Financial Math

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1 Introduction

1.1 Multidimensional Ito-Formula for continuous semimartingales

:
If an n variable function F is countinously differentiable two times and $(X_k)_{k=1}^n$ are continuous semimaritngales, then for arbitrary t time:

$$F(X(t))-F(X(0)) = \sum_{k=1}^{n} \int_0^t \frac{\partial F}{\partial x_k}(X(s))\, dX_k(s) + \frac{1}{2} \sum_{i,j} \int_0^t \frac{\partial^2 F}{\partial x_i \partial x_j}(X(s))\, d[X_i, X_j](s).$$

holds true. This is an integral. We can see that if the second part wasn't there, then this would be the classic Newton-Leibniz formula.
We will see, that the second order part is the most important part in financial math, and it is absolutely needed for the no arbitrage criteria.
We should also remember, that the first part is an integral with respect to $dX_k(s)$, so it will be a local martingale. The second part is an integral with respect to a finite variation process $d[X_i, X_j](s)$. So it will be a drift term.

Proof of the theorem:
Let's fix an $[a, b]$ interval, and partition it by (t_k) -s. Take

$$F(X(b)) - F(X(a)) = \sum_k \big(F(X(t_k)) - F(X(t_{k-1}))\big)$$

telescopic decomposition.
As Newton-Leibniz's theorem says:

$$F(X(t_{k+1}))-F(X(t_k)) \approx F'(X(\tau_k))\big(X(t_{k+1})-X(t_k)\big) = \sum_{i=1}^{n} \frac{\partial F}{\partial x_i}(X(\tau_k))\big(X_i(t_{k+1})-X_i(t_k)\big),$$

where $\tau_k \in [t_k, t_{k+1}]$, which we can choose according to Lagrange's mean value theorem.

Observe that the sum obtained in this way

$$\sum_k \sum_{i=1}^{n} \frac{\partial F}{\partial x_i}\big(X(\tau_k)\big)\big(X_i(t_k) - X_i(t_{k-1})\big)$$

in general, does not converge to the corresponding stochastic integrals, since the approximation points τ_k are not taken as the left endpoints of the intervals $[t_{k-1}, t_k]$.
We could also estimate the telescopic sum in a different way. Although this has no meaning in the classical case, it is now natural to use the second-order approximation provided by the Taylor formula:

$$F'(X(t_{k-1}))\big(X(t_k) - X(t_{k-1})\big) + \frac{1}{2}F''(X(\tau_k))\big(X(t_k) - X(t_{k-1})\big)^2.$$

The second derivative forms a quadratic term, so the second derivative F'' evaluated at $X(\tau_k)$ and applied to $X(t_k) - X(t_{k-1})$ can be written as

$$\big(X(t_k) - X(t_{k-1})\big)^T H \left(X(t_k) - X(t_{k-1})\right),$$

where

$$H = \left(\frac{\partial^2 F}{\partial x_i \partial x_j}(X(\tau_k))\right)$$

is the Hessian matrix consisting of the second partial derivatives. Because of that, the second-order term is

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 F}{\partial x_i \partial x_j}(X(\tau_k))\, \Delta X_i(t_k)\, \Delta X_j(t_k).$$

According to the Taylor formula, the point τ_k appearing in the second-order term is still some intermediate point in the interval $[t_{k-1}, t_k]$, but τ_k does not appear in the first-order term, only in the second-order one. The first-order approximation tends to the corresponding stochastic integrals:

$$\int_a^b F'(X)\, dX = \sum_{i=1}^{n} \int_a^b \frac{\partial F}{\partial x_i}(X)\, dX_i.$$

Since, based on the estimation of the increments of the cross-variation,

$$\Delta X_i(t_k)\, \Delta X_j(t_k) \approx \Delta[X_i, X_j](t_k),$$

we obtain

$$\frac{\partial^2 F}{\partial x_i \partial x_j}(X(\tau_k))\, \Delta X_i(t_k)\, \Delta X_j(t_k) \approx \frac{\partial^2 F}{\partial x_i \partial x_j}(X(t_k))\, \Delta[X_i, X_j](t_k).$$

In the second-order term, three factors can be neglected, hence we obtain

$$F(b, X(b)) - F(a, X(a)) = \int_a^b \frac{\partial F}{\partial s}(s, X(s))\, ds + \int_a^b \frac{\partial F}{\partial x}(s, X(s))\, dX(s) + \frac{1}{2}\int_a^b \frac{\partial^2 F}{\partial x^2}(s, X(s))\, d[X](s).$$

the proof can be found in my 2025 June document.
Economists and physicists like to write it in the

$$df(X) = f'(X)dX + \frac{1}{2}f''(X)d[X]$$

way so it represents some kind of dynamics.

