Team Round

GREATER BOSTON MATHEMATICS LEAGUE

MEET 4 – JANUARY 1999

TEAM ROUND
Problem submitted by Maimonides

3 pts. 1. ____________________

3 pts. 2. ____________________

4 pts. 3. ____________________

SAT APPROVED CALCULATORS ALLOWED ON THIS ROUND
except for the **TI-89 Calculator**, which is not allowed on the Team Round

1. Four letters are chosen randomly from the word *MATHEMATICS*. What is the probability the letters chosen can be used to spell the word *MATH* ? [For example, *TAHM* can be used to spell *MATH*.] Write the answer in the form $\frac{a}{b}$ where a and b are relatively prime whole numbers.

2. Given the five positive numbers, 17, 4, 28, 23, and x, such that their mean equals their median, find all possible values for x.

3. From a standard deck of playing cards (no jokers), two cards are chosen at random and from a box containing four red, three blue and two white marbles, two marbles are chosen at random. What is the probability that at least one of the cards is a face card and the two marbles chosen are of different colors? Write the answer in the form $\frac{a}{b}$ where a and b are relatively prime whole numbers.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 4 – JANUARY 2000

TEAM ROUND

3 pts. 1. ____________________

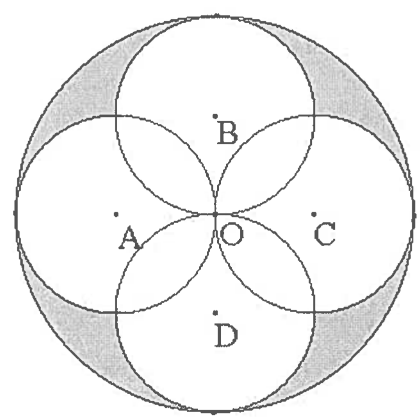
3 pts. 2. ____________________

4 pts. 3. ____________________

SAT APPROVED CALCULATORS ALLOWED ON THIS ROUND
except for the **TI-89 Calculator**, which is not allowed on the Team Round

1. What is the probability if three dice are shaken well and thrown, that the sum of the pips (numbers) on the three top faces either are less than five or greater than fourteen? Express the probability in the rational form, $\frac{a}{b}$, where a and b are relatively prime whole numbers or if estimated, rounded to **4 decimal places**.

2. A circle, centered at O, has a radius of 4 cm. and congruent circles, centered at A, B, C, and D, all contain point O and are tangent internally to circle O. Points A, B, C, and D form a square. (See the figure.) Find the exact shaded area of the figure or if estimated, then rounded to **four decimal places**.



3. Five cards are chosen at random from a standard deck of playing cards containing no jokers. What is the probability that at least 3 out of 5 are of the same suit? Write the answer in decimal form rounded to **4 decimal places**.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 4 – JANUARY 2001

TEAM ROUND

3 pts. 1. ________________

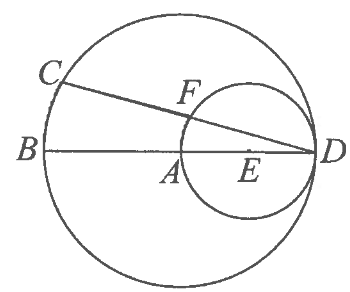
3 pts. 2. ________________

4 pts. 3. ________________

SAT APPROVED CALCULATORS ARE ALLOWED IN THE TEAM ROUND, EXCEPT FOR CALCULATORS WITH SYMBOLIC MANIPULATION PROGRAMS, (FOR EXAMPLE THE TI-89) WHICH ARE NOT ALLOWED IN THIS ROUND.

1. In a box are 6 red, 5 blue, 4 green and 3 yellow marbles. If 4 marbles are drawn at random from the box, what is the probability that that there are not three or four matching in color? Express the probability either as a rational number in reduced form or if estimated, round off to four decimal places.

2. Given circles centered at points A and E such that circle E contains point A and is internally tangent to circle A at point D. If $\overline{BAED}, \overline{CFD}$, $m\angle D = 15°$, and $BD = 12$, find the area bounded by $\overline{AB}, \overset{\frown}{BC}, \overline{CF}$, and $\overset{\frown}{AF}$, as **boldly outlined** on the diagram. If estimating the area, round off the result to four decimal places.



3. An urn contains 6 red, 3 blue and 1 white marble. A regular decahedron has on its faces the numbers from 1 to 10, one number per face. In a game 2 marbles are picked at random from the urn and the decahedron is rolled. If both marbles are the same color and a prime number comes up on the top face, you win \$20. If different colored marbles are picked and the number on the top face is not prime, you win \$5. Otherwise, you win nothing. How many dollars is your expectation if you play one game?

