NEW ENGLAND ASSOCIATION OF MATHEMATICS LEAGUES

PLAYOFFS – 2012

Round 1: Arithmetic and Number Theory

1. $a =$______ ; $b =$________

2. ________________

3. ________________

1. Find a and b, relatively prime, such that $\dfrac{0.\overline{36} \cdot 1.\overline{2}}{0.\overline{07}} = \dfrac{a}{b}$

2. How many times in this century is the sum of the digits of the year a perfect square?

3. Bob was numbering the pages of a book, but he made a mistake. After writing page number 169, he then wrote 160 for the next page and continued numbering the remaining pages consecutively based on that mistake. Without making any additional mistakes, he finished at page 220. If he can't change the order of the pages, what is the fewest number of digits that he can erase and replace to obtain a correct numbering scheme?

NEW ENGLAND ASSOCIATION OF MATHEMATICS LEAGUES

PLAYOFFS – 2012

Round 2: Algebra 1

1. _______________ %

2. _________:__________

3. ____________________

1. Gina was given a raise of 25%. She felt pretty good. But then the next week she was told that her salary was being cut by 25%. She was upset. By what percent does her latest salary differ from her salary before the raise?

2. Compute the ratio of the positive root of $9x^2 + 6x - 1 = 0$ to the positive root of $x^2 + 2x - 1 = 0$. Express the answer as a ratio of whole numbers.

3. Compute the <u>largest</u> of the three positive reals a, b, and c such that

$$ab = 100$$
$$bc = 150$$
$$ac = 135$$

NEW ENGLAND ASSOCIATION OF MATHEMATICS LEAGUES

PLAYOFFS – 2012

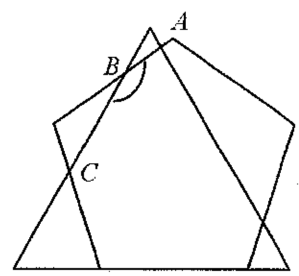
Round 3: Geometry

1. ____________________

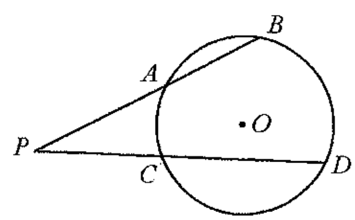
2. ____________________

3. ____________________

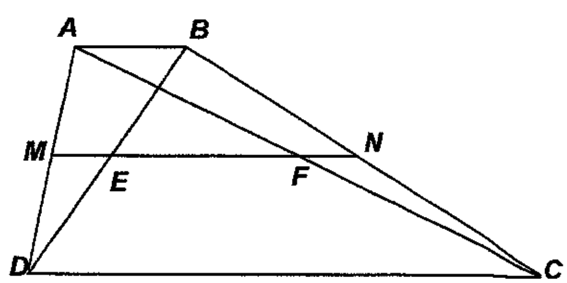
1. The diagram shows an equilateral triangle and a regular pentagon. A side of the pentagon lies on a side of the triangle. What is the degree-measure of angle ABC?



2. From point P secants $\overline{PB}$ and $\overline{PD}$ are drawn to circle O. If the lengths of $\overline{PA}$, $\overline{AB}$, $\overline{PC}$, and $\overline{CD}$ are selected from $\{5,6,7,8\}$, without replacement, what are the possible lengths of $\overline{AB}$?



3. In trapezoid $ABCD$, $\overline{AB} \parallel \overline{CD}$, $\overline{MN}$ is a median, and both AB and DC are integers with $AB < DC$ and $AB + DC = 48$. If $EF > AB$, find the number of ordered pairs (AB, DC).



NEW ENGLAND ASSOCIATION OF MATHEMATICS LEAGUES

PLAYOFFS – 2012

Round 4: Algebra 2

1. ____________________

2. ____________________

3. ____________________

1. If $\log_b 2 = m$, $\log_b 9 = n$, and $\log_b 125 = p$, determine $\log_b\left(1\frac{23}{27}\right)$ as a single reduced fraction in terms of *m*, *n*, and *p*.

2. If the real solutions of the equation $(x+8)^4 + 5x^2 + 80x + 320 = 14$ are denoted by *m* and *n*, determine $|m-n|$.