1.2 Black-Scholes, and Samuelson model

Samuelsondescribed the stock price's change:

$$dS = \mu S dt + \sigma S dW$$

Which is a semimartingale where the first part is the drift and the second is a local martingale because it has the dW part
Black and assumed that there is an option price

$$f(t, S(t)) = c(t)$$

This assumes that the stock market processes are Markov processes.
They applied the multidimensional Ito's lemma to f:

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial S}dS + \frac{1}{2}\frac{\partial^2 f}{\partial s^2}d[S] =$$

Then for the second part they applied the $dS = \mu S dt + \sigma S dW$ equality.

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial S}\mu S dt + \frac{\partial f}{\partial S}\sigma S dW + \frac{1}{2}\frac{\partial^2 f}{\partial s^2}d[S] =$$

for the 3rd part:

$$d[S] = [\mu S dt] + 2[\sigma S dW, \mu S dt] + [\sigma S dW]$$

where the 1st and the 2nd parts are 0 because of the dt part has bounded change so the first part has 0 bounded variation, the second has 0 bounded cross variation. so this means:

$$d[S] = [\sigma S dW] = \sigma^2 S^2 d[W] \overset{[W]=t}{=} \sigma^2 S^2 dt$$

inserting the equality above:

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial S}\mu S dt + \frac{\partial f}{\partial S}\sigma S dW + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial s^2}dt.$$

Let's look at the original

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial S}dS + \frac{1}{2}\frac{\partial^2 f}{\partial s^2}d[S]$$

equation!
The first plus the second part equals to $df - \frac{\partial f}{\partial S}dS$, so

$$\frac{\partial f}{\partial t}dt + \frac{1}{2}\frac{\partial^2 f}{\partial s^2}d[S] = df - \frac{\partial f}{\partial S}dS \overset{c=df}{=} c - \frac{\partial f}{\partial S}dS$$

so the right hand side can be interpreted as a portfolio consisting one c and $\frac{\partial f}{\partial S}$ pieces of S's value change.

so we can see that the left hand side is a differentiable function, so at time t it is predictable. And for this exact reason there can be no risk premium so the right hand side can be rewritten as

$$rf * (-\frac{\partial f}{\partial S})Sdt$$

we divide both sides by dt so we get a differential equation:

$$\frac{\partial f}{\partial t} + r\frac{\partial f}{\partial S}S + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial s^2} = rf$$

which is the Black-Scholes differential equation with

$$f(T, S) = \Phi(s)$$

How to solve this differential equation? The Feyman-Kac formula gives the answer:

If i have a partial differential equation:

$$\frac{\partial f}{\partial t} + \mu\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial^2 x^2} = 0$$

$$dX = \mu(t, x)dt + \sigma(t, x)dW \text{ , where } X(t) = x$$

$$E(\Phi(X(T)_{(x,t)})) = f(t, x)$$

Lets solve the heat partial differential equation!

$$\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2}{\partial x^2} = 0$$

$$f(T, x) = x^2$$

$$0dt + \sigma dW = dX$$

We know that:

$$dX = \sigma dW$$

is equivalent to

$$X(T) - X(t) = X(T) - x = \sigma\int_t^T 1dW$$

$$X(t) = x + W(T) - W(t)$$

(Remember, a parabolic differential equation can have multiple solutions: Consider the partial differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad u(0,x) = 0, \quad t \geq 0.$$

Clearly,

$$u(t,x) \equiv 0$$

is a solution of this equation. We show that the function

$$u(t,x) = \sum_{n=0}^{\infty} \varphi^{(n)}(t) \, \frac{x^{2n}}{(2n)!}$$

is also a solution, where

$$\varphi(t) = \begin{cases} \exp(-1/t^2), & t > 0, \\ 0, & t = 0. \end{cases}$$

This is the well-known C^∞ function whose every derivative vanishes at $t = 0$, i.e. $\varphi^{(n)}(0) = 0$ for all $n \in N$.

Hence $u(t,x)$ also satisfies the initial condition $u(0,x) = 0$, showing that the Cauchy problem does not have a unique solution on $t \geq 0$ in the class of smooth (C^∞) solutions.)

