EXPANDING FAMILIES, THE ZIG-ZAG AND REPLACEMENT PRODUCTS

A Thesis Presented to the Faculty of San Diego State University

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Purpose of Thesis

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TO FINISH!

Purpose of Thesis

Three Objects:

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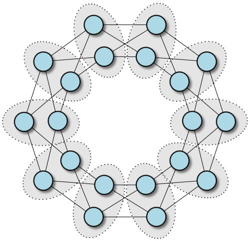
- *Zig-Zag* Graph Product.



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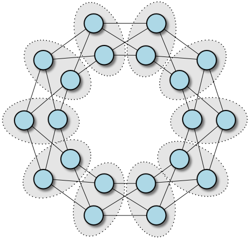

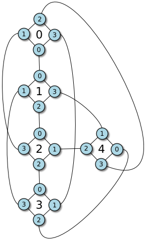

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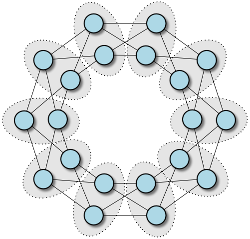
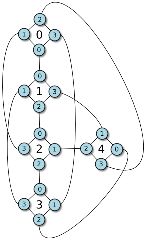
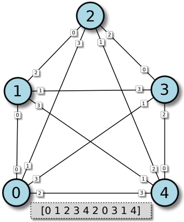
Three Objects:

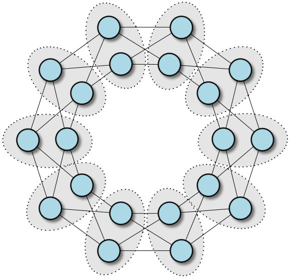
- *Zig-Zag* Graph Product.
- *Replacement* Graph Product.
- *Enumerations* that generate them.





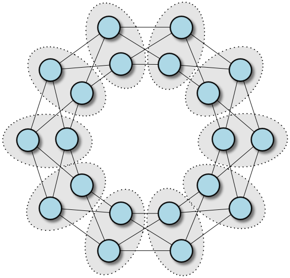


The Zig-Zag Product



The Zig-Zag Product

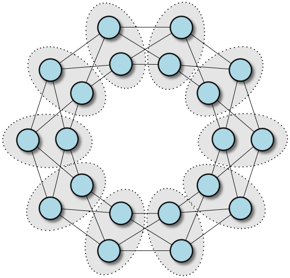
- Developed by Reingold, Vadhan and Widgerson [8] in 2000.



History

The Zig-Zag Product

- Developed by Reingold, Vadhan and Widgerson [8] in 2000.
- Principally to facilitate the explicit construction of graphs with good *expansion* properties.



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What are *expander graphs* good for?

They have been used in fields such as:

- Network Design [7, 6]
- Complexity Theory [11, 10]
- Topology [2] and Measure Theory [5]

How do we measure expansion?

Definition

The **edge boundary** of $S \subseteq V$ is the set $\partial S \subseteq E$ defined to be:

$$\partial S = \{(u, v) \mid u \in S \text{ and } v \in V - S\}$$

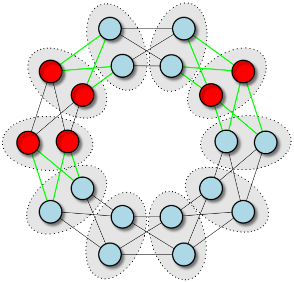


Figure: Example ∂S for 2-trellis over C_10

How do we measure expansion?

Definition

Let G be a graph and $S \subseteq V(G)$. The **isoperimetric constant** of G is

$$i(G) = \inf_{\substack{S \subseteq V \\ 0 < |S| < \infty}} \frac{|\partial S|}{\min\{|S|, |V - S|\}}$$

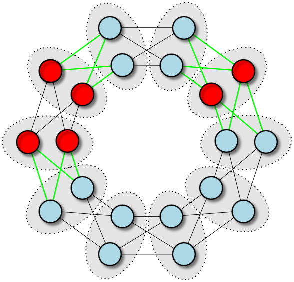


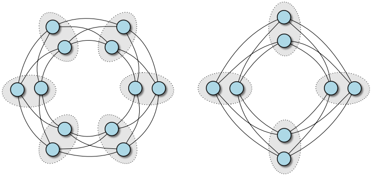
Figure: Example ∂S for 2-trellis over C_10

Example

Disconnected Graph, G

$$i(G) = 0.$$

Since $\partial S = \emptyset$ for any *component* S.



