Round 4 Algebra 2

Quadratic Equations and problems involving them, Theory of Quadratics

GREATER BOSTON MATHEMATICS LEAGUE

MEET 2 – NOVEMBER 1998

ROUND 4 – Algebra 2– Quadratic Equations, Problems Involving Them, Theory of Quadratics

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Find all solutions for real x for the following equation: $\dfrac{3}{x-3} - \dfrac{x^2+3}{4x-12} = \dfrac{x-2}{4}$

2. Given a quadratic equation in standard form with leading coefficient equaling 1, with roots 2 and r, and the value of its discriminant is 25, find all solutions for r.

3. Kaitlin's one hundred mile road trip by bicycle included exactly twenty-five miles uphill, fifteen miles downhill, and the rest on level ground. If her speed downhill was three times her speed uphill and her speed on level ground was eight miles per hour faster than her speed uphill, find her speed uphill in miles per hour if her travelling time totaled five hours and thirty minutes.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 2 – NOVEMBER 1999

ROUND 4 – Algebra 2– Quadratic Equations, Problems Involving Them, Theory of Quadratics

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Find all real solutions for x for the following equation: $\dfrac{2}{x+6}+\dfrac{12}{x^2+6x}=3$

2. Given the quadratic equation, $4x^2+kx+7=0$, which has two positive roots whose difference is 3, solve for k.

3. A motor boat travels 27 miles downstream helped by a current of 9 mph. A shorter water route of 21 miles is then found and the boat moves upstream against a 1 mph current. If the entire trip took 6 hours, and the boat maintains a constant speed, find this speed in miles per hour.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 2 – NOVEMBER 2000

ROUND 4 – Algebra 2– Quadratic Equations, Problems Involving Them, Theory of Quadratics

1. ____________________

2. ____________________

3. (,)

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Kaitlin can paint a house in 6 hours less time than Abbe. Kaitlin paints the house alone for 6 hours then stops. Now Abbe comes in and finishes painting the house in 16 hours. How many hours does it take Kaitlin to paint the entire house working alone?

2. The following quadratic equation in x has the property that the product of its roots is 8 more than the sum of its roots. Find all possible values for k.

   $kx^2 + k^2x - x + 8 = 0$

3. Given the quadratic equations, $x^2 - 3x - 5 = 0$ and $x^2 + bx + c = 0$, such that the second equation has solutions which are the squares of the solutions to the first equation, find the ordered pair (b, c).

GREATER BOSTON MATHEMATICS LEAGUE

MEET 2 – NOVEMBER 2001

ROUND 4 – Algebra 2– Quadratic Equations, Problems Involving Them, Theory of Quadratics

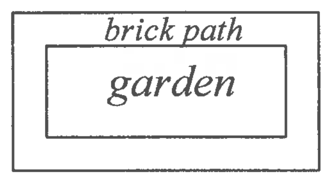
1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Mr. Green buys fencing to enclose a garden 4 times longer than it is wide. After fencing in the garden, he builds a brick path 2 feet wide around the garden. (See the figure below.) If the total area of the garden and brick path is 280 square feet, find how many feet of fencing Mr. Green bought.



2. Solve the following equation for x:

$$\frac{x}{3x-6}-\frac{2}{2x+10}=\frac{7}{x^2+3x-10}$$

3. Given the quadratic equation $x^2+bx+c=0$ has real roots whose difference is 3. If $\frac{c}{b}=-\frac{20}{3}$, find all possible values for the smaller of the two roots.

GREATER BOSTON MATHEMATICS LEAGUE
MEET 2 – NOVEMBER 2006

ROUND 4 – Algebra 2 – Quadratic Equations, Problems involving Them and Theory of Quadratics

1. ______________________

2. ______________________ (20)

3. ______________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND.

1. If $x = 2\sqrt{3}$ is a root of the equation $x^2 - k^2 - k = 0$, find all possible value(s) of k.

2. Given: $344_{(c)} = 163_{(c+5)}$, where $c > 0$.
   Find c^2 in base 20.

3. Given: $3x^2 + 4(x - 2) = 6mx - 2m^2$

   For what value(s) of m is the ratio of the product of the roots to the sum of the roots 5 : 7?

GREATER BOSTON MATHEMATICS LEAGUE

MEET 2 – NOVEMBER 2007

ROUND 4 – Algebra 2 – Quadratic Equations, Problems involving Them and Theory of Quadratics

1. ______________________________

2. ______________________________

3. ______________________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND.

