Round 4 Algebra 2

Factoring; Equations involving Factoring

GREATER BOSTON MATHEMATICS LEAGUE

MEET 1 – OCTOBER 1998

ROUND 4 – Algebra 2– Factoring

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Factor the following into the product of two polynomials: $a^2 + b^2 - c^2 - d^2 - 2ab - 2cd$

2. Factor the following into the product of two polynomials: $x^3 - 8y^3 + 3x^2 + 3x + 1$

3. Factor the following: $4^x - x^4 + 4^2 - 2^{x+3}$

GREATER BOSTON MATHEMATICS LEAGUE

MEET 1 – SEPTEMBER 1999

ROUND 4 – Algebra 2– Factoring

1. ______________________

2. ______________________

3. ______________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Factor the following completely: $12x^3 - 46x^2 + 42x$

2. Factor the following into the product of 2 polynomials: $4x^3 - 9xy^2 + 10x + 15y$

3. Factor the following into the product of 2 polynomials: $x^2 - a - ax - 3x - 4$

GREATER BOSTON MATHEMATICS LEAGUE

MEET 1 – SEPTEMBER 2000

ROUND 4 – Algebra 2– Factoring

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND.

1. Factor the following completely: $x^5 - 9x^3 - 8x^2 + 72$

2. Factor the following into the product of 2 polynomials: $2a^2 + 2b^2 - 5a + 5b - 4ab - 12$

3. Factor the following into the product of 2 polynomials: $x^2 - 3a^2 - xy - ay - 2ax$

GREATER BOSTON MATHEMATICS LEAGUE

MEET 1 – OCTOBER 2001

ROUND 4 – Algebra 2– Factoring

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND.

1. Factor the following completely: $12x^4 - 19x^2 - 18$

2. Factor the following completely: $4^x - 2^{x+3} + 2^4 - 9^x$

3. Factor the following completely: $3x^4 - 3x^3 - 102x^2 - 168x$

GREATER BOSTON MATHEMATICS LEAGUE

MEET 1 – OCTOBER 2006

ROUND 4 – Algebra 2 – Factoring

1. ______________________________

2. ______________________________

3. ______________________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND.

1. Factor completely: $x^2 - 25 - y(y + 10)$

2. Factor completely: $27^x + 4 \cdot 9^x - 25 \cdot 3^x - 100$

3. If

$$x^2(y - z) + y^2(z - x) + z^2(x - y)$$

is factored completely over the integers, determine a nonzero <u>sum</u> of the factors? (There are three possible answers.)

GREATER BOSTON MATHEMATICS LEAGUE

MEET 1 – OCTOBER 2007

ROUND 4 – Algebra 2 – Factoring

1. ______________________

2. ______________________

3. ______________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND.

1. Factor completely: $4A^4 + 7x^2A^2 - 2x^4$

2. Find all numerical values of t for which the expression $3tx^2 + 10x + 3t$ has equal binomial factors.

3. Factor over the integers: $3x^2 + yz + y^2 - xz - 4xy$

GREATER BOSTON MATHEMATICS LEAGUE

MEET 1 – OCTOBER 2008

ROUND 4 – Algebra 2 – Factoring

1. ______________________________

2. ______________________________

3. ______________________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND.

1. Factor completely over the integers: $8x^4 + 2x^2y^2 - y^4$

2. Determine all real solutions of $x^4 + 5x^2 - 36 = 0$

3. Factor completely over the integers: $5x^6 - 20x^3y^3 - 160y^6$

GREATER BOSTON MATHEMATICS LEAGUE

MEET 1 – OCTOBER 2009

ROUND 4 – Algebra 2 – Factoring

1. ______________________

2. ______________________

3. ______________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND.

Factor each of the following completely.

1. $10ac - 3b - 2a + 15bc$

2. $(x^2 - 4x - 12)^2 - (2x^2 + 3x - 2)^2$

3. $9x^4 + 11x^2a^2 + 4a^4$

GREATER BOSTON MATHEMATICS LEAGUE

MEET 1 – OCTOBER 2010

ROUND 4 – Algebra 2 – Factoring

1. ______________________

2. ______________________

3. ______________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND.

Factor each of the following completely over the integers.

1. $49\left(1-y^2\right)-70x+25x^2$

2. $3y^5-27y-24y^3$

3. $4x^4-4x-16+16x^3$

MASSACHUSETTS MATHEMATICS LEAGUE
JANUARY 2004
ROUND 2: FACTORING & APPLICATIONS

ANSWERS

A)_______________

B)_______________

C)_______________

A) The base of a triangle is five more than twice the altitude to that base. If the area of the triangle is 84, calculate the length of the base.

