All Rounds

May, 1979 NEAML

Round 1: Arithmetic

1. ____________________

2. ____________________

3. ____________________

(1 point) 1. One afternoon brothers David and John sat down to share all or part of $1 billion. Their conversation went as follows:

John: What I get plus three times what you get should be at most $1.5 billion.

David: That's O.K. as long as I get as much as you get.

John: Agreed, so long as I get as much as possible under the circumstances.

How many dollars did each man get?

(2 points) 2. How many of the positive integers less than 100 have an even number of positive integral divisors?

(3 points) 3. Find the least positive remainder obtained when $30!$ is divided by $5^8$.
(n! is the symbol for n-factorial)

May, 1979 NEAML

Round 2: Algebra(no transcendentals)

1. ____________________

2. ____________________

3. ____________________

(1 point) 1. Express as a product of binomials the positive square root of

$$(a^2 + ab + bc + ac)(b^2 + bc + ac + ab)(c^2 + ac + ab + bc)$$

where a, b and c are positive numbers.

(2 points) 2. A man travels a certain distance at the rate of 5 miles per hour, but returns over the same route at the rate of 5 hours per mile. Find the man's average rate in miles per hour.

(3 points) 3. If each of four numbers is added to the average of the other three, the respective sums are 27, 29, 33 and 37. Find the largest of the four numbers.

May, 1979 NEAML

Round 3: Geometry

1. ____________________

2. ____________________

3. ____________________

(1 point) 1. The lengths of the medians of a triangle are 30, 30 and 48. Find the length of the shortest side of this triangle.

(2 points) 2. Two different equilateral triangles are inscribed in a circle, the sides of the triangles being parallel in pairs. A second smaller circle is inscribed in the region common to the interiors of these two triangles. If the area of the first circle is $144\pi$, find the area of the smaller circle.

(3 points) 3. The lengths of the sides of a triangle are 3, 4 and 5. Find the length of the angle bisector drawn to the side of length 4.

May, 1979 NEAML

Round 4: Algebra

1. ____________________

2. ____________________

3. ____________________

(1 point) 1. Let $f(x) = ax^5 + bx^3 + cx + 5$ define the function f for all real values of x. If $f(3) = 9$, find the value of $f(-3)$.

(2 points) 2. If $b = \log_3 x$, find all real values of x which satisfy $\log_b(\log_3 x^2) = 2$.

(3 points) 3. If a and b are the roots of $x^2 - x + 1 = 0$, find the value of $a^3 + 3a^3b^3 + b^3$.

May, 1979 NEAML

Round 5: Analytic Geometry

1. ____________________

2. ____________________

3. ____________________

(1 point) 1. Find all values of m for which the line $y = mx + 5$ will not intersect the circle $x^2 + y^2 = 9$ in the real plane.

(2 points) 2. When the plane is rotated about the origin through an angle of 45°, the image of the line $y = 2x - 2$ is the line $y = mx + b$. Find the ordered pair $(m,b)$.

(3 points) 3. How many points, all of whose coordinates are integers, are interior to the sphere with equation $x^2 + y^2 + z^2 = 9$ ?

May, 1979 NEAML

Round 6: Trig and Complex Numbers

1. ______________________

2. ______________________

3. ______________________

(1 point) 1. Find, in simplified form, the value of $\sin(2\,\mathrm{Arcsec}\,\sqrt{5})$, where Arc denotes principal value.

(2 points) 2. Find the least positive value of x for which $|\cos x| = \cos x + 1$.

(3 points) 3. If a and b represent positive integers and if i represents the imaginary unit, then the square root of $-1 + i\sqrt{3}$ may be represented as $(\sqrt{a} + i\sqrt{b})/2$. Find the ordered pair (a,b).

May, 1979 NEAML

Team Round

1. ____________________

2. ____________________

3. ____________________

4. ____________________

5. ____________________

6. ____________________

1. How many integers between 100 and 15,000 are themselves 11 times the square of an integer?

2. A merchant buys a number of yards of cloth for a total price of \$45. If he is forced to sell the cloth at 65¢ per yard, he will take a net loss equal to his cost for 8 yards of the cloth. The original cost of the cloth was y¢ per yard. Find both possible values of y.

3. Arcs ABC and ADC are different semicircles of the same circle of which $\overline{AB}$ and $\overline{AD}$ are chords. Also, AB = 2 and AD = 5. From points B and D, perpendiculars are drawn to $\overline{AC}$, intersecting $\overline{AC}$ at points E and F respectively. If AE = x, EF = 3 and CF = y, find the sum $x + y$.

4. Find all ordered triples of integers (x,y,z) which satisfy

$z^x = y^{2x}$ , $2^z = (2)(4^x)$ , and $x + y + z = 16$ .

5. From the point (15,-4) tangent lines are drawn to the ellipse $16x^2 + 25y^2 = 400$. The line $ax + by = 20$ passes through the points of contact of the tangents with the ellipse. Find the ordered pair (a,b).

6. Determine the simplified numerical value of $(\sin 10°)(\sin 30°)(\sin 50°)(\sin 70°)$ .

May, 1979 NEAML Solutions

**ONE:** 1. $J + 3D \le 1.5$, $D \ge J$
max. $J \Rightarrow 4D \le 1.5$
$\therefore J \le D \le 3/8$

2. Only perfect squares have an odd number.
$\therefore 99 - 9 = 90.$

3. $30! = 5^7 x = 5^8 g + r$
$\therefore r = 5^7(x - 5g)$
$x - 5g < 5$ and is div. by 4.
$\therefore r = 5^7 \times 4.$

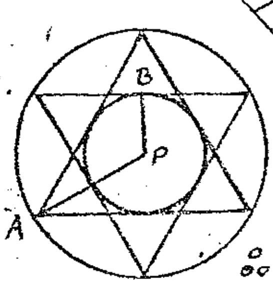
**TWO:** 1. Factor and result is obvious.

2. Go: $D \quad 5 \quad \frac{D}{5}$
Ret: $D \quad \frac{1}{5} \quad 5D$
$\therefore \frac{2D}{5\frac{1}{5}D} = \frac{5}{13}$

3. $S = a - b + c + d$
$$\left.\begin{array}{l} a + \frac{1}{3}(b+c+d) \\ b + \frac{1}{3}(a+c+d) \\ c + \frac{1}{3}(a+b+d) \\ d + \frac{1}{3}(a+b+c) \end{array}\right\} = 25 = 126 \qquad s = 63$$
$\therefore \frac{2}{3}d + \frac{1}{3}s = 37$
$\frac{2}{3}d + 21 = 37 \Rightarrow d = 24$

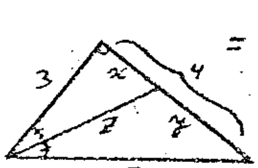
**THREE:** 1.


2.




$\frac{PB}{PA} = \frac{1}{2}$
$\therefore \frac{1}{4}(144\pi)$
$= 36\pi$

3.




$\frac{x}{3} = \frac{y}{5}$ and $x + y = 4 \Rightarrow x = \frac{3}{2}$
$\therefore z^2 = 9 + \frac{9}{4} = \frac{45}{4}$
and $z = \frac{3}{2}\sqrt{5}$

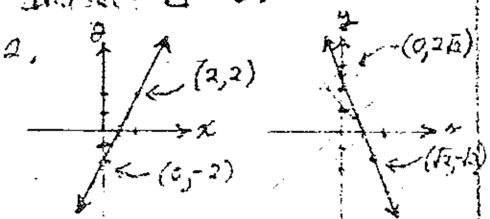
**FOUR:** 1. $f(3) = 9$ implies
$g(a,b,c) + 5 = 9$ and
$g(a,b,c) = 4.$
$f(-3) = -g(a,b,c) + 5 = -4 + 5$

2. $\log_3 x^2 = b^2 = (\log_3 x)^2$
or $2L = L^2 \Rightarrow L = 0, 2$
$L = 0 \Rightarrow b = 0$ is not possible.
$L = 2 \Rightarrow x = 3^2 = 9.$

3. $(x+1)(x^2 - x + 1) = x^3 + 1 = 0$
has roots where $a^3 = b^3 = -1.$
$\therefore (-1) + 3(-1)^2 + (-1) = 1.$

**FIVE:** 1. Solve $\begin{cases} y = m x + 5 \\ x^2 + y^2 = 9 \end{cases}$
and set $\Delta < 0.$

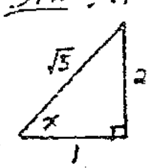
2.




$\therefore m = \frac{2\sqrt{2} + \sqrt{2}}{0 - \sqrt{2}} = -3$, $b = 2\sqrt{2}$

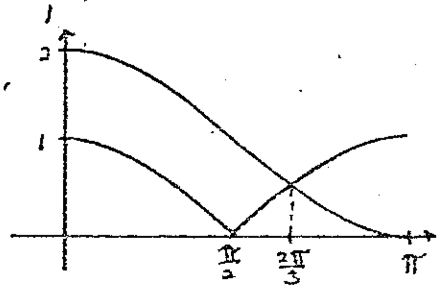
3. Use symmetry and vary $z$ from $-2$ to $2$. Set
$x^2 + y^2 = r^2 = 9 - z^2$ and count
$13 + 21 + 25 + 21 + 13 = 93$

**SIX:** 1.




$\sin 2x$
$= 2\sin x \cos x$
$= 2\left(\frac{2}{\sqrt{5}}\right)\left(\frac{1}{\sqrt{5}}\right) = \frac{4}{5}$

2.

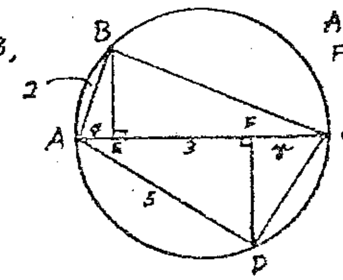



3. $\left(\frac{\sqrt{a} + i\sqrt{b}}{2}\right)^2 = \frac{a-b}{4} + \frac{2i\sqrt{ab}}{4}$
$= -1 + i\sqrt{3}$ implies
$a - b = -4$ and $ab = 12.$
$\therefore (a, b) = (2, 6).$

**TEAM:** 1. $\frac{100}{11} = 9^+$
$\frac{15000}{11} = 1363^+$
There are 33 squares from $4^2$ to $36^2$.

2. $\begin{cases} my = 4500 \\ 4500 - 65m = 8y, \end{cases}$
implies
$2y^2 - 1125y + 73125 = 0$
Let $y = 25x$ and
$2x^2 - 45x + 117 = 0$
$(2x - 39)(x - 3) = 0$
$\therefore y = \frac{39}{2}(25), 3(25).$

3.




