GREATER BOSTON MATHEMATICS LEAGUE

MEET 2 – NOVEMBER 2002

ROUND 1 – Arithmetic-Open

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. The number m is the smallest positive multiple of 17 such that 3 more than m is a multiple of 7. Find the value of m.

2. The $\lozenge$ operation on pairs of numbers is defined as follows: $a \lozenge b = \dfrac{ab}{a+b}$. Find all possible values of a such that a and $a \lozenge 3$ are both whole numbers.

3. Given $75 \times 196 \times 567 = 18^a \times 21^b \times 35^c$, where a, b, and c are positive integers, find the value of $a^2 + b^2 + c^2$.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 2 – NOVEMBER 2002

ROUND 2 – Simultaneous Linear Equations, Word Problems, Matrices

1. (,)

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Find the ordered pair (n, p) which is the solution to the following system of equations.

$$\begin{cases} 5(n-2)-7=8p \\ 2(p-4)+3n=14-n \end{cases}$$

2. On the first day of a trip, Albert spent 20% of his money and Sophia spent 30% of hers, leaving them with a total of \$820. Had, instead, Albert spent 40% of his money and Sophia spent 65% of hers, Albert would then have \$90 more to spend than Sophia. What was the total number of dollars that Albert and Sophia started with at the beginning of their trip?

3. Given the following matrix equation, $\begin{bmatrix} x & 3x \\ 2x & z \end{bmatrix} + y\begin{bmatrix} 2 & w \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} -4 & z \\ -5 & w-2 \end{bmatrix}$,

find the sum $w+z$.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 2 – NOVEMBER 2002

ROUND 3 – Geometry: Angles and Triangles

1. ____________________

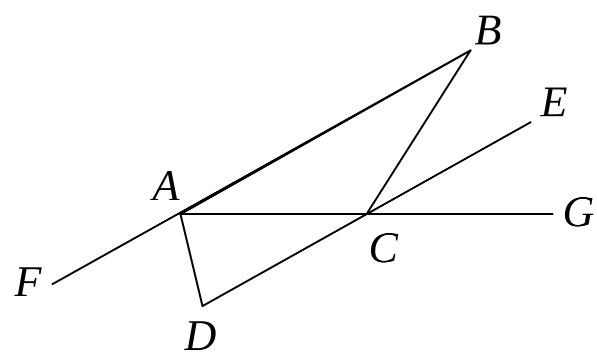
2. ____________________

3. ____________________

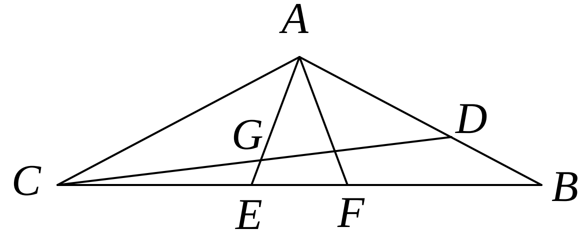
DIAGRAMS ARE NOT NECESSARILY DRAWN TO SCALE
CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Given ΔABC with the measures of $\angle A$, an exterior angle at B, and an exterior angle at C in the ratio of 1:4:5 respectively, find the number of degrees in the measure of $\angle A$.

2. Given the diagram on the right in which $\overline{BF} \parallel \overline{DE}$, $\overrightarrow{CE}$ bisects $\angle BCG$, $\overrightarrow{AD}$ bisects $\angle GAF$, and $m\angle B + m\angle D = 112°$, find the number of degrees in $m\angle B$.



3. Given $AC = AB$, $\overline{AE}$ and $\overline{AF}$ trisect $\angle BAC$, $m\angle ACD : m\angle BCD = 3:1$, and $m\angle ADC = 45°$, find the number of degrees in $m\angle DGE$.



GREATER BOSTON MATHEMATICS LEAGUE

MEET 2 – NOVEMBER 2002

ROUND 4 – Algebra 2– Quadratic Equations, Problems Involving Them, Theory of Quadratics

1. ____________________

2. ____________________

3. ____________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. The ratio of rows to seat per row in an auditorium was 5:2. After renovations, 10 rows were added, 2 seats were added to every row, and the auditorium now has 1820 seats. How many seats were in the auditorium before renovations?

2. The quadratic equation in x, $x^2 + bx + c = 0$, has roots $-3 \pm 3\sqrt{11}$. Find the roots to the equation $x^2 + bx + c = -18$.