Let's prove feymann kac So we have to prove that: If

$$\frac{\partial f}{\partial t} + \mu \frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial^2 x^2} = 0$$

,

$$dX = \mu(t,x)dt + \sigma(t,x)dW \text{ , where } X(t) = x$$

, then

$$E(\Phi(X(T)_{(x,t)})) = f(t,x)$$

if the differential equation has exactly one solution. Using Ito's lemma

$$f(T,X) - f(t,x) = \int_t^T \frac{\partial f}{\partial t} dt + \int_t^T \frac{\partial f}{\partial x} dX + \frac{1}{2} \int_t^T \frac{\partial^2 f}{\partial x^2} d[X]$$

because inserting

$$[X] = \sigma^2 dt$$

and $dX = \mu(t,x)dt + \sigma(t,x)dW$, we get :

$$\int_t^T \frac{\partial f}{\partial t} dt + \int_t^T \frac{\partial f}{\partial x} \mu dt + \int_t^T \frac{\partial f}{\partial x} \sigma dW + \frac{1}{2} \int_t^T \frac{\partial^2 f}{\partial x^2} \sigma^2 dt = \int_t^T \left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x} \mu + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} \sigma^2\right) dt + \int_t^T \frac{\partial f}{\partial x} \sigma dW$$

, where in the dt integral, the intergrand is 0 because of the condition. So only

$$f(T,X) - f(t,x) = f \int_t^T \frac{\partial f}{\partial x} \sigma dW$$

stays. If this is a martingale then it is solved. Since the right side is a martingale with zero mean:

$$E\left[\int_t^T \frac{\partial f}{\partial x}(s, X(s))\sigma(s, X(s))dW \mid \mathcal{F}_t\right] = 0$$

Therefore:

$$E[f(T, X_T) - f(t, X_t) \mid \mathcal{F}_t] = 0$$

Which gives:

$$E[f(T, X_T) \mid \mathcal{F}_t] = f(t, X_t)$$

So With terminal condition $f(T, x) = \phi(x)$:

$$f(t, X) = E[\phi(X(T))]$$

and this is what we wanted to prove.
Let's start with geometric Brownian motion:

$$dS = \mu S dt + \sigma S dW, where$$

$$S(0) = s_0$$

Note that only those solutions are achieveable using Feymann-Kac's formula if the integral is a martingale

Dalang Morton Willinger theorem: We have times:

$$t = 0, 1, ..., T$$

, and

$$S(0), S(1), ..., S(T), \text{ where } S(T) \in R^m$$

We have the Stochastic integral:

$$\sum_{t=1}^{T} \theta(t), (S(t) - S(t-1))$$

, where $\theta(t) \in \mathcal{F}_{t-1}, S(t) \in \mathcal{F}_t$

these $\theta(t)$ processes we call predictable processes.

Let $K = \sum_{t=1}^{T} \theta(t), (S(t) - S(t-1))$ We have the free disposal condition:

$$C = K - L^0_+$$

, where L^0_+ is the set of random variables with nonnegative payoffs almost surely. It is a metric space so for its topology it means it defines convergence. L^0_+ has stochastic convergence.
The following two are equivalent:

1. $C \cap L^0_+ = 0$

2. there exists a Q measure so S(t) process is a martingale

We know, that Q is equivalent to P, and $\frac{dQ}{dP} \in L^\infty$, so it is bounded.
where, $0 = \text{Q(N)} \Leftrightarrow \text{P(N)}$
in this theorem we only change the measure, not the process.

$1 \Rightarrow 2.$: **Using Kreps-Yan theorem:** If $K \subset L^1$ set is closed. Then there exists a $z \in L^\infty$ where $< K, z > \leq 0$ holds true. If we knew that K is closed then we are done, with DMW theorem, because C is a subspace $< c, z >= 0$ According to Gyula Magyarkuti's Mértékelmélet on page 171, statement 6.1.4, there exist only one functional where The dual of L^1 is L^∞. Therefore, continuous linear functionals on the space L^1 can be represented as integrals with the help of a suitable L^∞ function. That is, to every continuous linear functional z defined on L^1 According to Gyula Magyarkuti's Mértékelmélet on page 171, statement 6.1.4, there exist only one functional which corresponds uniquely an element of L^∞ (also denoted by z) such that for any $c \in L^1$,

$$\langle c, z\rangle = \int_\Omega z \cdot c \, dP.$$

so z would be equal to $\frac{dP}{dQ}$ So

$$E_Q(c) = \int_\Omega c \, dQ = \int_\Omega c \frac{dQ}{dP} \, dP = E_P\left(c \frac{dQ}{dP}\right) = \int_\Omega z \cdot c \, dP = 0$$

Classical Kreps- Yan only states that

$$E_Q(c) \leq 0$$

, but, if we take

$$c = \pm\big[S(t) - S(t-1)\big]\theta(t),$$

where $\theta(t)$ is $\mathcal{F}_{t-1}$-measurable and bounded, then $c \in C$.

