Problem Set 5

Released: September 20, 2021

1. **Equivalence Closure**

   (a) Show that the transitive closure of the symmetric closure of the reflexive closure of a relation R is the smallest equivalence relation that contains R.

   (b) Give an example such that the symmetric closure of the transitive closure of the reflexive closure of a relation R is not an equivalence relation.

2. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined as $f((x, y)) = (y, y - x)$. Then define f^{-1}, or show that there is no unique inverse for f.

3. Define a relation $\sim$ on the set of all functions from $\mathbb{R}$ to $\mathbb{R}$ by the rule $f \sim g$ if and only if there is a $z \in \mathbb{R}$ such that $f(x) = g(x)$ for every $x \geq z$. Prove that $\sim$ is an equivalence relation.

4. If functions $f : A \to B$ and $g : B \to C$ are such that $g \circ f$ is onto, then prove that g is onto. Use precise mathematical notation to prove this, starting from the definitions of onto and composition.

5. Suppose $f : A \to B$ and $g : B \to C$ are such that $g \circ f$ is one-to-one. Is f necessarily one-to-one? Is g necessarily one-to-one? Justify.

6. Suppose $f : A \to A$ is a function and $f \circ f$ is a bijection. Is f necessarily a bijection?

7. Given a function $f : A \to B$, define another function $f' : \mathcal{P}(A) \to \mathcal{P}(B)$ (where $\mathcal{P}(A)$ stands for the power-set of A), as follows: for any $S \subseteq A$, $f'(S) = \{f(x) | x \in S\}$. Show that $f'(S \cap T) \subseteq f'(S) \cap f'(T)$. Give an example of f and S, T such that $f'(S \cap T) \neq f'(S) \cap f'(T)$.

8. Given a function $f : A \to B$, we define another function $\text{inv}_f : \mathcal{P}(B) \to \mathcal{P}(A)$ as follows: for any $S \subseteq B$, $\text{inv}_f(S) = \{x | f(x) \in S\}$. Now, given functions $f : A \to B$ and $g : B \to C$, express $\text{inv}_{g \circ f}$ in terms of inv_f and inv_g. Justify.

9. Construct a bijection $f : \mathbb{Z} \to \mathbb{Z}^+$.

10. Construct a bijection $f : \mathbb{Z}^2 \to \mathbb{Z}$.