Round 1 – Arithmetic

Number Theory

GREATER BOSTON MATHEMATICS LEAGUE

MEET 2 – NOVEMBER 1998

ROUND 1 – Arithmetic-Open

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Given the following summation in base 6, $b34_6 + aac_6 = a0ba_6$, find the sum, $a + b + c$ **in base 6** .

2. How many perfect square factors does 4,000,000 have?

3. Find the 1998th counting number divisible by 4 but not by 5.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 2 – NOVEMBER 1999

ROUND 1 – Arithmetic-Open

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. If n % of 55 is 25% of 88, find n.

2. Given that the following multiplication of two digit base ten numbers produces a result which is divisible by nine, compute $x+y+z$.
Note that x, y, and z are not necessarily distinct digits.

$$\begin{array}{r} 2\;\,x \\ \underline{4\;\,y} \\ 9\;\,4\;\,z \end{array}$$

3. If $\sqrt{\sqrt{6!a}}$ is a positive integer, find the smallest possible integral value for a.
Note that $n! = n(n-1)(n-2)....(2)(1)$.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 2 – NOVEMBER 2000

ROUND 1 – Arithmetic-Open

1. (, ,)

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Given x, y, and z are distinct, non-zero digits in base 8 satisfying the following addition in base 8 with $x > y$, find the ordered triple (x, y, z).

$$\begin{array}{r} x\ z_8 \\ +\ y\ z_8 \\ \hline 1\ 4\ 0_8 \end{array}$$

2. How many natural (counting) numbers less than 199 are divisible by 3 or 5, but not by both 3 and 5?

3. How many counting (natural) numbers less than 100 have 12 positive integral factors?

GREATER BOSTON MATHEMATICS LEAGUE

MEET 2 – NOVEMBER 2001

ROUND 1 – Arithmetic-Open

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Find the smallest whole number which has a remainder of 3 when divided by 7 and has a remainder of 8 when divided by 13.

2. Let *M* equal the smallest positive multiple of five which is one less than a perfect cube. Let *N* equal the largest positive integer which is less than one thousand with exactly three factors. Find the sum of *M* and *N*.

3. Given $X4Y_{(9)} = Y4X_{(10)}$, find all possible ordered pairs (X, Y).

GREATER BOSTON MATHEMATICS LEAGUE
MEET 2 – NOVEMBER 2006

ROUND 1 – Arithmetic - Open

1. ______________________________ feet

2. ______________________________

3. ______________________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND.

1. A disabled car is pushed 5/8 of a mile in 1 hour.
   In 24 more minutes, at the same rate, the car will have reached its destination.
   How many feet is it to its destination from its original starting point?
   Give an <u>exact</u> answer.
   Note: 1 mile = 5280 feet

2. How many integers between 200 and 300 inclusive are divisible by 2 and 3, but not by 4?

3. Find the smallest natural number that is the product of three consecutive integers and is divisible by 7, 12 and 15.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 2 – NOVEMBER 2007

ROUND 1 – Arithmetic - Open

1. ______________________

2. ______________________

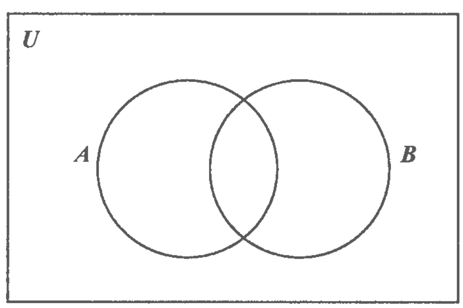
3. ______________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND.

1. x and y are positive integers and $x+\dfrac{2}{3y+1}=\dfrac{25}{8}$.
   Find all ordered pairs (x, y) that satisfy the given conditions.

2. Given the sum $\begin{array}{r} 132_4 \\ \underline{132_5} \\ 132_x \end{array}$ and $x \geq 2$, find the value of x.

3. $n(U)=30$, $n\left(\overline{A \cap B}\right)=23$, and $n\left(\overline{B}\right)=17$. Find $n\left(A \cup \overline{B}\right)$.



Note: $\overline{A}$ denotes the set of elements not in set A
$A \cap B$ denotes the set of elements in both set A and set B (i.e. the intersection)
$A \cup B$ denotes the set of elements that are either in set A, in set B or in both (i.e. the union)

GREATER BOSTON MATHEMATICS LEAGUE

MEET 2 – NOVEMBER 2008

ROUND 1 – Arithmetic - Open

1. ______________________

2. ______________________ (base 5)

3. ______________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND.