GREATER BOSTON MATHEMATICS LEAGUE

MEET 4 – JANUARY 2002

TEAM ROUND (<u>12 MINUTES LONG</u>)

3 pts. 1. ________________

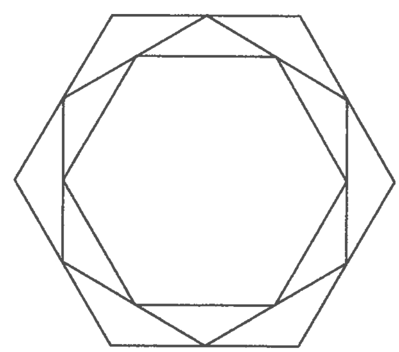
3 pts. 2. ________________

4 pts. 3. ________________

SAT APPROVED CALCULATORS ARE ALLOWED IN THE TEAM ROUND, EXCEPT FOR CALCULATORS WITH SYMBOLIC MANIPULATION PROGRAMS, (FOR EXAMPLE THE TI-89) WHICH ARE NOT ALLOWED IN THIS ROUND.

1. From a box containing 10 red, 8 white, and 7 blue marbles, 6 are chosen at random. What is the probability that exactly 4 are the same color? Express the result in reduced rational form or if estimated round off to exactly 4 decimal places.

2. The midpoints of the sides of a regular hexagon are connected forming a second regular hexagon. Then the midpoints of the sides of this second hexagon are connected forming a third regular hexagon. (See the figure to the right.) If this process continues forever, the sum of the areas of <u>all</u> the hexagons equals $\sqrt{3}$ square centimeters. Find the exact number of centimeters (simplest radical form) in the sum of the perimeters of <u>all</u> the hexagons.



3. How many different 4-letter permutations are possible using any of the letters in the word *MINIMUM*.

GREATER BOSTON MATHEMATICS LEAGUE
MEET 4 – JANUARY 2006

TEAM ROUND: Time Limit – 12 minutes

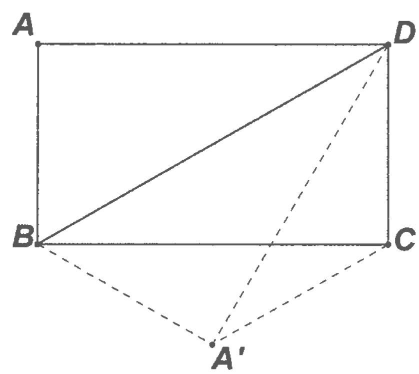
(3 pts) 1. _________ : _________

(3 pts) 2. ____________________

(4 pts) 3. ____________________

SAT APPROVED CALCULATORS ARE ALLOWED IN THE TEAM ROUND, EXCEPT FOR CALCULATORS WITH SYMBOLIC MANIPULATION PROGRAMS, (FOR EXAMPLE THE TI-89) WHICH ARE NOT ALLOWED IN THIS ROUND.

1. Rectangle $ABCD$ is such that when A is reflected over diagonal BD, its reflection A' satisfies $A'C = AB$. If $AB < BC$ and $BD = 10$, find $AB:BC$.



2. The base $\overline{AB}$ of a scalene triangle ΔABC has length 8. The sum of the lengths of the other two sides is 10. If $\underline{a}$ and $\underline{b}$ denote the lengths of the other two sides $\overline{BC}$ and $\overline{AC}$ respectively and $a < b$, what range of values for $\underline{a}$ guarantee that $m\angle ABC$ is obtuse?

3. Solve the following inequality over the reals: $\left|\frac{x+3}{x}\right| < x - 1$

GREATER BOSTON MATHEMATICS LEAGUE
MEET 4 – JANUARY 2007

TEAM ROUND: Time Limit – 12 minutes

(3 pts) 1. ______________________

(3 pts) 2. ______________________

(4 pts) 3. ______________________

SAT APPROVED CALCULATORS ARE ALLOWED IN THE TEAM ROUND, EXCEPT FOR CALCULATORS WITH SYMBOLIC MANIPULATION PROGRAMS, (FOR EXAMPLE THE TI-89) WHICH ARE NOT ALLOWED IN THIS ROUND.

1. Given: $A(2,2)$, $B(4,0)$, and $C(4,6)$

   Find the volume of the solid formed when ΔABC is rotated 360° about the y-axis.

2. The following are the first seven terms of an arithmetic sequence:

   $$1-8x\ ,\ 2x-4y\ ,\ 4z-2x\ ,\ 2y-6z\ ,\ 6x-2\ ,\ 5y+2z\ ,\ 10x+4.$$

   Find the value of $x+y+z$.

3. Determine the sum $\frac{1}{i^5}+\frac{4}{i^8}+\frac{9}{i^{11}}+\frac{16}{i^{14}}+\cdots\frac{1600}{i^{122}}$ in $a+bi$ form.

   This sum is equivalent to $\sum_{n=1}^{40}\frac{n^2}{i^{3n+2}}$.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 4 – JANUARY 2008

TEAM ROUND

3 pts.	1.	______________
3 pts.	2.	______________
4 pts.	3.	______________

SAT APPROVED CALCULATORS ARE ALLOWED IN THIS ROUND.

1. Find all values of x which satisfy $\dfrac{5}{|x-1|}-\dfrac{3}{x-1}\geq\dfrac{3}{2}$

2. There are K natural numbers less than or equal to 999 that are divisible by 8. Of these, J of them are also divisible by 3 or 5. Find the value of $\dfrac{K}{J}$ where J and K are relatively prime natural numbers.