B) Find three consecutive odd integers such that the product of the first and the third added to the sum of all three is 234.

C) Factor: $2x^5 - 3x^4 - 16x^2 + 24x$

MASSACHUSETTS MATHEMATICS LEAGUE
JANUARY 2005
ROUND 2 ALGEBRA ONE: FACTORING & EQUATIONS

ANSWERS

A)______________**ft**

B)______________

C)______________

A) A rectangular playground of area 560 square feet is built on a vacant lot 32 feet wide by 40 feet long. The playground is placed an equal distance from all four sides of the lot. Find the perimeter of the playground.

B) Find all real values of x for which: $2x = \dfrac{2 - x - x^2}{2 + x}$

C) Find all real values of x (no approximations!) for which $x^3 + 1 = x^2 + 4x + 3$

MASSACHUSETTS MATHEMATICS LEAGUE
JANUARY 2006
ROUND 2 ALGEBRA ONE: FACTORING & EQUATIONS

ANSWERS

A)_______________

B)_______________

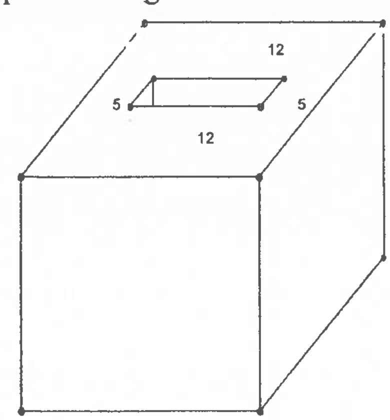
C)_______________ **cm**

All Factoring Is Over The Polynomials With Integer Coefficients

A) Find the largest integer g for which $2x^2 + gx - 15$ will be factorable.

B) Find the greatest common factor of $12x^2 - 42x + 18$ and $8x^2 + 20x - 12$.

C) A rectangular hole is cut all the way through a <u>cube</u> leaving side borders of 5 cm each and front and back borders of 12 cm as shown. If creating the hole removes exactly half of the volume of the cube, find all possible lengths for the side of the original cube.



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 – JANUARY 2007
ROUND 2 ALG 1: FACTORING AND/OR EQUATIONS INVOLVING FACTORING

ANSWERS

A) _______________________________

B) _______________________________

C) _______________________________

A) 1741 is a prime number.
It does not factor as the product of two integers (except the trivial $(1 \cdot 1741)$.
Find the ordered pair of consecutive positive integers (a, b), where $a > b$, for which the product ab is closest to 1741.

B) Mersenne Numbers are numbers of the form $2^n - 1$, for integers $n > 2$. If n is even, this formula always generates numbers that are composite. If n is odd, this is not necessarily the case. Find the sum of all prime factors of the smallest composite Mersenne number generated by an odd value of n.
Note: 1 is neither prime nor composite.

C) Determine all values of x for which $\left(6\left(\frac{x-3}{x-7}\right)-4\right)^2 - 5\left(2-3\left(\frac{x-3}{x-7}\right)\right) = 21$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 – JANUARY 2008
ROUND 2 ALG 1: FACTORING AND/OR EQUATIONS INVOLVING FACTORING

ANSWERS

A) ______________________________

B) ______________________________

C) ______________________________

A) Find all values of <u>a</u> so the expression $4x^2 + 8ax + 25$ is a perfect trinomial square.

B) For some integer values of <u>a</u>, the expression $x^2 + ax - 15$ may be written as the product of two binomials with integer coefficients.
For which of these values of <u>a</u>, does the expression $ax^2 + 98$ have two distinct linear factors with integer coefficients?

Note: A linear factor has the form $mx + b$, where $m \neq 0$.

