Assignment 2 - Advanced Image Processing

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February 2022

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1 Problem 1

1.1 Part A

Suppose $\delta_{2s} = 1$, then by the Restricted Isometry Property (RIP) which states that

$$(1-\delta_s)\cdot ||\theta||_2^2 \leq ||A\theta||_2^2 \leq (1+\delta_s)\cdot ||\theta||_2^2$$

for an s-sparse vector θ, we have $0 \leq ||Ax||_2^2 \leq 2\cdot ||x||_2^2$ for any 2s-sparse vector x. Now, consider two s-sparse vectors x_1, x_2 such that $x_1 - x_2$ has exactly 2s non-zero elements and is thus, 2s-sparse(x_1, x_2 have each s non-zero elements at distinct indices).
Now, we know that $0 \leq ||Ax||_2^2$ for a 2s-sparse vector, thus, it is possible for $x_1 - x_2$, which is 2s-sparse to attain the lower bound of 0. In this case, $A(x_1 - x_2) = 0$
Let S be the set of indices i such that $x_i = x_{1i} - x_{2i} \neq 0$, hence we can write $Ax = A(x_1 - x_2) = \sum_{i\in S} x_i A_i = 0$ where A_i is the ith column of A. Thus, 2s columns of A may be linearly dependent

1.2 Part B

Consider triangle inequality

$$|A+B| \leq |A| + |B|$$

Thus,

$$||\Phi(x^* - x)|| = ||\Phi(x^* - x) + y - y|| \leq ||\Phi x^* - y|| + ||y - \Phi x|| \tag{1}$$

Now, $y - \Phi x$ equals the noise which is upper bounded(by definition) by ϵ. Thus, $||y - \Phi x|| \leq \epsilon$.
Also, x^* is the solution the optimization problem whose constraint is that $||y - \Phi x|| \leq \epsilon$. Thus,

$$||\Phi x^* - y|| + ||y - \Phi x|| \leq \epsilon + \epsilon = 2\epsilon$$

1.3 Part C

Let S be the set of indices at which h_{T_j} has a non-zero element. Since, h_{T_j} is a s-sparse vector, cardinality of set S = s.
Consider the L2 norm of vector h_{T_j}.

$$||h_{T_j}||_2^2 = \sum_{i\in S} |x_i|^2 \leq |S| \max_i(|x_i|^2) = s||h_{T_j}||_\infty^2$$

where $|S|$ is the cardinality of set S(equals s). Thus,

$$||h_{T_j}||_2 \leq s^{1/2}||h_{T_j}||_\infty$$

Consider the l1 norm of $h_{T_{j-1}}$. Also, we note that every non-zero element in $h_{T_{j-1}}$ is greater than every non-zero element in h_{T_j} by definition, in terms of magnitude, Thus, we can express

$$||h_{T_{j-1}}||_{l_1} = \sum_{i=1}^{s} |h_{T_{j-1}i}| \geq s \max_k(h_{T_j k}) \implies ||h_{T_{j-1}}||_{l_1} \geq s\cdot ||h_{T_j}||_\infty$$

Thus,

$$||h_{T_j}||_2 \leq s^{1/2}||h_{T_j}||_\infty \leq s^{-1/2}||h_{T_{j-1}}||_{l_1}$$

1.4 Part D

Borrowing from the previous subpart,

$$||h_{T_j}||_2 \leq s^{-1/2}||h_{T_{j-1}}||_{l_1}$$

hence,

$$\sum_{j\geq 2}||h_{T_j}||_{l_2} \leq \sum_{j\geq 2} s^{-1/2}||h_{T_{j-1}}||_{l_1}$$

which concludes the first part of the inequality.

Now, $h_{T_0^c}$ has all but the s greatest elements of h. Thus, $||h_{T_0^c}||$ equals the sum of absolute values of all but the s-greatest elements of h. Meanwhile, $||h_{T_1}||$ is the sum of the next s-greatest elements of h and $||h_{T_2}||$ is the sum of the next to next s-greatest elements of h and so on. Thus,

$$\sum_{j\geq 2} s^{-1/2}||h_{T_{j-1}}||_{l_1} = s^{-1/2}(||h_{T_1}||_{l1} + ||h_{T_2}||_{l1} + \ldots ||h_{T_{k-1}}||_{l_1}) \leq s^{-1/2}||h_{T_0^c}||_{l_1}$$

where k is the sum of absolute values of the next s smallest values of h. Thus,

$$\sum_{j\geq 2}||h_{T_j}||_{l_2} \leq s^{-1/2}(||h_{T_1}||_{l_1} + ||h_{T_2}||_{l_1} + \ldots) \leq s^{-1/2}||h_{T_0^c}||_{l_1}$$

1.5 Part E

Let us consider

$$\left(\sum_{j\geq 2}||h_{T_j}||_2\right)^2 = \sum_{j\geq 2}||h_{T_j}||^2 + \sum_{i\neq j}||h_{T_i}||||h_{T_j}|| \geq \sum_{j\geq 2}||h_{T_j}||^2$$

And since, the non-zero elements of h_{T_i}, h_{T_j} are at disjoint indices(non-zero indices do not have common index) for $i \neq j$, we have

$$\sum_{j\geq 2}||h_{T_j}||^2 = ||\sum_{j\geq 2} h_{T_j}||_2^2$$

which implies that

$$\sum_{j\geq 2}||h_{T_j}||_2 \geq ||\sum_{j\geq 2} h_{T_j}||_2$$

Also, borrowing from the previous sub-part

$$\sum_{j\geq 2}||h_{T_j}||_2 \leq ||h_{T_0^c}||_{l_1}$$

. Hence,

$$||\sum_{j\geq 2} h_{T_j}||_2 \leq \sum_{j\geq 2}||h_{T_j}||_2 \leq s^{-1/2}||h_{T_0^c}||_{l_1}$$

