Team Round

GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 – DECEMBER 1998

TEAM ROUND

3 pts. 1. ________________

3 pts. 2. ________________

4 pts. 3. ________________

SAT APPROVED CALCULATORS ALLOWED ON THIS ROUND
except for the **TI-89 Calculator**, which is not allowed on the Team Round

1. The reciprocal of one more than the reciprocal of a number is k more than twice the number. If this number is **unique**, then the values for k can be put in the form $a \pm \sqrt{b}$. Find the sum $a + b$.

2. Given points O, the origin, A(0, 4), B(0, 6), and C(8, 0), the bisector of $\angle$ OBC and $\overline{AC}$ intersect at point D. Find the area of Δ BCD.

3. An isosceles triangle has two of its vertices on the positive x axis and its third vertex at (6, 4). If the slope of one of its legs is $\frac{4}{3}$, find all possible lines in the form Ax + By = C, where A, B, and C are relatively prime integers and **A > 0**, that contain (6, 4), do **not** have a slope of $\frac{4}{3}$, and contain a side of the isosceles triangle.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 – DECEMBER 1999

TEAM ROUND

3 pts. 1. ________________

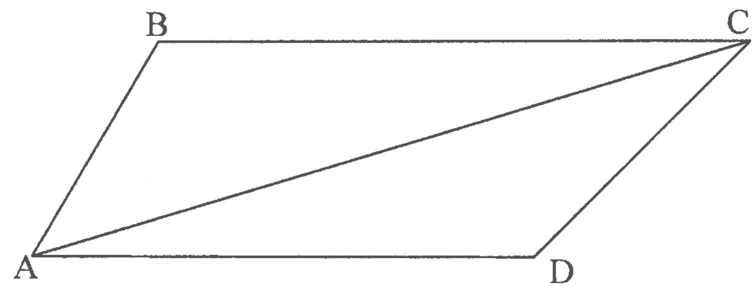
3 pts. 2. ________________

4 pts. 3. ________________

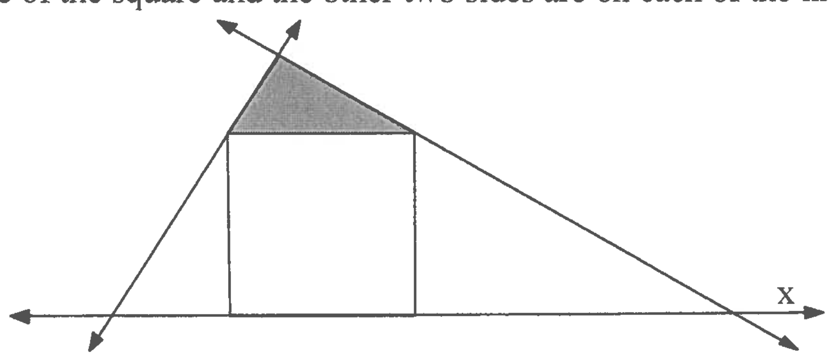
SAT APPROVED CALCULATORS ALLOWED ON THIS ROUND
except for the **TI-89 Calculator**, which is not allowed on the Team Round

1. If $\log_3 12 - \log_2 18 = a$ and $\log_2 3 = b$, find a in terms of b.

2. Given trapezoid ABCD, m $\angle$ BAD = 60°, m $\angle$ D = 135°, AD = 30, and CD = $8\sqrt{6}$, find the exact area of Δ ABC.



3. Given the lines $2x - y + 9 = 0$ and $x + 3y - 6 = 0$, a square is constructed with one side along the x axis and the other sides as shown. Find the area of the shaded triangle one of whose sides is a side of the square and the other two sides are on each of the lines.



GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 – DECEMBER 2000

TEAM ROUND

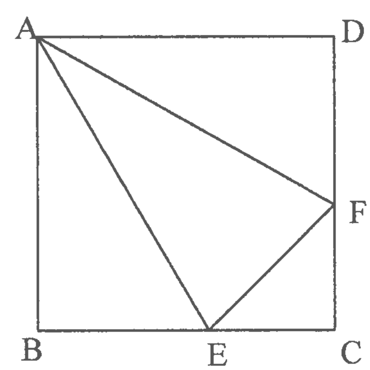
3 pts. 1. ____________________

3 pts. 2. ____________________

4 pts. 3. ____________________

SAT APPROVED CALCULATORS ARE ALLOWED IN THE TEAM ROUND, EXCEPT FOR CALCULATORS WITH SYMBOLIC MANIPULATION PROGRAMS, (FOR EXAMPLE THE TI-89) WHICH ARE NOT ALLOWED IN THIS ROUND.

1. Given square ABCD and isosceles ΔAEF with base $\overline{EF}$ such that $m\angle EAF = 30°$ as indicated on the diagram on the right. If CE = 2, find the exact area of ΔAEF.



