Linear Algebra-A

Assignments - Week 13

Assignments from the Textbook (*Hardcover*)

Section 5.6: 17,20,24,30,36,39,41,43,44.

#36 Hint: What is $(\boldsymbol{J}^2)^2$?

Section 6.1: 13,16.

Supplementary Problem Set

1. Suppose there exist a 3×3 matrix $\boldsymbol{A}$ and a 3-dimensional column vector $\boldsymbol{x}$ such that the set of vectors $\boldsymbol{x}, \boldsymbol{Ax}, \boldsymbol{A}^2\boldsymbol{x}$ are linearly independent, and
$$\boldsymbol{A}^3\boldsymbol{x} = 3\boldsymbol{Ax} - 2\boldsymbol{A}^2\boldsymbol{x}$$
(1) Let $\boldsymbol{P} = [\boldsymbol{x},\ \boldsymbol{Ax},\ \boldsymbol{A}^2\boldsymbol{x}]$. Find a matrix $\boldsymbol{B}$, such that $\boldsymbol{A} = \boldsymbol{PBP}^{-1}$.
(2) Compute the determinant $|\boldsymbol{A}^2 + \boldsymbol{A} + \boldsymbol{I}|$.
【**Hint**: You may use the following fact:
Please show that if $\boldsymbol{P}^{-1}\boldsymbol{AP} = \boldsymbol{B}$, then $\boldsymbol{P}^{-1}f(\boldsymbol{A})\boldsymbol{P} = f(\boldsymbol{B})$, i.e., if $\boldsymbol{A}$ is similar to $\boldsymbol{B}$, then $f(\boldsymbol{A})$ is similar to $f(\boldsymbol{B})$, where $f(x)$ is a polynomial of degree n: $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, where $a_n, a_{n-1}, \cdots a_1, a_0$ are constants.
Please prove it before applying it.】

2. If $\boldsymbol{A}$ is a 3×3 real symmetric matrix and has eigenvalues $\lambda_1 = -1, \lambda_2 = \lambda_3 = 1$, and $\boldsymbol{\alpha}_1 = (0,1,1)^{\mathrm{T}}$ is an eigenvector corresponding to $\lambda_1 = -1$. Please find the matrix $\boldsymbol{A}$.

3. (1) Find an orthogonal matrix $\boldsymbol{Q}$ (and a unitary matrix $\boldsymbol{U}$) to diagonalize the following matrix $\boldsymbol{A}$ (and $\boldsymbol{B}$):
$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 1 & -1 \\ 1 & 0 & -1 & 1 \\ 1 & -1 & 0 & 1 \\ -1 & 1 & 1 & 0 \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} 0 & -i & 0 \\ i & 1 & i \\ 0 & -i & 0 \end{bmatrix}.$$
(2) Find all the eigenvalues of the matrix $\boldsymbol{C} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{B} \end{bmatrix}$, and a unitary matrix to diagonalize $\boldsymbol{C}$.
【**Hint**: You may use the following fact:
For a block matrix

$$\boldsymbol{C}=\begin{bmatrix}\boldsymbol{C}_{11} & \mathbf{0} & \cdots & \mathbf{0}\\ \boldsymbol{C}_{21} & \boldsymbol{C}_{22} & \cdots & \mathbf{0}\\ \vdots & \vdots & \ddots & \vdots\\ \boldsymbol{C}_{m1} & \boldsymbol{C}_{m2} & \cdots & \boldsymbol{C}_{mm}\end{bmatrix},\quad \text{(the block } \boldsymbol{C}_{ii} \text{ is a matrix of order } r_i\text{)}$$

the eigenvalues of $\boldsymbol{C}$ come from the union set of the eigenvalues of $\boldsymbol{C}_{ii}$ $(i=1,2,\cdots,m)$. Please prove it before applying it.】

4. Let $\boldsymbol{A}=[a_{ij}]$ be a square matrix of degree $n(n\geq 2)$, and λ is an eigenvalue of $\boldsymbol{A}$. Please show that:

(1) There exists $1\leq k\leq n$ such that $|\lambda-a_{kk}|\leq\sum_{j=1,j\neq k}^{n}|a_{kj}|$.

【**Hint**: Denote $\boldsymbol{x}=(x_1,x_2,\cdots,x_n)^T$ as one of the eigenvectors of $\boldsymbol{A}$ corresponding to the eigenvalue λ, we let $\max\limits_{1\leq j\leq n}|x_j|=|x_k|$, then …】

(2) If the matrix $\boldsymbol{A}$ satisfies $|a_{ii}|>\sum_{j=1,j\neq i}^{n}|a_{ij}|, i=1,2,\cdots,n$ ($\boldsymbol{A}$ is called a *strictly diagonally dominant matrix*, or "严格对角占优矩阵" in Chinese), then $\boldsymbol{A}$ must be invertible.

5. Let $\mathbb{R}^{2\times2}$ be the vector space of all 2×2 matrices, and define $T:\mathbb{R}^{2\times2}\to\mathbb{R}^{2\times2}$ by $T(\boldsymbol{X})=\boldsymbol{AX},\forall\boldsymbol{X}\in\mathbb{R}^{2\times2}$, where $\boldsymbol{A}=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$.

(1) Show that T is a linear transformation.

(2) Find its representing matrix with respect to the basis $\boldsymbol{e}_1=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix},\boldsymbol{e}_2=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix},\boldsymbol{e}_3=\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix},\boldsymbol{e}_4=\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}$.

(3) If $\boldsymbol{A}=\begin{bmatrix}1 & 1\\ 0 & 2\end{bmatrix}$, find a basis for $\mathbb{R}^{2\times2}$ with the property that the representation matrix for T is a diagonal matrix and find the diagonal matrix.