C) Find all real values of x for which $\dfrac{2x^2 + x - 1}{x^2 - x - 2} = 1 - 2x$

GBML 1998

ROUND 4

1. $a^2+b^2-c^2-d^2-2ab-2cd=\left(a^2-2ab+b^2\right)-\left(c^2+2cd+d^2\right)=$
$(a-b)^2-(c+d)^2=(a-b-c-d)(a-b+c+d)$

2. $x^3-8y^3+3x^2+3x+1=x^3+3x^2+3x+1-8y^3=(x+1)^3-(2y)^3=$
$(x+1-2y)\left((x+1)^2+2y(x+1)+(2y)^2\right)=(x+1-2y)\left(x^2+2x+1+2xy+2y+4y^2\right)$

3. $4^x-x^4+4^2-2^{x+3}=2^{2x}-8\cdot 2^x+4^2-\left(x^2\right)^2=\left(2^x-4\right)^2-\left(x^2\right)^2=\left(2^x-x^2-4\right)\left(2^x+x^2-4\right)$

GBML 1999

ROUND 4

1. $12x^3-46x^2+42x=2x\left(6x^2-23x+21\right)=2x(3x-7)(2x-3)$

2. $4x^3-9xy^2+10x+15y=x\left(4x^2-9y^2\right)+5(2x+3y)=x(2x+3y)(2x-3y)+5(2x+3y)=$
$(2x+3y)\left(2x^2-3xy+5\right)$

3. $x^2-a-ax-3x-4=x^2-3x-4-ax-a=(x-4)(x+1)-a(x+1)=(x+1)(x-4-a)$

GBML 2000

ROUND 4

1. $x^4-x^2y^2-2xy^3+2y^4=$
$x^3\left(x^2-9\right)-8\left(x^2-9\right)=\left(x^2-9\right)\left(x^3-8\right)=(x-3)(x+3)(x-2)\left(x^2+2x+4\right)$

2. $2a^2+2b^2-5a+5b-4ab-12=$
$2a^2-4ab+2b^2-5a+5b-12=2(a-b)^2-5(a-b)-12=$
$(2(a-b)+3)((a-b)-4)=(2a-2b+3)(a-b-4)$

3. $x^2-3a^2-xy-ay-2ax=x^2-2ax-3a^2-xy-ay=(x-3a)(x+a)-y(x+a)$
$=(x-3a-y)(x+a)$

GBML 2001

ROUND 4

1. $12x^4-19x^2-18=\left(4x^2-9\right)\left(3x^2+2\right)=(2x-3)(2x+3)\left(3x^2+2\right)$

2. $4^x-2^{x+3}+2^4-9^x=2^{2x}-8\cdot 2^x+4^2-3^{2x}=\left(2^x-4\right)^2-3^{2x}=\left(2^x-3^x-4\right)\left(2^x+3^x-4\right)$

3. $3x^4-3x^3-102x^2-168x=3x\left(x^3-x^2-34x-56\right)$; now use synthetic division to factor the cubic polynomial: $\begin{array}{r|rrrr} 7 & 1 & -1 & -34 & -56 \\ \hline & 1 & 6 & 8 & 0 \end{array}$ $\Rightarrow$ the polynomial $=3x(x-7)\left(x^2+6x+8\right)=$
$3x(x-7)(x+2)(x+4)$

GBML 2006

ROUND 4

1.
$x^2 - 25 - y(y + 10) = x^2 - (y^2 + 10y + 25) = x^2 - (y + 5)^2$
As the difference of perfect squares, this factors to $\underline{(x + y + 5)(x - y - 5)}$.

2.
$27^x + 4\cdot 9^x - 25\cdot 3^x - 100 = 3^{3x} + 4\cdot 3^{2x} - 25\cdot 3^x - 100 = 3^{2x}(3^x + 4) - 25(3^x + 4)$
$= (3^x + 4)(3^{2x} - 25) = \underline{(3^x + 4)(3^x + 5)(3^x - 5)}$

3.
$\underline{x^2(y - z)} + \underline{y^2(z - x)} + z^2(x - y) = \underline{x^2y - x^2z + y^2z - y^2x} + z^2(x - y)$
$= \underline{(x^2y - y^2x)} - \underline{(x^2z - y^2z)} + z^2(x - y) = \underline{xy(x - y)} - \underline{z(x^2 - y^2)} + z^2(x - y)$
$= (x - y)[xy - z(x + y) + z^2] = (x - y)[xy - zx - zy + z^2] = (x - y)[y(x - z) - z(x - z)]$
$= (x - y)(x - z)(y - z)$
Finally, the sum of the three factors is $\underline{2x - 2z \text{ or } 2(x - z)}$.