3. The real roots of $x^3 - ax^2 + nax - k = 0$ form a geometric sequence; *a*, *n*, and *k* are real numbers; $a \neq 0$, $n \neq 1$ and $k \neq 0$. If the product nk equals n^t, find *t*.

NEW ENGLAND ASSOCIATION OF MATHEMATICS LEAGUES

PLAYOFFS – 2012

Round 5: Analytic Geometry

1. ______________

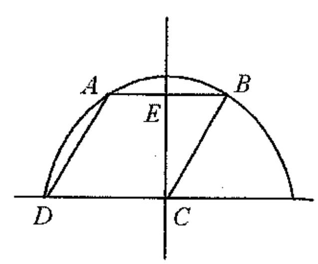
2. ______________

3. ______________

1. An ellipse whose axes are either vertical or horizontal is centered at the origin. The graph of $2x - 7y = 14$ passes through an endpoint of the major axis and an endpoint of the minor axis. What is the smallest distance between a focus and one end of the major axis?

2. Given $M(2a+1, 3b+2)$, $N(5a+2, 6b+1)$, and $P(8a+3, 8b)$, with a an integer from –5 to 5, inclusive. If N is the midpoint of non-vertical segment $\overline{MP}$, find the slope for $\overline{MP}$ that is numerically the largest.

3. $ABCD$ is a rhombus. Points A, B and D lie on a parabola whose equation is $f(x) = ax^2 + k$, C lies on the origin and E lies on the y-axis. If $B = (p, f(p))$, compute the numerical value of the product ap.



NEW ENGLAND ASSOCIATION OF MATHEMATICS LEAGUES

PLAYOFFS – 2012

Round 6: Trig and Complex Numbers

1. ____________________

2. ____________________

3. ____________________

1. Let A = least positive value of y where $y = 5\sec x - 2$ for x in radians. Find, in radians, $\mathrm{Tan}^{-1}\sqrt{A} + \mathrm{Cos}^{-1}\dfrac{\sqrt{A}}{2}$.

2. Determine the number of solutions in $[0, 2\pi)$ to $\dfrac{\sin 5x}{\sin x} + \dfrac{\cos 5x}{\cos x} = 0$.

3. If $\sin x \cos x = \dfrac{1 - \cos 4x}{2} = k$, for $k > 0$, find k.

NEW ENGLAND ASSOCIATION OF MATHEMATICS LEAGUES

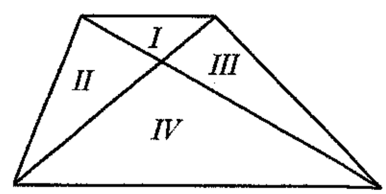
NEW ENGLAND PLAYOFFS – 2012

Team Round

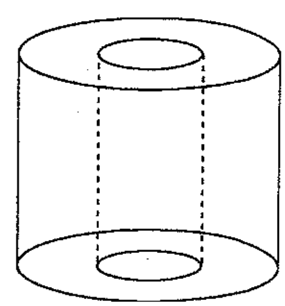
1. ____________	**4.** ____________
2. ____________	**5.** ____________
3. ____________	**6.** ____________

1. In how many ways can 6 different people sit in a row of 6 chairs if person A refuses to sit in the first chair and person B refuses to sit in the last chair?

2. In the trapezoid the Roman numerals I, II, III, and IV represent the areas of the four triangular regions that make up the trapezoid, with IV being the largest. If I, II, and III are integers and $IV = 256$, find the largest possible area of the trapezoid.



3. Starting with a right circular cylinder of height 12 and radius r, a hole is drilled completely through the cylinder and both bases, creating a pipe whose internal radius is x. If the total surface area of the new figure equals the total surface area of the original cylinder, determine all possible values of r.



4. The points $-2 + 2i$, $\frac{-2 + 2i}{1+i}$, $\frac{-2 + 2i}{(1 + i)^2}$, . . . , $\frac{-2 + 2i}{(1 + i)^8}$ form the consecutive vertices of a polygon in the complex plane. Determine the area of the figure.