1. How many two-digit prime numbers are such that the sum of their digits equals 8 ?

2. What is the base 5 value of $16_7 + 25_8 - 11_3$?

3. In a group of 134 high school sophomores, 11 take biology and geometry but not history, 30 take biology and history but not geometry, and 16 take history and geometry but not biology. 10 students take all three of these subjects. How many take only one of these subjects, if there are 15 who take none of them?

GREATER BOSTON MATHEMATICS LEAGUE

MEET 2 – NOVEMBER 2009

ROUND 1 – Arithmetic - Open

1. ______________________

2. ______________________

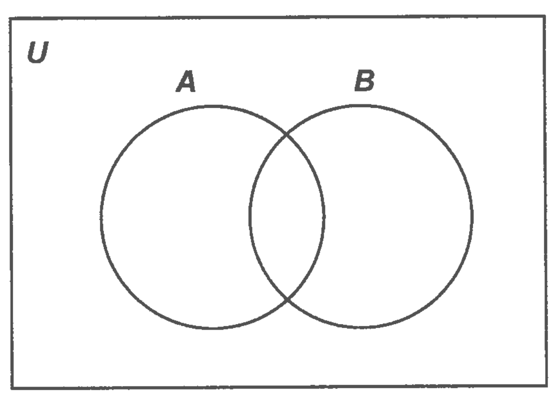
3. ______________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND.

1. The product of 165 and an integer K is the 5 digit number $2PP75$.
   Find all possible value(s) of P.

2. How many positive integral divisors does the number 13192 have?

3. $n(U) = 45$, $n(B) = 4n(A \cap B)$, $n(A) = 3n(A \cap B)$ and $n(\overline{A} \cap B) = 2n(\overline{A \cup B})$
   Find the $n(A \cap \overline{B}) + n(\overline{A} \cap B)$



GREATER BOSTON MATHEMATICS LEAGUE

MEET 2 – NOVEMBER 2010

ROUND 1 – Arithmetic - Open

1. ______________________

2. ______________________

3. ______________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND.

1. For <u>how many</u> integer values of x such that $2 \le x \le 10$ does $144_{(2x-1)} = 441_{(x)}$

2. For what real value(s) of x will the complex fraction $6-\cfrac{5}{4-\cfrac{3}{2-\cfrac{1}{x}}}$ be undefined?

3. The Math Club found it could achieve a membership ratio of 2 females for each male either by adding 24 females as new members <u>or</u> by dropping x males from the club, but not both. Compute x.

GBML 98

ROUND 1

1. $b34_6 + aac_6 = a0ba_6 \Rightarrow a = 1 \Rightarrow b34_6 + 11c_6 = 10b1_6 \Rightarrow 4 + c = 11_6 \Rightarrow c = 3 \Rightarrow$
$b34_6 + 113_6 = 10b1_6 \Rightarrow 1 + 3 + 1 = c \Rightarrow c = 5 \Rightarrow a + b + c = 9 = 13_6$

2. $4{,}000{,}000 = 2^8 \cdot 5^6$. The perfect square factors of 4,000,000 have an even power of 2 and an even power of 5. The even powers of 2 are 0,2,4,6, and 8. The even powers of 5 are 0,2,4, and 6. $\Rightarrow$There are $5 \cdot 4 =$ **20** factors that are perfect squares.

3. Consider the set of counting numbers from 1 to 20. Of these only 4, 8, 12, and 16 are divisible by 4, but not by 5; i.e. 4 out of every 20. $1998 \div 4 = 499$ R 2; 8 is the 2nd number divisible by 4 and not by 5 $\Rightarrow$ answer is $499 \times 20 + 8 =$ **9988**.

GBML 99

ROUND 1

1. $n \cdot 55 = 25 \cdot 88 \Rightarrow n = 40$

2. Since $94z$ is divisible by 9, then $z = 5$. If $z = 5$, then either x or $y = 5$. Since 945 is not divisible by 25, then $y = 5$ and $945 \div 45 = 21$. Therefore $x = 1$ and $x + y + z = 11$

3. $\sqrt{\sqrt{6!a}} = \sqrt[4]{6!a}$ and so $6!a$ must be a perfect 4th power. Since $6! = 2^4 \cdot 3^2 \cdot 5$, then $a = 3^2 \cdot 5^3 = 1125$

GBML 00

ROUND 1

1. Since $z \neq 0$, $z + z = 8 \rightarrow z = 4 \rightarrow x + y + 1 = 14_8 \rightarrow x + y = 11 \rightarrow x = 6,\ y = 5$. [Note since the digits are distinct $\rightarrow x = 7,\ y = 4$ is not possible.] The triple is $(6,5,4)$.