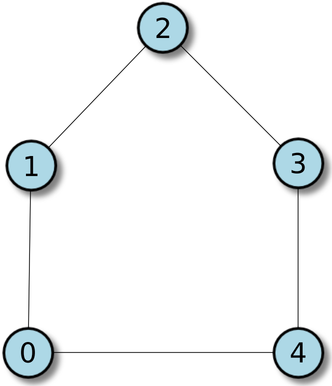
Examples

Example

Cycle Graph, C_n

$$i(G) = \frac{2}{\left\lfloor \frac{n}{2} \right\rfloor}$$

Since every chain of length m has a boundary of size 2.



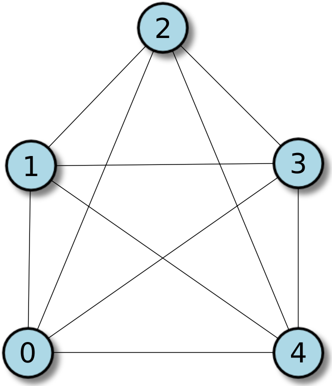
Examples

Example

The Complete Graph, K_n

$$i(G) = n - \left\lfloor \frac{n}{2} \right\rfloor = \left\lceil \frac{n}{2} \right\rceil .$$

Since for all $S \subset \mathrm{V}(K_n)$, $|\partial S| = m\,(n-m)$



Expanding Families of Graphs

Definition

Let $\mathcal{F} = \{G_k \mid k \in \mathbb{N}\}$ be a collection of graphs with $|\mathrm{V}(G_k)| = n_k$ and $n_k \to \infty$ as $k \to \infty$. $\mathcal{F}$ is called an **expanding family** of graphs if

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$\mathcal{K} = \{K_n \mid n \in \mathbb{N}\}$

$$\lim_{n\to\infty} \left\lceil \frac{n}{2} \right\rceil = \infty$$

Spectral Measure of Expansion

Spectrum of a Regular Undirected Graph

Let G is a d-regular undirected graph on n vertices. Then $A(G)$ is a real and symmetric matrix, and thus has real eigenvalues which can be placed in descending order

$$d = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} \geq \lambda_n$$

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Many times the *normalized adjacency matrix*, $\hat{A}(G) = \frac{1}{d}A(G)$ is used and expansion is measured in terms of convergence of a *random walk* on G.

Expansion and the Zig-Zag Product

Theorem

Let G be a m-regular graph on n vertices and let H be a d-regular graph on m vertices and let α, β be such that $\hat{\lambda}_2(G) \leq \alpha$ and $\hat{\lambda}_2(H) \leq \beta$. Then $G \textcircled{z} H$ is a d^2-regular graph on $n \cdot m$ vertices where the function $\hat{\lambda}_2(G \textcircled{z} H)$ satisfies the following:

- *If $\alpha < 1$ and $\beta < 1$ then $\hat{\lambda}_2(G \textcircled{z} H) < 1$.*
- *$\hat{\lambda}_2(G \textcircled{z} H) \leq \alpha + \beta$.*

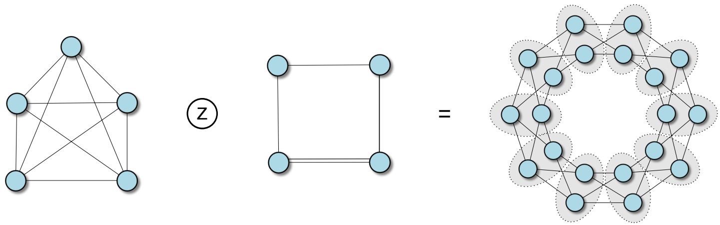
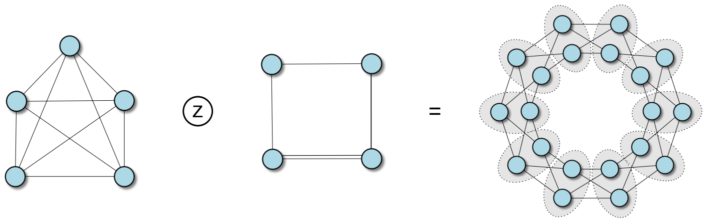


Figure: Generic Example of Zig-Zag Product

Expansion and the Zig-Zag Product

What does the last theorem prove?

If G and H have good expansion then $G \textcircled{z} H$ does also.



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Why?

Guarantees that the *spectral gap* is bounded away from 0.