2. If $\log_6 12 = k$, find $\log_2 3$ as a simplified expression in terms of k.

3. Given $0° < x < 45°$, $a > 4$ and $\tan x + \cot x = \sqrt{a}$, find in <u>simplest radical form</u> the value for $\cos 2x$ in terms of a.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 – DECEMBER 2001

TEAM ROUND (<u>12 MINUTES LONG</u>)

3 pts. 1. ________________

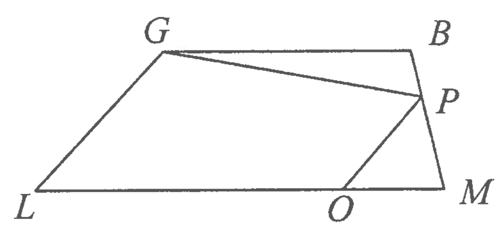
3 pts. 2. ________________

4 pts. 3. ________________

SAT APPROVED CALCULATORS ARE ALLOWED IN THE TEAM ROUND, EXCEPT FOR CALCULATORS WITH SYMBOLIC MANIPULATION PROGRAMS, (FOR EXAMPLE THE TI-89) WHICH ARE NOT ALLOWED IN THIS ROUND.

1. Given $L_1 : \{(x,y) \mid ax - 4y = -11\}$ and $L_2 : \{(x,y) \mid 5x + 6y = -4a\}$ intersect at point $P(-a,b)$, find all possible values for a.

2. Given $\overline{GB} \parallel \overline{ML}$, $\overline{BPM}$, $\overline{LOM}$, $GB : ML = 3:5$, $BP : PM = 1:2$, $LO : OM = 4:1$, and the area of quadrilateral $GLOP$ = 114, find the area of trapezoid $GBML$.



3. Given $a > 0$, find in terms of a the area of region $\Re$, $\{(x,y) \mid y \ge |2x - 4a| + a \text{ and } y \le x + 2a\}$.

GREATER BOSTON MATHEMATICS LEAGUE
MEET 3 – DECEMBER 2005

TEAM ROUND: Time Limit – 12 minutes

(3 pts) 1. ______________________

(3 pts) 2. ______________________

(4 pts) 3. ______________________

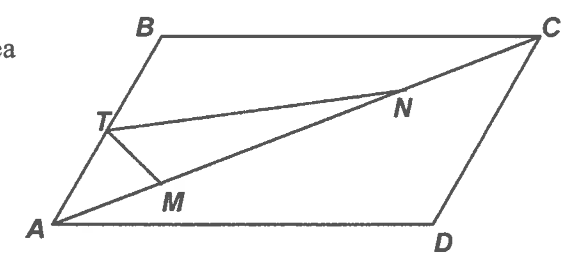
SAT APPROVED CALCULATORS ARE ALLOWED IN THE TEAM ROUND, EXCEPT FOR CALCULATORS WITH SYMBOLIC MANIPULATION PROGRAMS, (FOR EXAMPLE THE TI-89) WHICH ARE NOT ALLOWED IN THIS ROUND.

1. Determine the numerical value of the expression $\frac{a}{b}+\frac{b}{a}$, given

$$\frac{1}{\log_{\frac{a+b}{2}} c} = \frac{\log_c(\frac{a}{2})+\log_c(\frac{b}{4})}{2}$$

2. Given: Parallelogram $ABCD$ with points M and N on $\overline{AC}$ such that $AM : MC = 3 : 13$ and $AN : NC = 17 : 7$ and $AT = TB$

Find the ratio of the area of ΔTMN to the area of ΔACD.



3. $ABCD$ is a square and ABE is an equilateral triangle
Point E is in the exterior of the square. P is the centroid of ABE
$AD = 3x(x-6)$
$DC = 8(2x+3)$

PD^2 can be written in the form $a(b + c\sqrt{d}\ )$, where a, b, c and d are integers and d contains no perfect squares (> 1). Find the sum $b + c + d$.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 – DECEMBER 2006

TEAM ROUND

3 pts.	1.	______________
3 pts.	2.	______________
4 pts.	3.	______________

SAT APPROVED CALCULATORS ARE ALLOWED IN THIS ROUND.

1. Find all ordered pairs (P, Q) that make the following statement true:

   The sum of the roots of the equation $2Px^2 - 4Px + 4 - Qx^2 = 0$ is twice the product of the roots, and one of the roots is P.

2. If x, y and t are each positive integers and $x^2 - y^2 + 16xt + 64t^2 = 0$, determine the smallest possible value of $x + y + t$.

3. The system of lines defined by $\begin{cases} L_1 : ax + 2y = 6 \\ L_2 : x - 3y = b \end{cases}$ intersect at the point $P(2a, -\frac{b}{2})$.

   Determine the slope of L_1.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 – DECEMBER 2007

TEAM ROUND

3 pts. 1. ____________________

3 pts. 2. (__________ , __________)

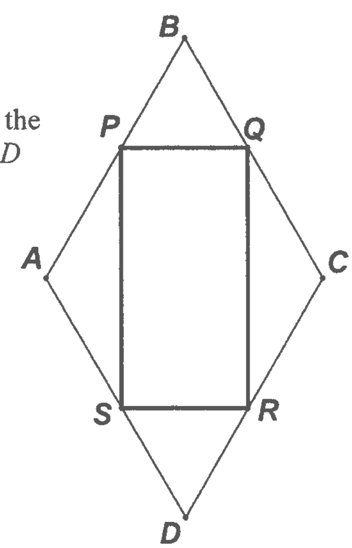
4 pts. 3. ____________________

SAT APPROVED CALCULATORS ARE ALLOWED IN THIS ROUND.

