A new generalized optimal linear quadratic tracker with universal applications－Part 2: Discrete-time systems

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Contrastive to Part 1, Part 2 presents a new generalized optimal linear quadratic digital tracker (LQDT) with universal applications for the discrete-time (DT) systems. This includes: (i) A novel optimal LQDT design for the system with the pre-specified trajectories of the output and the control input and additionally with both the input-to-output direct-feedthrough term and the known/estimated system disturbances or compensatory signals; (ii) A new optimal filter-shaped proportional plus integral state-feedback LQDT design for non-square non-minimum phase DT systems to achieve a minimum phase-like tracking performance; (iii) A new approach for computing the control zeros of the given non-square DT systems; (iv) A new input-constrained iterative learning LQDT design for the repetitive DT systems.

Keywords: optimal linear quadratic digital tracker; frequency shaping; optimal iterative learning control; model predictive control; PID control; control zeros

1. Introduction

In Part 2, we proceed the development of a novel generalized optimal linear quadratic digital tracker (LQDT) for the discrete-time systems with universal applications to the input-constrained LQDT design, PID filter-shaped PI state-feedback LQDT for the non-minimum phase system, and optimal LQDT design for iterative learning control (ILC). Thus, the continuous-time version of Part 1 (Ebrahimzadeh, Tsai, Liao, Chung, Guo, Shieh, & Wang, 2015) can be extended to the corresponding discrete-time version in this paper. Those preliminary introductions mentioned in Part 1 and their corresponding references are omitted here. Instead, some introductions on the model predictive control (MPC), one of most popular developments in modern control engineering, and its applications to predictive PID control for non-minimum phase systems, as well as other related research works on the command tracking control of non-minimum phase systems are given in this section.

Due to the simplicity and effectiveness of the PID control structure, from the fundamental and up to the current state-of-the-art in control engineering, the PID controller-based methodologies for process of control system design and industrial applications still steadily attract great attention of numerous researchers. One of the reasons is the fact that the simple structure of the PID controller is very suitable for control of the simple dynamic model of the first or second order (Valery & Yurkevich, 2011)., which can fully describe the nature of dynamical processes. In addition, many effective tuning rules are available to tuning the parameters of the PID controller to improve the performances of the industrial dynamical processes. Control engineers are often encountered with practical systems

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exhibiting a non-minimum phase behavior, particularly for the sampled-data systems, where digital controllers are implemented for controlling analog systems (MacFariance & Karcanias, 1976; Astrom, Hagander, & Sternby, 1984; Schrader & Sain, 1989; Clarke, 1984; Shieh, Wang, & Tsay, 1983; Tsay & Shieh, 1981; Wellstead, Edmunds, Prager, & Zanker, 1979; Wellstead, Prager, & Zanker, 1979; Sirisena & Teng, 1986). Non-minimum phase systems exhibit either inverse response (undershoot) or time-delay characteristics for some considerable time. The non-minimum phase behavior degrades the performance of a PID controlled system. A number of literatures investigate this issue from a predictive control point of view (Uren & Schoor, 2011), such as the Smith predictor and the internal model control (IMC) (Katebi & Moradi, 2001; Morari & Zafiriou, 1989; Tan, Lee, & Leu, 2001; Matausek, Mićić, & Dacić, 2002), as well as the generalized predictive control (GPC) or model-based predictive control (MPC) (Johnson & Moradi, 2005; Miller, Shah, Wood, & Kwok, 1999; Moradi, Katebi, & Johnson, 2001; Sato, 2010; Tan, Huang, & Lee, 2000; Wang, 2009). These methods often utilize a plant model to predict the future output of the controlled system, which results in a control law that acts immediately to the class of step-like command input to avoid instability and sluggish control.

Basically, the derivative component of the PID controller can be regarded as a predictive controller, however, this kind of prediction is not suitable for non-minimum phase systems. Predictive PID controls of non-minimum phase systems have been surveyed by Uren and Schoor (Uren & Schoor, 2011). The method in (Uren & Schoor, 2011) shows the MPC controller (Uren & Schoor, 2011; Wang, 2009) together with a PID controller demonstrates predictive properties. An improvement in control performance of the GPC can be observed compared to the Smith predictive controller and the IMC, due to the fact that the control law of the GPC is computed via the optimization of a quadratic cost function. Theoretically, the GPC ensures the type-one servomechanism performance, so it works well for the class of step-like command input provided that the time duration of each varying step command is long enough for the controlled system response to reach the steady state. However, it is not appropriate when addressing the arbitrary command input tracking problem for minimum and/or non-minimum phase plants.

Other related works to control the non-minimum phase systems are addressed by Balas and Frost (Balas & Frost, 2011a; Balas & Frost, 2011b), where the adaptive control of the non-minimum phase modal systems using residual model filter (RMF) is presented. In the literature (Balas & Frost, 2011a; Balas & Frost, 2011b), the RMF is able to deal with the overall system with the almost strictly positive realness (ASPR) for a non-minimum phase system, so that the proposed adaptive model-reference-based tracking mechanism can be applied to the non-minimum phase continuous-time system with persistent disturbances of known form but unknown amplitude. The ASPR can be briefly described as follows. Given a system represented by a triple $(A,\ B,\ C)$ in the state-space setting, the ASPR means $CB > 0$ and the transfer function $P(s) = C(sI - A)^{-1}B$ is a minimum phase system. In (Balas & Frost, 2011a; Balas & Frost, 2011b), an assumption is restricted to the given plant, where the troublesome non-minimum phase modal subsystem has to be known and open-loop stable. Besides, the stability and/or tracking performance of the system are determined by using the Lyapunov theory, but not the optimization one. Therefore, the existing method is still not appropriate to be applied to track the general non-minimum phase plants with arbitrary command inputs.

A quasi-perfect tracking control (QPTC) of non-minimum phase systems was developed by Swevers and Schutter (Torfs, Swevers, & Schutter, 1991). Their algorithm determines a feedforward compensatory signal in two steps. The first step is to determine a feedforward signal based on the zero phase error tracking control (ZPETC) of Tomizuka (Tomizuka, 1987). The second step is to compensate the remaining tracking error with an additional feedforward signal. It also allows to predict the effect of the sampling rate and the location of the unstable zeros in the z-plane. For non-minimum phase systems, by taking the inverse of the open-loop system model to construct the feedforward signal is practically impossible, because unstable zeros become unstable poles in the inverse system model. The ZPETC algorithm proposes a substitution scheme, which is able to convert a non-minimum phase system with unstable zeros into a modified stable inverse closed-loop system mode and gives the feedforwarded signal. However, the method in (Torfs, Swevers, & Schutter, 1991) is restricted to be applied to the single-input single-output systems represented in the frequency domain. There still have a lot of room to develop effective algorithms for the universal multi-input multi-output (MIMO) systems.

This paper is organized as follows. Section 2 shows the development of the novel optimal LQDT with pre-specified measurement output and control input trajectories for the discrete-time controllable, observable, and non-degenerate system, which has an input-to-output direct-feedthrough term and known/estimated system disturbances or compensatory signals. A new optimal PID filter-shaped PI state-feedback linear quadratic digital tracker for non-square non-minimum phase discrete-time systems is proposed in Section 3. This section proposes a closed-loop output-zeroing control system for the given non-square MIMO system integrated with the two degrees of freedom optimal LQDT, so that the reliable algorithm presented by Emami-Naeini and Dooren in (Emami-Naeini & Dooren, 1982) can be used for computing the control zeros of the square closed-loop MIMO system (squaring down due to the use of the two degrees of freedom optimal LQDT for the system with an input number (m) greater than the output number (p) or integrated with a $p \times m$ transformation matrix ξ for the case of $p > m$). Section 4 presents a new iterative learning LQDT with input constraint for the repetitive discrete-time system with a direct-feedthrough term and unknown disturbances. All above applications constitute the so-called universal application of the proposed generalized optimal LQDT. Illustrative examples are demonstrated in Section 5 to show the effectiveness of the proposed design methodologies, and conclusions are finally given in Section 6.

2. A novel optimal linear quadratic digital tracker for the discrete-time system with a direct-feedthrough term and known system disturbances

A novel optimal LQDT with pre-specified measurement output and control input trajectories and its corresponding steady-state version for the discrete-time controllable, observable, and non-degenerate system with both an input-to-output direct-feedthrough term and known system disturbances are presented in this section.

Consider the controllable and observable linear discrete-time system with an input-to-output direct-feedthrough term and known/estimated system disturbances or compensatory signals $d(k)$ and $s(k)$

$$x_d(k+1) = Gx_d(k) + Hu_d(k) + d(k), \tag{1a}$$

$$y_d(k) = Cx_d(k) + Du_d(k) + s(k), \tag{1b}$$

where $G \in \Re^{n\times n}$, $H \in \Re^{n\times m}$, $C \in \Re^{p\times n}$, and $D \in \Re^{p\times m}$ are state, input, output, and direct-feedthrough matrices, respectively. $x_d(k) \in \Re^n$ is the state vector, $u_d(k) \in \Re^m$ is the control input, and $y_d(k) \in \Re^p$ is the measurable output. The design goal is to determine the optimal control sequence $u_d(0)$, $u_d(1)$, $u_d(2)$, $\cdots$, $u_d(N_f-1)$ that minimizes the linear quadratic performance index for a finite time process

$$\begin{aligned} J(x_d,u_d) = {} & \frac{1}{2}\left[y_d(N_f)-r(N_f)\right]^T S\left[y_d(N_f)-r(N_f)\right] \\ & + \frac{1}{2}\sum_{k=0}^{N_f-1}\left\{\left[y_d(k)-r(k)\right]^T Q_d\left[y_d(k)-r(k)\right] + \left[u_d(k)-u_d^*(k)\right]^T R_d\left[u_d(k)-u_d^*(k)\right]\right\}, \end{aligned} \tag{2}$$

where Q_d is a $p\times p$ positive definite or positive semi-definite real symmetric matrix, R_d is an $m\times m$ positive definite real symmetric matrix, S is a $p\times p$ positive definite or positive semi-definite real symmetric matrix, $r(k)$ is a pre-specified output trajectory, and $u_d^*(k)$ is a pre-specified input trajectory.

The initial state of the discrete-time system is at some arbitrary state $x_d(0)=x_0$. The final state $x_d(N_f)$ and output $y_d(N_f)$ may be fixed, in which case the term $\frac{1}{2}\left[y_d(N_f)-r(N_f)\right]^T S\left[y_d(N_f)-r(N_f)\right]$ is removed from the performance index (2) and instead the terminal condition $y_d(N_f)=y_f=Cx_d(N_f)+Du_d(N_f)+s(N_f)$ is imposed, where $x_d(N_f)=x_f$ is the fixed terminal state. If the final state x_f is not fixed, then the first term in (2) represents the weight of the performance measure due to the final state and output. Note that in the minimization problem, the inclusion of the term $\frac{1}{2}\left[y_d(N_f)-r(N_f)\right]^T S\left[y_d(N_f)-r(N_f)\right]$ in the performance index $J(x_d,u_d)$ implies that desire the final output $y_d(N_f)$ to be as close to the pre-specified $r(N_f)$ as possible.