Other possible factorizations are:
$(-x + y)(-x + z)(y - z) \rightarrow$ sum $= \underline{2y - 2x \text{ or } 2(y - x)}$
$(x - y)(-x + z)(-y + z) \rightarrow$ sum $= \underline{2z - 2y \text{ or } (2(z - y)}$
$(-x + y)(x - z)(-y + z) \rightarrow$ sum $= 0$ (rejected)

GBML 2007

ROUND 4

1. $4A^4 + 7x^2A^2 - 2x^4 = (4A^2 - x^2)(A^2 + x^2) = \underline{(2A - x)(2A + x)(A^2 + 2x^2)}$

2. The roots of $3tx^2 + 10x + 3t$ are $\dfrac{-10 \pm \sqrt{100 - 36t^2}}{6t}$. To have equal binomial factors, the roots must be equal. This only happens if the discriminant $100 - 36t^2$ is zero.

   Thus, $t^2 = \dfrac{100}{36} \rightarrow t = \underline{\pm\dfrac{5}{3}}$ $[\rightarrow 5(x + 1)^2$ or $-5(x - 1)^2]$

3. Regrouping, $3x^2 + yz + y^2 - xz - 4xy = (3x^2 - 4xy + y^2) + (yz - xz)$
   $= (3x - y)(x - y) - z(x - y) = \underline{(x - y)(3x - y - z)}$

GBML 2008

ROUND 4

1. $8x^4 + 2x^2y^2 - y^4 = (4x^2 - y^2)(2x^2 + y^2) = \underline{(2x + y)(2x - y)(2x^2 + y^2)}$

2. $x^4 + 5x^2 - 36 = 0 \rightarrow (x^2 + 9)(x^2 - 4) = 0 \rightarrow x = \underline{\pm 2}$
3. $5x^6 - 20x^3y^3 - 160y^6 = 5(x^6 - 4x^3y^3 - 32y^6) = 5(x^3 - 8y^3)(x^3 + 4y^3)$
   $= \underline{5(x - 2y)(x^2 + 2xy + 4y^2)(x^3 + 4y^3)}$

GBML 2009

ROUND 4

1. $10ac - 3b - 2a + 15bc = 2a(5c - 1) + 3b(5c - 1) = \underline{(5c - 1)(2a + 3b)}$

2. $(x^2 - 4x - 12)^2 - (2x^2 + 3x - 2)^2 = ((x - 6)(x + 2))^2 - ((x + 2)(2x - 1))^2 =$
   $(x + 2)^2((x - 6)^2 - (2x - 1)^2) = (x + 2)^2(35 - 8x - 3x^2) = \underline{(x + 2)^2(7 - 3x)(5 + x)}$

3. Add/subtract a fudge factor so that the given trinomial becomes a perfect square trinomial.
   $(9x^4 + 12x^2a^2 + 4a^4) - x^2a^2 = (3x^2 + 2a^2)^2 - (xa)^2 = \underline{(3x^2 + xa + 2a^2)(3x^2 - xa + 2a^2)}$

GBML 2010

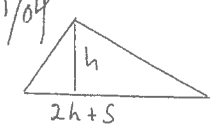
ROUND 4

1. $49(1 - y^2) - 70x + 25x^2 \rightarrow (49 - 70x + 25x^2) - 49y^2 = (7 - 5x)^2 - (7y)^2$
   As the difference of perfect squares, we have $\underline{(7 - 5x + 7y)(7 - 5x - 7y)}$.

2. $3y^5 - 27y - 24y^3 = 3y(y^4 - 8y^2 - 9) = 3y(y^2 - 9)(y^2 + 1) = \underline{3y(y + 3)(y - 3)(y^2 + 1)}$
3. $4x^4 - 4x - 16 + 16x^3 =$
   $4((x^4 + 4x^3) - (x + 4)) = 4(x^3(x + 4) - (x + 4)) = 4(x + 4)(x^3 - 1) = \underline{4(x + 4)(x - 1)(x^2 + x - 1)}$

MML 1/04

A) The base of a triangle is five more than twice the altitude to that base If the area of the triangle is 84, calculate the length of the base



$\frac{1}{2}h(2h + 5) = 84$
$2h^2 + 5h - 168 = 0$
$(2h + 21)(h - 8) = 0$
$h = 8,\ 2h + 5 = \boxed{21}$

B) Find three consecutive odd integers such that the product of the first and the third added to the sum of all three is 234

$x,\ x + 2,\ x + 4$
$x(x + 4) + (3x + 6) = 234$
$x^2 + 7x - 228 = 0$
$(x + 19)(x - 12) = 0$
$x = -19$
$\boxed{\underline{\text{ANS}}\ -19, -17, -15}$

C) Factor $2x^5 - 3x^4 - 16x^2 + 24x$

$x^4(2x - 3) - 8x(2x - 3) = x(2x - 3)(x^3 - 8) =$
$\boxed{x(2x - 3)(x - 2)(x^2 + 2x + 4)}$

MML 1/05

Round Two:

A. (32-2x)(40-2x)=560 becomes $x^2 - 36x + 180 = 0$. x=30 or 6. Nearest edge is 6. Playground is 20 X 28 so perimeter is 96 ft.