1. In ancient times, two merchants offered their bondsmen the following wages, in compliance with feudal minimum wage law.
   Merchant A offered his bondsman a starting salary of \$50.00 for the first 6 months.
   After that the bondsman would receive a \$5.00 increase in salary every 6 months.
   Merchant B offered his bondsman \$50.00 for the first year.
   After that the bondsman got a \$25.00 increase in salary once a year.
   At the end of k years both bondsmen's salaries were equal.
   In present day California, a minimum wage worker earns \$6.60 per hour and time and a half for hours over 40.
   Let the integer N denote the minimum number of <u>hours</u> required for a California minimum wage worker to earn at least as much as each bondsmen after k years.
   Determine the value of N.

2. The polar coordinates of points A and B are $(6, 60°)$ and $(8, 150°)$ respectively.
   Compute the polar coordinates (r, θ) of point C, the midpoint of $\overline{AB}$, where $0° < \theta < 360°$ and, if necessary, $r > 0$ is expressed as a simplified radical.

3. $ABCD$ is a rhombus and $PQRS$ is a rectangle. If $m\angle BAD = 120°$ and the area of $PQRS$ is half the area of $ABCD$, express the perimeter of $ABCD$ strictly in terms of PQ. Assume $PQ < PS$.



GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 – DECEMBER 2008

TEAM ROUND

3 pts. 1. ______________________________

3 pts. 2. _____, _____, _____, _____, _____

4 pts. 3. ______________________________

SAT APPROVED CALCULATORS ARE ALLOWED IN THIS ROUND.

1. I am thinking of integers greater than 30 that are each equal to the product of their proper divisors. Determine the sum of the five smallest such integers.

   Note: Proper divisors are positive divisors excluding the number itself.
   Ex: The proper divisors of 6 are 1, 2 and 3.

2. Determine the five smallest natural numbers that have exactly 18 positive factors.

3. Let $\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right)$ denote an ordered triple of "unit fractions", where a, b and c are integers and $a \le b \le c$. Determine all such ordered triples whose elements sum to $\frac{35}{48}$.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 – DECEMBER 2009

TEAM ROUND

3 pts.	1.	______________________
3 pts.	2.	______________________
4 pts.	3.	______________________

1. Find all values of x, $0° \le x < 360°$ which make the following statement true:

$$\sin 40° \sin 50° = \cos^2 x - \cos 60°$$

2. In ΔABC, $\angle B$ is bisected by $\overline{BD}$ (D on $\overline{AB}$), and the ratio of $AB : BC = 4 : 5$. If $DC = 16\frac{2}{3}$ and the perimeter of ΔABC is 66 inches, find the number of square inches in the area of ΔABD.

3. If $x > 1$, find all real values of t for which

$$\left(\log_t x^2\right)\left(\log_x 2t\right) = \log_8 t - \log_8 \frac{1}{2}$$

GREATER BOSTON MATHEMATICS LEAGUE

MEET 3 – DECEMBER 2010

TEAM ROUND

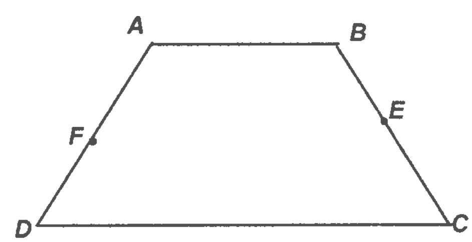
3 pts.	1.	______________________ years
3 pts.	2.	______________________
4 pts.	3.	______________________

CACULATORS ARE NOT ALLOWED IN THIS ROUND.

1. The ratio of the ages of father : mother : son is 16 : 12 : 5. They all have the same birthday. The ratio of father's age 8 years from today to the son's age 13 years from today will be 2 : 1. When the ratio of the mother's age to the son's age is 3 : 2, how old will the father be?

2. If $\left(\sqrt{2}cis195°\right)^7$ is written in $a + bi$ form, compute $a + b$.

3. In isosceles trapezoid $ABCD$, $AB = 6$, $DC = 12$, $BE : EC = 5 : 7$, $AF : FD = 5 : 4$. Compute the ratio of the area of ΔABF to the area of ΔDEC.

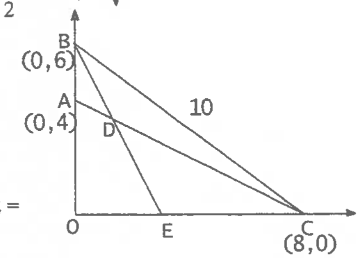


GBML '98

TEAM ROUND

1. $\frac{1}{\frac{1}{x}+1} = 2x + k \Rightarrow \frac{x}{1+x} = 2x + k \Rightarrow 2x^2 + (k+1)x + k = 0$. For there to be one solution for $x \Rightarrow (k+1)^2 - 8k = 0 \Rightarrow k^2 - 6k + 1 = 0 \Rightarrow k = \frac{6 \pm \sqrt{32}}{2} = 3 \pm \sqrt{8} \Rightarrow a + b = \mathbf{11}$