1.6 Part F

We will use triangle inequality in another form

$$|A - B| \geq ||A| - |B||$$

Thus,

$$\begin{aligned}
\sum_{i \in T_0} |x_i + h_i| + \sum_{i \in T_0^c} |x_i + h_i| &= \sum_{i \in T_0} |x_i - (-h_i)| + \sum_{i \in T_0^c} |h_i - (-x_i)| \\
&\geq \sum_{i \in T_0} |x_i| - \sum_{i \in T_0} |h_i| + \sum_{i \in T_0^c} |h_i| - \sum_{i \in T_0^c} |x_i| \\
&= ||x_{T_0}||_{l_1} - ||h_{T_0}||_{l_1} + ||h_{T_0^c}||_{l_1} - ||x_{T_0^c}||_{l_1}
\end{aligned}$$

This, this completes the inequality

1.7 Part G

Using Triangle inequality, we have

$$||x||_{l_1} = ||x_{T_0} + x_s||_{l_1} \leq ||x_{T_0}||_{l_1} + ||x_s||_{l_1}$$

From previous subpart, we have

$$||x||_{l_1} \geq ||x_{T_0}||_{l_1} - ||h_{T_0}||_{l_1} + ||h_{T_0^c}||_{l_1} - ||x_{T_0^c}||_{l_1}$$

Thus, from above 2 inequalities

$$\cancelto{0}{||x_{T_0}||_{l_1}} + ||x_s||_{l_1} \geq \cancelto{0}{||x_{T_0}||_{l_1}} - ||h_{T_0}||_{l_1} + ||h_{T_0^c}||_{l_1} - ||x_{T_0^c}||_{l_1}$$

Rearranging terms,

$$||h_{T_0^c}||_{l_1} \leq ||h_{T_0}||_{l_1} + 2||x_{T_0^c}||_{l_1}$$

which is the required inequality

1.8 Part H

From a previous subpart(E)

$$||h_{(T_0 \cup T_1)^c}||_{l_2} = ||\sum_{j \geq 2} h_{T_j}||_2 \leq \sum_{j \geq 2} ||h_{T_j}||_2 \leq s^{-1/2} ||h_{T_0^c}||_{l_1}$$

and from previous subpart(G)

$$||h_{T_0^c}||_{l_1} \leq ||h_{T_0}||_{l_1} + 2||x_{T_0^c}||_{l_1}$$

Thus, combining the above 2

$$||h_{(T_0 \cup T_1)^c}||_{l_2} \leq s^{-1/2}(||h_{T_0}||_{l_1} + 2||x_{T_0^c}||_{l_1}) = s^{-1/2}(||h_{T_0}||_{l_1}) + 2e_0$$

Sub-Result: To show that $s^{-1/2}(||h_{T_0}||_{l_1}) \leq ||h_{T_0}||_{l_2}$

$$||x||_{l_1} = a^T b, a_i = 1, b_i = |x_i|$$

Using Cauchy Schwarz Inequality

$$\begin{aligned} &\leq ||a||_2 ||b||_2 \\ &= \sqrt{(s)}|x||_{l_2} \end{aligned}$$

where x is a vector having s dimensions. Thus,

$$s^{-1/2}(||h_{T_0}||_{l_1}) + 2e_0 \leq ||h_{T_0}||_{l_2} + 2e_0$$

which concludes the inequality

$$||h_{(T_0 \cup T_1)^c})||_{l_2} \leq ||h_{T_0}||_{l_2} + 2e_0$$

1.9 Part I

The first part of the inequality is a direct consequence of the Cauchy Schwarz Inequality

$$|\langle \Phi h_{(T_0 \cup T_1)}, \Phi h \rangle| \leq ||\Phi h_{(T_0 \cup T_1)}||_{l_2} ||\Phi h||_{l_2}$$

From a previous subpart(B)

$$||\Phi h||_{l_2} \leq 2\epsilon$$

From the RIP property

$$(1 - \delta_{2s})||\theta||_{l_2}^2 \leq ||\Phi\theta||_{l_2}^2 \leq (1 + \delta_{2s})||\theta||_{l_2}^2 \implies ||\Phi\Phi h_{(T_0 \cup T_1)}||_{l_2} \leq \sqrt{(1 + \delta_{2s})}||\Phi h_{(T_0 \cup T_1)}||_{l_2}$$

Thus,

$$||\Phi h_{(T_0 \cup T_1)}||_{l_2} ||\Phi h||_{l_2} \leq 2\epsilon\sqrt{(1 + \delta_{2s})}||\Phi h_{(T_0 \cup T_1)}||_{l_2}$$

Which completes the required inequality

1.10 Part J

Lemma 2.1 in the paper states that

$$|\langle \Phi x, \Phi x' \rangle| \leq \delta_{s+s'} ||x||_{l_2} ||x'||_{l_2}$$

for disjoint subsets T, T' having cardinality less than or equal to s and s' respectively.
Now, we know that T_0 refers to the indices of the s-largest elements of h and T_j belongs to T_0^c and thus are disjoint sets having cardinality s each(By the same logic, T_1 and T_j are disjoint too). Hence, we can apply lemma 2.1 here and we obtain

$$|\langle \Phi h_{T_0}, \Phi h_{T_j} \rangle| \leq \delta_{2s} ||h_{T_0}||_{l_2} ||h_{T_j}||_{l_2}$$

which is the required inequality

1.11 Part K

Since T_0, T_1 are disjoint sets by definition, we have

$$||h_{T_0 \cup T_1}||_{l_2}^2 = \sum_{i \in T_0} h_i^2 + \sum_{k \in T_1} h_k^2 = ||h_{T_0}||_{l_2}^2 + ||h_{T_1}||_{l_2}^2$$