5. Find the smallest integer N greater than 10 for which $12N$ has three times as many factors as N.

6. If $.\overline{21}_3 = .\overline{ab}_5$, determine the ordered pair (a, b).

NEW ENGLAND ASSOCIATION OF MATHEMATICS LEAGUES

PLAYOFFS – 2012

Answer Sheet

Round 1

1. $a = 44$, $b = 7$
2. 16
3. 71

Round 2

1. 6.25%
2. 1 : 3
3. $\frac{9}{2}\sqrt{10}$

Round 3

1. 156
2. 5 or 8
3. 11

Round 4

1. $\frac{6m+4p-9n}{6}$
2. $2\sqrt{2}$
3. 4

Round 5

1. $7-3\sqrt{5}$
2. $\frac{1}{2}$
3. $-\frac{\sqrt{3}}{3}$

Round 6

1. $\frac{\pi}{2}$
2. 8
3. $\frac{1}{4}$

Team

1. 504
2. 961
3. $r > 12$
4. $\frac{255}{64}$ or $3\frac{63}{64}$
5. 30
6. (4, 1)

NEW ENGLAND ASSOCIATION OF MATHEMATICS LEAGUES

NEW ENGLAND PLAYOFFS – 2012 - SOLUTIONS

Round 1 Arithmetic and Number Theory

1. $$\frac{\frac{36}{99}\cdot\frac{11}{9}}{\frac{7}{99}}=\frac{44}{7}$$

2. For the sum of the digits to be a perfect square, since 2 and 0 are fixed, the only possible squares are 4, 9, and 16. Therefore, the sum of the last 2 digits must be 2, 7, or 14. There are 3 possibilities for 2, 8 for 7, and 5 for 14, giving a total of $\boxed{16}$.
The years are: 2002, 2011, 2020 | 2007, 2016, 2025, 2034, 2043, 2052, 2061, 2070 | 2059, 2068, 2077, 2086, 2095.

3. He'll have to change each of the numbers from 160 to 169 by replacing the 6 with a 7. He'll change each of the numbers from 170 to 179 by replacing the 7 with an 8 and each of the numbers from 180 to 189 by replacing each 8 with a 9. That makes for 30 changes. Then he'll change the numbers from 190 to 199 to 200 to 209. That's 20 changes. To change the numbers from 200 to 209 to 210 to 219 he'll replace the middle 0 with a 1. To change 210 to 219 to 220 to 229 he'll replace the middle 1 with a 2. That makes for 20 changes. Finally, he'll change 220 to 230 for 1 change. Total: $30 + 20 + 20 + 1 = \boxed{71}$.

Round 2 Algebra 1

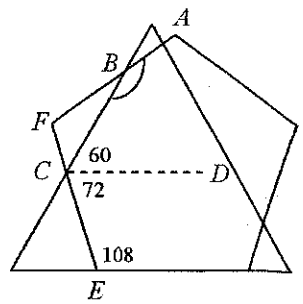
1. Let x be his salary before the raise. Then $x\left(\frac{5}{4}\right)\left(\frac{3}{4}\right) = \frac{15x}{16}$. So his latest salary is one-sixteenth less than his original salary, a reduction of $\frac{1}{16}$ of 100% or $\boxed{6.25\%}$.

2. Note that $9x^2 + 6x - 1 = 0$ can be written as $(3x)^2 - 2(3x) - 1 = 0$. Thus, if $x = a$ is a solution to $x^2 + 2x - 1 = 0$, then $3x = a$ will give $x = \frac{a}{3}$ as a solution to $9x^2 + 6x - 1 = 0$. The ratio is $\boxed{\frac{1}{3} \text{ or } 1:3}$. Or one could just solve both equations, obtaining $\frac{-1+\sqrt{2}}{3}$ for the first equation and $-1+\sqrt{2}$ for the second giving a ratio of 1 : 3.

