# Assignment 1 - Advanced Image Processing

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## Contents



**Note -** To obtain the results reported in this document, run the MATLAB files specified alongside the section headers above.

# 1 Problem 1

## 1.1 Part A

We claim that as s increases, so does $\delta_{2s}$. Note that the measurement matrix $A = \Phi\Psi$ being $2s$-sparse means that any $2s$ columns of $A$ are linearly independent. Otherwise, there will exist some $2s$-sparse vector $\theta$ that is in the null space of $A$, in which case $||A\theta|| = 0$ will never comply with the condition for RIP. Now, as $s$ increases, the probability of $2s$ columns being linearly dependent also increases as we are choosing more columns from $A$ (here we are assuming that $m, n$ are fixed). So, as $s$ increases, there is a lesser probability that $A$ satisfies RIP, or in other words, the value of the restricted isometry constant (i.e. $\delta_{2s}$) will increase with high probability.

Now, as $s$ increases, $\delta_{2s}$ increases, and since the constants $C_1$ and $C_2$ are increasing functions of $\delta_{2s}$, so they also increase. This means that the actual effect of increasing $C_1$ and $C_2$ in the numerator and increasing $\sqrt{s}$ in denominator compensates for each other in such a way that the overall bound on the error actually increases. Thus, there is no discrepancy involved, and the bound increases as one would expect from compressed sensing theory.

## 1.2 Part B

The claim made by the student that the bound is independent of $m$ is incorrect. $m$ actually affects various terms in the error bound. As $m$ decreases, the chances of $2s$ columns of the measurement matrix $A = \Phi\Psi$ being linearly independent reduces. So the probability of $A$ satisfying RIP decreases, or equivalently the constant $\delta_{2s}$ might increase. For example, we have seen that if $m \geq CS \log \frac{n}{S}$ then some random matrices have a high probability of satisfying RIP of order $S$. In particular, large $m$ allows for smaller $\delta_{2s}$. The error term $\epsilon$ also generally depends on the number of measurements $m$.

## 1.3 Part C

As given by Theorem 3, the error can be upper bounded as

$$||\theta^* - \theta||_2 \leq \frac{C_0}{\sqrt{S}}||\theta - \theta_s||_1 + C_1\epsilon$$

Now, $C_1$ and $C_2$ are both increasing functions of $\delta_{2S}$. Thus, by reducing the value of $\delta_{2s}$, we lower the value of the upper bound on the error of reconstruction. Hence, Theorem 3A is more useful than Theorem 3 as it ensures and satisfies all conditions for valid reconstruction but also improves the bound on the error making it more useful.

## 1.4 Part D

Making the error term $\epsilon = 0$ and solving problem BP gives us a minimizer of $||\theta||_1$ for the exact equation $y = \Phi\Psi\theta$. However, the actual measurements are noisy with noise $\eta$, and so the estimated $\theta$ from solving this optimization problem is not a correct estimate for the actual $\theta$. So the error bound does not signify anything of use, and reducing it is not of our concern either.

In general, taking $\epsilon$ to be some parameter independent of $\eta$ is not the correct way, and so the premise of taking zero $\epsilon$ for non-zero $\eta$ is itself incorrect.

# 2 Problem 2

Coherence is given by

$$\mu(\Phi, \Psi) = \sqrt{n}\max(|\Phi_i^t \Psi_j|)$$

Note that the Cauchy Schwarz inequality says that for two vectors $A, B$, we have $|\langle A, B\rangle| \leq ||A|| \cdot ||B||$. Let $i_0$ and $j_0$ be the indices where the following maximum is attained.

$$\sqrt{n}\max_{i,j}(|\Phi_i^t \Psi_j|) = \sqrt{n} \cdot |\Phi_{i_0}^t \Psi_{j_0}|$$

Now, using Cauchy Schwarz, and the fact that the rows of the measurement matrix are unit normalised and the basis matrix represents an orthonormal basis, we get

$$\sqrt{n} \cdot |\Phi_{i_0}^t \Psi_{j_0}| \leq \sqrt{n} \cdot ||\Phi_{i_0}^t|| \cdot ||\Psi_{j_0}|| = \sqrt{n}$$