Now, consider

$$\begin{aligned}\left(\sqrt{2}||h_{T_0 \cup T_1}||_{l_2}\right)^2 &= 2(||h_{T_0}||_{l_2}^2 + ||h_{T_1}||_{l_2}^2) \\ &= ||h_{T_0}||_{l_2}^2 + ||h_{T_1}||_{l_2}^2 + 2\frac{||h_{T_0}||_{l_2}^2 + ||h_{T_1}||_{l_2}^2}{2}\end{aligned}$$

Using AM-GM inequality on the second term

$$\begin{aligned}&\geq ||h_{T_0}||_{l_2}^2 + ||h_{T_1}||_{l_2}^2 + 2||h_{T_0}||_{l_2} \cdot ||h_{T_1}||_{l_2} \\ &= \left(||h_{T_0}||_{l_2}^2 + ||h_{T_1}||_{l_2}\right)^2\end{aligned}$$

which completes the inequality

$$\sqrt{2}||h_{T_0 \cup T_1}||_{l_2} \geq ||h_{T_0}||_{l_2}^2 + ||h_{T_1}||_{l_2}$$

1.12 Part L

The first part of the inequality comes from the restricted isometry property

$$(1 - \delta_{2s})||\theta||_{l_2}^2 \leq ||\Phi\theta||_{l_2}^2$$

where $\theta = h_{T_0 \cup T_1}$ having sparsity 2s.
For the second inequality, we first note that

$$||h_{T_0 \cup T_1}||_{l_2}^2 = \langle \Phi h_{(T_0 \cup T_1)}, \Phi h \rangle - \langle \Phi h_{(T_0 \cup T_1)}, \sum_{j \geq 2} \Phi h_{T_j} \rangle$$

Borrowing from a previous subpart (I)

$$||\Phi h_{(T_0 \cup T_1)}||_{l_2}||\Phi h||_{l_2} \leq 2\epsilon\sqrt{(1 + \delta_{2s})}||\Phi h_{(T_0 \cup T_1)}||_{l_2}$$

Borrowing from a previous subpart (J)

$$\begin{aligned}|\langle \Phi h_{T_0}, \Phi h_{T_j} \rangle| &\leq \delta_{2s}||h_{T_0}||_{l_2}||h_{T_j}||_{l_2} \\ |\langle \Phi h_{T_1}, \Phi h_{T_j} \rangle| &\leq \delta_{2s}||h_{T_1}||_{l_2}||h_{T_j}||_{l_2}\end{aligned}$$

Adding the above 2 equations

$$\implies |\langle \Phi h_{T_0 \cup T_1}, \Phi h_{T_j} \rangle| \leq \delta_{2s}(||h_{T_0}||_{l_2} + ||h_{T_1}||_{l_2})||h_{T_j}||_{l_2}$$

Using the bound obtained in subpart (K)

$$\implies |\langle \Phi h_{T_0 \cup T_1}, \Phi h_{T_j} \rangle| \leq \sqrt{2}\delta_{2s}||h_{T_0 \cup T_1}||_{l_2}||h_{T_j}||_{l_2}$$

Thus, this implies

$$-\sum_{j \geq 2} \sqrt{2}\delta_{2s}||h_{T_0 \cup T_1}||_{l_2}||h_{T_j}||_{l_2} \leq \langle \Phi h_{(T_0 \cup T_1)}, \sum_{j \geq 2} \Phi h_{T_j} \rangle \leq \sum_{j \geq 2} \sqrt{2}\delta_{2s}||h_{T_0 \cup T_1}||_{l_2}||h_{T_j}||_{l_2}$$

Now, we have bounded both terms, Hence

$$||h_{T_0 \cup T_1}||_{l_2}^2 \leq ||h_{T_0 \cup T_1}||_{l_2} \left(2\epsilon\sqrt{(1+\delta_{2s})} + \sqrt{2}\delta_{2s} \sum_{j \geq 2} ||h_{T_j}||_{l_2} \right)$$

This completes the inequality

1.13 Part M

Borrowing from the previous subpart

$$\begin{aligned}
(1-\delta_{2s})||h_{T_0 \cup T_1}||_{l_2}^2 &\leq ||h_{T_0 \cup T_1}||_{l_2} \left(2\epsilon\sqrt{(1+\delta_{2s})} + \sqrt{2}\delta_{2s} \sum_{j \geq 2} ||h_{T_j}||_{l_2} \right) \\
\implies (1-\delta_{2s})||h_{T_0 \cup T_1}||_{l_2} &\leq 2\epsilon\sqrt{(1+\delta_{2s})} + \sqrt{2}\delta_{2s} \sum_{j \geq 2} ||h_{T_j}||_{l_2} \\
\implies ||h_{T_0 \cup T_1}||_{l_2} &\leq \frac{2\epsilon\sqrt{(1+\delta_{2s})} + \sqrt{2}\delta_{2s} \sum_{j \geq 2} ||h_{T_j}||_{l_2}}{1-\delta_{2s}}
\end{aligned}$$

Borrowing from subpart (E)

$$\sum_{j \geq 2} ||h_{T_j}||_2 \leq s^{-1/2} ||h_{T_0^c}||_{l_1}$$

Thus

$$\begin{aligned}
\implies ||h_{T_0 \cup T_1}||_{l_2} &\leq \frac{2\epsilon\sqrt{(1+\delta_{2s})} + \sqrt{2}\delta_{2s} s^{-1/2} ||h_{T_0^c}||_{l_1}}{1-\delta_{2s}} \\
&= \alpha\epsilon + \rho s^{-1/2} ||h_{T_0^c}||_{l_1}
\end{aligned}$$

which is the required inequality.

