A new generalized optimal linear quadratic tracker with universal applications－Part 1: Continuous-time systems

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This paper presents a new generalized optimal linear quadratic analog tracker (LQAT) with universal applications for the continuous-time (CT) systems. This includes: (i) A novel optimal LQAT design for the system with the pre-specified trajectories of the output and the control input and additionally with both the input-to-output direct-feedthrough term and the known/estimated system disturbances or compensatory signals; (ii) A new optimal filter-shaped proportional plus integral state-feedback LQAT design for non-square non-minimum phase CT systems to achieve a minimum phase-like tracking performance. (iii) A new approach for computing the control zeros of the given non-square CT system (iv) A new input-constrained iterative learning LQAT design for the repetitive CT system.

Keywords: optimal linear quadratic tracker; optimal iterative learning control; frequency shaping; PID controller; input constraint ;non-minimum phase system; control zeros

1. Introduction

Despite the great progress of advanced control theory, the frequency shaping design methodology makes the designers to revisit the classical frequency-domain viewpoints, such as: (i) Achieving the specific gain margins of $(\frac{1}{2}, \infty)$, guaranteed 60-degree phase margin, or its multivariable equivalents (Gupta, 1980; Anderson & Mingori, 1985; Moore, Glover, & Telford, 1990; Thompson, Coleman, & Blight, 1987; Gangsaas, Bruce, Blight, & Ly, 1986; Anderson & Moore, 1989; Huang & Wang, 2001; Tibaldi & Capitani, 1990) y assuming the given plant is square and minimum phase; (ii) Setting proper bandwidths for effective control actions; (iii) Good partial pole-zero assignment based upon the observation of high-gain approach. The afore-mentioned design objectives in the frequency domain might be achievably by using proportional plus integral (PI) compensators or PI plus differential (PID) compensators, under the assumption that the given plant is square and minimum phase (Anderson & Moore, 1989).

The PI state feedback LQAT presented in this paper for the command tracking can be achieved by applying the optimal linear quadratic tracker (LQT) methods (not the optimal linear quadratic regulator (LQR), represented by an optimal state-feedback controller plus a feed-forward constant-reference tracking gain, which is often determined by the final value theory) to the plants augmented with PID filters (at the output terminal), which were driven from a linear combination of the plant outputs (see Figure 1). It is well-known that a well-designed traditional optimal regulator together

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with a feedforward gain determined by the final value theory works well for the constant set-point command tracking required by most chemical processes. However, if the system is subjected to time-varying set-point changes with huge variations required by some other scenarios, the proposed tracker design outperforms the traditional regulator-based command tracking design.

Essentially, (i) Augmenting a cascaded/series square PID filter to a plant induces some extra zeros, however, the original zeros of the plant are invariant, (ii) A high-gain linear quadratic optimal tracker in terms of some appropriately assigned weighting matrices $(C^T Q_t C,\ R_t)$, where $Q_t \in \Re^p,\ R_t \in \Re^m,$ m is the input number, p is the output number, and $m \geq p,$ results in the closed-loop poles approach to the stable control zeros (see Sec. 3) of the original plant and the augmented PID filter as well as the mirror images of the unstable control zeros of the original plant; therefore, the arbitrarily optimal pole-placement design is not permitted by the optimal LQT, (iii) A high-gain linear quadratic optimal regulator in terms of some appropriately assigned weighting matrices $(Q_r,\ R_r)$, where $Q_r \in \Re^n,\ R_r \in \Re^m, n$ is the system state number, and m=p is required, integrated with a feed-forward constant gain determined by the final value theory for command tracking does not have the above-mentioned characteristic of the optimal LQT, however, the arbitrarily optimal pole-placement design can be achieved by the optimal LQR, and (iv) A high-gain LQR in terms of some pre-assigned weighting matrices $(C^T Q_t C,\ R_t)$ integrated with a feed-forward constant gain determined by the final value theory for tracking purpose demonstrates a poor performance for command tracking in general, even for the square system. Notice that the selection of weighting matrices with appropriate dimensions plays an important role for the above fact. Therefore, the PID filter can be well-selected to give some specified extra zeros between the controls and frequency-shaped output variables and give open-loop responses which are like those of a low-order easy-to-control system (Anderson & Moore, 1989; Sebakhy, Singaby, & Arabawy, 1986; Tanaka, Shibasaki, Ogawa, Murakami, & Ishida, 2013). The assigned extra zeros then become closed-loop system poles in a high-gain state feedback tracking design (not regulation design) for the augmented plant. As a result, they shape the closed-loop response. The Riccati theory could take care of plant input phase margins (Anderson & Moore, 1989).

The utilization of frequency shaping method to a plant enables to convert the performance index in the time domain into that in the frequency domain. As a result, the optimization methodology in the time domain can be interpreted as that in the frequency domain. For example, when a plant is augmented with an integrator at its input terminal, the augmented input $u_a(t)$ is $\dot{u}(t)$, the time derivative of the original plant input. Thus, penalizing the augmented plant input by the quadratic term $u_a^T(t) R_a u_a(t)$ induces a penalty on the rate of change of control input $u(t)$, where the cost term $u_a^T(t) R_a u_a(t) = \dot{u}^T(t) R_a \dot{u}(t)$ is really a time domain version of $u^T(-jw) R(jw) u(jw) = u(-jw)$ $(w^2 R_a u(jw),$ where the frequency-shaped penalty matrix is $R(jw) = w^2 R_a$, but not a constant penalty matrix R. As pointed out by Anderson (Anderson & Moore, 1989), the disadvantage of this particular augmentation is that the return difference inequality is satisfied at the augmented plant input $u_a(t) = \dot{u}(t)$, but not the original control input $u(t)$. Hence, the control input robustness properties usually associated with an LQ design could well be lost (Anderson & Moore, 1989).

In this paper, the terminologies of PID filter and PID controller are alternatively mentioned. Basically, the PID filter is same as the PID controller. When the cascaded state-space PID controller is augmented with the multiple time-delayed multi-input multi-output (MIMO) plant at the plant input (Madsen, Shieh, & Guo, 2006; Zhou, Shieh, Liu, & Wang, 2006; Tsai, Lin, Shieh, Chandra, & Guo, 2008; Zhang, Shieh, Liu, & Guo, 2004), the parameters of the state-space MIMO optimal PID controller can then be determined by tuning the weighting matrices in the LQR performance indices in the time domain, and the closed-loop stability is guaranteed during the tuning process. However, the forward gain matrix in the afore-mentioned state-feedback control law is determined via the final value theory for tracking the constant set-point command. Nevertheless, if the command signal is a time-varying one, or large set-point step change, the development of the practically applicable optimal PID controller to avoid the integral windup and large set-point step change is still under investigation. To develop such a practically applicable optimal PID controller, we consider the state-space PID controller augmented at plant output as the PID filter in the frequency domain by emphasizing the role of the derivative term in the PID controller to relax the effects of integral windup and large set-point changes to the plant. More precisely, the PID filter is installed at the output of the plant to eliminate the effects of the huge discontinuity caused by the derivative of the large set-point change at the input of the plant. As a result, the derivative of the output produced by the derivative term of the PID filter becomes the continuous-time signal as $\dot{y}_c(t) = C\dot{x}_c(t) = C(Ax_c(t) + Bu_c(t))$. Indeed, the derivative term of the newly presented PID filter is theoretically merged to the modified state-space augmented system model to determine the optimal PI controller. In this paper, the PID filters can be arranged in such a way that the high-gain LQAT can be developed to optimally track the time-varying command signal or large set-point step change by tuning the weighting matrices in the optimal LQAT performance indices. In addition, both the closed-loop stability and the high-performance command tacking can be achieved during the tuning process.

Some limitations of the traditional PID-filter shaped control design methodology and corresponding improvements presented in this paper are listed sequentially.

(i) In order to achieve a strictly proper augmented system/controller without having the input-to-output feed-through term for the system represented by (A, B, C), there is a requirement that $CB = 0$ (i.e., a special case of the newly proposed augmented system model in (30)). The traditional PID filter can be placed at the output of the system and acts as a differential filter to achieve the above objective. The above-mentioned structure gives the time derivative of output as $\dot{y}_c(t) = C\dot{x}_c(t) = C(Ax_c(t) + Bu_c(t)) = CAx_c(t)$, in which the input-to-output feed-through term $Du_c(t) = CBu_c(t)$ is missing with the requirement that $CB = 0$. In this paper, the unusual constraint $CB = 0$ is relaxed and utilized to solve an unsolved problem in the literature; that is, given a proper system (A, B, C, D) with known system disturbance $d(t)$ and output disturbance $s(t)$ how to solve for the optimal LQAT for an arbitrary reference $r(t)$. Nevertheless, a novel optimal LQAT has been proposed in this paper to solve for the above-mentioned issue.

(ii) The PI state feedback regulation/tracking in a linear quadratic optimization context is able to automatically reject the unknown constant disturbances $u_{ext}(t)$ appeared at the plant input and achieves the set point regulation/tracking goal at the plant output terminal based upon the internal model principle (IMP) (Skogestad & Postlethwaite, 2005). However, if the arbitrary time-varying reference input $r(t)$ with huge variations at some isolated time instants is involved in the PID-filter shaping, it induces the augmented system model with an extra system disturbance term $d(t)$ in terms of $r(t)$, an output measurement disturbance term $s(t)$ in terms of $r(t)$ and its time derivative $\dot{r}(t)$ (may not exist theoretically), and an input-to-output feed-through term $Du_c(t)$ if $CB \neq 0$. To the best of our knowledge, the optimal linear quadratic tracker design for the above-mentioned generalized mathematical model has not yet been fully explored and will be developed in this paper.

(iii) In order to achieve the control specifications such as the gain margins of $(\frac{1}{2}, \infty)$, the guaranteed 60-degree phase margin, their multivariable equivalents, the set point regulation for tracking purpose, and the rejection of disturbance $u_{ext}(t)$ having arbitrary constant values, it requires that the given plant is often assumed to be square and minimum phase. It is well-known that a common response in the feedback system design with non-minimum phase zeros is of undershoot, which causes a delay in the system response. Unfortunately, a square non-minimum plant is still non-minimum phase, even though through appending PID filter(s)/controller(s) at either the input terminal, output terminal, or both terminals. The minimum phase property for the given system is often desirable if the output measurements are used to recover the state through an estimator and its robustness associated with exact state feedback is not to be lost using an estimator. In addition, for a minimum phase system, the feedback design with the estimated states can inherit the phase margins of the feedback design with the exact states (Anderson & Moore, 1989; Lewis, 1992). Nevertheless, the problems to improve and design a non-minimum phase system are still under investigation in the literature. In this paper, the PID filter-based frequency shaping approach to achieve a new optimal PI state feedback LQAT for the non-square non-minimum phase plants has been proposed. For tracking objective, it is required that the number of available control signals is greater or equal to that of the output signals. The proposed two degrees of freedom LQAT (integrated with some newly presented techniques given in Sec. 3) can make the square minimum phase closed-loop system with stable control zeros (see Sec. 3) from the non-square non-minimum phase plant, so that the controlled closed-loop system demonstrates a minimum phase-like performance for arbitrarily pre-specified reference $r(t)$.

In addition to the above-mentioned improvements, the newly proposed generalized optimal LQAT could be applied to the input-constrained continuous-time repetitive system with both the input-to-output feed-through term and unknown system and output disturbances.