1. If $243_C = 322_7$ and $C > 0$, determine the value of C.

2. One root of a quadratic equation is 3 times the other and their product is 75. If the equation is written in the form $ax^2 + bx + c = 0$, where $a > 0$ and the GCF of a, b, and c is 1, determine all possible ordered triples (a,b,c).

3. Let P = the larger root of $x^2 - 5x - 2A = 0$ and Q = the larger root of $x^2 - 8x + A = 0$. Find all possible values of A for which $P = Q$.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 2 – NOVEMBER 2008

ROUND 4 – Algebra 2 – Quadratic Equations, Problems involving Them and Theory of Quadratics

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND.

1. Find the absolute value of the difference of the roots of the following equation:

$$\frac{4x+1}{2x-1}+\frac{11x+3}{x-2x^2}=\frac{x-1}{x}$$

2. Find <u>all</u> values of k such that the square of the sum of the roots will be equal to the product of the roots in the equation $3x^2+(k+2)x+3k=0$.

3. The sum of the roots of the equation $kx^2-1-4k-6x=0$ multiplied by the reciprocal of the product of the roots equals $-\frac{2}{3}$. Find <u>all</u> possible values of k.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 – NOVEMBER 2009

ROUND 4 – Algebra 2: Quadratic Equations and theory

1. ______________________

2. ______________________

3. $h =$ ______________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND.

1. If the equation, $x^2 + 3x + K = 0$ has a root equal to K, find all possible values of K.

2. One root of the equation $2x^2 + j^2 + 3j = 15x$ is four times the other root. Find all values of j which make this true.

3. Given: $(h+10)x^2 - \left(4h^2 + 9\right)x + 6h = 0$, an equation in x.
   Compute all possible values of h for which the sum of the roots divided by the product of the roots gives a result of $\frac{25}{12}$.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 2 – NOVEMBER 2010

ROUND 4 – Algebra 2 – Quadratic Equations, Problems involving Them/Theory of Quadratics

1. ______________________

2. ______________________

3. ______________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND.

1. The roots of $x^2 - 3x + k = 0$ are a and b. Find the value of $a^2 - ab + b^2$ in terms of k.

2. Compute all ordered pairs (a, b) for which $\begin{matrix} a + 4b^2 = 4 \\ a^2 - 8b^2 = 7 \end{matrix}$.

3. The equation $x^2 - kx + 2k = 2x$ has one root that is 3 greater than the other root. If the two roots are called R_1 and R_2, where $R_1 > R_2$, find all possible ordered triples (R_1, R_2, k) which satisfy the given conditions.

MASSACHUSETTS MATHEMATICS LEAGUE
JANUARY 2004
ROUND 4: QUADRATICS

ANSWERS

A)______________

B)______________

C)______________

A) For what values of k will the equation $2x^2 - kx + 8 = 0$ have two equal real roots?

B) The area of a square piece of tin is 625 sq. in. Squares of equal size are cut out of the two top corners. Larger squares, each four times the area of a top corner square, are cut out of the two bottom corners. Calculate the perimeter of the resulting figure if its area is 535 sq. in.

C) If one root of $ax^2 + bx + c = 0$ is x = -2, b + c = 0, and a + b = 7; find the value of b.

MASSACHUSETTS MATHEMATICS LEAGUE
JANUARY 2005
ROUND 4 ALGEBRA TWO: QUADRATIC EQUATIONS

ANSWERS

A)______________

B)(____,_____,____)

C)______________

A) Find all real x for which $x^2\sqrt{6} - 4x - 2\sqrt{6} = 0$

B) For integers a, b, and c the solutions for $ax^2 = 2x + b$ are the same as those for $(x + 0.5)(cx - 3) = 0$. Find the ordered triple (a, b, c).

C) If $N = 1 - \cfrac{1}{4 - \cfrac{1}{4 - \cfrac{1}{4 - \cdots}}}$ then N may be expressed in the form $A + \sqrt{B}$ Find A+B.