3. A boat travels a certain distance upstream and the same distance downstream. If the stream's current is $4\sqrt{3}$ miles per hour and boat averages 13 miles per hour for the entire trip upstream and downstream, find the number of miles per hour in the boat's speed without a current.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 2 – NOVEMBER 2002

ROUND 5 – Trig. Equations

1. ______________________

2. ______________________

3. ______________________

CALCULATORS ARE NOT ALLOWED ON THIS ROUND

1. Given $0° \le x < 360°$, $\cos x > 0$, and $3\tan x = \cot x$, find all solutions for *x*.

2. Given $0° \le x < 360°$ and $\sin 2x + 2\cos x - \cos^2 x = \sin x + \sin^2 x$, find all solutions for *x*.

3. Given $0° \le x < 360°$ and $\sin x = \frac{1}{2}\sqrt{8\cos x + 7}$, find all solutions for *x*.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 2 – NOVEMBER 2002

TEAM ROUND

3 pts. 1. ___________________

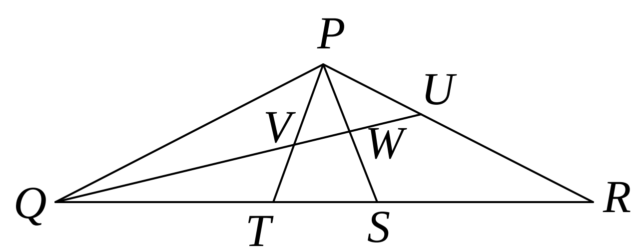
3 pts. 2. ___________________

4 pts. 3. ___________________

SAT APPROVED CALCULATORS ARE ALLOWED IN THE TEAM ROUND, EXCEPT FOR CALCULATORS WITH SYMBOLIC MANIPULATION PROGRAMS, (FOR EXAMPLE THE TI-89) WHICH ARE NOT ALLOWED IN THIS ROUND.

1. Given n is a composite (non-prime) whole number and $72n$ has exactly 24 whole number factors, find all possible values for n.

2. Given $PQ = PR$, $\overrightarrow{PT}$ and $\overrightarrow{PS}$ trisect $\angle QPR$, $\overrightarrow{QU}$ bisects $\angle PQR$, and $m\angle QWS = 105°$, find the number of degrees in $m\angle UVT$.



3. Al has \$7.85 in nickels, dimes and quarters. There are 9 more quarters than nickels. If Al has at least one of each type of coin, what is the difference between the most number of coins and least number of coins Al can have?

Detailed Solutions of GBML MEET 2 – NOVEMBER 2002

ROUND 1

1. $17+3=20,\ 34+3=37,\ 51+3=54,\ 68+3=71,\ 85+3=88,\ 102+3=105$, which is a multiple of 7. Answer is 102.
Alternative solution: $20\equiv -1 \bmod 7, 17\equiv 3 \bmod 7,\ -1+5(3)\equiv 0 \bmod 7 \Rightarrow 20+5(17)=105$ is the multiple of 7 $\Rightarrow$ 102 is the multiple of 17.

2. $a\Diamond 3=\dfrac{3a}{a+3}=3-\dfrac{9}{a+3}$; since a is a whole number, the only possibilities are 0 and 6.

3. $75\times196\times567=18^a\times21^b\times35^c \Rightarrow 3\times5^2\times2^2\times7^2\times7\times3^4=\left(2\cdot3^2\right)^a(3\cdot7)^b(5\cdot7)^c \Rightarrow$
$2^2\times3^5\times5^2\times7^3=2^a\times3^{2a+b}\times5^c\times7^{b+c} \Rightarrow a=2,\ c=2,\text{and } b=1 \Rightarrow a^2+b^2+c^2=9.$

ROUND 2

1. $\begin{cases}5(n-2)-7=8p\\2(p-4)+3n=14-n\end{cases}\Rightarrow\begin{cases}5n-17=8p\\2p-8+3n=14-n\end{cases}\Rightarrow\begin{cases}5n-8p=17\\4n+2p=22\end{cases}\Rightarrow\begin{cases}5n-8p=17\\16n+8p=88\end{cases}\Rightarrow$
$21n=105\Rightarrow n=5\Rightarrow 20+2p=22\Rightarrow p=1\Rightarrow$ solution is $(5,1)$.