$$E_Q\big(\langle S(t) - S(t-1), \theta(t)\rangle\big) = 0.$$

If $\theta(t) = \chi_F$ for some $F \in \mathcal{F}_{t-1}$, then

$$\int_F \big(S(t) - S(t-1)\big) \, dQ = 0.$$

Since S is P-integrable and the Radon–Nikodym derivative $\frac{dQ}{dP}$ is essentially bounded, S is also Q-integrable. Hence we may split the integral, and we get

$$\int_F S(t) \, dQ = \int_F S(t-1) \, dQ, \qquad F \in \mathcal{F}_{t-1}.$$

By the definition of conditional expectation this implies

$$E_Q\big(S(t) \mid \mathcal{F}_{t-1}\big) = S(t-1),$$

so S is a martingale under Q. In particular, Q is an equivalent martingale measure for S.
We assumed no arbitrage, so we can safely say that there is no arbitrage for every time interval. Let's construct two martingale measures Q_1 for t= 0, 1. And Q_2 for t = 1, 2, ... T

$$\frac{dQ}{dP} = \frac{dQ_1}{dQ_2}\frac{dQ_2}{dP}.$$

And using backward induction we can prove that $(S(t))_{t=0}^{T}$ is a martingale.
$2 \Rightarrow 1.$: The main problem comes from the fact, that we don't know if $\theta(t)$ has expected value, and it is not neccesarily bounded. So we have to use a trick. We have to prove that.:

$$0 \leq h \leq \sum_{t=1}^{T} \big\langle S(t) - S(t-1),\, \theta(t) \big\rangle.$$

Multiply the upper inequality by $\chi(\|\theta(1)\| \leq n)$. Now we may assume that $\theta(1)$ is bounded. For each n, the function $\chi(\|\theta(1)\| \leq n)$ is $\mathcal{F}_0$-measurable, hence the new strategy θ remains predictable.

By the fact that S is a Q-martingale,

$$\begin{aligned} E_Q\big(\langle S(1) - S(0), \theta(1)\rangle\big) &= E_Q\big(E_Q\big(\langle S(1) - S(0), \theta(1)\rangle \mid \mathcal{F}_0\big)\big) \\ &= E_Q\big(\big\langle E_Q(S(1) - S(0) \mid \mathcal{F}_0),\, \theta(1)\big\rangle\big) \\ &= E_Q\big(\langle 0, \theta(1)\rangle\big) = 0. \end{aligned}$$

Note that we were allowed to get $\theta(1)$ out of the expected value, because ($\mathcal{F}_0$-measurable because of predictability and, moreover, bounded.

Consequently, we may split the sum inside the Q-expectation and obtain

$$0 \leq E_Q(h) \leq E_Q\big(\langle S(1) - S(0), \theta(1)\rangle\big) + E_Q\left(\sum_{t=2}^{T} \langle S(t) - S(t-1), \theta(t)\rangle\right),$$

where the first expectation equals 0. Hence we are left with

$$0 \leq E_Q(h) \leq E_Q\left(\sum_{t=2}^{T} \langle S(t) - S(t-1), \theta(t)\rangle\right).$$

Now multiply the main expression by $\chi(\|\theta(2)\| \leq n)$. By the dominated convergence theorem,

$$\lim_{n\to\infty} E_Q\Big(\big\langle S(1) - S(0),\, \theta(1)\,\chi(\|\theta(2)\| \leq n)\big\rangle\Big) = 0.$$

Thus there exists $n \in N$ such that

Continuing this procedure, one can show that for a suitable $n \in N$,

$$E_Q\left(h \prod_{t=1}^{T} \chi(\|\theta(t)\| \leq n)\right) \leq \varepsilon.$$

By the monotone convergence theorem,

$$E_Q(h) = \lim_{n\to\infty} E_Q\left(h \prod_{t=1}^{T} \chi(\|\theta(t)\| \leq n)\right) \leq \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, it follows that

$$E_Q(h) = 0.$$

Second theorem of fundamental asset pricing: Assume that the market defined by the asset price process $(S(t))_{t=0,1,\ldots,T}$ is free of arbitrage. Then the model is complete if and only if the martingale measure on $(\Omega, \mathcal{F}_T)$ is unique. Let

$$H_T = \lambda + \overline{\theta} \bullet S$$

then $\Pi(H_T) = \lambda$ because we will see that $\overline{\theta} \bullet S$ is free, so :

$$0 = \lambda - \Pi(H_T) + \theta \bullet S$$

So if there is such a λ, and θ. Completeness means:

$$\forall H_T \in L^0$$

H_T can be constructed by $\lambda + \overline{\theta} \bullet S$

Let's assume that the martingale measures are not unique so there are 2 martingales (Q and R) that satisfy the condition. So $Q(F) \neq R(F)$

$$H_F = \lambda + \theta \bullet S, where\ E(\theta \bullet S) = 0$$

The problem once again arises, where we are worried, of $\theta \bullet S$ being only a local martingale, because we don't know anything about θ, but according to the last theorem we should not be, because at finite time horizon, $\theta \bullet S$ can be integrated, so its' expected value is 0.

martingale measure is unique $\Rightarrow$ model is complete.

