Discrete Structures :: CS 207 :: Autumn 2021

Problem Set 2

Released: August 13, 2021

1. Prove that $((p \to r) \wedge (r \to q)) \to (p \to q)$, by three methods:
   (a) First, prove it by expanding this expression using distributive properties and conclude that it is equivalent to True.
   (b) Secondly, prove it by analysing two cases based on the truth value of r.
   (c) Finally, prove it by analysing 8 cases based on the truth values of p, q, r.

2. **Contrapositive.** Prove each of the following by stating and proving its contrapositive.
   (a) If x and y are real numbers such that the product xy is an irrational number, then either x or y must be an irrational number.
   (b) If x and y are two integers whose product is odd, then both must be odd.
   (c) If n is a positive integer such that n leaves a remainder of 2 when divided by 3, then n is not a perfect square.
   (d) If n is a positive integer such that n leaves a remainder of 2 or 3 on division by 4, then n is not a perfect square.

3. **Proof by Contradiction.**
   (a) There are no positive integer solutions to the equation $x^2 - y^2 = 10$. (Such a problem, when an integral solution is sought for a polynomial equation, the equations is called a *Diophantine equation.*)
   (b) There is no rational solution to the equation $x^5 + x^4 + x^3 + x^2 + 1 = 0$.
   [Hint: A rational number can be written as $\frac{p}{q}$ where p, q are integers which have no common factors.]
   (c) We say that a point $P = (x, y)$ in the Cartesian plane is rational if both x and y are rational. More precisely, P is rational if $P = (x, y) \in \mathbb{Q}^2$. An equation $F(x, y) = 0$ is said to have a rational point if there exists $x_0, y_0 \in \mathbb{Q}$ such that $F(x_0, y_0) = 0$. For example, the equation $x^2 + y^2 - 1 = 0$ has rational points $(0, \pm 1)$ and $(\pm 1, 0)$. Show that the equation $x^2 + y^2 - 3 = 0$ has no rational points.
   [Hint: Prove by contradiction. It would be useful to consider whether the largest power of 3 that divides an integer is even or odd. Also, it will be useful to know what values can appear as the remainder of a perfect square when divided by 3.]
   (d) Use (c) to show that $\sqrt{3}$ is irrational.

4. **Weak Induction**. Prove by induction that the following hold for every positive integer n:
   (a) $1^2 - 2^2 + 3^2 - \cdots + (-1)^{n-1} n^2 = (-1)^{n-1} \dfrac{n(n+1)}{2}$.
   (b) if $h > -1$, then $1 + nh \leq (1 + h)^n$.
   (c) 12 divides $n^4 - n^2$.

5. **Strong induction.** An $a \times b$ chocolate bar is a rectangular piece of chocolate consisting of ab square pieces of chocolate. Your job is to break this chocolate into the ab individual square pieces. At any point during this task, you will have one or more pieces of the chocolate bar; you can pick any piece and break it into two, along a vertical or horizontal line separating the square pieces. For instance, if you start with a 2×2 bar, you can first break it vertically to get two 2×1 bars; then each of them you can break once horizontally, to end up with all 4 individual squares. In this process you made 3 breaks in all (one vertical, two horizontal).

   Show that to completely break an $a \times b$ bar into individual squares, you need exactly $ab - 1$ breaks, no matter which breaks you make.

   [Hint: Induct on the number of squares. A single break splits a piece of chocolate into two smaller pieces with the same total number of squares.]

6. **Well Ordering Principle.** Prove the Well-Ordering Principle – that every non-empty subset of $\mathbb{Z}^+$ has a minimum element – using mathematical induction.

   [Hint: Use *strong induction* to prove the contrapositive of the above statement, i.e. if a subset of $\mathbb{Z}^+$ does not have a least element, then it must be empty.]

7. Suppose that 9 bits – five ones and four zeros – are arranged around a circle in some order. Between any two equal bits you insert a 0 and between any two unequal bits you insert a 1 to produce nine new bits. Then you erase the nine original bits. Show that when you iterate this procedure, you can never get nine zeros. [Hint: Prove using the well ordering principle, or by mathematical induction.]

8. Let $a_1, a_2, ...$ be a sequence of real numbers satisfying $a_{i+j} \leq a_i + a_j$ for all positive integers i and j. Use strong induction to prove :

$$a_n \leq a_1 + \frac{a_2}{2} + \frac{a_3}{3} + \ldots + \frac{a_n}{n}.$$

[Hint: You can write $a_n \leq a_i + a_{n-i}$ for $i = 1, \ldots, n-1$. You will want to use all $n-1$ of these inequalities. Use the strong inductive hypothesis to reason about a_i or a_{n-i}. It may help to work out examples for small values of n.]

9. There are n identical cars on a circular track, at arbitrary distances from each other. All of them together have just enough petrol required for one car to complete a lap. Show, using induction, that there is a car which can complete a lap by collecting petrol from the other cars on its way around.

   [Hint: It will be helpful to prove a stronger statement, that there is a car which can complete a lap in the clockwise direction. Your proof in the induction step may have following steps:

   - Consider an arbitrary configuration of $k+1$ cars (satisfying the given condition).
   - First argue that there is a car who can reach its clockwise neighbouring car with the petrol it has. (Use proof by contradiction.)
   - Use these two cars to change the given instance of the problem into an instance with k cars.
   - Use the induction hypothesis to get some solution of the smaller instance; translate it into a solution for the original instance of the problem.]