1.14 Part N

From the previous subpart

$$||h_{T_0 \cup T_1}||_{l_2} \leq \alpha\epsilon + \rho s^{-1/2} ||h_{T_0^c}||_{l_1}$$

From subpart (G)

$$\begin{aligned}
||h_{T_0^c}||_{l_1} &\leq ||h_{T_0}||_{l_1} + 2||x_{T_0^c}||_{l_1} \\
\implies ||h_{T_0 \cup T_1}||_{l_2} &\leq \alpha\epsilon + \rho s^{-1/2} (||h_{T_0}||_{l_1} + 2||x_{T_0^c}||_{l_1}) \\
\implies ||h_{T_0 \cup T_1}||_{l_2} &\leq \alpha\epsilon + \rho s^{-1/2} ||h_{T_0}||_{l_1} + 2s^{-1/2}\rho||x_{T_0^c}||_{l_1}
\end{aligned}$$

Using upper bound on L1-norm by $\sqrt{size} \times L2$-norm of vector

$$\implies ||h_{T_0 \cup T_1}||_{l_2} \leq \alpha\epsilon + \rho||h_{T_0}||_{l_2} + 2\rho e_0$$

Since every element in the union $T_0 \cup T_0$ is greater than just T_1

$$\implies ||h_{T_0 \cup T_1}||_{l_2} \leq \alpha\epsilon + \rho||h_{T_0 \cup T_1}||_{l_2} + 2\rho e_0$$

1.15 Part O

The first part of the inequality comes from triangle inequality

$$h = h_{T_0 \cup T_1} + h_{(T_0 \cup T_1)^c} \implies ||h||_{l_2} \le ||h_{T_0 \cup T_1}||_{l_2} + ||h_{(T_0 \cup T_1)^c}||_{l_2}$$

The next part borrows from subpart H

$$||h_{(T_0 \cup T_1)^c)}||_{l_2} \le ||h_{T_0}||_{l_2} + 2e_0 \le ||h_{T_0 \cup T_1}||_{l_2} + 2e_0$$

Thus,

$$||h||_{l_2} \le 2||h_{T_0 \cup T_1}||_{l_2} + 2e_0$$

. Borrowing from the previous subpart

$$\begin{aligned}
||h_{T_0 \cup T_1}||_{l_2} &\le \alpha\epsilon + \rho||h_{T_0 \cup T_1}||_{l_2} + 2\rho e_0 \\
\implies ||h_{T_0 \cup T_1}||_{l_2} &\le (1-\rho)^{-1}(a\epsilon + 2\rho e_0) \\
\implies 2||h_{T_0 \cup T_1}||_{l_2} + 2e_0 &\le 2(1-\rho)^{-1}(a\epsilon + 2\rho e_0) + 2e_0 \\
\implies 2||h_{T_0 \cup T_1}||_{l_2} + 2e_0 &\le 2(1-\rho)^{-1}(a\epsilon + (1+\rho)e_0)
\end{aligned}$$

Hence,

$$||h||_{l_2} \le 2(1-\rho)^{-1}(a\epsilon + (1+\rho)e_0)$$

which completes the inequality

1.16 Part P

We know $x_{T_0} + x_{T_0^c} = x$ for any vector x Hence,

$$\sum_i |x_i| = \sum_{i \in T_0} |x_i| + \sum_{i \in T_0^c} |x_i| \implies ||x||_{l_1} = ||x_{T_0}||_{l_1} + ||x_{T_0^c}||_{l_1}$$

From lemma 2

$$||h_{T_0}||_{l_1} \le \rho||h_{T_0^c}||_{l_1}$$

Using subpart G

$$\begin{aligned}
||h_{T_0^c}||_{l_1} &\le \rho(||h_{T_0}||_{l_1} + 2||x_{T_0^c}||_{l_1}) \\
\implies ||h_{T_0^c}||_{l_1} &\le 2(1-\rho)^{-1}||x_{T_0^c}||_{l_1}
\end{aligned}$$

Also

$$\begin{aligned}
||h||_{l_1} &= ||h_{T_0}||_{l_1} + ||h_{T_0^c}||_{l_1} \\
\implies ||h||_{l_1} &\le (1+\rho)||h_{T_0^c}||_{l_1} \le 2(1+\rho)(1-\rho)^{-1}||x_{T_0^c}||_{l_1}
\end{aligned}$$

which is the required inequality

2 Problem 2

First we define some variables. Let $|S| = k$, where S is the set of indices of the non-zero elements of x. Take the elements of this set as $a_1, a_2, \ldots, a_k$, which are arranged in strictly increasing order. Also consider the $n \times k$ matrix U (n is the size of signal x) given by

$$U_{ij} = \begin{cases} 1 & i = a_j \\ 0 & \text{otherwise} \end{cases}$$

Finally, consider the $k \times 1$ column vector x_S formed by taking the k non-zero entries of x, and note that we have $x = Ux_s$.

2.1 Part A

Since Φx only has non-zero elements due to the columns of Φ belonging to indices in S, so we can say that $\Phi x = \Phi_S x_S$. So the solution to $y = \Phi x$ also satisfies $y = \Phi_S x_S$. In the oracle's solution, we know S, and so we can find out Φ_S from Φ. Then

$$y = \Phi_S x_S \implies \Phi_S^\dagger y = \Phi_S^\dagger \Phi_S x_S = (\Phi_S^T \Phi_S)^{-1} \Phi_S^T \Phi_S x_S = x_S$$

So the k non-zero values in $\tilde{x}$ is given by $\Phi_S^\dagger y$. To get $\tilde{x}$ from this, we simply pre-multiply this $k \times 1$ vector with the matrix U. Thus, the oracular solution is $\tilde{x} = U\Phi_S^\dagger y$.