In this paper, we use both of the classical controller design methodologies and the newly proposed generalized linear quadratic tracker design approach (newly presented) to improve the performances of the controlled square/non-square minimum phase systems and non-square non-minimum phase systems. The rest of the paper is organized as follows. Section 2 details the development of the novel optimal LQAT with pre-specified measurement output and control input trajectories for the continuous-time controllable, observable, and non-degenerate (Ferreira, 1976) system, which has an input-to-output direct-feedthrough term and known/estimated system disturbances or compensatory signals. A new optimal PID filter-shaped PI state-feedback linear quadratic tracker for non-square non-minimum phase systems is proposed in Section 3. Notice that the PID filters are not the resultant controllers so that their design is much straightforward than a classical controller design (Tsai, Lin, Shieh, Chandra, & Guo, 2008; Zhang, Shieh, Liu, & Guo, 2004; Skogestad & Postlethwaite, 2005; Lewis, 1992) and (Wang, 2014). Section 3 shows that one cannot find a non-singular $p \times p$ transformation matrix ξ to ensure that the system $(A, B, \tilde{C})$ is minimum phase, neither nor the case $(A, \tilde{B}, \tilde{C}) = (A, B\eta, \xi C)$. Hence, it points that the PID-filter shaped PI state-feedback LQAT for the non-minimum phase square system is left as a future research topic. Section 3 proposes a closed-loop output-zeroing control system for the given non-square MIMO system integrated with the two degrees of freedom optimal LQAT, so that the reliable algorithm presented by Emami-Naeini and Dooren in (Emami-Naeini & Dooren, 1982) can be used for computing the control zeros of the square closed-loop MIMO system (squaring down due to the use of the two degrees of freedom optimal LQAT for the system with an input number (m) greater than the output number (p) or integrated with a $p \times m$ transformation matrix ξ for the case of $p > m$). Section 4 presents a new iterative learning LQAT with input constraint for the repetitive system with a direct-feedthrough term and unknown disturbances. All above applications constitute the so-called universal application of the proposed generalized optimal LQAT. Illustrative examples are demonstrated in Section 5 to show the effectiveness of the proposed design methodologies, and conclusions are finally given in Section 6.

2. A novel optimal linear quadratic analog tracker for the system with a direct-feedthrough

term and known system disturbances

This section presents a novel optimal LQAT with pre-specified measurement output and control input trajectories and with its corresponding steady-state version for the continuous-time controllable, observable, and non-degenerate system, which has both an input-to-output direct-feedthrough term and known system disturbances.

Consider the controllable and observable linear continuous-time system with an input-to-output direct-feedthrough term and known/estimated system disturbances or compensatory signals $d(t)$ and $s(t)$

$$\dot{x}_c(t) = Ax_c(t) + Bu_c(t) + d(t), \tag{1a}$$

$$y_c(t) = Cx_c(t) + Du_c(t) + s(t), \tag{1b}$$

where $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $C \in \Re^{p\times n}$ and $D \in \Re^{p\times m}$ are state, input, output, and direct feedthrough matrices, respectively. $x_c(t) \in \Re^n$ is the state vector, $u_c(t) \in \Re^m$ is the control input, and $y_c(t) \in \Re^p$ is the measurable output of system at time t. Let the quadratic performance index function for tracking problem be

$$\begin{aligned} J(y_c, u_c) = &\frac{1}{2}\left[y_c(t_f) - r(t_f)\right]^T S\left[y_c(t_f) - r(t_f)\right] \\ &+ \frac{1}{2}\int_0^{t_f} \left\{\left[y_c(t) - r(t)\right]^T Q_c\left[y_c(t) - r(t)\right] + \left[u_c(t) - u_c^*(t)\right]^T R_c\left[u_c(t) - u_c^*(t)\right]\right\} dt, \end{aligned} \tag{2}$$

where $r(t)$ denotes the pre-specified output trajectory, t_f is the final time duration, Q_c is a $p\times p$ positive definite or positive semi-definite real symmetric matrix, R_c is an $m\times m$ positive definite real symmetric matrix, S is a $p\times p$ semi-definite real symmetric matrix, and $u_c^*(t)$ is a pre-specified control input trajectory. Substituting (1) into (2) and inducing the undetermined Lagrange multiplier $\lambda(t) \in \Re^n$,

$$\lambda(t) = P(t)x_c(t) - v(t), \tag{3}$$

where $P(t) > 0$ and $v(t) \in \Re^n$ (to be determined later), one has

$$\begin{aligned} J(x_c, u_c, \lambda) = &\frac{1}{2}\left[y_c(t_f) - r(t_f)\right]^T S\left[y_c(t_f) - r(t_f)\right] + \int_0^{t_f}\left\{\frac{1}{2}\left[y_c(t) - r(t)\right]^T Q_c\left[y_c(t) - r(t)\right]\right. \\ &\left. + \frac{1}{2}\left[u_c(t) - u_c^*(t)\right]^T R_c\left[u_c(t) - u_c^*(t)\right] + \lambda^T(t)\left[Ax_c(t) + Bu_c(t) + d(t) - \dot{x}_c(t)\right]\right\} dt \\ = &\frac{1}{2}\left[y_c(t_f) - r(t_f)\right]^T S\left[y_c(t_f) - r(t_f)\right] + \int_0^{t_f}\left[\mathbb{H}\left(x_c(t), u_c(t), \lambda(t)\right) - \lambda^T(t)\dot{x}_c(t)\right] dt \\ = &\frac{1}{2}\left[y_c(t_f) - r(t_f)\right]^T S\left[y_c(t_f) - r(t_f)\right] + \int_0^{t_f}\mathbb{F}\left(x_c(t), u_c(t), \lambda(t)\right) dt, \end{aligned} \tag{4a}$$

where

$$\begin{aligned} \mathbb{H}\left(x_c(t), u_c(t), \lambda(t)\right) \triangleq &\frac{1}{2}\left\{\left[y_c(t) - r(t)\right]^T Q_c\left[y_c(t) - r(t)\right] + \left[u_c(t) - u_c^*(t)\right]^T R_c\left[u_c(t) - u_c^*(t)\right]\right\} \\ &+ \lambda^T\left[Ax_c(t) + Bu_c(t) + d(t)\right], \end{aligned} \tag{4b}$$

$$\mathbb{F}\left(x_c(t), u_c(t), \lambda(t)\right) = \mathbb{H}\left(x_c(t), u_c(t), \lambda(t)\right) - \lambda^T(t)\dot{x}_c(t). \tag{4c}$$

The stationary condition is

$$0=\frac{\partial \mathbb{F}}{\partial u_c(t)}=\frac{\partial \mathbb{H}}{\partial u_c(t)}=D^T Q_c\left[Cx_c(t)+Du_c(t)+s(t)-r(t)\right]$$
$$+R_c\left[u_c(t)-u_c^*(t)\right]+B^T\lambda(t). \tag{5}$$

Solving (5) yields the optimal control in terms of the costate $\lambda(t)$ as

$$\begin{aligned} u_c(t) &= \left(R_c+D^T Q_c D\right)^{-1} R_c u_c^*(t) \\ &= -\left(R_c+D^T Q_c D\right)^{-1}\left\{B^T\lambda(t)+D^T Q_c\left[Cx_c(t)+s(t)-r(t)\right]\right\}. \end{aligned} \tag{6}$$

From (3) and (6), one has

$$\begin{aligned} u_c(t) &= \left(R_c+D^T Q_c D\right)^{-1} R_c u_c^*(t)-\left(R_c+D^T Q_c D\right)^{-1}\left\{B^T\left[Px_c(t)-v(t)\right]\right. \\ &\quad \left.+D^T Q_c\left[Cx_c(t)+s(t)-r(t)\right]\right\}. \end{aligned} \tag{7}$$

The state and costate equations are respectively obtained by forcing $\frac{\partial \mathbb{F}}{\partial \lambda(t)}=0$ and $\frac{\partial \mathbb{F}}{\partial x_c(t)}=0$ as

$$\dot{x}_c(t)=\frac{\partial \mathbb{H}}{\partial \lambda(t)}=Ax_c(t)+Bu_c(t)+d(t) \tag{8}$$

and

$$\dot{\lambda}(t)=-\frac{\partial \mathbb{H}}{\partial x_c(t)}=-A^T\lambda(t)-C^T Q_c\left[Cx_c(t)+Du_c(t)+s(t)-r(t)\right], \tag{9}$$

with boundary conditions $x_c(0)$ given and $\lambda(t_f)=C^T Q_{c0}\left[Cx_c(t_f)+s(t_f)-r(t_f)\right]$.

Taking the time derivative of (3) yields

$$\begin{aligned} \dot{\lambda}(t) &= P(t)\dot{x}_c(t)+\dot{P}(t)x_c(t)-\dot{v}(t) \\ &= P(t)\left[Ax_c(t)+Bu_c(t)+d(t)\right]+\dot{P}(t)x_c(t)-\dot{v}(t) \\ &= P(t)\left\{Ax_c(t)+B\left(R_c+D^T Q_c D\right)^{-1} R_c u_c^*(t)\right. \\ &\quad -B\left(R_c+D^T Q_c D\right)^{-1}\left\{B^T\left[Px_c(t)-v(t)\right]+D^T Q_c\left[Cx_c(t)+s(t)-r(t)\right]\right\} \\ &\quad \left.+d(t)\right\}+\dot{P}(t)x_c(t)-\dot{v}(t), \end{aligned} \tag{10}$$

where (1a) and (7) are used in the above derivation. Equalizing (9) and (10) yields

$$\begin{aligned} &\left\{\dot{P}(t)+A^T P(t)+P(t)A-\left[P(t)B+N\right]\bar{R}_c^{-1}\left[B^T P(t)+N^T\right]+C^T Q_c C\right\}x_c(t) \\ &+\left[N\bar{R}_c^{-1}R_c+P(t)B\bar{R}_c^{-1}R_c\right]u_c^*(t)-\left[A^T-P(t)B\bar{R}_c^{-1}B^T-N\bar{R}_c^{-1}B^T\right]v(t) \\ &+\left[P(t)B\bar{R}_c^{-1}D^T Q_c-C^T Q_c+N\bar{R}_c^{-1}D^T Q_c\right]r(t)-\dot{v}(t)+Pd(t) \\ &-\left[N\bar{R}_c^{-1}D^T Q_c+P(t)B\bar{R}_c^{-1}D^T Q_c-C^T Q_c\right]s(t)=0, \end{aligned} \tag{11a}$$

where

$$\bar{R}_c=R_c+D^T Q_c D, \tag{11b}$$
$$N=C^T Q_c D. \tag{11c}$$

Equation (11) holds for any arbitrary $x_c(t)$, which implies

$$-\dot{P}(t)=A^T P(t)+P(t)A-\left[P(t)B+N\right]\bar{R}_c^{-1}\left[B^T P(t)+N^T\right]+C^T Q_c C \tag{12}$$

and

$$-\dot{v}(t)=\left[A^T-\left(PB+N\right)\bar{R}_c^{-1}B^T\right]v(t)-\left[\left(PB+N\right)\bar{R}_c^{-1}D^TQ_c-C^TQ_c\right]r(t)$$
$$+\left[\left(PB+N\right)\bar{R}_c^{-1}D^TQ_c-C^TQ_c\right]s(t)-Pd(t)-\left(PB+N\right)\bar{R}_c^{-1}R_cu_c^*(t). \qquad (13)$$

To determine the boundary conditions of (12) and (13), let

$$P(t_f)=\begin{cases}C^TSC, \text{ for a finite } t_f \text{ and the pre-specified } S\\ 0_{n\times n} \ \ , \text{ for } t_f\to\infty\end{cases}$$

and

$$v(t_f)=\begin{cases}C^TS\left[r(t_f)-Du_c(t_f)-s(t_f)\right], \text{ for a finite } t_f \text{ and the pre-specified } S\\ 0_{n\times 1} \qquad\qquad , \text{ for } t_f\to\infty\end{cases}.$$

The matrix $P(t)$ is independent of the state trajectory, so the Riccati equation can be solved off-line in a backward manner. As a result, $P(t)$ and the feedback gain in (7)

$$K_c(t)=\bar{R}_c^{-1}\left[B^TP(t)+N^T\right] \qquad (14)$$

can be stored. If the tracking reference $r(t)$ is known a priori, the auxiliary function $v(t)$ can also be pre-computed and stored. The only work left to do during the actual control run is then to compute the control law in (7).

Suppose $v(t)$ has been determined by integrating backward the closed-loop adjoint system in (13) with $v(t_f)=C^TS\left[r(t_f)-s(t_f)\right]$. Then, $v(0)$ is known. During the actual control run, we could use this $v(0)$ and the following equation

$$-\dot{v}(t)=\left[A-BK_c(t)\right]^Tv(t)+\left[C-DK_c(t)\right]^TQ_cr(t)-\left[C-DK_c(t)\right]Q_cs(t)-P(t)d(t)-K_c^TR_cu_c^*(t) \qquad (15)$$

to compute $v(t)$. This avoids storage of the auxiliary function $v(t)$.