MASSACHUSETTS MATHEMATICS LEAGUE
JANUARY 2006
ROUND 4 ALGEBRA TWO: QUADRATIC EQUATIONS

ANSWERS

A)______________

B)(____,____,____)

C)______________

A) Find both real solutions for x: $x^2 + (x+1)^2 = (x+2)^2$

B) Find all real values of z if $2z^2 + yz + 3 = 0$ and $2y = y^2 - 35$

C) Shalomar owns several clothing stores. If she sells her sweaters at a price of \$100, her stores average 102 sales per month. Shalomar finds that for every \$5.00 she drops her price her stores sell on average 3 additional sweaters; each \$5 increase in price loses her three sales, however. What price will provide the greatest income for Shalomar?

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 – JANUARY 2007
ROUND 4 ALG 2: QUADRATIC EQUATIONS

ANSWERS

A) (_______ , _______)

B) a b c d e f

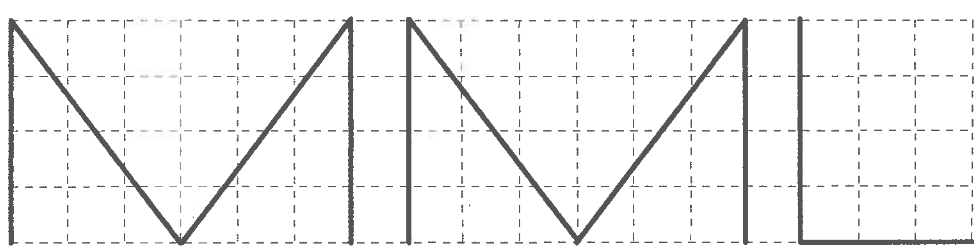
C) ________________________ oz.

A) The vertex of the parabola $y = (x - 1)(x - a) + b$ is located at (3, 7).
Determine the ordered pair (a, b).

B) Let a, b, and c, in some order, denote three consecutive positive integers.
Indicate which of the following orders of (a, b, c) guarantees that the sum of the roots of $ax^2 - \underline{bx} + c = 0$ is as <u>large</u> as possible.
(S, M and L denote the smallest, the middle and the largest integer.)

a) (S, M, L) b) (S, L, M) c) (M, S, L) d) (M, L, S) e) (L, M, S) f) (L, S, M)

C) 24 ounces of copper are drawn into a wire of uniform cross section. If the wire had been one foot longer (and still of the same uniform cross section), the wire would have weighed 0.1 oz. less per foot.
This wire is used to make 4" high letters, as shown below.
The grid squares are 1 inch on side.
If no extra wire is used to make the bends, find the weight of the wire used.
Give an exact answer or an approximation accurate to the nearest 0.01 ounce.



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 – JANUARY 2008
ROUND 4 ALG 2: QUADRATIC EQUATIONS

ANSWERS

A) Equation: ______________________________

B) ______________________________

C) ______________________________

A) Find a quadratic equation of the form $x^2 + Bx + C = 0$, where B and C are integers, given that $2 - i\sqrt{5}$ is one of its roots.

B) The sum of the squares of two positive real numbers L and W is 81. Twice the larger number is 9 more than the smaller number. Determine $|L - W|$.

C) $x^2 + Ax + B = 0$ and $x^2 + px + q = 0$ are different equations.
Each of the roots of the equation $x^2 + Ax + B = 0$ are 3 more than twice the corresponding roots of $x^2 + px + q = 0$. If $A : B = -2 : 3$, compute the ratio of $p : q$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2011
ROUND 4 ALG 2: QUADRATIC EQUATIONS

ANSWERS

A) ______________________________

B) _________________________ inches

C) ______________________________

***** NO CALCULATORS ON THIS ROUND *****

A) Given: $x+\frac{1}{x}=\frac{13}{6}$

Compute the numerical value of $(x+1)\left(\frac{1}{x}+1\right)$.

B) A rectangular piece of cardboard is 13 inches longer than it is wide. If squares whose perimeter is 16 inches are cut from each corner and the resulting figure is folded up to form an open-topped box, the volume will be 1200 cubic inches. Find the <u>sum</u> of the three dimensions of the box.

C) For what value(s) of m are the roots of $x^2 - mx + m + 3 = 0$ nonzero and real?