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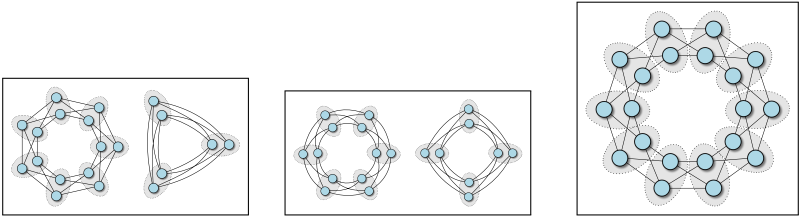
- Even with two fixed constituent graphs there are really many zig-zag products, some of which are non-isomorphic.

Example of three different $K_5 \textcircled{z} C_4$



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3. Present a generalization of the zig-zag product that will be called the *sandwich product*. (which will allow for many of the restrictions on the constituent graphs to be removed)

The Replacement and Zig-Zag Products

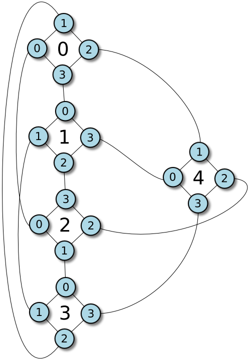
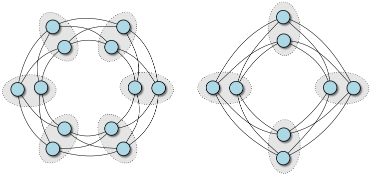


Figure: Example of $K_5 \textcircled{r} C_4$



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- For each $u \in \mathrm{V}(G)$, N_u be the set of vertices that are adjacent to u.
- For each $u \in \mathrm{V}(G)$, define a bijection $\eta_u : N_u \to \mathrm{V}(H)$ which we will call the **local-enumeration** of u with respect to H

The Enumeration of G with respect to H

Each local enumeration provides a one-to-one correspondence between the *vertices* adjacent to a vertex and the vertices of the other graph.

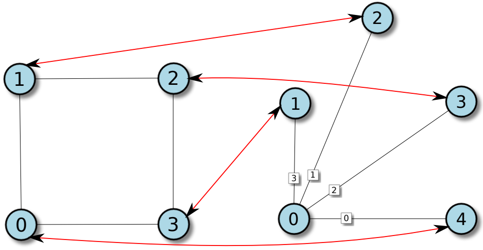

Figure: Example of a 0-enumeration of K_5 with respect to C_4

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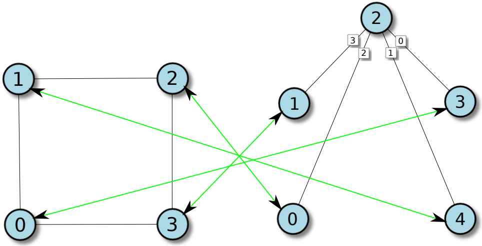

Figure: An example of a 2-enumeration of K_5 with respect to C_4

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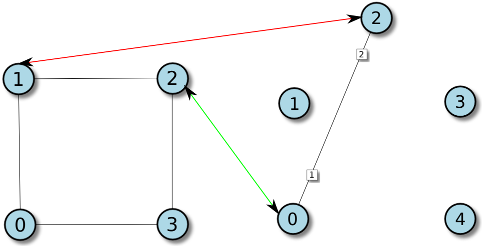

Figure: How two enumerations effect a single edge

Enumerations of G with respect to H

We call the collection of local-enumerations of G with respect to H

$$\mathcal{E} = \{\eta_u \mid u \in \mathrm{V}(G)\}$$

the *enumeration* of G.

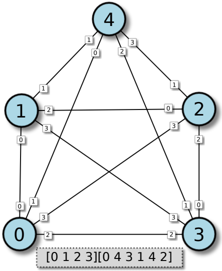

Figure: K_5 with an enumeration with respect to C_4

Definition: The Replacement Product

Definition

The **replacement** product of G with H and enumeration $\mathcal{E}$, denoted $G \circledR_{\mathcal{E}} H$, is the graph with vertex set $\mathrm{V}(G) \times \mathrm{V}(H)$ for which (u, a) is adjacent to (v, b) if either

1. $u = v$ and $(a, b) \in E(H)$, or
2. $u \neq v$ and $(u, v) \in \mathrm{E}(G)$ where $\eta_u(v) = a$ and $\eta_v(u) = b$.

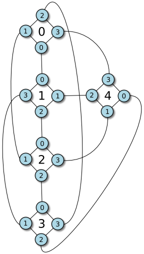

Figure: Example of $K_5 \circledR C_4$

The Replacement Product: Clouds and Bridges

The **replacement product** is the disjoint union of two graphs.

