Problem Set 7b

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1. **Matching Number.** For a graph G, its *matching number* is the size of a maximum matching in G. For each of the following graphs, compute its matching number: C_5, K_5, W_5, $K_{4,5}$.

2. How many different perfect matchings exist in each of the following graphs.

   (a) C_{2n}

   (b) $K_{n,n}$

   (c) K_{2n} *Hint: Any ordering of the $2n$ vertices as $v_1, v_2, \ldots, v_{2n-1}, v_{2n}$ can be interpreted as describing a matching, consisting of all edges of the form $\{v_{2i-1}, v_{2i}\}$. Account for the number of different orderings which result in the same matching.*

   (d) W_{2n-1}

3. A pack of $m \times n$ cards with m *values* and n *colours* consists of one card of each value and colour. The cards are arranged in an array with n rows and m columns, such that no two cards in a column have the same colour. Then, show that there exists a set of m cards, one in each column, such that they all have distinct values.

4. A *doubly stochastic matrix* is a square matrix with non-negative real numbers such that the entries in each row adds up to 1, as does the entries in each column. A *permutation matrix* is a square matrix in which each row and each column has a single entry that is *equal to* 1, and all the other entries are 0. Show that any doubly stochastic matrix M can be written as $M = p_1Q_1 + \ldots p_tQ_t$, where $Q_1, \ldots, Q_t$ are permutation matrices and $p_1, \ldots, p_t$ are positive real numbers that add up to 1.

   Hint: It is enough to show that $M = \sum_i p_iQ_i$ for some positive real numbers p_i and permutation matrices Q_i. (Argue separately that $\sum_{i=1}^{t} p_t = 1$ must hold.) A special case of this problem was solved in the lecture, when M was the adjacency matrix of a d-regular bipartite graph (scaled by $1/d$, to make it doubly stochastic). Here, use induction on the number of positive entries in M. As base case, consider an $n \times n$ doubly stochastic matrix with exactly n non-zero entries.

5. Let $G = (X, Y, E)$ be a bipartite graph such that $\deg(x) \geq 1 \ \forall\, x \in X$ and $\deg(x) \geq \deg(y) \ \forall\, (x, y) \in E$ where $x \in X$ and $y \in Y$. Show that G has a complete matching from X into Y.

6. Suppose that $G = (X, Y, E)$ is a bipartite graph. For each $S \subseteq X$, define $\text{shrinkage}(S) = \max\{0, |S| - |\Gamma(S)|\}$. Show that the the size of the largest subset of X which has a complete matching into Y is $|X| - \max_{S \subseteq X} \text{shrinkage}(S)$.

   Hint: Hall's theorem is a special case when $\text{shrinkage}(S) = 0$ for all $S \subseteq X$. To prove the above general formulation, if the largest subset of X which has a complete matching is of size $|X| - t$, consider applying Hall's theorem to a larger graph formed by adding t new vertices to Y which are all connected to every vertex in X.

7. Let $G = (X, Y, E)$ be a bipartite graph. Suppose that $S \subseteq X$, $T \subseteq Y$ such that G has a complete matching from S to Y, and also has a complete matching from T to X. Prove that there exists a matching which contains a complete matching from S to Y as well as a complete matching from T to X.

   Hint: Start with an arbitrary complete matching from T to X. Can you extend this to a matching which contains a complete matching from S to Y?

8. Given a graph $G = (V, E)$, its *line graph* (or "edge graph") $L(G)$ is defined as the graph whose vertices are the edges of G, and two such vertices are connected if they share a vertex in G. That is, $L(G) = (E, E')$, where $E' = \{\{e_1, e_2\} \mid e_1, e_2 \in E,\ e_1 \cap e_2 \neq \emptyset\}$.

   (a) Describe $L(C_n)$ and $L(K_{1,n})$. What is $L(C_3)$ and $L(K_{1,3})$?

   (b) Show that if G is a d-regular bipartite graph, then $L(G)$ has a d-colouring.

   Hint: Argue that a colouring of $L(G)$ corresponds to a partition of G into matchings.

(c) Show that the size of the largest matching in G is the same as the size of the largest independent set in $L(G)$.

(d) Given the above problem, one may wonder if algorithms for finding the size of the largest matching in a graph can be used to find the size of the largest independent sets in graphs. Unfortunately, this does not always work. (Indeed, finding the size of the largest matching is an "easy" problem, while finding the size of the largest independent set is a "hard" problem.) In particular, there are graphs which are not line graphs for any graph. Show that $K_{1,3}$ (the "claw graph") is not the line graph of any graph.

9. In a graph, if every vertex has degree at least 2, then it must have a cycle (as it has no leaves). Show that if every vertex has degree at least 3, then it must have a cycle of even length.

   Hint: Consider a maximal path.

10. **An application of Kőnig's theorem.**

    (a) Show that every bipartite graph with m edges and maximum degree d has a matching of size at least m/d.

    (b) Show that a bipartite graph (X, Y, E) with $|X| = |Y| = n$ and $|E| > n(k-1)$ should have a matching of size at least k.

    Hint: Use the previous part.

11. Give an alternate proof for Hall's Theorem, using Kőnig's theorem.

    (In class, we proved Kőnig's theorem using Hall's theorem; so this may appear to be circular reasoning. But it is possible to prove Kőnig's theorem directly, by induction.)

    Hint: If a bipartite graph $G = (X, Y, E)$ does not have a complete matching from X to Y, the largest matching has size $< |X|$, and hence, by Kőnig, G has a vertex cover C of that size. Can you show that $X - C$ must be shrinking?

12. A maximal matching can be smaller than a maximum matching. In this problem we explore how much smaller it can be.

    (a) Show that for any graph G, a maximal matching is at least half as large as a maximum matching. Prove this directly (by contradicting the maximality of a matching which is less than half the size of the maximum matching), and then repeat the proof using the connections between matchings and vertex cover.

    (b) Give examples of graphs which have a maximal matching which is exactly half the size of a maximum matching. Specifically, for any n, describe a connected graph G with $2n$ nodes which has a perfect matching and also a maximal matching of size $\lceil n/2 \rceil$.

    Hint: For $n = 2$, consider the "path graph" P_4.

    (c) What is the size of the smallest maximal matching in C_6? What about in C_n?

13. An **edge cover** is a concept closely related to a matching. An edge cover of a graph $G = (V, E)$ is a set of edges such that every vertex in the graph is part of some edge in the set. That is, $L \subseteq E$ is an edge cover if $V = \bigcup_{e \in L} e$.

    (a) What is the condition on a graph $G = (V, E)$ so that E is an edge cover?

    (b) Suppose $G = (V, E)$ is a graph which has an edge cover. Show that if L is the smallest edge cover for G, then the graph (V, L) is a forest, in which each connected component is a "star graph" – i.e., a tree in which every edge is incident on a leaf.

    (c) Suppose G and L are as above. Then describe a matching M in G such that $|M| = |V| - |L|$.

    Hint: How many connected components are there in (V, L)?

    (d) Suppose $G = (V, E)$ is a graph which has an edge cover. Also, suppose M is a matching in G. Then show that G has an edge cover L such that $|L| \leq |V| - |M|$.

    (e) Conclude from the above that if G is a graph which has an edge cover, then the size of a maximum matching and the size of a minimum edge cover add up to the number of nodes in the graph.