GBML 98

ROUND 4

1. $\frac{3}{x-3}-\frac{x^2+3}{4x-12}=\frac{x-2}{4} \Rightarrow x\neq 3$ and $12-(x^2+3)=(x-3)(x-2) \Rightarrow$
$x\neq 3$ and $12-x^2-3=x^2-5x+6 \Rightarrow 2x^2-5x-3=0 \Rightarrow (2x+1)(x-3)=0 \Rightarrow x=-\mathbf{\frac{1}{2}}$

2. Since a, the coefficient of x^2, is 1, and the discriminant is 25, the roots of the quadratic equation are $\frac{-b\pm 5}{2}$, where b is the coefficient of x. $\Rightarrow$ Difference of the roots is 5 $\Rightarrow$ the second root is either 2 + 5 or 2 – 5. $\Rightarrow$ The second root is **–3 or 7.**

3.

	speed	distance	time
uphill	x	25	$25/x$
downhill	$3x$	15	$5/x$
level	$x+8$	60	$60/(x+8)$

Equation: $\frac{30}{x}+\frac{60}{x+8}=\frac{11}{2} \Rightarrow 60(x+8)+120x=11x(x+8) \Rightarrow 11x^2-92x-480=0$
$\Rightarrow (11x+40)(x-12)=0 \Rightarrow x=$ **12** mph

GBML 99

1. $\frac{2}{x+6}+\frac{12}{x^2+6x}=3 \Rightarrow 2x+12=3x^2+18x \Rightarrow 3x^2+16x-12=0 \Rightarrow (3x-2)(x+6)=0$
$\Rightarrow x=\frac{2}{3}$ since $x=-6$ is extraneous

2. Call the roots r and s, with $r>s$; $r-s=3$ and $rs=\frac{7}{4} \Rightarrow r(r-3)=\frac{7}{4} \Rightarrow 4r^2-12r-7=0$
$\Rightarrow (2r+1)(2r-7)=0 \Rightarrow r=\frac{7}{2} \Rightarrow s=\frac{1}{2};\ \frac{k}{4}=-(r+s) \Rightarrow k=-16$

3. Time down the first waterway $=\frac{27}{x+9}$; time up the second waterway $=\frac{21}{x-1}$
Equation: $\frac{27}{x+9}+\frac{21}{x-1}=6$
$\Rightarrow 27x-27+21x+189=6x^2+48x-54 \Rightarrow 6x^2=216 \Rightarrow x^2=36 \Rightarrow x=6$

GBML 00

1. Let x = number of hours Kaitlin takes to paint the house $\rightarrow x+6$ = number of hours Abbe takes to paint the house; $\frac{6}{x}+\frac{16}{x+6}=1 \rightarrow 6x+36+16x=x^2+6x \rightarrow$
$x^2-16x-36=0 \rightarrow (x-18)(x+2)=0 \rightarrow x=18$

2. $kx^2+k^2x-x+8=0 \rightarrow kx^2+(k^2-1)x+8=0 \rightarrow$ product of its roots $=\frac{8}{k}$ and the sum of its roots $=-\frac{k^2-1}{k}=\frac{1-k^2}{k} \rightarrow \frac{8}{k}=\frac{1-k^2}{k}+8 \rightarrow 8=1-k^2+8k \rightarrow k^2-8k+7=0 \rightarrow$
$(k-1)(k-7)=0 \rightarrow k=1,7$

3. Call the roots of the first equation r and $s \rightarrow r^2$ and s^2 are the roots of the 2nd equation $\rightarrow r+s=3,\ rs=-5,\ r^2+s^2=-b,\ r^2s^2=c;\ r^2+s^2=(r+s)^2-2rs=3^2-2(-5)=19 \rightarrow$
and $r^2s^2=(rs)^2=(-5)^2=25 \rightarrow (b,c)=(-19,25)$

GBML 01

1. Let x = width of garden $\Rightarrow 4x$ = length of garden $\Rightarrow$ length of fence $=10x$;
$(x+4)(4x+4)=280 \Rightarrow (x+4)(x+1)=70 \Rightarrow x^2+5x+4=70 \Rightarrow x^2+5x-66=0 \Rightarrow$
$(x+11)(x-6)=0 \Rightarrow x=6 \Rightarrow$ fencing = 60 ft.

2. $\frac{x}{3x-6}-\frac{2}{2x+10}=\frac{7}{x^2+3x-10} \Rightarrow \frac{x}{3(x-2)}-\frac{1}{x+5}=\frac{7}{(x-2)(x+5)} \Rightarrow$
$x(x+5)-3(x-2)=21 \Rightarrow x^2+5x-3x+6=21 \Rightarrow x^2+2x-15=0 \Rightarrow$
$(x+5)(x-3)=0 \Rightarrow x=3$ (since $x=-5$ is extraneous to the equation.)

