Round 3 Geometry

Polygons: Area and Perimeter

GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 – DECEMBER 1998

ROUND 3 – Geometry: Polygons: Area and Perimeter

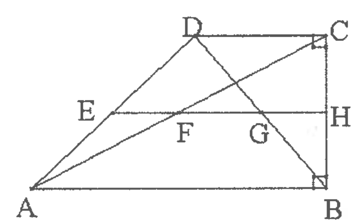
1. ____________________

2. ____________________

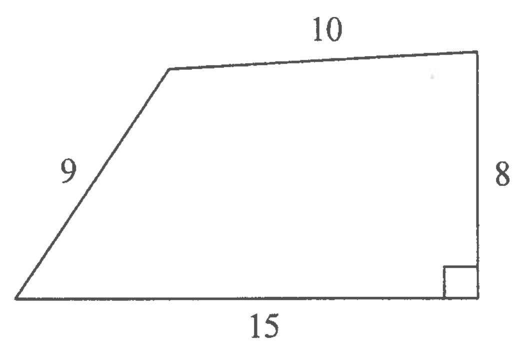
3. ____________________

DIAGRAMS ARE NOT NECESSARILY DRAWN TO SCALE
CALCULATORS ARE NOT ALLOWED ON THIS ROUND

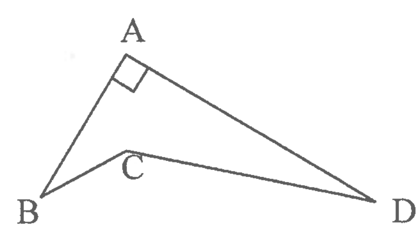
1. Given quadrilateral ABCD with right angles at vertices B and C as indicated on the figure, BC = 8, E and H midpoints of $\overline{AD}$ and $\overline{BC}$ respectively, EF = 2.5, and FG = 3, find the area of quadrilateral ABCD.



2. Find the area of this quadrilateral which has only 1 right angle as indicated on the diagram.



3. Find the area of this quadrilateral ABCD such that AB = 6, AD = $6\sqrt{3}$, BC = $2\sqrt{3}$, $\angle$ A is right, and m $\angle$ B = 30°.



GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 – DECEMBER 1999

ROUND 3 – Geometry: Polygons: Area and Perimeter

1. ____________________

2. ____________________

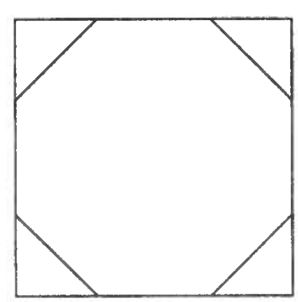
3. ____________________

DIAGRAMS ARE NOT NECESSARILY DRAWN TO SCALE
CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. A regular hexagon has a perimeter of $12\sqrt{3}$ inches. An equilateral triangle is constructed that has a side equal in length to one of the longest diagonals of the hexagon. Find the number of square inches in the area of this equilateral triangle.

2. A triangle has sides of length 8, 15, and 17 centimeters. From a point in the interior of the triangle perpendiculars are drawn to all three sides. If the perpendicular drawn to the longest side is 2 cm., and the perpendicular drawn to the shortest side is 4 cm., find the length in centimeters of the perpendicular drawn to the remaining side.

3. Four congruent right isosceles triangles are sliced off each corner of a square leaving a regular octagon. If the area of the octagon is 4 square units, find the area of the original square.



GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 – DECEMBER 2000

ROUND 3 – Geometry: Polygons: Area and Perimeter

1. ____________________

2. ____________________

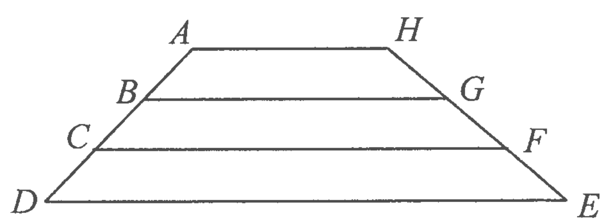
3. ____________________

DIAGRAMS ARE NOT NECESSARILY DRAWN TO SCALE
CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Given the ratio of the lengths of the diagonals of a rhombus is 2:3 and its area is 48 square inches, find the number of inches in the perimeter of this rhombus.

2. Find the number of square centimeters in the area of a right triangle with hypotenuse of length 10 cm and the lengths of its legs are in the ratio of 1:3.

3. Segments $\overline{AH}, \overline{BG}, \overline{CF}$, and $\overline{DE}$ are parallel, $\overline{EFGH}$, with points B and C trisecting $\overline{AD}$. If $AH = 3$ and $DE = 7$, find the ratio of the area of trapezoid $ABGH$ to the area of trapezoid $ADEH$.



GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 – DECEMBER 2001

ROUND 3 – Geometry: Polygons: Area and Perimeter

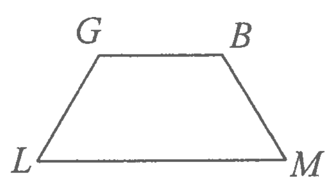
1. ______________________

2. ______________________

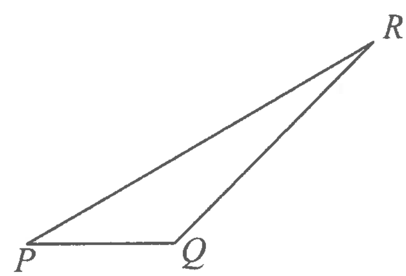
3. ______________________

DIAGRAMS ARE NOT NECESSARILY DRAWN TO SCALE
CALCULATORS ARE NOT ALLOWED ON THIS ROUND

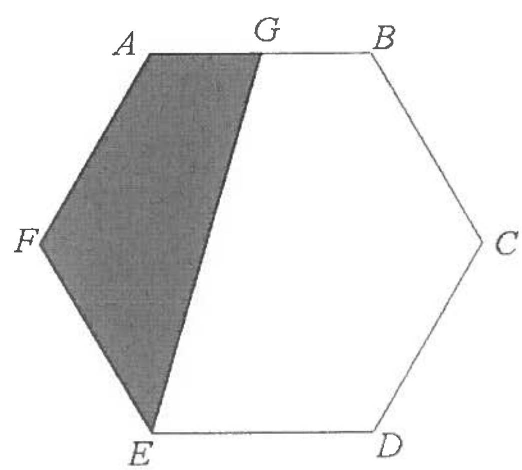
1. Given $LG = GB = BM$, $m\angle G = m\angle B = 120°$, and the area of quadrilateral $GBML = 48\sqrt{3}$, find its perimeter.



2. Given $m\angle P = 30°$, $m\angle Q = 135°$, and $QR = 2$, find the area of ΔPQR.



3. Given $ABCDEF$ is a regular hexagon and G is the midpoint of $\overline{AB}$, find the ratio of the shaded area $AFEG$ to the unshaded area $GBCDE$.



