Modular arithmetic

- Much of modern number theory, and many practical problems (including problems in cryptography and computer science), are concerned with *modular arithmetic*. While this is probably familiar to most people taking this course, I will review it briefly.
- In arithmetic modulo N, we are concerned with arithmetic on the integers, where we identify all numbers which differ by an exact multiple of N. That is, $x \equiv y \bmod N$ if $x = y + mN$ for some integer m.
- This identification divides all the integers into N equivalence classes. We usually denote these by their "simplest" members, that is, the numbers $0, 1, \ldots, N-1$. (Usually. In the case of clock arithmetic (modulo 12), we use $1, \ldots, 12$ instead.)

Arithmetic Operations

- Most ordinary arithmetic operations extend to modular arithmetic straightforwardly.

$$x + y \to x + y \bmod N,$$

$$xy \to xy \bmod N,$$

$$x^y \to x^y \bmod N.$$

- This does lead to some things which are strange based on the intuition from *ordinary* integer arithmetic, but that make sense for this finite case.

- For instance, all numbers have additive inverses, but these are now represented by positive numbers: $(-x) \equiv N - x$, so the additive inverse of 3 modulo 7 is 4.
- And unlike ordinary arithmetic, it is possible for a non-zero integer to have a *multiplicative* inverse, as well: $3 \cdot 5 = 15 = 1 \bmod 7$.
- In fact, for *prime* N, all numbers $1, \ldots, N-1$ have multiplicative inverses. If N is composite, then all numbers that have no common factor with N have multiplicative inverses.

Order-finding

- For integers x and N with no common factor, the *order* of x modulo N is the least positive integer r such that

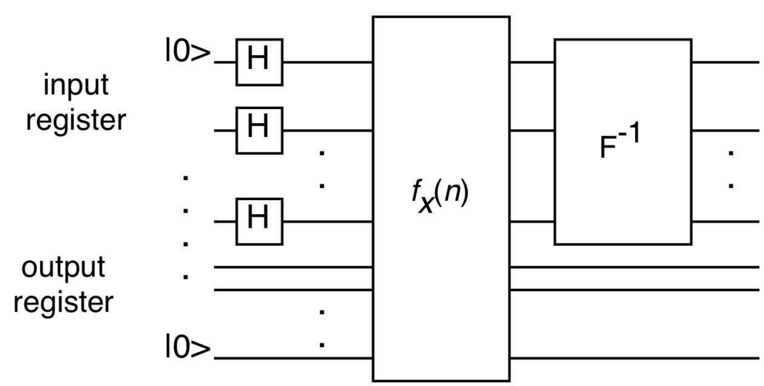
$$x^r = 1 \bmod N.$$

Obviously, $r \leq N$. (If not, the sequence of numbers $x^n \bmod N$ for $n = 1, \ldots, r$ must all be distinct modulo N, which is impossible, since there are only N equivalence classes.)

- Our problem is, given x and N, to find the order r of x modulo N. The description of the problem just requires the statement of x and N; we parametrize the size of the problem by $L = \log_2 N$, the number of bits needed to state N. No classical algorithm for order-finding is known which is polynomial in L.

Building a quantum algorithm

- The first thing to notice is that we can define a function $f_x(n) = x^n \bmod N$. Since the order r means that $x^r = 1 \mod N$, this means that $f_x(n+r) = x^{n+r} \mod N = x^n x^r \mod N = x^n \mod N = f_x(n)$. So the function is *periodic* with period r. Moreover, the $f(n)$ must all be distinct for $0 \leq n < r$.
- We can therefore build a circuit for order-finding based on our circuit for period-finding:



- We have divided the problem into two sub-circuits. The first peforms the unitary $\hat{U}_{f_x}(|n\rangle|y\rangle) = |n\rangle|y \oplus f_x(n)\rangle$. The second performs the inverse Fourier transform on the input register. Both the input and the output registers start in the state $|0\rangle$, and the input register is put into a superposition of all $|n\rangle$ by Hadamard gates.
- In fact, for this problem, it is better to have the second register start in the state $y = 1$, and create the unitary $\hat{U}'_{f_x}(|n\rangle|y\rangle) = |n\rangle|y \cdot f_x(n) \mod N\rangle$. Note that this multiplication is invertible as long as x and N have no common factors; we would just multiply $yx^n \mod N$ by x^{r-n} where r is the order of x. (Of course, we don't know what r is, but that doesn't change the fact that the multiplication is invertible in principle.) We call this unitary operator the circuit for *modular exponentiation*.