3. Call the smaller root of the equation $r \Rightarrow$ larger root $=r+3$;
$b=-(2r+3)$ and $c=r(r+3) \Rightarrow \frac{r(r+3)}{-(2r+3)}=-\frac{20}{3} \Rightarrow 3r^2+9r=40r+60 \Rightarrow$
$3r^2-31r-60=0 \Rightarrow (3r+5)(r-12)=0 \Rightarrow r=-\frac{5}{3},12$

GBML 06

1. Since $2\sqrt{3}$ is a root, substituting $x=2\sqrt{3}$ must satisfy the equation.
$\rightarrow 12-k^2-k=0 \rightarrow k^2+k-12=(k-3)(k+4)=0 \rightarrow k=$ <u>3, -4</u>

1. The base equation $344_{(c)}=163_{(c+5)}$ is equivalent to $3c^2+4c+4=(c+5)^2+6(c+5)+3$ $=c^2+16c+58 \rightarrow 2c^2-12c-54=0 \rightarrow c^2-6c-27=(c-9)(c+3)=0 \rightarrow c=9$
Thus, $c^2=81_{(10)}=$ <u>$41_{(20)}$</u>.

3. Rewriting $3x^2+4(x-2)=6mx-2m^2$ as $3x^2-(6m-4)x+2(m^2-4)=0$ and normalizing (i.e. making the lead coefficient 1), we have the sum of the roots is $\frac{6m-4}{3}$ and the product of the roots is $\frac{2}{3}(m^2-4)$. The simplified ratio of product to the sum is then $\frac{m^2-4}{3m-2}=\frac{5}{7}$. Cross multiplying, we have $7m^2-28=15m-10 \rightarrow 7m^2-15m-18=(m-3)(7m+6)=0$
$\rightarrow m=$ <u>3, -6/7</u>

GBML 07

ROUND 4

1. $2C^2 + 4C + 3 = 3(7^2) + 2(7) + 2 = 163 \rightarrow 2(C^2 + 2C - 80) = 2(C-8)(C+10) = 0 \rightarrow C = \underline{8}$

2. Let the roots be r_1 and r_2. Then $r_2 = 3(r_1)$ and $r_1r_2 = 75 \rightarrow r_1(3r_1) = 75 \rightarrow (r_1, r_2) = \pm(5, 15)$
$(x-5)(x-15) = x^2 - 20x + 75 = 0 \rightarrow (a, b, c) = \underline{\mathbf{(1, -20, 75)}}$
$(x+5)(x+15) = x^2 + 20x + 75 = 0 \rightarrow (a, b, c) = \underline{\mathbf{(1, 20, 75)}}$

3. $P = \dfrac{5+\sqrt{25+8A}}{2}$ and $Q = \dfrac{8+\sqrt{64-4A}}{2}$ Then $P = Q \rightarrow \sqrt{25+8A} = 3 + \sqrt{64-4A}$
Squaring both sides, $25 + 8A = 9 + 6\sqrt{64-4A} + 64 - 4A = 73 - 4A + 6\sqrt{64-4A}$
$12A - 48 = 6 \rightarrow 2A - 8 = \sqrt{64-4A} \rightarrow 4A^2 - 32A + 64 = 64 - 4A$
$\rightarrow 4A^2 - 28A = 4A(A-7) = 0$
$A = 0$ is rejected (roots of the equations are 0, 5 and 0, 7 – the smaller roots are equal)
Thus, $A = \underline{7}$ only.
Check: $x^2 - 5x - 14 = (x-7)(x+2) = 0 \rightarrow x = 7, -2$
$x^2 - 8x + 7 = (x-7)(x+1) = 0 \rightarrow x = 7, -1$
The larger roots are in fact equal.

GBML 08

1. $\dfrac{4x+1}{2x-1} + \dfrac{11x+3}{x-2x^2} = \dfrac{x-1}{x} \rightarrow \dfrac{4x+1}{2x-1} + \dfrac{11x+3}{x(2x-1)} = \dfrac{x-1}{x} \rightarrow$
$4x^2 + x - (11x+3) = (x-1)(2x-1) \rightarrow 2x^2 - 7x - 4 = 0 \rightarrow$
$(x-4)(2x+1) = 0 \rightarrow$
$x = 4$ or $-\dfrac{1}{2} \rightarrow \left|4 - \left(-\dfrac{1}{2}\right)\right| = \underline{\dfrac{9}{2}}$