GREATER BOSTON MATHEMATICS LEAGUE
MEET 3 – DECEMBER 2005

ROUND 3 – Geometry: Polygons – Area and Perimeter

1. ____________ units²

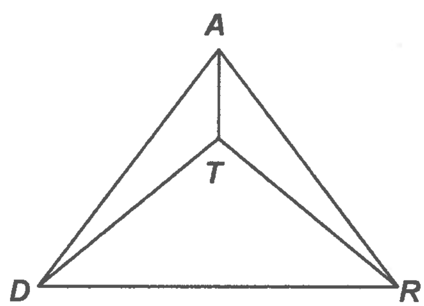
2. ____________ units

3. __________ : ________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND.

1. The perimeter of a regular polygon is 120. If the polygon has at most 6 sides, determine the maximum possible area.

2. In the following diagram $AD = AR = DR = 6$, $DT = TR$ and $\overline{DT} \perp \overline{TR}$. Line $\overleftrightarrow{XY}$ passes through point T parallel to $\overline{DR}$, intersecting sides $\overline{DA}$ and $\overline{AR}$ in M and N respectively. Exactly how long is MN?



3. Given right triangle ACB, with $\angle C$ the right angle. Altitude $\overline{CD}$ is drawn to side $\overline{AB}$. $AC = 6\sqrt{6}$ and $DB = 12\sqrt{2}$. Find the ratio of $AD : BC$ in simplified form.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 – DECEMBER 2006

ROUND 3 – Geometry – Polygons: Area and Perimeter

1. ______________________________

2. ______________________________

3. ______________________________

DIAGRAMS ARE NOT NECESSARILY DRAWN TO SCALE.
CALCULATORS ARE NOT ALLOWED ON THIS ROUND.

1. The measures of consecutive angles of a rhombus are in the ratio of 1 : 2. The length of the smaller diagonal is $4\sqrt{6}$. Find the exact number of square units in area of the rhombus.

2. In isosceles ΔABC, where $AB = AC$ and the vertex angle measures 120°, the numerical value of the perimeter is equal to the numerical value of its area. Find the number of units in the exact length of the altitude drawn from the vertex A to the side $\overline{BC}$.

3. Equilateral triangles ABE and CDF are such that points E and F are in the interior of square $ABCD$. If the area of square $ABCD$ is 16 units^2, determine, as a single fraction in exact form, the number of square units in the area of the rhombus formed by the triangles.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 – DECEMBER 2007

ROUND 3 – Geometry – Polygons: Area and Perimeter

1. ______________________________

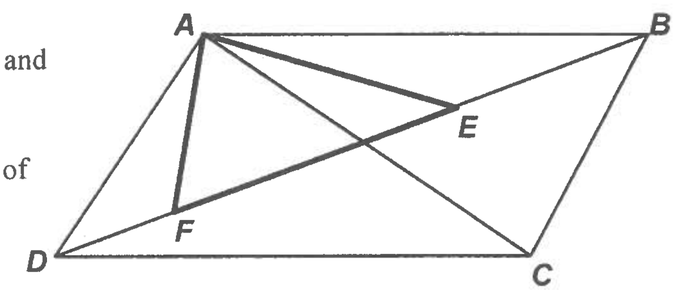
2. ____________ : ____________

3. __________________________ units

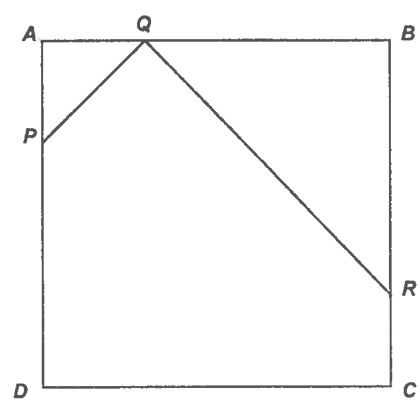
DIAGRAMS ARE NOT NECESSARILY DRAWN TO SCALE.
CALCULATORS ARE NOT ALLOWED ON THIS ROUND.

1. The perimeter of a square is numerically equal to the area of an equilateral triangle whose side is S. If the diagonal of the square equals $4\sqrt{6}$, what is the numerical value of S?

2. In the parallelogram $ABCD$, $DE : EB = 11 : 5$ and $DF : DB = 5 : 24$
Points E and F lie on diagonal $\overline{DB}$.
Find the ratio of the area of ΔAFE to the area of $\square ABCD$.



3. $ABCD$ is a square with side of length 1.
$AP = AQ$ and $BQ = BR$
If the ratio of the area of ΔPAQ to the area of ΔQBR is 1 : 3, determine the exact perimeter of pentagon $PQRCD$.



GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 – DECEMBER 2008

ROUND 3 – Geometry – Polygons: Area and Perimeter

1. ______________________

2. ____________ : ____________

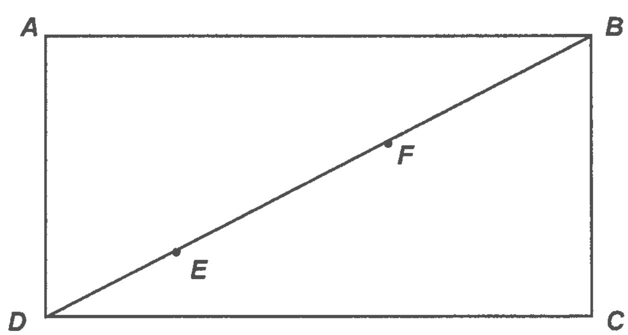
3. ____________ : ____________

DIAGRAMS ARE NOT NECESSARILY DRAWN TO SCALE.
CALCULATORS ARE NOT ALLOWED ON THIS ROUND.

1. The hypotenuse and a leg of a $30° - 60° - 90°$ right triangle have the same lengths as the long and short diagonals in a regular hexagon with side of length s. The ratio of the perimeter of the hexagon to the perimeter of the triangle can be expressed as $A : 1$. Determine a simplified value of A.

2. The ratio of the length of a diagonal of a square to the altitude of an equilateral triangle is $5\sqrt{6} : 4$. What is the ratio of the area of the triangle to the area of the square in simplified form? The denominator of the fraction must be rationalized.

3. A rectangle $ABCD$ has points E and F located on diagonal $\overline{DB}$ such that $\text{Area}(\Delta ADE) : \text{Area}(\Delta CEB) = 1:3$ and $DF : FB = 3 : 2$.
Determine the ratio of the area of ΔECF to the area of rectangle $ABCD$.



GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 – DECEMBER 2009

ROUND 3 – Geometry – Polygons: Area and Perimeter

1. ______________________________

2. ______________________________

3. ______________________________

DIAGRAMS ARE NOT NECESSARILY DRAWN TO SCALE.