The minimization problem subjected to equality constraint may be solved by adjoining the constraint (1a) to the function to be minimized by use of Lagrange multiplier (Anderson & Moore, 1989). Now, by using a set of Lagrange multipliers $\lambda(1), \lambda(2), \ldots, \lambda(N_f)$, define a new performance index J' as follows

$$\begin{aligned}
J'(x_d,u_d,\lambda) &= \frac{1}{2}\left[y_d(N_f)-r(N_f)\right]^T S\left[y_d(N_f)-r(N_f)\right] \\
&\quad+\sum_{k=0}^{N_f-1}\left\{\frac{1}{2}\left\{\left[y_d(k)-r(k)\right]^T Q_d\left[y_d(k)-r(k)\right]+\left[u_d(k)-u_d^*(k)\right]^T R_d\left[u_d(k)-u_d^*(k)\right]\right\}\right. \\
&\quad\left.+\lambda^T(k+1)\left[Gx_d(k)+Hu_d(k)+d(k)-x_d(k+1)\right]\right\} \\
&= \frac{1}{2}\left[y_d(N_f)-r(N_f)\right]^T S\left[y_d(N_f)-r(N_f)\right] \\
&\quad+\sum_{k=0}^{N_f-1}\left[\mathbb{H}\left(x_d(k),u_d(k),\lambda(k)\right)-\lambda^T(k+1)x_d(k+1)\right] \\
&= \frac{1}{2}\left[y_d(N_f)-r(N_f)\right]^T S\left[y_d(N_f)-r(N_f)\right]+\sum_{k=0}^{N_f-1}\mathbb{F}\left(x_d(k),u_d(k),\lambda(k)\right),
\end{aligned} \tag{3a}$$

where

$$\begin{aligned}
\mathbb{H} &= \frac{1}{2}\left\{\left[y_d(k)-r(k)\right]^T Q_d\left[y_d(k)-r(k)\right]+\left[u_d(k)-u_d^*(k)\right]^T R_d\left[u_d(k)-u_d^*(k)\right]\right\} \\
&\quad+\lambda^T(k+1)\left[Gx_d(k)+Hu_d(k)+d(k)\right],
\end{aligned} \tag{3b}$$

$$\mathbb{F}=\mathbb{H}-\lambda^T(k+1)x_d(k+1). \tag{3c}$$

Then, differentiating $\mathbb{F}$ with respect to each component of vectors $u_d(k)$, $x_d(k)$, and $\lambda(k)$ and setting the results equal to zero yields

$$
\begin{aligned}
\frac{\partial \mathbb{F}}{\partial u_d(k)} &= D^T Q_d \left[y_d(k) - r(k) \right] + R_d \left[u_d(k) - u_d^*(k) \right] + H^T \lambda(k+1) \\
&= D^T Q_d \left[C x_d(k) + D u_d(k) + s(k) - r(k) \right] + R_d u_d(k) - R_d u_d^*(k) \\
&\quad + H^T \lambda(k+1) \\
&= 0,\ k = 0,1,\ldots, N_f - 1,
\end{aligned} \tag{4}
$$

$$
\begin{aligned}
\frac{\partial \mathbb{F}}{\partial x_d(k)} &= C^T Q_d \left[C x_d(k) + D u_d(k) + s(k) - r(k) \right] + G^T \lambda(k+1) - \lambda(k) \\
&= 0,\ k = 0,1,\ldots, N_f - 1,
\end{aligned} \tag{5}
$$

$$
\begin{aligned}
\frac{\partial \mathbb{F}}{\partial x_d(N_f)} &= C^T S \left[y_d(N_f) - r(N_f) \right] - \lambda(N_f) \\
&= C^T S \left[C x_d(N_f) + D u_d(N_f) + s(N_f) - r(N_f) \right] - \lambda(N_f) \\
&= 0,\ k = N_f,
\end{aligned} \tag{6}
$$

and

$$
\begin{aligned}
\frac{\partial \mathbb{F}}{\partial \lambda(k)} &= G x_d(k) + H u_d(k) + d(k) - x_d(k+1) \\
&= 0,\ k = 0,1,\ldots, N_f.
\end{aligned} \tag{7}
$$

Equation (6) specifies the final value of the Lagrange multiplier. Note that the Lagrange multiplier $\lambda(k)$ is often called a covector or adjoint vector. From (4) and (5), one has

$$
u_d(k) = \bar{R}_d^{-1} R_d u_d^*(k) - \bar{R}_d^{-1} N_d^T x_d(k) + \bar{R}_d^{-1} D^T Q_d \left[r(k) - s(k) \right] - \bar{R}_d^{-1} H^T \lambda(k+1) \tag{8}
$$

and

$$
\lambda(k) = C^T Q_d \left[C x_d(k) + s(k) - r(k) \right] + G^T \lambda(k+1) + N_d u_d(k), \tag{9}
$$

where

$$
\begin{aligned}
\bar{R}_d &= R_d + D^T Q_d D, \\
N_d &= C^T Q_d D.
\end{aligned}
$$

Substituting (8) into (7) results

$$
\begin{aligned}
x_d(k+1) = G x_d(k) + H \Big\{ & \bar{R}_d^{-1} R_d u_d^*(k) - \bar{R}_d^{-1} N_d^T x_d(k) \\
& + \bar{R}_d^{-1} D^T Q_d \left[r(k) - s(k) \right] - \bar{R}_d^{-1} H^T \lambda(k+1) \Big\} + d(k),
\end{aligned} \tag{10}
$$

with the initial condition $x_d(0) = x_0$ In order to obtain the solution to the minimization problem, we need to solve (9) with the final condition and (10) with the initial condition $x_d(0) = x_0$ simultaneously. Thus, the problem here becomes a two-point boundary-value problem.

If the two-point boundary-value problem is solved, then the optimal values for the state vector $x_d(k)$ and Lagrange multiplier vector $\lambda(k)$ may be determined and the optimal control vector $u_d(k)$ may be obtained in the open-loop form. However, if we employ the Riccati transformation, the optimal control vector $u_d(k)$ can be obtained in the following closed-loop, or feedback, form:

$$
u_d(k) = -K_d(k) x_d(k) + E_d(k) r(k) + C_d(k) + C_{u_d}(k) u_d^*(k), \tag{11}
$$

where $K_d(k)$ is the $m \times n$ state-feedback gain, $E_d(k)$ is the $m \times p$ feed-forward gain, $C_d(k)$ is the $m \times 1$ feed-forward gain in terms of signals $d(k)$ and $s(k)$, and $C_{u_d}(k)$ is the $m \times m$ feedforward term. All the above-mentioned gains in (11) are to be determined later.

To obtain the above-mentioned Riccati equation, assume that $\lambda(k)$ can be written in the following form

$$\lambda(k) = P(k)x_d(k) - v(k), \tag{12}$$

where $P(k)$ is an $n \times n$ real symmetric matrix and $v(k) \in \Re^n$. Both $P(k)$ and $v(k)$ are to be determined. Substituting (1a) and (12) into (8) yields

$$\begin{aligned} u_d(k) = {} & \bar{R}_d^{-1} R_d u_d^*(k) - \bar{R}_d^{-1} N_d^T x_d(k) + \bar{R}_d^{-1} D^T Q_d \left[r(k) - s(k)\right] - \bar{R}_d^{-1} H^T \\ & \times \left[P(k+1)Gx_d(k) + P(k+1)Hu_d(k) + P(k+1)d(k) - v(k+1)\right], \end{aligned} \tag{13a}$$

which results in

$$\begin{aligned} u_d(k) = {} & -\tilde{R}_d^{-1} \bar{P}(k+1) x_d(k) + \tilde{R}_d^{-1} D^T Q_d \left[r(k) - s(k)\right] \\ & - \tilde{R}_d^{-1} H^T \left[P(k+1)d(k) - v(k+1)\right] + \tilde{R}_d^{-1} R_d u_d^*(k), \end{aligned} \tag{13b}$$

where

$$\tilde{R}_d = \bar{R}_d + H^T P(k+1) H, \tag{13c}$$

$$\bar{P}(k+1) = H^T P(k+1) G + N_d^T. \tag{13d}$$

By substituting (1a), (12), and (13b) into (9), one obtains

$$\begin{aligned} P(k)x_d(k) - v(k) = {} & C^T Q_d \left[Cx_d(k) + s(k) - r(k)\right] + G^T \left[P(k+1)x_d(k+1) \right. \\ & \left. - v(k+1)\right] + N_d u_d(k) \\ = {} & C^T Q_d \left[Cx_d(k) + s(k) - r(k)\right] + G^T \left[P(k+1)Gx_d(k) \right. \\ & \left. + P(k+1)Hu_d(k) + P(k+1)d(k) - v(k+1)\right] + N_d u_d(k) \\ = {} & \left[G^T P(k+1)G + C^T Q_d C\right] x_d(k) - C^T Q_d r(k) + C^T Q_d s(k) \\ & + G^T P(k+1)d(k) - G^T v(k+1) + \bar{P}^T(k+1) u_d(k) \\ = {} & \left[G^T P(k+1)G + C^T Q_d C - \bar{P}^T(k+1)\tilde{R}_d^{-1}\bar{P}(k+1)\right] x_d(k) \\ & + \left[\bar{P}^T(k+1)\tilde{R}_d^{-1}D^T - C^T\right] Q_d r(k) + \left[C^T - \bar{P}^T(k+1)\tilde{R}_d^{-1}D^T\right] \\ & \times Q_d s(k) + \left[G^T - \bar{P}^T(k+1)\tilde{R}_d^{-1}H^T\right] P(k+1)d(k) \\ & + \left[\bar{P}^T(k+1)\tilde{R}_d^{-1}H^T - G^T\right] v(k+1) + \bar{P}^T(k+1)\tilde{R}_d^{-1} R_d u_d^*(k). \end{aligned} \tag{14}$$

Equation (14) holds for arbitrary $x_d(k)$, which induces

$$P(k) = G^T P(k+1) G + C^T Q_d C - \bar{P}^T(k+1)\tilde{R}_d^{-1}\bar{P}(k+1) \tag{15}$$

and

$$\begin{aligned} v(k) = {} & -\left[\bar{P}^T(k+1)\tilde{R}_d^{-1}D^T - C^T\right] Q_d r(k) - \left[C^T - \bar{P}^T(k+1)\tilde{R}_d^{-1}D^T\right] \\ & \times Q_d s(k) - \left[G^T - \bar{P}^T(k+1)\tilde{R}_d^{-1}H^T\right] P(k+1)d(k) \\ & - \left[\bar{P}^T(k+1)\tilde{R}_d^{-1}H^T - G^T\right] v(k+1) - \bar{P}^T(k+1)\tilde{R}_d^{-1} R_d u_d^*(k). \end{aligned} \tag{16}$$

Equation (15) is a Riccati equation. Referring to (6) and (12) at $k = N_f$, one has

$$\begin{aligned} \lambda(N_f) & = P(N_f)x_d(N_f) - v(N_f) \\ & = C^T S\left[Cx_d(N_f) + Du_d(N_f) + s(N_f) - r(N_f)\right], \end{aligned} \tag{17}$$

$$P(N_f) = C^T S C, \tag{18}$$

and

$$v(N_f) = C^T S\left[r(N_f) - Du_d(N_f) - s(N_f)\right]. \tag{19}$$

Hence, $P(k)$ in (15) and $v(k)$ in (16) as well as $\lambda(k)$ in (9) can be solved uniquely backward from $k = N_f$ to $k = 0$.

Now, let $K_d(k+1) = \tilde{R}_d^{-1}\bar{P}(k+1)$, then the optimal control vector $u_d(k)$, given by (13b), is represented as

$$\begin{aligned} u_d(k) = &-K_d(k+1)x_d(k) + \tilde{R}_d^{-1}D^T Q_d\left[r(k) - s(k)\right] \\ &- \tilde{R}_d^{-1}H^T\left[P(k+1)d(k) - v(k+1)\right] + \tilde{R}_d^{-1}R_d u_d^*(k). \end{aligned} \tag{20}$$

Equation (20) gives the closed-loop form, or feedback form, for the optimal control vector $u_d(k)$. Note that a property of the state-feedback gain $K_d(k+1)$ is that it is almost constant, except near the end of the process at $k = N_f$.