$\frac{x}{2} = \frac{2}{x+3+y}$, $\frac{x+3}{5} = \frac{5}{x+3+y}$
$\therefore \frac{x}{4} = \frac{x+3}{25}$ and $x = \frac{4}{7}$
$\frac{4/7}{2} = \frac{2}{x+3+y} \Rightarrow x + y = 4.$

4. $2^z = (2)(4^x) \Rightarrow z = 2x + 1$
$z^x = y^{2x} \Rightarrow z = y^2$ or $x = 0$
$x = 0$ yields $(0, 15, 1)$, but
$y = z^2$ yields $(4, 3, 9)$.

5. $\left.\begin{array}{l} y = m(x - 15) - 4 \\ 16x^2 + 25y^2 = 400 \end{array}\right\}$ with $\Delta = 0$
gives $y = -4$ at $(0, -4)$ and
$3x + 5y = 25$ at $(3, \frac{16}{5})$.
$\therefore 12x - 5y = 20.$

6. $N = \frac{1}{2}\sin 10°[\sin 70° \sin 50°]$
$= \frac{1}{2}\sin 10°\left[\frac{1}{2}(\cos 20° - \cos 120°)\right]$
$= \frac{1}{4}\left[\cos 20° \sin 10° + \frac{1}{2}\sin 10°\right]$
$= \frac{1}{4}\left[\frac{1}{2}(\sin 30° - \sin 10°) + \frac{1}{2}\sin 10°\right]$
$= \frac{1}{8}\left[\sin 30° - \sin 10° + \sin 10°\right]$
$= \frac{1}{16}.$

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ROUND 1 - Arithmetic

1. ______________________

2. ______ t = ______ u = ______

3. ______________________

1. Find the least common multiple of 4004 and 25025. (1 point)

2. $7^{43}$ has 54 digits when written in standard form. Find the ten's digit and the units digit. (2 points)

3. How many base 10 counting numbers will have a three digit representation in bases 4, 5, and 7. (3 points)

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ROUND 2 - Algebra I

1. ____________________

2. ____________________

3. ____________________

1. If $\sqrt{x} = \sqrt[5]{y}$ and $\sqrt{y} = 2 \cdot \sqrt[4]{2}$, find the value of x. (1 point)

2. For what value(s) of x, $x \in$ Reals, is the following statement true:

$$1 - \frac{3}{2 - \frac{1}{1 - \frac{1}{x}}} = 3 - 2(x+1)$$

(2 points)

3. A six gallon radiator was $\frac{2}{3}$ full of a 55% antifreeze mixture. 100% antifreeze was added to fill the radiator. How many gallons of this new mixture must be drained off and replaced with water to make the mixture 25% antifreeze? (3 points)

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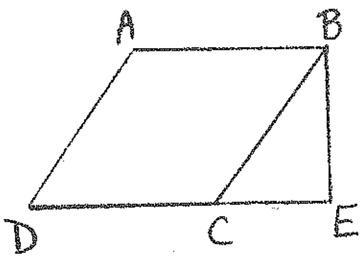
ROUND 3 - Geometry

1. ______ : ______

2. ______

3. ______

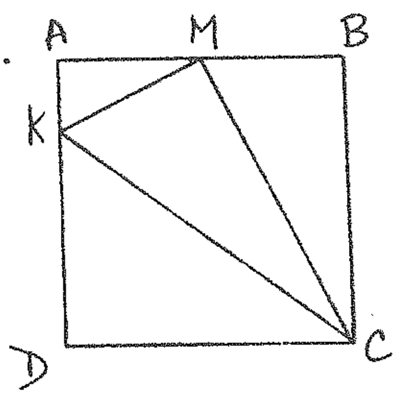
1.




In rhombus ABCD, $\overline{BE} \perp \overline{DE}$, CE:BE = 1:2 . Find the ratio of the area of rhombus ABCD to the area of triangle BEC .

(1 point)

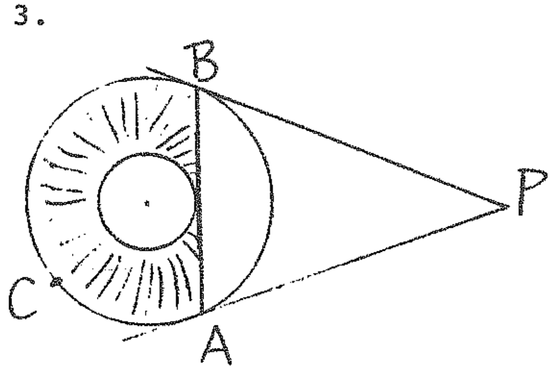
2.




In square ABCD, M is the mdpt. of $\overline{AB}$ and $\overline{MK} \perp \overline{MC}$ . Find KC if the area of the square is 16.

(2 points)

3.




Given: Concentric circles as shown

M $\overset{\frown}{ACB}$ = 240

Chord $\overline{AB}$ is tangent to the smaller circle. $\overline{PB}$ and $\overline{PA}$ are tangents to the larger circle. If PB = $8\sqrt{3}$, find the area of the shaded region.

(3 points)

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ROUND 4 - Algebra II

1. ______________________

2. ______________________

3. ______________________

1. For how many integers, $x$, is it true that the product $(x + 3)^3 \cdot (x + 1)^2 \cdot (x - 2)^4 \cdot (x - 5)^5 \leq 0$ ?

(1 point)

2. Find the sum of the natural numbers from 1 to 200 that are divisible by 3 and not by 7.

(2 points)

3. Find the value(s) of $x$ for which $x^{\log_2 x} = 16x^3$ .

(3 points)

NEAML PLAYOFFS - 1986

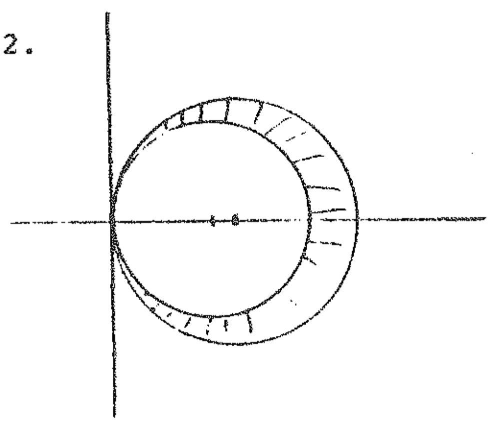
ROUND 5 - Coordinate Geometry

1. ______________________

2. ______________________

3. ______________________

1. The x and y intercepts of line L are both positive with the x-intercept 4 times the y-intercept. The area of the triangle determined by L and the coordinate axes is $4\frac{1}{2}$ . Find the equation of line L in the form $ax + by = c$, where a, b, c are relatively prime integers. (1 point)

2. 

The shaded portion of the graph has area $\frac{\pi}{9}$ and is bounded on the inside by the graph of $x^2-2x+y^2 = 0$ and on the outside by the graph of $x^2-2px+y^2 = 0$. Find the value of p. (2 points)

3. The point $(10, 2\sqrt{3})$ is one endpoint of a chord through the center of an ellipse whose equation is $x^2-8x+9y^2-r = 0$, $r \in R$. Find the coordinates of the other endpoint. (3 points)

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ROUND 6 - Trig., Complex Numbers

1. ______________________

2. ______________________

3. ______________________

1. For what value(s) of $x$, $0^\circ \leq x < 360^\circ$ is $\sin 37^\circ \cos 7^\circ + \cos 37^\circ \sin 7^\circ = \cos x$. (1 point)

2. For what values of $k$ will the expression $(2+ki)^3$ be equivalent to a real number? (2 points)

3. In $\triangle ABC$ with sides $a$, $b$, and $c$, if $\cos A = \frac{2a}{c}$, find $\cos C$ in terms of 'a' and 'b'. (3 points)

NEAML PLAYOFFS - 1986

TEAM ROUND - (Page 1)

(Large Schools 4 points each Medium and Small Schools 3 points each)

1. ____________________ 4. ____________________

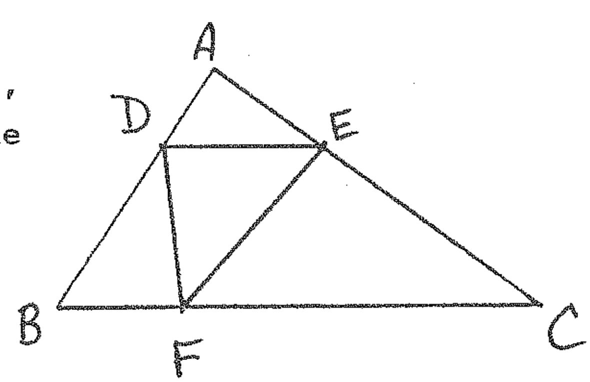
2. ____________________ 5. ____________________

3. ____________________ 6. ____________________

1. Find all three digit numbers of the form ABA (A,B distinct whole numbers) which are divisible by 3 and 11.

2. The polynomial $x^4 + 5x^3 + 11x^2 + 14x + 4$ can be factored as $(x^2 + ax + b) \cdot (x^2 + cx + d)$ . Find the value of $(a+b) \cdot (c+d)$ , given that all are integers.

3. Given $\triangle ABC$, BD = 2AD, EC = 2AE, FC = 3BF. Find the ratio of the area of $\triangle DEF$ to the ratio of $\triangle ABC$ .