1. The sides of a triangle are in the ratio 6 : 10 : 12. The area of the triangle is $18\sqrt{14}$. Find the perimeter of the triangle.

2. The area of a regular hexagon equals the sum of the areas of an equilateral triangle and a square. The length of a side of the hexagon (s) is half the length of a side of the equilateral triangle. Find the area of the square in terms of s.

3. The longer diagonal of a rhombus has the same length as the base of an isosceles triangle. The shorter diagonal of the rhombus has the same length as the legs of the isosceles triangle. In the rhombus, the ratio of the length of the longer diagonal to the shorter diagonal is 4 : 3. Find the ratio of the area of the isosceles triangle to the area of the rhombus.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 – DECEMBER 2010

ROUND 3 – Geometry – Polygons: Area and Perimeter

1. ______________________________

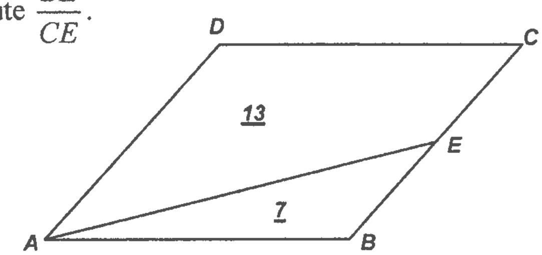
2. ______________________________

3. ______________________________

DIAGRAMS ARE NOT NECESSARILY DRAWN TO SCALE.
CALCULATORS ARE NOT ALLOWED ON THIS ROUND.

1. Three identical equilateral triangles are cut from the corners of a larger equilateral triangle whose perimeter is 24 units, thus reducing the area of the original equilateral triangle by 25%. Compute the length of a side of one of the smaller equilateral triangles.

2. Point E lies on side $\overline{BC}$ of rhombus $ABCD$. The rhombus is subdivided into two regions by segment $\overline{AE}$ whose areas are in the ratio of 13 : 7. Compute $\dfrac{BE}{CE}$.



3. An isosceles trapezoid has an area of 96 square inches. The length of its altitude is $6\sqrt{2}$ inches and the length of its upper base is $5\sqrt{2}$ inches. For a certain rhombus, one of its diagonals has the same length as the height of the trapezoid and the other diagonal has the same length as the diagonal of the trapezoid.
Compute the perimeter of the rhombus (in inches).

MASSACHUSETTS MATHEMATICS LEAGUE
NOVEMBER 2003
ROUND 3: GEOMETRY AREAS

ANSWERS

A)______________

B)______________

C)______________

A) The shorter diagonal of a rhombus is equal to the diagonal of a square, while the longer diagonal is equal to twice the side of the same square. Calculate in simplified radical form the ratio of the area of the rhombus to the area of the square.

B) A regular hexagon and a square share a common side. Calculate the ratio in simple radical form of the area of the hexagon to the area of the square.

C) In regular hexagon ABCDEF, the area of triangle ABD is 9. Calculate the area of the hexagon.

MASSACHUSETTS MATHEMATICS LEAGUE
NOVEMBER 2005
ROUND 3 GEOMETRY: AREA

ANSWERS

A)______________

B)______________

C)______________

A) Square ABCD has area 400. Its diagonals intersect at E. Find the exact perimeter of triangle ABE.

B) Rectangle SROH has SR = 40 and SH = 30. Point E is on $\overline{RH}$ so that $\overline{SE} \perp \overline{RH}$ Find the area of concave pentagon SHORE.

C) The area of a kite is 168. The shorter diagonal is the axis of symmetry; the other diagonal has length 24. If the kite has integral sides, find its perimeter.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2006
ROUND 3 PLANE GEOMETRY: AREAS OF RECTILINEAR FIGURES

ANSWERS

A) ______________________

B) __________ : __________

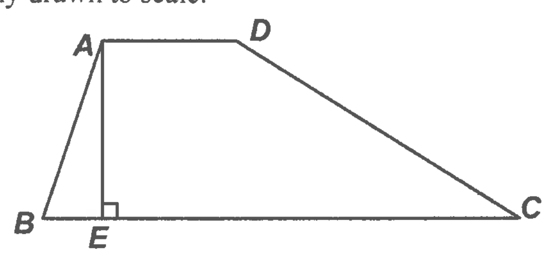
C) ______________________

A) Find the exact area of trapezoid $ABCD$, with bases $\overline{AD}$ and $\overline{BC}$, given:

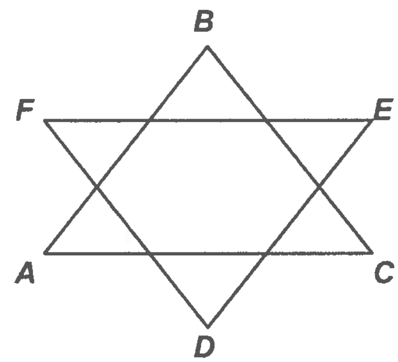
$AB = 25$, $BC = DC = 40$, $AE = 24$

$AD < BC$ and E is between B and C

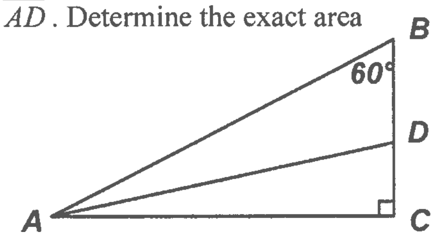
The diagram is not necessarily drawn to scale.



B) A six-pointed star is formed by taking equilateral ΔABC, flipping it over a horizontal line to form ΔDEF, and placing it on top of the ΔABC so that all of its sides are trisected by the intersection points. Express (in simplest form) the ratio of the area of the entire star to the area of the original ΔABC.



C) The area of ΔABC is 6 units^2. The 30° angle is bisected by $\overline{AD}$. Determine the exact area of ΔADC.



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2007
ROUND 3 PLANE GEOMETRY: AREAS OF RECTILINEAR FIGURES

ANSWERS

A) ______________________

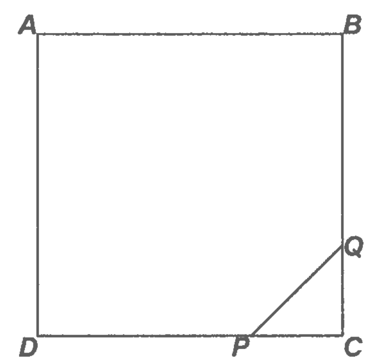
B) ______________________

C) ______________________

A) A right triangle has an area of 60 and its legs have lengths in a 2 : 3 ratio. Compute the length of the hypotenuse.