2.2 Part B

Since $\tilde{x}$ and x have non-zero elements at the same positions (i.e. along the set of indices S), so the 2-norm of $\tilde{x} - x$ is the same as the 2-norm of $\Phi_S^\dagger y - x_S$, i.e. we only need the difference of the non-zero elements. This gives

$$||\tilde{x} - x||_2 = \left|\left|\Phi_S^\dagger y - x_S\right|\right|_2 = \left|\left|\Phi_S^\dagger(\Phi_S x_S + \eta) - x_S\right|\right|_2 = \left|\left|\Phi_S^\dagger \eta\right|\right|_2$$

Here we use the fact that $y = \Phi x + \eta = \Phi_S x_S + \eta$. Now we show the inequality, starting with the following claim:

CLAIM For any $m \times n$ real matrix A, the largest singular value of A is equal to $\sup\limits_{||x||_2=1} ||Ax||_2$ over all vectors $x \in \mathbb{R}^n$.

Proof. Let $A = U\Sigma V^T$ be the singular value decomposition of A, where U and V are orthogonal matrices, and Σ is a rectangular diagonal matrix with diagonal elements as the singular values. Now, for an orthogonal matrix M and any vector z, we have

$$||Mz||_2^2 = (Mz)^T(Mz) = z^T M^T M z = z^T z = ||z||_2^2 \implies ||Mz||_2 = ||z||_2$$

Using the SVD for A and the fact that V^T is also orthogonal, we get

$$||Ax||_2 = ||U\Sigma V^T x||_2 = ||\Sigma V^T x||_2 \text{ and } ||V^T x||_2 = ||x||_2$$

Let $y = V^T x$, so that we want to find out $\sup\limits_{||y||_2=1} ||\Sigma y||_2$. Take the diagonal elements of Σ as $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$ (assuming $m \geq n$, since the case of $m < n$ is the same except that we take m elements instead of n in the derivation ahead). Then we have

$$||\Sigma y||_2^2 = \sum_{i=1}^{n} (\sigma_i y_i)^2 \leq \sigma_1^2$$

where the last inequality follows using $\sigma_i \leq \sigma_1$ for all $i \in [n]$ and $||y||_2 = 1$. So the supremum of $||\Sigma y||_2$ is σ_1, and is attained when $y_1 = 1$ and all other y_i's are 0. $\square$

In the particular, the above claim gives that $||Ax||_2$ is lesser than or equal to the maximum singular value of A for all unit vectors x, i.e. $||Ax||_2 \leq ||A||_2$. Return back to the problem at hand. Let v be the unit vector in the direction of η, so that $\eta = ||\eta||_2 v$. Then we have

$$||\tilde{x} - x||_2 = \left|\left|\Phi_S^\dagger \eta\right|\right|_2 = \left|\left|\Phi_S^\dagger ||\eta||_2 v\right|\right|_2 = ||\eta||_2 \left|\left|\Phi_S^\dagger v\right|\right|_2 \leq ||\eta||_2 \left|\left|\Phi_S^\dagger\right|\right|_2$$

2.3 Part C

Since Φ satisfies RIP with constant δ_{2k}, so we have the inequality

$$(1 - \delta_{2k}) \cdot ||\theta||_2^2 \leq ||\Phi\theta||_2^2 \leq (1 + \delta_{2k}) \cdot ||\theta||_2^2$$

for all $2k$-sparse $\theta \in \mathbb{R}^{n\times 1}$. Now, consider any $k \times 1$ unit vector x_S, and let $x = Ux_S$ be the $(n \times 1)$ k-sparse vector with S as the support. Then x is also $2k$-sparse with $||x||_2 = ||x_S||_2 = 1$, and $\Phi x = \Phi_S x_S$, and so substituting $\theta = x$ in the above inequality gives

$$1 - \delta_{2k} \leq ||\Phi_S x_S||_2^2 \leq 1 + \delta_{2k}$$

So $||\Phi_S x_S||_2$ is bounded between $\sqrt{1 - \delta_{2k}}$ and $\sqrt{1 + \delta_{2k}}$ for all $||x_S||_2 = 1$.

Now, if we take the singular value decomposition of Φ_S as $\Phi_S = U\Sigma V^T$, then $||\Phi_S x_S||_2 = ||\Sigma V^T x_S||_2$. So taking the i^{th} coordinate of $V^T x_S$ as 1 and remaining coordinates as 0, we get the i^{th} singular value of Φ_S. Since $||x_S||_2 = ||v^T x_S||_2 = 1$, so we can say that all singular values of Φ_S also lie in the range $[\sqrt{1 - \delta_{2k}}, \sqrt{1 + \delta_{2k}}]$.

Next, we find the singular value decomposition of $\Phi_S^\dagger$.

$$\begin{aligned}\Phi_S^\dagger &= (\Phi_S^T \Phi_S)^{-1} \Phi_S^T = ((V\Sigma^T U^T)(U\Sigma V^T))^{-1}(V\Sigma^T U^T) \\ &= (V\Sigma^2 V^T)^{-1}(V\Sigma^T U^T) = (V\Sigma^{-2}V^T)(V\Sigma U^T) = V\Sigma^{-1}U^T\end{aligned}$$

Here we use the facts that $\Sigma^T = \Sigma$, and U and V are orthogonal. So the singular values of $\Phi_S^\dagger$ are the diagonal elements of Σ^{-1}, which are in turn reciprocal of the diagonal elements of Σ. Thus, the singular values of $\Phi_S^\dagger$ are reciprocal of the singular values of Φ_S. Since the singular values of Φ_S lie in the range $[\sqrt{1 - \delta_{2k}}, \sqrt{1 + \delta_{2k}}]$, so the singular values of $\Phi_S^\dagger$ (and consequently the largest singular value of $\Phi_S^\dagger$) satisfy the inequality

$$\frac{1}{\sqrt{1 + \delta_{2k}}} \leq \sigma(\Phi_S^\dagger) \leq \frac{1}{\sqrt{1 - \delta_{2k}}} \implies \frac{1}{\sqrt{1 + \delta_{2k}}} \leq \left|\left|\Phi_S^\dagger\right|\right|_2 \leq \frac{1}{\sqrt{1 - \delta_{2k}}}$$

