Linear Algebra-A

Assignments - Week 6

Please write down your answers to the assignments from the textbook and supplementary problems on different answer sheets because they will go to different graders.

Assignments from the Textbook (*Hardcover*)

Section 2.2: 24.
Section 2.4: 3,6,18,25,27,33,35,38.
Note: In Ex. 35, $\boldsymbol{u}$,$\boldsymbol{v}$,$\boldsymbol{w}$,$\boldsymbol{z}$ are column vectors.
Section 2.6: 1,3,5,6,15,19,33,45,48.

Supplementary Problem Set

1. Let $E = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ and $F = \{\boldsymbol{b}_1,\ \boldsymbol{b}_2\}$, where

$$\boldsymbol{u}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \boldsymbol{u}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \boldsymbol{u}_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

and $\boldsymbol{b}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \boldsymbol{b}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

For each of the following linear transformations $L: \mathbf{R}^3 \to \mathbf{R}^2$, find the matrix representing L with respect to the ordered bases E and F:

(a) $L(\boldsymbol{x}) = (x_3, x_1)^{\mathrm{T}}$;
(b) $L(\boldsymbol{x}) = (x_1 + x_2, x_1 - x_3)^{\mathrm{T}}$;
(c) $L(\boldsymbol{x}) = (2x_2, -x_1)^{\mathrm{T}}$.

2. Let $\mathbb{R}^{2\times 2}$ be the vector space of all 2×2 real matrices, and define

$$T: \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$$

by $T(\boldsymbol{A}) = \boldsymbol{A} + \boldsymbol{A}^{\mathrm{T}}$, where $\boldsymbol{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

(a) Show that T is a linear transformation.

(b) Find its matrix with respect to the basis $\left\{\boldsymbol{b}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \boldsymbol{b}_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \boldsymbol{b}_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \boldsymbol{b}_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$.

3. Prove that:
   (a) $\text{rank}(\boldsymbol{A}+\boldsymbol{B}) \le \text{rank}(\boldsymbol{A}) + \text{rank}(\boldsymbol{B})$, where $\boldsymbol{A}$ and $\boldsymbol{B}$ are matrices of same size.
   (b) If $\boldsymbol{A}$ is a square matrix of order n, and $\boldsymbol{A}^2 - \boldsymbol{I} = \boldsymbol{0}$, then $\text{rank}(\boldsymbol{A}-\boldsymbol{I}) + \text{rank}(\boldsymbol{A}+\boldsymbol{I}) = n$. (*Hint*: Apply (a) and the result of problem 38 in Section 2.4.)

4. Prove that: If $\boldsymbol{P}$ and $\boldsymbol{Q}$ are $m \times m$ and $n \times n$ *invertible* matrices respectively, and $\boldsymbol{A}$ is an $m \times n$ matrix, then $\text{rank}(\boldsymbol{PAQ}) = \text{rank}(\boldsymbol{A})$.

5. Let $\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 1 & 2 & 2 & 2 & 3 \\ -1 & -2 & 0 & 2 & 3 \end{bmatrix}$.

(a) Find the complete solution to $\boldsymbol{Ax} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

(b) Find the complete solution to $\boldsymbol{Ax} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$.

(c) Find the rank of $\boldsymbol{A}$ and dimensions of the four fundamental subspaces of $\boldsymbol{A}$.
(d) Find bases of the four fundamental subspaces of $\boldsymbol{A}$.