If the final time t_f goes to infinity, we have the infinite-horizon tracker problem, in which we also let $P(t_f)=0_{n\times n}$ and $v(t_f)=0_{n\times 1}$. Then, $P(t)$ and $K_c(t)$ reach their respective steady-state values P and K_c. For the infinite-horizon tracker problem, the optimal tracker is given by

$$-\dot{v}(t)=\left(A-BK_c\right)^Tv(t)+\left(C-DK_c\right)^TQ_cr(t)-\left(C-DK_c\right)^TQ_cs(t)$$
$$-P(t)d(t)-K_c^TR_cu_c^*(t), \qquad (16)$$
$$u_c(t)=-K_cx_c(t)+R_c^{-1}B^Tv(t)+\bar{R}_c^{-1}R_cu_c^*(t)-\bar{R}_c^{-1}D^TQ_c\left[s(t)-r(t)\right]. \qquad (17)$$

A steady-state tracker can be devised for a finite control interval $\left[0,\ t_f\right]$ by using the steady-state gain K_c. The initial condition $v(0)$ can be determined off-line using (16), and then during the simulation, we need use only (15) and (17). Experience shows that this simplified tracker is often satisfied if t_f is large (Lewis & Syrmos, 1995).

Similarly, a steady-state tracker can also be derived for an infinite control interval $\left[0,\ \infty\right]$ by using the steady-state P and v. Under these circumstances, (12) and (13) are simplified to

$$A^TP+PA-\left(PB+N\right)\bar{R}_c^{-1}\left(B^TP+N^T\right)+C^TQ_cC=0 \qquad (18)$$

and

$$v(t)=-\left[\left(A-BK_c\right)^T\right]^{-1}\left(C-DK_c\right)^TQ_cr(t)+\left[\left(A-BK_c\right)^T\right]^{-1}\left(C-DK_c\right)^TQ_cs(t)$$

$$+\left[\left(A-BK_c\right)^T\right]^{-1} Pd(t)+\left[\left(A-BK_c\right)^T\right]^{-1} K_c^T R_c u_c^*(t). \tag{19}$$

Substituting (19) into (7) yields

$$u_c(t)=-K_c x_c(t)+E_c r(t)+C_c(t)+C_{u_c}^* u_c^*(t), \tag{20a}$$

where

$$K_c=\bar{R}_c^{-1}\left(B^T P+N^T\right), \tag{20b}$$

$$E_c=-\bar{R}_c^{-1}\left[\left(C-DK_c\right)\left(A-BK_c\right)^{-1} B-D\right]^T Q_c, \tag{20c}$$

$$C_c(t)=\bar{R}_c^{-1}\left[\left(C-DK_c\right)\left(A-BK_c\right)^{-1} B-D\right]^T Q_c s(t)+\bar{R}_c^{-1}B^T\left[\left(A-BK_c\right)^T\right]^{-1} Pd(t), \tag{20d}$$

$$C_{u_c}^*=\bar{R}_c^{-1}\left\{I_m+B^T\left[\left(A-BK_c\right)^T\right]^{-1} K_c^T\right\}R_c, \tag{20e}$$

$$\bar{R}_c=R_c+D^T Q_c D, \tag{20f}$$

$$N=C^T Q_c D. \tag{20g}$$

3. A new optimal PI state-feedback linear quadratic tracker for non-square non-minimum phase systems: PID filter-based frequency shaping approach

The objectives of the following demonstration are: (i) A closed-loop output-zeroing control system for the given non-square open-loop MIMO system integrated with the control-zero-related two degrees of freedom optimal LQAT is newly developed, so that the reliable algorithm in (Emami-Naeini & Dooren, 1982) can be used for computing the control zeros of the square closed-loop MIMO system; (ii) The newly developed optimal PID filter-based frequency shaped PI state-feedback linear quadratic tracker is able to be applied to non-square non-minimum phase systems, so that the controlled system demonstrates a minimum phase-like performance.

The poles and zeros of a system must share a certain property in order for the system to be strongly stabilizable. In the following remark, the point at $s=\infty$ is included among the real zeros of the system (Doyle, Francis, & Tannenbaum, 1990).

Remark 1 (Doyle, Francis, & Tannenbaum, 1990): A system is strongly stabilizable if and only if it has an even number of real poles between every pair of zeros in Re $s \geq 0$.

The reason for the controlled system of a non-minimum phase system demonstrates a minimum phase-like performance can be explained in the following.

As pointed out by Latawiec et al. (Latawiec, Banka, & Tokarzewski, 2000) that an input blocking the transmission is nothing but simply an open-loop control signal $u_c(t)$ driving the output $y_c(t)$ to the pre-specific reference $r(t)=0$. However, an input zeroing the output can represent either open-loop or closed-loop control, depending on whether 'no pole' (finite impulse response) or 'pole-involving' models of a plant are respectively employed. New multivariable zeros, i.e. the so-called 'control zeros', are co-generated by a plant and a part of a controller. A closer inspection reveals that 'control zeros' are in fact transmission zeros of a square open/closed-loop output-zeroing control system (Latawiec, Banka, & Tokarzewski, 2000). Some counterexamples were demonstrated by Latawiec et al. (Latawiec, Banka, & Tokarzewski, 2000) to show that some existing definitions of multivariable transmission zeros fail to detect certain important zeros (of non-square MIMO system) which contribute to zeroing the system output.

For the high-gain optimal linear quadratic two-degrees-of-freedom state-feedback tracker design, as many closed-loop poles as number of open-loop control zeros are close to stable open-loop control zeros or the stable reflections of the non-minimum phase open-loop control zeros of the plant. The remaining poles approach infinity in a manner such that they and their reflections across the imaginary axis have asymptotes that are evenly distributed.

With the above-mentioned viewpoints in mind, a closed-loop output-zeroing control system for the given non-square MIMO system integrated with the two degrees of freedom optimal LQAT is newly proposed in this paper, so that the reliable algorithm in (Emami-Naeini & Dooren, 1982) for computing the control zeros of the square closed-loop MIMO system (squaring down due to the use of the two degrees of freedom optimal LQAT for the system with an input number greater than the output number) can be found. Syntax ‘tzero (sys)’ in MATLAB implements the algorithm in (Emami-Naeini & Dooren, 1982) for finding transmission zeros of a square and/or non-square MIMO system. However, it is worth noticing that Syntax ‘tzero (sys)’ in MATLAB belongs to one of some existing definitions of multivariable transmission zeros which fails to detect certain important zeros of non-square MIMO system which contribute to zeroing the system output.

Consider the continuous-time controllable, observable, non-degenerate non-square non-minimum phase system described by

$$\dot{x}_c(t) = Ax_c(t) + Bu_c(t), \;\; x_c(0) = x_0, \tag{21a}$$

$$y_c(t) = Cx_c(t), \tag{21b}$$

where $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, and $C \in \Re^{p\times n}$ denote the system, input, and output matrices, respectively, and $x_c(t) \in \Re^{n}$, $u_c(t) \in \Re^{m}$, and $y_c(t) \in \Re^{p}$ represent the state, input, and output vectors, respectively. The design procedure of the new PID filter-based frequency shaping approach is described as follows for two cases.

Case 1: The number of inputs is less than the number of outputs

Step 1: Transform the non-square non-minimum phase system to a square minimum phase system.

i) Define the new output variables, being a linear combination of the original output measurements, as

$$y_\xi(t) = \xi y_c(t) = \xi\left[Cx_c(t)\right] = \tilde{C}x_c(t), \tag{22}$$

where $\xi \in \Re^{m\times p}$ is a transformation matrix, such that $(A, B, \tilde{C})$ is observable, in which $\tilde{C} = \xi C$.

Remark 2: The tracking problem only makes sense if the number of output signals does not exceed the number of available control signals, i.e., $m \geq p$.

Remark 3: When A is nonsingular, $\tilde{C}A^{-1}B$ is nonsingular. That is, the poles of the integrator (at the origin) must not be cancelled by zeros in the transfer function $\tilde{C}(sI - A)^{-1}B$. The non-singularity of $\tilde{C}A^{-1}B$ permits infinite loop gains at the origin with use of integral feedback. Whenever A is singular, the plant itself contains a pure integration, and it may not be necessary to use the full integral feedback (Anderson & Moore, 1989).

ii) Randomly choose an above-mentioned $m \times p$ transformation matrix ξ to ensure that the system $(A - BK_c,\ BE_c,\ \tilde{C})$ is square minimum phase, where a control-zero-related two degrees of freedom LQAT in terms of (K_c, E_c) with the high-gain property (see Lemma 1 at the end of this section) is used.

Remark 4: If the number of inputs is equal to the number of outputs, i.e., $m = p$, then we cannot find an $m \times p$ non-singular transformation matrix ξ to ensure that the system $(A - BK_c,\ BE_c,\ \tilde{C})$ is minimum phase. This is due to the fact that the row rank of the square closed-loop pencil of $(n+m)\times(n+m)$

$$H_c(z_i) = \begin{bmatrix} A_{n\times n} - B_{n\times m}K_{c\ m\times n} - z_i I_n & B_{n\times m}E_{c\ m\times m} \\ \xi_{m\times p}C_{p\times n} & 0_{m\times m} \end{bmatrix}$$
$$= \begin{bmatrix} I_n & 0_{n\times p} \\ 0_{m\times n} & \xi_{m\times p} \end{bmatrix} \begin{bmatrix} A_{n\times n} - z_i I_n & B_{n\times m} \\ C_{p\times n} & 0_{p\times m} \end{bmatrix} \begin{bmatrix} I_n & 0_{n\times m} \\ -K_{c\ m\times n} & I_m \end{bmatrix} \begin{bmatrix} I_n & 0_{n\times m} \\ 0_{m\times n} & E_{c\ m\times m} \end{bmatrix} \tag{23}$$

is smaller than the row rank of the non-square open-loop pencil of $(n+p)\times(n+m)$

$$H_o(z_i) = \begin{bmatrix} A_{n\times n} - z_i I_n & B_{n\times m} \\ C_{p\times n} & 0_{p\times m} \end{bmatrix} \quad \text{for} \quad p > m, \tag{24}$$

where a two degrees of freedom LQAT in terms of (K_c, E_c) is used. Similarly, the column rank of the square closed-loop pencil of $(n+p)\times(n+p)$

$$H_c(z_i) = \begin{bmatrix} A_{n\times n} - B_{n\times m}K_{c\ m\times n} - z_i I_n & B_{n\times m}E_{c\ m\times p} \\ \xi_{p\times p}C_{p\times n} & 0_{p\times p} \end{bmatrix} \quad \text{for} \quad m > p \tag{25}$$

is smaller than the column rank of the non-square open-loop pencil of $(n+p)\times(n+m)$, where E_c is a function of ξ. However, for the square open-loop system where $m = p$, (23) shows that the row- and column-ranks of $H_c(z_i)$ are same as those of $H_o(z_i)$, respectively; therefore, Remark 4 holds.

Step 2: Assign some extra target zeros (without open-loop pole-zero cancellation) to attract some of closed-loop poles in a closed-loop design.