The Replacement Product: Clouds and Bridges

1. $u = v$ and $(a, b) \in E(H)$ (Clouds)

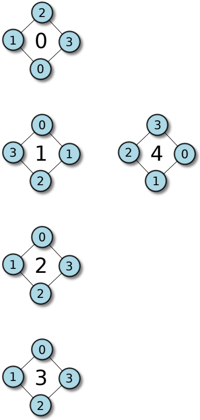

Figure: *clouds* in $K_5 \circledR C_4$

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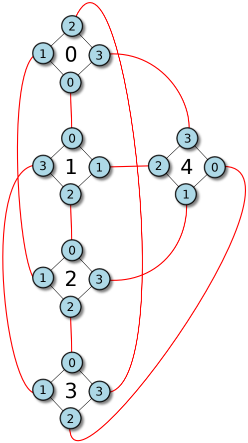

Figure: *bridges* in $K_5 \textcircled{r} C_4$

The Replacement Product: Different Enumerations

- The *clouds* are fixed, independent of enumeration.

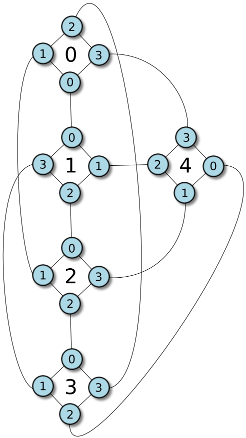

Figure: $K_5 \textcircled{r} C_4$ with a $3, 7$ enumeration

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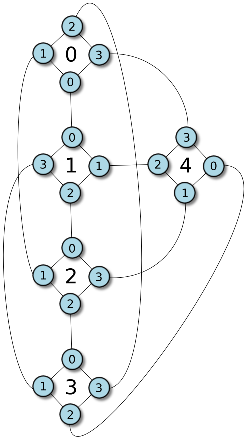

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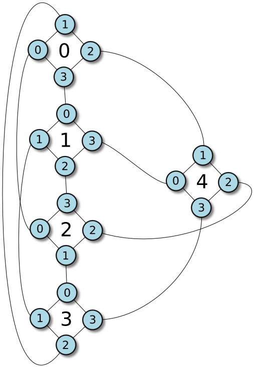

Figure: $K_5 \textcircled{r} C_4$ with a $4, 6$ enumeration

Bridges and Enumerations

Each bridge edge is of the form

$$((u, \eta_u(v)), (v, \eta_v(u)))$$

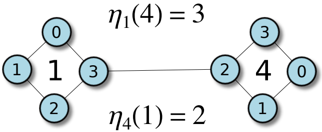

Figure: A *bridge* and the local-enumeration.

Non Isomorphic Replacement Products

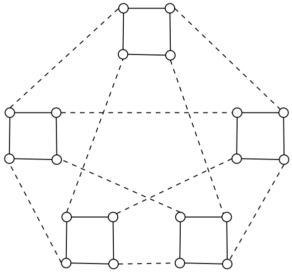

Figure: An example $K_5 \circledR C_4$ which is bipartite.

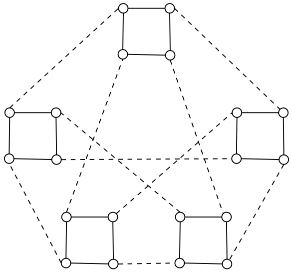

Figure: Example $K_5 \circledR C_4$ with odd cycles.

The Replacement Product Isomorphism

Definition

Let $\mathcal{E}$ and $\mathcal{E}'$ be two enumerations and let $G \textcircled{r} H$ and $G \textcircled{r}' H$ be the associated replacement products.

- Let $f \in \mathrm{Aut}(G)$ and, for each $u \in \mathrm{V}(G)$, let $g_u \in \mathrm{Aut}(H)$.
- Define the function F mapping $\mathrm{V}(G) \times \mathrm{V}(H)$ to itself, by $F(u, a) = (f(u), g_u(a))$.
- $\eta'_{f(u)}(f(v)) = g_u(\eta_u(v))$

If F satisfies these properties we call it a **replacement product isomorphism** (rp-isomorphism).