4. How many integral value(s) of $x$ make the following statement true? $\frac{x-5}{x+3} \leq \frac{3}{x}$

5. For what values of $x$, $0 \leq x < 2\pi$, is $(\tan 2x)(\csc x) = \sqrt{2}\,(\sin 2x)(\sec 2x)$.

6. Find all three roots of the equation $3x^3 - kx^2 + 8x - 15 = 0$ given one root is the reciprocal of another root.

NEAML PLAYOFFS - 1986

ANSWER SHEET

ROUND 1

1. 100,100

2. 4,3

3. 15

ROUND 2

1. 2

2. $\frac{1}{2}$ , 3

3. $\frac{27}{7}$ or $3\frac{6}{7}$

ROUND 3

1. $2\sqrt{5} : 1$

2. 5

3. $\frac{80\pi}{3} + 16\sqrt{3}$

ROUND 4

1. 9

2. 5688

3. $\frac{1}{2}$ , 16

ROUND 5

1. $x + 4y = 6$

2. $\frac{\sqrt{10}}{3}$

3. $(-2, -2\sqrt{3})$

ROUND 6

1. $46°$ , $314°$

2. 0, $\pm 2\sqrt{3}$

3. $\frac{b-2a}{a}$

TEAM ROUND

1. 363, 858

2. 10

3. 2:9 or $\frac{2}{9}$

4. 11

5. $\frac{\pi}{2}$ , $\frac{3\pi}{2}$

6. 5, $\frac{1 \pm i\sqrt{35}}{6}$

# NEAML Playoffs, 1996 — Solutions Outline

## Round 1

1. $4004 = 4 \times 1001$ $\quad 25025 = 25 \times 1001$

2. Power of 7 $\quad$ 1 $\quad$ 2 $\quad$ 3 $\quad$ 4 $\quad$ 5
   last 2 digits $\quad$ 7 $\quad$ 49 $\quad$ 43 $\quad$ 01 $\quad$ 07

3. $333_4 = 63_{10}$ $\quad 100_7 = 49_{10}$

## Round 2

1. $x = y^{4/5}$ $\quad y = 2^{5/4}$

2. $1 - \dfrac{3}{2 - \frac{x}{x-1}} = 1 - 2x$

   $1 - \dfrac{3x-3}{x-2} = 1 - 2x$

   $-3x + 3 = -2x^2 + 4x$

   $2x^2 - 7x + 3 = 0$ $\quad (2x-1)(x-3)$

3. $4(.55) + 2(1.0) = 4.2$ gal antifreeze
   ∴ 70% antifr or 30% $H_2O$
   $.30(6 - x) + x = .75(6)$

## Round 3

1. $BE : BC = 2 : \sqrt{5}$

   $\dfrac{\triangle BEC}{\square ABCD} = \dfrac{\frac{1}{2}\cdot 1 \cdot 2}{\sqrt{5}\cdot 2} = \dfrac{1}{2\sqrt{5}}$

2. $AB = 4$, $MB = 2$, $MC = 2\sqrt{5}$
   $\triangle AKM \sim \triangle BMC \Rightarrow KM = \sqrt{5}$
   $\therefore KC = 5$

3. Draw $\overline{OA}$ and $\overline{OB}$, $m\angle AOB = 120$
   $AB = 8\sqrt{3}$, $R = 8$, $r = 4$

   $\frac{2}{3}(64\pi - 16\pi) = 32\pi$

   $A_{\triangle AOB} = 16\sqrt{3}$ $\quad A_{\text{small } 120° \text{ sector}} = \dfrac{16\pi}{3}$

   $32\pi + (16\sqrt{3} - \frac{1}{3}16\pi)$

## Round 4

1. equivalent to $(x+3)(x-5) \le 0$

2. $(3 + 6 + 9 + \cdots + 198) - (21 + 42 + \cdots + 189)$
   $\frac{66}{2}(201) - \frac{9}{2}(210)$

3. $(\log_2 x)^2 = \log_2 16 + 3\log_2 x$
   $(\log_2 x)^2 - 3\log_2 x - 4 = 0$
   $(\log_2 x - 4)(\log_2 x + 1) = 0$

## Round 5

1. $\frac{1}{2}a \cdot 4a = \frac{9}{2}$ $\quad (0, \frac{3}{2})$ $\quad (6, 0)$

2. $(x-1)^2 + y^2 = 1$ $\quad (x-p)^2 + y^2 = p^2$
   $\pi p^2 - \pi = \dfrac{\pi}{9}$

3. $100 - 80 + 108 - r = 0 \Rightarrow r = 128$
   $(x-4)^2 + 9y^2 = 144$ $\quad C(4, 0)$

## Round 6

1. $\sin(3x + x) = \cos x$ $\quad x + 4x = 90$

2. $(2 + ki)^3 = 8 + 12ki - 6k^2 - k^3 i$
   $12k - k^3 = 0$

3. $a^2 = b^2 + c^2 - 2bc\cos A$
   $= b^2 + c^2 - 4ab$

   $a^2 - b^2 + 4ab = c^2 = a^2 + b^2 - 2ab\cos C$

   $-2b^2 + 4ab = -2ab\cos C$

   $b - 2a = a\cos C$

# NEAML 1986 Team Solutions

1. Divisible by 11: 121, 242, $\underline{363}$, 484, 616, 737, $\underline{858}$, 979

2. $a+c=5 \quad b+d+ac=11 \quad bc+ad=14 \quad bd=4$

3. let $x = AD \quad y = AE \quad z = BF$
h is alt. of $\triangle ADE$; $H$ is alt. of $\triangle ABC$

$\frac{3x}{4z} = \frac{x}{DE} \qquad \triangle DEF = \frac{4}{3}z \cdot \frac{1}{3}H \qquad \triangle ABC = 2zH$

$DE = \frac{4}{3}z \qquad = \frac{4}{9}Hz \qquad \frac{4}{9} \cdot \frac{1}{2} = \frac{2}{9}$

$\frac{h}{H} = \frac{1}{3} \quad h = \frac{1}{3}H \quad PQ = \frac{2}{3}H$

4. $\frac{x-5}{x+3} - \frac{3}{x} \le 0 \qquad \frac{x^2-5x-3x-9}{x(x+3)} \le 0 \qquad \frac{x^2-8x-9}{x(x+3)} \le 0$

$\frac{(x-9)(x+1)}{x(x+3)} \le 0 \qquad + \; -3 \; - \; -1 \; + \; 0 \; - \; 9 \; +$

$-2, -1, 1-9$

5. $\tan 2x \csc x = \sqrt{2} \tan 2x \qquad \tan 2x = 0$ or $\csc x = \sqrt{2}$
$\sin x = \frac{\sqrt{2}}{2}$

soln of $\sin x = \frac{\sqrt{2}}{2}$ not allowed since $\tan 2x$ not defined.

6. denote roots by $a, \frac{1}{a}, b$
product of roots is $-\frac{-15}{3} = 5 = a \cdot \frac{1}{a} \cdot b = b$
$\therefore b = 5$ is one root $\Rightarrow 375 - 25k + 40 - 15 = 0 \Rightarrow k = 16$

$3x^3 - 16x^2 + 8x - 15 = 0 \qquad 5 \,)\; 3 \quad -16 \quad 8 \quad -15$
$\qquad 15 \quad -5 \quad 15$
$\qquad 3 \quad -1 \quad 3 \quad 0$

$3x^2 - x + 3 = 0$

N.E.A.M.L - PLAYOFFS - 1993

ROUND 1 - Arithmetic, Number Theory

1.________________________

2.________________________

3. ________________________

1. For p a prime number less than or equal to 200, determine all values of p such that 3 + p is a perfect square.

2. Using the digits 0,2,3,6, and 9, without repetition, 96 distinct 5-digit numbers between 10,000 and 100,000 can be formed. If they are arranged from the smallest to the largest, name the 62nd number.

3. Let E, N, P and Q represent distinct digits. If the sum of the two three-digit numbers, QP3 and QNE is PEEP, compute the product of Q, N, and E.

N.E.A.M.L - PLAYOFFS - 1993

Round 2 - Algebra I

1. ____________________

2. ____________________

3. ____________________

1. Factor completely: $4x(x+1) - (y^4 - 1)$

2. A scientist created two new scales for measuring temperature. On her P scale water boils at 80° and freezes at 0°. On her Q scale water boils at 120° and freezes at −20°. What is the equivalent on the P scale of a temperature of 29° on the Q scale?

3. If x and y are positive integers, and M is the largest value of y satisfying $xy + 1883 = 1993x$, express $\sqrt{M+8}$ in simplest form

N.E.A.M.L - PLAYOFFS - 1993

Round 3 - Geometry

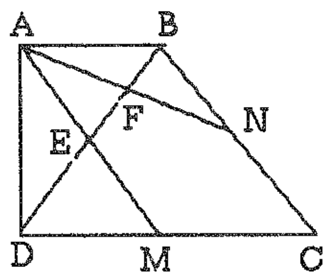
1. ______________________

2. ______________________

3. ______________________

1. Three concentric circles are such that the radius of the inner circle is 6 cm. and the radius of the outer circle is 12 cm. The middle circle bisects the area between the inner and outer circles. Find the number of centimeters in the radius of the middle circle.

2.




In trapezoid ABCD, $\overline{AB} \perp \overline{AD}$ and $\overline{DC} \perp \overline{AD}$. IF AB = 6, DC = 12, and N and M are midpoints of BC and DC respectively, determine the numerical value of EF:DF.

3. Triangle ABC has a right angle at C. A circular arc with center A and radius AC is drawn intersecting AB in E. With B as center and radius BE, a circular arc is drawn intersecting BC in F. If BF:FC = 5:2, determine the lengths of the sides of $\Delta ABC$ if the lengths of the sides are relatively prime.

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Round 4 - Algebra II

1. ____________________

2. ____________________

3. ____________________

1. Using only odd digits, all possible three digit numbers are formed. What is the sum of all such numbers?

135 137 139

999

357 359

579

2. If $\dfrac{\dfrac{4x+5}{\sqrt{2x+7}} - x\sqrt{2x+7}}{x-1} = \dfrac{ax+b}{\sqrt{2x+7}}$ , determine (a,b).

3. The sides of a box are 2, m and n, where m and n are the roots of $x^2 - 5x + 2 = 0$. Find the length of the longest diagonal of the box.

N.E.A.M.L - PLAYOFFS - 1993

Round 5 - Analytic Geometry

1. ______________________

2. ______________________

3. ______________________

1. The point P(4, -1) is reflected with respect to the line $y = x$ to become Point Q. Point Q is reflected with respect to the x-axis to become point R. Find the length of $\overline{PR}$.