2. Find how many multiples of 3, 5, and 15 are less than 199:
$199 \div 3 = 66\frac{1}{3}$; $199 \div 5 = 39\frac{4}{5}$; $199 \div 15 = 13\frac{4}{15}$; therefore the result = 66 + 39 – 13 – 13 = 79.

3. Since the number has 12 factors it is of four types: $(i)\,p^{11}$ $(ii)\,p^5q$ $(iii)\,p^3q^2$ $(iv)\,p^2qr$, where p, q, and r are prime. Since the number is less than 100 $\rightarrow$ none of type (i); $2^5 \cdot 3$ of type (ii); $2^3 \cdot 3^2$ of type (iii); $2^2 \cdot 3 \cdot 5,\ 2^2 \cdot 3 \cdot 7,\ 3^2 \cdot 2 \cdot 5$ of type (iv); therefore there are 5 numbers less than 100 with 12 factors.

GBML 01

1. The number is of the forms $7y + 3$ or $13x + 8 \Rightarrow y = \dfrac{13x+5}{7} \Rightarrow$ when $x = 5$, $y = 10 \Rightarrow$ the number $= 7 \times 10 + 3 = 73$.

2. $M = 6^3 - 1 = 215$; N must be the square of a prime to have exactly 3 factors $\Rightarrow$
$N = 31^2 = 961 \Rightarrow M + N = 215 + 961 = 1176$.

3. $X4Y_{(9)} = Y4X_{(10)} \Rightarrow 81X + 36 + Y = 100Y + 40 + X \Rightarrow 80X = 99Y + 4 \Rightarrow$ when $Y = 4$ then $80X = 99 \cdot 4 + 4 = 400 \Rightarrow X = 5 \Rightarrow (X,Y) = (5,4)$.

GBML 06

ROUND 1

1. Since 24 min = 24/60 = 2/5 hr. and (rate)(time) = distance, (2/5)(5/8) = ¼ mile = (1/4)(5280) = **1320** feet
2. Integers divisible by 2 and 3, but not by 4 are multiples of 6 that are not multiples of 12.
The multiples of 6 are 204 = 6(34), 210, 216, … , 300 = 6(50) → 17 values
Starting with 204, every other number in the sequence is a multiple of 12 → 9 values
Thus, there are **8** integers divisible by 2 and 3, but not 4.
3. Since the number we seek must be divisible by 7,12 and 15, we must examine triples of consecutive numbers that contain multiples of 7. 5, 6, 7 fails – not divisible by 12; 6, 7, 8 fails – not divisible by 15; 7, 8, 9 fails – not divisible by 15; 12, 13, 14 fails – not divisible by 15; 13, 14, 15 fails – not divisible by 12; Thus, 14, 15, 16 is the smallest triple producing a product of **3360**.

GBML 07

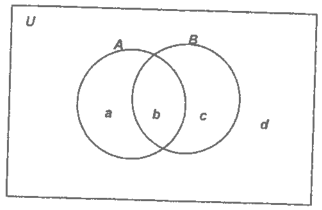
1. For any positive integer y, $\frac{2}{3y+1}$ is always a rational non-integer fraction.

   Therefore, $x + \frac{2}{3y+1} = \frac{25}{8} = 3\frac{1}{8} = 3 + \frac{1}{8} \rightarrow x = 3$ and $\frac{2}{3y+1} = \frac{1}{8}$

   Cross multiplying, $3y + 1 = 16 \rightarrow y = 5 \rightarrow (x, y) = \underline{\mathbf{(3, 5)}}$

2. $132_4 + 132_5 = (16 + 12 + 2) + (25 + 15 + 2) = 72 = x^2 + 3x + 2$
   $\rightarrow x^2 + 3x - 70 = (x - 7)(x + 10) = 0 \rightarrow x = \underline{7}$ ($x = -10$ is rejected, since $x \geq 2$)

3. $a + b + c + d = 30$
   $c + d = 11$
   $a + c + d = 23$
   $a + d = 17$
   Subtracting, $c = 6$
   $n(A \cup \overline{B}) = a + b + d = 30 - 6 = \underline{\mathbf{24}}$