2. $-\left(\dfrac{k+2}{3}\right)^2 = k \rightarrow k^2 - 5k + 4 = 0 \rightarrow (k-4)(k-1) = 0 \rightarrow k = \underline{1}$ or $\underline{4}$

3. The sum of the roots is $\dfrac{6}{k}$ and the reciprocal of the product of the roots is $\dfrac{k}{-4k-1}$.
The required product is $\dfrac{6}{k}\dfrac{k}{(-4k-1)} = -\dfrac{2}{3} \rightarrow \dfrac{3}{4k+1} = \dfrac{1}{3}$
$\rightarrow k = \underline{2}$

GBML 09

ROUND 4

1. Suppose the roots of $x^2 + 3x + K = 0$ are K and r.
Then $Kr = K \rightarrow K(r-1) = 0 \rightarrow K = 0$ or $r = 1$
$r = 1$ and $K + r = -3 \rightarrow K = -4$ and there are two possible values of K, $\underline{\mathbf{0, -4}}$

2. Let the roots of $2x^2 - 15x + (j^2 + 3j) = 0$ are r and $4r$. Then $5r = 15/2 \rightarrow r = 3/2$
The two roots are 3/2 and 6, producing a product of 9.
Thus, $\dfrac{(j^2+3j)}{2} = 9 \rightarrow j^2 + 3j - 18 = (j-3)(j+6) = 0 \rightarrow j = \underline{\mathbf{3, -6}}$

3. $(h+10)x^2 - (4h^2+9)x + 6h = 0 \rightarrow$ the sum of the roots is $\dfrac{4h^2+9}{h+10}$ and the product is $\dfrac{6h}{h+10}$
Thus, provided $h \neq -10$ $\dfrac{4h^2+9}{6h} = \dfrac{25}{12} \rightarrow \dfrac{4h^2+9}{h} = \dfrac{25}{2}$.
$h \neq 0 \rightarrow 8h^2 - 25h + 18 = (8h-9)(h-2) = 0 \rightarrow h = \underline{\dfrac{9}{8}, 2}$
The only extraneous values of x would be values which caused division by 0 and neither of these values do that. Both answers will check! However, for those who must see to believe:
[$h = 2 \rightarrow 12x^2 - 25x + 12 = (3x-4)(4x-3) = 0 \rightarrow$ roots of 4/3, 3/4 $\rightarrow \dfrac{\frac{4}{3}+\frac{3}{4}}{\frac{4}{3}\cdot\frac{3}{4}} = \dfrac{\frac{16+9}{12}}{1} = \dfrac{25}{12}$

$h = 9/8 \rightarrow 178x^2 - 225x + 108 = 0 \rightarrow$ roots of $\dfrac{225 \pm 3i\sqrt{2919}}{356}$

$\dfrac{\frac{225}{178}}{\frac{225^2+9(2919)}{356^2}} = \dfrac{225}{178}\cdot\dfrac{356^2}{76896} = \dfrac{225\cdot 2\cdot 356}{76896} = \dfrac{225\cdot 2}{216} = \dfrac{225}{108} = \dfrac{25}{12}$]

GBML 10

1. Since $\begin{cases} a+b=3 \\ ab=k \end{cases}$, $(a+b)^2 = a^2 + 2ab + b^2 = 9$. Subtracting $3ab$ from both sides, we have
$a^2 - ab + b^2 = 9 - 3ab = \underline{\mathbf{9 - 3k}}$.

2. $\left.\begin{array}{l} 2a + 8b^2 = 8 \\ a^2 - 8b^2 = 7 \end{array}\right\} \rightarrow a^2 + 2a - 15 = (a+5)(a-3) = 0 \rightarrow a = -5, 3$
$a = -5 \rightarrow -5 + 4b^2 = 4 \rightarrow b^2 = \dfrac{9}{4} \rightarrow b = \pm\dfrac{3}{2}$.
$a = 3 \rightarrow 3 + 4b^2 = 4 \rightarrow b^2 = \dfrac{1}{4} \rightarrow b = \pm\dfrac{1}{2}$. Thus, $(a, b) = \underline{\left(-5, \pm\dfrac{3}{2}\right), \left(3, \pm\dfrac{1}{2}\right)}$

3. Let $(R_1, R_2) = (A, A-3)$. Rewriting the given equation as $x^2 + (-k-2)x + 2k = 0$, we note that the sum of the roots is $2A - 3 = k + 2$ or $k = 2A - 5$ and the product of the roots is $A(A-3) = 2k$. Substituting for k, $A(A-3) = 2(2A-5) \rightarrow A^2 - 7A + 10 = (A-5)(A-2) = 0$
$A = 5 \rightarrow (R_1, R_2, k) = \underline{\mathbf{(5, 2, 5)}}$, $A = 2 \rightarrow (R_1, R_2, k) = \underline{\mathbf{(2, -1, -1)}}$.