2. Given A(-7,10) and D(11, 4), points B and C trisect $\overline{AD}$. If B and C are the endpoints of a diameter of a circle, find the equation of the circle.

3. Given, a parabola whose line of symmetry is vertical, whose vertex is the center of $4x^2 + 9(y-2)^2 = 36$ and which passes through the upper points of intersection of the given curve and the graph of $y = |x|$. If the equation of the parabola is $y = ax^2 + c$, find the value of a.

N.E.A.M.L - PLAYOFFS - 1993

Round 6 - Complex Numbers, Trigonometry

1. ______________________

2. ______________________

3. ______________________

1. Determine all x in radians, $0 \le x < 2\pi$ such that $\tan x + 1 = \frac{1}{\sqrt{2}} \sec x$.

2. If $\sin^2 x = \frac{5}{8}$, $\cot x > 0$, and $\sec x < 0$, find $\tan 2x$.

3. In rectangular form, find the root of $x^5 = -4 + 4i$ which is in quadrant 4.

N.E.A.M.L - PLAYOFFS - 1993

TEAM ROUND - Page 1

Large Schools: 4 points each Medium and Small Schools: 3 points each

ANSWERS:

| | | | |
|---|---|---|---|
| 1. | ________________ | 4. | ________________ |
| 2. | ________________ | 5. | ________________ |
| 3. | ________________ | 6. | ________________ |

1. A population P is growing according to the formula $P = ab^t + c$ where t is time. If, when $t = 1,2$, and 3 then $P = 5$, 17, and 53 respectively, determine P when $t = 5$.

2. Determine all real x, $x \neq 1$, such that $(\log x)^{\log x} = (\log_x 10)^3$.

3. The point (3, 4) is a vertex of a regular octagon centered at the origin. Moving in a counterclockwise direction, determine the sum of the x- and y-coordinates of the next vertex point of the octagon.

4. In trapezoid ABCD, diagonals $\overline{AC}$ and $\overline{BD}$ intersect at P. If the area of $\Delta ABP = 7$ and the area of $\Delta BPC = 21$, determine the area of the trapezoid.

5. Given $f(x) = |x+3| + |x-5|$. Find all values of x, x $\varepsilon$ {even integers}, such that $f(x) = 8$.

6. Given the points A(8, 6), B(2, 2), and C(1, t). Find the value of t for which AC + BC is a minimum.

N.E.A.M.L. Playoffs - 1993

ANSWER SHEET

ROUND 1

1. 13, 61, 97, 193

2. 63092

3. 432

ROUND 2

1. $(2x + 1 - y^2)(2x + 1 + y^2)$

2. $28°$

3. $20\sqrt{5}$

ROUND 3

1. $3\sqrt{10}$

2. $\frac{1}{3}$

3. 12, 35, 37

ROUND 4

1. 69375

2. $(-2, -5)$

3. 5

ROUND 5

1. $\sqrt{34}$

2. $(x-2)^2 + (y-7)^2 = 10$

$x^2 + y^2 - 4x - 14y + 43 = 0$

3. $\frac{65}{648}$

ROUND 6

1. $\frac{\pi}{12}, \frac{5\pi}{12}$ $\frac{7\pi}{12}, \frac{23\pi}{12}$

2. $-\sqrt{15}$

3. $1 - i$

TEAM ROUND

1. 485

2. $\frac{1}{1000}, \frac{1}{10}, 10$

3. $3\sqrt{2}$

4. 112

5. $-2, 0\ 2, 4$

6. $\frac{5}{2}$

# New England Meet 1993 Solutions Outline

## Round 1

1. Primes 1 6 13 22 33 46 78 61

Squares 4 9 16 25 36 49 81 64

97 118 141 166 193

100 121 144 169 196

2. 2 as 1st digit 4! numbers
same for 3 as 1st digit = 48 #s
60___ 62___ 12 number
63029 61st 63092 62nd

3. $\begin{array}{r} QP3 \\ QNE \\ \hline PEEP \end{array}$ $P=1$ $E=8$ $N=6$ $Q=9$

## Round 2

1. $4x^2+4x-y^4+1 = (2x+1)^2-y^4$

2. $P=xQ+y$ $80=120x+y$
$0=-20x+y$

3. $y=1993-\frac{1883}{x}$ $M=1992$

## Round 3

1. Area mid $O = (144\pi-36\pi)/2$

2. Draw $\overline{BM}$ and $\overline{EN}$
ABMD rectangle $\overline{EN} \parallel \overline{AB}$
ABNE parallelogram
$EF = \frac{1}{2}BE = \frac{1}{2}DE$

3. $AC=m$ $BF=5n$ $FC=2n$
$(7n)^2+m^2=(m+5n)^2$

## Round 4

1. A.P. with $a=111$ $l=999$ $n=125$

2. mult lhs by $\frac{\sqrt{2x+7}}{\sqrt{2x+7}}$. get

$$\frac{(4x+5)-x(2x+7)}{(x-1)\sqrt{2x+7}} = \frac{ax+b}{\sqrt{2x+7}}$$

3. $x=\frac{5\pm\sqrt{17}}{2}$ $m=\frac{5+\sqrt{17}}{2}$ $n=\frac{5-\sqrt{17}}{2}$

## Round 5

1. $Q(-1,4)$ $R(-1,-4)$

2. $B(-1,8)$ $C(5,6)$
center $(2,7)$ $r=\sqrt{10}$

3. $4x^2+9(x-2)^2=36 \rightarrow x=0, \frac{36}{13}$
$y=ax^2+2 \rightarrow a=\frac{65}{648}$

## Round 6

1. Mult by $\cos x \rightarrow \sin x+\cos x=\frac{1}{\sqrt{2}}$
sq both sides $1+\sin 2x=\frac{1}{2}$
$\sin 2x=-\frac{1}{2}$

2. $\sin x=\frac{-\sqrt{10}}{4}$ $\cos x=-\frac{\sqrt{6}}{4}$

3. $x^5=4\sqrt{2}\ \text{cis}\ \frac{3\pi}{4}$

$x=\sqrt{2}\ \text{cis}\ \frac{3\pi}{20}\ \frac{11\pi}{20}\ \frac{19\pi}{20}\ \frac{27\pi}{20}\ \frac{35}{20}$

use $\frac{35\pi}{20}$

# NEAML 93 Team Round solutions

1. $5 = ab + c \quad 17 = ab^2 + c \quad 53 = ab^3 + c$

$$\left.\begin{aligned} 12 &= ab^2 - ab = ab(b-1) \\ 36 &= ab^3 - ab^2 = ab^2(b-1 \end{aligned}\right] b = 3$$

2. $(\log x)^{\log x} = \left(\dfrac{1}{\log x}\right)^3 = (\log x)^{-3}$

$(\log x)^{(\log x + 3)} = 1 \Rightarrow \log x = 1$ or $\log x = -1$

3. Coordinates are $(5\cos\theta, 5\sin\theta)$ $\theta = 45 + \alpha$

$5[\cos(45+\alpha)] + 5[\sin(45+\alpha)]$ $\cos\alpha = \frac{3}{5}$ $\sin\alpha = \frac{4}{5}$

4. $\dfrac{a(\triangle APB)}{a(\triangle BPC)} = \dfrac{AP}{PC} = \dfrac{1}{3}$ $\dfrac{a(\triangle APD)}{a(\triangle CPD)} = \dfrac{1}{3} = \dfrac{21}{a(\triangle CPD)}$ $a(\triangle CPD) = 63$

5. if $x > 5$ $f(x) = 2x - 2$ $f(5) = 8$ 5 not even

$-3 < x < 5$ $f(x) = 8$ $-2, 0, 2, 4$

$x < -3$ $f(x) = 2 - 2x$ $f(-3) = 8$ $-3$ not even

6. Let $B' = (0, 2)$ $AC + BC = AC + CB'$ $AB' = \min.$

$m(AB') = \frac{1}{2}$ $y - 2 = \frac{1}{2}x$ $\left(1, \frac{5}{2}\right)$

## N.E.A.M.L - PLAYOFFS - 1994

Round 1 - Arithmetic, Number Theory

1.______________________

2.______________________

3.______________________

1. Find 6.25% of 3 less than the first prime number greater than 175.

2. If $N = .\overline{10}$, then determine the number X such that $NX = .\overline{1}$.

3. How many natural numbers less than 1000 are divisible by 6 and have a units digit of 4.

N.E.A.M.L - PLAYOFFS - 1994

Round 2 - Algebra I

1. ____________________

2. ____________________

3. ____________________

1. Solve over the set of real numbers: $4 + \sqrt{x} = \sqrt{16x}$.

2. The sum of three numbers is 150. The largest number is ten more than the sum of the other two and 10 less than three times the difference of the other two. What are the three numbers?

3. Given:

$$\begin{aligned} A + B &= 1 \\ B + C &= 2 \\ C + D &= 3 \\ D + E &= 5 \\ E + F &= 8 \\ F + G &= 13 \\ G + H &= 21 \\ H + I &= 34 \\ I + J &= 55 \end{aligned}$$

determine the numerical value of $A + J$

N.E.A.M.L - PLAYOFFS - 1994

**Round 3 - Geometry**

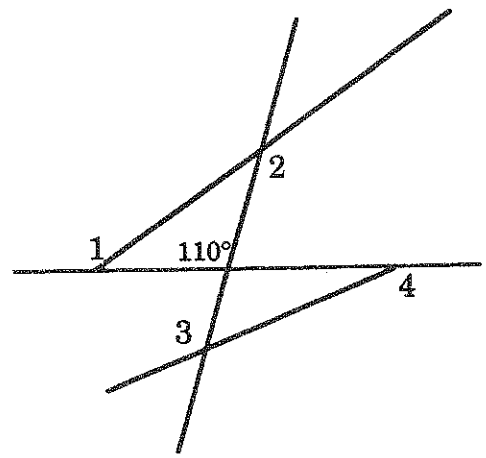
1. ______________________

2. ______________________

3. ______________________

1. A triangle with sides of lengths 10 cm., 24 cm, and 26 cm. is inscribed in a circle. What is the exact number of square centimeters in the area of the region interior to the circle and exterior to the triangle?