2.4 Part D

From part B, and using $||\eta||_2 \leq \epsilon$, we get

$$||\tilde{x} - x||_2 \leq ||\eta||_2 \left|\left|\Phi_S^\dagger\right|\right|_2 \leq \epsilon \left|\left|\Phi_S^\dagger\right|\right|_2$$

So, using the bounds on $\left|\left|\Phi_S^\dagger\right|\right|_2$ from part C, the worst case error $||\tilde{x} - x||_2$ follows the inequality given in the statement for part D. Now, Theorem 3 gives a solution x^* which satisfies the error bound

$$||x^* - x||_2 \leq \frac{C_0}{\sqrt{k}} ||x - \tilde{x}||_1 + C_1\epsilon$$

Now, since x is purely k-sparse, and $\tilde{x}$ consists of all non-zero elements at the indices in set S of x, so the L1 norm above is 0, which gives $||x^* - x||_2 \leq C_1\epsilon$. The bound $C_1\epsilon$ is a linear multiple of the maximum error bound for the oracular solution. So this solution is only constant times worse than the oracle's solution.

3 Problem 3

Sparsity: An s-sparse vector is one which has *at most* s non-zero elements

Thus, consider a s-sparse vector with $k \leq s$ non-zero elements. And since, $s < t$, we have $k < t$. Thus, any s-sparse vector is also a t-sparse vector under the condition $s < t$.
Now, we have the restricted isometry property (RIP) which states that

$$(1 - \delta_s) \cdot ||\theta||_2^2 \leq ||A\theta||_2^2 \leq (1 + \delta_s) \cdot ||\theta||_2^2$$

where δ_s is the smallest such constant which satisfies the above equation for all vectors θ having sparsity s.
Hence, consider the restricted isometry property for a t-sparse vector.

$$(1 - \delta_t) \cdot ||\theta||_2^2 \leq ||A\theta||_2^2 \leq (1 + \delta_t) \cdot ||\theta||_2^2$$

where θ has sparsity t. Now, consider vectors having sparsity s. Since, as shown, every s-sparse vector is also a t-sparse vector, every s-sparse vector also satisfies the above equation. Thus, δ_t satisfies the RIP for s-sparse vectors. Hence, the smallest value of the constant which satisfies RIP for s-sparse vectors must be smaller than or equal to δ_t. Hence, $\delta_s \leq \delta_t$

4 Problem 4

4.1 Details of the Paper

Title: **Sensing Matrix Design via Mutual Coherence Minimization for Electromagnetic Compressive Imaging Applications**
Venue: IEEE Transactions on Computational Imaging, VOL. 3, NO. 2
Date: 17 February 2017
The paper can be accessed here.

4.2 Imaging System

The authors consider a scenario in which a single transmitting and receiving antenna is used to excite a region of interest with a single frequency. In these scenarios, there are two possibilities, one in which the positions of the antenna are restricted to lie along a circle centered about the imaging region, and one in which the positions of the antenna are restricted to lie on a two-dimensional plane parallel to the imaging region.

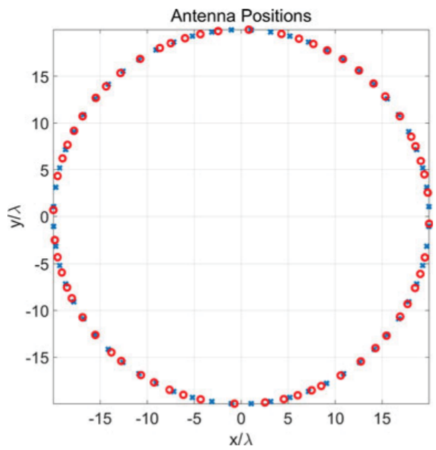

Fig. 1. Antenna positions of the baseline (blue) and optimized (red) designs in the circular configuration.

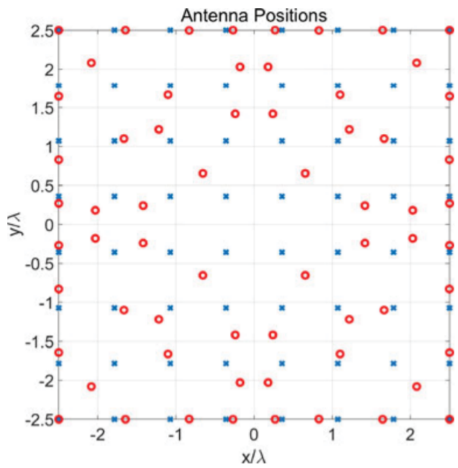

Fig. 14. Antenna positions of the baseline (blue) and optimized (red) designs in the planar configuration.

In the circular optimization problem, the antenna was constrained to operate at $M = 60$ positions along the circle of radius 20λ relative to the center of the imaging region. The initial positions were selected by distributing the points uniformly over the circle. The sensing matrix was computed at $N = 121$ imaging points, uniformly located on a 2D $5\lambda \times 5\lambda$ grid centered at the origin. This corresponds to a grid size of approximately 0.5λ.

The design parameters and constraints imposed on the planar optimization problem were as follows: The antenna was constrained to operate at $M = 64$ positions within a $5\lambda \times 5\lambda$ grid centered at $x = y = 0$ on the $z = 5\lambda$ plane. The initial positions were selected by distributing the points uniformly over the rectangular grid, and the sensing matrix was computed at the same positions as in the circular example. Once again, the optimized design distributes the antenna positions symmetrically.