If the final time $k = N_f$ goes to infinity, we have the infinite-horizon tracker, where we also let $P(N_f) = 0$ and $v(N_f) = 0$. Then, $P(k)$, $v(k)$, and $K_d(k)$ reach their steady-state values P, v, and K_d, respectively. Under these circumstances, one can solve for the steady-state v from (16) as

$$\begin{aligned} v = &-\left[\left(G - HK_d\right)^T - I_n\right]^{-1}\left(C - DK_d\right)^T Q_d r(k) + \left[\left(G - HK_d\right)^T - I_n\right]^{-1} \\ &\times\left(C - DK_d\right)^T Q_d s(k) + \left[\left(G - HK_d\right)^T - I_n\right]^{-1}\left(G - HK_d\right)^T Pd(k) \\ &+\left[\left(G - HK_d\right)^T - I_n\right]^{-1} K_d^T R_d u_d^*(k). \end{aligned} \tag{21}$$

Consequently, the optimal control vector $u_d(k)$, given in (20), is reduced to

$$u_d(k) = -K_d x_d(k) + E_d r(k) + C_d(k) + C_{u_d} u_d^*(k), \tag{22a}$$

where

$$K_d = \tilde{R}_d^{-1}\bar{P}, \tag{22b}$$

$$E_d = \tilde{R}_d^{-1}\left\{D^T + H^T\left[I_n - \left(G - HK_d\right)^T\right]^{-1}\left(C - DK_d\right)^T\right\}Q_d, \tag{22c}$$

$$\begin{aligned} C_d(k) = &\tilde{R}_d^{-1}\left\{H^T\left[\left(G - HK_d\right)^T - I_n\right]^{-1}\left(C - DK_d\right)^T - D^T\right\}Q_d s(k) \\ &+ \tilde{R}_d^{-1}H^T\left\{\left[\left(G - HK_d\right)^T - I_n\right]^{-1}\left(G - HK_d\right)^T - I_n\right\}Pd(k), \end{aligned} \tag{22d}$$

$$C_{u_d} = \tilde{R}_d^{-1}\left\{H^T\left[\left(G - HK_d\right)^T - I_n\right]^{-1} K_d^T + I_m\right\}R_d, \tag{22e}$$

in which

$$\bar{R}_d = R_d + D^T Q_d D, \tag{22f}$$

$$N_d = C^T Q_d D, \tag{22g}$$

$$\tilde{R}_d = \bar{R}_d + H^T PH, \tag{22h}$$

$$\bar{P} = H^T PG + N_d^T, \tag{22i}$$

and P satisfies the algebraic Riccati equation

$$
\begin{aligned}
P &= G^T P G + C^T Q_d C - \bar{P}^T \tilde{R}_d^{-1} \bar{P} \\
&= G^T P G + C^T Q_d C - \left(H^T P G + N_d^T\right)^T \left[\bar{R}_d + H^T P H\right]^{-1} \left(H^T P G + N_d^T\right).
\end{aligned} \tag{22j}
$$

It is worth to mention that the optimal LQDT for the discrete-time control system without an input-to-output direct-feedthrough term is proposed as follows

$$u_d(k) = -K_d x_d(k) + E_d r(k+1) + C_d(k) + C_{u_d} u_d^*(k). \tag{23}$$

The reason for replacing $r(k)$ by $r(k+1)$ is that the control input $u_d(k)$ is to be determined based on the design goal $y_d(k+1) \cong r(k+1)$ for $k = 0, 1, \cdots, N_f - 1$, but not $y_d(k) \cong r(k)$ for $k = 0, 1, \cdots, N_f - 1$. To implement the control law in (23), it is required to have the available current state $x_d(k)$ a prior, which implies the current output $y_d(k) = C x_d(k)$ was existed already, so it cannot be changed by the current control law in (23).

3. A new optimal PI state-feedback linear quadratic digital tracker for non-square non-minimum phase systems: PID filter-based frequency shaping approach

The objectives of the following demonstration are: (i) A closed-loop output-zeroing control system for the given non-square open-loop MIMO system integrated with the control-zero-related two degrees of freedom optimal LQDT is newly proposed in this paper, so that the reliable algorithm in (Torfs, Swevers, & Schutter, 1991) can be used for computing the control zeros (Latawiec, Banka, & Tokarzewski, 2000) of the square closed-loop MIMO system; (ii) Propose a new optimal PID filter-based frequency shaped PI state-feedback linear quadratic digital tracker for non-square non-minimum phase systems, so that the controlled system demonstrates a minimum phase-like performance.

For the high-gain discrete-time optimal linear quadratic two degrees of freedom state-feedback tracker design, as many closed-loop poles as number of open-loop control zeros are close to stable open-loop control zeros or the inverse of the non-minimum phase open-loop control zeros of the plant. The remaining poles approach origin in a manner such that they and their reflections across the unit circle have asymptotes that are evenly distributed.

With above notion in mind, a closed-loop output-zeroing control system for the given non-square MIMO system integrated with the two degrees of freedom optimal LQDT is newly proposed in this paper, so that the reliable algorithm in (Emami-Naeini & Dooren, 1982) for computing the control zeros of the square closed-loop MIMO system (squaring down due to the use of the two degrees of freedom optimal LQDT for the system with an input number greater than the output number) can be found. Syntax ‘tzero (sys)’ in MATLAB implements the algorithm in (Emami-Naeini & Dooren, 1982) for finding transmission zeros of a square and/or non-square MIMO system. However, it is worth noticing that Syntax ‘tzero (sys)’ in MATLAB belongs to one of some existing definitions of multivariable transmission zeros which fails to detect certain important zeros of non-square MIMO system which contribute to zeroing the system output.

Consider the discrete-time controllable, observable, non-degenerate non-square non-minimum phase system described by

$$x_d(k+1) = G x_d(k) + H u_d(k), \tag{24a}$$

$$y_d(k) = C x_d(k), \tag{24b}$$

where $G \in \Re^{n \times n}$, $H \in \Re^{n \times m}$, and $C \in \Re^{p \times n}$ denote the system, input, and output matrices, respectively, and $x_d(k) \in \Re^n$, $u_d(k) \in \Re^m$, and $y_d(k) \in \Re^p$ represent the state, input, and output vectors, respectively. The design procedure of the new PID filter-based frequency shaping approach is described in the following two cases.

Case 1: The number of inputs is less than the number of outputs

Step 1: Transform the non-square non-minimum phase system to a square minimum phase system.

i) Define the new output variables, being a linear combination of the original output measurements, as

$$y_\xi(k) = \xi y_d(k) = \xi\left[Cx_d(k)\right] = \tilde{C}x_d(k), \tag{25}$$

where $\xi \in \Re^{m\times p}$ is a transformation matrix, such that $(G, H, \tilde{C})$ is observable, in which $\tilde{C} = \xi C$.

Remark 1: The poles of the integrator (at the unit) must not be cancelled by zeros in the transfer function $\tilde{C}(zI_n - G)^{-1}H$, so that it permits infinite loop gains at the unit with use of integral feedback. Whenever the plant itself contains a pure integration, it may not be necessary to use the full integral feedback (Tomizuka, 1987).

ii) Randomly choose an above-mentioned $m\times p$ transformation matrix ξ to ensure that the system $(G - HK_d,\ HE_d,\ \tilde{C})$ is square minimum phase, where a control-zero-related two degrees of freedom LQDT in terms of (K_d, E_d) with the high-gain property (Ebrahimzadeh, Tsai, Liao, Chung, Guo, Shieh, & Wang, 2015; Tsai, Hsu, Tsai, Lin, Guo, & Shieh, 2014) is used.

Remark 2: If the number of inputs is equal to the number of outputs, i.e., $m = p$, then we cannot find an $m\times p$ non-singular transformation matrix ξ to ensure that the system $(G - HK_d,\ HE_d,\ \tilde{C})$ is minimum phase (Ebrahimzadeh, Tsai, Liao, Chung, Guo, Shieh, & Wang, 2015).

Step 2: Assign some extra target zeros (without open-loop pole-zero cancellation) to attract some of closed-loop poles in a closed-loop design.

A PID filter $\left(k_p + k_i\dfrac{T_s}{z-1} + k_d\dfrac{z-1}{T_s}\right)$ has a pair of zeros at specified locations and a pole at the unit. The PID filter gains $\{\ k_p,\ k_i,\ k_d\ \}$ are selected to achieve minimum phase target zeros intended to attract the closed-loop poles in a closed-loop design.

i) Append the PID filter to each element of $\left[Y_\xi(z) - R_\xi(z)\right]$. The output of the PID filter is given by

$$\begin{aligned}
Y_f(z) &= F(z)\left[Y_\xi(z) - R_\xi(z)\right] \\
&= \left(K_P + K_I\frac{T_s}{z-1} + K_D\frac{z-1}{T_s}\right)\left[Y_\xi(z) - R_\xi(z)\right] \\
&= \frac{1}{z-1}\left[(K_D/T_s)z^2 + (K_P - 2K_D/T_s)z + (K_D/T_s + K_I T_s - K_P)\right]\left[Y_\xi(z) - R_\xi(z)\right] \\
&= \frac{1}{z-1}diag\left[(k_d^{(j)}/T_s)z^2 + (k_p^{(j)} - 2k_d^{(j)}/T_s)z + (k_d^{(j)}/T_s + k_i^{(j)}T_s - k_p^{(j)});\ j = 1,\ldots,m\right]\left[Y_\xi(z) - R_\xi(z)\right].
\end{aligned} \tag{26}$$

The integrator emphasizes low frequencies, so one can select a large value of integral gain to improve the performance. The differentiator emphasizes high frequencies, thus one can select a small value of derivative gain to improve the robustness. Moreover, if the bandwidth is too low to achieve a practical design, then increase the proportional gain to tune the cross-over frequency.

ii) Choose the PID filter gains $\left\{\ k_p^{(j)},\ k_i^{(j)},\ k_d^{(j)}\ |\ j = 1 \sim m\ \right\}$ to assign target zeros. To simplify analysis, let all derivative gains $\{ k_d^{(j)} \mid j = 1 \sim m \}$ be the sampling time T_s. Thus, the numerator polynomial of the PID filter can be written as

$$z^2 + (k_p^{(j)} - 2)z + (1 + k_i^{(j)} T_s - k_p^{(j)}) = (z - z_1^{(j)})(z - z_2^{(j)}) = z^2 - (z_1^{(j)} + z_2^{(j)})z + z_1^{(j)} z_2^{(j)}. \tag{27}$$

It shows the sum and product of target zeros $\{ z_1^{(j)}, z_2^{(j)} \}$ can be expressed in terms of the PID filter gains $\{ k_p^{(j)}, k_i^{(j)}, k_d^{(j)} \}$ as $z_1^{(j)} + z_2^{(j)} = -(k_p^{(j)} - 2)$ and $z_1^{(j)} z_2^{(j)} = (1 + k_i^{(j)} T_s - k_p^{(j)})$, respectively. Designer can assign target zeros at desired locations to obtain the PID filter gains.

Step 3: Construct the augmented plant.

i) Define the integral of $\left[Y_\xi(z) - R_\xi(z) \right]$ as another state variable

$$X_\xi(z) = \frac{T_s}{z-1}\left[Y_\xi(z) - R_\xi(z) \right], \tag{28}$$

where T_s is the sampling time. By taking the inverse z-transform, one has

$$\begin{aligned} x_\xi(k+1) &= x_\xi(k) + T_s\left[y_\xi(k) - r_\xi(k) \right] \\ &= x_\xi(k) + T_s\left[\tilde{C} x_d(k) - r_\xi(k) \right] \\ &= \begin{bmatrix} T_s \tilde{C} & I_m \end{bmatrix} \begin{bmatrix} x_d(k) \\ x_\xi(k) \end{bmatrix} - T_s r_\xi(k). \end{aligned} \tag{29}$$

Hence, the output of the PID filter can be written as follows

$$Y_f(z) = K_P\left[Y_\xi(z) - R_\xi(z) \right] + K_I X_\xi(z) + \frac{K_D}{T_s}(z-1)\left[Y_\xi(z) - R_\xi(z) \right]. \tag{30}$$

Similarly, by taking the inverse z-transform, one has

$$\begin{aligned} y_f(k) &= K_P\left[y_\xi(k) - r_\xi(k) \right] + K_I x_\xi(k) + \frac{K_D}{T_s}\left\{ \left[y_\xi(k+1) - r_\xi(k+1) \right] - \left[y_\xi(k) - r_\xi(k) \right] \right\} \\ &= K_P\left[\tilde{C} x_d(k) - r_\xi(k) \right] + K_I x_\xi(k) + \frac{K_D}{T_s}\left\{ \tilde{C}\left[G x_d(k) + H u_d(k) \right] - \tilde{C} x_d(k) - r_\xi(k+1) + r_\xi(k) \right\}. \end{aligned} \tag{31}$$

ii) The state-space model of the augmented plant is given by

$$\begin{aligned} \begin{bmatrix} x_d(k+1) \\ x_\xi(k+1) \end{bmatrix} &= \begin{bmatrix} G & 0_{n\times m} \\ T_s \tilde{C} & I_m \end{bmatrix} \begin{bmatrix} x_d(k) \\ x_\xi(k) \end{bmatrix} + \begin{bmatrix} H \\ 0_{m\times m} \end{bmatrix} u_d(k) + \begin{bmatrix} 0_{n\times 1} \\ -T_s r_\xi(k) \end{bmatrix} \\ &= G_{aug} x_{aug}(k) + H_{aug} u_d(k) + d_{aug}(k), \end{aligned} \tag{32}$$

$$\begin{aligned} y_f(k) &= \begin{bmatrix} K_P \tilde{C} + \frac{K_D}{T_s}\left(\tilde{C} G - \tilde{C} \right) & K_I \end{bmatrix} \begin{bmatrix} x_d(k) \\ x_\xi(k) \end{bmatrix} + \frac{K_D}{T_s} \tilde{C} H u_d(k) \\ &\quad + \left(\frac{K_D}{T_s} - K_P \right) r_\xi(k) - \frac{K_D}{T_s} r_\xi(k+1) \\ &= C_{aug} x_{aug}(k) + D_{aug} u_d(k) + s_{aug}(k). \end{aligned} \tag{33}$$

Step 4: Formulate the performance index.