B) In square $ABCD$, $AB = 6$, $PC = QC$ and $\dfrac{\text{Area}(PQC)}{\text{Area}(PDABQ)} = \dfrac{1}{5}$

P and Q lie on $\overline{DC}$ and $\overline{BC}$ respectively. Compute PQ.



C) Compute the area of the region bounded by $\begin{cases} y = |x-1| + |x-2| + |x-4| \\ x = 0 \\ x = 8 \\ y = 0 \end{cases}$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2009
ROUND 3 PLANE GEOMETRY: AREAS OF RECTILINEAR FIGURES

******* NO CALCULATORS IN THIS ROUND *******

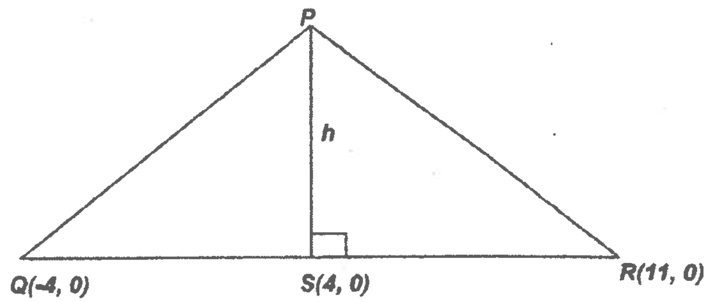
ANSWERS

A) ___________________________ units^2

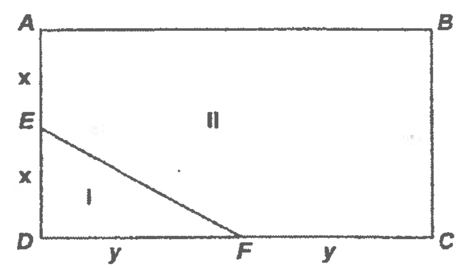
B) ___________________________ units^2

C) ___________________________ units

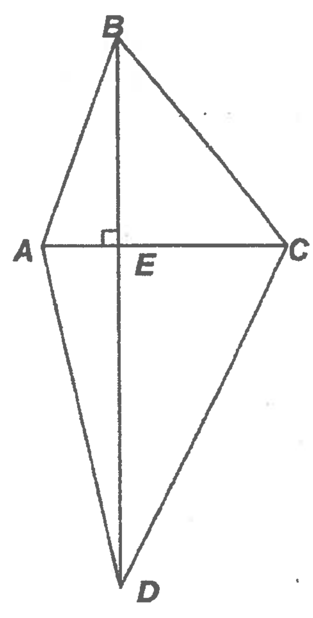
A) The area of ΔPQR is 45 square units.
The areas of ΔPQS and ΔPSR are unequal.
Determine the smaller of the two areas.



B) Rectangle $ABCD$ has an area of 500 square units.
E and F are midpoints of two adjacent sides.
Determine the area of the larger of the two regions inside $ABCD$ created by $\overline{EF}$.



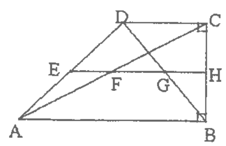
C) Given: quadrilateral $ABCD$ with perpendicular diagonals and
$AB = 13$, $BC = 15$, $BD = 52$, $AC = 14$
To the nearest integer, what is the perimeter of ΔADE?



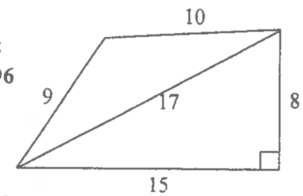
GBML '98

ROUND 3

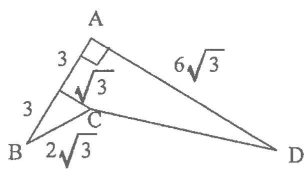
1. ABCD is a trapezoid $\Rightarrow$ Area = $m\,h$, where m is the median and h is its height, EF = $\frac{1}{2}$DC = GH $\Rightarrow$ EH = 2.5 + 3 + 2.5 = 8 = m $\Rightarrow$ Area = **64**



2. The area of the right triangle = $\left(\frac{1}{2}\right)(15)(8) = 60$
Use Hero's formula to find the other triangle's area:
$s = 18 : \sqrt{18 \cdot 1 \cdot 8 \cdot 9} = 36 \Rightarrow$ Area = 60 + 36 = **96**



3. Draw a perpendicular from C to $\overline{AB}$. The lengths of the sides of the 30-60-90° Δ are indicated on the diagram. The quadrilateral is now divided into a trapezoid and a triangle. Area of triangle = $\frac{3\sqrt{3}}{2}$; Area of trapezoid = $\frac{21\sqrt{3}}{2} \Rightarrow$ Total area = $\mathbf{12\sqrt{3}}$

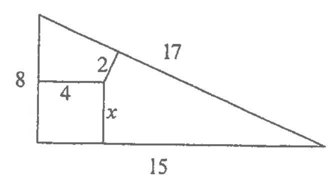


GBML '99

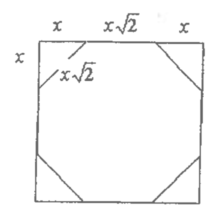
ROUND 3

1. The side of the regular hexagon is $2\sqrt{3} \Rightarrow$ longest diagonal is $4\sqrt{3} \Rightarrow$ area of the equilateral triangle = $\dfrac{(4\sqrt{3})^2\sqrt{3}}{4} = 12\sqrt{3}$

2. The triangle is right. $\Rightarrow$ its area = $\frac{1}{2}\cdot 8 \cdot 15 = 60$;
$60 = \frac{1}{2}2\cdot 17 + \frac{1}{2}4\cdot 8 + \frac{1}{2}15x \Rightarrow x = 3.6\left(\frac{18}{5} \text{ or } 3\frac{3}{5}\right)$



3. The side of the square = $2x + x\sqrt{2} = x\left(2+\sqrt{2}\right)$;
The area of the square = $x^2\left(2+\sqrt{2}\right)^2 = x^2\left(6+4\sqrt{2}\right)$;
The area of the 4 rt. iso. Δ's = $2x^2$; therefore the area of octagon = $x^2\left(6+4\sqrt{2}\right) - 2x^2 = x^2\left(4+4\sqrt{2}\right) = 4 \Rightarrow$
$x^2 = \dfrac{4}{4+4\sqrt{2}} = \dfrac{1}{1+\sqrt{2}} = \sqrt{2}-1 \Rightarrow$ area of the square = $\left(\sqrt{2}-1\right)\left(4\sqrt{2}+6\right) = 2+2\sqrt{2}$

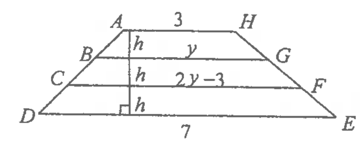


ROUND 3 GBML '00

1. Let the diagonal be 2x and 3x inches long. $\frac{1}{2}(2x)(3x) = 48 \to 3x^2 = 48 \to x = 4$;
the length of one side = $\sqrt{4^2+6^2} = 2\sqrt{13} \to$ Perimeter = $8\sqrt{13}$ inches.