A PID filter $(k_p + k_i s^{-1} + k_d s)$ has a pair of zeros at specified locations and a pole at the origin. The PID filter gains $\{\ k_p,\ k_i,\ k_d\ \}$ are selected to achieve minimum phase target zeros intended to attract the closed-loop poles in a closed-loop design. This means that the zero assignments are approximate closed-loop pole assignments.

i) Append the PID filter to each element of $\left[y_\xi(t) - r_\xi(t)\right]$. The output of the PID filter is given by

$$Y_f(s) = (K_P + K_I s^{-1} + K_D s)\left[Y_\xi(s) - R_\xi(s)\right]$$
$$= \begin{bmatrix} k_p^{(1)} + k_i^{(1)}s^{-1} + k_d^{(1)} s & & 0 \\ & \ddots & \\ 0 & & k_p^{(m)} + k_i^{(m)}s^{-1} + k_d^{(m)} s \end{bmatrix} \left[Y_\xi(s) - R_\xi(s)\right]. \tag{26}$$

The integrator emphasizes low frequencies, so one can select a large value of the integral gain to improve the performance. The differentiator emphasizes high frequencies, thus one can select a small value of derivative gain to improve the robustness. Moreover, if the bandwidth is too low to achieve a practical design, then increase the proportional gain to tune the cross-over frequency.

ii) Choose the PID filter gains $\left\{\ k_p^{(j)},\ k_i^{(j)},\ k_d^{(j)}\ |\ j = 1 \sim m\ \right\}$ to assign target zeros

$$\left\{ \frac{-k_p^{(j)} \pm \sqrt{(k_p^{(j)})^2 - 4k_d^{(j)} k_i^{(j)}}}{2k_d^{(j)}} \;\middle|\; j = 1 \sim m \right\}. \tag{27}$$

On the other hand, the numerator polynomial of the PID filter can be written as

$$s^2 + \frac{k_p^{(j)}}{k_d^{(j)}} s + \frac{k_i^{(j)}}{k_d^{(j)}} = (s - z_1^{(j)})(s - z_2^{(j)}) = s^2 - (z_1^{(j)} + z_2^{(j)})s + z_1^{(j)} z_2^{(j)}. \tag{28}$$

Thus, the sum and product of target zeros can be expressed in terms of the PID filter gains { $k_p^{(j)}$, $k_i^{(j)}$, $k_d^{(j)}$ } as $z_1^{(j)} + z_2^{(j)} = -\frac{k_p^{(j)}}{k_d^{(j)}}$ and $z_1^{(j)} z_2^{(j)} = \frac{k_i^{(j)}}{k_d^{(j)}}$, respectively. Designer can assign target zeros at desired location to obtain the PID filter gains.

Step 3: Construct the augmented plant.

i) Define the integral of $\left[y_\xi(t) - r_\xi(t) \right]$ as another state variable

$$x_\xi(t) = \int_0^t \left[y_\xi(\tau) - r_\xi(\tau) \right] d\tau. \tag{29}$$

ii) The state-space model of the augmented plant is given by

$$\begin{aligned} \begin{bmatrix} \dot{x}_c(t) \\ \dot{x}_\xi(t) \end{bmatrix} &= \begin{bmatrix} A & 0_{n\times m} \\ \tilde{C} & 0_{m\times m} \end{bmatrix} \begin{bmatrix} x_c(t) \\ x_\xi(t) \end{bmatrix} + \begin{bmatrix} B \\ 0_{m\times m} \end{bmatrix} u_c(t) + \begin{bmatrix} 0_{n\times 1} \\ -r_\xi(t) \end{bmatrix} \\ &= A_{aug} x_{aug}(t) + B_{aug} u_c(t) + d_{aug}(t), \end{aligned} \tag{30a}$$

$$\begin{aligned} y_f(t) &= \left[K_p \tilde{C} + K_d \tilde{C} A \quad K_i \right] \begin{bmatrix} x_c(t) \\ x_\xi(t) \end{bmatrix} + K_d \tilde{C} B u_c(t) - \left[K_p r_\xi(t) + K_d \dot{r}_\xi(t) \right] \\ &= C_{aug} x_{aug}(t) + D_{aug} u_c(t) + s_{aug}(t). \end{aligned} \tag{30b}$$

Remark 5: If the reference input $r_\xi(t)$ is discontinuous, then we can construct a square minimum phase model $(A_{r_\xi}, B_{r_\xi}, C_{r_\xi})$ to replace $\dot{r}_\xi(t)$ by

$$\dot{r}_\xi(t) \approx \dot{y}_{r_\xi}(t) = C_{r_\xi} \dot{x}_{r_\xi}(t) = C_{r_\xi} \left[A_{r_\xi} x_{r_\xi}(t) + B_{r_\xi} u_{r_\xi}(t) \right], \tag{31}$$

where

$$u_{r_\xi}(t) = -K_{r_\xi} x_{r_\xi}(t) + E_{r_\xi} r_\xi(t), \tag{32a}$$

in which

$$K_{r_\xi} = R_{r_\xi}^{-1} B_{r_\xi}^T P_{r_\xi}, \tag{32b}$$

$$E_{r_\xi} = -R_{r_\xi}^{-1} B_{r_\xi}^T [(A_{r_\xi} - B_{r_\xi} K_{r_\xi})^{-1}]^T C_{r_\xi}^T Q_{r_\xi}, \tag{32c}$$

with the high-gain property, and P_{r_ξ} satisfies the algebraic Riccati equation

$$A_{r_\xi}^T P_{r_\xi} + P_{r_\xi} A_{r_\xi} - P_{r_\xi} B_{r_\xi} R_{r_\xi}^{-1} B_{r_\xi}^T P_{r_\xi} + C_{r_\xi}^T Q_{r_\xi} C_{r_\xi} = 0. \tag{32d}$$

Step 4: Formulate the performance index.

The first goal of this control problem is to minimize the performance index associated with the augmented plant with input $u_c(t)$ and output $y_f(t)$ as

$$J = \int_0^\infty \left\{ \left[y_f(t) - r_\xi(t) \right]^T Q_c \left[y_f(t) - r_\xi(t) \right] + u_c^T(t) R_c u_c(t) \right\} dt, \tag{33a}$$

so that $y_f(t)$ well tracks the command input $r_\xi(t)$. By Parseval's theorem, the performance index can be written in the frequency domain as

$$J=\frac{1}{2\pi}\int_{-\infty}^{\infty}\Big\{\Big[\left(K_P+K_I s^{-1}+K_D s\right)\left(Y_\xi(s)-R_\xi(s)\right)-R_\xi(s)\Big]^* Q_c \times\Big[\left(K_P+K_I s^{-1}+K_D s\right)\left(Y_\xi(s)-R_\xi(s)\right)-R_\xi(s)\Big]+U_c^*(s)R_cU_c(s)\Big\}\,d\omega, \tag{33b}$$

where the superscript $*$ denotes the complex conjugate transpose and $s=j\omega$. The second goal of this control problem is to minimize the other performance index as

$$J'=\frac{1}{2\pi}\int_{-\infty}^{\infty}\Big\{\left(Y_\xi(s)-R_\xi(s)\right)^*\Big[\left(K_P+K_I s^{-1}+K_D s\right)^* Q_c\left(K_P+K_I s^{-1}+K_D s\right)\Big] \times\left(Y_\xi(s)-R_\xi(s)\right)+U_c^*(s)R_cU_c(s)\Big\}\,d\omega, \tag{34}$$

so that $y_\xi(t)$ well tracks the command input $r_\xi(t)$. Some notable remarks are listed in the following.

i) From (33b), one observes that

$$\left(K_P+K_I s^{-1}+K_D s\right)\left[Y_\xi(s)-R_\xi(s)\right]-R_\xi(s)=\left(K_P+K_I s^{-1}+K_D s\right)\Big\{Y_\xi(s)-\Big[I_m+\left(K_P+K_I s^{-1}+K_D s\right)^{-1}\Big]R_\xi(s)\Big\}$$

$$=\left(K_P+K_I s^{-1}+K_D s\right)\left\{Y_\xi(s)-\begin{bmatrix}\dfrac{k_d^{(1)}s^2+(k_p^{(1)}+1)s+k_i^{(1)}}{k_d^{(1)}s^2+k_p^{(1)}s+k_i^{(1)}} & & 0\\ & \ddots & \\ 0 & & \dfrac{k_d^{(m)}s^2+(k_p^{(m)}+1)s+k_i^{(m)}}{k_d^{(m)}s^2+k_p^{(m)}s+k_i^{(m)}}\end{bmatrix}R_\xi(s)\right\}$$

$\rightarrow\left(K_P+K_I s^{-1}+K_D s\right)\left[Y_\xi(s)-R_\xi(s)\right]$ as $k_p^{(j)}\approx k_p^{(j)}+1$ or $k_p^{(j)}\rightarrow\infty$ for indices $j=1,2,\cdots,m,$

or $\dfrac{k_d^{(j)}s^2+(k_p^{(j)}+1)s+k_i^{(j)}}{k_d^{(j)}s^2+k_p^{(j)}s+k_i^{(j)}}\rightarrow 1$ for $j=1,2,\cdots,m,$ in general, which reveals a criterion for the selection of PID filters so that the performance index in (33b) approaches to the one in (34).

ii) Performance indices (33) and (34) show that $y_f(t)\rightarrow r_\xi(t)$ and $y_\xi(t)\rightarrow r_\xi(t)$, respectively, provided that $\dfrac{k_d^{(j)}s^2+(k_p^{(j)}+1)s+k_i^{(j)}}{k_d^{(j)}s^2+k_p^{(j)}s+k_i^{(j)}}\rightarrow 1$ for $j=1,2,\cdots,m,$ as well as the weighting function pair $\left\{\left(K_P+K_I s^{-1}+K_D s\right)^* Q_c\left(K_P+K_I s^{-1}+K_D s\right),\ R_c\right\}$ has the high-gain property in the interested frequency range.

Step 5: Perform the linear quadratic PI state-feedback tracker design.
Use (18) and (20) with an appropriate weighting matrix pair $Q_c \gg R_c$ to have the optimal control law

$$\begin{aligned}u_c(t)&=-K_c x_{aug}(t)+E_c r_\xi(t)+C_c(t)\\&=-\left[K_{c1}\ \vdots\ K_{c2}\right]\begin{bmatrix}x_c(t)\\x_\xi(t)\end{bmatrix}+E_c r_\xi(t)+C_c(t),\end{aligned} \tag{35}$$

for the augmented system model in (30), where

$$
\begin{aligned}
K_c &= \bar{R}_{aug}^{-1}\left(B_{aug}^T P_{aug} + N_{aug}^T\right), \\
E_c &= -\bar{R}_{aug}^{-1}\left[\left(C_{aug} - D_{aug}K_c\right)\left(A_{aug} - B_{aug}K_c\right)^{-1} B_{aug} - D_{aug}\right]^T Q_c, \\
C_c(t) &= \bar{R}_{aug}^{-1}\left[\left(C_{aug} - D_{aug}K_c\right)\left(A_{aug} - B_{aug}K_c\right)^{-1} B_{aug} - D_{aug}\right]^T Q_c\, s_{aug}(t) \\
&\quad + \bar{R}_{aug}^{-1} B_{aug}^T\left[\left(A_{aug} - B_{aug}K_c\right)^T\right]^{-1} P_{aug}\, d_{aug}(t),
\end{aligned}
$$

in which $\bar{R}_{aug} = R_c + D_{aug}^T Q_c D_{aug}$ and $N_{aug} = C_{aug}^T Q_c D_{aug}$, and P_{aug} satisfies the algebraic Riccati equation

$$A_{aug}^T P_{aug} + P_{aug}A_{aug} - \left(B_{aug}^T P_{aug} + N_{aug}^T\right)^T \bar{R}_{aug}^{-1}\left(B_{aug}^T P_{aug} + N_{aug}^T\right) + C_{aug}^T Q_c C_{aug} = 0.$$

The architecture of the linear quadratic PI state-feedback tracker for the non-square non-minimum phase system is shown in Figure 1.