The first goal of this control problem is to minimize the performance index associated with the augmented plant with input $u_d(k)$ and output $y_f(k)$ as

$$J = \frac{1}{2}\sum_{k=0}^{N-1}\left\{\left[y_f(k) - r_\xi(k)\right]^T Q_d\left[y_f(k) - r_\xi(k)\right] + u_d^T(k) R_d u_d(k)\right\}, \tag{34a}$$

so that $y_f(k)$ well tracks the command input $r_\xi(k)$. By Parseval's theorem, the performance index can be written in the frequency domain as

$$\begin{aligned} J &= \frac{1}{2}\sum_{f=0}^{N-1}\left\{\left[Y_f(z) - R_\xi(z)\right]^* Q_d\left[Y_f(z) - R_\xi(z)\right] + U_d^*(z) R_d U_d(z)\right\} \\ &= \frac{1}{2}\sum_{f=0}^{N-1}\left\{\left[F(z)\left(Y_\xi(z) - R_\xi(z)\right) - R_\xi(z)\right]^* Q_d\left[F(z)\left(Y_\xi(z) - R_\xi(z)\right) - R_\xi(z)\right] + U_d^*(z) R_d U_d(z)\right\}, \end{aligned} \tag{34b}$$

where the superscript $*$ denotes the complex conjugate transpose and $z = e^{j\omega T_s} = e^{j2\pi f T_s}$. The second goal of this control problem is to minimize the other performance index as

$$J' = \frac{1}{2}\sum_{f=0}^{N-1}\left\{\left[Y_\xi(z) - R_\xi(z)\right]^*\left[F^*(z) Q_d F(z)\right]\left[Y_\xi(z) - R_\xi(z)\right] + U_d^*(z) R_d U_d(z)\right\}, \tag{35}$$

so that $y_\xi(k)$ well tracks the command input $r_\xi(k)$. Some notable remarks are listed in the following.

i) From another point of view, one has

$$\begin{aligned} &\left(K_P + K_I\frac{T_s}{z-1} + K_D\frac{z-1}{T_s}\right)\left[Y_\xi(z) - R_\xi(z)\right] - R_\xi(z) \\ &= \left(K_P + K_I\frac{T_s}{z-1} + K_D\frac{z-1}{T_s}\right)\left\{Y_\xi(z) - \left[I_m + \left(K_P + K_I\frac{T_s}{z-1} + K_D\frac{z-1}{T_s}\right)^{-1}\right]R_\xi(z)\right\} \\ &= \left(K_P + K_I\frac{T_s}{z-1} + K_D\frac{z-1}{T_s}\right)\left\{Y_\xi(z) - diag\left(w_{pid}^{(j)};\ j = 1, 2, \cdots, m\right)R_\xi(z)\right\}, \end{aligned}$$

where $w_{pid}^{(j)} = \dfrac{(k_d^{(j)}/T_s)z^2 + (k_p^{(j)} - 2k_d^{(j)}/T_s + 1)z + (k_d^{(j)}/T_s + k_i^{(j)}T_s - k_p^{(j)} - 1)}{(k_d^{(j)}/T_s)z^2 + (k_p^{(j)} - 2k_d^{(j)}/T_s)z + (k_d^{(j)}/T_s + k_i^{(j)}T_s - k_p^{(j)})}$,

$\rightarrow \left(K_P + K_I\dfrac{T_s}{z-1} + K_D\dfrac{z-1}{T_s}\right)\left[Y_\xi(z) - R_\xi(z)\right]$ as $k_p^{(j)} \approx k_p^{(j)} + 1$ or $k_p^{(j)} \rightarrow \infty$ for indices $j = 1, 2, \cdots, m$,

or $\dfrac{(k_d^{(j)}/T_s)z^2 + (k_p^{(j)} - 2k_d^{(j)}/T_s + 1)z + (k_d^{(j)}/T_s + k_i^{(j)}T_s - k_p^{(j)} - 1)}{(k_d^{(j)}/T_s)z^2 + (k_p^{(j)} - 2k_d^{(j)}/T_s)z + (k_d^{(j)}/T_s + k_i^{(j)}T_s - k_p^{(j)})} \rightarrow 1$ for $j = 1, 2, \cdots, m$, in general, which reveals a criterion for the selection of digital PID filters so that the performance index in (34b) approaches to the one in (35).

ii) Performance indices (34) and (35) show that $y_f(k) \rightarrow r_\xi(k)$ and $y_\xi(k) \rightarrow r_\xi(k)$, respectively, provided that $\dfrac{(k_d^{(j)}/T_s)z^2 + (k_p^{(j)} - 2k_d^{(j)}/T_s + 1)z + (k_d^{(j)}/T_s + k_i^{(j)}T_s - k_p^{(j)} - 1)}{(k_d^{(j)}/T_s)z^2 + (k_p^{(j)} - 2k_d^{(j)}/T_s)z + (k_d^{(j)}/T_s + k_i^{(j)}T_s - k_p^{(j)})} \rightarrow 1$ for $j = 1, 2, \cdots, m$, as well as the weighting function pair $\left\{F^*(z) Q_d F(z),\ R_d\right\}$ has the high-gain property in the interested frequency range as mentioned above.

Step 5: Perform the linear quadratic PI state-feedback tracker design.
Use (22) with an appropriate weighting matrix pair $Q_d \gg R_d$ to have the optimal control law

$$u_d(k) = -K_d x_{aug}(k) + E_d r_\xi(k) + C_d(k),$$

$$= -\left[K_{d1} \mid K_{d2}\right]\begin{bmatrix} x_d(k) \\ x_\xi(k) \end{bmatrix} + E_d r_\xi(k) + C_d(k), \tag{36}$$

for the augmented system model in (32)-(33), where

$$K_d = \tilde{R}_{aug}^{-1}\bar{P}_{aug},$$

$$E_d = \tilde{R}_{aug}^{-1}\left\{D_{aug}^T + H_{aug}^T\left[I_{n+m} - \left(G_{aug} - H_{aug}K_d\right)^T\right]^{-1}\left(C_{aug} - D_{aug}K_d\right)^T\right\}Q_d,$$

$$C_d(k) = \tilde{R}_{aug}^{-1}\left\{H_{aug}^T\left[\left(G_{aug} - H_{aug}K_d\right)^T - I_{n+m}\right]^{-1}\left(C_{aug} - D_{aug}K_d\right)^T - D_{aug}^T\right\}Q_d s_{aug}(k)$$
$$+ \tilde{R}_{aug}^{-1}H_{aug}^T\left\{\left[\left(G_{aug} - H_{aug}K_d\right)^T - I_{n+m}\right]^{-1}\left(G_{aug} - H_{aug}K_d\right)^T - I_{n+m}\right\}P_d d_{aug}(k),$$

in which

$$\bar{R}_{aug} = D_{aug}^T Q_d D_{aug} + R_d,$$

$$N_{aug} = C_{aug}^T Q_d D_{aug},$$

$$\tilde{R}_{aug} = \bar{R}_{aug} + H_{aug}^T P_{aug} H_{aug},$$

$$\bar{P}_{aug} = H_{aug}^T P_{aug} G_{aug} + N_{aug}^T,$$

and P_{aug} satisfies the algebraic Riccati equation

$$P_{aug} = G_{aug}^T P_{aug} G_{aug} + C_{aug}^T Q_d C_{aug} - \bar{P}_{aug}^T \tilde{R}_{aug}^{-1}\bar{P}_{aug}$$
$$= G_{aug}^T P_{aug} G_{aug} + C_{aug}^T Q_d C_{aug} - \left(H_{aug}^T P_{aug} G_{aug} + N_{aug}^T\right)^T\left[\bar{R}_{aug} + H_{aug}^T P_{aug} H_{aug}\right]^{-1}\left(H_{aug}^T P_{aug} G_{aug} + N_{aug}^T\right).$$

The architecture of the linear quadratic PI state-feedback tracker for the non-square non-minimum phase system is shown in Figure 1.

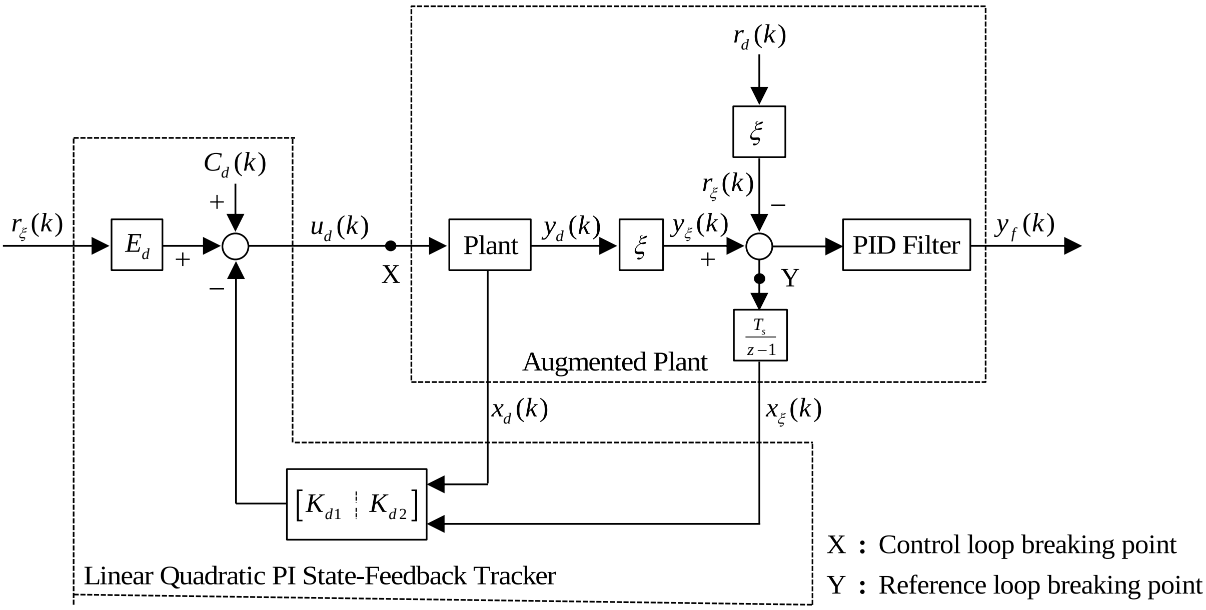


Figure 1. Optimal linear quadratic PI state-feedback tracker for the augmented plant.

Step 6: Examine the open-loop frequency response and adjust weights.
Open the control loop at point X shown in Figure 1 at the first input and examine the cross-over frequency, which we recall approximates the closed-loop control system bandwidth. If this bandwidth

is too high (low) to achieve a practical design, then increase (decrease) the first component of the diagonal matrix R_d. Repeat for all inputs. Likewise, when considering response to reference signals $r_\xi(k)$, the reference loop is opened at point Y shown in Figure 1, and the following rule applied. Increase (decrease) the *i*th diagonal element of Q_d, if the reference loop bandwidth is too low (high) for a practical design. Repeat for all *i*. In short, the bandwidth is proportional to the ratio of Q_d to R_d. Tune the ratio of Q_d to R_d to have a desired bandwidth and satisfy the low-frequency and high-frequency conditions. The control loop and reference loop can be respectively derived as follows

$$L_X(z) = \left(K_{d1} + K_{d2}\tilde{C}\frac{T_s}{z-1}\right)\left(zI_n - G\right)^{-1} H \tag{37}$$

and

$$L_Y(z) = \tilde{C}\left[zI_n - \left(G - HK_{d1}\right)\right]^{-1} HK_{d2}\frac{T_s}{z-1}. \tag{38}$$

Case 2: The number of inputs is greater than the number of outputs

Step 1: Transform the non-minimum phase system to a minimum phase system.

i) Define the new output variables, being a linear combination of the original output measurements as

$$y_\xi(k) = \xi y_d(k) = \xi\left[Cx_d(k)\right] = \tilde{C}x_d(k), \tag{39}$$

where $\xi \in \Re^{p\times p}$ is a transformation matrix, such that $(G, H, \tilde{C})$ is observable, in which $\tilde{C} = \xi C$.

ii) Randomly choose an above-mentioned $p\times p$ non-singular transformation matrix ξ to ensure that the system $(G - HK_d,\ HE_d,\ \tilde{C})$ is minimum phase, where a two degrees of freedom LQDT in terms of (K_d, E_d) with the high-gain property is used.

