Linear Algebra-A

Assignments - Week 10

(Class 3 - 线性代数 A 中英双语 3 班)

Assignments from the Textbook (*Hardcover*)

Section 4.3: 11,36,40.
Section 4.4: 1,2,4,5,8,13,16,23,28,35,42.
Section 5.1: 3,6,7.

Supplementary Problem Set

1. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be $n \times n$ matrices. Please prove that
$$|\boldsymbol{I}_n - \boldsymbol{AB}| = |\boldsymbol{I}_n - \boldsymbol{BA}|.$$
As a corollary, $\boldsymbol{I}_n - \boldsymbol{AB}$ is invertible if and only if $\boldsymbol{I}_n - \boldsymbol{BA}$ is invertible.
[**Hint**: start from the block matrix: $\begin{bmatrix} \boldsymbol{I}_n & \boldsymbol{A} \\ \boldsymbol{B} & \boldsymbol{I}_n \end{bmatrix}$.]

2. Calculate the following determinant:
$$D = \begin{vmatrix} a_0 & 1 & 2 & \cdots & n \\ 1 & a_1 & 0 & \cdots & 0 \\ 2 & 0 & a_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ n & 0 & 0 & \cdots & a_n \end{vmatrix}.$$
[注：此行列式又称作“爪形行列式”或“箭头形行列式”。
提示：按照$a_1, a_2, \ldots, a_n$中0的个数分别进行讨论.]

3. （抽象型行列式）
a) Let $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3$ be 3-dimensional column vectors which are linearly independent. If $\boldsymbol{A} = [\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3]$, $\boldsymbol{B} = [\boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2 + \boldsymbol{\alpha}_3, \boldsymbol{\alpha}_1 - \boldsymbol{\alpha}_2 + \boldsymbol{\alpha}_3, \boldsymbol{\alpha}_1 + 3\boldsymbol{\alpha}_2 + 9\boldsymbol{\alpha}_3]$, and $|\boldsymbol{A}| = 1$. Find $|\boldsymbol{B}|$.
b) Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be $n \times n$ matrices, and $|\boldsymbol{A}| = 3$, $|\boldsymbol{B}| = 2$, $|\boldsymbol{A}^{-1} + \boldsymbol{B}| = 2$. Find $\left|\boldsymbol{A} + \boldsymbol{B}^{-1}\right|$.

4. Let $\boldsymbol{A} = \begin{bmatrix} 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & -1 & 2 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$. Find the adjoint matrix (伴随矩阵) $\boldsymbol{A}^*$.

5. Let $\boldsymbol{A}$ be a square matrix of order n $(n \geq 2)$, with $\boldsymbol{A}^*$ as its adjoint matrix. Please prove the following statement about the rank of $\boldsymbol{A}^*$:

$$\operatorname{rank}(\boldsymbol{A}^*) = \begin{cases} n, & \text{if } \operatorname{rank}(\boldsymbol{A}) = n, \\ 1, & \text{if } \operatorname{rank}(\boldsymbol{A}) = n-1, \\ 0, & \text{if } \operatorname{rank}(\boldsymbol{A}) < n-1. \end{cases}$$