2. $x^2 + (3x)^2 = 10^2 \to 10x^2 = 100 \to x = \sqrt{10} \to$ area = $\frac{1}{2}\left(\sqrt{10}\right)\left(3\sqrt{10}\right) = 15\text{cm}^2$

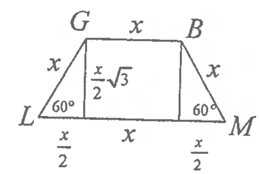
3. Because of the median property of trapezoids, if $BG = y$, $CF = 2y-3$ and $7 + y = 4y - 6 \to y = \frac{13}{3}$.
The ratio of areas = $\dfrac{\frac{1}{2}\left(\frac{22}{3}\right)(h)}{\frac{1}{2}(10)(3h)} = 11:45$



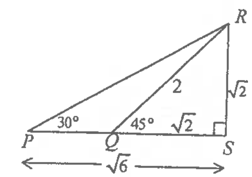
GBML '01

ROUND 3 – Geometry: Polygons: Area and Perimeter

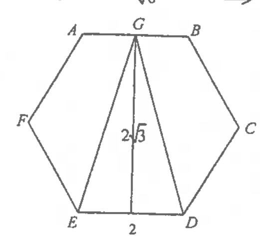
1. Let $LG = x \Rightarrow$ perimeter of $GBML$ = 5x. The area of $GBML = x\cdot\frac{x}{2}\sqrt{3} + 2\cdot\frac{1}{2}\cdot\frac{x}{2}\cdot\frac{x}{2}\sqrt{3} = \frac{3}{4}x^2\sqrt{3} = 48\sqrt{3} \Rightarrow$
$x^2 = 64 \Rightarrow x = 8 \Rightarrow 5x = 40$.



2. Extend $\overline{PQ}$ until it meets the altitude $\overline{RS}$. Since $RQ = 2$ $\Rightarrow RS = QS = \sqrt{2} \Rightarrow PS = \sqrt{6} \Rightarrow PQ = \sqrt{6}-\sqrt{2}$. The area of $\Delta PQR = \frac{1}{2}\left(\sqrt{6}-\sqrt{2}\right)\sqrt{2} = \frac{2\sqrt{3}-2}{2} = \sqrt{3}-1$.



3. Let the side of the hexagon = 2 . Draw $\overline{GD}$ and a perpendicular from G to $\overline{DE}$. The length of this perpendicular = $2\sqrt{3}$ (same length as $\overline{AE}$) $\Rightarrow$ area of ΔDEG = $2\sqrt{3}$. The area of the hexagon = $6\cdot\frac{2^2\sqrt{3}}{4} = 6\sqrt{3}$. Since $AGEF \cong BGDC \Rightarrow$ area of $AGEF = 2\sqrt{3} \Rightarrow$ ratio of area of $AGEF$: area of $GBCDE$ = 1:2.

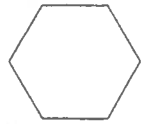


GBML '05

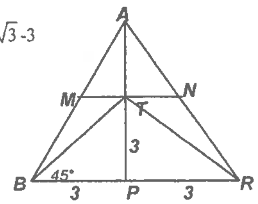
ROUND 3 Geometry: Polygons – Area and Perimeter

1. Maximum area occurs for the 6-sided polygon.
With a fixed perimeter the circle bounds the largest possible area.
The more sides a polygon has the harder it is to distinguish the polygon from the circle.
Per = 120 → side = 30

A_{hex} = 6 equilateral triangles = $6 \cdot \frac{20^2}{4}\sqrt{3} = \underline{600\sqrt{3}}$



2. ΔBAP = 30-60-90 → $AP=3\sqrt{3}$ → $AT=3\sqrt{3}-3$

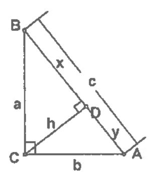


$\Delta MAT \sim \Delta BAP$

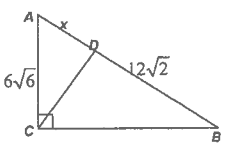
$\frac{3\sqrt{3}-3}{3\sqrt{3}} = \frac{MT}{3} \rightarrow MT = 3 - \frac{3}{\sqrt{3}} = 3-\sqrt{3}$

Therefore, $MN = \underline{6-2\sqrt{3}}$.

3. Three important identities whenever you have an altitude drawn to the hypotenuse.
Know them!
Leg² = Adjacent segment · Hypotenuse or
Altitude² = Segment₁ · Segment₂



$a^2 = xc$
$b^2 = yc$
$h^2 = xy$



$(6\sqrt{6})^2 = 216 = x(x+12\sqrt{2}) \rightarrow x^2 + 12\sqrt{2}x - 216 = 0$

Using the Q.F. carefully, $x = 6\sqrt{2}$.

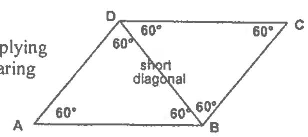
Now use the Pythagorean theorem on ΔADC, $CD = 12$.
Finally, using the Pythagorean theorem on ΔBDC, $BC = 12\sqrt{3}$

Thus, the ratio $AD : BC = 6\sqrt{2} : 12\sqrt{3} = \underline{\sqrt{6}:6}$

GBML '06

ROUND 3

1. Let the measures of consecutive angles be x and $2x$.
Then $6x = 360 \rightarrow x = 60$ and the angles are 60° and 120°, implying that the rhombus is comprised of two equilateral triangles sharing the shorter diagonal as a common side.

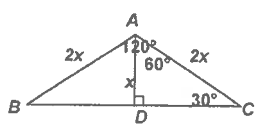


Thus, the area is given by $2 \cdot \frac{s^2\sqrt{3}}{4} \rightarrow \frac{(4\sqrt{6})^2\sqrt{3}}{2} = \underline{48\sqrt{3}}$

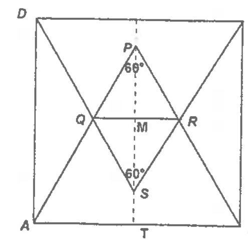
2. Clearly ΔADC is a 30-60-90 triangle and $BC = 2x\sqrt{3}$.