3. Dividing equation (2) by equation (1) gives $\frac{c}{a} = \frac{3}{2} \to c = \frac{3a}{2}$. Note that this tells us that $c > a$, so we must choose between b and c. Substituting for c into $ac = 135$ gives $\frac{3}{2}a^2 = 135 \to a^2 = 90 \to a = 3\sqrt{10}$. Then $c = \frac{9}{2}\sqrt{10}$ and from $b = \frac{100}{a}$ we obtain $b = 100 \cdot \frac{1}{3\sqrt{10}} = \frac{100}{3\sqrt{10}} \cdot \frac{\sqrt{10}}{\sqrt{10}} = \frac{10}{3}\sqrt{10}$. Thus, the largest is $\boxed{\frac{9}{2}\sqrt{10}}$.

Round 3 – Geometry

1. Draw $\overline{CD}$ parallel to the base of the pentagon. Since $m\angle E = 108$, then $m\angle DCE = 72$. Since $\overline{CD}$ is parallel to the base of the triangle, $m\angle BCD = 60$, making $m\angle FCB = 48$. Since angle ABC is an exterior angle of triangle BFC it equals $m\angle FCB + m\angle F = 48 + 108 = \boxed{156}$.



Alternate Approach: Since the base angle of the equilateral triangle measures 60 and the exterior angle of the pentagon at E measures 72, the small angle at C measures 48.

2. Let $(PA, AB, PC, CD) = (a, b, c, d)$ The secant relationship tells us that $a(a+b) = c(c+d)$ or $a^2 - c^2 = cd - ab$. Examining possible squares and pair-products of the available numbers (25, 36, 49, 64) and (30, 35, 40, 42, 48, 56), we see that only common difference is $13 = 49 - 36 = 48 - 35$. Thus, we have $(a,b,c,d) = (7,5,6,8)$ and $7(7+5) = 6(6+8)$, so $AB = \boxed{5 \text{ or } 8}$.

3. From $EF = \frac{DC - AB}{2}$, we have $\frac{DC - AB}{2} > AB \to DC > 3(AB)$. If $DC = 3AB$, then $4AB = 48 \to AB = 12$ and $DC = 36$. Thus, $1 \le AB \le 11$, so there are $\boxed{11}$ ordered pairs.

Round 4 – Algebra 2

1. Since $\log_b 9 = n$, then $\log_b 3 = \frac{n}{2}$; similarly, since $\log_b 125 = p$, then $\log_b 5 = \frac{p}{3}$.
$\log_b 1\frac{23}{27} = \log_b \frac{50}{27} = \log_b 50 - \log_b 27 = \log_b 2 + 2\log_b 5 - 3\log_b 3 = \; m + \frac{2}{3}p - \frac{3}{2}n =$
$\frac{6m+4p-9n}{6}$.

2. The given equation can be written as $(x+8)^4 + 5(x+8)^2 - 14 = 0$ which factors as $\left[(x+8)^2+7\right]\cdot\left[(x+8)^2-2\right]=0$. Obviously, the first factor leads to complex roots. From the second factor the real roots are $-8+\sqrt{2}$ and $-8-\sqrt{2}$. Subtracting and taking the absolute value produces $2\sqrt{2}$.

3. Let b be the first term in the sequence and let r be the common ratio. Then the roots are $b, br,$ and br^2. From the relationship between the roots and coefficients we have $b + br + br^2 = a$, $b(br) + b(br^2) + (br \cdot br^2) = na$, and $b \cdot br \cdot br^2 = b^3r^3 = k$. From $b(br) + b(br^2) + (br \cdot br^2) = na$, we obtain $br\left(b + br + br^2\right) = na$ which simplifies to $abr = na \rightarrow br = n$ provided $a \neq 0$. Thus, $nk = (br)(br)^3 = (br)^4 = n^4$. So, as long as br and n don't equal 1, then $\boxed{t=4}$.

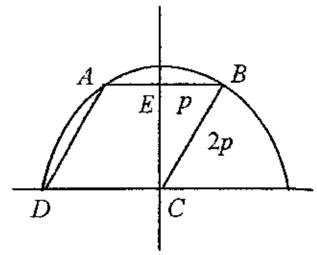
Round 5 – Analytic Geometry

1. The graph of $2x - 7y = 14$ has intercepts at (7, 0) and (0, –2), therefore (7, 0) is an end of the major axis and (0, –2) is an end of the minor axis of the ellipse. That makes $a = 7$ and $b = 2$ which makes $c = 3\sqrt{2}$. $\therefore a - c = 7 - 3\sqrt{5}$.