2. By the angle bisector theorem,
$OE:EC = 6:10 = 3:5 \Rightarrow E = (3, 0) \Rightarrow$
line $\overline{BE}$ is $y = -2x + 6$ and line $\overline{AC}$ is $y = -\frac{1}{2}x + 4$
Finding coordinates of D:
$-2x + 6 = -\frac{1}{2}x + 4 \Rightarrow x = \frac{4}{3}$ and $y = \frac{10}{3}$
Area of Δ BCD = area of Δ BCE – area of Δ DCE =
$\left(\frac{1}{2}\right)(5)(6) - \left(\frac{1}{2}\right)(5)\left(\frac{10}{3}\right) = \mathbf{\frac{20}{3}}$



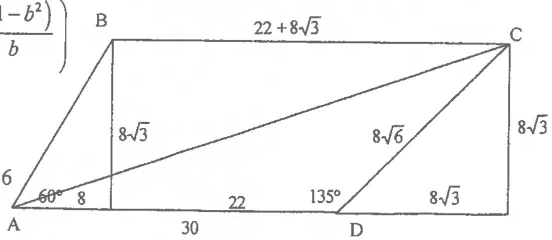
3. If the base of the isosceles triangles is along the x axis then the line would have slope $-\frac{4}{3}$ $\Rightarrow y - 4 = -\frac{4}{3}(x - 6) \Rightarrow 4x + 3y = 36$. To find another possibility $y - 4 = \frac{4}{3}(x - 6) \Rightarrow$ when $y = 0, x = 3$. The distance from (3, 0) to (6, 4) = 5 $\Rightarrow$ the line would intersect the x axis at (8, 0) $\Rightarrow$ the line through (8, 0) and (6,4) is $2x + y = 16$ $\Rightarrow$ Lines are
$\mathbf{4x + 3y = 36}$ or $\mathbf{2x + y = 16}$

GBML '99

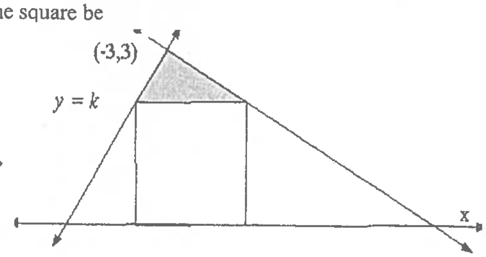
TEAM ROUND

1. Since $\log_3 12 - \log_2 18 = a$ and $\log_2 3 = b$,
$\frac{2\log 2 + \log 3}{\log 3} - \frac{2\log 3 + \log 2}{\log 2} = a \Rightarrow a = \frac{2\log 2}{\log 3} - \frac{2\log 3}{\log 2} = 2\left(\frac{\log 2}{\log 3}\right) - 2\left(\frac{\log 3}{\log 2}\right) \Rightarrow$
$a = \frac{2}{b} - 2b\left(\frac{2-2b^2}{b} \text{ or } \frac{2(1-b^2)}{b}\right)$

2. BC = $22 + 8\sqrt{3}$
height of Δ ABC = $8\sqrt{3}$
area of Δ ABC =
$4\sqrt{3}(22 + 8\sqrt{3}) = 88\sqrt{3} + 96$
or $8(11\sqrt{3} + 12)$



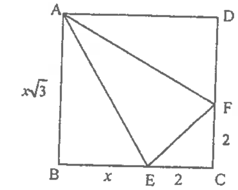
3. $2x - y + 9 = 0$ and $x + 3y - 6 = 0 \Rightarrow 6x - 3y + 27 = 0 \Rightarrow 7x + 21 = 0 \Rightarrow x = -3$ *and* $y = 3$;
let the equation of the top side of the square be
$y = k$: $x = \frac{k-9}{2}$ and on the other
line: $x = 6 - 3k$; $6 - 3k - \frac{k-9}{2} = k$
$\Rightarrow 12 - 6k - k + 9 = 2k \Rightarrow k = \frac{7}{3} \Rightarrow$
area of triangle = $\frac{1}{2} \cdot \frac{7}{3}\left(3 - \frac{7}{3}\right) = \frac{7}{9}$



GBML '00

TEAM ROUND

1. ΔABE is a 30-60-90° triangle; $BE = x$, $AB = x\sqrt{3}$;
$x + 2 = x\sqrt{3} \rightarrow x = \frac{2}{\sqrt{3}-1} = \sqrt{3} + 1$; area of ΔAEF =
$(x+2)^2 - x^2\sqrt{3} - 2 = (3+\sqrt{3})^2 - (1+\sqrt{3})^2\sqrt{3} - 2 =$
$12 + 6\sqrt{3} - \sqrt{3}(4 + 2\sqrt{3}) - 2 = 4 + 2\sqrt{3}$



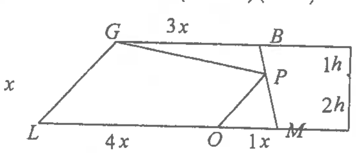
2. $\log_6 12 = k \rightarrow \frac{\log_2 12}{\log_2 6} = k \rightarrow \frac{\log_2 4 + \log_2 3}{\log_2 2 + \log_2 3} = k \rightarrow \frac{2 + \log_2 3}{1 + \log_2 3} = k \rightarrow$
$2 + \log_2 3 = k + k\log_2 3 \rightarrow \log_2 3(1-k) = k - 2 \rightarrow \log_2 3 = \frac{k-2}{1-k}$