Lower Bound can be proved by representing the rows of the measurement matrix in the basis used in the representation matrix. Consider a unit vector $g \in \mathbb{R}^n$. We compute its coherence with $\Psi$. As $\Psi$ is an orthonormal basis, so we can write $g$ as follows.

$$g = \sum_{k=1}^{n} \alpha_k \Psi_k$$

$$\implies \mu(g, \Psi) = \sqrt{n}\max_j\left(\left|\sum_{k=1}^{n} \alpha_k \Psi_k^t \cdot \Psi_j\right|\right)$$

From the orthonormality of the representation matrix, the inner product simply reduces to 0 when the indices are not the same and 1 when they are

$$\implies \mu(g, \Psi) = \sqrt{n}\max_j(|\alpha_j|)$$

Now, we know sum of squares of all $\alpha_i s = 1$ and g is a unit vector. Under this condition, we must evaluate the minimum value of the maximum $\alpha_i$.

$$\max_i(\alpha_i^2) \geq \frac{\Sigma_k \alpha_k^2}{n} \implies \max_i(|\alpha_i|) \geq \frac{1}{\sqrt{n}}$$

And thus, the coherence is lower bounded by 1.

# 3 Problem 3

## 3.1 Part A

Without knowing the index of the non-zero element, it is not possible to uniquely estimate x. For instance, for an observation y = [y], consider the measurement matrix

$$\Phi = [a_1\ a_2\ a_3 \dots a_n]$$

Now, if this measurement matrix has at least 2 non-zero values (WLOG assume they are $a_1$ and $a_2$), then both $x_1 = [y/a_1\ 0 \dots 0]$ and $x_2 = [0\ y/a_2 \dots 0]$ satisfy $\Phi x = y$. Hence, we cannot uniquely determine x in the general case.

Note that for any 2 distinct solutions $x_1, x_2$ satisfying $y = \Phi x$, we can have $x_1 - x_2$ in the null space of the measurement matrix $\Phi$. This follows from the fact that $x_1 - x_2$ is a 2-sparse vector, and any 2 columns in the measurement matrix (for $m = 1$) are linearly dependent, so there exists such a non-zero solution for $x_1 - x_2$ which satisfies $\Phi(x_1 - x_2) = 0$.

On the other hand, if we know the index of the non-zero element in x (say i), then we can uniquely determine its value as $x_i = y/a_i$, where $a_i \neq 0$ is the value at the $i^{th}$ index in the measurement matrix.

## 3.2 Part B

For any general measurement matrix, it is not possible to uniquely determine x. Consider

$$\begin{bmatrix} 1 & 2 & \dots \\ 2 & 4 & \dots \end{bmatrix}$$

and $y = [1\ 2]^T$, we can have the non-zero element of x in either index 1 or 2 (1 or 0.5 respectively). Thus, this counter-example shows that it is not possible to uniquely determine x.

However, suppose we have a condition on the measurement matrix that no 2 columns are linearly dependent, i.e. any 2 columns are linearly independent, then x can be uniquely determined. This follows from contradiction. Since any 2 columns of $\Phi$ are linearly independent, so the 2-sparse vector $x_1 - x_2$ can not be in the null space of the measurement matrix for $x_1 \neq x_2$. So if instead there were 2 solutions $x_1, x_2$ to $\Phi x = y$, then this would contradict the above statement (since 2 solutions means that $\Phi(x_1 - x_2) = 0$). Thus, x can be uniquely determined in this case.

## 3.3 Part C

For m=3 and a vector $x$ with 2 non-zero elements, we cannot uniquely determine x. Consider

$$\begin{bmatrix} 1 & 1 & 2 & 0 & \dots \\ 1 & 2 & 3 & 0 & \dots \\ 1 & 3 & 0 & 4 & \dots \end{bmatrix}$$

and say $y = [2\ 3\ 4]^T$. Thus $x_1 = [1\ 1\ 0\ 0\ \dots]^T$ and $x_2 = [0\ 0\ 1\ 1\ \dots]^T$ are both possible solutions. Hence, by counter-example, we cannot uniquely determine x.