"Perimeter = Area" → $4x + 2x\sqrt{3} = \frac{1}{2} \cdot 2x\sqrt{3} \cdot x = x^2\sqrt{3}$

Since $x \neq 0$, → $x = \frac{4+2\sqrt{3}}{\sqrt{3}} = \underline{2 + \frac{4\sqrt{3}}{3}}$



3. $AB = 4 \rightarrow PT = 2\sqrt{3} \rightarrow PM = 2\sqrt{3} - 2 = 2(\sqrt{3} - 1)$
Since $\overline{PM}$ is an altitude in equilateral triangle PQR,



$PQ = \frac{4(\sqrt{3}-1)}{\sqrt{3}} \rightarrow$ A(rhombus) = $\frac{\left(\frac{4(\sqrt{3}-1)}{\sqrt{3}}\right)^2\sqrt{3}}{2}$

$= \frac{16(4-2\sqrt{3})\sqrt{3}}{6} = \boxed{\frac{8(4\sqrt{3}-6)}{3}}$ or $\boxed{\frac{32\sqrt{3}-48}{3}}$

An alternate method might use $\frac{1}{2}d_1 \cdot d_2 \rightarrow \frac{1}{2}(PS)(QR)$, where PS

$= 2PM = 4(\sqrt{3} - 1)$ and $QR = PQ = \frac{4(\sqrt{3}-1)}{\sqrt{3}}$

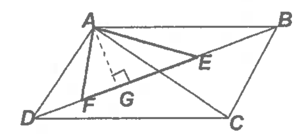
GBML '07

ROUND 3

1. Let x denote the length of a side of the square. Then:

$d = x\sqrt{2} = 4\sqrt{6} \rightarrow x = 4\sqrt{3}$ and Per(□) = $16\sqrt{3} = \frac{S^2\sqrt{3}}{4}$

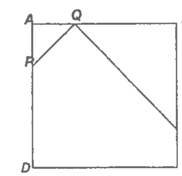
$\rightarrow S^2 = 64 \rightarrow S = \underline{8}$



2. $DE : EB = 11 : 5$ ∴ Let $DE = 11a$ and $EB = 5a$
$DF : DB = 5 : 24 \rightarrow DF : FB = 5 : 19$ ∴ Let $DF = 5b$ and $FB = 19b$
Diagonal $BD = DF + FB = DE + EB \rightarrow 16a = 24b \rightarrow b = 2a/3$, $FE = 11a - 5b = 23a/3$

$\frac{\text{Area}(\Delta AFE)}{\text{Area}(\square ABCD)} = \frac{\frac{1}{2}(AG)(FE)}{2 \cdot \frac{1}{2}(AG)(BD)} = \frac{FE}{2(BD)} = \frac{23a/3}{2(16a)} \rightarrow \underline{23 : 96}$

3. Let $AP = x$. Then:
Perimeter($PQRCD$) = $PQ + QR + RC + CD + DP$
$= x\sqrt{2} + (1-x)\sqrt{2} + x + 1 + (1-x) = \underline{2+\sqrt{2}}$
Thus, regardless of the value of x, the perimeter of $PQRCD$ is invariant.



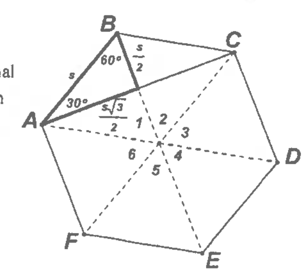
$\frac{\frac{1}{2}x^2}{\frac{1}{2}(1-x)^2} = \frac{x^2}{(1-x)^2} = \frac{1}{3} \rightarrow 3x^2 = 1 - 2x + x^2 \rightarrow 2x^2 + 2x - 1 = 0 \rightarrow x =$

$\frac{-2 \pm \sqrt{4+8}}{4}$ and $x > 0 \rightarrow \frac{\sqrt{3}-1}{2}$ Thus, x exists satisfying the ratio requirement, but it was unnecessary to determine its specific value. If Q is the midpoint of $\overline{AB}$, then $\Delta APQ \cong \Delta BRQ$ and the ratio of the areas would be 1 : 1. As Q moves closer to A, the ratio becomes arbitrarily small and, conversely, as Q moves closer to B, the ratio becomes arbitrarily large.

GBML '08

ROUND 3

1. From the diagram at the right, it's clear that a short diagonal ($\overline{AC}$) in a regular hexagon with side of length s has length $s\sqrt{3}$ and a long diagonal ($\overline{AD}$) has length $2s$. The sides of the triangle are s, $s\sqrt{3}$ and $2s$. Thus, the ratio of the required perimeters is



$\frac{\text{Per}(hex)}{\text{Per}(\Delta)} = \frac{6s}{(3+\sqrt{3})s} = \frac{6}{(3+\sqrt{3})} \cdot \frac{3-\sqrt{3}}{3-\sqrt{3}} = \frac{6(3-\sqrt{3})}{9-3}$

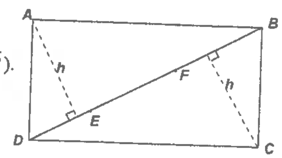
$\rightarrow A = \underline{3-\sqrt{3}}$

2. If x and $2y$ denote the sides of the square and the equilateral triangle respectively, then the diagonal of the square will have length $x\sqrt{2}$ and the altitude of the equilateral triangle will have length $y\sqrt{3}$. Therefore, $\frac{x\sqrt{2}}{y\sqrt{3}} = \frac{5\sqrt{6}}{4} \rightarrow \frac{x\sqrt{6}}{3y} = \frac{5\sqrt{6}}{4}$

$\rightarrow \frac{y}{x} = \frac{4}{15}$ and the required ratio of the areas is

$$\frac{A(\Delta)}{A(\square)} = \frac{\frac{1}{2}\cdot 2y\cdot y\sqrt{3}}{x^2} = \frac{y^2\sqrt{3}}{x^2} = \left(\frac{y}{x}\right)^2\sqrt{3} = \underline{\frac{16\sqrt{3}}{225}}$$

3. Let h denote the length of the altitude from A to $\overline{DB}$ (which is also the length of the altitude from C to $\overline{DB}$). The area of the rectangle $ABCD$ is twice the area of ΔADB.



area(ΔADE) : area$(\Delta CEB) = 1:3 \rightarrow \frac{1}{2}\cdot DE\cdot h : \frac{1}{2}\cdot BE\cdot h = 1:3 \rightarrow \frac{DE}{BE} = \frac{1}{3}$

$\rightarrow BD = DE + EB = x + 3x = 4x.$

$DF : FB = 3:2 \rightarrow DB = DF + FB = 3y + 2y = 5y$ Thus, $4x = 5y$ or $\frac{y}{x} = \frac{4}{5}$

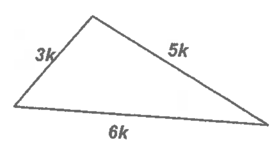
Now, the required ratio $\frac{area(\Delta ECF)}{area(ABCD)} = \frac{\frac{1}{2}EF\cdot h}{2\cdot\frac{1}{2}\cdot h\cdot BD} = \frac{EF}{2BD} = \frac{DF-DE}{2DB} = \frac{3y-x}{2(4x)}$