3. A full house is three of a kind and two of another kind in any order. For ex: 11122, XOXXO. There are k ways you can draw 5 cards from a standard deck of 52 cards and have a full house. There are j ways you can throw 5 dice and get a full house. Find the value of $\dfrac{j}{k}$ in simplified form as the ratio of two relatively prime natural numbers.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 4 – JANUARY 2009

TEAM ROUND

3 pts.	1.	________________
3 pts.	2.	________________
4 pts.	3.	________________

SAT APPROVED CALCULATORS ARE ALLOWED IN THIS ROUND.

1. There are as many integers that do not satisfy the inequality $|4x-9| \geq 7$ as there are integers that do satisfy the inequality $|9x-4| < a$. Determine the minimum integer value of a for which this is true.

2. In a right triangle with sides of lengths 3, 4 and 5 units respectively, segments are drawn through the center of the inscribed circle parallel to the legs. Each segment has one endpoint on a leg and the other on the hypotenuse. Determine the positive difference between their lengths.

3. Let $A = \left\{(x, y) \mid \frac{x^2}{9} - \frac{y^2}{4} = 0\right\}$, $B = \{(x, y) \mid x = 6\}$ and $C = \{(x, y) \mid y = k\}$.
   Determine the **positive** value of k for which the area of the region bounded by A and B is the same as the area of the region bounded by A and C.

GREATER BOSTON MATHEMATICS LEAGUE

MEET 4 – JANUARY 2010

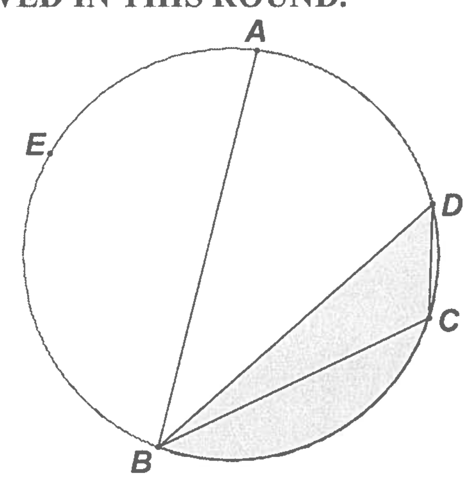
TEAM ROUND

3 pts. 1. ______________________

3 pts. 2. ______________________

4 pts. 3. (____________ , ____________)

CALCULATORS ARE NOT ALLOWED IN THIS ROUND.



1. Given: $m\widehat{AEB} = (3x)°$
   $m\widehat{AD} = x°$
   $m\angle BCD = 120°$
   $BD = 9$
   Compute the area of the shaded region.

2. Find all values of x which satisfy $\left\{ x \mid \dfrac{x^3 + 5x^2 + 10x + 8}{x^2 + 2x} \le 0 \text{ and } |2x+1| \le |3x+2| \right\}$.

3. Given: conics $C_1 : x^2 + y^2 - 10x + 4y - 36 = 0$, $C_2 : x^2 + y^2 + 6x - 20y + 44 = 0$
   $P, Q \in C_1 \cap C_2$
   Find the (x, y) coordinates of center of the circle which passes through the points P, Q and $K(1, 0)$.

GREATER BOSTON MATHEMATICS LEAGUE
MEET 4 – JANUARY 2011

TEAM ROUND

3 pts. 1. ______________________________

3 pts. 2. $C =$ ______________________

4 pts. 3. { ______________________________ }

CALCULATORS ARE NOT ALLOWED ON THIS ROUND.

1. The areas of the faces of a rectangular solid are in a ratio of 3 : 4 : 6. If the lengths of the edges are all integers, compute <u>all</u> possible volumes of this solid less than 1000.

2. For all values of m, the line defined by $(y+4)=m(x-1)$ divides the region bounded by $x^2+y^2+Ax+By+C=0$ into two regions of equal area. Compute C, if the point $P(10, 8)$ lies on the boundary of the region.

3. Find $\left\{ x \,|\, \dfrac{\left(x^2+6x+9\right)\left(4x^2-3x-1\right)}{x^2+4x-5} \geq 0 \right\}$

'99 GBML

TEAM ROUND

1. The probability the letters will be chosen in any order $MATH = \dfrac{2\cdot 2\cdot 2\cdot 1}{\binom{11}{4}} = \mathbf{\dfrac{4}{165}}$

2. The mean of 4, 17, 23, 28, and x is $\dfrac{72+x}{5}$; if $x \le 17$, then $\dfrac{72+x}{5} = 17 \Rightarrow x = 13$;
If $17 < x \le 23$, then $\dfrac{72+x}{5} = x \Rightarrow x = 18$; If $x > 23$, then $\dfrac{72+x}{5} = 23 \Rightarrow x = 43. \Rightarrow$
Answer is $x =$ **13, 18, or 43**

3. Probability of choosing at least 1 face card $= \dfrac{\binom{12}{1}\binom{40}{1}+\binom{12}{2}}{\binom{52}{2}}$

Probability of choosing 2 different colored marbles $= \dfrac{\binom{4}{1}\binom{3}{1}+\binom{4}{1}\binom{2}{1}+\binom{3}{1}\binom{2}{1}}{\binom{9}{2}}$

$$\frac{\binom{12}{1}\binom{40}{1}+\binom{12}{2}}{\binom{52}{2}} \times \frac{\binom{4}{1}\binom{3}{1}+\binom{4}{1}\binom{2}{1}+\binom{3}{1}\binom{2}{1}}{\binom{9}{2}} = \mathbf{\frac{91}{306}}$$

'00 GBML

Team Round

1.