Step 2: Assign some extra target zeros (without open-loop pole-zero cancellation) to attract some of closed-loop poles in a closed-loop design.

i) Append the PID filter to each element of $\left[Y_\xi(z) - R_\xi(z)\right]$. The output of the PID filter is given by

$$Y_f(z) = \frac{1}{z-1}diag\left[w_{pid}^{(1)},\ w_{pid}^{(2)}, \cdots,\ w_{pid}^{(p)}\ \right]\left[Y_\xi(z) - R_\xi(z)\right], \tag{40}$$

where $w_{pid}^{(j)} = (k_d^{(j)}/T_s)z^2 + (k_p^{(j)} - 2k_d^{(j)}/T_s)z + (k_d^{(j)}/T_s + k_i^{(j)}T_s - k_p^{(j)})$ for $j = 1,\ \cdots,\ p$.

ii) Choose the PID filter gains $\{k_p^{(j)},\ k_i^{(j)},\ k_d^{(j)} \mid j = 1 \sim p\}$ to assign target zeros. To simplify analysis, let all derivative gains $\{\ k_d^{(j)}\ \mid j = 1 \sim p\ \}$ be the sampling time T_s. Thus, the sum and product of target zeros $\{\ z_1^{(j)}\ ,\ z_2^{(j)}\ \}$ can be expressed in terms of the PID filter gains $\{\ k_p^{(j)},\ k_i^{(j)},\ k_d^{(j)}\ \}$ as $z_1^{(j)} + z_2^{(j)} = -(k_p^{(j)} - 2)$ and $z_1^{(j)}z_2^{(j)} = (1 + k_i^{(j)}T_s - k_p^{(j)})$, respectively. Designer can assign target zeros at desired locations to obtain the PID filter gains.

Step 3: Construct the augmented plant.

i) Define the integral of $\left[Y_\xi(z) - R_\xi(z)\right]$ as another state variable

$$X_\xi(z) = \frac{T_s}{z-1}\left[Y_\xi(z) - R_\xi(z)\right], \tag{41}$$

where T_s is the sampling time.

ii) The state-space model of the augmented plant is given by

$$\begin{bmatrix} x_d(k+1) \\ x_\xi(k+1) \end{bmatrix} = \begin{bmatrix} G & 0_{n\times p} \\ T_s\tilde{C} & I_p \end{bmatrix} \begin{bmatrix} x_d(k) \\ x_\xi(k) \end{bmatrix} + \begin{bmatrix} H \\ 0_{p\times m} \end{bmatrix} u_d(k) + \begin{bmatrix} 0_{n\times 1} \\ -T_s r_\xi(k) \end{bmatrix} \tag{42}$$
$$= G_{aug}x_{aug}(k) + H_{aug}u_d(k) + d_{aug}(k),$$

$$y_f(k) = \begin{bmatrix} K_P\tilde{C} + \frac{K_D}{T_s}\left(\tilde{C}G - \tilde{C}\right) & K_I \end{bmatrix} \begin{bmatrix} x_d(k) \\ x_\xi(k) \end{bmatrix} + \frac{K_D}{T_s}\tilde{C}Hu_d(k)$$
$$+\left(\frac{K_D}{T_s} - K_P\right) r_\xi(k) - \frac{K_D}{T_s} r_\xi(k+1) \tag{43}$$
$$= C_{aug}x_{aug}(k) + D_{aug}u_d(k) + s_{aug}(k).$$

Step 4: Formulate the performance index.

The first goal of this control problem is to minimize the performance index associated with the augmented plant with input $u_d(k)$ and output $y_f(k)$ as

$$J = \frac{1}{2}\sum_{k=0}^{N-1}\left\{\left[y_f(k) - r_\xi(k)\right]^T Q_d \left[y_f(k) - r_\xi(k)\right] + u_d^T(k)R_d u_d(k)\right\}, \tag{44a}$$

so that $y_f(k)$ well tracks the command input $r_\xi(k)$. By Parseval's theorem, the performance index can be written in the frequency domain as

$$J = \frac{1}{2}\sum_{f=0}^{N-1}\left\{\left[F(z)\xi\left(Y_d(z) - R_d(z)\right) - \xi R_d(z)\right]^* Q_d \right.$$
$$\left. \times\left[F(z)\xi\left(Y_d(z) - R_d(z)\right) - \xi R_d(z)\right] + U_d^*(z)R_d U_d(z)\right\}, \tag{44b}$$

where the superscript * denotes the complex conjugate transpose and $z = e^{j\omega T_s} = e^{j2\pi f T_s}$. The second goal of this control problem is to minimize the other performance index as

$$J' = \frac{1}{2}\sum_{f=0}^{N-1}\left\{\left[Y_d(z) - R_d(z)\right]^*\left[\xi^* F^*(z) Q_d F(z)\xi\right]\left[Y_d(z) - R_d(z)\right] \right.$$
$$\left. + U_d^*(z)R_d U_d(z)\right\}, \tag{45}$$

so that $y_d(k)$ well tracks the command input $r_d(k)$. Similarly, one can prove $y_f(k) \to r_\xi(k)$, $y_\xi(k) \to r_\xi(k)$, and $y_d(k) \to r_d(k)$, provided that $\frac{(k_d^{(j)}/T_s)z^2 + (k_p^{(j)} - 2k_d^{(j)}/T_s + 1)z + (k_d^{(j)}/T_s + k_i^{(j)}T_s - k_p^{(j)} - 1)}{(k_d^{(j)}/T_s)z^2 + (k_p^{(j)} - 2k_d^{(j)}/T_s)z + (k_d^{(j)}/T_s + k_i^{(j)}T_s - k_p^{(j)})} \to 1$, for $j = 1, 2, \cdots, p,$ as well as the weighting function pair $\left\{\xi^* F^*(z) Q_d F(z)\xi,\ R_d\right\}$ has the high-gain property in the interested frequency range as mentioned above. Notice that $y_\xi(k) \to r_\xi(k)$ implies $y_d(k) \to r_d(k)$, since ξ is a non-singular square matrix for this case.

Step 5: Perform the linear quadratic PI state-feedback tracker design.

Repeat Step 5 in Case 1 to have the optimal control law

$$u_d(k) = -K_d x_{aug}(k) + E_d r_\xi(k) + C_d(k). \tag{46}$$

Step 6: Examine the open-loop frequency response and adjust weights.
Repeat Step 6 in Case 1. The bandwidth is proportional to the ratio of Q_d to R_d. Tune the ratio of Q_d to R_d to have a desired bandwidth and satisfy the low-frequency and high-frequency conditions. The control loop and reference loop can be respectively given as

$$L_X(z) = \left(K_{d1} + K_{d2}\tilde{C}\frac{T_s}{z-1} \right)\left(zI_n - G\right)^{-1} H \tag{47}$$

and

$$L_Y(z) = \tilde{C}\left[zI_n - \left(G - HK_{d1}\right)\right]^{-1} HK_{d2}\frac{T_s}{z-1}. \tag{48}$$

As mentioned in Remark 2, for the square plant, we cannot find a non-singular $p \times p$ transformation matrix ξ to ensure that the transformed system $(G,\ H,\ \tilde{C})$ is minimum phase, also for the transformed system $(G,\ \tilde{H},\ \tilde{C}) = (G,\ H\eta,\ \xi C)$. Due to this issue, the PID-filter shaped PI state-feedback LQDT for the square non-minimum phase system is left as a future research topic. Notice that our proposed approach still works well for the square and/or non-square minimum phase plants. Also, to the best of our knowledge, no literature solves the optimal linear quadratic tracker for the system model in (32)-(33) or (42)-(43).

4. A new iterative learning LQDT with input constraint for the repetitive system with a direct-feedthrough term and unknown disturbances

Consider the discrete-time repetitive minimum phase system with unknown deterministic disturbances

$$x_{dj}(k+1) = Gx_{dj}(k) + Hu_{dj}(k) + d(k), \quad x_{dj}(0) = x_0, \quad \text{for } j = 0,1,2,..., \tag{49a}$$

$$y_{dj}(k) = Cx_{dj}(k) + Du_{dj}(k) + s(k), \tag{49b}$$

where j is the learning iteration number, $d(k) \in \Re^n$ and $s(k) \in \Re^p$ are unknown deterministic disturbances, and $G,\ H,\ C,\ D$ are constant matrices with appropriate dimensions. Since $d(k)$ and $s(k)$ are unknown disturbances, one can imagine there exists an equivalently undetermined 'artificial' system model

$$x_a(k+1) = Gx_a(k) + Hu_a(k), \tag{50a}$$

$$y_a(k) = Cx_a(k) + Du_a(k) + s_a(k), \tag{50b}$$

where $s_a(k)$, however, denotes the 'actual' steady-state error signal (to be determined later) between the actual output of the system $y_{dj}(k)$ in (49b) and the pre-specified trajectory $r(k)$ as $j \to \infty$. That is, the whole influence to the system induced by disturbances $d(k)$ and $s(k)$ can be seen as the influence induced by $s_a(k)$. Consequently, the artificial control $u_a(k)$ can be determined from Section 2, such as

$$u_a(k) = -K_a x_a(k) + E_a r(k) + C_a(k) + C_{u_a} u_a^*(k), \tag{51a}$$

where

$$K_a = \tilde{R}_a^{-1}\bar{P}_a, \tag{51b}$$

$$E_a = \tilde{R}_a^{-1}\left\{ D^T + H^T\left[I_n - \left(G - HK_a\right)^T\right]^{-1}\left(C - DK_a\right)^T \right\} Q_a, \tag{51c}$$

$$C_a(k) = \tilde{R}_a^{-1}\left\{H^T\left[\left(G-HK_a\right)^T - I_n\right]^{-1}\left(C-DK_a\right)^T - D^T\right\}Q_a s_a(k), \tag{51d}$$

$$C_{u_a} = \tilde{R}_a^{-1}\left\{H^T\left[\left(G-HK_a\right)^T - I_n\right]^{-1}K_a^T + I_m\right\}R_a, \tag{51e}$$

in which

$$\bar{R}_a = R_a + D^T Q_a D, \tag{51f}$$

$$N_a = C^T Q_a D, \tag{51g}$$

$$\tilde{R}_a = \bar{R}_a + H^T P_a H, \tag{51h}$$

$$\bar{P}_a = H^T P_a G + N_a^T, \tag{51i}$$

and P_a satisfies the algebraic Riccati equation

$$\begin{aligned} P_a &= G^T P_a G + C^T Q_a C - \bar{P}_a^T \tilde{R}_a^{-1}\bar{P}_a \\ &= G^T P_a G + C^T Q_a C - \left(H^T P_a G + N_a^T\right)^T\left[\bar{R}_a + H^T P_a H\right]^{-1}\left(H^T P_a G + N_a^T\right), \end{aligned} \tag{51j}$$

for the infinite-horizon tracking problem, and $s_a(k)$ is to be determined in the following.