$= \frac{3}{8}\cdot\frac{y}{x} - \frac{1}{8} = \frac{3}{8}\cdot\frac{4}{5} - \frac{1}{8} = \underline{\frac{7}{40}}$

GBML '09

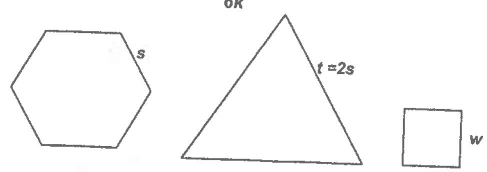
ROUND 3

1. Let the sides of the triangle be $3k$, $5k$ and $6k$.
The semi-perimeter is $7k$. Using Heron's formula,
$A = \sqrt{(7k)(4k)(2k)(k)} = 2\sqrt{14}k^2 = 18\sqrt{14} \rightarrow k = 3$
$\rightarrow$ sides are 9, 15, and 18 $\rightarrow$ perimeter = $\underline{42}$

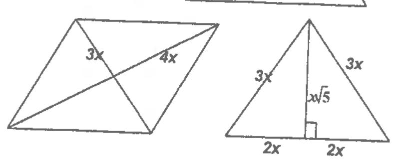


2. $\frac{6s^2\sqrt{3}}{4} = \frac{(2s)^2\sqrt{3}}{4} + w^2$

$\rightarrow 6s^2\sqrt{3} - 4s^2\sqrt{3} = 4w^2 \rightarrow w^2 = \underline{\frac{s^2\sqrt{3}}{2}}$



3. $\frac{\frac{1}{2}(4x)(x\sqrt{5})}{\frac{1}{2}(3x)(4x)} = \frac{2x^2\sqrt{5}}{6x^2} = \underline{\frac{\sqrt{5}}{3}}$



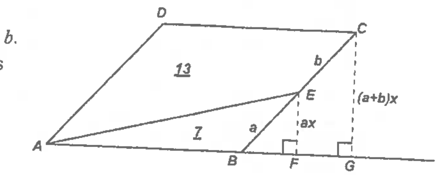
GBML '10

ROUND 3

1. $AB = 8 \rightarrow$ area$(\Delta ABC) = \frac{8^2\sqrt{3}}{4} = 16\sqrt{3} \rightarrow$ reduced area = $12\sqrt{3}$

Subtracting the three corners, $16\sqrt{3} - \frac{3x^2\sqrt{3}}{4} = 12\sqrt{3} \rightarrow 16 - \frac{3x^2}{4} = 12 \rightarrow x^2 = \frac{16}{3} \rightarrow x = \underline{\frac{4\sqrt{3}}{3}}$

2. Let the $BE = a$ and $EC = b$.
Since $ABCD$ is a rhombus, $AB = CD = DA = a + b$.
Extend $\overline{AB}$ thru point B and drop perpendiculars from points C and E.
$\Delta BEF \sim \Delta BCG \rightarrow EF = ax$ and $CG = (a+b)x$.



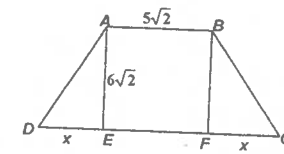
Then:

$$\frac{area(\Delta AEB)}{area(AECD)} = \frac{7}{13} = \frac{\frac{1}{2}(a+b)ax}{(a+b)(a+b)x - \frac{1}{2}(a+b)ax} = \frac{ax(a+b)}{2x(a+b)^2 - ax(a+b)} = \frac{x(a+b)a}{x(a+b)(2(a+b)-a)} = \frac{a}{a+2b}$$

$\rightarrow 13a = 7a + 14b \rightarrow 6a = 14b \rightarrow \frac{a}{b} = \frac{BE}{CE} = \underline{\frac{7}{3}}$

3. Area$(ABCD) = 96 =$
$\frac{1}{2}(6\sqrt{2})(2(5\sqrt{2}) + 2x) = 6\sqrt{2}(5\sqrt{2} + x) = 60 + 6\sqrt{2}x$

$\rightarrow 6\sqrt{2}x = 36 \rightarrow x = \frac{6}{\sqrt{2}} = 3\sqrt{2}$



$\overline{AC}$, a diagonal in $ABCD$ is the hypotenuse in ΔAEC.
$AC^2 = (6\sqrt{2})^2 + (8\sqrt{2})^2 = 72 + 128 = 200 \rightarrow AC = 10\sqrt{2}$

In a rhombus with diagonals d_1 and d_2 and side s, $d_1^2 + d_2^2 = 4s^2$. Given $d_1 = h_{trap}$, $d_2 = d_{trap}$ and substituting, $4s^2 = 36\cdot 2 + 100\cdot 2 = 272 \rightarrow s^2 = 68 \rightarrow s = 2\sqrt{17} \rightarrow$ Per = $\underline{8\sqrt{17}}$

MML 11/03

MASSACHUSETTS MATHEMATICS LEAGUE
NOVEMBER 2003
ROUND 3: GEOMETRY AREAS

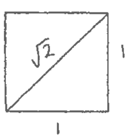
ANSWERS

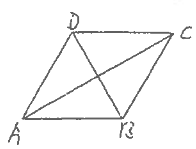
A) $\sqrt{2}:1$

B) $3\sqrt{3}/2$

C) 27

A) The shorter diagonal of a rhombus is equal to the diagonal of a square, while the longer diagonal is equal to twice the side of the same square. Calculate in simplified radical form the ratio of the area of the rhombus to the area of the square.



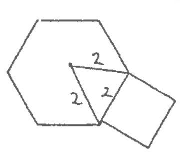


$BD = \sqrt{2}$ Area rhombus $= \frac{1}{2} \cdot 2\sqrt{2} = \sqrt{2}$

$AC = 2$ Area square $= 1$

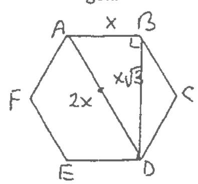
ANS $\sqrt{2}:1$

B) A regular hexagon and a square share a common side. Calculate the ratio in simple radical form of the area of the hexagon to the area of the square.



$\frac{A_H}{A_S} = \frac{6 \cdot \frac{4\sqrt{3}}{4}}{4} = \frac{6\sqrt{3}}{4} = \frac{3\sqrt{3}}{2}$

C) In regular hexagon ABCDEF, the area of triangle ABD is 9. Calculate the area of the hexagon.