3. $0° < x < 45°$ and $\tan x + \cot x = \sqrt{a} \rightarrow \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} = \sqrt{a} \rightarrow \frac{\sin^2 x + \cos^2 x}{\sin x \cos x} = \sqrt{a} \rightarrow$
$2\sin x \cos x = \frac{2}{\sqrt{a}} \rightarrow \sin 2x = \frac{2}{\sqrt{a}}$ and
$0° < 2x < 90° \rightarrow \cos 2x = \sqrt{1 - \sin^2 2x} = \sqrt{1 - \frac{4}{a}} = \frac{\sqrt{a^2 - 4a}}{a}$;

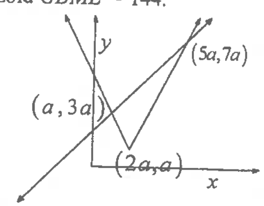
GBML '01

1. ~~L_1~~ and L_2 intersect at
point $P(-a,b) \Rightarrow 3ax - 12y = -33$ and $10x + 12y = -8a \Rightarrow 3ax + 10x = -33 - 8a \Rightarrow$
$x = \frac{-33 - 8a}{3a + 10} = -a \Rightarrow -33 - 8a = -3a^2 - 10a \Rightarrow 3a^2 + 2a - 33 = 0 \Rightarrow (3a + 11)(a - 3) = 0$
$\Rightarrow a = -\frac{11}{3}, 3$.

2. Let $GB = 3x \Rightarrow ML = 5x \Rightarrow LO = 4x$ and $OM = x$
Let h = distance from P to $\overline{GB} \Rightarrow 2h$ =
distance from P to $\overline{LM}$ $\Rightarrow$ distance between
$\overline{GB}$ and $\overline{ML} = 3h$. The area of trapezoid $GBML = \frac{1}{2} \cdot 3h \cdot (3x + 5x) = 12hx$. The area of
$\Delta GBP = \frac{1}{2}3hx = \frac{3}{2}hx$ and the area of $\Delta OMP = \frac{1}{2}x \cdot 2h = hx \Rightarrow$ area of quadrilateral $GPOL$
$= 12hx - \frac{3}{2}hx - hx = \frac{19}{2}hx = 114 \Rightarrow hx = 12$ $\Rightarrow$ area of trapezoid $GBML$ = 144.



3. $\{(x, y) \mid y \geq |2x - 4a| + a$ and $y \leq x + 2a\}$. The
vertex of the absolute value inequality = $(2a, a)$.
When $x \geq 2a \Rightarrow y = 2x - 4a + a = 2x - 3a$. When
$x < 2a \Rightarrow y = 4a - 2x + a = 4 = 5a - 2x$.
$x + 2a = 2x - 3a \Rightarrow x = 5a \Rightarrow y = 7a$.
$x + 2a = 5a - 2x \Rightarrow x = a \Rightarrow y = 3a$. The area of the triangle
formed by these 3 points = $\frac{1}{2}\text{abs}\begin{vmatrix} 2a & a & 1 \\ a & 3a & 1 \\ 5a & 7a & 1 \end{vmatrix} = \frac{a^2}{2}\text{abs}\begin{vmatrix} 2 & 1 & 1 \\ 1 & 3 & 1 \\ 5 & 7 & 1 \end{vmatrix} = 6a^2$



GBML '05

TEAM ROUND

1. The left side simplifies to $\log_c(\frac{a+b}{2})$

The right side simplifies to $\frac{1}{2}\log_c(\frac{ab}{8})$ or $\log(\sqrt{\frac{ab}{8}})$

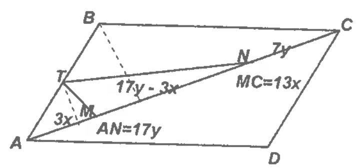
Since the log is a one-to-one function, the arguments can be equated.

$$\frac{a+b}{2}=\sqrt{\frac{ab}{8}} \rightarrow \frac{a^2+2ab+b^2}{4}=\frac{ab}{8} \rightarrow a^2+2ab+b^2=\frac{ab}{2} \rightarrow a^2+b^2=\frac{-3}{2}ab$$

$\rightarrow \frac{a^2+b^2}{ab}=\frac{-3}{2}$ Breaking the left side into separate fractions, $\frac{a}{b}+\frac{b}{a}=\underline{\frac{-3}{2}}$

2. $AC = 16x = 24y \rightarrow x = 3x/2$

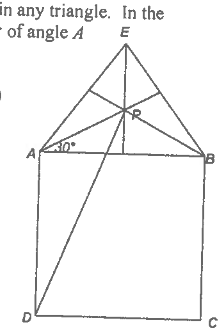
$AM : MN : NC = 9y/2 : 17y - 9y/2 : 7y$
$= 9 : 34 - 9 : 14 = 9 : 25 : 14$



$\frac{\text{Altitude from } T \text{ to } \overline{AC}}{\text{Altitude from } B \text{ to } \overline{AC}}=\frac{1}{2}$ (WHY?)

$\therefore \quad \frac{Area(\triangle TMN)}{Area(\triangle ABC)}=\frac{\frac{1}{2}(25x)(\frac{1}{2}h)}{\frac{1}{2}(48x)(h)}=\underline{\frac{25}{96}}$

3. The centroid is located at the point of intersection of the medians in any triangle. In the equilateral triangle the median from point A and the angle bisector of angle A are identical. Thus, $m\angle PAD = 120°$.