L $= \lambda + \theta \bullet S \subset L^0$
Let's assume it is not true, so there exists a $H_T \neq \lambda + \theta \bullet S$.
The key is to be able to separate H_T from $\theta \bullet S$ in L_1.

H_T is $\mathcal{F}_T$-measurable, but that alone does not imply that $H_T \in L^1(Q)$. In the proof we may assume $H_T \in L^1(Q)$ because (in the finite discrete-time model) we can replace P by an equivalent measure $P' \sim P$ under which H_T (and all $S(t)$) are integrable. Since the market remains arbitrage-free under P, there exists an equivalent martingale measure $Q \sim P$. If moreover $\frac{dQ}{dP} \in L^\infty$ (the Radon-Nikodym derivative is bounded), then integrability transfers from P to Q, hence $H_T \in L^1(Q)$.

Since $H_T \notin L$ is also integrable, there exists an element of the space L^1 that is not contained in the closed subspace L. By the Hahn-Banach separation theorem, there exists some $z \in L^\infty(\Omega, \mathcal{F}_T, Q)$ that separates the subspace L and the random variable H_T. Since L is a subspace, for the function $z \in L^\infty$ defining the separating hyperplane we have

$$\langle z, l \rangle := \int_\Omega z \cdot l \, dQ = E_Q(z \cdot l) = 0, \qquad l \in L.$$

Since $\varphi(t) = 0$ and $\lambda = 1$ is an admissible predictable strategy, it follows that

$$\langle z, 1 \rangle := \int_\Omega z \cdot 1 \, dQ = \int_\Omega z \, dQ = 0.$$

Let

$$g := 1 + \frac{z}{2\|z\|_\infty} > 0,$$

and define the measure

$$R(A) := \int_A g \, dQ.$$

The density $g = \frac{dR}{dQ}$ is bounded from above and is greater than or equal to a positive constant, hence the integrable random variables under the two measures coincide. Clearly $g > 0$, and

$$R(\Omega) = E_Q(1) + \frac{E_Q(z)}{2\|z\|_\infty} = 1,$$

therefore R is an equivalent probability measure. Since for any predictable process θ, with $\lambda = 0$ we have

$$\sum_{t=1}^{T} \langle S(t) - S(t-1), \theta(t) \rangle \in L,$$

it follows that if θ is bounded, then using $\langle z, l \rangle := \int_\Omega z \cdot l \, dQ = E_Q(z \cdot l) = 0, \qquad l \in L.$ we obtain

$$E_R\left(\sum_{t=1}^{T} \langle S(t) - S(t-1), \theta(t) \rangle \right) = E_Q\left(\sum_{t=1}^{T} \langle S(t) - S(t-1), \theta(t) \rangle \left(1 + \frac{z}{2\|z\|_\infty}\right) \right) = E_Q\left(\sum_{t=1}^{T} \langle S(t) - S(t-1),$$

Since S is a martingale under Q and θ is predictable, the expression on the right-hand side is zero for every bounded θ, hence the left-hand side is also zero. If θ is identically zero except at time $t-1$, where it equals χ_F for some $F \in \mathcal{F}_{t-1}$, then

$$E_R\big((S(t)-S(t-1))\chi_F\big) = 0,$$

which is nothing but

$$\int_F S(t)\,dR = \int_F S(t-1)\,dR,$$

that is, by the definition of conditional expectation,

$$E_R\big(S(t) \mid \mathcal{F}_{t-1}\big) = S(t-1).$$

Thus the process S is a martingale also under the measure R (as well as under Q), and consequently the martingale measure is not unique.

Theorem of lost illusions:
If the model defined on a finite and discrete time horizon contains no arbitrage and the model is complete, then the underlying probability space $(\Omega, \mathcal{F}_T, P)$ consists finitely many atoms.

Proof: Let H_T be an arbitrary $\mathcal{F}_T$-measurable financial payoff. Proceeding as shown earlier, by replacing the measure P we may assume that H_T is integrable under P. When passing to Q, the Radon–Nikodym derivative is bounded, hence H_T remains integrable under Q as well. Since completeness implies that Q is unique, it follows that every $\mathcal{F}_T$-measurable random variable H_T is integrable under the unique common measure Q (2nd law of asset pricing).

Consequently, $(\Omega, \mathcal{F}_T, Q)$ can contain only finitely many pairwise disjoint sets of positive measure. Since P and Q are equivalent, the space $(\Omega, \mathcal{F}_T, P)$ has analogous properties.

Definition: The (X, Y, R, S) vector process is called self-financing if

$$XR + YS \overset{\circ}{=} V = V(0) + X \bullet R + Y \bullet S.$$

Statement: The introduction of the new numeraire does not modify the self-financing nature of the portfolio.