To determine the actual steady-state error signal $s_a(k)$, one can modify (21) as

$$\begin{aligned} v_j(k) &= -\left[\left(G-HK_a\right)^T - I_n\right]^{-1}\left(C-DK_a\right)^T Q_a\left[r(k) - s_{aj}(k)\right] \\ &\quad + \left[\left(G-HK_a\right)^T - I_n\right]^{-1} K_a^T R_a u_a^*(k) \\ &= -\left[\left(G-HK_a\right)^T - I_n\right]^{-1}\left(C-DK_a\right)^T Q_a\left[r(k) - \sum_{i=0}^{j} e_{i-1}(k)\right] \\ &\quad + \left[\left(G-HK_a\right)^T - I_n\right]^{-1} K_a^T R_a u_a^*(k), \text{ for } j = 0,1,2,\ldots, \end{aligned} \tag{52}$$

for the repetitive system (49), where

$$s_{aj}(k) = \sum_{i=0}^{j} e_{i-1}(k) \tag{53}$$

and

$$e_{i-1}(k) = y_{d(i-1)}(k) - r(k), \tag{54}$$

for $e_{-1}(k) = 0$. For simplicity, consider the infinite-horizon tracker problem to induce the optimal error compensation iterative learning digital tracker (OECILDT) as

$$u_{dj}(k) = -K_d x_{dj}(k) + E_d r(k) + C_{aj}(k) + C_{u_d} u_d^*(k), \tag{55a}$$

where

$$K_d = \tilde{R}_d^{-1}\bar{P}_d, \tag{55b}$$

$$E_d = \tilde{R}_d^{-1}\left\{D^T + H^T\left[I_n - \left(G-HK_d\right)^T\right]^{-1}\left(C-DK_d\right)^T\right\}Q_d, \tag{55c}$$

$$C_{aj}(k) = \tilde{R}_d^{-1}\left\{H^T\left[\left(G-HK_d\right)^T - I_n\right]^{-1}\left(C-DK_d\right)^T - D^T\right\}Q_d s_{aj}(k), \tag{55d}$$

$$C_{u_d} = \tilde{R}_d^{-1}\left\{H^T\left[\left(G-HK_d\right)^T - I_n\right]^{-1}K_d^T + I_m\right\}R_d, \tag{55e}$$

in which

$$\bar{R}_d = R_d + D^T Q_d D, \tag{55f}$$

$$N_d = C^T Q_d D, \tag{55g}$$

$$\tilde{R}_d = \bar{R}_d + H^T P_d H, \tag{55h}$$

$$\bar{P}_d = H^T P_d G + N_d^T, \tag{55i}$$

$$\begin{aligned} s_{aj}(k) &= \sum_{i=0}^{j} e_{i-1}(k) \\ &= \sum_{i=0}^{j} \left[y_{d(i-1)}(k) - r(k) \right], \end{aligned} \tag{55j}$$

and P_d satisfies the algebraic Riccati equation

$$\begin{aligned} P_d &= G^T P_d G + C^T Q_d C - \bar{P}_d^T \tilde{R}_d^{-1} \bar{P}_d \\ &= G^T P_d G + C^T Q_d C - \left(H^T P_d G + N_d^T \right)^T \left[\bar{R}_d + H^T P_d H \right]^{-1} \left(H^T P_d G + N_d^T \right). \end{aligned} \tag{55k}$$

The framework of the proposed OECILDT is demonstrated in Figure 2, where the transformation matrix form $s_{aj}(k)$ to $C_{aj}(k)$ is given as follows

$$Z_{s_{aj}}^{C_{aj}} = \tilde{R}_d^{-1} \left\{ H^T \left[\left(G - HK_d \right)^T - I_n \right]^{-1} \left(C - DK_d \right)^T - D^T \right\} Q_d .$$

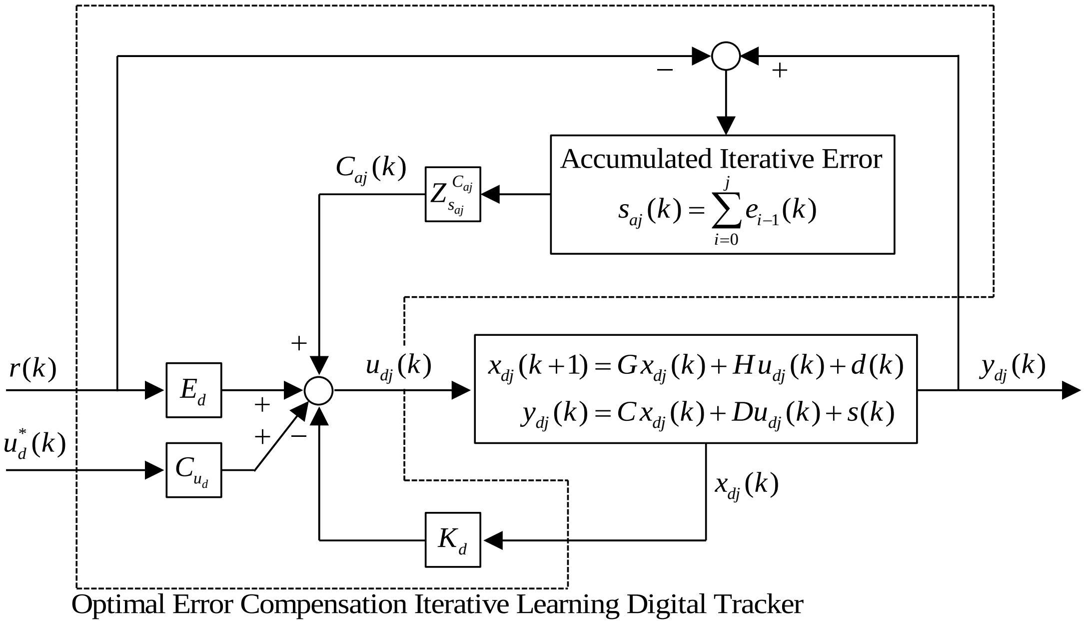


Figure 2. Optimal error compensation iterative learning digital tracker for the repetitive minimum phase system.

Some remarkable observations are given as follows. For $j = 0,\ 1,\ 2, \ldots,$ one has

$$u_{d0}(k) = -K_d x_{d0}(k) + E_d r(k) + C_{u_d} u_d^*(k),$$

$$u_{d1}(k) = -K_d x_{d1}(k) + E_d r(k) + Z_{s_{aj}}^{C_{aj}} e_0(k) + C_{u_d} u_d^*(k),$$

$$u_{d1}(k) - u_{d0}(k) = -K_d \left[x_{d1}(k) - x_{d0}(k) \right] + Z_{s_{aj}}^{C_{aj}} e_0(k),$$

$$u_{d2}(k) = -K_d x_{d2}(k) + E_d r(k) + Z_{s_{aj}}^{C_{aj}} \left[e_0(k) + e_1(k) \right] + C_{u_d} u_d^*(k),$$

$$u_{d2}(k) - u_{d1}(k) = -K_d \left[x_{d2}(k) - x_{d1}(k) \right] + Z_{s_{aj}}^{C_{aj}} e_1(k),$$

$$\vdots$$

$$u_{dj}(k) = -K_d x_{dj}(k) + E_d r(k) + Z_{s_{aj}}^{C_{aj}} \sum_{i=0}^{j} e_{i-1}(k) + C_{u_d} u_d^*(k),$$

$$u_{dj}(k) - u_{d(j-1)}(k) = -K_d \left[x_{dj}(k) - x_{d(j-1)}(k) \right] + Z_{s_{aj}}^{C_{aj}} e_{j-1}(k),$$

which implies

$$\begin{aligned} u_{dj}(k) = u_{d(j-1)}(k) &- \left[R_d + D^T Q_d D + H^T P(k) H \right]^{-1} \left[H^T P(k) G + D^T Q_d C \right] \left[x_{dj}(k) - x_{d(j-1)}(k) \right] \\ &+ \left[R_d + D^T Q_d D + H^T P(k) H \right]^{-1} \left\{ H^T \left[\left(G - HK_d \right)^T - I_n \right]^{-1} \left(C - DK_d \right)^T - D^T \right\} Q_d e_{j-1}(k). \end{aligned} \tag{56}$$

To compare our proposed optimal error compensation ILC for the direct-feedthrough term-free case with the optimal ILC for the discrete-time minimum phase system presented in (Amann, Owens, & Rogers, 1996), we briefly summarize the optimal ILC presented in (Amann, Owens, & Rogers, 1996) as follows.

Consider the same repetitive minimum phase system given in (49). It is desired to determine the optimal ILC

$$u_j(k) = u_{j-1}(k) + \Delta u_j(k) \tag{57}$$

to minimize the quadratic cost function

$$J_j = \frac{1}{2} \sum_{k=0}^{N_f - 1} \left\{ e_j^T(k) Q_d e_j(k) + \left[\Delta u_j(k) \right]^T R_d \left[\Delta u_j(k) \right] \right\}, \tag{58}$$

where

$$e_j(k) = y_j(k) - r(k). \tag{59}$$

Physical interpretation of the given cost function is that we wish to closely track the system output to the desired output trajectory without many changes in input of the system.

Finally, the optimal input of the system at iteration j is obtained as

$$\begin{aligned} u_j(k) = u_{j-1}(k) &- \left[R_d + H^T P(k) H \right]^{-1} H^T P(k) G \left[x_j(k) - x_{j-1}(k) \right] \\ &+ R_d^{-1} H^T v_j(k), \end{aligned} \tag{60}$$

where $P(k)$ and $v_j(k)$ satisfy the following equations, respectively,

$$\begin{aligned} P(k) = G^T P(k+1) G + C^T Q_d C &- \left[H^T P(k+1) G \right]^T \left[R_d + H^T P(k+1) H \right]^{-1} \\ &\times \left[H^T P(k+1) G \right] \end{aligned} \tag{61}$$

and

$$v_j(k) = \left[I_n + P(k) H R_d^{-1} H^T \right]^{-1} \left[G^T v_j(k+1) + C^T Q_d e_j(k+1) \right]. \tag{62}$$

Substituting (62) into (56) and setting $D = 0_{p \times m}$ shows

$$\begin{aligned} u_{dj}(k) = u_{d(j-1)}(k) &- \left[R_d + H^T P(k) H \right]^{-1} H^T P(k) G \left[x_{dj}(k) - x_{d(j-1)}(k) \right] \\ &+ R_d^{-1} H^T v_j(k), \end{aligned} \tag{63}$$

which is identical to (60). This implies our proposed ILC is essentially identical to the optimal ILC (Amann, Owens, & Rogers, 1996) for the minimum phase plant, although both formulations are not same for this case.

To implement the optimal ILC presented in (Amann, Owens, & Rogers, 1996), the control input of the initial learning epoch is generally assumed to be

$$u_{d0}(k) = 0, \text{ for } k \text{ in } \left[0,\ N_f - 1\right]. \tag{64}$$

However, it may not be acceptable for some practical systems. Nevertheless, the initialization of the control in the first learning epoch of our approach is determined by

$$u_{d0}(k) = -K_d x_{d0}(k) + E_d r(k).$$

Experience shows that by this initialization, it significantly reduces the learning epochs for the pre-specified tracking performance (Chen, Tsai, Liao, Guo, Ho, Shaw, & Shieh, 2014; Tsai, Dua, Huanga, Guo, Shieh, & Chen, 2011).

Notice that whenever the given non-square plant is non-minimum phase, integrating with the presented approach in Sec.3 is suggested for the design goal. It is worth to mention that the direct-feedthrough term-free and disturbance-free optimal ILC presented in (Amann, Owens, & Rogers, 1996) for the discrete-time minimum phase system without input constraint has now been extended to the case for the (non-minimum phase non-square) system with a direct-feedthrough term, unknown disturbances, and input constraint, as well as with a pre-specified control input if it is available.

5. Illustrative examples

In this section, some numerical simulations are given to illustrate some applications for the proposed generalized optimal digital tracker.

5.1. A new optimal PI state-feedback linear quadratic digital tracker for non-square non-minimum phase systems: PID-based frequency shaping approach

Consider the continuous-time, non-square, and non-minimum phase system described by

$$x_d(k+1) = \begin{bmatrix} 1.0660 & 0.0171 & -0.0143 & 0.0196 & -0.0361 & 0.0163 \\ 0.0075 & 1.0897 & 0.0211 & 0.1202 & 0.0114 & -0.0360 \\ 0.0574 & 0.1456 & 1.0249 & 0.0081 & -0.0076 & 0.1479 \\ -0.0064 & -0.0167 & 0.0575 & 0.9641 & 0.0333 & -0.0647 \\ 0.0017 & 0.1253 & -0.0681 & -0.0267 & 1.0889 & 0.0366 \\ -0.0323 & 0.0505 & -0.0247 & -0.0247 & 0.0581 & 1.0405 \end{bmatrix} x_d(k)$$

$$+ \begin{bmatrix} -0.1098 & -0.0805 & 0.0298 \\ 0.0342 & 0.0707 & 0.0270 \\ 0.0028 & 0.0037 & 0.0827 \\ 0.0500 & -0.0400 & -0.0825 \\ -0.0123 & -0.0100 & -0.0103 \\ 0.0745 & 0.0749 & 0.0360 \end{bmatrix} u_d(k),\ x_d(0) = 0_{6\times 1},$$

$$y_d(k) = \begin{bmatrix} 2.2397 & -2.8967 & -0.9704 & -2.6084 & -0.9583 & -1.8260 \\ -0.2340 & 3.1711 & -0.9539 & 0.3118 & 0.2554 & 2.8133 \end{bmatrix} x_d(k),$$

with the sampling time $T_s = 0.03$ sec. It is desired to determine an optimal LQDT so that the controlled system demonstrates a good tracking performance for the desired reference signal. The desired output trajectory is given by $r_\xi(k)=\left[r_{\xi,1}(k) \quad r_{\xi,2}(k)\right]^T$, where

$$r_{\xi,1}(k)=\begin{cases}\cos(2\pi k) & ,\ 0\le k<1 \text{ sec}\\ 0.5\ k^2(1-k) & ,\ 1\le k<2 \text{ sec}\\ 0.5\ \cos(4\pi k)+1 & ,\ 2\le k\le 3 \text{ sec}\end{cases} \text{ and } r_{\xi,2}(k)=\begin{cases}1.2\ k^2(1-k) & ,\ 0\le k<1 \text{ sec}\\ \cos(2\pi k) & ,\ 1\le k<2 \text{ sec}\\ 0.2\ \sin(4\pi k)-0.5 & ,\ 2\le k\le 3 \text{ sec}\end{cases}.$$

Poles and finite control zeros of the original system are $\{0.9699\pm j0.071,\ 1.528,\ 1.0255,\ 1.078\pm j0.051\}$ and $\{2.091,\ 0.855,\ 0.9258,\ 0.997\}$, respectively. It shows that the system has a control zero outside the unit circle; however, the syntax 'tzero (sys)' in MATLAB fails to detect any finite control zeros of the given non-square system.