$AD = DC \rightarrow 3x^2 - 18x = 16x + 24 \rightarrow 3x^2\text{-}34x\text{-}24=0 \rightarrow (3x+2)(x-12)=0$
$\rightarrow x = 12 \rightarrow AD = 216 = 6^3$

PA = 2/3(length of the median = $(2/3)108\sqrt{3} = 72\sqrt{3} = 2\cdot 6^2\cdot\sqrt{3}$

Using the law of cosine,

$PD^2 = 6^6 + 2^2\cdot 6^4\cdot 3 - 2(6^3)(2\cdot 6^2\sqrt{3}\ )\cos(120°)$
$= 6^6 + 2\cdot 6^5 + 6^5\cdot 2\sqrt{3} = 6^5(6 + 2 + 2\sqrt{3}) = 6^5(8 + 2\sqrt{3})$
$= 2\cdot 6^5(4 + 1\sqrt{3}) \rightarrow$ B+C+D = 4+1+3 = $\underline{8}$

GBML '06

TEAM ROUND

1. $2Px^2 - 4Px + 4 - Qx^2 = 0$ is equivalent to $(2P - Q)x^2 - 4Px + 4 = 0$ which is quadratic provided $Q \neq 2P$. Normalizing, we have $x^2 - \frac{4P}{2P-Q}x + \frac{4}{2P-Q} = 0$

sum = twice product $\rightarrow \frac{4P}{2P-Q} = \frac{8}{2P-Q}$ and this is only true when $P = 2$ and $Q \neq 4$

Note that the original equation would become linear, if $P = 2$ and $Q = 4$.
Substituting $P = 2$, we have $(4 - Q)x^2 - 8x + 4 = 0$ and one of its roots must be 2.
Substituting $x = 2$, we have $(4 - Q)4 - 16 + 4 = 0 \rightarrow 4 - Q = 3 \rightarrow Q = 1$
Thus, the only ordered pair (P, Q) = $\underline{(2, 1)}$.

2. Rearranging the terms and factoring the difference of perfect squares,
$x^2 + 16xt + 64t^2 - y^2 = (x + 8t)^2 - y^2 = (x + 8t + y)(x + 8t - y) = 0$
Since all the variable represent positive integers, only the second factor can produce a zero factor. $x + 8t - y = 0 \rightarrow t = (y - x)/8$
If $x = y$, $t = 0$ and this violates the condition that all variables are positive.
If $y - x = 8$, then $t = 1$ and to minimize the sum we take $y = 9$ and $x = 1 \rightarrow x + y + t = 11$

3. Solving simultaneously, $(x, y) = \left(\frac{18+2b}{3a+2}, \frac{6-ab}{3a+2}\right) = \left(2a, -\frac{b}{2}\right)$.

The slope of L_1 is $-\frac{a}{2}$.

Equating the 2[nd] coordinates, $12 - 2ab = -3ab - 2b \rightarrow b = \frac{-12}{a+2}$

Equating the 1[st] coordinates, $18 + 2b = 6a^2 + 4a$

Substituting for b, $18 - \frac{24}{a+2} = 6a^2 + 4a \rightarrow 18a + 12 = 6a^3 + 16a^2 + 8a$

$\rightarrow 3a^3 + 8a^2 - 5a - 6 = (a - 1)(3a^2 + 11a + 6) = (a - 1)(3a + 2)(a + 3) = 0 \rightarrow a = 1, -2/3, -3$
However, $a = -2/3$ does not produce two distinct lines.
[$a = -2/3 \rightarrow b = -9$ and the system degenerates to coincident lines $-x + 3y = 9$.]
Thus, the possible slopes of L_1 are: $\underline{-1/2, 3/2}$

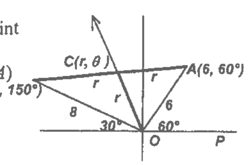
GBML '07

TEAM ROUND

1. Merchant A: (year, wage paid) = (1,105), (2, 125), (3, 145), … (n, $20n + 85$)
Merchant B: (year, wage paid) = (1, 50), (2, 75), (3, 100), … (n, $25n + 25$)
Equating, $20n + 85 = 25n + 25 \rightarrow n = 12 \rightarrow$ year's wages = \$325.

At \$6.60, 40 hours earns \$264. Still to be earned – at least \$61
Let t denote the number of hours of overtime required to earn the differential.
$9.90t > 61 \rightarrow t > 6 \rightarrow t_{\min} = 7 \rightarrow t_{total} = \underline{47}$ hours

2. Clearly, $\angle AOB$ is a right angle and $AB = 10$. Since C, as the midpoint of the hypotenuse, is equidistant from points A, B and O, $r = 5$.
Using the Law of Cosines, in ΔACO, $5^2 = 5^2 + 6^2 - 2\cdot 5\cdot 6\cdot\cos(\angle COA)$
$\rightarrow 60\cos(\angle COA) = 36 \rightarrow \cos(\angle COA) = 3/5$
Thus, $C\underline{(\mathbf{5}, \mathbf{60° + Arccos(3/5)})}$



3. Perimeter($ABCD$) = $4\left(2x+\frac{2y\sqrt{3}}{3}\right)$

$2\cdot$Area(ΔBPQ) + $2\cdot$Area(ΔAPS) = Area($PQRS$)