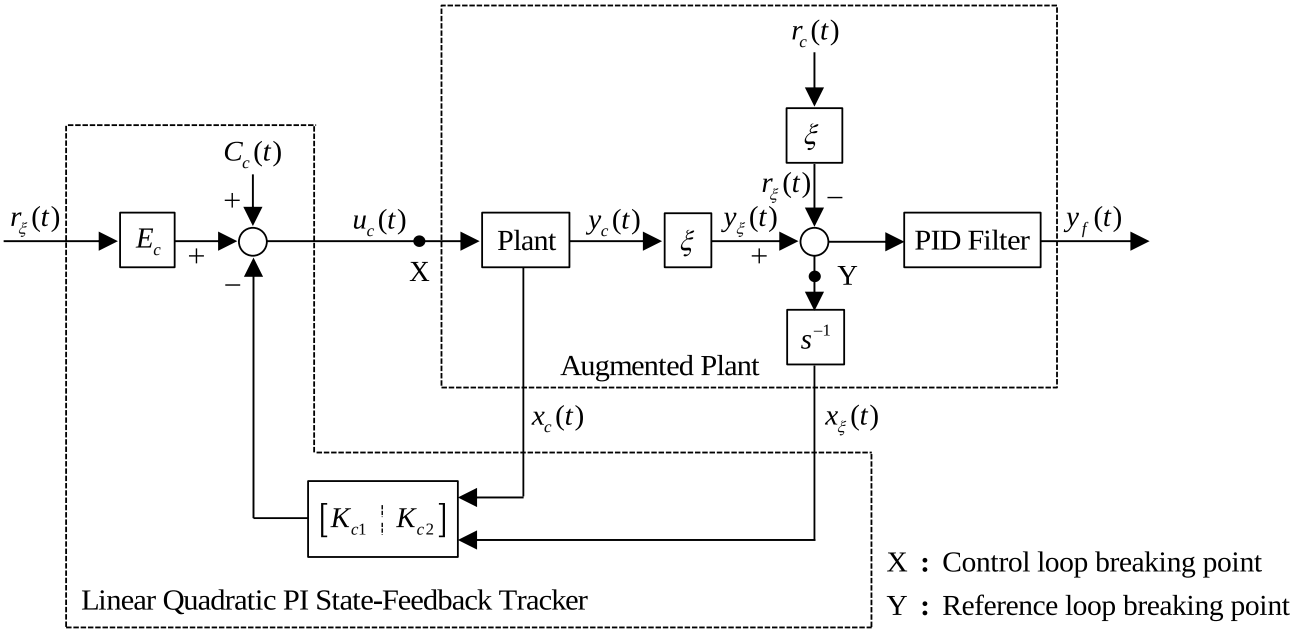


Figure 1. Linear quadratic PI state-feedback tracker.

Step 6: Examine the open-loop frequency response and adjust weights.
Open the control loop at point X shown in Figure 1 at the first input and examine the cross-over frequency, which we recall approximates the closed-loop control system bandwidth. If this bandwidth is too high (low) to achieve a practical design, then increase (decrease) the first component of the diagonal matrix R_c. Repeat for all inputs. Likewise, when considering response to reference signals $r_\xi(t)$, the reference loop is opened at point Y shown in Figure 1, and the following rule is applied. Increase (decrease) the *i*th diagonal element of Q_c, if the reference loop bandwidth is too low (high) for a practical design. Repeat for all *i*. The control loop and reference loop can be respectively derived as follows

$$L_X(s) = \left(K_{c1} + K_{c2}\tilde{C}s^{-1}\right)\left(sI_n - A\right)^{-1} B \tag{36}$$

and

$$L_Y(s) = \tilde{C}\left[sI_n - \left(A - BK_{c1}\right)\right]^{-1} BK_{c2}s^{-1}. \tag{37}$$

Case 2: The number of inputs is greater than the number of outputs

Step 1: Transform the non-minimum phase system to a minimum phase system.

i) Define the new output variables, being a linear combination of the original output measurements as

$$y_\xi(t) = \xi y_c(t) = \xi\left[Cx_c(t)\right] = \tilde{C}x_c(t), \tag{38}$$

where $\xi \in \Re^{p\times p}$ is a transformation matrix, such that $(A, B, \tilde{C})$ is observable, in which $\tilde{C} = \xi C$.

ii) Randomly choose an above-mentioned $p \times p$ non-singular transformation matrix ξ to ensure that the system $(A - BK_c,\ BE_c,\ \tilde{C})$ is minimum phase, where a two degrees of freedom LQAT in terms of (K_c, E_c) with the high-gain property is used.

Step 2: Assign some extra target zeros (without open-loop pole-zero cancellation) to attract some of closed-loop poles in a closed-loop design.

i) Append the PID filter to each element of $\left[y_\xi(t) - r_\xi(t)\right]$. The output of the PID filter is given by

$$Y_f(s) = \begin{bmatrix} k_p^{(1)} + k_i^{(1)}s^{-1} + k_d^{(1)}s & & 0 \\ & \ddots & \\ 0 & & k_p^{(p)} + k_i^{(p)}s^{-1} + k_d^{(p)}s \end{bmatrix}\left[Y_\xi(s) - R_\xi(s)\right]. \tag{39}$$

ii) Choose the PID filter gains $\left\{ k_p^{(j)},\ k_i^{(j)},\ k_d^{(j)} \mid j = 1 \sim p \right\}$ to assign target zeros

$$\left\{ \frac{-k_p^{(j)} \pm \sqrt{(k_p^{(j)})^2 - 4k_d^{(j)}k_i^{(j)}}}{2k_d^{(j)}} \;\middle|\; j = 1 \sim p \right\}. \tag{40}$$

Step 3: Construct the augmented plant.

i) Define the integral of $\left[y_\xi(t) - r_\xi(t)\right]$ as another state variable

$$x_\xi(t) = \int_0^t \left[y_\xi(\tau) - r_\xi(\tau)\right] d\tau. \tag{41}$$

ii) The state-space model of the augmented plant is given by

$$\begin{aligned} \begin{bmatrix} \dot{x}_c(t) \\ \dot{x}_\xi(t) \end{bmatrix} &= \begin{bmatrix} A & 0_{n\times p} \\ \tilde{C} & 0_{p\times p} \end{bmatrix}\begin{bmatrix} x_c(t) \\ x_\xi(t) \end{bmatrix} + \begin{bmatrix} B \\ 0_{p\times m} \end{bmatrix} u_c(t) + \begin{bmatrix} 0_{n\times 1} \\ -r_\xi(t) \end{bmatrix} \\ &= A_{aug}x_{aug}(t) + B_{aug}u_c(t) + d_{aug}(t), \end{aligned} \tag{42a}$$

$$\begin{aligned} y_f(t) &= \left[K_p\tilde{C} + K_d\tilde{C}A \quad K_i\right]\begin{bmatrix} x_c(t) \\ x_\xi(t) \end{bmatrix} + K_d\tilde{C}Bu_c(t) - \left[K_p r_\xi(t) + K_d\dot{r}_\xi(t)\right] \\ &= C_{aug}x_{aug}(t) + D_{aug}u_c(t) + s_{aug}(t). \end{aligned} \tag{42b}$$

Step 4: Formulate the performance index.

The first goal of this control problem is to minimize the performance index associated with the augmented plant with input $u_c(t)$ and output $y_f(t)$ as

$$J = \int_0^\infty \left\{\left[y_f(t) - r_\xi(t)\right]^T Q_c\left[y_f(t) - r_\xi(t)\right] + u_c^T(t)R_cu_c(t)\right\} dt, \tag{43a}$$

so that $y_f(t)$ well tracks the command input $r_\xi(t)$. By Parseval's theorem, the performance index can be written in the frequency domain as

$$J = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Big\{\Big[\big(K_P + K_I s^{-1} + K_D s\big)\xi\big(Y_c(s) - R_c(s)\big) - \xi R_c(s)\Big]^* Q_c \\ \times\Big[\big(K_P + K_I s^{-1} + K_D s\big)\xi\big(Y_c(s) - R_c(s)\big) - \xi R_c(s)\Big] + U_c^*(s) R_c U_c(s)\Big\} d\omega, \tag{43b}$$

where the superscript $*$ denotes the complex conjugate transpose and $s = j\omega$. The second goal of this control problem is to minimize the other performance index as

$$J' = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Big\{\big(Y_c(s) - R_c(s)\big)^* \Big[\xi^*\big(K_P + K_I s^{-1} + K_D s\big)^* Q_c \big(K_P + K_I s^{-1} + K_D s\big)\xi\Big] \\ \times\big(Y_c(s) - R_c(s)\big) + U_c^*(s) R_c U_c(s)\Big\} d\omega, \tag{44}$$

so that $y_c(t)$ well tracks the command input $r_c(t)$. Similarly, one can prove $y_f(t) \to r_\xi(t)$, $y_\xi(t) \to r_\xi(t)$, and $y_c(t) \to r_c(t)$, provided that $\dfrac{k_d^{(j)} s^2 + (k_p^{(j)} + 1)s + k_i^{(j)}}{k_d^{(j)} s^2 + k_p^{(j)} s + k_i^{(j)}} \to 1$, for $j = 1, 2, \cdots, p$, as well as the weighting function pair $\left\{\xi^*\big(K_P + K_I s^{-1} + K_D s\big)^* Q_c \big(K_P + K_I s^{-1} + K_D s\big)\xi,\ R_c\right\}$ has the high-gain property in the interested frequency range as mentioned above. Notice that $y_\xi(t) \to r_\xi(t)$ implies $y_c(t) \to r_c(t)$, since ξ is a non-singular square matrix for this case.

Step 5: Perform the linear quadratic PI state-feedback tracker design.
Repeat Step 5 in Case 1 to have the optimal control law

$$u_c(t) = -K_c x_{aug}(t) + E_c r_\xi(t) + C_c(t). \tag{45}$$

Step 6: Examine the open-loop frequency response and adjust weights.
Repeat Step 6 in Case 1. The control loop and reference loop can be respectively derived as

$$L_X(s) = \big(K_{c1} + K_{c2}\tilde{C}s^{-1}\big)\big(sI_n - A\big)^{-1} B \tag{46}$$

and

$$L_Y(s) = \tilde{C}\big[sI_n - \big(A - BK_{c1}\big)\big]^{-1} BK_{c2} s^{-1}. \tag{47}$$

Remark 6 (Anderson & Moore, 1989)**:** For rejection of disturbances u_{ext} having arbitrary constant values at the control input point, the dimension of $y_\xi(t)$ must be no less than that of u_{ext}. Notice that the dimension of disturbances u_{ext} may be less than that of control input $u_c(t)$. For tracking problem, a necessary condition is that the dimension of the control input $u_c(t)$ must be no less than that of the system output $y_c(t)$. These two dimensional constraints suggest that one can achieve fully the goals of arbitrary set point regulations only in the presence of arbitrary constant disturbance inputs, when the plant is square and with no zero at the origin.

As mentioned in Remark 4, for the square plant, we cannot find a non-singular $p \times p$ transformation matrix ξ to ensure that the transformed system $(A, B, \tilde{C})$ is minimum phase, also for the transformed system $(A, \tilde{B}, \tilde{C}) = (A, B\eta, \xi C)$. Due to this issue, the PID-filter shaped PI state-feedback LQAT for the square non-minimum phase system is left as a future research topic.

Notice that our proposed approach still works well for the square and/or non-square minimum phase plants. Also, to the best of our knowledge, no literature solves the optimal linear quadratic tracker for the system model in (30) or (42).

The quantitative statement for the high-gain property controller mentioned in this paper is given as follows.