2. The slope of $\overline{MN}$ is $\frac{(6b+1)-(3b+2)}{(5a+2)-(2a+1)} = \frac{3b-1}{3a+1}$. The slope of $\overline{NP}$ is
$\frac{(8b)-(6b+1)}{(8a+3)-(5a+2)} = \frac{2b-1}{3a+1}$. Setting the slopes equal we have $3b - 1 = 2b - 1$,
making $b = 0$. The slope of the line is therefore $\frac{-1}{3a+1}$. The largest value occurs
when $a = -1$, giving an answer of $\boxed{\frac{1}{2}}$.

3. Let $BC = 2p$, then since E is the midpoint of $\overline{AB}$, $EB = p$, and $EC = p\sqrt{3}$. The coordinates of B are therefore $(p, p\sqrt{3})$. Thus $p\sqrt{3} = ap^2 + k$. Since $D = (-2p, 0)$ also lies on the parabola, $0 = a(-2p)^2 + k$. Subtracting the first from the second cancels the k's and gives $-p\sqrt{3} = 3ap^2$.

Thus $ap = -\frac{1}{\sqrt{3}}$, so $\boxed{ap = -\frac{\sqrt{3}}{3}}$.



Round 6 – Trig and Complex Numbers

1. The least positive value of $y = \sec x$ is 1, so the least positive value of A is $5 \cdot 1 - 2 = 3$.

$Tan^{-1}\sqrt{3} = \frac{\pi}{3}$ and $Cos^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{6}$; $\frac{\pi}{3} + \frac{\pi}{6} = \frac{\pi}{2}$

2. Adding we have $\frac{\sin 5x \cos x + \cos 5x \sin x}{\sin x \cos x} = \frac{\sin 6x}{\sin x \cos x} = 0$. Thus,

$6x = 0 + \pi k \to x = 0 + \frac{\pi}{6}k$. For $k = 0$ to 11 we have solutions in $[0, 2\pi)$.

That's a total of 12 solutions. But since the denominator can't equal zero, we eliminate those values of k that give multiples of $\pi/2$, namely $k = 0, 3, 6$, and 9. Answer: $\boxed{8}$.

3. $2\sin x \cos x = 2k \to \sin 2x = 2k \to \sin^2 2x = 4k^2$.

$\frac{1 - \cos 4x}{2} = \frac{1 - (1 - 2\sin^2 2x)}{2} = \sin^2 2x$. Thus, $k = 4k^2$, giving $\boxed{k = \frac{1}{4}}$.

Team Round

1. Put B in the first position. Then any one of 5 letters goes in the second, any one of 4 in the third and so on giving $5! = 120$ possibilities. Now put B in any other position besides the last. Then C, D, E, or F can go in the first position—4 choices. B can go in the 2nd, 3rd, 4th, or 5th. So we have 4 choices for B. For each choice of B, there are 4! arrangements of the remaining 4 letters; so, $4 \cdot 4 \cdot 4! = 384$. Total: $\boxed{504}$.

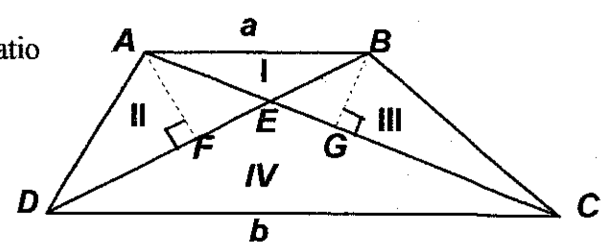
2, It helps to know that the areas form a geometric sequence and that $II = III$. Thus, we seek integers such that $(II)^2 = 256 \cdot I$. Note that I must be a perfect square. Here are a couple of candidates for the areas and their sums:

1-16-16-256	289	4-32-32-256	324
9-48-48-256	361	16-64-64-256	400

Since I must be an integer and a perfect square, it is clear that the largest area will occur when $I = 15^2$, giving 225-240-240-256 and a sum of $\boxed{961}$.