In general, for any solutions $x_1, x_2$ satisfying $y = \Phi x$, we have $x_1 - x_2$ is in the null space of the measurement matrix $\Phi$. Since $x_1 - x_2$ is a 4-sparse vector, it can belong to the null space of the measurement matrix for any measurement matrix as 4 vectors in 3 dimensions are always linearly dependent. Hence, no special matrix exists either.

### 3.4 Part D

Given that m=4, we still cannot uniquely determine x in the general case. Consider

$$\begin{bmatrix} 1 & 1 & 2 & 0 & \dots \\ 1 & 2 & 3 & 0 & \dots \\ 1 & 3 & 0 & 4 & \dots \\ 1 & 4 & 2 & 3 & \dots \end{bmatrix}$$

and say $y = [2\ 3\ 4\ 5]^T$. Thus $x_1 = [1\ 1\ 0\ 0\ \dots]^T$ and $x_2 = [0\ 0\ 1\ 1\ \dots]^T$ are both possible solutions. Hence, by counter-example, we cannot uniquely determine x.

In general, for any solutions, $x_1, x_2$ satisfying $y = \Phi x$, we have $x_1 - x_2$ is in the null space of the measurement matrix $\Phi$. Since $x_1 - x_2$ is a 4-sparse vector, suppose we impose a condition on the measurement matrix that any 4 columns have to be linearly independent, Then no 4-sparse vector can belong to the null space of the measurement matrix as their product is the linear combination of 4 columns, which only can be 0 when all numbers are 0, i.e. $x_1 = x_2$.

In particular, consider 2 solutions $x_1, x_2$ and $\Phi(x_1 - x_2) = 0$. Say $x_1 - x_2$ has 4 elements which are possibly non-zero, $a_1, a_2, a_3, a_4$. Thus,

$$a_1C_1 + a_2C_2 + a_3C_3 + a_4C_4 = 0 \implies a_1 = a_2 = a_3 = a_4 = 0 \implies x_1 = x_2$$

where $C_1$ is the column of the measurement matrix corresponding to the index at which $a_1$ is present. The linear combination of any 4 columns is 0 only when all 4 coefficients are 0 (By linear independence). Thus, we can determine a unique x in the case that the measurement matrix has all possible combinations of 4 columns as linearly independent.

# 4 Problem 4

Let $x_0$ be the unique minimizer for the P1 optimization problem. Now, let us consider Q1 where we want to find

$$\min_x ||y - Ax||_2 \text{ under the constraint } ||x||_1 \leq \epsilon$$

Let $t = ||x_0||_1$. Now, to show that $x_0$ is also the unique minimizer of Q1, we assume for the sake of contradiction that $x_1$ not equal to $x_0$ is a minimizer of Q1. Hence,

$$||x_1||_1 \leq t = ||x_0||_1 \text{ and } ||y - Ax_1||_2 \leq ||y - Ax_0||_2 \leq \epsilon$$

Thus, $x_1$ is also a minimizer for P1, which contradicts the fact that $x_0$ is a unique minimizer for P1.

Hence, $x_0$ is also a unique minimizer for Q1 - By Contradiction

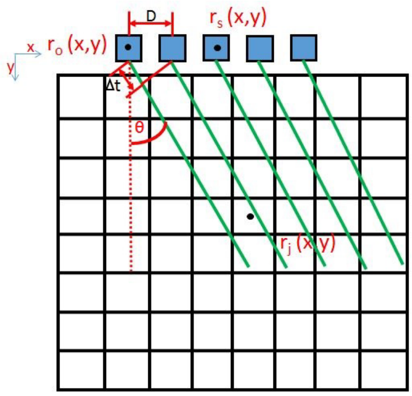
# 5 Problem 5

## 5.1 Details of the Paper

Title: **Compressive sensing ultrasound beamformed imaging in time and frequency domain**
Venue: 17th International Conference on E-health Networking, Application & Services (HealthCom)
Date: October 2015
The paper can be accessed here

## 5.2 Hardware Architecture

The main hardware components in the paper is a transducer and an ultrasound. The paper uses the fact that an ultrasound array transmits small pulses of ultrasound echo to ROI, and due to the difference in acoustic impedance along the path of transmission some echos are reflected back to transducer. The image is generated using these collected reflected signals, which is modeled as a Linear Time Invariant (LTI) system using impulse response. They assume delays in transmitting beamforming pattern, and so the authors have modelled the received beamforming in time, as well as derived its equivalent frequency domain beamforming model. They presented both time and frequency domain ultrasound beamforming matrix based on spatial impulse response of transducers.