Step 1: Transform the non-minimum phase system to a minimum phase system.
Let the selected 2×2 transformation matrix be $\xi=\begin{bmatrix}-16.5242 & -104.5251\\ 23.7903 & -8.7017\end{bmatrix}$ to ensure that the system $(G-HK_d,\ HE_d,\ \tilde{C})$ is minimum phase, where a two degrees of freedom LQDT in terms of (K_d,E_d) is used for $Q_d=10^8 I_2$ and $R_d=I_3$. The resultant minimum phase finite control zeros are $\{0.8547,\ 0.8806,\ 0.9259,\ 0.9970\}$.

Step 2: Assign target zeros to attract the closed-loop poles in a closed-loop design.
Let extra target zeros be $\{\ 0.3,\ 0.5,\ 0.2\pm j0.2\ \}$, then one can select the PID filter gains as

$$K_p=\begin{bmatrix}1.2 & 0\\ 0 & 1.6\end{bmatrix}, K_i=\frac{1}{T_s}\times\begin{bmatrix}0.35 & 0\\ 0 & 0.68\end{bmatrix}, \text{ and } K_d=\begin{bmatrix}T_s & 0\\ 0 & T_s\end{bmatrix}.$$

The singular value plot of the augmented system $(G_{aug},H_{aug},C_{aug},D_{aug})$ are shown in Figure 3.

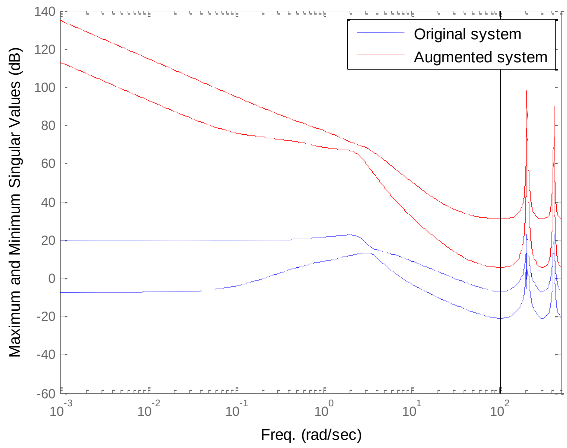


Figure 3**.** Singular value plot of the augmented plant.

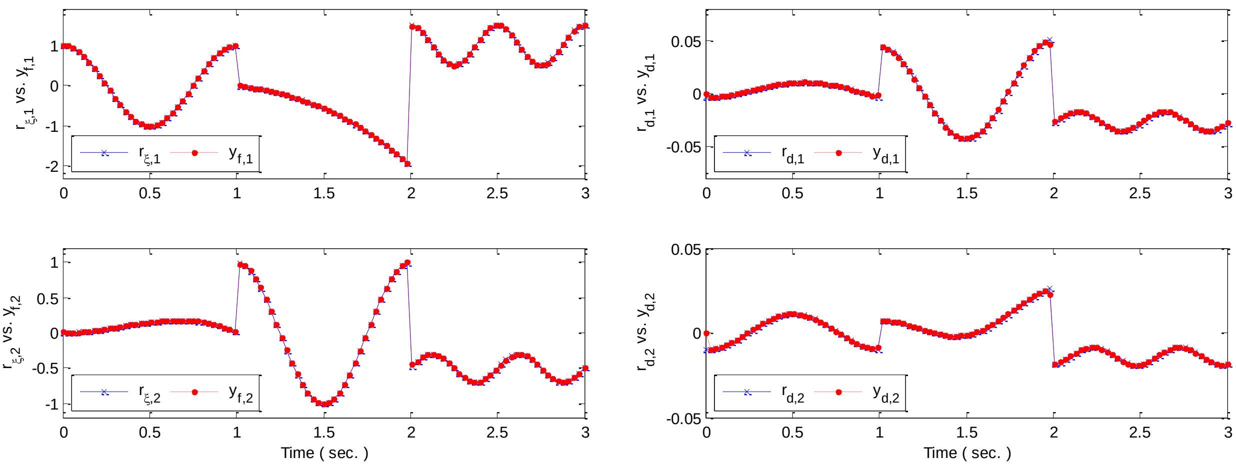
Step 3: Perform the linear quadratic PI state-feedback tracker design.
In the LQDT design, choose an appropriate weighting matrix pair $\{\ Q_d,\ R_d\ \}=\{\ 10^6 I_2,\ I_3\ \}$ to have the optimal control law

$$u_d(k)=-K_d\, x_{aug}(k)+E_d\, r_\xi(k)+C_d(k),$$

where the PI state-feedback and feed-forward gains are

$$K_d = [K_{d1} \,\vdots\, K_{d2}]$$
$$= \left[\begin{array}{cccccc:cc} -1.8957 & -3.2801 & -0.2916 & -1.9833 & 0.1236 & 0.2572 & 0.0253 & -0.0760 \\ 2.1527 & 3.7267 & 0.3313 & 2.2533 & -0.1401 & -0.2914 & -0.0287 & 0.0862 \\ -3.8830 & -6.7246 & -0.5997 & -4.0680 & 0.2523 & 0.5268 & 0.0517 & -0.1550 \end{array}\right],$$
$$E_d = \begin{bmatrix} 21.9119 & -33.2884 \\ -24.8948 & 37.7636 \\ 44.8140 & -67.8524 \end{bmatrix}.$$

The tracking performances of the closed-loop system are shown in Figure 4. It shows that $y_f(k) \to r_\xi(k)$ and $y_d(k) \to r_d(k)$ as expected.

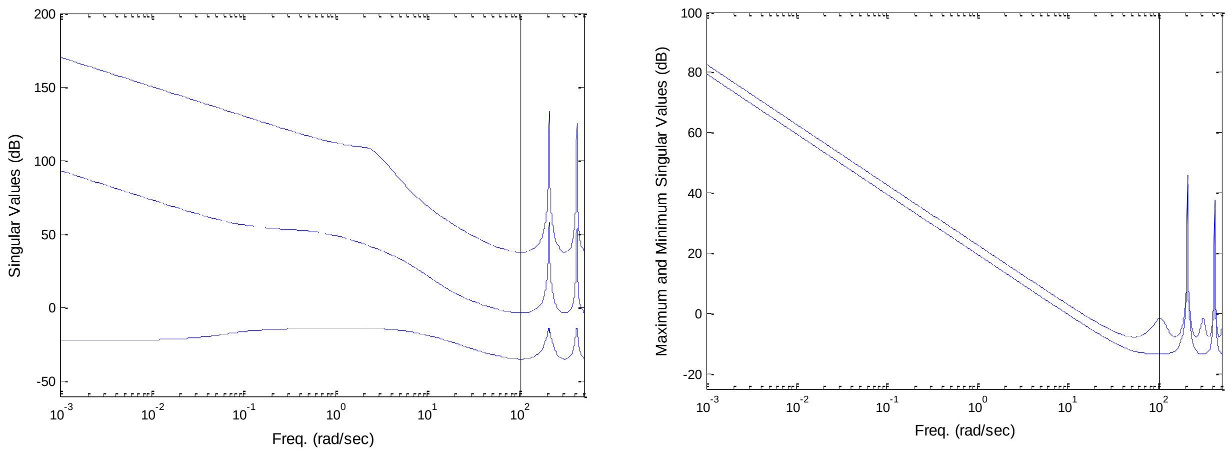


(a) (b)

Figure 4. Time responses of the controlled non-minimum phase system via the proposed approach: (a) $r_\xi(k)$ vs. $y_f(k)$, (b) $r_d(k)$ vs. $y_d(k)$.

Step 4: Examine the open-loop frequency response.

Open-loop frequency responses for $Q_d = 10^6 I_2$ and $R_d = I_3$ are shown in Figure 5. The control loop shows that the closed-loop system has good external constant disturbance rejection if the external constant disturbance occurs at the first and second control inputs. Moreover, the bandwidth of the reference loop is very low so the closed-loop system has good sensor noise rejection.



(a) (b)

Figure 5. Open-loop frequency responses: (a) control loop, (b) reference loop.

For comparison, the predictive PID control in (Uren & Schoor, 2011) is used for our simulation study. In this example, the control and prediction horizons are selected to be $(N_c, N_p) = (3, 20)$ used in (Uren & Schoor, 2011). Then, the predicted output is given by

$$\mathbf{Y} = \mathbf{F}\mathbf{X}(k) + \mathbf{\Phi}\Delta\mathbf{U}. \qquad \big((70) \text{ in } (\text{Uren \& Schoor, 2011})\big)$$

By selecting the weight $r_\omega = 10^{-6}$, the optimal control law is given by

$$\Delta\mathbf{U} = \mathbf{K}^T \mathbf{e}(k). \qquad \big((93) \text{ in } (\text{Uren \& Schoor, 2011})\big)$$

Figure 6 shows the tracking performances of the controlled system $(G, H, \tilde{C})$ for the arbitrary command input via the proposed PID filter-based frequency shaping approach and the predictive PID control (Uren & Schoor, 2011), respectively. It shows that our proposed approach outperforms the predictive PID control for the time-varying reference trajectory with drastic variations. Our other simulation (precisely same as the one demonstrated in (Uren & Schoor, 2011), omitted at here due to a consideration of the paper length) shows the predictive PID control (Uren & Schoor, 2011) works well for the class of step-like command input. Besides, based on our other pre-study work, the strength of the predictive PID control is not limited to current performance, however, it will be regarded as a future research work.

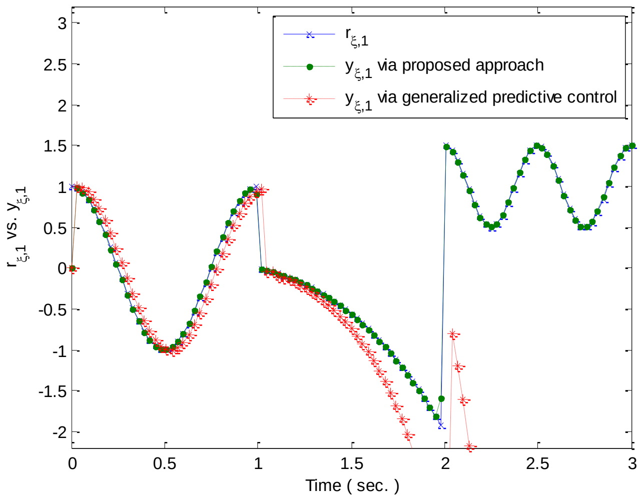


Figure 6. Tracking performances of the controlled system $(G, H, \tilde{C})$ via GPC (Uren & Schoor, 2011) and the proposed approach (shown by parts).