2. Find the value of
   $m\angle 1 + m\angle 2 + m\angle 3 + m\angle 4$.




3. The parallel sides of a trapezoid measure 3 and 9 inches while the non-parallel sides measure 4 and 6 inches. A line parallel to the base divides the trapezoid into two trapezoids having equal perimeters. Find the ratio in which the non-parallel sides are divided, long segment : short segment.

# N.E.A.M.L - PLAYOFFS - 1994

## Round 4 - Algebra II

1. ______________________

2. ______________________

3. ______________________

1. If $4 \log m + \log n = \dfrac{\log w - \log 32}{5}$, express w in terms of m and n.

2. Let a, b and c be positive integers which are not necessarily distinct. If $a < 2b$, $b > 3c$ and $a + b + c = 100$, determine the greatest possible value for c.

3. The sequence 1, 2, 5, 10, 17, 26, 37, 50, . . . is such that $A_{n+1} = A_n + (2n - 1)$. What is the 100th number of the sequence?

# N.E.A.M.L - PLAYOFFS - 1994

## Round 5 - Analytic Geometry

1. ______________________

2. ______________________

3. ______________________

1. Given A(–9, 3), B(11, 8), C(7, –5), and D(–1, –10). M is the midpoint of $\overline{AC}$ and N is on $\overline{BD}$ such that BN = $\frac{1}{5}$ ND. Find the value of MN.

M(–1, –1)  $10^2 + 6^2 = 136$
N(9, 5)  $\sqrt{136}$  $2\sqrt{34}$

2. Find the equation of the hyperbola which passes through (6,–2), whose center is the vertex of $x + y^2 - 4 = 0$ and whose asymptotes are parallel to the axes.

$y^2 = -x + 4 = -1(x - 4)$  $y(x-4) = k$
V(0,4)

3. A circle whose center lies in the first quadrant is tangent to the y-axis at P(0,k). The circle intersects the x-axis at points whose x values are the roots of the quadratic equation $ax^2 + bx + c = 0$. Express the value of k in terms of the coefficients of the quadratic equation.

## N.E.A.M.L - PLAYOFFS - 1994

**Round 6 - Complex Numbers, Trigonometry**

1. ______________

2. ______________

3. ______________

1. Find all values of x, $0° \le x < 360°$ such that $\sin(x - 90°) + \cos(180° - x) = \tan 135°$.

2. Let $z_1$, $z_2$, and $z_3$ be complex numbers such that $|z_1| = |z_2| = |z_3| = 1$ and $z_1 + z_2 + z_3 = 0$. Determine the numerical value of $|z_1 - z_2|$.

3. For x in radians, determine the number of solutions in [0, 2] to the equation

$$\sin^2 x^2 = \frac{1}{4}$$

# N.E.A.M.L - PLAYOFFS - 1994

## TEAM ROUND

Large Schools: 4 points each       Medium and Small Schools: 3 points each

ANSWERS:

1. ______________________     4. ______________________

2. ______________________     5. ______________________

3. ______________________     6. ______________________

1. There are more than two solutions to the problem:

$$\begin{array}{r} F\ O\ U\ R \\ -\ T\ W\ O \\ \hline T\ W\ O \end{array}$$

Where each letter represents a distinct one-digit whole number . Neither F nor T equals 0. Find two of the solutions.

2. There are 24 different four digit natural numbers that can be formed using the digits 3, 5, 7 and 8 once in each number. If the numbers were arranged from smallest to largest, what would be the positive difference between the 21st and 10th numbers in this arrangement

3. $\Delta$ ABC has a right angle at B. Let BC = 10, AB = 18 and let point D be between A and B such that $\Delta$DBC is isosceles. If m$\angle$ACD = arctan k, determine the value of k.

4. Determine the number of integer solutions for $0 \le x \le 10$ of the inequality:

$$\left|2\cos\frac{\pi x}{5}\right| > 1$$

5. Given rhombus ABCD with sides of length 4 and m$\angle$D = 72. Let P be the center of the circle passing through points A, D, and C. If twice the ratio of DP to PB = $\sqrt{a} + b$, determine integers (a, b)

6. Find all integral value(s) of w so that $w^2 + 11w + 5$ will be a perfect integral square

N.E.A.M.L. Playoffs - 1994

# ANSWER SHEET

## ROUND 1

1. 11

2. 1.1

3. 33

## ROUND 2

1. $\frac{16}{9}$

2. 20, 50, 80

3. 34

## ROUND 3

1. $169\pi - 120$

2. 580

3. 4 : 1

## ROUND 4

1. $w = 32m^{20}n^5$

2. 24

3. 9802

## ROUND 5

1. $2\sqrt{34}$

2. $xy - 4y = -4$

3. $\sqrt{\frac{c}{a}}$

## ROUND 6

1. 60°, 300°

2. $\sqrt{3}$

3. 3

## TEAM ROUND

1. $\begin{array}{cccc} 1876 & 1672 & 1530 & 1468 \\ \underline{938} & \underline{836} & \underline{765} & \underline{734} \\ 938 & 836 & 765 & 734 \end{array}$ (any two)

2. 2754

3. $\frac{2}{7}$

4. 7

5. $(5, -1)$

6. $20, -31$

# NEAML '94 Solutions Outline

Rd 1

1. $\frac{1}{16}(179-3)$

2. $N = 10/99 \quad .\overline{1} = 1/9$

3. 24, 54, 84, 114, ...

Rd 2

1. $\sqrt{16x} = 4\sqrt{x} \quad 3\sqrt{x} = 4$

2. $a+b+c=150 \quad a = b+c+10$
   $a = 3(b-c) - 10$

3. $A - C = -1, \rightarrow A + D = 2 \rightarrow$
   $A - E = -3$ etc.

Rd 3

1. rt $\triangle$ $r = 13 \quad \pi 13^2 - \frac{1}{2} 10 \cdot 24$

2. $m\measuredangle 1 = 100 + b \quad m\measuredangle 2 = 110 + a$
   $m\measuredangle 3 = 110 + d \quad m\measuredangle 4 = 110 + c$

3. $m + n + 3 + c =$
   $4 - m + 6 - n + 9 + c$

Rd 4

1. $m^4 n = \sqrt[5]{w}/2$

2. Let $a = 1 \quad b + c = 99, \ b > 3c$
   $b = 75 \quad c = 25 - 1 = 24$

3. 1 2 5 10 17 26
   1 3 5 7 9
   2 2 2 2

Rd 5

1. $M(-1,-1) \quad N(9,5)$

2. $V(4,0) \quad (x-4)y = -4$

3. $r_1, r_2$ are roots
   $k^2 = r_1 r_2 = \frac{c}{a}$

Rd 6

1. $-\cos x - \cos x = -1$

2. $z_1, z_2, z_3$ equally spaced on unit circle

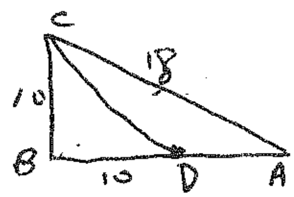
3. $\sin x^2 = \frac{1}{2}$ or $\sin x^2 = -\frac{1}{2}$

Team Round

1. trial & error

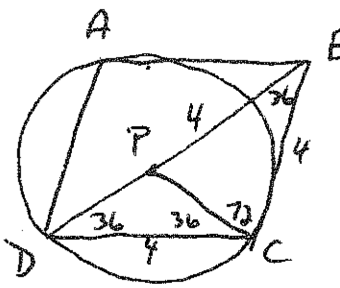
2. 10th #1, 5783
   21st #15 8573

3.




4. period = 10
   $\frac{\pi x}{5} = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

5.




$\frac{DB}{DC} = \frac{BC}{DP}$

$\frac{4+DP}{4} = \frac{4}{DP}$

6. $w^2 + 11w = 20$ or $220$ or $620$ etc.

N.E.A.M.L. PLAYOFFS - 1991

Round 1 - Arithmetic

1. ____________________

2. ____________________

3. ____________________

1. On a scale drawing of a playground, .75 inch = 2 yards. If the actual distance between the swing and the sandbox is 47 feet, how many inches apart are they on the drawing?

2. Find the sum of all natural numbers N, $425 \le N \le 475$, which are divisible by 6 but by neither 4 nor 5.

3. A book has 201 pages, Following page 1, pages 2 and 3 are facing pages, pages 4 and 5 are facing, and so on. Let n be the number of times the product of two facing pages is 2 more than a multiple of 6. Determine the value of n.

N.E.A.M.L. PLAYOFFS - 1991

Round 2 - Algebra 1

1. ______________________

2. ______________________

3. ______________________

1. Find all multiples of 5 which satisfy the statement: $|2.5z - 3.7| \leq 22.8$

2. Factor completely: $4^{2a} - 21 - 5(8^{2a/3} + 3)$

3. At a certain time, Alina, Brett, and Carlos start to solve a certain number of problems . Alina solved 6 a day and finished them 4 days after Brett. Carlos solved 3 more a day than Brett and finished 2 days before Brett. How many problems, p, were there, and how many days did it take each person? Answer as (p, A, B, C) where A, B, C, represent numbers of days taken by the people whose names begin with those letters, respectively.

$p = 6n$

$6x$
$6(x+1)$

$\cancel{A = 6x}$
$n = 6A$

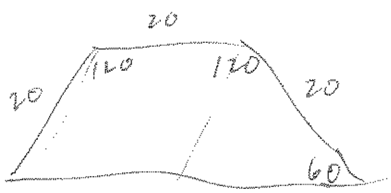
N.E.A.M.L. PLAYOFFS - 1991

Round 3 - Geometry

1. ____________________

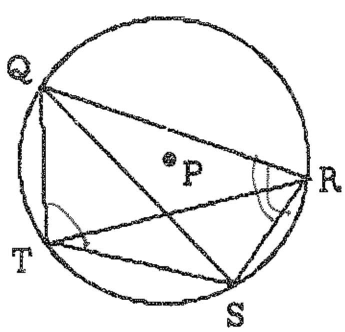
2. ____________________

3. ____________________




1. In an isosceles trapezoid, two of the angles measure 120° and the lengths of three of the sides are 20. Find the length of the fourth side.