## 5.3 Reconstruction Technique

Time Series representation of each transducer is obtained by

$$y_{(s,t)} = K_{(s,t)\times(m,n)} o_{(m,n)}$$

and the frequency step of each transducer can be obtained by

$$y_{(s,f)} = \tilde{K}_{(s,f)\times(m,n)} o_{(m,n)}$$

The first goal of the paper was to express the data in a sparse basis, since CS reconstruction is applicable for nearly sparse signals. Vectors $\hat{o}$ defined in the equations above are assumed to be sparse. Since the ultrasound images are not sparse in natural domain, a good sparsifier was found based on its ability to represent ultrasound image as sparse image.

Now, the above equations were simplified to

$$y = K\hat{o}$$

, and so recovering $\hat{o}$ from measurement y became a problem of $l_0$ minimization. The problem being NP-hard was solved using greedy algorithms. The authors have used Orthogonal Matching Pursuit (OMP) to reconstruct the sparse solution $\hat{o}$. The constraints are

$$\min ||\hat{o}||_1 \text{ such that } ||\phi\hat{o} - y||_1 \leq \epsilon$$

# 6 Problem 6

## 6.1 Part B - cars.m

The coded snapshot for $T = 3$ frames is as shown below:

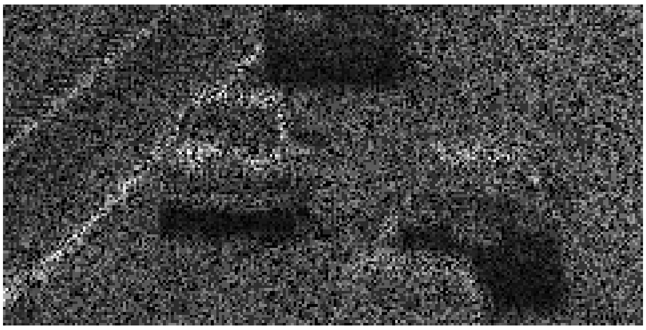


Figure 1: Coded Snapshot for 3 Frames with Gaussian noise

## 6.2 Part C

Note that the coded snapshot has been defined as follows

$$E_u = \sum_{t=1}^{T} C_t \cdot F_t$$

We rewrite this equation in the form $Ax = b$ for $A \in \mathbb{R}^{HW \times HWT}$, $x \in \mathbb{R}^{HWT}$ and $b \in \mathbb{R}^{HW}$. These are defined as follows:

$x$ is the vectorized form of the original video sequence $F$

$b$ is the vectorized form of the coded snapshot$E_u$

$A$ satisfies $A = [S_1|S_2|\ldots|S_T]$ for $S_t = \text{diag}(C_t)$

Here $\text{diag}(C_t)$ is the diagonal matrix formed by taking the vectorized form of $C_t$ as the diagonal.

### 6.3 Part D

We first perform part C for a patch of the video of size $8 \times 8 \times T$. Let $f$ represent this patch, and $c$ represent the corresponding $8 \times 8 \times T$ patch in the random code pattern. Also, take $e_u$ as the corresponding $8 \times 8$ patch in the coded snapshot $E_u$. Then we can write the original definition of $E_u$ patchwise as

$$e_u = \sum_{t=1}^{T} c_t \cdot f_t$$

As shown in part C, if we take $b$ as vectorized $e_u$ and $x_1$ as vectorized $f$, then for $\Phi = [s_1|s_2|\ldots|s_T]$ where $s_t = \text{diag}(c_t)$, we have $\Phi x_1 = b$. Here $b$ is vector of length 64, $x_1$ is a vector of length $64T$, and $\Phi$ is a matrix of size $64 \times 64T$.