Alternate Solution:

$\Delta ABE \sim \Delta CDE$, so their areas are in an $a^2 : b^2$ ratio and, as corresponding sides, $BE : DE = a : b$
Since ΔADE and ΔABE share the same altitude from A, their areas are in the same ratio as their bases. Similarly, for ΔBAE and ΔBCE.



Thus, the area of ΔI is $\frac{a^2}{b^2} \cdot 256$

$$\frac{area(I)}{area(II)} = \frac{a}{b} \Rightarrow \frac{\frac{a^2}{b^2} \cdot 256}{II} = \frac{a}{b} \Rightarrow \text{area of } \Delta\text{II is } \frac{a}{b} \cdot 256.$$

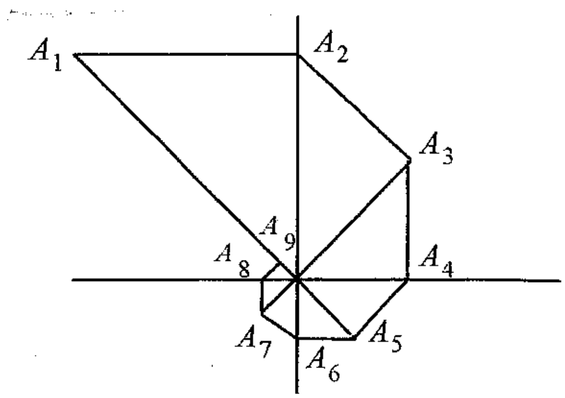
Interestingly, the area of ΔIII is also $\frac{a}{b} \cdot 256$.

Therefore, the area trapezoid $ABCD$ is $\left(1 + 2\frac{a}{b} + \frac{a^2}{b^2}\right)256 \Leftrightarrow 256\left(1 + \frac{a}{b}\right)^2$ or $256\left(\frac{a+b}{b}\right)^2$.

To maximize the area we want the "fudge factor" to be as large as possible while maintaining integer areas for all four triangles. Thus, the largest possible value of b is 16 and, since we were given $a < b$, we take a to be 15.
The areas of (I, II, III) are (225, 240, 240), resulting in a total area of $\boxed{961}$.

3. The total area of the cylinder is $2\pi rh + 2\pi r^2$. The total area of the pipe is the outside lateral area plus the inside lateral area plus the area of the rings on the top and bottom. Let x be the internal radius. The pipe's area can be written as $2\pi rh + 2\pi xh + 2\left(\pi r^2 - \pi x^2\right)$. Setting the two areas equal gives $2\pi rh + 2\pi r^2 = 2\pi rh + 2\pi xh + 2\left(\pi r^2 - \pi x^2\right)$ which simplifies to $0 = 2\pi xh - 2\pi x^2 \rightarrow x^2 = xh \rightarrow x = h$. So as long as the external radius is greater than the height, then if a hole is drilled through a cylinder with radius equal to the height, the total surface area of the cylinder will equal the total surface area of the pipe. Thus, the answer is $\boxed{r > 12}$.

4. Since multiplying a point in the complex plane by $1 + i$ rotates the point by $45°$ counterclockwise and increases the distance of the point from the origin by a factor of $\sqrt{2}$, dividing by $1 + i$ should rotate by $45°$ in a clockwise manner and reduce the distance from the origin by a factor of the reciprocal of $\sqrt{2}$. Thus we see the points rotate around the plane as shown in the diagram at the right. Let O represent the origin. In triangle A_1A_2O, $A_1O = 2\sqrt{2}$ while $A_1A_2 = A_2O = 2$, so the area of A_1A_2O is 2. Each successive triangle is similar to the original and since the scale factor in going from one triangle to the next smaller triangle is $\frac{1}{\sqrt{2}}$, the smaller triangle has half the area. The area of the figure is therefore $2 + 1 + \frac{1}{2} + \cdots + \frac{1}{64} = \boxed{3\frac{63}{64} \text{ or } \frac{255}{64}}$.