$\rightarrow 2x^2\sqrt{3}+2\left(\frac{1}{2}\cdot 2y\cdot\frac{y}{\sqrt{3}}\right)=4xy$

$\rightarrow 2x^2\sqrt{3}+\frac{2y^2\sqrt{3}}{3}=4xy$

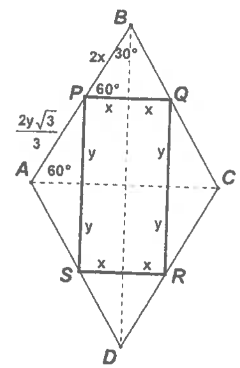
$\rightarrow 2\sqrt{3}\left(3x^2+y^2\right)=12xy$

$\rightarrow 3x^2-2\sqrt{3}xy+y^2=0$

$\rightarrow \left(\sqrt{3}x-y\right)^2=0 \rightarrow y=x\sqrt{3}$

$\therefore$ Perimeter($ABCD$) = $4(2x+2x)=4(2PQ)=\mathbf{8PQ}$

The assumption that $PQ < PS$ was unnecessary.



GBML '08

TEAM ROUND

1. Note that any integers (and equally important <u>only</u> integers) which are the product of two distinct primes are always equal to the product of their proper factors.
(Ex: $21 = 3\cdot 7$ and has 3 proper factors: 1, 3 and 7 and their product is 21)
Starting with 31:
31 is prime and is rejected
32 is not the product of two distinct primes and is rejected
Clearly we want $33 = 3\cdot 11$, $34 = 2\cdot 17$, $35 = 5\cdot 7$, $38 = 2\cdot 19$ and $39 = 3\cdot 13$
Adding, the required sum is <u>**179**</u>.

2. Note that $360 = 2^3\cdot 3^2\cdot 5^1$ and when the product $\left(1+2^1+2^2+2^3\right)\left(1+3^1+3^2\right)\left(1+5^1\right)$ is multiplied out we get 24 terms in the expansion and each term represents a positive factor of 360. Thus, the total number of positive factors is determined by adding 1 to each exponent in the prime factorization of 360 and taking the product of these sums, i.e. $(3+1)(2+1)(1+1) = 24$ factors.

   Since the numbers we seek have 18 factors, we look at all possible factorizations of 18, namely $18\cdot 1$, $9\cdot 2$, $6\cdot 3$ or $3\cdot 2\cdot 2$. These factorizations give potential exponents of <u>17, 0</u>, <u>8, 1</u>, <u>5, 2</u> and <u>2, 1, 1</u>. By associating these exponents systematically with small prime factors, we can produce the smallest five natural numbers with 18 factors:
$2^2\cdot 3^2\cdot 5^1 = \underline{\mathbf{180}}$, $2^2\cdot 3^2\cdot 7^1 = \underline{\mathbf{252}}$, $2^5\cdot 3^2 = \underline{\mathbf{288}}$, $2^2\cdot 3\cdot 5^2 = \underline{\mathbf{300}}$, $2^2\cdot 3^2\cdot 11^1 = \underline{\mathbf{396}}$

3. Unit fractions which sum to $\frac{35}{48}$ must have denominators which are factors of 48.

   Specifically, the possible unit fractions are: $\frac{1}{2},\frac{1}{3},\frac{1}{4},\frac{1}{6},\frac{1}{8},\frac{1}{12},\frac{1}{16},\frac{1}{24},\frac{1}{48}$

   The same list with a common denominator of 48: $\frac{24}{48},\frac{16}{48},\frac{12}{48},\frac{8}{48},\frac{6}{48},\frac{4}{48},\frac{3}{48},\frac{2}{48},\frac{1}{48}$

   Thus, we must find three numbers (not necessarily distinct) from the numerators <u>1, 2, 3, 4, 6, 8, 12, 16 and 24</u> that sum to 35. All three numbers can't be the same, since 35 is not a multiple of 3. Therefore, either all three numbers are different or exactly two of them are the same. In the first case, we need $a+b+c=35$ and the only combination that works is 24, 8 and 3 which corresponds to the ordered triple $\underline{\left(\frac{1}{2},\frac{1}{6},\frac{1}{16}\right)}$. In the second case, we need $2a+b=35$ and the only combination that works is $2(16)+3$ which corresponds to $\underline{\left(\frac{1}{3},\frac{1}{3},\frac{1}{16}\right)}$.