Sum of pips	possibilities	permutations
3	1-1-1	1
4	1-1-2	3
15	5-5-5	1
	4-5-6	6
	6-6-3	3
16	4-6-6	3
	5-5-6	3
17	5-6-6	3
18	6-6-6	1

probability $= \dfrac{24}{6^3} = \dfrac{1}{9}$

2. Each of the four smaller circles have a radius = 2cm. The shaded area = area of circle O $- 4\times$ area of smaller circle + area where circles A, B, C, and D overlap, which consists of 8 - 90° segments $= \pi\cdot 4^2 - 4\cdot\pi\cdot 2^2 + 8\left(\frac{1}{4}\pi\cdot 2^2 - \frac{1}{2}2\cdot 2\right) = 16\pi - 16\pi + 8(\pi-2) = 8\pi - 16$

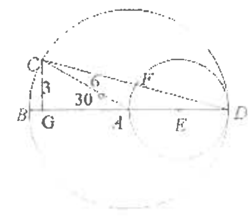
3. probability $= \dfrac{\binom{4}{1}\binom{13}{3}\binom{39}{2}+\binom{4}{1}\binom{13}{4}\binom{39}{1}+\binom{4}{1}\binom{13}{5}}{\binom{52}{5}} \approx 0.3711$

'01 GBML

Team Round

1. Number of elements in the sample space $= \binom{18}{4} = 3060$; event having 3 of 1 color =
$\binom{6}{3}\binom{12}{1}+\binom{5}{3}\binom{13}{1}+\binom{4}{3}\binom{14}{1}+\binom{3}{3}\binom{15}{1} = 441$; event having 4 of 1 color =
$\binom{6}{4}+\binom{5}{4}+\binom{4}{4} = 21$; probability neither event occurs $= \dfrac{3060-441-21}{3060} = \dfrac{433}{510} \approx 0.8490$

2. $m\angle BAC = 30° \rightarrow CG = 3$; area of sector $ABC =$
$\dfrac{1}{12}(36\pi) = 3\pi$; area of $\Delta ACD = \dfrac{1}{2}(6)(3) = 9 \rightarrow$
area bounded by $\overline{DB}, \overline{DC}, \widehat{BC} = 3\pi + 9$; by similarity
area bounded by $\overline{DA}, \overline{DF}, \widehat{AF} = \dfrac{1}{4}(3\pi+9)$; by subtraction,
the area bounded by $\overline{AB}, \overline{CF}, \widehat{AF}, \widehat{BC} = \dfrac{3}{4}(3\pi+9)$



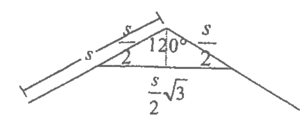
3. Probability 2 marbles of same color $= \dfrac{{}_6C_2 + {}_3C_2}{{}_{10}C_2} = \dfrac{2}{5} \rightarrow$ probability 2 marbles of different colors $= \dfrac{3}{5}$; probability of a prime $= \dfrac{2}{5} \rightarrow$ probability a non-prime $= \dfrac{3}{5}$;
expectation $= \left(\dfrac{2}{5}\right)\left(\dfrac{2}{5}\right)(20) + \left(\dfrac{3}{5}\right)\left(\dfrac{3}{5}\right)(5) = 5$

'02 GBML

TEAM ROUND

1. The sample space has ${}_{25}C_6$ elements in it. The successful events are choosing 4 reds and 2 non-reds, 4 whites and 2 non-whites, 4 blues and 2 non-blues.
Therefore the probability $= \dfrac{{}_{10}C_4\cdot{}_{15}C_2 + {}_8C_4\cdot{}_{17}C_2 + {}_7C_4\cdot{}_{18}C_2}{{}_{25}C_6} = \dfrac{211}{1012} \approx 0.2085$

2. The ratio of the sides of each hexagon to the previous one $= \dfrac{\sqrt{3}}{2}$ (See the figure on the right.) $\Rightarrow$ ratio of perimeters $= \dfrac{\sqrt{3}}{2}$ and the ratio of area $= \left(\dfrac{\sqrt{3}}{2}\right)^2 = \dfrac{3}{4}$; since the sum



of areas $= \sqrt{3}$ and if s = side of the first, then $\dfrac{\frac{3}{2}s^2\sqrt{3}}{1-\frac{3}{4}} = \sqrt{3} \Rightarrow 6s^2\sqrt{3} = \sqrt{3} \Rightarrow s^2 = \dfrac{1}{6} \Rightarrow s = \dfrac{\sqrt{6}}{6}$
$\Rightarrow P = \sqrt{6} \Rightarrow$ sum of perimeters $= \dfrac{\sqrt{6}}{1-\frac{\sqrt{3}}{2}} = \dfrac{2\sqrt{6}}{2-\sqrt{3}} = 2\sqrt{6}\left(2+\sqrt{3}\right) = 4\sqrt{6}+6\sqrt{2}$.