5. To minimize N, we assume the prime factorization of N contains only factors of 2, 3 and/or 5. Try $N=2^m$, a number with $m+1$ factors. Then $2^m \cdot 12 = 2^{m+2} \cdot 3$ has $(m+3)2$ factors. Then $2(m+3) = 3(m+1) \Rightarrow m = 3$ and the number $2^3 = 8$ would solve the problem, but it is too small. Try $N = 3^m$. Then: $3^m \cdot 12 = 3^{m+1} \cdot 2^2$ and we require that $(m+2)3 = 3(m+1)$ and there is no solution. Try $N = 2^m \cdot 3^n$ Then: $2^m \cdot 3^n \cdot 12 = 2^{m+2} \cdot 3^{n+1}$, and we require that $(m+3)(n+2) = 3(m+1)(n+1)$, giving $2mn + m - 3 = 0 \Rightarrow m(2n+1) = 3$. Thus, $m = n = 1$, making $2^m \cdot 3^n = 6$ and that is too small. Try $N = (2^m \cdot 5)$ Then: $(2^m \cdot 5) \cdot 12 = 2^{m+2} \cdot 3 \cdot 5$, and we require that $(m+3) \cdot 2 \cdot 2 = 3(m+1)2$ or $4m + 12 = 6m + 6 \Rightarrow m = 3$, making $2^m \cdot 5 = 40$.
Note: 40 has $4(2) = 8$ factors and $40 \cdot 12 = 2^5 \cdot 3 \cdot 5 = 480$ has $6(2)(2) = 24$ factors, 3 times as many!
Finally, we try $N = 2 \cdot 3 \cdot 5 = 30$ which has $2^3 = 8$ factors. Then: $2 \cdot 3 \cdot 5 \cdot 12 = 2^3 \cdot 3^2 \cdot 5$.
It has $4 \cdot 3 \cdot 2 = 24$ factors. Since this is less than 40 and no other product of three primes can be less than 30, we have our answer, namely $\boxed{30}$.

6. $$0.212121\ldots_3 = \frac{2}{3} + \frac{1}{3^2} + \frac{2}{3^3} + \frac{1}{3^4} + \cdots = \left(\frac{2}{3} + \frac{2}{3^3} + \cdots\right) + \left(\frac{1}{3^2} + \frac{1}{3^4} + \cdots\right)$$

This simplifies to $2\left(\dfrac{\frac{1}{3}}{1-\left(\frac{1}{3}\right)^2}\right) + \left(\dfrac{\frac{1}{3^2}}{1-\left(\frac{1}{3}\right)^2}\right) = 2\left(\frac{1}{3} \cdot \frac{9}{8}\right) + \left(\frac{1}{9} \cdot \frac{9}{8}\right) = \frac{7}{8}$.

Similarly, $\overline{ab}_5 = \left(\frac{a}{5} + \frac{a}{5^3} + \frac{a}{5^5} + \cdots\right) + \left(\frac{b}{5^2} + \frac{b}{5^4} + \frac{b}{5^6} + \cdots\right) =$

$a\left(\dfrac{\frac{1}{5}}{1-\frac{1}{5^2}}\right) + b\left(\dfrac{\frac{1}{5^2}}{1-\frac{1}{5^2}}\right) = \dfrac{5a+b}{24}$. Setting $\dfrac{5a+b}{24} = \dfrac{7}{8}$ gives $5a + b = 21$.

Then $a = 4$ and $b = 1$ making the answer $\boxed{(4,1)}$.

Alternate solution:
Let $N_{10} = 0.\overline{21}_3$ Multiply the right side by $100_{(3)}$ and the left side by the equivalent base 10 value, namely 9. This gives us $9N_{10} = 21.\overline{21}_3$. Subtracting, $8N_{10} = 21_3 = 7_{10}$ or $N = \frac{7}{8}$. Similarly, if $M_{(10)} = 0.\overline{ab}_{(5)}$, multiplying by $100_{(5)}$ and subtracting gives us $24M_{(10)} = ab_{(5)} = 5a+b$ or $M = \dfrac{5a+b}{24}$. Equating, $\dfrac{5a+b}{24} = \dfrac{7}{8} \Leftrightarrow 5a+b = 21$.
Since a and b are digits in base 5, we are limited to $\{0,1,2,3,4\}$ and only $(a, b) = \boxed{(4, 1)}$ gives us 21.