Lemma 1 (Tsai, Hsu, Tsai, Lin, Guo, & Shieh, 2014) Given the analog system in terms of the pair of system matrices $\{A,B,C,D\}$, let a pair of weighting matrices $\{Q,R\}$ be given as diagonal matrices $Q=qI_P \gg R$ and $R=rI_m>0$. There exists the lower bound of the weighting matrix pair $\{Q^*,R^*\}$, i.e. $Q^*=q^*I_p$ and $R^*=r^*I_m$, determined by

$$\kappa^*=\sqrt{\frac{\|B^TB\|\|C^TC\|}{\|A^TA\|}\left(\frac{q^*}{r^*}\right)},$$

as long as the property of the high-gain control still holds, that is $P_2 \approx P_1$ for $\zeta=\kappa_2/\kappa_1=\sqrt{\frac{\|B^TB\|\|C^TC\|}{\|A^TA\|}\left(\frac{q_2}{r_2}\right)}\Bigg/\sqrt{\frac{\|B^TB\|\|C^TC\|}{\|A^TA\|}\left(\frac{q_1}{r_1}\right)}=\sqrt{\frac{q_2}{r_2}}\Bigg/\sqrt{\frac{q_1}{r_1}}$ and $\kappa_2>\kappa_1\geq\kappa^*$, where P_1 and P_2 are the symmetric positive-definite solutions of the following Riccati equations, respectively,

$$A^TP_1+P_1A+C^TQ_1C-(P_1B+C^TQ_1D)(R_1+D^TQ_1D)^{-1}(B^TP_1+D^TQ_1C)=0, \tag{48}$$

$$A^TP_2+P_2A+C^TQ_2C-(P_2B+C^TQ_2D)(R_2+D^TQ_2D)^{-1}(B^TP_2+D^TQ_2C)=0. \tag{49}$$

It is remarkable to notice that the high-gain property $P_2 \approx P_1$ for the proper system $\{A,B,C,D\}$ shall be revised as $P_2 \approx \kappa_2 P_1$ for the strictly proper system $\{A,B,C\}$ (Tsai, Du, Zhuang, Guo, Chen, & Shieh, 2011; Tsai, Hsu, Lin, Guo, & Tan, 2014). □

4. A new iterative learning LQAT with input constraint for the repetitive system with a direct-feedthrough term and unknown disturbances

Consider the continuous-time minimum phase repetitive system with unknown deterministic disturbances

$$\dot{x}_{cj}(t)=Ax_{cj}(t)+Bu_{cj}(t)+d(t),\quad x_{cj}(0)=x_0,\quad \text{for } j=0,1,2,..., \tag{50a}$$

$$y_{cj}(t)=Cx_{cj}(t)+Du_{cj}(t)+s(t), \tag{50b}$$

where j is the learning iteration number, $d(t)\in\Re^n$ and $s(t)\in\Re^p$ are unknown deterministic disturbances, and A, B, C, D are constant matrices with appropriate dimensions. Since $d(t)$ and $s(t)$ are unknown disturbances, one can imagine there exists an equivalently undetermined 'artificial' system model

$$\dot{x}_{ac}(t)=Ax_{ac}(t)+Bu_{ac}(t), \tag{51a}$$

$$y_{ac}(t)=Cx_{ac}(t)+Du_{ac}(t)+s_a(t), \tag{51b}$$

where $s_a(t)$, however, denotes the 'actual' steady-state error signal (to be determined later) between the actual output of the system $y_{cj}(t)$ in (50b) and the pre-specified track $r(t)$ as $j\to\infty$. That is, the whole influence to the system induced by disturbances $d(t)$ and $s(t)$ can be seen as the influence induced by $s_a(t)$. Consequently, the artificial control $u_{ac}(t)$ can be determined from Section 2, such as

$$u_{ac}(t) = -K_c x_{ac}(t) + E_c r(t) + C_{ac}(t) + C_{cu} u_{ac}^*(t), \tag{52a}$$

where

$$K_c = \bar{R}_c^{-1}\left(B^T P + N^T\right), \tag{52b}$$

$$E_c = -\bar{R}_c^{-1}\left[\left(C - DK_c\right)\left(A - BK_c\right)^{-1} B - D\right]^T Q_c, \tag{52c}$$

$$C_{ac}(t) = \bar{R}_c^{-1}\left[\left(C - DK_c\right)\left(A - BK_c\right)^{-1} B - D\right]^T Q_c s_a(t), \tag{52d}$$

$$C_{cu} = \bar{R}_c^{-1}\left\{I_m + B^T\left[\left(A - BK_c\right)^T\right]^{-1} K_c^T\right\} R_c, \tag{52e}$$

$$\bar{R}_c = R_c + D^T Q_c D, \tag{52f}$$

$$N = C^T Q_c D, \tag{52g}$$

for the infinite-horizon tracker problem, and $s_a(t)$ is to be determined in the following.

To determine the actual steady-state error signal $s_a(t)$, let us modify (13) as

$$\begin{aligned}-\dot{v}_j(t) &= \left(A - BK_c\right)^T v_j(t) + \left(C - DK_c\right)^T Q_c\left[r(t) - s_{aj}(t)\right] - K_c^T R_c u_{ac}^*(t)\\ &= \left(A - BK_c\right)^T v_j(t) + \left(C - DK_c\right)^T Q_c\left[r(t) - \sum_{i=0}^{j} e_{i-1}(t)\right] - K_c^T R_c u_{ac}^*(t), \text{ for } j = 0,\ 1,\ 2, \ldots,\end{aligned} \tag{53}$$

for the repetitive system (50), where

$$s_{aj}(t) = \sum_{i=0}^{j} e_{i-1}(t) \tag{54}$$

and

$$e_{i-1}(t) = y_{c(i-1)}(t) - r(t), \tag{55}$$

for $e_{-1}(t) = 0$. For simplicity, let us consider the infinite-horizon tracker problem to induce the optimal error compensation ILC as

$$u_{cj}(t) = -K_c x_{cj}(t) + E_c r(t) + C_{acj}(t) + C_{cu} u_c^*(t), \tag{56a}$$

where

$$K_c = \bar{R}_c^{-1}\left(B^T P + N^T\right), \tag{56b}$$

$$E_c = -\bar{R}_c^{-1}\left[\left(C - DK_c\right)\left(A - BK_c\right)^{-1} B - D\right]^T Q_c, \tag{56c}$$

$$C_{acj}(t) = \bar{R}_c^{-1}\left[\left(C - DK_c\right)\left(A - BK_c\right)^{-1} B - D\right]^T Q_c \sum_{i=0}^{j} e_{i-1}(t), \tag{56d}$$

$$C_{cu} = \bar{R}_c^{-1}\left\{I_m + B^T\left[\left(A - BK_c\right)^T\right]^{-1} K_c^T\right\} R_c, \tag{56e}$$

$$\bar{R}_c = R_c + D^T Q_c D, \tag{56f}$$

$$N = C^T Q_c D, \tag{56g}$$

$$e_{i-1}(t) = y_{c(i-1)}(t) - r(t) \ \left(= \left[C x_{c(i-1)}(t) + D u_{c(i-1)}(t) + s(t)\right] - r(t)\right). \tag{56h}$$

The framework of the proposed OECILC is demonstrated in Figure 2, where

$$Z_3 = \bar{R}_c^{-1}\left[\left(C - DK_c\right)\left(A - BK_c\right)^{-1} B - D\right]^T Q_c.$$

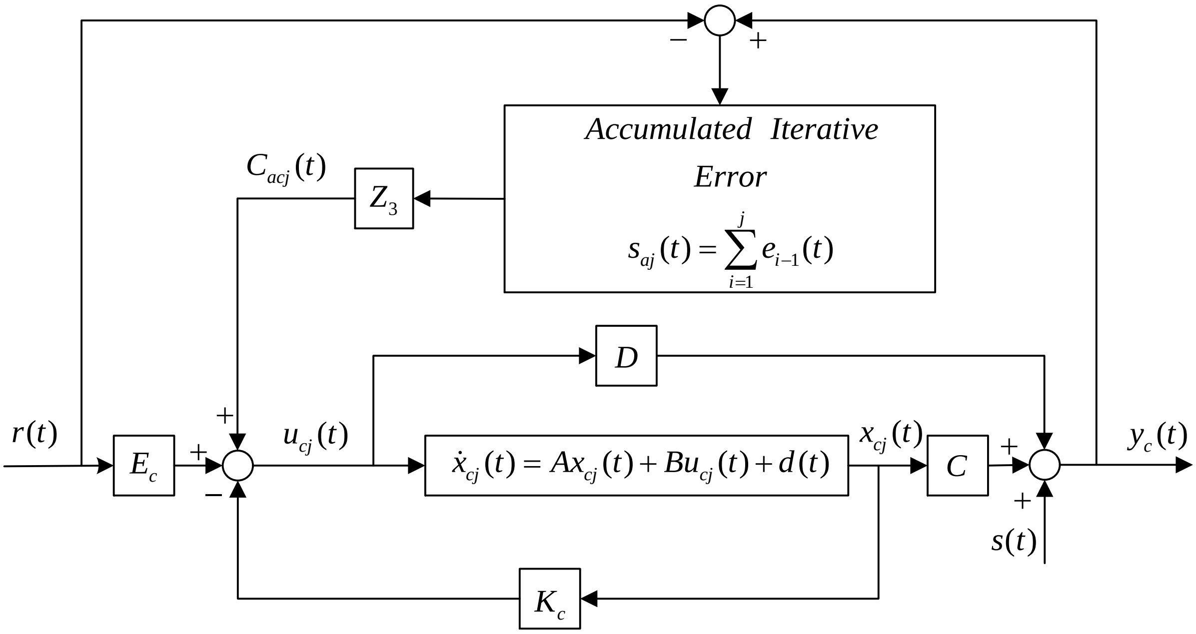


Figure 2. Framework of the OECILC minimum phase system.

Some remarkable observations are given as follows. For $j = 0,\ 1,\ 2,...,$ one has

$$u_{c0}(t) = -K_c x_{c0}(t) + E_c r(t) + C_{cu} u_c^*(t),$$

$$u_{c1}(t) = -K_c x_{c1}(t) + E_c r(t) + Z_3 e_0(t) + C_{cu} u_c^*(t),$$

$$u_{c1}(t) - u_{c0}(t) = -K_c\left[x_{c1}(t) - x_{c0}(t)\right] + Z_3 e_0(t),$$

$$u_{c2}(t) = -K_c x_{c2}(t) + E_c r(t) + Z_3\left[e_0(t) + e_1(t)\right] + C_{cu} u_c^*(t),$$

$$u_{c2}(t) - u_{c1}(t) = -K_c\left[x_{c2}(t) - x_{c1}(t)\right] + Z_3 e_1(t),$$

$$\vdots$$

$$u_{cj}(t) = -K_c x_{c(j-1)}(t) + E_c r(t) + Z_3 \sum_{i=0}^{j} e_{i-1}(t) + C_{cu} u_c^*(t),$$

$$u_{cj}(t) - u_{c(j-1)}(t) = -K_c\left[x_{cj}(t) - x_{c(j-1)}(t)\right] + Z_3 e_{j-1}(t),$$

which implies

$$u_{cj}(t) = u_{c(j-1)}(t) - \bar{R}_c^{-1}\left(B^T P + N^T\right)\left[x_{cj}(t) - x_{c(j-1)}(t)\right]$$
$$+ \bar{R}_c^{-1}\left[\left(C - DK_c\right)\left(A - BK_c\right)^{-1} B - D\right]^T Q_c e_{j-1}(t). \tag{57}$$

To compare our proposed optimal error compensation ILC for the direct-feedthrough term-free case with the optimal ILC for the continuous-time minimum phase system presented in (Nasiri, 2006), let's briefly summary the optimal ILC presented in (Nasiri, 2006) as follows.

Consider the same repetitive minimum phase system given in (50). It is desired to determine the optimal ILC

$$u_j(t) = u_{j-1}(t) + \Delta u_j(t), \tag{58}$$

to minimize the quadratic cost function

$$J_j = \frac{1}{2}\int_0^{t_f} \left\{ e_j^T(t) Q_c e_j(t) + \left[\Delta u_j(t) \right]^T R_c \left[\Delta u_j(t) \right] \right\} dt, \tag{59}$$

where

$$e_j(t) = y_j(t) - r(t). \tag{60}$$

Physical interpretation of the given cost function is that we wish to close the system output to desired output trajectory without many changes in input of the system. Finally, the optimal input of the system at iteration j is obtained as

$$u_j(t) = u_{j-1}(t) - R_c^{-1} B^T \left\{ P(t) \left[x_j(t) - x_{j-1}(t) \right] + v_j(t) \right\}, \tag{61}$$

where $P(t)$ and $v(t)$ satisfy the following equations, respectively,

$$-\dot{P}(t) = A^T P(t) + P(t) A - P(t) B R_c^{-1} B^T P(t) + C^T Q C \tag{62}$$

and

$$-\dot{v}_j(t) = -\left[A - B R_c^{-1} B^T P(t) \right]^T v_j(t) + C^T Q e_{j-1}(t). \tag{63}$$

Consider the same infinite-horizon tracker problem, where $v_j(t) = v_{j-1}(t)$. Then, (63) implies

$$v_j(t) = \left[(A - B R_c^{-1} B^T P(t))^T \right]^{-1} C^T Q_c e_{j-1}(t). \tag{64}$$

Substituting (64) to (57) shows

$$u_{cj}(t) = u_{c(j-1)}(t) - R^{-1} B^T \left\{ P(t) \left[x_{cj}(t) - x_{c(j-1)}(t) \right] + v_j(t) \right\} \tag{65}$$

is identical to (61). This implies our proposed ILC is essentially identical to the optimal ILC (Nasiri, 2006) for the minimum phase plant, although both formulate are not same for this case.