3. Consider 5 cases: (i) *MMM* and a 4th letter (ii) *MMII* (iii) *MM* and 2 different letters (iv) *II* and 2 different letters (same number as (iii)) (v) 4 different letters;
therefore the number of permutations $= 3\cdot\dfrac{4!}{3!} + \dfrac{4!}{2!2!} + 3\cdot\dfrac{4!}{2!} + 3\cdot\dfrac{4!}{2!} + 4! = 114$.

'06
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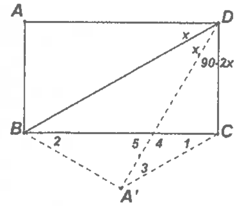
Detailed Solutions to GBML Meet 4 – 2006

TEAM ROUND



1. The answer does not depend on the fact that $BD = 10$.
   $BA'D$ is a right triangle and $BADA'$ is a kite since reflections preserve angle measure and length.
   Let x denote $m\angle ADB$ and $m\angle BDA'$.
   Then $m\angle CDA' = 90 - 2x$.
   $\Delta BA'C \cong \Delta A'CD$ (by SSS)
   $\rightarrow m\angle 1 = m\angle 2 = m\angle 3 = 90 - 2x$
   $\rightarrow m\angle 4 = 4x \rightarrow m\angle 5 = 180 - 4x$
   Angles 2 and 5 must be complementary $\rightarrow$
   $90 - 2x + (180 - 4x) = 90 \rightarrow x = 30$
   Thus, ΔBAD is a 30-60-90 right triangle and $AB : BC = \underline{1:\sqrt{3}}$
2. Suppose $A(-4, 0)$ and $B(4, 0)$ and the third vertex is $C(x, y)$.
   Stated conditions $\rightarrow$ locus of third vertex $C(x, y)$ is a horizontal ellipse w/ $c = 4$, $a = 5$ and $b = 3$.

   Sketch the ellipse and the focal chord at B. The equation is $\frac{x^2}{25} + \frac{y^2}{9} = 1$.

   At point C ($x = 4$), ΔABC is a right triangle and $\underline{a} = BC = ½$ length of a focal chord, i.e. $b^2/a = 9/5$. As point C rotates CW towards the rightmost vertex, $m\angle ABC$ increases and $\underline{a}$ decreases to 1. As point C rotates CCW towards the uppermost vertex, $m\angle ABC$ becomes acute and $\underline{a}$ increases to 5 when ΔABC becomes isosceles.)
   Thus, $\underline{1 < a < 9/5}$ guarantees $m\angle ABC$ is obtuse.
3. The absolute value expression is equivalent to either $\frac{x+3}{x}$ or $-\frac{x+3}{x} = \frac{-x-3}{x}$
   If the quotient is positive, use the first expression, otherwise use the second.

   (I) positive ______ -3 ______ (II) negative ______ 0 ______ (III) positive

   Answers outside the domain of definition are extraneous and must be rejected.
   Case 1: Domain of definition I: $x < -3$ or III: $x > 0$

   Equivalent inequality: $\frac{x+3}{x} < x - 1 \Rightarrow \frac{x+3}{x} - x + 1 < 0 \Rightarrow \frac{-x^2+2x+3}{x} < 0$

   $$\Rightarrow \frac{x^2-2x-3}{x} > 0 \Rightarrow \frac{(x-3)(x+1)}{x} > 0$$

   Critical values are at -1, 0 and 3. At the extreme left, all three expressions are negative producing a negative quotient. As we cross each critical value, exactly one more factor becomes positive. Thus, the quotient is positive for $-1 < x < 1$ or $\underline{x > 3}$. Only the latter is within the domain of definition.

   Case 2: Domain of definition II: $-3 < x < 0$

   Equiv. inequality: $\frac{-x-3}{x} < x - 1 \Rightarrow \frac{-x-3}{x} - x + 1 < 0 \Rightarrow \frac{-x-3-x^2+x}{x} < 0 \Rightarrow \frac{x^2+3}{x} > 0$

   Since the numerator is always positive, only $x > 0$ satisfies this inequality, but all these values are outside the domain of definition and are extraneous.