MASSACHUSETTS MATHEMATICS LEAGUE
JANUARY 2004
ROUND 4: QUADRATICS

ANSWERS

A) ± 8

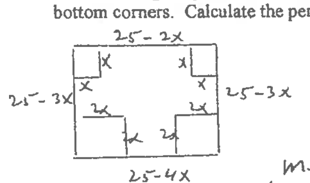
B) 100

C) 4

A) For what values of k will the equation $2x^2 - kx + 8 = 0$ have two equal real roots?

$k^2 - 4(2)(8) = 0$
$k^2 - 64 = 0$
$k = \pm 8$

B) The area of a square piece of tin is 625 sq in. Squares of equal size are cut out of the two top corners. Larger squares, each four times the area of a top corner square, are cut out of the two bottom corners. Calculate the perimeter of the resulting figure if its area is 535 sq. in.



$$P = \left.\begin{matrix} 50 - 6x \\ 50 - 6x \\ 4x \\ 8x \end{matrix}\right\} = 100$$

method 2! Solve to get $x = 3$,
Then add: $24 + 13 + 32 + 12 + 19 = 100$

C) If one root of $ax^2 + bx + c = 0$ is x = -2, b + c = 0, and a + b = 7; find the value of b

$x = -2, \quad 4a - 2b + c = 0, \quad c = -b, \quad a = 7 - b$
$4(7 - b) - 2b - b = 0$
$28 - 7b = 0$
$b = 4$

MML 05

Round Four:

A. Use the quadratic formula or factor as $(\sqrt{6}x + 2)(x - \sqrt{6}) = 0$ so $x = \frac{-2}{\sqrt{6}} = \frac{-\sqrt{6}}{3}$ or $x = \sqrt{6}$

B. For integer coefficients use $(2x+1)(cx-3) = 2cx^2 + (c-6)x - 3 = ax^2 - 2x - b$ so b=3 and c–6 = –2 so c = 4 and a = 2c so a = 8.

C. View the expression as 1 – x and notice $x = \frac{1}{4-x}$ so $x^2 - 4x + 1 = 0$ and $x = 2 + \sqrt{3}$ so $1 - x = -1 + \sqrt{3}$ and the answer is **2**

MML 06

Round Four:

A. $x^2 + x^2 + 2x + 1 = x^2 + 4x + 4$ so $x^2 - 2x - 3 = 0 = (x-3)(x+1)$

B. Second equation gives $y = 7$ or $y = -5$. $2z^2 + 7z + 3 = (2z+1)(z+3)$ so $z = -0.5$ or $z = -3$. $2z^2 - 5z + 3 = (2z-3)(z-1)$ so $z = 1.5$ or $z = 1$.

C. For n increases of \$5, price is $100 + 5n$ while sales is $102 - 3n$. Zeroes are at $n = -20$ and $n = 34$, so vertex is at their average, $n = 7$.

MML 07

A) The minimum occurs at the vertex which lies on the axis of symmetry of this upward opening parabola. The axis of symmetry occurs at $x = \frac{1+a}{2} = 3 \rightarrow a = 5$
Substituting, $7 = (3-1)(3-5) + b \rightarrow 7 = -4 + b \rightarrow b = 11$
Thus $(a, b) = \underline{(5, 11)}$.

