Linear Algebra-A

Assignments - Week 8

Assignments from the Textbook (*Hardcover*)

Section 3.3: 22,24,27
Section 3.4: 1,2,3,4,5,6,8,13,15,16,17,27,28,30.
Note: #30(a): correction:

$$B = column\ 2 + \frac{1}{2}(column\ 1) \quad and \quad C = column\ 3 + \frac{2}{3}(column\ 2)$$

Supplementary Problem Set

1. Let $\boldsymbol{A} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 1 & -1 & 2 \\ 1 & -1 & -2 \end{bmatrix}$, $\boldsymbol{b} = \begin{bmatrix} 2 \\ -2 \\ -1 \\ 3 \end{bmatrix}$.
   a) Explain why $\boldsymbol{Ax} = \boldsymbol{b}$ is inconsistent.
   b) Find the least squares solution to $\boldsymbol{Ax} = \boldsymbol{b}$.
   c) Split $\boldsymbol{b}$ into a column space component $\boldsymbol{b}_c$ and a left nullspace component $\boldsymbol{b}_l$, i.e., $\boldsymbol{b} = \boldsymbol{b}_c + \boldsymbol{b}_l$.

2. Let $\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_m$ be linearly independent vectors in $\mathbf{R}^n$ $(n > m)$, and

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{v}_1^T \\ \boldsymbol{v}_2^T \\ \vdots \\ \boldsymbol{v}_m^T \end{bmatrix}.$$

   It follows that $\boldsymbol{A}$ is an $m \times n$ matrix with rank m.
   Let $\boldsymbol{w}_1, \boldsymbol{w}_2, \cdots, \boldsymbol{w}_{n-m}$ be a set of linearly independent vectors in $\mathbf{R}^n$ satisfying

$$\boldsymbol{Aw}_j = \boldsymbol{0}, \qquad j = 1,2,\cdots,n-m.$$

   Show that the vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_m, \boldsymbol{w}_1, \boldsymbol{w}_2, \cdots, \boldsymbol{w}_{n-m}$ are linearly independent.
   (Note: This is to say, *the basis for the row space* $C(\boldsymbol{A}^{\mathrm{T}})$ *and the basis for the nullspace* $N(\boldsymbol{A})$ *together form a basis for* $\mathbf{R}^n$.)

3. Let $\boldsymbol{A}$ be an $m \times n$ real matrix and $\boldsymbol{A}^T$ be its transpose. Show that the column spaces of $\boldsymbol{A}^T\boldsymbol{A}$ and $\boldsymbol{A}^T$ are the same, i.e., $C(\boldsymbol{A}^T\boldsymbol{A}) = C(\boldsymbol{A}^T)$.
   (Note: This is another way to prove that for the least square method, the normal equation $\boldsymbol{A}^T\boldsymbol{Ax} = \boldsymbol{A}^T\boldsymbol{b}$ is always solvable.)

4. Let $\mathbf{0} \neq \boldsymbol{v} \in \mathbf{R}^n$. Please give a matrix $\boldsymbol{P}$ such that

$$\begin{cases} \boldsymbol{P}\boldsymbol{v} = \mathbf{0} \\ \boldsymbol{P}\boldsymbol{x} = \boldsymbol{x}, \forall \boldsymbol{x} \in N(\boldsymbol{v}^T) \end{cases},$$

where $N(\boldsymbol{v}^T)$ is the nullspace of $\boldsymbol{v}^T$. In addition, please show that

a) $\boldsymbol{P}^T = \boldsymbol{P}$ and $\boldsymbol{P}^2 = \boldsymbol{P}$.

b) Please show that $\boldsymbol{P}\boldsymbol{b}$ is the projection of $\boldsymbol{b}$ onto the column space of $\boldsymbol{P}$. The error vector $\boldsymbol{b} - \boldsymbol{P}\boldsymbol{b}$ is orthogonal to the space. In other words, please show that the inner product $(\boldsymbol{b} - \boldsymbol{P}\boldsymbol{b})^{\mathrm{T}}\boldsymbol{P}\boldsymbol{c} = \mathbf{0}$.

5. Let $\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 1 & 2 & 1 & 2 \\ 3 & 5 & 4 & 6 \end{bmatrix}$.

Please give a 4 by 4 orthogonal matrix $\boldsymbol{Q} = [\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3, \boldsymbol{q}_4]$, such that $\boldsymbol{q}_1, \boldsymbol{q}_2 \in C(\boldsymbol{A}^T)$ and $\boldsymbol{q}_3, \boldsymbol{q}_4 \in N(\boldsymbol{A})$.