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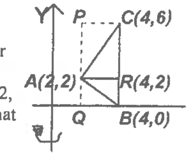
TEAM ROUND



1. Rotating about the Y-axis, the rectangle $CRAP$ generates a washer shaped region of height 4, inner radius of 2 and outer radius of 4. The rectangle $RBQA$ generates a washer shaped region of height 2, and the same inner and outer radii. From the diagram it is clear that the volume of the region generated by ΔABC equals half the sum of the volumes of the regions generated by these two rectangles.
   Thus, $V = \frac{1}{2}\,[(\pi(4^2)4 - \pi(2^2)4) + (\pi(4^2)2 - \pi(2^2)2)] = \frac{1}{2}(48\pi + 24\pi) = \underline{36\pi}$
   Note that relocating point A __anywhere__ along the vertical line $x = 2$ does not change the answer.
2. Let d denote the difference between successive terms of the arithmetic sequence. Then:
   $t_7 - t_5 = 2d = 4x + 6 \rightarrow d = 2x + 3$
   $t_2 = t_1 + d \rightarrow 2x - 4y = 4 - 6x \rightarrow y = 2x - 1$
   $t_4 - t_3 = d \rightarrow 2y - 10z + 2x = 2x + 3 \rightarrow z = (4x - 5)/10$
   Thus, the 7 terms (in terms of x) are:
   $$1 - 8x,\ 4 - 6x,\ \frac{-2}{5}x - 2,\ \frac{8}{5}x + 1,\ 6x - 2,\ \frac{54}{5}x - 6 \text{ and } 10x + 4$$
   $t_5 - t_4 = d \rightarrow 6x - 2 - (\frac{8}{5}x + 1) = 2x + 3 \rightarrow 4x - 6 = \frac{8}{5}x \rightarrow 12x = 30$

   $\rightarrow (x, y, z) = \left(\frac{5}{2}, 4, \frac{1}{2}\right) \rightarrow x + y + z = \underline{7}$
3. Let T denote the required sum. $T = i^3 + 4i^4 + 9i^5 + 16i^6 + \ldots + 1600i^{42}$
   $iT = \quad i^4 + 4i^5 + 9i^6 + \ldots + 1521i^{42} + 1600i^{43}$
   Subtracting, $(1 - i)T = i^3 + 3i^4 + 5i^5 + 7i^6 + \ldots + 79i^{42} - 1600i^{43}$
   $i(1 - i)T = \quad i^4 + 3i^5 + 5i^6 + \ldots + 77i^{42} + 79i^{43} - 1600i^{44}$
   Subtracting, $(1 - i)T - i(1 - i)T = T(1 - i)^2 = i^3 + 2i^4 + 2i^5 + 2i^6 + \ldots 2i^{42} - 1679i^{43} + 1600i^{44}$
   $\rightarrow T(1 - i)^2 - i^3 + 1679i^{43} - 1600i^{44} = 2(i^4 + i^5 + \ldots + i^{42})$ (39 terms = 9 groups of 4 plus 3)
   $\rightarrow -2iT + i - 1679i - 1600 = 2[9(1 + i - 1 - i) + 1 + i - 1] = 2i$
   $\rightarrow T = \frac{1680i + 1600}{-2i} = \underline{-840 + 800i}$

'08
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1. The equivalent inequalities:
   For $x < 1$, $\frac{-5}{x-1} - \frac{3}{x-1} \geq \frac{3}{2} \rightarrow \frac{8}{1-x} - \frac{3}{2} \geq 0 \rightarrow \frac{13+3x}{2(1-x)} \geq 0 \rightarrow -13/3 \leq x < 1$
   For $x > 1$, $\frac{5}{x-1} - \frac{3}{x-1} \geq \frac{3}{2} \rightarrow \frac{2}{x-1} - \frac{3}{2} \geq 0 \rightarrow \frac{7-3x}{2(x-1)} \geq 0 \rightarrow 1 < x < 7/3$
   Thus, $\underline{-\frac{13}{3} \leq x \leq \frac{7}{3} \text{ and } x \neq 1}$
2. $\frac{999}{8} = 124,\ \frac{999}{24} = 41,\ \frac{999}{40} = 24,\ \frac{999}{120} = 8 \rightarrow 41 + 24 - 8 = 57 \rightarrow \frac{K}{J} = \frac{57}{124}$
3. Fullhouse (cards) Pick a denomination – 13 ways
   Pick 3 cards from that denomination – 4 ways
   Pick a different denomination – 12 ways
   Pick 2 cards from that denomination - ${}_4C_2 = 6$ ways
   $k = 13(4)(12)(6)$

   Full house (dice) _ _ _ _ _ Pick 3 position - ${}_5C_3 = 10$ ways
   Pick a filler number – 6 ways
   Fill the remaining 2 positions with a different filler number – 5 ways
   $j = 10(6)(5)$
   $\rightarrow \frac{j}{k} = \underline{\frac{25}{312}}$

GBML '09

TEAM ROUND

1.
$|4x-9| \geq 7 \rightarrow 4x-9 \geq 7$ or $4-9 \leq -7 \rightarrow x \geq 4$ or $x \leq ½ \rightarrow$ non-solutions: 1, 2, 3

$|9x-4| < a \rightarrow -a < 9x-4 < a \rightarrow \frac{4-a}{9} < x < \frac{4+a}{9}$

Unless the value of a is positive, the inequality $|9x-4| < a$ has no solutions.

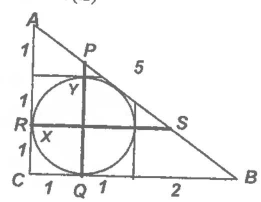
$a = 1 \rightarrow \frac{1}{3} < x < \frac{5}{9} \rightarrow$ no integer solutions

$a = 13 \rightarrow -1 < x < \frac{17}{9} \rightarrow$ only 2 integer solutions, namely 0 and 1

$a = \underline{\mathbf{14}} \rightarrow \frac{-10}{9} < x < 2 \rightarrow$ 3 integer solutions, namely -1, 0 and 1

2.

