. $abcabc = abc\,(1001) = abc(11)(7)(13)$. $13 + 11 = 24$.

B. $15©21 = 16 + 18 + 20 = 54$; $28©33 = 30 + 32 = 62$;
$54©62 = 55 + 56 + 57 + 58 + 60 = 286$

C. $2xy - y = 4x + 1$ so $y = \dfrac{4x+1}{2x-1} = \dfrac{2(2x-1)+3}{(2x-1)}$ so $2x - 1$ is a factor of 3 (±1 or ±3).
Thus, x must be 0, ±1 or 2. This yields the ordered pairs (1, 5), (0, -1), (2, 3) and (-1, 1). The first of these is furthest from the origin.

MML 2/07

Round 2

A) Substituting A = 1,2,3,…produces the sequence 8, 15, 22, ….
The first multiple of 13 in this sequence occurs when A = 11, B = 78/13 = 6
Thus, (A, B) = **(11, 6)**.

B) The expressions $7n + 2$ and $11n + 4$ generate the sequences
2, 9, 16, 23, 30, 37, … and 4, 15, 26, 37, …
Clearly, the two-digit integer they have in common is 37.
The next common integer can be found by adding 77, the least common multiple of 7 and 11.
To find the largest three-digit integer A that they have in common, solve the inequality
$A = 37 + 77k < 1000$ over the integers. $k < 963/77 = 12^+ \rightarrow k = 12 \rightarrow A = 37 + 924 =$ **961**

C) **111** = 3(37) – ok
222 = 2(3)(37) – ok
333 = 3(111) = 3^2(37) – fails because of the repeated prime.
444 = 4(111) = 2^2(3)(37) fails
555 = 3(5)(37) - ok
666 = (2)3^2(37) – fails
777 = 3(7)(37) – ok
888 = 2^3(3)(37) – fails
999 = 3^3(37) – fails
Thus, the required sum is (1 + 2 + 5 + 7)(111) = (15)(111) = **1665**.

MML 12/03

A) On a particular section of Route 314 there were three exits. During rush hour on December fourth, 20% of the cars on Route 314 took exit 1, 25% of those remaining took exit 2, and 10% remaining after that took exit 3. If 162 cars continued on Route 314, how many cars traveled the route during rush hour that day?

$X(.80)(.75)(.90) = 162 \quad X = 300$

B) A palindrome reads the same from right to left or vice versa. For example, 37673 is a palindrome. How many palindromes are there between 10,000 and 20,000?

Must be of the form 1 A B A 1 ANS $10 \cdot 10 = 100$

C) The length of each side of a triangle is a prime, and its perimeter is also a prime. What is the smallest possible perimeter that the triangle could have?

2, 2, 3 $P = 7$

MML 12/04

Round Two:

A. 26,390=2x5x7x13x29 so 2x13 + 5x7 + 29 = 90.
B. The desired numbers must be the squares of primes so we have 2^2=4 up to 19^2=361 or 8 such numbers
C. $8n^3 - 2197 = (2n)^3 - 13^3 = (2n - 13)(4n^2 + 26n + 169)$ which is prime only if one of the factors is one. Thus n=7 and the number is 547.

MML 12/05

Round Two:

A. $160 = 2^5 5^1$ so there are (5+1)(1+1) = 12 factors.
B. 123_4 in base ten is 27; 567_8 is 375 so product is 10,125 which in base 9 is 14800_9.
C. Increase is (10u + t) – (10t +u) = 9(u – t). % increase is $\frac{9(u-t)}{10t+u} = \frac{108}{100} = \frac{27}{25}$ so $225u - 225t = 270t + 27u$ so $198u = 495t$ or $2u=5t$ thus u=5, t=2.

MML 12/06

Round 2

A) The rightmost digit of positive integer powers of 4 are alternately 4 and 6.
The rightmost digit of positive integer powers of 9 are alternately 9 and 1
2 → 2,4,8,6 3 → 3,9,7,1 7 → 7,9,3,1 8 → 8,4,2,6 0 → 0 1 → 1 5 → 5 6 → 6
Thus, d may be <u>4 or 9</u>.

B) $77 = 7 \cdot 11$ and $119 = 7 \cdot 17$

If $N = p_1^{e_1} \cdot p_2^{e_2} \cdot p_3^{e_3} \cdot \ldots \cdot p_k^{e_k}$, then the number of factors of N is given by $(e_1+1)(e_2+1) \cdot \ldots \cdot (e_k+1)$.
Note: The number of positive factors of N does not depend on what its prime factors are, only how many of each there are.