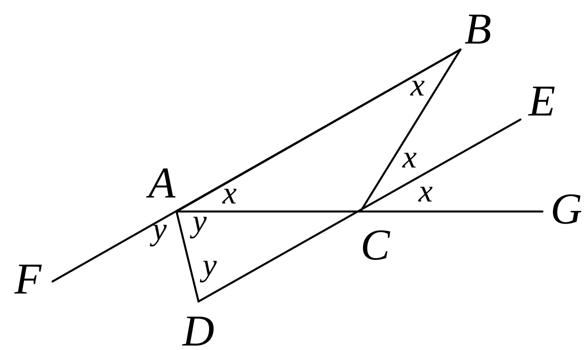
2 Let x = Albert's money, y = Sophia's money $\Rightarrow$
$\begin{cases}0.8x+0.7y=820\\0.6x-0.35y=90\end{cases}\Rightarrow\begin{cases}0.8x+0.7y=820\\1.2x-0.7y=180\end{cases}\Rightarrow$
$2x=1000\Rightarrow x=500\Rightarrow 400+.7y=820\Rightarrow .7y=420\Rightarrow y=600\Rightarrow x+y=1100$

3. $\begin{cases}x+2y=-4\\2x+3y=-5\end{cases}\Rightarrow\begin{cases}-2x-4y=8\\2x+3y=-5\end{cases}\Rightarrow -y=3\Rightarrow y=-3\Rightarrow x-6=-4\Rightarrow x=2\Rightarrow$
$\begin{cases}6-3w=z\\z-12=w-2\end{cases}\Rightarrow 6-3w-12=w-2\Rightarrow 4w=-4\Rightarrow w=-1\Rightarrow z=9\Rightarrow w+z=8$

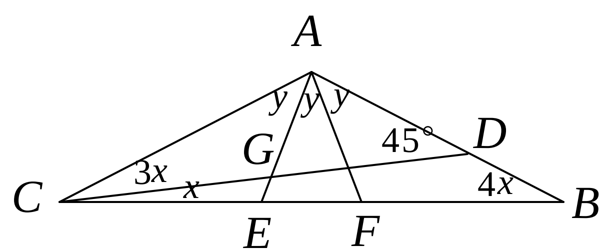
ROUND 3

1. Let $x = m\angle A \Rightarrow 4x =$ measure of exterior angle at B and $5x =$ measure of exterior angle at $C \Rightarrow 5x = x + 180 - 4x \Rightarrow 8x = 180 \Rightarrow x = 22.5$

2. Let $m\angle BCE = m\angle GCE = x \Rightarrow m\angle BAG = m\angle ABC = x$ by corresponding and alternate interior angles, respectively.
Let $m\angle GAD = m\angle FAD = y \Rightarrow m\angle ADE = y$ by alternate interior angles. $x + y = 112$ and $x + 2y = 180 \Rightarrow y = 68 \Rightarrow x = 44$.



3. Let $m\angle BCD = x \Rightarrow m\angle ACD = 3x \Rightarrow m\angle B = 4x$; let $m\angle CAE = m\angle EAF = m\angle BAF = y$; $5x = 45 \Rightarrow x = 9 \Rightarrow 4x = 36 \Rightarrow 3y + 72 = 180 \Rightarrow y = 36$; $m\angle DGE = m\angle AGC = 2y + 45 = 72 + 45 = 117$.



ROUND 4

1. Let $5x$ = number of rows $\Rightarrow 2x$ = seats per row: $(5x+10)(2x+2) = 1820 \Rightarrow (x+2)(x+1) = 182 \Rightarrow x^2 + 3x - 180 = 0 \Rightarrow (x+15)(x-12) = 0 \Rightarrow x = 12 \Rightarrow$ the auditorium original number of seats was $10x^2 = 1440$.

2. sum of the roots $= -b = -6 \Rightarrow b = 6$; product of the roots $= c = 9 - 99 = -90$; new equation is: $x^2 + 6x - 90 = -18 \Rightarrow x^2 + 6x + 9 = 81 \Rightarrow (x+3)^2 = 9^2 \Rightarrow x + 3 = \pm 9 \Rightarrow x = -3 \pm 9 = -12, 6$.

3. Let r = rate of the boat without any current and d = distance one way $\Rightarrow$
$$\frac{d}{r+4\sqrt{3}} + \frac{d}{r-4\sqrt{3}} = \frac{2d}{13} \Rightarrow \frac{1}{r+4\sqrt{3}} + \frac{1}{r-4\sqrt{3}} = \frac{2}{13} \Rightarrow \frac{2r}{r^2-48} = \frac{2}{13} \Rightarrow \frac{r}{r^2-48} = \frac{1}{13} \Rightarrow$$
$13r = r^2 - 48 \Rightarrow r^2 - 13r - 48 = 0 \Rightarrow (r-16)(r+3) = 0 \Rightarrow r = 16$ miles per hour.