Now, let $\psi_{8\times8}$ be the 1D DCT basis matrix of size $8 \times 8$. We find the 2D basis matrix by taking the kronecker product of $\psi_{8\times8}$ with itself, and transposing the obtained $64 \times 64$ matrix. In particular, we write

$$\psi_{2D} = (\psi_{8\times8} \otimes \psi_{8\times8})^T$$

Finally, to club these 2D DCT bases for the $T$ different frames, we make a block diagonal matrix $\Psi$ of size $64T \times 64T$ which has $T$ instances of the 2D DCT matrix in the following way

$$\Psi = \begin{pmatrix} \psi_{2D} & 0 & \ldots & 0 \\ 0 & \psi_{2D} & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \ldots & 0 & \psi_{2D} \end{pmatrix}_{64T\times64T}$$

Note that, we can write the vectorized form of $f$ (i.e. $x_1$) in the DCT basis as $x_1 = \Psi x$, where $x$ is a column vector of length $64T$. Then we take $A = \Phi\Psi$, to get the required formula $Ax = b$ (since $b = \Phi x_1$). Here $x$ is a sparse representation of the $8 \times 8 \times T$ patch in the video.

We explain the error term now. We have the equation $y = \Phi\Psi x + \eta$, where $\eta_i \sim \mathcal{N}(0, \sigma^2)$ for $\sigma = 2$. Then the squared magnitude of $\eta$ is a chi-squared random variable, and so the magnitude of $\eta$ has a high probability to lie within a distance of 3 standard deviations from the mean. Since each patch has 64 elements, so the variance is 64 times $\sigma^2$. Thus, the error term in OMP (to be compared inside the while loop with the squared norm) is taken as $\epsilon = 9 \times 64\sigma^2$.

## 6.4 Part E

The relative mean squared error for $T = 3$ frames is 0.10765.

Figure 2: $t = 1$

Figure 3: $t = 2$

Figure 4: $t = 3$

## 6.5 Part F

### 6.5.1 5 Frames

The relative mean squared error for $T = 5$ frames is 0.1424. Below are the reconstructed images for each of the 5 frames.

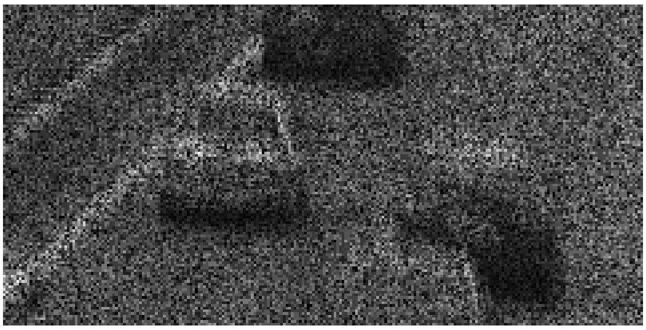


Figure 5: Coded snapshot for 5 Frames with Gaussian noise

Figure 6: $t = 1$

Figure 7: $t = 2$

Figure 8: $t = 3$

Figure 9: $t = 4$

Figure 10: $t = 5$

#### 6.5.2 7 Frames

The relative mean squared error for $T = 7$ frames is 0.18225. Below are the reconstructed images for each of the 7 frames.

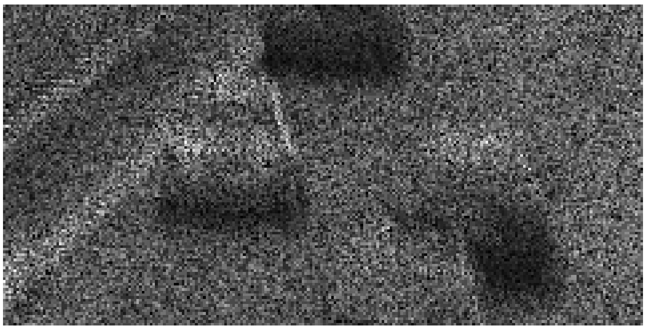


Figure 11: Coded snapshot for 7 Frames with Gaussian noise

Figure 12: $t = 1$

Figure 13: $t = 2$

Figure 14: $t = 3$

Figure 15: $t = 4$

Figure 16: $t = 5$

Figure 17: $t = 6$

Figure 18: $t = 7$

### 6.6 Part G

All reconstruction has been done on the bottom right $120 \times 240$ area of each frame.