To implement the optimal ILC presented in (Nasiri, 2006), the control input of the initial learning epoch is generally assumed to be

$$u_{c0}(t) = 0, \text{ for } t \text{ in } \left[0,\ t_f \right]. \tag{66}$$

However, it may not be acceptable for some practical systems. Nevertheless, the initialization of the control in the first learning epoch of our approach is determined by

$$u_{c0}(t) = -K_c x_{c0}(t) + E_c r(t). \tag{67}$$

Experience shows that by this initialization, it significantly reduces the learning epochs for the pre-specified tracking performance (Chen, Tsai, Liao, Guo, Ho, Shaw, & Shieh, 2014).

Notice that whenever the given non-square plant is non-minimum phase, integrating with the presented approach in Sec. 3 is suggested for the desired goal. It is worth to mention that the direct-feedthrough term-free and disturbance-free optimal ILC presented in (Nasiri, 2006) for the continuous-time minimum phase system without input constraint has now been extended to the case for the (non-minimum phase non-square) system with a direct-feedthrough term, unknown disturbances, and input constraint, as well as with a pre-specified control input if it is available.

5. Illustrative examples

In this section, some numerical simulations are given to illustrate some applications for the proposed generalized optimal tracker.

5.1. A new optimal PI state-feedback linear quadratic tracker for non-square non-minimum phase systems: PID-based frequency shaping approach

Consider the continuous-time, non-square, and non-minimum phase system described by

$$\dot{x}_c(t)=\begin{bmatrix} 2.1538 & 0.6194 & -0.5075 & 0.5989 & -1.1474 & 0.6040 \\ 0.2146 & 2.8620 & 0.5545 & 3.8924 & 0.3211 & -1.0499 \\ 1.8895 & 4.4859 & 0.8324 & 0.0378 & -0.3618 & 4.8398 \\ -0.2987 & -0.6874 & 1.9427 & -1.1916 & 1.1511 & -2.3237 \\ 0.1131 & 3.9398 & -2.1418 & -1.0836 & 2.7937 & 1.3274 \\ -1.0133 & 1.5181 & -0.7346 & -0.8742 & 1.8056 & 1.3479 \end{bmatrix} x_c(t)$$

$$+\begin{bmatrix} -3.5975 & -2.6397 & 0.9812 \\ 1.0390 & 2.3712 & 1.0171 \\ -0.0588 & -0.3828 & 2.5450 \\ 1.9103 & -1.0776 & -2.8154 \\ -0.4696 & -0.5244 & -0.3671 \\ 2.3904 & 2.3477 & 1.1726 \end{bmatrix} u_c(t),\ x_c(0)=0_{6\times 1},$$

$$y_c(t)=\begin{bmatrix} 2.2397 & -2.8967 & -0.9704 & -2.6084 & -0.9583 & -1.8260 \\ -0.2340 & 3.1711 & -0.9539 & 0.3118 & 0.2554 & 2.8133 \end{bmatrix} x_c(t).$$

The desired output trajectory is given by $r_\xi(t)=\begin{bmatrix} r_{\xi,1}(t) & r_{\xi,2}(t)\end{bmatrix}^T$, where

$$r_{\xi,1}(t)=\begin{cases}\cos(2\pi t) & ,\ 0\le t<1 \text{ sec} \\ 0.5\ t^2(1-t) & ,\ 1\le t<2 \text{ sec} \\ 0.5\ \cos(4\pi t)+1 & ,\ 2\le t\le 3 \text{ sec}\end{cases} \text{ and } r_{\xi,2}(t)=\begin{cases}1.2\ t^2(1-t) & ,\ 0\le t<1 \text{ sec} \\ \cos(2\pi t) & ,\ 1\le t<2 \text{ sec} \\ 0.2\ \sin(4\pi t)-0.5 & ,\ 2\le t\le 3 \text{ sec}\end{cases},$$

with system poles $\{-0.9306\pm 2.4346i,\ 0.8395,\ 2.5397\pm 1.5767,\ 4.7407\}$ and control zeros $\{-0.0994, -2.5631,\ 4.2808,\ -5.2184,\ -2.967\times 10^5,\ -5.9432\times 10^4\}$ (obtained by the proposed approach in Sec. 3). A simple test shows that the controlled system has the inverse response (i.e. undershoot), which implies the system has control zero in the right-half plane; however, the syntax ‘tzero (sys)’ in MATLAB fails to detect any finite zeros of the given non-square system. Also, the traditional optimal LQAT-based closed-loop non-minimimum phase system demonstrates a poor tracking performance in general. To overcome this issue, the design procedure presented in Sec. 3 is demonstrated as follows.

Step 1: Transform the non-minimum phase system to a minimum phase system.
Let the selected 2×2 transformation matrix be $\xi=\begin{bmatrix} -24.3750 & -79.2337 \\ -89.7601 & -95.2975\end{bmatrix}$ to ensure that the system $(A-BK_c,\ BE_c,\ \tilde{C})$ is minimum phase, where a two degrees of freedom LQAT in terms of (K_c, E_c) is used for $Q_c=10^6 I_2$ and $R_c=I_3$. There results minimum phase finite control zeros $\{-0.0990,\ -2.5685,\ -4.2324,\ -5.2161\}$.

Step 2: Assign some extra target zeros to attract some of closed-loop poles in a closed-loop design.
Let extra target zeros be $\{-0.1,\ -0.1,\ -1\times 10^5,\ -1\times 10^5\}$, then one can select the PID filter gains as

$$K_p=\begin{bmatrix} 10 & 0 \\ 0 & 10\end{bmatrix}, K_i=\begin{bmatrix} 1 & 0 \\ 0 & 1\end{bmatrix}, \text{ and } K_d=10^{-4}\times\begin{bmatrix} 1 & 0 \\ 0 & 1\end{bmatrix}.$$

The singular value plot of the augmented system $(A_{aug}, B_{aug}, C_{aug}, D_{aug})$ is shown in Figure 3.

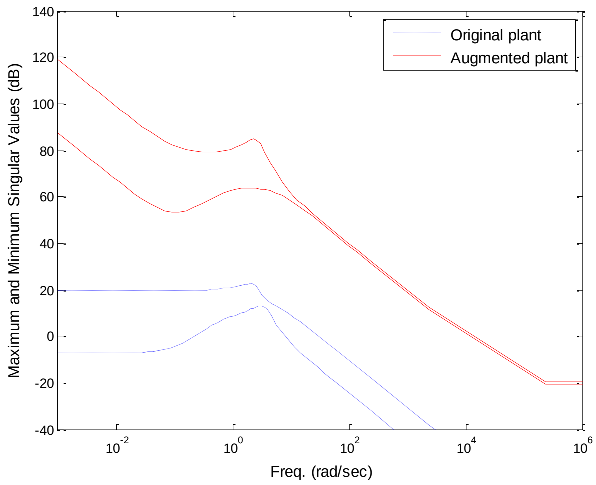


Figure 3. Singular value plot of the augmented plant.

Step 3: Perform the linear quadratic PI state-feedback tracker design.
In the LQAT design, choose an appropriate weighting matrix pair $\{Q_c, R_c\} = \{10^6 I_2, I_3\}$ to have the optimal control law

$$u_c(t) = -K_c x_{aug}(t) + E_c r_\xi(t) + C_c(t),$$

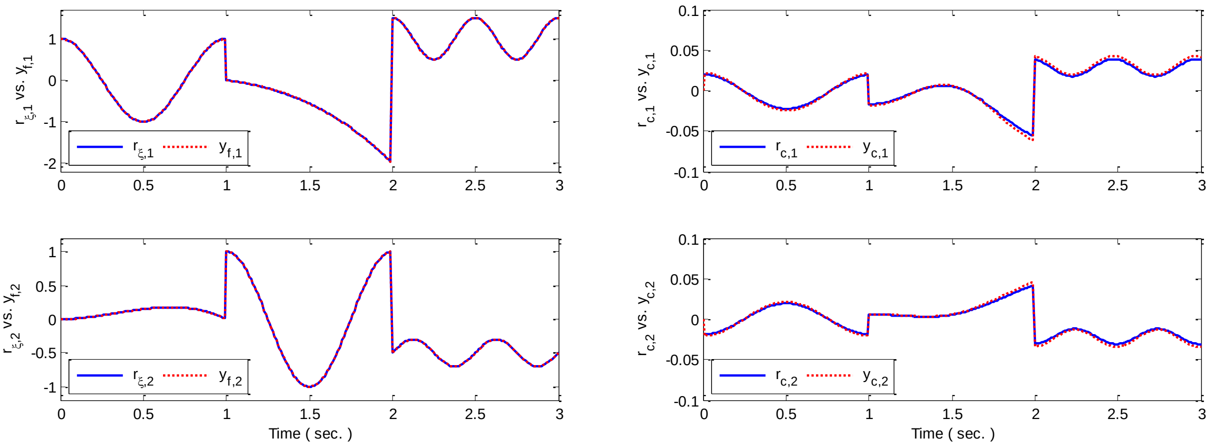
where the PI state-feedback and feed-forward gains are

$$K_c = [K_{c1} \mid K_{c2}]$$

$$= 10^5 \times \left[\begin{array}{cccccc:cc} 0.1737 & 0.6129 & 0.1767 & 0.5237 & 0.0375 & -0.0636 & -0.0001 & 0.0000 \\ -0.2971 & -0.4915 & -0.1961 & -0.4915 & 0.0029 & 0.2361 & 0.0000 & 0.0000 \\ 0.7372 & 1.1676 & -0.0332 & 0.5311 & -0.0740 & 0.0285 & -0.0002 & -0.0002 \end{array}\right],$$

$$E_c = 10^1 \times \begin{bmatrix} -1.2429 & 0.4364 \\ 0.0458 & 0.3157 \\ -1.8125 & -1.5547 \end{bmatrix}.$$

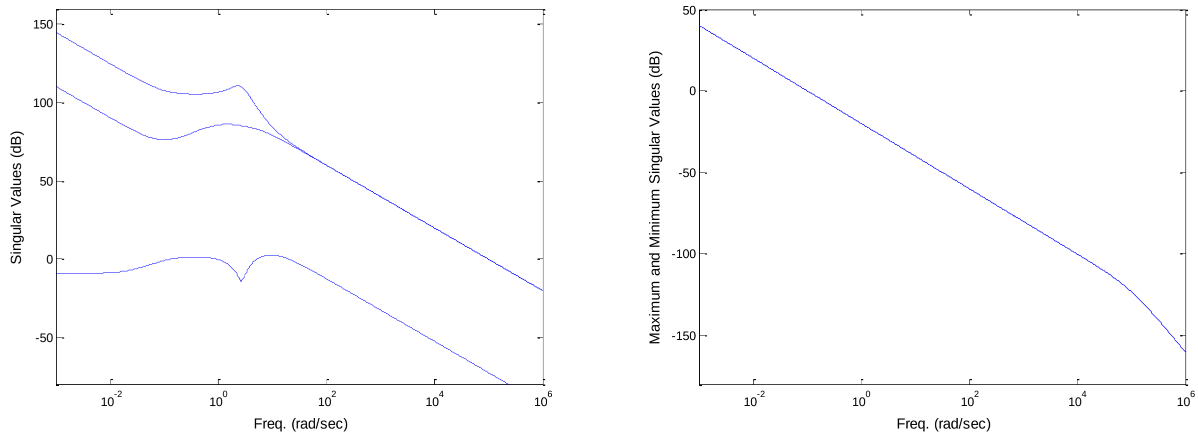
The tracking performances of the closed-loop system are shown in Figure 4. It shows that $y_f(t) \to r_\xi(t)$ and $y_c(t) \to r_c(t)$ as expected.