4.3 Matrix Quality Measure

The paper minimizes mutual coherence to find the sensing matrix, since it is a more practical measure for deterministic matrices, such as those used in electromagnetic imaging applications. Using random matrices is not a possibility in these areas, as the sensing matrix is constrained by a number of practical factors, such as the positions of the transmitting and receiving antennas and the excitation frequencies. The mutual coherence of a complex sensing matrix A is defined as follows:

$$\mu(A) = \max_{1 \leq i \neq j \leq N} \frac{|a_i^H a_j|}{||a_i||_{l_2} \cdot ||a_j||_{l_2}}$$

where a_i is the i^{th} column of A. The method described in the paper minimizes the mutual coherence above in applications where the sensing matrix is some differentiable non-linear function of the design variables, i.e. $A = f(p)$, where $A \in \mathbb{C}^{M \times N}$ and $p \in \mathbb{C}^L$.

The discretized measurement process for this system can be modeled as follows:

$$y_m = \sum_{n=1}^{N} x_n e^{-2jk||r_m - r_n||_{l_2}} = \sum_{n=1}^{N} A_{mn} x_n$$

where y_m is the measured scattered field, r_m is the position of the antenna, r_n is a position in the imaging region, k is the wavenumber, and x_n is the reflectivity at the position in the imaging region. Keeping the wavenumber fixed, the objective is to select the antenna positions r_m such that the coherence is minimized.

4.4 Optimization Technique

The design algorithm seeks the minimum of the following optimization problem:

$$\min_{p,A} \mu(A) \text{ subject to } A = f(p), p \in Q_p$$

where Q_p is the set of feasible values that the design variables can take. The Augmented Lagrangian method has been used in order to represent this constrained optimization problem in an unconstrained form. The Augmented Lagrangian method finds a local optimal solution to the constrained form by solving a sequence of unconstrained subproblems. These subproblems are solved using an alternating minimization procedure. The whole process is summarized in the 2 algorithms given below.

Algorithm 1: Summary of the Augmented Lagrangian update procedure for the coherence minimization problem of Eq.(11)

Choose the initial values for $p^{(0)}$, $\rho^{(1)}$;

Set $u_{i,j}^{(0)} = \frac{f_i(p^{(0)})^H f_j(p^{(0)})}{\|f_i(p^{(0)})\|_{\ell_2} \|f_j(p^{(0)})\|_{\ell_2}}$, $\beta_{i,j}^{(1)} = 0$;

for k = *1,2,3...* **do**

  Solve the unconstrained subproblem

$$\left(p^{(k)}, u^{(k)}\right) = \underset{p,u}{\text{argmin}}\ \mathcal{L}_{\mathcal{A}}\left(p, u, \beta^{(k)}; \rho^{(k)}\right)$$

  Update the dual variables

$$\beta_{i,j}^{(k+1)} = \beta_{i,j}^{(k)} + \rho^{(k)} \left(u_{i,j}^{(k)} - \frac{f_i(p^{(k)})^H f_j(p^{(k)})}{\|f_i(p^{(k)})\|_{\ell_2} \|f_j(p^{(k)})\|_{\ell_2}}\right)$$

  Compute $\rho^{(k+1)}$ using the method described in [15]

Algorithm 2: Summary of the alternating minimization procedure for solving the Augmented Lagrangian subproblem of Eq.(13)

Given $p_{(0)}^{(k)}$, $u_{(0)}^{(k)}$, $\beta^{(k)}$, $\rho^{(k)}$;

for m = *0,1,2,...* **do**

  Update u while holding p fixed

$$u_{(m+1)}^{(k)} = \underset{u}{\text{argmin}}\ \mathcal{L}_{\mathcal{A}}\left(p_{(m)}^{(k)}, u, \beta^{(k)}; \rho^{(k)}\right)$$

  Update p while holding u fixed

$$p_{(m+1)}^{(k)} = \underset{p}{\text{argmin}}\ \mathcal{L}_{\mathcal{A}}\left(p, u_{(m+1)}^{(k)}, \beta^{(k)}; \rho^{(k)}\right)$$

4.5 Improvements

In the circular imaging system, the optimized design has a mutual coherence of approximately 0.1943, which is an improvement over the 0.3232 coherence of the baseline design, even though it does not achieve the theoretical minimum coherence of 0.0920. Given the restrictive nature of the design constraints in this problem, it is not surprising that the minimum coherence was not achieved. The red ellipses in the figures below highlight the regions where the off-diagonal elements differ significantly between the two configurations.

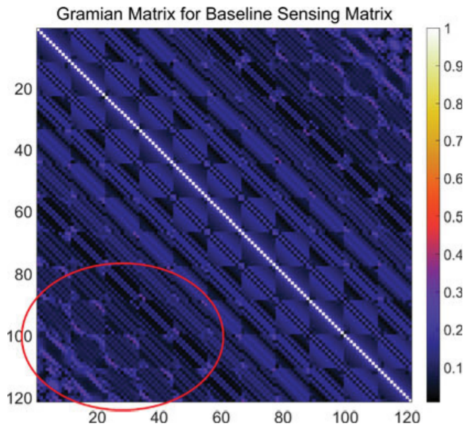

Fig. 2. Gramian matrix of the baseline design sensing matrix for the circular configuration. The mutual coherence, given by the maximum off-diagonal element, is approximately 0.3232.