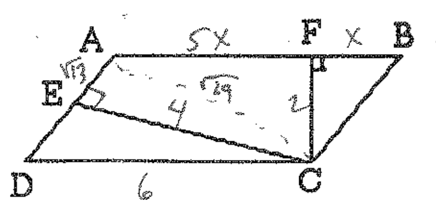
2.




Quadrilateral QRST is inscribed in ⊙ P.

$m\angle QTS = x^2 + 2x + 115$ and $m\angle QRS = 3x + 71$.
Find the number of degrees in $m\angle QTS$.

3. In parallelogram ABCD, $\overline{CF} \perp \overline{AB}$ and $\overline{CE} \perp \overline{AD}$. If $CF = 2$, $CE = 4$, and FB is one-sixth of AB, then determine the ratio of the area of AFCE to the area of ABCD.




$(AC)^2 = 25x^2 + 4 \qquad AC = \sqrt{29}$

$(AE)^2 = 25x^2 - 12$

$AE = \sqrt{13}$

N.E.A.M.L. PLAYOFFS - 1991

Round 4 - Algebra 2

1. ____________________

2. ____________________

3. ____________________

1. Find the value of x which satisfies: $\log_2 [\log_4 (\log_3 x)] = -1$

2. A side of an equilateral triangle is 24 inches. The midpoints of its sides are joined to form an inscribed equilateral triangle. If this process is continued without end, find the number of inches in the sum of the lengths of the altitudes of the triangles.

3. a, b, c, d represent four numbers in geometric progression whose sum is 65. $a - b$ is 2.25 times $c - d$. Find the four numbers.

N.E.A.M.L. PLAYOFFS - 1991

Round 5 - Analytic Geometry

1. ____________________

2. ____________________

3. ____________________

1. Given A(7, −2) and B(25, 10), find the coordinates of point C such that AC = 5/6 AB and AC + CB = AB.

2. Find the distance between the positive intercepts in the graph of $4x^2 + 16x + y^2 - 12y + 16 = 0$

3. Determine the coordinates of the points in the first quadrant whose distance from $3x + 4y = 12$ is 5 and whose distance from (4,0) is $\frac{5\sqrt{13}}{3}$.

N.E.A.M.L. PLAYOFFS - 1991

TEAM ROUND

Large Schools: 4 points each    Medium and Small Schools: 3 points each

ANSWERS:

1. ______________________    4. ______________________

2. ______________________    5. ______________________

3. ______________________    6. ______________________

1. Determine all positive integral values of n such that the equation:

$$\frac{1}{x-7}+\frac{n}{x-1}=1$$

has 2 distinct rational solutions.

2. When $kz^2 + 3z + 4$ is divided by $z - 2$, the remainder is p, the quotient is f(z). When $kz^2 + 3z + 4$ is divided by $z + 3$, the remainder is q, the quotient is g(z). If $p = q$, find the value(s) of k.

3. $\Delta$ ABC has a right angle at C. If $\tan A - \tan B = 1.5$, then $n \le m\angle A \le n + 1$ for n an integer. Determine n. Note: $\tan 36° = .7265$ and $\tan 37° = .7536$.

4. On a sphere of radius 8, six congruent circles are drawn so that each is externally tangent to 4 others. Determine the radius of each circle.

5. Determine the range of y in the expression: $\sqrt{y} = 3\sqrt{x} - x$

6. In $\Delta$ ABC, AC = AB = m and BC = n with n > m. Point D is chosen on $\overline{BC}$ so that $m\angle ADB = 60$. If m, n, and AD are integers with BD = 15, determine all possible ordered pairs of integers (m, n).

N.E.A.M.L. PLAYOFFS - 1991

ANSWER SHEET

ROUND 1

1. 5 7/8 or 5. 875

2. 1800

3. 33

ROUND 2

1. –5, 0, 5, 10

2. $4(2^a + 3)(2^a - 3)(4^{a-1} + 1)$
$(4^a + 4)$

3. (72, 12, 8, 6)

ROUND 3

1. 40

2. 115, 118

3. 7/12

ROUND 4

1. 9

2. $24\sqrt{3}$

3. 27, 18, 12, 8 or 135, –90, 60, –40

ROUND 5

1. (22,8)

2. $4\sqrt{5}$

3. (29/3, 2) (13/3, 6)

ROUND 6

1. $(48 - 25\sqrt{3})/39$

2. $-.5 - .5i\sqrt{3}$

3. 75/7

TEAM ROUND

1. 4, 6, 10

2. 3

3. 63

4. $4\sqrt{2}$

5. $0 \le y \le 81/16$

6. (13, 22) and (13, 23)

# NEAML PLAYOFFS 1991 SOLUTIONS OUTLINE

## Rd. 1

1. $\frac{3''}{4} = 6' \rightarrow \frac{1}{8}'' = 1'$

2. 426, 438, 462, 474

3. $(4+6n)(5+6n) = 2 + 6(3+9n+6n^2)$

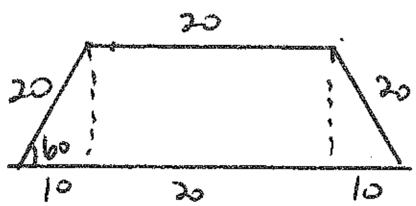
## Rd 2

1. $-22.8 \le 2.5z - 3.7 \le 22.8$

2. $4^{2a} - 5 \cdot 4^a - 36$

3. $c(c+6) = x(c+2)$  C = days - Carlos
   $c(x+3) = x(c+2$  x = rate - Brett

## Rd 3

1. 


2. $x^2 + 2x + 115 + 3x + 71 = 180$
   $x^2 + 5x + 6 = 0$

3. $AF = 5x$  $BF = x$  $\Delta BFC \sim \Delta DEC$
   $ED = 2x$  $BC = 3x$
   $$\frac{a(ABCD) - a(FBC) - a(CED)}{a(ABCD}$$
   $$1 - \frac{4x + x}{12x}$$

## Rd 4

1. $\log_4(\log_3 x) = \frac{1}{2}$

2. $s = 24 \rightarrow h = 12\sqrt{3}$
   $S = \frac{12\sqrt{3}}{1/2}$

3. $a - ar = \frac{9}{4}(ar^2 - ar^3)$
   $r = \pm\frac{2}{3}$
   $a + \frac{2}{3}a + \frac{4}{9}a + \frac{8}{27}a = 65$

## Rd 5

1. $\frac{5}{6} \cdot 18 = 15$  $\frac{5}{6} \cdot 12 = 10$

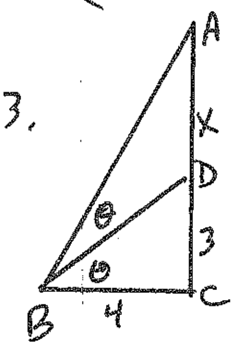
2. $4(x+2)^2 + (y-6)^2 = 36$
   $V(-2, 6)$
   $(0, 6+2\sqrt{5})$ $(0, 6-2\sqrt{5})$

3. $\frac{|3a + 4b - 12|}{5} = 5$  $a = \frac{37 - 4b}{3}$
   $(a-4)^2 + b^2 = \frac{25 \cdot 13}{9}$

## Rd 6

1. $\sin x = \frac{4}{5}$  $\tan x = -\frac{4}{3}$

2. $\left(1 \operatorname{cis} \frac{2\pi}{3}\right)^{1991} = \frac{3982\pi}{3}$

3. 


   from $\tan 2\theta$
   $$\frac{x+3}{4} = \frac{3/2}{1 - 9/16}$$

NEAML Playoffs - 1991- Team Round Soln. Outline

1. $x=\frac{(n+9)\pm\sqrt{(n+9)^2-4(8+7n)}}{2}$ Let D = $t^2$

$(n+9)^2-4(8+7n)=t^2 \to n^2-10n+49=t^2$

or $(n-5)^2+24=t^2$ or $t^2-(n-5)^2=24$

$[t+(n-5)][t-(n-5)]=24$

2. $4k+6+4=9k-9+4$

3. $\tan B=\frac{1}{\tan A}$ $2\tan^2 A-3\tan A-2=0$ $\tan A=\frac{3\pm5}{4}$

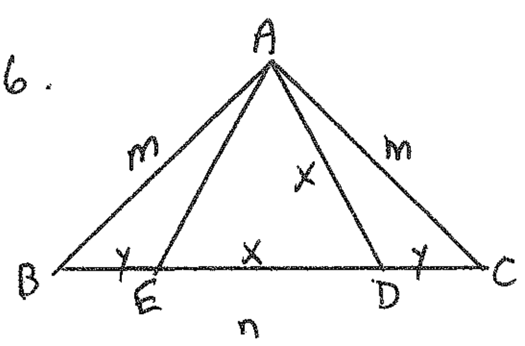
$\theta = m\angle ABC = m\angle BAD$ $90-m\angle DAC=2\theta$

if $m\angle DAC=36 \to 2\theta=54$ $\theta=27 \to m\angle BAC=63$

4. Assume circles are on the six faces of a cube. If O is center of sphere, P is a point of tangency and R is center of one circle, then OPR is an isos rt $\Delta$, $OP=8$, $\therefore PR=4\sqrt{2}$

5 $x-3\sqrt{x}+\sqrt{y}=0$ is quadratic in $\sqrt{x}$

$\sqrt{x}=\frac{3\pm\sqrt{9-4\sqrt{y}}}{2}$

6. 


Choose E: $m\angle AEC=60$, set $y=BE$

$x+y=15$ $n=x+2y$

$m^2=x^2+y^2-2xy\cos 120=225-xy$

$xy$ can be 29, 56, 81, 104, 125, 144, 161

$x=7$ $y=8$ or $x=8$ $y=7$

MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

NEW ENGLAND PLAYOFFS – 2004

**Round 1 Arithmetic and Number Theory**

1. ____________________

2. ____________________

3. ____________________

1. If $N = 9{,}998^2 + 4(9{,}998)$, determine the number of digits in $N$.

2. The number $1 - 0.66_9$ is what number in base 3.

3. Given $n$ is a positive integer, $n \leq 2004$, $n$ equals the sum of 3 consecutive positive integers, and $n$ equals the sum of 4 consecutive positive integers. How many different values are there for $n$?

MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

NEW ENGLAND PLAYOFFS – 2004

**Round 2 Algebra 1**

1. ______________________

2. ______________________

3. ______________________

1. $\dfrac{x^{-3/2}\cdot\sqrt[5]{y}}{x^{-7/3}\cdot\sqrt[10]{y}}$ can be written as $\sqrt[n]{x^a y^b}$ where $a$ and $b$ are integers. What is the minimum possible sum of $a$, $b$, and $n$?

2. There are 24 students in a classroom. Six move from the left side of the classroom to the right side and now the right side has as many students as the left side used to have. How many did the left side have originally?

3. Given $2^{\pi}x+\left(2^{\pi}+5\right)y=3^{\sqrt{2}}x+\left(3^{\sqrt{2}}+5\right)y$, determine the value of $\dfrac{x}{y}$.

**MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES**

**NEW ENGLAND PLAYOFFS – 2004**

**Round 3 – Geometry**

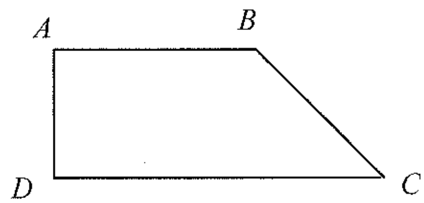
**1.** ______________________

**2.** ______________________

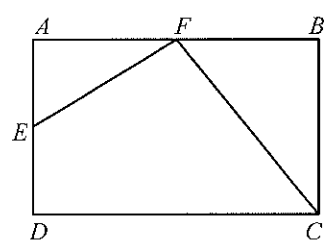
**3.** ______________________

**Diagrams are not drawn to scale.**

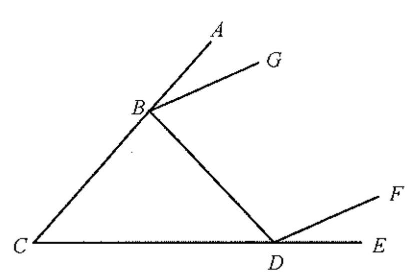
1. In quadrilateral $ABCD$, $m\angle D = m\angle A = 90°$ and $m\angle C = 45°$. If $AB = 17$ and $AD = 12$, determine the number of square units in the area of $ABCD$.




2. $ABCD$ is a rectangle, $E$ and $F$ are midpoints of $\overline{AD}$ and $\overline{AB}$ respectively. If $EF = 2\sqrt{3}$ and $FC = \sqrt{13}$, compute $\dfrac{AB}{AD}$.




3. If $m\angle ABG = \dfrac{1}{6} m\angle ABD$, $m\angle FDE = \dfrac{1}{6} m\angle BDE$ and $\overline{BG} \| \overline{DF}$, find the number of degrees in the measure of $\angle C$.



**MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES**

**NEW ENGLAND PLAYOFFS – 2004**

**Round 4 – Algebra 2**

**1.** ____________________

**2.** ____________________

**3.** ____________________

1. For $a$, $b > 0$, compute the minimum value of $\left(\sqrt[3]{\frac{a}{b}}+\sqrt[3]{\frac{b}{a}}\right)^3$.

2. Let $[\,x\,]$ = the greatest integer less than or equal to $x$. Determine the domain of the function $y=\sqrt{1-\left[x^2\right]}$. Write your answer in inequality form.

3. Solve for real $x$: $\sqrt{\frac{x-1}{x}} > \sqrt{\frac{x}{x-1}}$.

MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES

NEW ENGLAND PLAYOFFS – 2004

**Round 5 – Analytic Geometry**

1. ______________________

2. ______________________

3. ______________________

1. Line $\ell$ passes through the origin and divides square $ABCD$ into two regions of equal area. Given $A(7, 11)$, $B(8, 11)$, $C(8, 10)$ and $D(7, 10)$, determine the slope of $\ell$.

2. Point $B$ is in the first quadrant and lies on the line $y = 7x + 4$. Given $A(0, 4)$, $C(0, 0)$, the area of $\Delta ABC = m$ square units, and the slope of $\overline{BC} = m$, find the coordinates of $B$.

3. Given $O(0, 0)$, $A(0,6)$, $B(k, 0)$ for $k > 0$, if $P$ lies on the positive $x$-axis so that $\overline{AP}$ is a bisector of $\angle OAB$ and $P$ is a trisection point of $\overline{OB}$, determine the slope of $\overline{AP}$.

**MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES**

**NEW ENGLAND PLAYOFFS – 2004**

**Round 6 – Trig and Complex Numbers**

**1.** ________________

**2.** ________________

**3.** ________________

1. Determine the value of $\cos(127.5°)\cos(7.5°)-\cos(37.5°)\sin(187.5°)$.

2. Determine the value of $i+2i^2+3i^3+\ldots+2004i^{2004}+\frac{2005}{i}+\frac{2006}{i^2}+\ldots+\frac{4008}{i^{2004}}$.

3. If $\frac{\sin 4x}{1+\cos 4x}=f(y)$ where $y=\tan x$, determine a formula for $f(y)$ in terms of $y$.

**MASSACHUSETTS ASSOCIATION OF MATHEMATICS LEAGUES**

**NEW ENGLAND PLAYOFFS – 2004**

**Team Round**

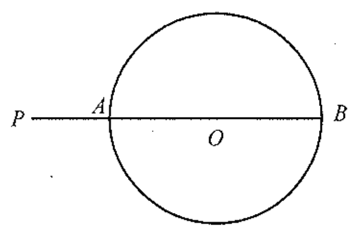
1. ______________________ 4. ______________________

2. ( ______ , ______ ) 5. ______________________

3. ______________________ 6. ______________________

1. Points A and B are on circle $O$ such that $\overline{AO} \perp \overline{OC}$, and $AC = 6$. Point $B$ is on minor arc $\widehat{AC}$. Determine the minimum value of the area interior to quarter circle AOC and exterior to quadrilateral ABCO.

2. Given $\overline{PAOB}$ with $A$ and $B$ lying on circle $O$, $PA$ and $PB$ are the roots of $x^2 - 8x + 13 = 0$. Determine the ordered pair $(a, b)$ where $a$ is the length of the tangent to the circle from $P$ and $b$ is the area of the circle.




3. If $\sin x - \cos x = n$, determine $\sin^3 x - \cos^3 x$ in terms of $n$.

4. If $S = 1 - 2 - 3 + 4 + 5 - 6 - 7 + 8 + 9 - 10 - 11 + 12 + 13 - 14 - 15 + \ldots$, then let $S_1 = 1$, $S_2 = 1 - 2$, $S_3 = 1 - 2 - 3$, $S_4 = 1 - 2 - 3 + 4$, $S_5 = 1 - 2 - 3 + 4 + 5$, and so on. Find $n$ so that $S_n = -1$ for the 2004th time. The series S has these two properties: taking the absolute value of each term produces the set of positive integers and the terms are signed in consecutive groups of 4 beginning with +, –, –, +.

5. Find the solution $(x, y)$ to the following system if $a_i = 2a_{i-1} + 3$ and $a_1 > 0$ for $i \in \{2,3,4,5,6\}$

$$a_1 x + a_2 y = a_3$$
$$a_4 x + a_5 y = a_6$$

6. Determine all ordered pairs $(x, y)$ that satisfy the following system on the set of real numbers:

$$\begin{cases} x + x|y| = 16 \\ x + y|x| = -9 \end{cases}$$

**NEW ENGLAND ASSOCIATION OF MATHEMATICS LEAGUES**

**PLAYOFFS – 2004**

*Answer Sheet*

**Round 1**

1. 8
2. $.021_3$ (or .021 with base 3 implied)
3. 166

**Round 2**

1. 15
2. 58
3. –1

**Round 3**

1. 276
2. $\sqrt{35}$
3. 36 (36°)

**Round 4**

1. 33
2. $-\sqrt{2} < x < \sqrt{2}$
3. $x < 0$

**Round 5**

1. $\frac{7}{5}$
2. (4, 32)
3. $-\sqrt{3}$

**Round 6**

1. $-\frac{1}{2}$
2. $f(y) = \frac{2y}{1-y^2}$
3. 167

**Team**

1. $9\left(\frac{\pi}{2} - \sqrt{2}\right)$ or equivalent
2. $\frac{492}{25}$
3. $n\left(\frac{3-n^2}{2}\right)$ $\left(\text{or } \frac{3n-n^3}{2} \text{ or } \frac{3}{2}n - \frac{1}{2}n^3\right)$
4. 8014
5. (-2, 3)
6. $\left(\frac{7}{2}, -\frac{25}{7}\right)$

# NEW ENGLAND ASSOCIATION OF MATHEMATICS LEAGUES

## PLAYOFFS – 2004 – Solutions

### Round 1

1. $N = \left(9{,}998^2 + 4 \cdot 9{,}998 + 4\right) - 4 = (9{,}998 + 2)^2 - 4 = 10{,}000^2 - 4 =$
$\left(10^4\right)^2 - 4 = 10^8 - 4$. Since $10^8$ has 9 digits, $N$ has 8 digits.

2. $1 - 0.66_9 = 1 - \left(\frac{6}{9} + \frac{6}{81}\right) = 1 - \left(\frac{2}{3} + \frac{2}{27}\right) = 1 - 0.202_3 = 0.021_3$

3. $n = 3x + 3 = 4y + 6$ for positive integers $x$ and $y$. Therefore $4y = 3(x-1)$. Since $4y + 6 \le 2004 \Rightarrow y \le 499.5$, then the possible values for $y$ are 3, 6, 9, … 498, which are 166 possibilities.

### Round 2

1. $$\frac{x^{-3/2} \cdot \sqrt[5]{y}}{x^{-7/3} \cdot \sqrt[10]{y}} = x^{5/6} \cdot y^{1/10} = \sqrt[30]{x^{25} y^3}$$

2. Let $x$ = the original number of students on the left side and $24 - x$ be the original number on the right side. If 6 students move from left to right then the right side has $30 - x$. Thus, $x = 30 - x$ so $x = 15$.