Since the radius of the inscribed circle in a triangle is given by $r_{ic} = \frac{A(\Delta)}{s(\Delta)}$, where s denotes the semi-perimeter, we have $r = \frac{\frac{1}{2}\cdot 3\cdot 4}{\frac{1}{2}(3+4+5))} = \frac{6}{6} = 1$.



Let x and y denote the required lengths of the horizontal and vertical segments in the diagram at the right. From similar triangles, we have

$(\Delta ARS \sim \Delta ACB)\ \frac{x}{4} = \frac{2}{3}$ and $(\Delta PQB \sim \Delta ACB)\ \frac{y}{3} = \frac{3}{4}$ and

$\rightarrow (x, y) = \left(\frac{8}{3}, \frac{9}{4}\right)$ and $|x-y| = \frac{32-27}{12} = \underline{\mathbf{\frac{5}{12}}}$

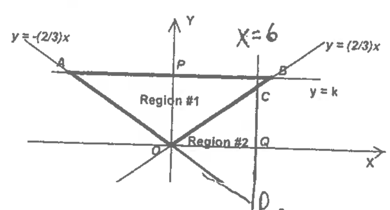
3.

A is equivalent to $y^2 = \frac{4}{9}x^2$ or $y = \pm\frac{2}{3}x$. B and C denote vertical and horizontal lines respectively.

$(OP, AB) = (k, 3k) \rightarrow$ the area of region #1 is $\frac{3k^2}{2}$

$(OQ, CD) = (6, 8) \rightarrow$ area of region #2 is 24.

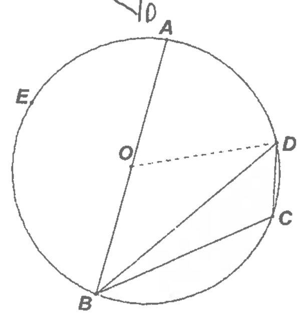
Thus, $\frac{3k^2}{2} = 24 \rightarrow k = \underline{\pm 4}$



TEAM ROUND

GBML '10

1. $\angle BCD$ is an inscribed angle with intercepted arc $\overparen{BEAD}$.
Therefore, $\frac{1}{2}(3x+x) = 120°$
$\rightarrow x = 60$ and $\overline{AB}$ must be a diameter
Let O denote the center of the circle. Then:
The shaded area equals the area of sector BOD minus the area of ΔBOD. Since $m\angle BOD = 120°$, dropping a perpendicular from O to $\overline{BD}$ creates two $30-60-90$ right triangles. $BD = 9 \rightarrow BO = OD = 3\sqrt{3}$



Thus, we have $\frac{\pi(3\sqrt{3})^2}{3} - \frac{1}{2}(3\sqrt{3})^2 \sin 120°$

$= 9\pi - \frac{27\sqrt{3}}{4} = \mathbf{\frac{36\pi - 27\sqrt{3}}{4}}$ (or equivalent)

GBML '10 (Cont.)

2. Rewrite the first condition $\frac{x^3+5x^2+10x+8}{x^2+2x} \leq 0$ as $\frac{(x^2+3x+4)(x+2)}{x(x+2)} \leq 0$

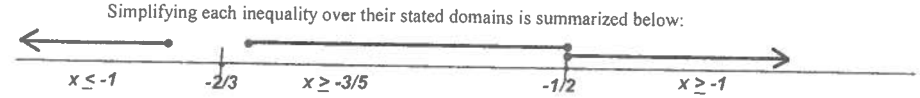
Completing the square, $\frac{\left(\left(x+\frac{3}{2}\right)^2+\frac{7}{4}\right)\cancel{(x+2)}}{x\cancel{(x+2)}} \leq 0$, we note the first term in the numerator is always positive. $(x+2)$ can be cancelled, provided $x \neq -2$.

Therefore, the solution set is equivalent to $\left\{x \mid \frac{1}{x} \leq 0 \text{ and } x \neq -2\right\} \rightarrow \underline{\{x \mid x < 0 \text{ and } x \neq -2\}}$

TEAM ROUND – continued

The second condition $|2x+1| \leq |3x+2| \rightarrow \begin{cases} -2x-1 \leq -3x-2 \text{ for } x \leq -\frac{2}{3} \\ -2x-1 \leq 3x+2 \text{ for } -\frac{2}{3} < x < -\frac{1}{2} \\ 2x+1 \leq 3x+2 \text{ for } x \geq -\frac{1}{2} \end{cases}$

Simplifying each inequality over their stated domains is summarized below:



Thus, the second condition has the solution set $\left\{x \mid x \leq -1 \text{ or } x \geq -\frac{3}{5}\right\}$

Taking the intersection, we have $\underline{\mathbf{x \leq -1 \text{ and } x \neq -2 \text{ or } -\frac{3}{5} \leq x < 0}}$