ROUND 5

1. $3\tan x = \cot x \Rightarrow \tan^2 x = \frac{1}{3} \Rightarrow \tan x = \pm\frac{1}{\sqrt{3}}$; since $\cos x > 0 \Rightarrow x$ is in quadrants I or IV $\Rightarrow$ $x = 30°, 330°$.

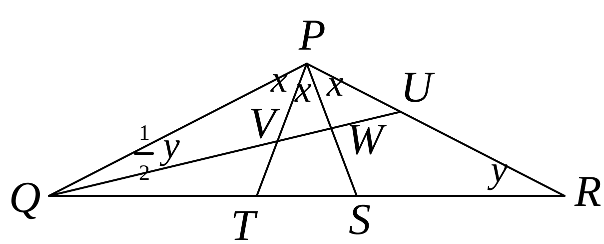
2. $\sin 2x + 2\cos x - \cos^2 x = \sin x + \sin^2 x \Rightarrow 2\sin x \cdot \cos x + 2\cos x = \sin x + \sin^2 x + \cos^2 x \Rightarrow$ $2\sin x \cdot \cos x + 2\cos x = \sin x + 1 \Rightarrow 2\cos x(\sin x + 1) = 1(\sin x + 1) \Rightarrow$ $(2\cos x - 1)(\sin x + 1) = 0 \Rightarrow \cos x = \frac{1}{2}$ or $\sin x = -1 \Rightarrow x = 60°, 270°, 300°$.

3. $\sin x = \frac{1}{2}\sqrt{8\cos x + 7} \Rightarrow \sin^2 x = \frac{1}{4}(8\cos x + 7) \Rightarrow 4(1 - \cos^2 x) = 8\cos x + 7 \Rightarrow$ $4\cos^2 x + 8\cos x + 3 = 0 \Rightarrow (2\cos x + 1)(2\cos x + 3) = 0 \Rightarrow \cos x = -\frac{1}{2}, \cancel{-\frac{3}{2}} \Rightarrow$ x = 120°, 240°, but $\sin 240° < 0 \Rightarrow$ only solution is 120°.

TEAM ROUND

1. $72 = 2^3 \cdot 3^2$; if n was prime $\Rightarrow$ 72n would have 4×3×2 = 24 factors; since n is composite $\Rightarrow$ n must just consist of factors of 2 and/or 3. $2^5 \cdot 3^3$ has 6×4 factors $\Rightarrow n = 2^2 \cdot 3 = 12$; $2^3 \cdot 3^5$ has 4×6 factors $\Rightarrow n = 3^3 = 27$; $2^7 \cdot 3^2$ has 8×3 factors $\Rightarrow n = 2^4 = 16$; therefore, n = 12, 16, or 27.

2. Let $m\angle QPT = m\angle TPS = m\angle RPS = x$ and let $m\angle R = y \Rightarrow m\angle PQU = \frac{1}{2}y$; $3x + 2y = 180$ and $2x + \frac{1}{2}y = 105 \Rightarrow$ $8x + 2y = 420 \Rightarrow 5x = 240 \Rightarrow x = 48 \Rightarrow$ $y = 18$ Þ $m\angle UVT = 180 - x - \frac{1}{2}y = 123$.



3. Let n = number of nickels, d = number of dimes, q = number of quarters $\Rightarrow$ $5n + 10d + 25q = 785$ and $q = n + 9 \Rightarrow n + 2d + 5q = 157$ and $q = n + 9 \Rightarrow$ $n + 2d + 5(n + 9) = 157 \Rightarrow 6n + 2d = 112 \Rightarrow 3n + d = 56 \Rightarrow$ If d = 2, n = 18, and q = 27 for a total of 47 coins; If d = 53, n = 1, and q = 10 for a total of 64 coins; $64 - 47 = 17$.

   Note: As the number of nickels go up by 1, the number of dimes go down by 3, and the number of quarters go up by one, giving you one less coin than before. So the extreme values for the number of nickels (1 and 18) give the largest and smallest number of coins.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 2 – NOVEMBER 2002

ANSWER SHEET:

ROUND 1

1. 102
2. 0, 6
3. 9

ROUND 4

1. 1440
2. –12, 6
3. 16 (16 miles per hour)

ROUND 2

1. $(5,1)$
2. 1100 ($1100)
3. 8

ROUND 5

1. 30°, 330°
2. 60°, 270°, 300°
3. 120°

ROUND 3

1. 22.5 or equivalent (22.5°)
2. 44 (44°)
3. 117 (117°)

TEAM ROUND

3 pts. 1. 12, 16, 27

3 pts. 2. 123 (123°)

4 pts. 3. 17