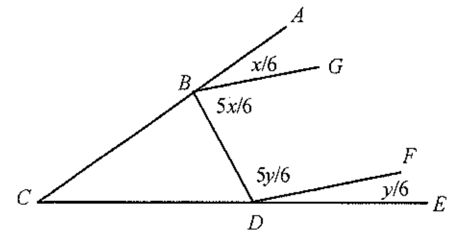
3. Let $2^\pi = a$ and $3^{\sqrt{2}} = b$. Then $ax + (a+5)y = bx + (b+5)y$
$ax + (a+5)y = bx + (b+5)y$ gives $(a-b)x = \left[(b+5) - (a+5)\right]y = (b-a)y$. Then
$\frac{x}{y} = -1$.

### Round 3

1. Draw $\overline{BE} \perp \overline{DC}$, then BE = 12 EC = 12, DC = 29

2. Let $AB = 2x$ and $AD = 2y$. Using $\Delta AEF$, $x^2 + y^2 = \left(2\sqrt{3}\right)^2 = 12$. Using $\Delta FBC$, $x^2 + (2y)^2 = \left(\sqrt{13}\right)^2 \rightarrow x^2 + 4y^2 = 13$. Subtracting the first from the second gives $3y^2 = 1 \rightarrow y = \frac{1}{\sqrt{3}}$. Then $x^2 + \frac{1}{3} = 12 \rightarrow x = \sqrt{\frac{35}{3}}$. Hence, $\frac{x}{y} = \frac{AB}{AD} = \sqrt{35}$.

3. Let $m\angle ABD = x \rightarrow m\angle GBD = \dfrac{5x}{6}$. Let $m\angle BDE = y \rightarrow m\angle BDF = \dfrac{5y}{6}$. Since $\overline{BG} \parallel \overline{DF}$, then $\dfrac{5x}{6} + \dfrac{5y}{6} = 180$ giving $x + y = 216$. Since $m\angle CBD = 180 - x$ and $m\angle BDC = 180 - y$, then $m\angle C = 180 - (180 - x + 180 - y) = x + y - 180 = 216 - 180 = 36$.

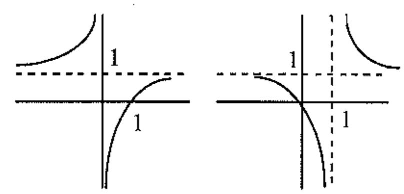



## Round 4

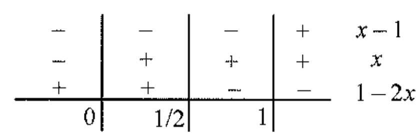
1. $M:N = 3:8 \rightarrow N = \dfrac{8}{3}M$. $M^2 - 3N = M \rightarrow M^2 - 8M = M$

2. For $\sqrt{1-[x^2]}$ to be real, $[x^2] \le 1 \rightarrow [x^2] = 0,1 \rightarrow 0 \le x^2 < 2 \rightarrow -\sqrt{2} < x < \sqrt{2}$.

3. Graphing $y = \dfrac{x-1}{x}$ and $y = \dfrac{x}{x-1}$ shows us that the domain of the inequality is $x < 0$ or $x > 1$. It is also shows that the solution set is $x < 0$.



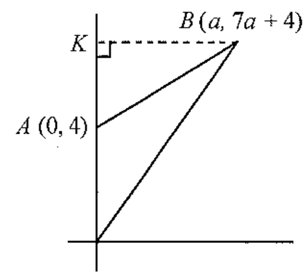

Alternate solution: Squaring both sides we obtain: $\dfrac{x-1}{x} > \dfrac{x}{x-1} \rightarrow \dfrac{x-1}{x} - \dfrac{x}{x-1} > 0 \rightarrow \dfrac{1-2x}{x(x-1)} > 0$. Solving using the sign graph we obtain $x < 0$ or $1/2 < x < 1$, but reject the latter set since it violates the domain of the inequality.




## Round 5

1. A line passing through the center of a square bisects the square. The center of $ABCD$ is $M\left(\dfrac{15}{2}, \dfrac{21}{2}\right)$ and the slope of $\ell$ is $\dfrac{21/2-0}{15/2-0} = \dfrac{7}{5}$.

2. Let the coordinates of $B$ be $(a, 7a+4)$. Then




$\frac{1}{2} \cdot AC \cdot BK = slope \rightarrow \frac{1}{2} \cdot 4a = \frac{7a+4}{a} \quad \rightarrow$

$2a^2 - 7a - 4 = 0 \;\rightarrow\; a = 4$ gives $B(4, 32)$.

3. By the Triangle Angle Bisector Theorem, if $\overline{AP}$ bisects $\angle OAB$ then $\frac{OA}{AB} = \frac{OP}{PB}$.

Since $P$ is a trisection point then $\frac{OP}{PB} = \frac{1}{2} \rightarrow \frac{OA}{AB} = \frac{1}{2}$. Thus, $\Delta OAB$ is a 30-60-90 right triangle making $AB = 12$ and $OB = 6\sqrt{3}$. Hence, the $x$-coordinate of $P$ is $2\sqrt{3}$, making the slope of $\overline{AP}$ equal $\frac{6-0}{0-2\sqrt{3}} = -\frac{3}{\sqrt{3}} = -\sqrt{3}$.

**Round 6**

1. $\cos(127.5)\cos(7.5) - \cos(37.5)\sin(187.5) = -\sin(37.5)\cos(7.5) - \cos(37.5)[-\sin(7.5)]$
$= -(\sin 37.5\cos 7.5 - \cos 37.5 \sin 7.5) = -\sin 30 = -\frac{1}{2}$.

2. Group the first 2004 terms by 4's obtaining
$\left(i + 2i^2 + 3i^3 + 4i^4\right) + \cdots + \left(2001^{2001} + 2002^{2002} + 2003^{2003} + 2004^{2004}\right) =$
$\left(i - 2 - 3i + 4\right) + \cdots + \left(2001 - 2002 - 2003 + 2004\right) = \left(-2i + 2\right) + \cdots + \left(-2i + 2\right)$.
There are 501 such terms with a sum total of $-1002i + 1002$. Grouping the next 2004 terms in the same way we obtain 501 terms of $2i + 2$ for a total of $1002i + 1002$. The sum of both groups is 2004.

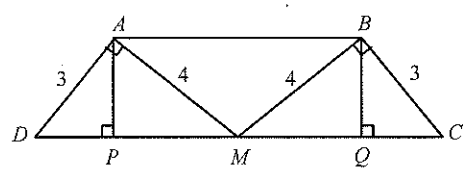
3. $\frac{sin 4x}{1 + cos\, 4x} = \frac{2\sin 2x \cos 2x}{1 + 2\cos^2 2x - 1} = \frac{\sin 2x}{\cos 2x} = \tan 2x = \frac{2\tan x}{1 - \tan^2 x}$. Thus, $f(y) = \frac{2y}{1 - y^2}$.

**NEW ENGLAND ASSOCIATION OF MATHEMATICS LEAGUES**

**PLAYOFFS – 2004 – Solutions – Team Round**

1. The minimum occurs when B bisects $\overset{\frown}{AC}$. The area of $\Delta$AOC = $\frac{1}{2}\left(3\sqrt{2}\right)^2$. The area of $\Delta$ABC is $\frac{1}{2}(6)\left(3\sqrt{2}-2\right)$.

2. Both $\Delta ADP$ and $\Delta BOQ$ are 3-4-5 triangles so $AD$ = 3 = $5x$ gives $x$ = 3/5. Then, $DP$ = $3x$ = 9/5 and $\underline{AP}$ = $4x$ = 12/5. The length of base $\overline{AB}$ = 10 – 2(9/5) = 32/5. The area of $ABCD$ equals $\frac{1}{2}\cdot\frac{12}{5}\left(10+\frac{32}{5}\right)=\frac{492}{25}$.




3. $\sin^3 x-\cos^3 x=(\sin x-\cos x)\left(\sin^2 x+\sin x\cos x+\cos^2 x\right)=n\left(1+\sin x\cos x\right)$. From $\left(\sin x-\cos x\right)^2=n^2$ we obtain $\sin^2 x-2\sin x\cos x+\cos^2 x=n^2\rightarrow$ $\sin x\cos x=\frac{n^2-1}{-2}=\frac{1-n^2}{2}$. Thus, $\sin^3 x-\cos^3 x$ = $n\left(1+\frac{1-n^2}{2}\right)$ = $n\left(\frac{3-n^2}{2}\right)$.

4. $S_1=1, S_2=-1, S_3=-4, S_4=0, S_5=5, S_6=-1, S_7=-8, S_8=0, S_9=9, S_{10}=-1$, Thus, $S_{2+4k}=-1$. Starting with $k$ = 0, The 2004$^{th}$ term occurs when $k$ = 2003, giving $n$ = 2 + 4(2003) = 8014.

5. The problem implies an invariant result. The following system leads to the answer:

$$\begin{aligned} x+5y&=13\\ 29x+61y&=125\end{aligned}\qquad \text{Solving gives } x = -2 \text{ and } y = 3.$$

More generally, starting with $a_1$ we obtain in turn $a_2=2a_1, a_3=4a_1+9$, . $a_4=8a_1+21, a_5=16a_1+45, and\ a_6=32a_1+93$ Using determinants, we have

$$x=\frac{\begin{vmatrix}a_3 & a_2\\ a_1 & a_5\end{vmatrix}}{\begin{vmatrix}a_1 & a_2\\ a_4 & a_5\end{vmatrix}}=\frac{(4a+9)(16a+45)-(32a+93)(2a+3)}{a(16a+45)-(2a+3)(8a+21)}=\frac{42a(a+3)}{-21a(a+3)}=-2.$$

$$y=\frac{\begin{vmatrix}a_1 & a_3\\ a_4 & a_6\end{vmatrix}}{-21a(a+3)}=\frac{-63a(a+3)}{-21a(a+3)}=3.$$

6. In the 1st quadrant the system becomes $x + xy = 16$ and $x + xy = -9$. That system has no solution. In the 2nd quadrant we have $x + xy = 16$ and $x - xy = -9$. Solving, we obtain $x = \frac{7}{2}$ which lies outside the quadrant. In the 3rd quadrant we obtain $x - xy = 16$ and $x - xy = -9$ which has no solution. Finally, in the 4th quadrant the system becomes $x - xy = 16$ and $x + xy = -9$. Solving, we obtain $x = \frac{7}{2}$ and $y = -\frac{25}{7}$ yielding the answer $\left(\frac{7}{2}, -\frac{25}{7}\right)$.