Due to the semimartingale property of the processes, the integration by parts formula is applicable, according to which

$$\overline{V} \overset{\circ}{=} \frac{V}{U} = \overline{V}(0) + V_- \bullet \frac{1}{U} + \frac{1}{U_-} \bullet V + \left[V, \frac{1}{U}\right].$$

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$$\frac{1}{U_-}\bullet V=\frac{1}{U_-}\bullet(X\bullet R)+\frac{1}{U_-}\bullet(Y\bullet S)=$$
$$=\left(\frac{1}{U_-}X\right)\bullet R+\left(\frac{1}{U_-}Y\right)\bullet S=X\bullet\left(\frac{1}{U_-}\bullet R\right)+Y\bullet\left(\frac{1}{U_-}\bullet S\right).$$

According to the polarity rule

$$\left[V,\frac{1}{U}\right]=X\bullet\left[R,\frac{1}{U}\right]+Y\bullet\left[S,\frac{1}{U}\right].$$

Substituting back, by elementary rearrangement

$$\overline{V}-\overline{V}(0)=V_-\bullet\frac{1}{U}+\left(X\bullet\left(\frac{1}{U_-}\bullet R\right)+X\bullet\left[R,\frac{1}{U}\right]\right)+\left(Y\bullet\left(\frac{1}{U_-}\bullet S\right)+Y\bullet\left[S,\frac{1}{U}\right]\right).$$

$$V_-\bullet\frac{1}{U}=XR_-\bullet\frac{1}{U}+YS_-\bullet\frac{1}{U}.$$

Substituting this in

$$\overline{V}-\overline{V}(0)=\left(XR_-\bullet\frac{1}{U}+\left(X\bullet\left(\frac{1}{U_-}\bullet R\right)+X\bullet\left[R,\frac{1}{U}\right]\right)\right)+$$
$$+\left(YS_-\bullet\frac{1}{U}+Y\bullet\left(\frac{1}{U_-}\bullet S\right)+Y\bullet\left[S,\frac{1}{U}\right]\right),$$

which is precisely

$$\overline{V}-\overline{V}(0)=X\bullet\left(R_-\bullet\frac{1}{U}\right)+X\bullet\left(\frac{1}{U_-}\bullet R\right)+X\bullet\left[R,\frac{1}{U}\right]+$$
$$+Y\bullet\left(S_-\bullet\frac{1}{U}\right)+Y\bullet\left(\frac{1}{U_-}\bullet S\right)+Y\bullet\left[S,\frac{1}{U}\right].$$

Substituting into these

$$\overline{S}=\frac{S}{U}=\overline{S}(0)+S_-\bullet\frac{1}{U}+\frac{1}{U_-}\bullet S+\left[S,\frac{1}{U}\right],\tag{1}$$
$$\overline{R}=\frac{R}{U}=\overline{R}(0)+R_-\bullet\frac{1}{U}+\frac{1}{U_-}\bullet R+\left[R,\frac{1}{U}\right]\tag{2}$$

we obtain

$$\overline{V}-\overline{V}(0)=X\bullet\overline{R}+Y\bullet\overline{S},$$

so $(X,Y,\overline{R},\overline{S})$ is self-financing.

If we consider the process R as the discount factor, then $\overline{R}=R/R=1$, thus the following statement is evident:

If $R > 0$, then (X, Y, R, S) is self-financing if and only if for the discounted value process

$$\overline{V} - \overline{V}(0) = X \bullet \overline{R} + Y \bullet \overline{S} = Y \bullet \overline{S}.$$

Radon-Nikodym Process:

Let $\Lambda(t) = E^P\left(\frac{dQ}{dP}\middle|\mathcal{F}_t\right)$ be the Radon-Nikodym process.

If dQ/dP is a Radon-Nikodym derivative between Q and P on the $\mathcal{A}$ σ-algebra, then it is a Radon-Nikodym derivative on the $\mathcal{F}_t$ σ-algebra.

$$\Lambda(t) = E^P\left(\frac{dQ}{dP}\middle|\mathcal{F}_t\right)$$

Let $F \in \mathcal{F}_t$. Then:

$$Q(F) = \int_F \frac{dQ}{dP} dP = \int_F E^P\left(\frac{dQ}{dP}\middle|\mathcal{F}_t\right) dP = \int_F \Lambda(t) dP$$

and the variable $\Lambda(t)$ is $\mathcal{F}_t$-measurable.

The relationship between conditional expectations:

$$E^Q(X|\mathcal{F}_s) = E^P(X\Lambda(t)|\mathcal{F}_s)\Lambda^{-1}(s)$$

Proof idea: If $F \in \mathcal{F}_s \subseteq \mathcal{F}_t$:

$$\int_F E^Q(X|\mathcal{F}_s) dQ = \int_F X dQ = \int_F X\Lambda(t) dP = \int_F E^P(X\Lambda(t)|\mathcal{F}_s) dP = \int_F E^P(X\Lambda(t)|\mathcal{F}_s)\Lambda^{-1}(s) dQ$$

Let dQ/dP be a Radon-Nikodym derivative and $\Lambda(t) = E^P(\frac{dQ}{dP}|\mathcal{F}_t)$ be the Radon-Nikodym process.