B. $2x = \dfrac{2-x-x^2}{2+x}$ becomes $2x = \dfrac{(2+x)(1-x)}{2+x}$ Since $x \neq -2$, $2x = 1 - x$ so x=1/3

C. Factoring each side: $(x+1)(x^2 - x + 1) = (x+1)(x+3)$ so $x = -1$ or $(x^2 - x + 1) = (x+3)$ which by quad formula gives $x = 1 \pm \sqrt{3}$

MML 1/06

Round Two:

A. $(2x - a)(x + b)$ maximizes g when b is maximum, a minimum so b=15, a=1.

B. Factoring gives $6(2x-1)(x-3)$ and $4(2x-1)(x+3)$ common is $2(2x-1)$

C. $\frac{1}{2}x^3 = x(x-24)(x-10)$ so $0 = \frac{1}{2}x^3 - 34x^2 + 240 = \frac{1}{2}x(x^2 - 68x + 480) = \frac{1}{2}x(x-8)(x-60)$. Only $x = 60$ gives a large enough cube.

MML 1/07

Round 2

A) Taking the square root of 1741 → 41.7^+
The product 40(41) = 1640 is obviously smaller then 1741, since both factors are smaller than the square root of 1741. Likewise 42(43) = 1806 is obviously larger then 1741, since both factors are larger than the square root of 1741. Thus, the product closest to 1741 is produced by the pair of integers that sandwich the square root, 41(42) = 1722. $a > b \rightarrow (a, b) = \underline{(42, 41)}$.

B) n = 3, 5 and 7 produces 7, 31 and 127 respectively, all of which are primes.
$n = 9 \rightarrow 511 = 7(73)$
7 + 73 = **80**

C) Let $A = \dfrac{x-3}{x-7}$. Then $\left(6\left(\dfrac{x-3}{x-7}\right)-4\right)^2 - 5\left(2-3\left(\dfrac{x-3}{x-7}\right)\right) = 21$ simplifies to

$(2(3A-2))^2 + 5(3A-2) - 21 = 0$
Letting $B = 3A - 2$, we have $4B^2 + 5B - 21 = (4B-7)(B+3) = 0$ or substituting back
$(4(3A-2)-7)(3A-2+3) = (12A-15)(3A+1) = 0 \rightarrow A = 5/4$ or $-1/3$
Finally, substituting for A,

$\dfrac{x-3}{x-7} = \dfrac{5}{4} \rightarrow 4x - 12 = 5x - 35 \rightarrow x = \underline{23}$

$\dfrac{x-3}{x-7} = \dfrac{-1}{3} \rightarrow 3x - 9 = -x + 7 \rightarrow 4x = 16 \rightarrow x = \underline{4}$

MML 1/08

A) $4x^2 + 8ax + 25 = (2x \pm 5)^2 = 4x^2 \pm 20x + 25 \rightarrow 8a = \pm 20 \rightarrow a = \underline{\pm \frac{5}{2}}$

B) -15 factors as (1)(-15), (-1)(15), (3)(-5), (-3)(5), $\rightarrow a = \pm 14$ or ± 2
The corresponding factorizations are: $14(x^2 + 7)$, $-14(x^2 - 7)$, $2(x^2 + 49)$ and $-2(x^2 - 49)$ and only the latter has two distinct linear factors over the integers. Thus, $a = \underline{-2}$

C) $\dfrac{2x^2 + x - 1}{x^2 - x - 2} = 1 - 2x \rightarrow \dfrac{(2x-1)(x+1)}{(x-2)(x+1)} = 1 - 2x$

Clearly, $x = -1$ is not a solution. Canceling, $\dfrac{(2x-1)}{(x-2)} = 1 - 2x \rightarrow 2x - 1 = (x-2)(1-2x)$

$2x - 1 = x - 2x^2 - 2 + 4x \rightarrow 2x^2 - 3x + 1 = (x-1)(2x-1) = 0 \rightarrow x = \mathbf{1, \frac{1}{2}}$