### 6.7 Part H - flame.m

The relative mean squared error for the first 5 frames on 'flame.avi' is 0.035586. Below are the reconstructed images for each of the 5 frames.

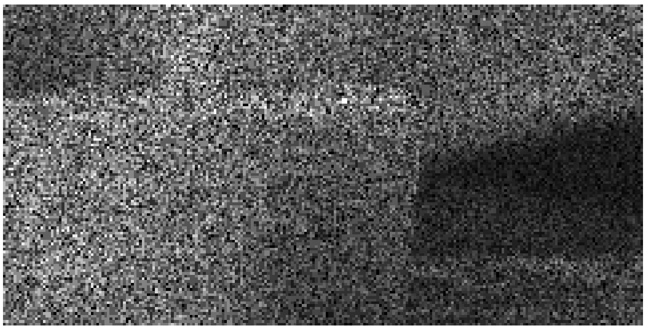


Figure 19: Coded snapshot for 5 Frames with Gaussian noise

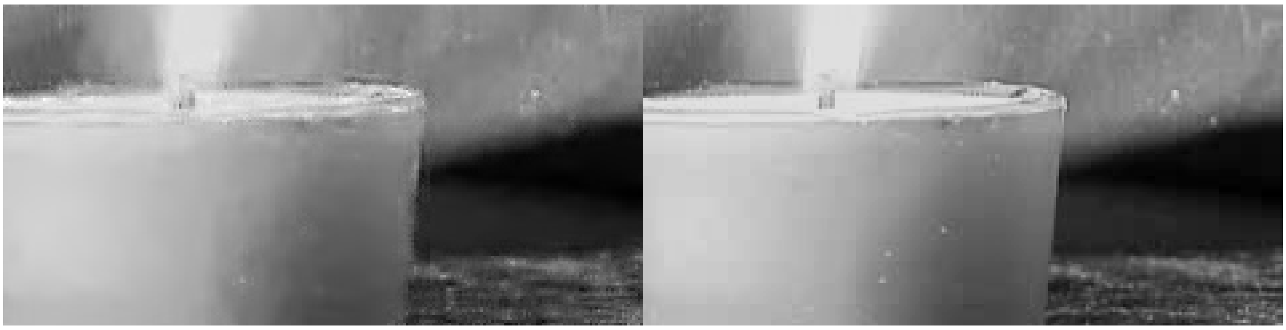


Figure 20: $t = 1$

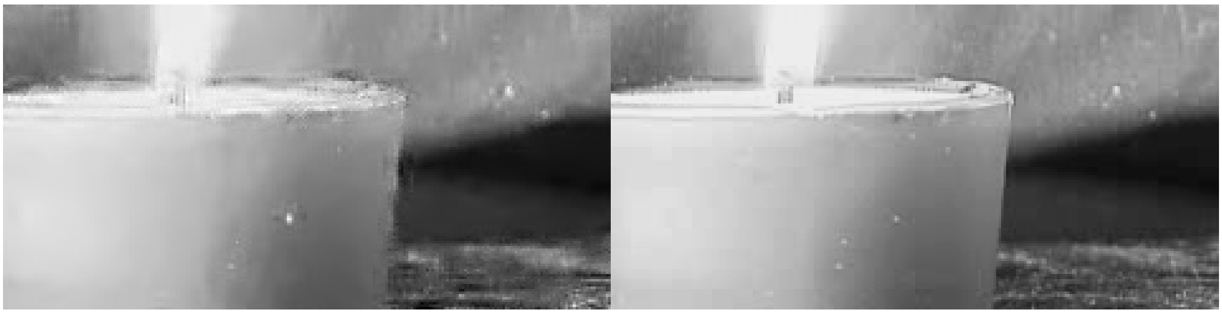


Figure 21: $t = 2$

Figure 22: $t = 3$

Figure 23: $t = 4$

Figure 24: $t = 5$