- We can build modular exponentiation out of repeated applications of modular multiplication. Define a unitary operator $\hat{U}_x$ such that

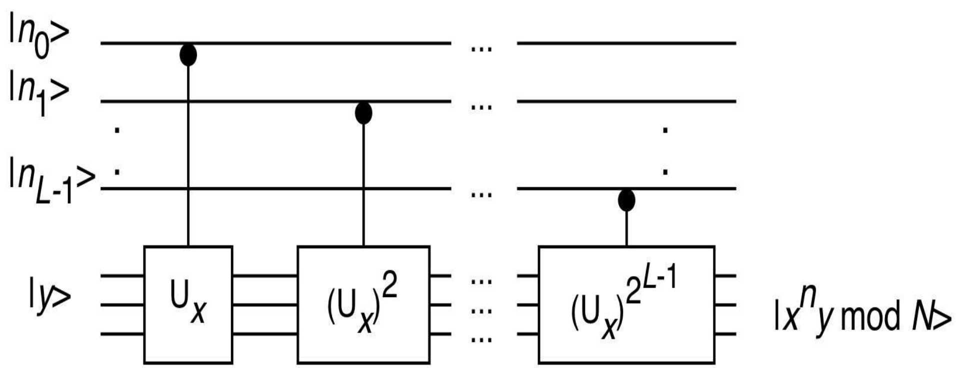
$$\hat{U}_x|y\rangle = |xy \mod N\rangle.$$

- Let n be the argument of the modular exponential function, with L-bit binary representation $n_{L-1}n_{L-2}\dots n_0 = n_{L-1}2^{L-1} + \cdots + n_0$.

$$x^n y \mod N = x^{n_{L-1}2^{L-1}} x^{n_{L-2}2^{L-2}} \cdots x^{n_0} y.$$

- That is, we successively multiply y by x^{2^j} if $n_j = 1$, and by 1 otherwise.

- Turning this into a quantum circuit we get:



- This circuit looks very familiar—it is the same circuit used in phase estimation. This is no surprise, since we have seen that period-finding is an instance of phase estimation.

- Of course, written in this form, the circuit is only efficient if we can do all these controlled-$\hat{U}_x^{2^j}$ operations efficiently. We will now see that this is the case.
- We build these out circuits out of two other circuits: *modular multiplication* and *modular squaring*. These work as follows:

$$\hat{U}_m(|x\rangle|y\rangle) = |x\rangle|xy \mod N\rangle,$$

$$\hat{U}_2|x\rangle = |x^2 \mod N\rangle.$$

- Again, these are invertible as long as x has no common factors with N. (We don't care what they do in other cases, so it is possible to construct unitaries that do what we want.)

- We also need a scratch register, which we start in the state $|1\rangle$. Let us write this first, so our full state is $|1\rangle|y\rangle$. The first thing we do is apply $\hat{U}_x$ to the scratch register:

$$|1\rangle|y\rangle \to |x\rangle|y\rangle.$$

- We now apply $\hat{U}_2$ j times to perform $\hat{U}_x^{2^j}$:

$$|x\rangle|y\rangle \to |x^2 \mod N\rangle|y\rangle \to |x^4 \mod N\rangle|y\rangle$$

$$\to |x^8 \mod N\rangle \to \cdots \to |x^{2^j} \mod N\rangle|y\rangle.$$

So we've calculated $x^{2^j} \mod N$.

- We then do modular multiplication between x^{2^j} and y using the unitary circuit $\hat{U}_m$:

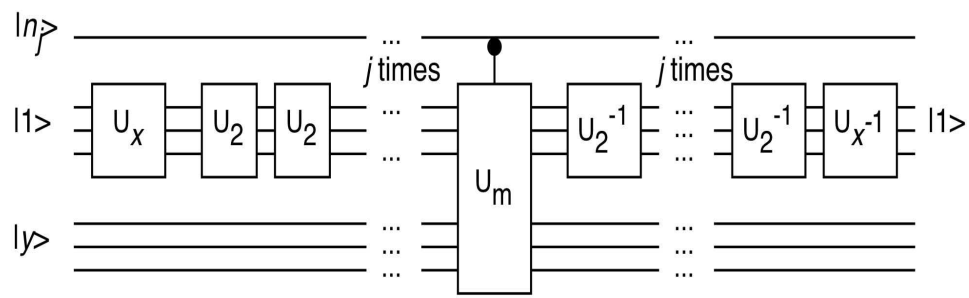
$$|x^{2^j} \mod N\rangle|y\rangle \rightarrow |x^{2^j} \mod N\rangle|x^{2^j} y \mod N\rangle.$$

- Finally, we want to re-use the scratch space, so we "uncompute" $x^{2^j} \mod N$ by applying $\hat{U}_2^\dagger$ j times and then $\hat{U}_x^\dagger$ once, making the whole transformation

$$|1\rangle|y\rangle \rightarrow |1\rangle|x^{2^j} y \mod N\rangle.$$

We've returned the scratch register to the state $|1\rangle$, so it can be reused.

- The circuit for controlled-$\hat{U}_x^{2^j}$ looks like this:



- (It would be more efficient to keep our partial results for $x^{2^{j-1}}$ for use with each successive n_j, but that does not alter the principle, or change the complexity of the circuit significantly.)
- Now we need to build the circuits for $\hat{U}_x$, $\hat{U}_2$, and $\hat{U}_m$.