TEAM ROUND

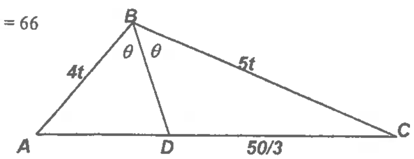
GBML '09

1. $\sin 40°\sin 50° = \cos^2 x - \cos 60° \rightarrow \sin 40°\cos 40° = \cos^2 x - \frac{1}{2} \rightarrow$

   $2\sin 40°\cos 40° = \sin 80° = 2\cos^2 x - 1 = \cos 2x$

   $\rightarrow \cos 2x = \cos 10° \rightarrow 2x = \begin{cases}10° + 360k\\ 350° + 360k\end{cases} \rightarrow x = \begin{cases}5° + 180k\\ 175° + 180k\end{cases} \rightarrow \underline{\mathbf{5°, 175°, 185°, 355°}}$

2. $AB : BC = 4 : 5$, $DC = \frac{25}{3}$ and perimeter(ΔABC) = 66



   $\frac{AD}{4t}=\frac{25/3}{5t} \rightarrow AD = \frac{50}{3}$

   $4t+5t+\frac{40}{3}+\frac{50}{3}=66 \rightarrow 9t = 36 \rightarrow t = 4$

   $\rightarrow AB = 16$, $BC = 20$ and $AC = 30$

   Area(ΔABC) = $\sqrt{33(17)(13)3} = 3\sqrt{11\cdot 13\cdot 17}$

   Since the area of ΔABD is $\frac{4}{9}$ the area of ΔABC, we have $\underline{\frac{4}{3}\sqrt{2431}}$.

3. $\left(\log_t x^2\right)\left(\log_x 2t\right) = \log_8 t - \log_8 \frac{1}{2}$ (for $x > 1$)

   $\rightarrow \left(\log_t x^2\right)\left(\log_x 2 + \log_x t\right) = \frac{1}{3}\log_2 t + \frac{1}{3}$

   $\rightarrow \log_t x^2\cdot\log_x 2 + \log_t x^2\cdot\log_x t = \frac{1}{3}\log_2 t + \frac{1}{3}$

   $\rightarrow 2\log_t x\cdot\log_x 2 + 2\log_t x\cdot\log_x t = \frac{1}{3}\log_2 t + \frac{1}{3}$

   Note that $\log_t x\cdot\log_x 2 = \frac{\log x}{\log t}\cdot\frac{\log 2}{\log x} = \frac{\log 2}{\log t} = \log_t 2$.

   Thus, the equation simplifies to: $2\log_t 2 + 2 = \frac{1}{3}\log_2 t + \frac{1}{3}$

   Let $A = \log_2 t$. Then: $\frac{2}{A} + 2 = \frac{1}{3}A + \frac{1}{3} \rightarrow 6 + 6A = A^2 + A$

   $\rightarrow A^2 - 5A - 6 = (A-6)(A+1) = 0 \rightarrow A = 6$ or -1

   and $\log_2 t = 6, -1 \rightarrow t = 2^6$ or 2^{-1} = $\underline{\mathbf{64, \frac{1}{2}}}$

TEAM ROUND

GBML '10

1. Given $f : m : s = 16 : 12 : 5$. Actual ages today: $16k$, $12k$ and $5k$

   $\frac{f+8}{s+13}=\frac{2}{1} \rightarrow f + 8 = 2s + 26 \rightarrow f - 2s = 18 \rightarrow 16k - 2(5k) = 18 \rightarrow 6k = 18 \rightarrow k = 3$

   Now $(f, m, s) = (48, 36, 15)$. In n years, $\frac{36+n}{15+n}=\frac{3}{2} \rightarrow n = 27 \rightarrow$ Father: $48 + 27 =$ <u>**75**</u>.

2. $\left(\sqrt{2}cis195°\right)^7 = \left(\sqrt{2}\right)^7 cis\left(7\cdot 195°\right) = 8\sqrt{2}cis1365° = 8\sqrt{2}cis285°$

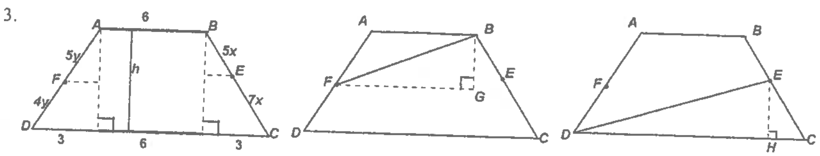
Thus, $a = 8\sqrt{2}\cos 285° = 8\sqrt{2}\cos 75°$ and $b = 8\sqrt{2}\sin 285° = -8\sqrt{2}\sin 75°$.

$\cos 75° = \cos(30°+45°) = \cos 30°\cos 45° - \sin 30°\sin 45° = \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} - \frac{1}{2}\cdot\frac{\sqrt{2}}{2} = \frac{\sqrt{6}-\sqrt{2}}{4}$

$\sin 75° = \sin(30°+45°) = \sin 30°\cos 45° + \sin 45°\cos 30° = \frac{1}{2}\cdot\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} = \frac{\sqrt{2}+\sqrt{6}}{4}$

$\rightarrow a+b = 8\sqrt{2}\left(\frac{\sqrt{6}-\sqrt{2}}{4} - \frac{\sqrt{2}+\sqrt{6}}{4}\right) = 8\sqrt{2}\left(-\frac{\sqrt{2}}{2}\right) = \underline{-8}$

3.



$BG = \frac{5}{9}h$, $EH = \frac{7}{12}h$. Note that, in obtuse ΔABF, the altitude from F to $\overline{AB}$ has the same length as $\overline{BG}$.

Thus, the required ratio is $\dfrac{\frac{1}{2}(AB)(BG)}{\frac{1}{2}(EH)(DC)} = \dfrac{6\left(\frac{5}{9}h\right)}{12\left(\frac{7}{12}h\right)} = \dfrac{\frac{10}{3}}{7} = \underline{\mathbf{\dfrac{10}{21}}}$.