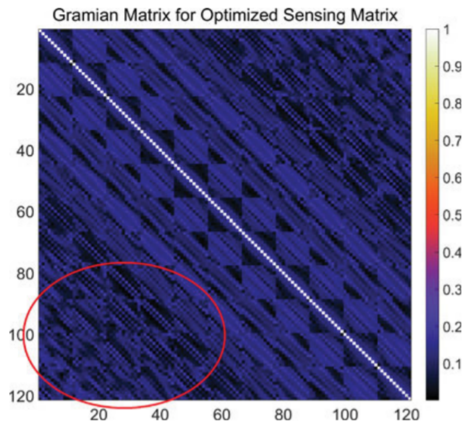

Fig. 3. Gramian matrix of the optimized design sensing matrix for the circular configuration. The mutual coherence, given by the maximum off-diagonal element, is approximately 0.1943.

A similar improvement over the baseline model is seen for the planar configuration also. In this example, the optimized design has a mutual coherence of approximately 0.2252, which is a significant improvement over the 0.8300 coherence of the baseline design, even though it does not achieve the minimum coherence of 0.0862. The figures below display the magnitude of the Gramian matrices computed from the baseline and optimized sensing matrices.

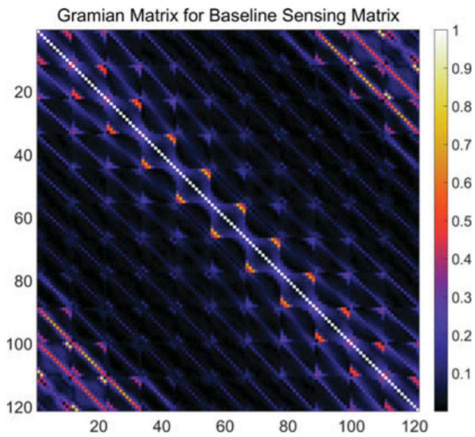

Fig. 15. Gramian matrix of the baseline design sensing matrix for the planar configuration. The mutual coherence, given by the maximum off-diagonal element, is approximately 0.8300.

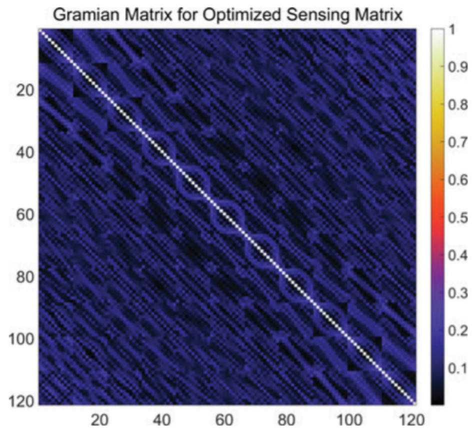

Fig. 16. Gramian matrix of the optimized design sensing matrix for the planar configuration. The mutual coherence, given by the maximum off-diagonal element, is approximately 0.2252.

5 Problem 5

Suppose x_0 is a minimizer of the LASSO problem for some $\lambda > 0$, i.e. for all vectors x we have

$$J(x_0) \leq J(x) \implies ||y - \Phi x_0||_2^2 + \lambda ||x_0||_1 \leq ||y - \Phi x||_2^2 + \lambda ||x||_1$$

Now, take $\epsilon = ||y - \Phi x_0||_2$. Then, for all vectors x with $||y - \Phi x||_2 \leq \epsilon$, we can say that

$$||y - \Phi x_0||_2^2 + \lambda ||x_0||_1 \leq ||y - \Phi x||_2^2 + \lambda ||x||_1 \leq ||y - \Phi x_0||_2^2 + \lambda ||x||_1 \implies ||x_0||_1 \leq ||x||_1$$

So x_0 is also a minimizer of the P1 problem with $\epsilon = ||y - \Phi x_0||_2$.

6 Problem 6

Note that, in the first round, we always perform $\frac{n}{g}$ tests. Suppose we do y tests in the second round. Then the total number of tests is $\frac{n}{g} + y$, and the average number of tests done is $\frac{n}{g} + \mathbb{E}[y]$.

Now, suppose for each i from 1 to $\frac{n}{g}$, the i^{th} group has an infected sample with a probability p. Then p is same as 1 minus the probability that the i^{th} group has no infected samples. This can happen in a total of $\binom{n-g}{k}$ ways out of the possible $\binom{n}{k}$ combinations. So, we get $p = 1 - \frac{\binom{n-g}{k}}{\binom{n}{k}}$.

Let σ_i be equal to 1 if the i^{th} pool is infected with at least 1 sample, and 0 otherwise. Then, using linearity of expectation, we have

$$y = \sum_{i=1}^{\frac{n}{g}} \sigma_i \times g \implies \mathbb{E}[y] = \sum_{i=1}^{\frac{n}{g}} \mathbb{E}[\sigma_i \times g] = \sum_{i=1}^{\frac{n}{g}} g \times \mathbb{E}[\sigma_i]$$

And, the expected value of σ_i is given by

$$\mathbb{E}[\sigma_i] = 1 \cdot P(i^{\text{th}} \text{ group is infected}) + 0 \cdot P(i^{\text{th}} \text{ group is not infected}) = p$$

Thus, we have

$$\mathbb{E}[y] = \frac{n}{g}(g \times p) = n \times p = n\left(1 - \frac{\binom{n-g}{k}}{\binom{n}{k}}\right)$$

So the average number of tests is $\frac{n}{g} + n\left(1 - \frac{\binom{n-g}{k}}{\binom{n}{k}}\right)$.

In the worst case, the k infected samples will each be present in a different pool. Now, the first round requires testing $\frac{n}{g}$ pools. And in the second round, since the infected samples are in k different pools, so we will need to check each person of these k pools, requiring $k \times g$ tests. So total number of tests in the worst case are $(\frac{n}{g} + k \times g)$ tests. By the AM-GM inequality, this is minimized when $\frac{n}{g} = k \times g$, and so the optimal group size for the worst case is $\sqrt{\frac{n}{k}}$.