3. $C_1 - C_2 \rightarrow -16x + 24y - 80 = 0 \rightarrow \boxed{x = \frac{3y-10}{2}\ ***}$

$P, Q \in C_1 \cap C_2 \rightarrow$ the coordinates of P and Q satisfy both equations. Therefore, substituting,

$*** \rightarrow \left(\frac{3y-10}{2}\right)^2 + y^2 - 10\left(\frac{3y-10}{2}\right) + 4y - 36 = 0$

$\left(\frac{9y^2-60y+100}{4}\right) + y^2 - 5(3y-10) + 4y - 36 = 0$

$9y^2 - 60y + 100 + 4y^2 - 60y + 200 + 16y - 144 = 0$

$13y^2 - 104y + 156 = 13(y^2 - 8y + 12) = 13(y-6)(y-2) = 0$

$\rightarrow y = 2, 6 \rightarrow x = -2, 4 \rightarrow P(-2, 2), Q(4, 6)$

The center of the circle lies on the intersection of the perpendicular bisectors of any two chords of the circle.

chord $\overline{PQ}$ - midpoint: (1, 4), slope: $\frac{2}{3} \rightarrow (y-4) = -\frac{3}{2}(x-1) \rightarrow$ (1) $3x + 2y - 11 = 0$

chord $\overline{PK}$ - midpoint: $\left(-\frac{1}{2}, 1\right)$, slope: $-\frac{2}{3} \rightarrow (y-1) = \frac{3}{2}\left(x+\frac{1}{2}\right) \rightarrow$ (2) $6x - 4y + 7 = 0$

Solving simultaneously, $(6x - 4y + 7 = 0) + 2(3x + 2y - 11 = 0) \rightarrow 12x - 15 = 0 \rightarrow x = \frac{5}{4}$

Substituting in (1), $\frac{15}{4} + 2y - 11 = 0 \rightarrow 15 + 8y - 44 = 0 \rightarrow 8y = 29 \rightarrow y = \frac{29}{8}$

The coordinates of the center is $\underline{\left(\frac{5}{4}, \frac{29}{8}\right)}$.

GBML '11

TEAM ROUND

1. Let the dimensions of the rectangular solid be L, W and H. Thrn:

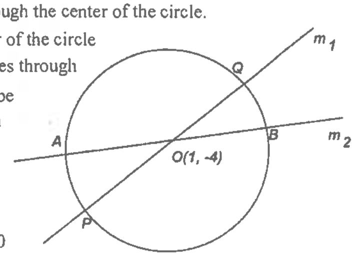
$LW : LH : WH = 3:4:5 \rightarrow \frac{L}{H} = \frac{3}{6} = \frac{1}{2}$ and $\frac{W}{H} = \frac{3}{4} \rightarrow H = 2L$ and $3H = 4W = 6L$

Thus, all dimensions can be expressed min terms of L.

$L : W : H = L : \frac{3}{2}L : 2L = 2:3:4$

Let $(L, W, H) = (2k, 3k, 4k)$ and the volume will be $24k^3$.
For $k = 4$, the volume exceed 1000. $k = 1, 2, 3 \rightarrow V =$ **<u>24, 192, 648</u>**.

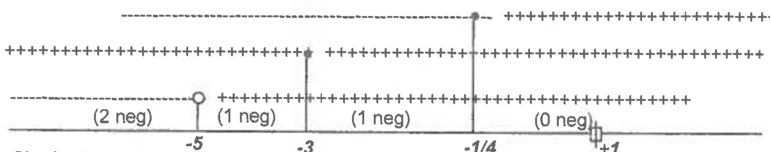
2. The line defined by $(y+4) = m(x-1)$ must pass through the center of the circle. If this must be true for all values of m, then the center of the circle must be $(1, -4)$. WHY? Let $\overleftrightarrow{AB}$ and $\overleftrightarrow{PQ}$ be two lines through $(1, -4)$ with different slopes. $\overline{AB}$ and $\overline{PQ}$ must both be diameters and the center of the circle must lie on both segments. The only common point is O(1, −4).
Thus, O is the center and $\overline{OC}$ is a radius.
Applying the distance formula, OC = 15
$x^2 + y^2 + Ax + By + C = 0$ becomes
$(x-1)^2 + (y+4)^2 = 225 \rightarrow x^2 + y^2 - 2x + 8y - 208 = 0$
$\rightarrow C =$ **<u>−208</u>**.



3. $$\frac{(x^2+6x+9)(4x^2-3x-1)}{x^2+4x-5} \ge 0 \rightarrow \frac{(x+3)^2(4x+1)(x-1)}{(x+5)(x-1)} \ge 0$$

$x \ne 1 \rightarrow \frac{(x+3)^2(4x+1)}{(x+5)} \ge 0 \rightarrow$ Critical points at $x = -5, -3$ and $-1/4$.

There is a sign change at $x = -5$ and $-1/4$, but not at $x = -3$.



Clearly, the quotient is ≥ 0 for: **<u>$x < -5$ or $x = -3$ or ($x \ge -1/4$ and $x \ne 1$)</u>**