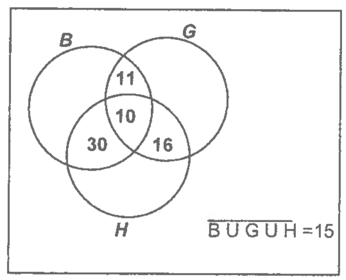
GBML 08

ROUND 1

1. Two digit prime numbers will end with a 1, 3, 5, 7, or 9. If the sum of the digits is 8, then we must inspect 17, 71, 35 and 53. Only 35 is not prime → Answer: $\underline{\mathbf{3}}$

2. $16_7 = 13_{10}$, $25_8 = 21_{10}$, $11_3 = 4_{10} \rightarrow 13 + 21 - 4 = 30_{10} = \underline{\mathbf{110_5}}$

3. Looking at the Venn Diagram, we can compute the answer as follows:
   $134 - (11 + 30 + 16 + 10 + 15) = 134 - 82 = \underline{\mathbf{52}}$



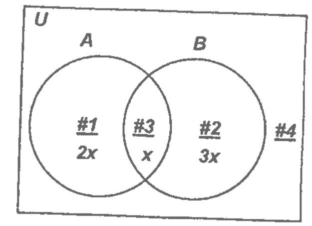
GBML 09

ROUND 1

1. The sum of the digits must be a multiple of 3.
   $14 + 2P \rightarrow 18$, 24 and 30 for $P = \underline{\mathbf{2, 5 \text{ and } 8}}$

2. $28712 = 4(7178) = 2^3(1649)$. Is this last factor prime?
   We must test prime factors up to the square root of 1649 which is approximately 40, i.e. we must test for divisibility by 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37.
   The shortcut tests for 3, 5 and 11 quickly fail. Brute force with other candidates finally gives us factors of 17(97), both of which are prime. Thus, $13192 = 2^3 17^1 97^1$
   $\rightarrow 4(2)(2) = \underline{\mathbf{16}}$ positive integer divisors.

3. The universal set (the rectangle) is divided into 4 region by the overlapping circles.
   Let these regions be #1, #2, #3 and #4 and let there be x elements in region #3.
   The second condition $n(B) = 4n(A \cap B) \rightarrow$ region #3 contains $3x$ elements.
   The third condition $n(A) = 3n(A \cap B) \rightarrow$ region #1 contains $2x$ elements.
   The fourth condition $n(\overline{A} \cap B) = 2n(\overline{A \cup B})$
   $\rightarrow 3x = 2(45 - 6x) \rightarrow x = 6$.
   $n(A \cap \overline{B}) + n(\overline{A} \cap B)$ denotes the number of elements in regions 1 and 3 combined.
   Thus, $5x = \underline{\mathbf{30}}$.



GBML 10

ROUND 1

1. $144_{(2x-1)} = 441_{(x)} \rightarrow (2x-1)^2 + 4(2x-1) + 4 = 4x^2 + 4x + 1$
   Expanding and collecting like terms on the left side of the equation, we see this equation is actually an identity, i.e. it is valid for all values of x. However, the original base equation establishes that 4 is a digit in base x. Therefore, $x > 4$ and, intersecting with the initial condition, $x = 5, 6, 7, 8, 9$ or $10 \rightarrow \underline{\mathbf{6}}$ values.

2. Clearly, $6 - \cfrac{5}{4 - \cfrac{3}{2 - \cfrac{1}{x}}}$ is undefined for $x = \underline{\mathbf{0}}$.

   $$6 - \cfrac{5}{4 - \cfrac{3}{2 - \cfrac{1}{x}}} = 6 - \cfrac{5}{4 - \cfrac{3}{\cfrac{2x-1}{x}}} = 6 - \cfrac{5}{4 - \cfrac{3x}{2x-1}}$$ is undefined for $x = \underline{\mathbf{\frac{1}{2}}}$.

   $$6 - \cfrac{5}{4 - \cfrac{3x}{2x-1}} = 6 - \cfrac{5}{\cfrac{5x-4}{2x-1}} = 6 - \frac{5(2x-1)}{5x-4}$$ is undefined for $x = \underline{\mathbf{\frac{4}{5}}}$.

3. Let (B, G, N) denote the number of boys, the number of girls and the total membership of the Math Club respectively. Thus, $B = N - G$.
   Inducting 24 females → (1) $\frac{G+24}{N-G} = \frac{2}{1}$ Expelling x males → (2) $\frac{G}{N-G-x} = \frac{2}{1}$
   $\rightarrow N - G = \frac{G+24}{2}$ Substituting in (2), $\cfrac{G}{\cfrac{G+24}{2} - x} = \frac{2}{1} \rightarrow G = G + 24 - 2x \rightarrow x = \underline{\mathbf{12}}$.