$\frac{1}{2}x^2\sqrt{3} = 9, \quad x^2\sqrt{3} = 18, \quad x^2 = 6\sqrt{3}$

$A_H = 6 \cdot \frac{x^2}{4}\sqrt{3} = \frac{6 \cdot 6\sqrt{3}\sqrt{3}}{4} = \frac{36 \cdot 3}{4} = 27$

MML 11/05

Round Three:

A. Side is 20, diagonal is $20\sqrt{2}$.

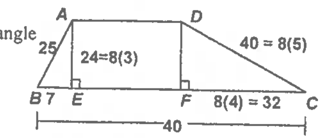
B. $\Delta SER \sim \Delta HSR$ ratio 4:5. Area ΔSER is 16/25 of $\Delta HSR = 384$. Subtract from 1200.

C. Area implies other diagonal is 14. Thus $12^2 + x^2 = a^2$ while $12^2 + (14\text{-}x)^2 = b^2$, Integer solutions suggest 9-12-15 and 5-12-13 triangles (14=9+5)

MML 11/06

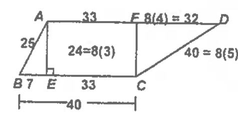
Round 3

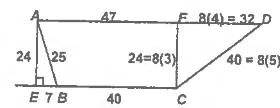
A) Drop perpendiculars from A and D to base $\overline{BC}$, creating a rectangle and two special right triangles as indicated in the diagram. $EF = 40 - (7 + 32) = 1 \rightarrow AD = 1$. Thus, Area(trapezoid) = $\frac{1}{2}h(b_1 + b_2) = \frac{1}{2}(24)[40+1] = \underline{492}$.

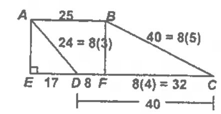


Failing to specify that $AD < BC$ and that $\overline{AD}$ and $\overline{BC}$ are bases, allows additional solutions.

$\frac{1}{2}(24)[40+33+32] = \underline{\mathbf{1260}}$ $\frac{1}{2}(24)[40+47+32] = \underline{\mathbf{1428}}$ $\frac{1}{2}(24)[25+40+17] = \underline{\mathbf{984}}$

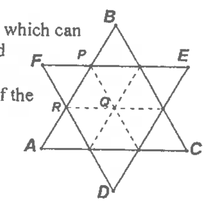






Are there others?

B) Note that the intersection of the two equilateral triangles is a regular hexagon, which can be subdivided into 6 congruent equilateral triangles by drawing the 3 indicated diagonals. It's easy to argue that $FPQR$ is a parallelogram and, therefore, $\Delta FPR \cong \Delta QRP$ and all 12 equilateral triangles are congruent. Thus, the ratio of the area of the entire star to the area of the original ΔABC is $12 : 9 = \underline{\mathbf{4:3}}$

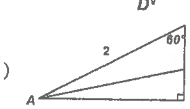


C) Triangles ADC and ADB have the same altitude from point A and, therefore, their areas are in the ratio of bases DC and DB.

By the angle bisector theorem, $\frac{DC}{\sqrt{3}} = \frac{DB}{2} \rightarrow \frac{DC}{DB} = \frac{\sqrt{3}}{2}$

Area(ΔADC) + Area(ΔADB) $= \sqrt{3}x + 2x = 6 \rightarrow x = \frac{6}{2+\sqrt{3}} = 6(2-\sqrt{3})$

and Area(ΔADC) $= \sqrt{3}x = \underline{\mathbf{12\sqrt{3} - 18}}$ or $\underline{\mathbf{6(2\sqrt{3} - 3)}}$



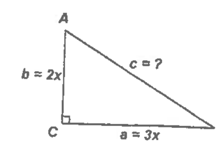
MML 11/07

Round 3

A) Area $= \frac{1}{2}bh = \frac{1}{2} \cdot 2x \cdot 3x = 60 \rightarrow x = 2\sqrt{5}$

and $(2x)^2 + (3x)^2 = c^2$

$\rightarrow$ hypotenuse $c = x\sqrt{13} = 2\sqrt{5} \cdot \sqrt{13} = \underline{\mathbf{2\sqrt{65}}}$



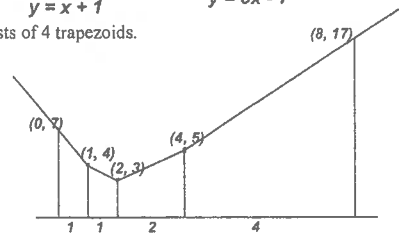
B) Let $PC = QC = x$. Then $PQ = x\sqrt{2}$ and

$\frac{1}{2}x^2 : (36 - \frac{1}{2}x^2) = 1:5 \rightarrow \frac{x^2}{72 - x^2} = \frac{1}{5} \rightarrow 6x^2 = 72$

$\rightarrow x = 2\sqrt{3}$ and $PQ = \underline{\mathbf{2\sqrt{6}}}$

C) The critical points occur at $x = 1$, 2 and 4.
The first equation may be expressed without absolute value over restricted domains as follows:

Thus, the region bounded by this system consists of 4 trapezoids.

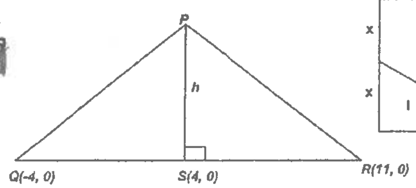


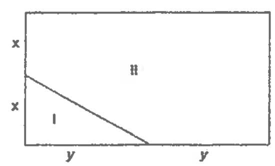
$$A = \frac{1}{2}(1(4+7)+1(4+3)+2(3+5)+4(5+17))$$

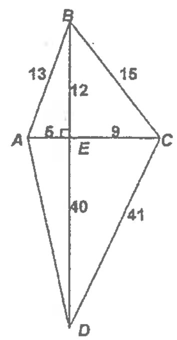
$$= \frac{1}{2}(11+7+16+88) = \frac{122}{2} = \underline{61}$$

MML
11/09

Round 3







A) $QR = 15$, $QS = 8$ and $SR = 7$
$\frac{1}{2}(15)h = 45 \rightarrow h = 6$

ΔPSR has the smaller area, $\frac{1}{2}\cdot 7\cdot 6 = \underline{21}$.

B) Area(I) $= \frac{1}{2}xy$, Area(II) $= 4xy - \frac{1}{2}xy = \frac{7}{2}xy$
Thus, regardless of the dimensions of the rectangle, region II has an area 7/8 that of the rectangle $\rightarrow \frac{7}{8}(500) = \underline{437.5}$ or $\underline{\left(\frac{875}{2}\right)}$.

C) Noting special right triangles 5 - 12 - 13, 3(3 - 4 - 5) and 9 – 40 - 41, the problem is almost done.
$AD = \sqrt{1625} = 5\sqrt{65}$
65 is only slightly bigger than the perfect square 64.
$8.1^2 = 65.61 \rightarrow \sqrt{65} < 8.1 \rightarrow 5\sqrt{65} < 40.5$
Thus, to the nearest integer, the perimeter of ΔADE is $\underline{85}$.