- We can build modular squaring ($\hat{U}_2$) and modular multiplication by x ($\hat{U}_x$) out of general modular multiplication ($\hat{U}_m$); and $\hat{U}_m$ can be built out of modular addition and modular multiplication by two.
- That is, if the binary expressions are $x = x_{L-1} \ldots x_0$ and $y = x_{L-1} \ldots y_0$, then $xy \mod N$ is just

$$xy \mod N = x_0 y + 2x_1 y + \cdots + 2^{L-1} x_{L-1} y \mod N.$$

Each term just multiplies y by a power 2^j, controlled by the bit x_j, and we then sum the terms to get $xy \mod N$.
- Simple reversible circuits exist for both modular addition and modular multiplication by two.

Summary of the Algorithm

We can now summarize the order-finding algorithm:

1. Prepare the input register in state $|0\rangle$ and the output register in state $|1\rangle$.
2. With Hadamard gates, put the input register in a superposition of all values $|n\rangle$.
3. Calculate the modular exponential function.
4. Do the inverse Fourier transform on the input register.
5. Measure the input register.
6. Find the order r using the continued fraction algorithm.

The circuit uses $O(L^3)$ gates. The input register needs $t = 2L + 1 + \log(1 + 1/2\epsilon)$ bits to succeed with $p > 1 - \epsilon$.

Order-finding and factoring

- While order-finding may seem of limited interest by itself, the problem of factoring large numbers reduces to order-finding for its most difficult cases.
- To state the problem concretely: given a composite number N, we want to find one of its prime factors. No efficient classical algorithm is known when N is large.
- The algorithm proceeds in several steps. Most values of N will use order-finding, but we must also eliminate special cases for which order-finding fails, but alternative efficient algorithms exist.

Steps of Factoring

1. Check if N is even. If it is, obviously 2 is a factor.
2. Check if N is a power a^b for integers a and b. An efficient algorithm for this exists.
3. Choose a random integer x, $1 < x < N - 1$. Calculate $\mathrm{GCD}(x, N)$ using Euclid's algorithm. If it is not 1, congratulations!
4. Use the order-finding algorithm to find the order r of x modulo N. If we know r, in many cases we can find a prime factor of N.

To see this, we need some number theory.

- If r is even, we calculate $x^{r/2} \mod N$. If this is not $N - 1 \equiv -1 \mod N$, then we calculate $\mathrm{GCD}(x^{r/2} \pm 1, N)$. If one of these gives a nontrivial factor, that is our answer. Otherwise, the algorithm fails.
- This may seem like a lot of conditions. But in fact, r has at least a 50% probability of being even and not having $x^{r/2} = -1 \mod N$. If the algorithm fails, we just pick a new value of x and try again. The probability is overwhelming that we will succeed after only a few repetitions.
- This algorithm is $O(L^3)$ (from modular exponentiation and the continued fraction). The best classical algorithm has a complexity $O(e^{L^{1/3}})$, which is superpolynomial.

RSA

Factoring is of interest largely because the difficulty of factoring is used to guarantee the security of popular public-key cryptography codes, such as Diffie-Hellman and RSA. Let's briefly look at RSA to see how factoring plays a role.

1. Alice picks two large prime numbers, p and q, and calculates $N = pq$. She also picks another number e which is mutually prime to $(p-1)(q-1)$. She publishes the numbers N and e; they form her *encryption key*.
2. Privately, Alice also calculates a number d such that $ed = 1 \mod (p-1)(q-1)$. (She can do this using Euclid's algorithm.) This is her *decryption key*.

3. Bob encrypts messages by first converting them to binary form and considering them to be a large number M. This is the *plaintext*.
4. Bob then calculates $C = M^e \mod N$. This new number C is the *ciphertext*.
5. Bob transmits C to Alice.
6. When Alice receives C, she calculates $M = C^d \mod N$. This will be Bob's original message.

The reason this works is as follows. Every number that is coprime with N has a multiplicative inverse; these numbers form a multiplicative group with $(p-1)(q-1)$ elements. Any number M that is coprime with N must obey $M^{(p-1)(q-1)} = 1 \mod N$. So $M^{ed} = M^1 = M \mod N$.

- The function $M^e \mod N$ is a *one-way function*; it is virtually impossible (using standard techniques) to invert the function without the value of d (or the numbers p and q). Obviously, an efficient factoring algorithm would destroy the security of RSA.
- There are other public-key cryptosystems which do not rely on factoring. However, several of these can also be broken by quantum computers. For instance, another system relies on the difficulty of the *discrete logarithm* problem: given a and b, finding the smallest s such that $a^s = b$. Shor found an efficient algorithm for the discrete logarithm at the same time as his factoring algorithm.

- No known classical algorithm can solve the discrete logarithm in polynomial time; but a quantum computer can do so in a time $O(L^3)$.
- Factoring, the discrete logarithm, and period-finding are all examples of a general problem known as the *Abelian hidden subgroup problem*, and the algorithms for solving them all have the same general structure.
- The problem is this: given an Abelian (commutative) group G and a function $f(g)$ on the group elements g, such that $f(g)$ is constant on cosets of an unknown subgroup $H \subseteq G$, find H. These problems are all efficiently solvable by a quantum computer.

Next time: Grover's search algorithm.