Linear Algebra-A

Assignments - Week 11

Assignments from the Textbook (*Hardcover*)

Section 5.1: 8,9,10,11,14,18,24,25,26,29,40.
Section 5.2: 2,6,7,8,11,19,22,23,33,38,40,41.

Supplementary Problem Set

1. (1) Let $\boldsymbol{A} = \begin{bmatrix} 0 & -2 & -2 \\ 2 & -4 & -2 \\ -2 & 2 & 0 \end{bmatrix}$, find all the eigenvalues and eigenvectors of $\boldsymbol{A}$. Is $\boldsymbol{A}$ diagonalizable? If so, write it as $\boldsymbol{S}^{-1}\boldsymbol{A}\boldsymbol{S} = \boldsymbol{\Lambda}$, where $\boldsymbol{\Lambda}$ is a diagonal matrix.

   (2) The matrix $\boldsymbol{A} = \begin{bmatrix} 1 & -1 & 1 \\ x & 4 & y \\ -3 & -3 & 5 \end{bmatrix}$ has 3 linearly independent eigenvectors, and $\lambda = 2$ is an eigenvalue of multiplicity 2 (i.e., its algebraic multiplicity = 2). Find an invertible matrix $\boldsymbol{P}$, such that $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}$ is a diagonal matrix.

2. Let $\lambda_1, \lambda_2, \cdots \lambda_m$ be m distinct eigenvalues of an $n \times n$ matrix $\boldsymbol{A}$. The vectors $\boldsymbol{x}_{i_1}, \boldsymbol{x}_{i_2}, \cdots, \boldsymbol{x}_{i_{r_i}}$ are independent eigenvectors corresponding to λ_i $(i = 1,2,\cdots,m)$. Let $\Phi_i = \{\boldsymbol{x}_{i_1}, \boldsymbol{x}_{i_2}, \cdots, \boldsymbol{x}_{i_{r_i}}\}$ $(i = 1,2,\cdots,m)$. Please show that the set of vectors $\bigcup_{i=1}^{m} \Phi_i$ is linearly independent.

3. (1) Let $\boldsymbol{A}, \boldsymbol{B}$ be square matrices of order n. Please show that if $\lambda_1 (\lambda_1 \neq 0)$ is an eigenvalue of $\boldsymbol{AB}$, then λ_1 is also an eigenvalue of $\boldsymbol{BA}$. (如果$\boldsymbol{A}, \boldsymbol{B}$是同阶方阵，则$\boldsymbol{AB}$的特征值也是$\boldsymbol{BA}$的特征值.)

   (2) Generally, let $\boldsymbol{A}$ be an $m \times n$ matrix and $\boldsymbol{B}$ be an $n \times m$ matrix, and $m \geq n$, then $\boldsymbol{AB}$ and $\boldsymbol{BA}$ have same nonzero eigenvalues. To prove this, please show that $|\lambda \boldsymbol{I} - \boldsymbol{AB}| = \lambda^{m-n} |\lambda \boldsymbol{I} - \boldsymbol{BA}|$. (如果$\boldsymbol{A}, \boldsymbol{B}$分别是$m \times n$和$n \times m$的矩阵，则$\boldsymbol{AB}$和$\boldsymbol{BA}$的非零特征值相同. 由此可将 3(1)中的结论进一步加强：如果$\boldsymbol{A}, \boldsymbol{B}$是同阶方阵，则$\boldsymbol{AB}$的特征值和$\boldsymbol{BA}$的特征值及其代数重数完全相同.)

4. ***True or false***: If true, please give a proof. Otherwise, please give a counterexample.
   (1) If $\boldsymbol{A}^2 = \boldsymbol{I}$, then $\boldsymbol{A}$ is always diagonalizable.
   (2) An idempotent matrix (幂等矩阵) is always diagonalizable. （If this is true, then as an example, a projection matrix is always diagonalizable.）
   (3) A 2×2 rotation matrix is always diagonalizable.
   (4) A rank-1 matrix (秩为 1 的矩阵) is always diagonalizable.

5. Let $\boldsymbol{\alpha} = [a_1, a_2, \cdots a_n]^{\mathrm{T}} \in \mathbf{R}^n$, and $\boldsymbol{\alpha} \neq \mathbf{0}$.
   (1) Find all the eigenvalues of $\boldsymbol{A} = \boldsymbol{\alpha}\boldsymbol{\alpha}^{\mathrm{T}}$;
   (2) Find an invertible matrix $\boldsymbol{P}$ to make $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}$ diagonal.
   [**Hint**: Without loss of generality (不失一般性), assume that $a_1 \neq 0$.]