5.2. A new optimal LQDT with input constraint for the repetitive system with a direct-feedthrough term and unknown disturbances

Consider the repetitive minimum phase system in (49) with unknown deterministic disturbances $d(k)$ and $s(k)$, where

$$G = \begin{bmatrix} 0.9048 & 0.0000 & 0.0000 & -0.0464 \\ 0.0000 & 0.7512 & -0.0238 & -0.0476 \\ 0.0000 & 0.0000 & 0.9112 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & -0.8712 \end{bmatrix}, \quad H = \begin{bmatrix} 0.0505 & -0.3196 & -1.3966 \\ -1.4935 & -0.5775 & -1.1478 \\ 1.4286 & -0.1346 & 0.1959 \\ -0.2338 & -1.4708 & -0.5057 \end{bmatrix},$$

$$C=\begin{bmatrix} 1.2061 & 0.9635 & -1.0511 & 0.2557 \\ -1.1513 & -0.3076 & -0.7540 & 0.0205 \end{bmatrix},\quad D=\begin{bmatrix} 0.5203 & 10.1414 & -18.5353 \\ 8.7760 & -0.8435 & -10.9682 \end{bmatrix},$$
$$x_0=\begin{bmatrix} 0.1 & -0.2 & 0.5 & -1.0 \end{bmatrix}^T.$$

The deterministic system disturbance $d(k)=\begin{bmatrix} d_1(k) & d_2(k) & d_3(k) & d_4(k) \end{bmatrix}^T$ and measurement disturbance $s(k)=\begin{bmatrix} s_1(k) & s_2(k) \end{bmatrix}^T$ are created by $d_1(k)=-0.2$, $d_2(k)=1$, $d_3(k)=0.5\sin(k)$, $d_4(k)=\begin{cases} 0.3 & 0.0\le k<1.5\,\text{sec} \\ 0.0, & 1.5\le k<2.5\,\text{sec} \\ 0.5, & 2.5\le k\le 3.0\,\text{sec} \end{cases}$, $s_1(k)=0.3\sin(5k)$, and $s_2(k)=0.7\sin(3k)$. The sampling time is $T_s=0.1$ sec.

Case 1. OECILC without input-constraint for the repetitive system with a direct-feedthrough term and unknown deterministic system and measurement disturbances

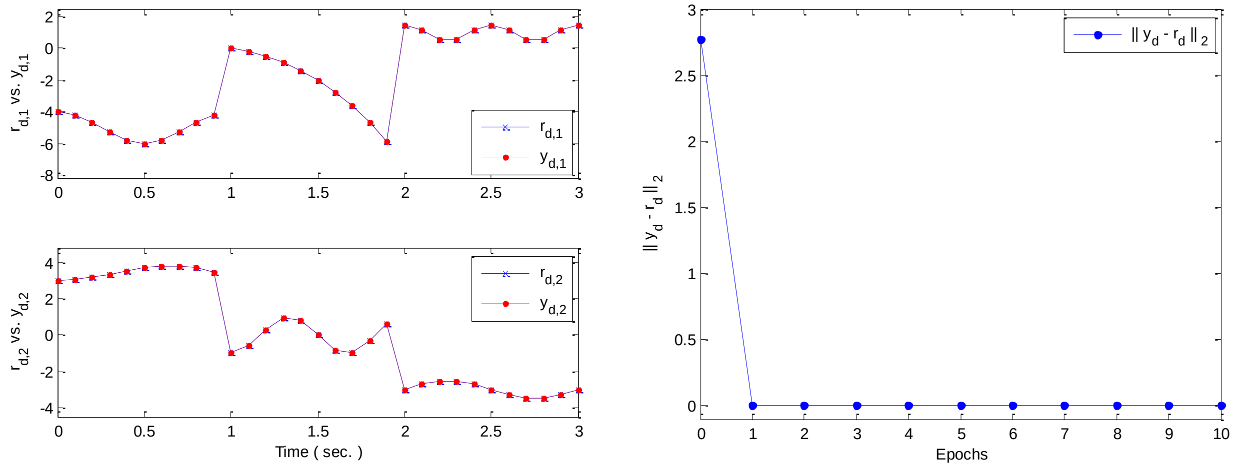
It is required to determine an iterative learning digital tracker for the input-constraint-free case so that the controlled system demonstrates a good tracking performance for

$$r(k)=\begin{cases} \begin{bmatrix} \cos(2\pi k)-5 \\ 5.5\,k^2(1-k)+3 \end{bmatrix}, & 0\le k<1\ \text{sec} \\ \begin{bmatrix} 1.8\,k^2(1-k) \\ \cos(3\pi k) \end{bmatrix}, & 0\le k<1\ \text{sec} \\ \begin{bmatrix} 0.5\ \cos(4\pi k)+1 \\ 0.5\ \sin(2\pi k)-3 \end{bmatrix}, & \text{otherwise} \end{cases}.$$

The proposed OECILC (49) is then determined, where

$$K_d=\begin{bmatrix} -0.2729 & -0.0824 & -0.0962 & 0.0003 \\ -0.0652 & 0.0262 & -0.0997 & 0.0195 \\ -0.1084 & -0.0399 & -0.0005 & -0.0031 \end{bmatrix},\ E_d=\begin{bmatrix} -0.0418 & 0.3952 \\ 0.0348 & 0.3429 \\ -0.0361 & 0.1987 \end{bmatrix},$$
$$C_{aj}(k)=\begin{bmatrix} 0.0418 & -0.3952 \\ -0.0348 & -0.3429 \\ 0.0361 & -0.1987 \end{bmatrix}\sum_{i=0}^{j} e_{i-1}(k),$$

for $Q_d=10^4 I_2$ and $R_d=I_3$. The tracking performance of the first learning epoch is shown in Figure 7 (a), and the learning curve is shown in Figure 7 (b), which demonstrates a satisfied tracking performance as expected.



(a) (b)

Figure 7. OECILC-based controlled system without input-constraint at the first learning epoch: (a) tracking responses, (b) learning curve.

Case 2. OECILC with input-constraint for the repetitive system with a direct-feedthrough term and unknown deterministic system and measurement disturbances

The objective of the classical tracking problem is to design an appropriate control law $u_d(k)$ so that the system output $y_d(k)$ can well track the pre-specified output target trajectory $r(k)$. In Section 2, a novel tracking problem is presented, so that a trade-off between the output tracking for a pre-specified output target trajectory $r(k)$ and the control input tracking for a pre-specified input target trajectory $u_d^*(k)$, in terms of each corresponding weighting matrices Q_d and R_d, is newly proposed. A new application of this problem on the input-constrained digital tracker design is to be given in the following. Let the discrete-time system have the above-mentioned disturbances. Based on the proposed input-saturation-free LQDT for $Q_d = 10^4 I_2$ and $R_d = I_3$, the tracking performance is quite satisfactory, while the proposed tracker-based control inputs have some acute responses at some time instants. Assume it is desired to narrow down the control-input magnitudes at those time instants with unexpected acute responses, and let $u_d^*(k)$ specify the desired time response of the control input. Here, based on the upper and lower bounds of the input-saturation-free case $u_{d,sat} = \begin{bmatrix} u_{d,1}^{\min} & u_{d,1}^{\max} \\ u_{d,2}^{\min} & u_{d,2}^{\max} \\ u_{d,3}^{\min} & u_{d,3}^{\max} \end{bmatrix} = \begin{bmatrix} -1.5507 & 1.4120 \\ -1.4781 & 1.3534 \\ -0.9385 & 0.7405 \end{bmatrix}$, the upper and lower bounds of $u_d^*(k)$ is pre-specified as $u_{d,sat}^* = 0.9 \times u_{d,sat} = \begin{bmatrix} -1.3957 & 1.2708 \\ -1.3303 & 1.2181 \\ -0.8446 & 0.6665 \end{bmatrix}$. Whenever some components of $u_d(k)$ exceed the upper and lower bounds of $u_{d,sat}^*$, the corresponding components of $u_d^*(k)$ are specified as the pre-specified upper and lower bounds and time-varying weighting matrices $\{ Q_d(k), R_d(k) \}$ are suggested at these sampling instants; otherwise, it is set to be the default value where $u_d^*(k) = 0_3$ for this particular application demonstrated in this example. Apply the proposed approach-based LQDT for $Q_d = 10^4 I_2$ and $R_d = I_3$ to the given system. Figure 8 shows that both tracking performance and learning curve are well-performed. Figure 9 shows those acute control inputs are restricted to the pre specified upper and lower bounds for this illustrative example.

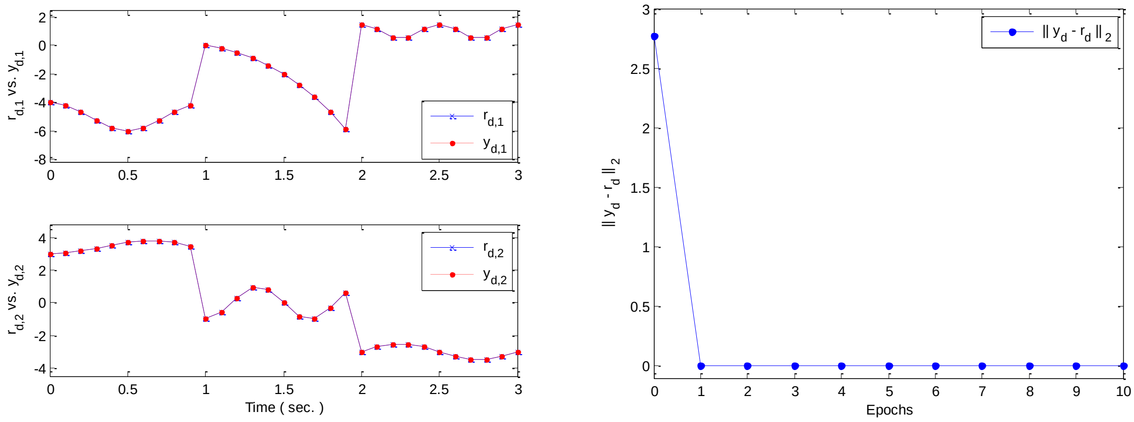


Figure 8. OECILC-based controlled system with input-constraint at the first learning epoch: (a) tracking responses, (b) learning curve.

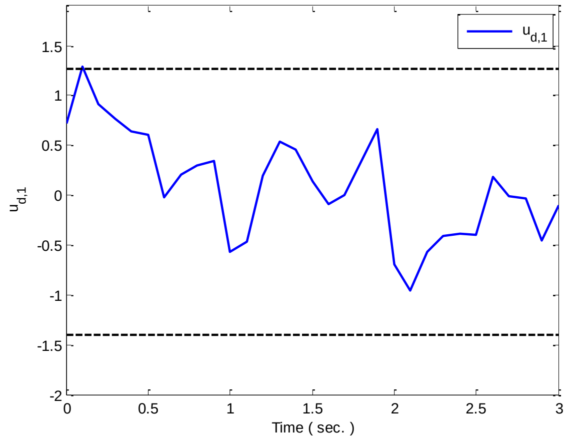


Figure 9. Control input of the OECILC-based controlled system with input-constraint for the repetitive system at the first learning epoch: $u_{d,1}(t)$ (shown by parts).

6. Conclusion

This paper has developed a generalized optimal linear quadratic digital tracker (LQDT) for minimum/non-minimum phase and square/non-square discrete-time systems. Their theoretical developments are summarized as follows: (i) A novel optimal LQDT with pre-specified measurement output and control input trajectories and its corresponding steady-state version for the discrete-time controllable, observable, and non-degenerate system, which has both an input-to-output direct-feedthrough term and the known/estimated system disturbances or compensatory signals, have been presented. (ii) A new optimal PID filter-shaped PI state-feedback linear quadratic digital tracker for non-square non-minimum phase discrete-time systems has been also proposed. As a result, the controlled non-minimum-phase non-square multivariable systems are able to achieve a minimum-phase-like tracking performance. In addition, the proposed tracker works well for the arbitrary reference trajectory with drastic variations. (iii) A closed-loop output-zeroing control system for the given non-square MIMO discrete-time system integrated with the two degrees of freedom optimal

LQDT has been newly proposed also, so that the reliable algorithm available in the literature can be used for computing the control zeros of the square closed-loop MIMO discrete-time system (squaring down due to the use of the two degrees of freedom optimal LQDT for the system with an input number (m) greater than the output number (p) or integrated with a transformation matrix for the case of $p > m$). Notice that the syntax ‘tzero (sys)’ in MATLAB fails to detect any finite control zeros of some non-square discrete-time systems demonstrated in this paper. (iv) A new iterative learning LQDT with input constraint for the repetitive discrete-time system with a direct-feedthrough term and unknown disturbances has been presented. Simulation experience shows that by the presented initialization of the proposed ILC, it could significantly reduce the learning epochs for the pre-specified tracking performance.

Acknowledgments

This work was supported by the National Science Council of Republic of China under Grants NSC102-2221-E-208-MY3 and NSC102-2221-E-006-199.

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