B) The sum of the roots is b/a. To maximize the value of this fraction you need to maximize the numerator and minimize the denominator. Thus, b = L, a = S and c = M → **b) (S, L, M)**

C) Assume the wire is x feet long. Thus, the weight per foot is $\frac{24}{x}$ and
$\frac{24}{x+1} = \frac{24}{x} - \frac{1}{10} \rightarrow 240x = 240(x+1) - x(x+1) \rightarrow 0 = 240 - x^2 - x$
$\rightarrow x^2 + x - 240 = (x+16)(x-15) = 0 \rightarrow x = 15$
Therefore, 15 feet of wire weighs 24 ounces → 2/15 oz per inch.
Wire needed is 5(4) + 4(5) + 3 = 43 inches → 86/15 = $\underline{5\frac{11}{15}}$ or $\underline{5.73}$

MML 08

Round 4

A) Integer coefficients → roots must occur in conjugate pairs.
Thus, the two roots are $2 \pm i\sqrt{5}$ → sum = 4 and product = 9 → $\underline{x^2 - 4x + 9 = 0}$

B) Let L denote the larger of the positive numbers.
$$\begin{cases} L^2 + W^2 = 81 \\ 2L = 9 + W \end{cases}$$
$\rightarrow L^2 + (2L - 9)^2 = 81 \rightarrow 5L^2 - 36L = L(5L - 36) = 0 \rightarrow L = \frac{36}{5}$ and $W = \frac{27}{5} \rightarrow |L - W| = \frac{9}{5}$

C) Assume the roots of the original quadratic are r_1 and r_2 and the corresponding roots of the new equation are s_1 and s_2. Then $s_1 = 2r_1 + 3$ and $s_2 = 2r_2 + 3$

According to the root/coefficient relationship for quadratics, $p = -(r_1 + r_2)$ and $q = r_1r_2$.
Also $A = -(s_1 + s_2) = -(2(r_1 + r_2) + 6) = \underline{2p - 6}$ or $\underline{2(p - 3)}$
$B = s_1s_2 = (2r_1 + 3)(2r_2 + 3) = 4r_2r_2 + 6(r_1 + r_2) + 9 = \underline{9 - 6p + 4q}$

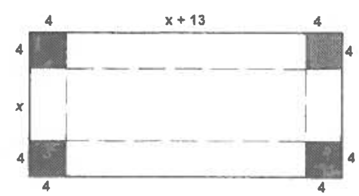
Continuing, $\frac{2p-6}{9-6p+4q} = \frac{-2}{3} \rightarrow 6p - 18 = 18 - 12p + 8q \rightarrow 6p = 8q \rightarrow \frac{p}{q} = \underline{\mathbf{4:3}}$

Note: If $A = 6$ and $B = 9$, then the first equation, $x^2 + 6x + 9 = 0$ has a double root of -3.
Since $2(-3) + 3 = -3$, the second equation would be identical.
In the above solution, $A = 6 - 2p = 6 \rightarrow p = 0$ and $B = 9 - 6p + 4q = 9 \rightarrow q = 0$
In this situation the ratio of $p : q$ would be indeterminant. Thus, it was necessary to require that the equations be different.

MML 11

Round 4

A) $(x+1)\left(\frac{1}{x}+1\right) = 1 + x + \frac{1}{x} + 1 = x + \frac{1}{x} + 2 = \frac{13}{6} + 2 = \underline{\frac{25}{6}}$



B) Let the dimensions of the cardboard be $(x + 8)$ by $(x + 21)$
→ dimensions of the box are 4 by x by $(x + 13)$
$V = 4x(x + 13) = 1200 \rightarrow 4(x^2 + 13x) - 1200 = 0$
$\rightarrow x^2 + 13x - 300 = 0$
$\rightarrow (x + 25)(x - 12) = 0 \rightarrow x = 12$, $\cancel{-25}$
→ dimensions: 4 x 12 x 25 → sum: **41**.

C) To have real roots the discriminant $B^2 - 4AC$ must be nonnegative.
$(-m)^2 - 4(m+3)(1) = m^2 - 4m - 12 = (m-6)(m+2) \geq 0$
The critical points on the number line are −2 and +6.
Testing in the 3 intervals on the number line, the product is positive when $m \leq -2$ or $m \geq 6$.
However, we also require that the roots be nonzero, that is
$\frac{m \pm \sqrt{m^2 - 4m - 12}}{2} \neq 0 \rightarrow m \neq \pm\sqrt{m^2 - 4m - 12} \rightarrow m^2 \neq m^2 - 4m - 12 \rightarrow 4m \neq -12 \rightarrow m \neq -3$
Therefore, the required set of m-values is: $\underline{\mathbf{m \leq -2 \text{ or } m \geq 6 \ (m \neq 3)}}$
Alternately, $m < -3$ or $-3 < m \leq -2$ or $m \geq 6$
Also acceptable: $(-\infty, -3) \cup (-3, -2] \cup [6, \infty)$