A process L is a martingale under Q if and only if ΛL is a martingale under P.

Proof: If ΛL is a P-martingale, then:

$$E^Q(L(t)|\mathcal{F}_s) = E^P(L(t)\Lambda(t)|\mathcal{F}_s)\Lambda^{-1}(s) = L(s)\Lambda(s)\Lambda^{-1}(s) = L(s)$$

Thus, L is a Q-martingale. Conversely, if L is a Q-martingale, then:

$$\Lambda(s)L(s) = \Lambda(s)E^Q(L(t)|\mathcal{F}_s) = E^P(L(t)\Lambda(t)|\mathcal{F}_s)$$

Semimartingales Let $S = \exp(V + L)$ be a semimartingale under P. Under what condition on Λ will it become a local martingale under Q?

Since Λ is positive, it is in the form of $\Lambda = \exp(X)$.

We are looking for when $S\Lambda$ becomes a P-martingale:

$$S\Lambda = \exp(V + L)\exp(X) = \exp(V + L + X)$$

If U is a local martingale, then

$$V = \exp\left(U - \frac{1}{2}[U]\right)$$

is a local martingale.

Using Itô's formula with the function $\exp(x)$:

$$\begin{aligned} V - V(0) &= V \bullet \left(U - \frac{1}{2}[U]\right) + \frac{1}{2} V \bullet \left[U - \frac{1}{2}[U]\right] \\ &= V \bullet \left(U - \frac{1}{2}[U]\right) + \frac{1}{2} V \bullet [U] = V \bullet U \end{aligned}$$

Thus, V is an integral with respect to a local martingale, so it is itself a local martingale.

Finding the Martingale Condition Based on the above, is there a P-martingale U such that:

$$V + L + X = U - \frac{1}{2}[U]$$

and $\exp(X)$ is a P-martingale?

Since L is a P-local martingale, we look for X in the form $U = \alpha \bullet L$.

$$V + L + X = \alpha \bullet L - \frac{1}{2}\alpha^2 \bullet [L]$$

From which:

$$X = (\alpha - 1) \bullet L - \frac{1}{2}\alpha^2 \bullet [L] - V$$

Determining X The local martingale part of X is $(\alpha - 1) \bullet L$. Therefore, for Λ to be a local martingale, the condition on its bounded variation part must hold.

Specifically:

$$\frac{1}{2}[(\alpha - 1) \bullet L] = \frac{1}{2}\alpha^2 \bullet [L] - V$$

Calculating the quadratic variation:

$$\frac{1}{2}(\alpha - 1)^2 \bullet [L] = \frac{1}{2}(\alpha^2 - 2\alpha + 1) \bullet [L] = \frac{1}{2}\alpha^2 \bullet [L] - V$$

Simplifying gives:

$$\frac{1}{2}(1 - 2\alpha) \bullet [L] = V$$

Itô Processes

A process of the form $X = \alpha dt + \beta dw$, where α and β are general functions of (t, ω), is called an Itô process (assuming integrals exist).

Suppose S is a solution to the equation:

$$dS = a(t, S)dt + b(t, S)dw$$

If $S \neq 0$, then:

$$dS = \frac{a(t,S)}{S} S dt + \frac{b(t,S)}{S} S dw = \mu S dt + \sigma S dw$$

where μ and σ can depend on t and ω.

Solution of the SDE Based on Itô's formula (assuming a unique solution exists):

$$S(t) - S(0) = \exp\left(\left(\mu - \frac{1}{2}\sigma\right) \bullet t + \sigma \bullet w\right)$$

Wait, usually written as $\exp((\mu - \frac{1}{2}\sigma^2)t + \sigma w)$.

Verification using $\exp(x)$:

$$\begin{aligned} S(t) - S(0) &= S \bullet \left(\left(\mu - \frac{1}{2}\sigma\right) \bullet t + \sigma \bullet w\right) + \frac{1}{2}[\sigma \bullet w] \\ &= S \bullet (\mu \bullet t) + S \bullet (\sigma \bullet w) = \mu S \bullet t + \sigma S \bullet w \end{aligned}$$

Since:

$$\frac{1}{2}[\sigma \bullet w] = \frac{1}{2}\sigma^2 \bullet [w] = \frac{1}{2}\sigma^2 \bullet t$$