(a) (b)

Figure 4. Time responses of the closed-loop system: (a) $r_\xi(t)$ vs. $y_f(t)$, (b) $r_c(t)$ vs. $y_c(t)$.

Step 4: Examine the open-loop frequency response.
Open-loop frequency responses for $Q_c = 10^6 I_2$ and $R_c = I_3$ are shown in Figure 5. The control loop shows that the system has good external disturbance rejection if the external disturbance occurs at the first and second control inputs. Moreover, the reference loop shows that the system has good sensor noise rejection.



(a) (b)

Figure 5. Open-loop frequency responses: (a) control loop, (b) reference loop.

5.2. A new optimal LQAT with input constraint for the repetitive system with a direct-feedthrough term and unknown disturbances

Consider the repetitive minimum phase system in (50) with unknown deterministic disturbances $d(t)$ and $s(t)$, where

$$A = \begin{bmatrix} -0.02 & 0.005 & 2.4 & -32 \\ -0.14 & 0.44 & -1.3 & -30 \\ 0 & 0.018 & -1.6 & 1.2 \\ 0 & 10 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0.14 & -0.12 & 0 \\ 0.36 & -8.6 & 0 \\ 0.35 & 0.01 & 0 \\ 0.6 & 0.5 & 1 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 27.3 \end{bmatrix}, \quad D = \begin{bmatrix} -0.1 & 0 & 1 \\ 0 & 0.2 & -2 \end{bmatrix}, \quad x_d(0) = \begin{bmatrix} -7 & 1 & -2 & -0.1 \end{bmatrix}^T,$$

the deterministic system disturbance $d(t) = \begin{bmatrix} d_1(t) & d_2(t) & d_3(t) & d_4(t) \end{bmatrix}^T$, where $d_1(t)$ is created by random distribution white signal $N(0,1)$ with zero mean and standard deviation 1, $d_2(t) = 1$,

$d_3(t) = 0.6\sin(t)$, $d_4(t) = \begin{cases} 0.4, & 0 \le t < 4\,\text{sec} \\ 0, & 4 \le t < 8\,\text{sec} \\ 0.4, & 8 \le t < 10\,\text{sec} \end{cases}$, and the deterministic measurement disturbance $s(t)$ is created by $s(t) = \begin{bmatrix} 0.1\sin(5t) & 0.1\sin(3t) \end{bmatrix}^T$.

Case 1. Input-constraint-free case for OECILC for the repetitive system with a direct-feedthrough term and unknown disturbances

It is required to determine an iterative learning tracker for the input-constraint-free case, so that the controlled system demonstrates a good tracking performance for

$$r_1(t)=\begin{cases}\cos(t), & 0\le t<3\sec\\ -1+\cos(t), & 3\le t<6\sec\\ 1+\cos(t), & 6\le t<10\sec\end{cases},\quad r_2(t)=\begin{cases}1+\sin(t), & 0\le k<3\sec\\ 1+\cos(t), & 3\le k<6\sec\\ -0.4+\sin(t), & 6\le k<10\sec\end{cases}.$$

The proposed OECILC (56) is then determined, where

$$K_c=\begin{bmatrix}0.1313 & -130.22 & -31.21 & -724.29\\ 0.1403 & -124.27 & -28.36 & -637.29\\ 0.0138 & -12.72 & -3.29 & -76.29\end{bmatrix},\quad E_c=\begin{bmatrix}-49.87 & -27.59\\ -43.27 & -24.49\\ -4.25 & -2.91\end{bmatrix},$$

$$C_{acj}(t)=\begin{bmatrix}49.87 & 27.59\\ 43.27 & 24.49\\ 4.25 & 2.91\end{bmatrix}\sum_{i=0}^{j}e_{i-1}(t),$$

for $Q_c=10^4 I_p$, and $R_c=I_m$. The convergence of learning errors is shown in Figure 6, which demonstrates a satisfied tracking performance as expected.

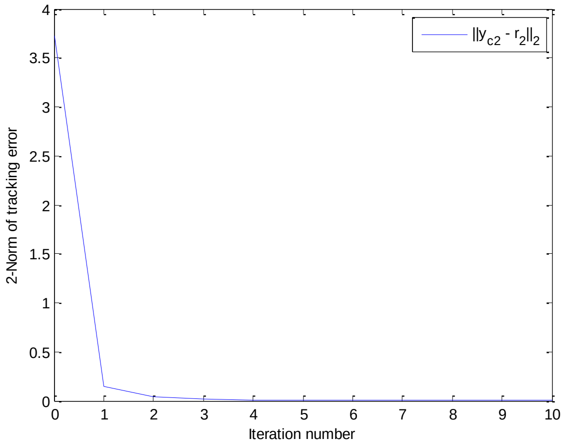


Figure 6. Convergence of the proposed OECILC-based tracking performance vs. iteration number $j=0,1,2,\cdots 5$: $\left\|y_{c2,j}-r_2\right\|_2$ (shown by parts).

Case 2. Input-constrained OECILC for the repetitive system with a direct-feedthrough term and unknown disturbances

The objective of the classical tracking problem is to design an appropriate control law $u_c(t)$ so that the system output $y_c(t)$ can well track the pre-specified output target trajectory $r(t)$. In Section 2, a novel tracking problem is presented, so that a trade-off between the output tracking for a pre-specified output target $r(t)$ and the control input tracking for a pre-specified input target $u_c^*(t)$, in terms of each corresponding weighting matrices Q_c and R_c, is newly proposed. A new application of this problem on the input-constrained tracker design is to be given in the following.

Let the continuous-time system have the above-mentioned disturbances. Based on the proposed input-saturation-free LQAT for $Q_c=10^4 I_2$ and $R_c=I_3$, the tracking performance is quite satisfactory. Figure 8 (a) shows the proposed tracker-based control input has some acute responses at some time instants. Assume it is desired to narrow down the control-input magnitudes at those time instants with unexpected acute responses, and let $u_c^*(t)$ specify the desired time response of the control input. Here, based on the bound of the input-saturation-free case $u_c^I(t)=\begin{bmatrix}[-123.67,\ 65.42]\\ [-99.99\ \ 36.32]\\ [-12.36,\ \ 6.65]\end{bmatrix}$, the saturation bound of $u_c^*(t)$ is pre-specified as $u_{c\_sat}^{I*}(t)=\begin{bmatrix}[-66.91,\ 60.54]\\ [-43.26,\ 31.13]\\ [-7.94,\ 6.68]\end{bmatrix}$. Whenever some components of $u_c(t)$ exceeds the upper/low bound of $u_{c\_sat}^*(t)$, the corresponding components of $u_c^*(t)$ are specified as the pre-specified upper/low bound, and time-varying weighting matrices $\{Q_c(t),\ R_c(t)\}$ are suggested at these sampling instants if it is necessary; otherwise, it is set to be the default where $u_c^*(t)=0$ for this particular application demonstrated in this example. Then, we re-apply the proposed disturbance-based LQAT for $Q_c=10^4 I_2$ and $R_c=I_3$ to the given system. Figure 7 and Figure 8 show that both output tracking and control input tracking are well-performed.

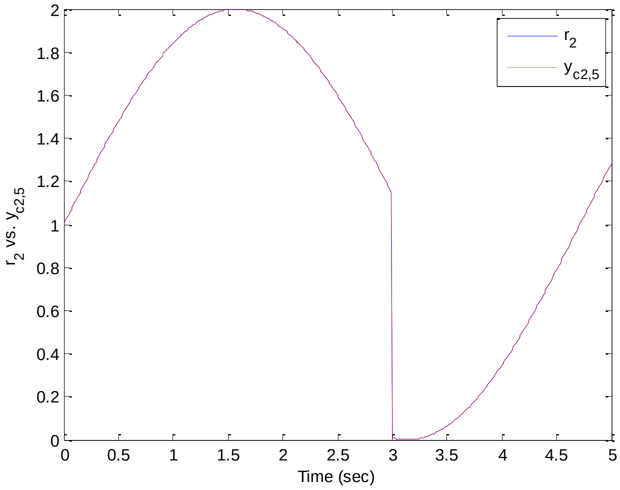
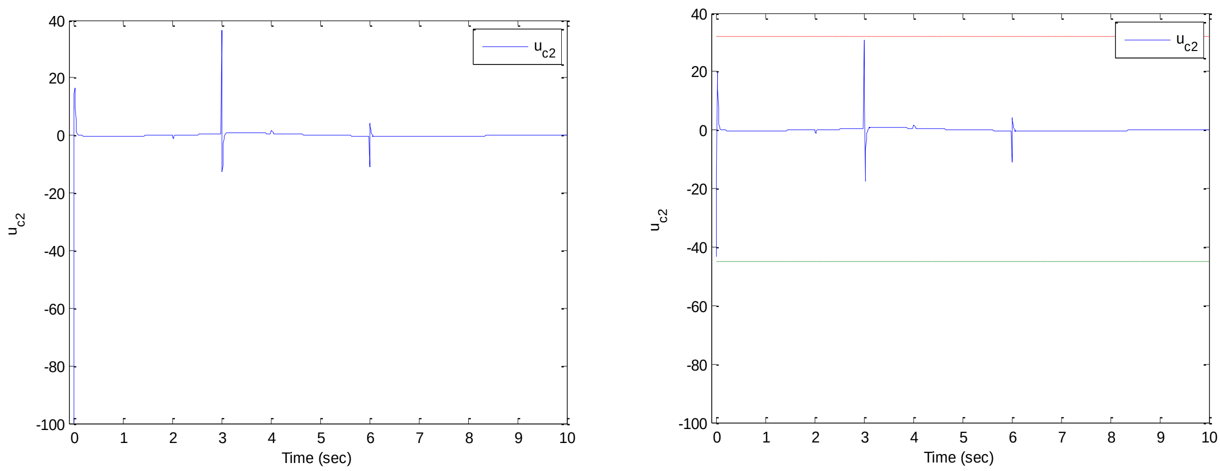


Figure 7. The proposed OECILC-based tracking responses for the system with unknown disturbances and a direct-feedthrough term under input constraint: $y_{c2,5}(t)$ vs. $r_2(t)$ (shown by parts).



(a) (b)

Figure 8. Control inputs of the proposed OECILC controlled system with a direct-feedthrough term and unknown disturbances: (a) control inputs $u_{c2}(t)$ without input constraint, (b) control inputs $u_{c2}(t)$ under input constraint (shown by parts).

Figure 8 also shows those acute control inputs are restricted to the pre-specified saturated bounds for this illustrative example.

6. Conclusion

This paper has developed a generalized optimal linear quadratic tracker for minimum/non-minimum phase and square/non-square systems. Their theoretical developments are summarized as follows: (i) A novel optimal LQAT with pre-specified measurement output and control input trajectories and its corresponding steady-state version for the continuous-time controllable, observable, and non-degenerate system, which has both an input-to-output direct-feedthrough term and the known/estimated system disturbances or compensatory signals, have been presented. (ii) A new optimal PID filter-shaped PI state-feedback linear quadratic tracker for non-square non-minimum phase systems has been also proposed. As a result, the controlled non-minimum phase non-square multivariable systems are able to achieve a minimum phase-like tracking performance. In addition, the proposed tracker works well for the arbitrary reference trajectory with drastic variations. (iii) A closed-loop output-zeroing control system for the given non-square MIMO system integrated with the two degrees of freedom optimal LQAT has been newly proposed also, so that the reliable algorithm available in the literature can be used for computing the control zeros of the square closed-loop MIMO system (squaring down due to the use of the two degrees of freedom optimal LQAT for the system with an input number (*m*) greater than the output number (*p*) or integrated with a transformation matrix for the case of $p > m$). Notice that the syntax 'tzero (sys)' in MATLAB fails to detect any finite zeros of some non-square systems demonstrated in this paper. (iv) A new iterative learning LQAT with input constraint for the repetitive system with a direct-feedthrough term and unknown disturbances has been presented. Simulation experience shows that by the presented initialization of the proposed ILC, it could significantly reduce the learning epochs for the pre-specified tracking performance.

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