Thus, $7 \cdot 11 \cdot 17 = 7^1 \cdot 11^1 \cdot 17^1 = \underline{1309}$ has $(1+1)(1+1)(1+1) = 8$ positive factors.
(The factors are: 1, 7, 11, 17, $7 \cdot 11 = 77$, $7 \cdot 17 = 119$, $11 \cdot 17 = 187$, $7 \cdot 11 \cdot 17 = 1309$)

C) Since $12 = 3 \cdot 4$, a number divisible by 12 is divisible by 3 and 4 and vice versa.
Divisibility Rules
÷ by 3: check the sum of the digits – it must be divisible by 3
÷ by 4: check the number formed by the rightmost 2 digits – it must be divisible by 4

The digit sum $2(A + B)$ must be divisible by 3 → $(A + B)$ must be divisible by 3
Thus, for some integer k, $A + B = 3k$ or $B = 3k - A$

The positive two-digit number $10B + A = 10(3k - A) + A = 30k - 9A = 3(10k - 3A)$ must be a multiple of 4, so $(10k - 3A)$ must be a multiple of 4 and $k \geq 1$.
Remember A and B denote digits in base 10 and, therefore, are restricted to 0, 1, …, 9.
$k = 1$ → $B = 3 - A$ and $10 - 3A = 4j$ → $A = 2, B = 1$ → **2112**
$k = 2$ → $B = 6 - A$ and $20 - 3A = 4j$ → $A = 4, B = 2$ → **4224**
$k = 3$ → $B = 9 - A$ and $30 - 3A = 4j$ → $A = 2$ or 6 and $B = 7$ or 3 → **2772** or **6336**
$k = 4$ → $B = 12 - A$ and $40 - 3A = 4j$ → $A = 4$ or 8 and $B = 8$ or 4 → **4884** or **8448**
$k = 5$ → $B = 15 - A$ and $50 - 3A = 4j$ → $A = 2$ or 6
and only $A = 6$ produces a legal value for B → **6996**
$k = 6$ → $B = 18 - A$ and $60 - 3A = 4j$ → $A = 4$ or 8 and neither produces a legal value for B
and the search stops. $2112 + 2772 + 4224 + 4884 + 6336 + 6996 + 8448 = \underline{35772}$.

MML 12/07

A) Since magic integers are 6^{th} powers of positive integers, we have
$1^6 = 1$, $2^6 = 64$, $3^6 = 729$ $4^6 = 2^{12} = 1024(4) = 4096$
$5^6 = 625(25) = \frac{625(100)}{4} = 15615$ $6^6 = 2^6 3^6 = 64(729) = 46656$
$1 + 64 + 729 + 4096 + 15625 + 46656 = \underline{67171}$

Note in the expansion of 5^6, instead of multiplying by 25, a two-digit number, we multiply by 100 and divide by 4, i.e. append two zeros and divide by a 1-digit multiplier, 4.

B) Summing the entries in the first three rows we have: 10, 20 and 40
The sum of the entries is apparently doubling from one row to the next; or, looking from another perspective, is given by the formula $10(2^{row-1})$.
→ $S_{16} = 10(2^{15}) = \underline{327680}$

$2^{10} = 1024 \to 2048 \to 4096 \to 8192 \to 16384 \to 32768$

C) $(12^{x+1}) \cdot (18^{x-1}) \cdot (75^3) = (2^2 \cdot 3)^{x+1} \cdot (2 \cdot 3^2)^{x-1} \cdot (5^2 \cdot 3)^3 = 2^{2x+2} \cdot 3^{x+1} \cdot 2^{x-1} \cdot 3^{2x-2} \cdot 5^6 \cdot 3^3$
$= 2^{3x+1} \cdot 3^{3x+2} \cdot 5^6$
Since the factor must be even, there are $(3x + 1)$ choices for the exponent of 2, namely 1, 2, …, 3x+ 1.
However, since 0 is an allowable exponent for 3 and 5, there are $(3x + 3)$ and 7 choices for the exponents of 3 and 5 respectively.
Thus, the number of even factors is $(3x + 1)(3x + 3)(7) = \underline{3 \cdot 7(x+1)(3x+1)}$ or $\underline{21(x+1)(3x+1)}$