Discrete Structures :: CS 207 :: Autumn 2021

Problem Set 1

Released: August 6, 2021

1. **Contrapositive.** Show that $p \to q \equiv \neg q \to \neg p$.

   This illustrates the equivalence of the statements "If today is a Sunday then today is a holiday" and "If today is not a holiday, then today is not a Sunday." (Note that these statements are **not equivalent** to "If today is not a Sunday, then today is not a holiday," or, $\neg p \to \neg q$.)

2. **Distributive Property.** To show the equivalences below, you can derive the truth table of the formulas on the LHS and RHS, and compare them. Alternately, for a quicker argument in the problems below, you can consider two cases, $p \equiv T$ and $p \equiv F$.

   (a) Show that $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$.

   (b) Show that $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$.

   (c) What is the condition on a binary operator $\star$ so that $\wedge$ distributes over $\star$ (i.e., $p \wedge (q \star r) \equiv (p \wedge q) \star (p \wedge r)$)? What is the condition for $\vee$ to distribute over $\star$?

   (d) Does $\wedge$ distribute over $\oplus$? Does $\vee$ distribute over $\oplus$?

3. **Simplifying formulas.**

   Every formula in two variables is equivalent to a binary operator. Identify the operator in the following cases, and write down an equivalent expression.

   (Thus your answer should be one of the 16 possibilities: T, F, p, q, $\neg p$, $\neg q$, $p \oplus q$, $p \leftrightarrow q$, $p \wedge q$, $p \vee q$, $p \uparrow q$, $p \downarrow q$, $p \to q$, $q \to p$, $p \not\to q$ and $q \not\to p$.)

   You could prepare a truth table for each formula to help with the task. You could also employ the distributive property, De Morgan's law and other equivalences from the lecture.

   (a) $p \wedge \neg q$

   (b) $(p \to q) \wedge \neg q$

   (c) $p \vee \neg(q \to p)$

   (d) $(p \wedge q) \to q$

   (e) $(p \wedge q) \leftrightarrow q$

   (f) $(p \leftrightarrow q) \leftrightarrow ((p \wedge q) \vee (\neg p \wedge \neg q))$

4. **Functional Completeness.** A set of operators is *functionally complete* if all n-ary logical operations, for any $n > 0$, can be expressed as formulas that use only operators from this set. In other words, all possible truth tables *over any number of inputs* can be produced by formulas that use only these operators.

   Show that the set $\{\neg, \wedge, \vee\}$ is functionally complete.

   [*Hint: First consider an n-ary operation which has a single row in its truth table evaluating to T. Can you design an equivalent formula with just $\neg$s and $\wedge$s? Next, if an operation's truth table has k rows that evaluate to T, can you design a formula with k terms of the above kind, combined using $\vee$s?*]

5. **A Tautology.** Prove that $\exists x \forall y\ P(x) \to P(y)$ is true no matter what the predicate P is (assuming that the domain is non-empty).

   [Hint: consider two cases, depending on whether $\forall y\ P(y)$ is true or false.]

6. **Pointless Games.** Suppose a game has the following structure: Alice specifies an integer a, then Bob specifies an integer b, and finally Alice specifies an integer c. Alice wins the game if $g(a, b, c) = 0$, where g is a function associated with the game; if $g(a, b, c) \neq 0$ Bob wins.

   Alice is said to have a *winning strategy* if there is some way for her to play the game (i.e., pick a and c) to ensure that she will win no matter how Bob plays (i.e., picks b). Note that Alice can pick c *after seeing* Bob's number b.

   (a) Suppose $g(a, b, c) = a + b + c$. Specify a winning strategy for Alice.

   (b) Suppose $g(a, b, c) = \max\{a + b, b + c\}$. Specify a winning strategy for Bob.

(c) Express the proposition that Alice has a winning strategy in the language of first-order predicate calculus.

(d) Express the proposition that Bob has a winning strategy.

(e) Argue that, irrespective of what function g is used, this is a "pointless game": either Alice or Bob has a winning strategy.