Finding α In this case $V = (\mu - \frac{1}{2}\sigma^2) \bullet t$. If the bond is R, then for the discounted process (log-drift):

$$v = (\mu - r - \frac{1}{2}\sigma^2) \bullet t$$

Also $L = \sigma \bullet w$, so $[L] = \sigma^2 \bullet t$. Using the condition $\frac{1}{2}(1 - 2\alpha) \bullet [L] = V$:

$$\begin{aligned} \frac{1}{2}(1-2\alpha)\sigma^2 \bullet t &= (\mu - r - \frac{1}{2}\sigma^2) \bullet t \\ \frac{1}{2}(1-2\alpha)\sigma^2 &= \mu - r - \frac{1}{2}\sigma^2 \\ -2\alpha\frac{\sigma^2}{2} &= \mu - r - \sigma^2 \\ \alpha &= \frac{r-\mu}{\sigma^2} + 1 \end{aligned}$$

Let $\theta = \frac{r-\mu}{\sigma}$. Then:

$$U = (\alpha - 1) \bullet (\sigma \bullet w) = \theta \bullet w$$

And the Radon-Nikodym process is:

$$\Lambda = \exp(X) = \exp\left(U - \frac{1}{2}[U]\right) = \exp\left(\theta \bullet w - \frac{1}{2}\theta^2 \bullet t\right)$$

Using the expected value of the lognormal distribution If μ, r, σ are constant:

$$1 = exp(0) = E^P(\Lambda(T)) = \exp\left(-\frac{1}{2}\theta^2 T + \frac{1}{2}\theta^2 T\right)$$

What will be the distribution of the Wiener process under $\mathbf{Q}$? Based on the formula for the expected value of transformed distributions, in the constant coefficient case:

$$\begin{aligned}
\mathbf{Q}(w(t) \in A) = \mathbf{E}^{\mathbf{Q}}(\chi_A(w(t))) = \mathbf{E}^{\mathbf{P}}(\chi_A(w(t))\Lambda(t)) = \\
= \mathbf{E}^{\mathbf{P}}\left(\chi_A(w(t)) \exp\left(\theta w(t) - \frac{1}{2}\theta^2 t\right)\right) = \\
= \frac{1}{\sqrt{2\pi t}} \int_A \exp\left(-\frac{x^2}{2t}\right) \exp\left(\theta x - \frac{1}{2}\theta^2 t\right) dx = \\
= \frac{1}{\sqrt{2\pi t}} \int_A \exp\left(-\frac{x^2 - 2\theta x t + \theta^2 t^2}{2t}\right) dx = \\
= \frac{1}{\sqrt{2\pi t}} \int_A \exp\left(-\frac{(x-\theta t)^2}{2t}\right) dx \Rightarrow N\left(\theta t, \sqrt{t}\right).
\end{aligned}$$

From which

$$\widehat{w}(t) = w(t) - \theta t$$

is a continuous process, whose expected value is zero at every time instant, its standard deviation is $\sqrt{t}$, and it is normally distributed. This implies that the process:

$$\hat{w}(t) = w(t) - \theta t$$

is a continuous process with expected value 0 and variance t (standard normal distribution) at every time point.

We must show that $\widehat{w}$ is a Wiener process under $\mathbf{Q}$. To do this, we must see that it has independent increments. Calculating with characteristic functions:

$$\mathbf{E}^{\mathbf{Q}}\left(\exp\left(itw(u) + is(w(v) - w(u))\right)\right) == \mathbf{E}^{\mathbf{P}}\left(\exp\left(itw(u) + is(w(v) - w(u))\right) \exp\left(-v\theta^2 + \theta w(v)\right)\right)$$

$$= \mathbf{E}^{\mathbf{P}}\left(\exp(itw(u)) \exp\left(-u\theta^2 + \theta w(u)\right)\right) \mathbf{E}^{\mathbf{P}}\left(\exp\left(is(w(v) - w(u))\right) \exp\left(-(v-u)\theta^2 + \theta(w(v) - w(u))\right)\right)$$

$$= \mathbf{E}^{\mathbf{Q}}\left(\exp(itw(u))\right) \mathbf{E}^{\mathbf{Q}}\left(\exp(is(w(v) - w(u))\right)$$

so they are independent.

If now $0 = t_0 < t_1 < t_2 < \cdots < t_n$ is a partition, then applying this to the variables $\Delta w(t_k)$, we get that they are independent.

But what is it good for?

Suppose we are a bank. Under the assumption of no arbitrage pricing, there is a unique measure Q which ensures, that markets are complete, so we can hedge every portfolio.

We get the fair price of a financial instrument, which has a payoff according to time T (Like S_T). Firstly we have to discount the values which determine the payoff, by r (risk free bond), and multiply by the Radon-Nikodym process .

This is great news, because we can now calculate the financial instrument price under risk neutral measure, in time 0.

So in real world if we are a bank, we may sell the option at a slightly higher price than it is worth, and if it gets bought, the bank always wins.