Uniqueness Theorems

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1 Introduction

A theorem stating the uniqueness of a mathematical object, also called a unicity theorem, which usually means that there is only one object fulfilling given properties, or that all objects of a given class are equivalent. This is often expressed by saying that the object is uniquely determined by a certain set of data.

2 Uniqueness Theorems in Electrodynamics

Laplace's equation does not by itself uniquely determine V. The question arises on what are suitable boundary conditions which would be sufficient to uniquely determine the potential without generating inconsistencies.

3 First Uniqueness Theorem

The First Uniqueness Theorem directly deals with providing the necessary and sufficient parameters which uniquely determine the potential field over a region of interest.

Statement: The solution to Laplace's equation in some Volume $\mathcal{V}$ is uniquely determined if V is specified on the boundary surface $\mathcal{S}$

Proof

We are given the potential over the entire boundary. Suppose 2 potential functions $V_1(\vec{r})$ and $V_2(\vec{r})$ satisfy the given boundary condition.By Laplace's equation over the region of interest.

$$\nabla^2 V_1(\vec{r}) = \nabla^2 V_2(\vec{r}) = 0 \tag{1}$$

Consider a third potential function $V_3(\vec{r})$ such that $V_3(\vec{r}) = V_2(\vec{r}) - V_1(\vec{r})$. Since both $V_1(\vec{r})$ and $V_2(\vec{r})$ have the same values at the boundary, $V_3(\vec{r}) = 0$ over the boundary. Also, $V_3(\vec{r})$ satisfies Laplace's equation.

$$\nabla^2 V_3(\vec{r}) = \nabla^2 V_2(\vec{r}) - \nabla^2 V_1(\vec{r}) = 0 \tag{2}$$

Since it obeys Laplace's equation, by Earnshaw's theorem, it must be identically equal to 0 everywhere. Hence, $V_2(\vec{r}) = V_1(\vec{r})$

Q.E.D.

Corollary: The potential in a volume $\mathcal{V}$ is uniquely determined if both charge density throughout the region and the value of V on all boundaries is specified

The corollary generalises the first Uniqueness theorem to include regions with non-zero charge density.

4 Second Uniqueness Theorem

By the first uniqueness theorem, specifying the value of potential on all the surfaces surrounding the region of interest is sufficient. However, in many situations, a part of the surface is conducting. The measurable quantity is the total charge. The charge distribution and potential on the surface are unknown.This leads to another set of parameters which are sufficient to uniquely determine the electric field in the form of the Second Uniqueness Theorem.

Statement: In a Volume $\mathcal{V}$ surrounded by conductors and containing a specified charge density ρ, the electric field is uniquely determined if the total charge on each conductor is given

Proof

In a similar manner to the previous problem, we consider that 2 electric fields, say E_1 and E_2 satisfy the given boundary conditions. By Gauss' Law:

$$\nabla \cdot E_1 = \nabla \cdot E_2 = \frac{\rho}{\epsilon_0} \tag{3}$$

Consider the i-th conductor. Say the charge on it is given as Q_i. Hence, by Gauss' Law,

$$\oiint E_1 \cdot d\mathbf{a} = \oiint E_2 \cdot d\mathbf{a} = \frac{Q_i}{\epsilon_0} \tag{4}$$

where the surface is that of the i-th conductor. Taking a summation over all conducting surfaces,

$$\oiint E_1 \cdot d\mathbf{a} = \oiint E_2 \cdot d\mathbf{a} = \frac{Q_{total}}{\epsilon_0} \tag{5}$$

where the integral is over the entire outer boundary.
Consider an electric field $E_3 = E_2 - E_1$

By (3) and (4)

$$\nabla \cdot E_3 = 0 \tag{6}$$

$$\oiint E_3 \cdot d\mathbf{a} = 0 \tag{7}$$

Let the potential due to electric field be some arbitrary V_3. Over the i-th conducting surface, we know the potential is a constant, say $V_3(i)$. Now consider

$$\nabla \cdot (V_3 E_3) = V_3(\nabla \cdot E_3) + E_3 \cdot (\nabla V_3) = -(E_3)^2 \tag{8}$$

We will now take the volume integral of the divergence of $V_3 E_3$ over the region of interest and use Gauss Law on the same.

$$\iiint\limits_{\mathcal{V}} \nabla \cdot (V_3 E_3) d\tau = \oiint V_3 E_3 \cdot d\mathbf{a} = -\iiint\limits_{\mathcal{V}} (E_3)^2 d\tau \tag{9}$$

Using the fact that $V_3(i)$ is a constant over the surface of the i-th conductor, we can rewrite the integral as a summation of integrals

$$\oiint V_3 E_3 \cdot d\mathbf{a} = \Sigma_{i=1}^{n} V_3(i) \oiint E_3 \cdot d\mathbf{a} = 0 \tag{10}$$

Hence, back-substituting in (9),

$$\iiint\limits_{\mathcal{V}} (E_3)^2 d\tau = 0 \tag{11}$$

which is possible only when $E_3 = 0$ everywhere in space, thus implying $E_1 = E_2$

Q.E.D.

Corollary: The electric field is uniquely defined if the charge density within a region and the normal derivative of the potential is stated on the entire boundary

5 Conclusion

The Uniqueness Theorems only state the parameters which once fixed uniquely determine the respective field. They do NOT prove existence, which is significantly harder to do but is taken as an assumption since the system is physically realisable. The Uniqueness Theorems just say that if a field exists satisfying the given boundary conditions, then it is unique.

This leads to some interesting applications. It doesn't matter how we solve for the field, mathematically or by some insightful guesswork, if we obtain some field which satisfies the given boundary conditions, then it is the correct(and the only correct) field.