Replacement Product Isomorphism

Facts:

- Every rp-isomorphism F is an *automorphism* of $G \circledR H$

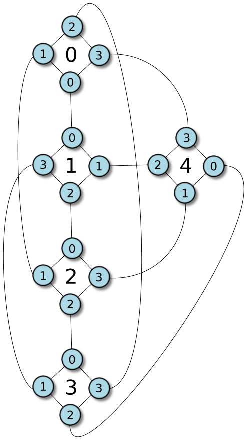

Figure: $K_5 \circledR C_4$

Replacement Product Isomorphism

Facts:

- Every rp-isomorphism F is an *automorphism* of $G \circledR H$
- F preserves the enumeration $\mathcal{E}$.

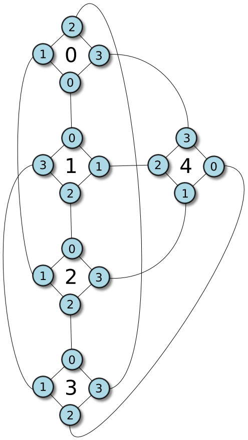

Figure: $K_5 \circledR C_4$

Replacement Product Isomorphism

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- Every rp-isomorphism F is an *automorphism* of $G \circledR H$
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Open Question:

Is the definition is too restrictive?

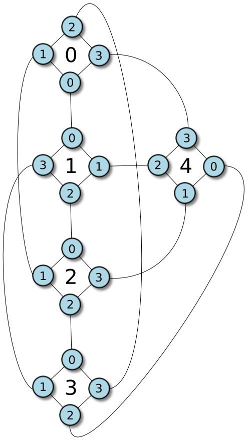

Figure: $K_5 \circledR C_4$

Definition: The Zig-Zag Product

Definition

The **zig-zag** product of G (with a given enumeration $\mathcal{E}$) with H, denoted $G \text{ⓩ} H$, is a graph with vertex set $V(G) \times V(H)$ for which (u, a) is adjacent to (v, b) if and only i

1. $(u, v) \in \mathrm{E}(G)$
2. both $(a, \eta_u(v))$ and $(\eta_v(u), b)$ are in $\mathrm{E}(H)$

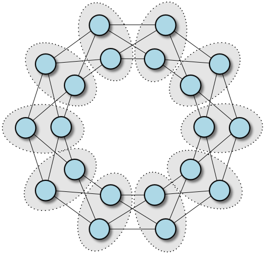

Figure: Example of $K_5 \text{ⓩ} C_4$

Relationship to Replacement Product

The *zig-zag* product is the graph whose adjacency's arise from walks of length three of a "zig-zag" nature in $G \textcircled{r} H$.

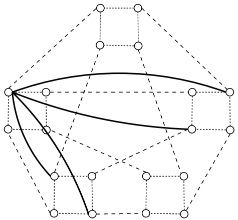

Figure: "zig-zag" edges incident in $K_5 \textcircled{r} C_4$

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1. An edge within one *cloud* (zig).

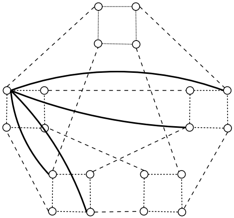

Figure: "zig-zag" edges incident in $K_5 \circledR C_4$

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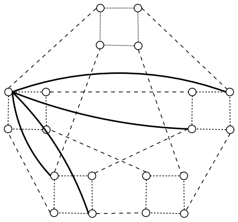

Figure: "zig-zag" edges incident in $K_5 \textcircled{r} C_4$

Relationship to Replacement Product

The *zig-zag* product is the graph whose adjacency's arise from walks of length three of a "zig-zag" nature in $G \textcircled{r} H$.

1. An edge within one *cloud* (zig).
2. An edge connecting one *cloud* to another (zag).
3. An edge within the final *cloud* (zig again).

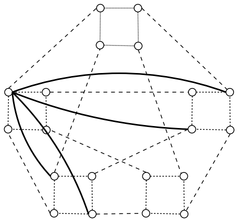

Figure: "zig-zag" edges incident in $K_5 \textcircled{r} C_4$

Zig-Zag Product and RP-Isomorphisms

Theorem

If $G \circledR H$ and $G \circledR' H$ are rp-isomorphic. Then $G \circled{z} H$ is isomorphic to $G \circled{z}' H$.

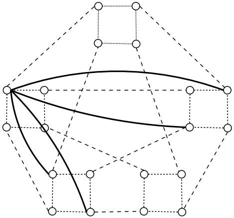

Figure: "zig-zag" edges incident in $K_5 \circledR C_4$

The size of rp-isomorphism classes:

Theorem

Let $n(G \circledR H)$ be the number of unique non-isomorphic replacement products. Then we have

$$n(G \circledR H) \leq \left(\frac{m!}{|\mathrm{Aut}(H)|} \right)^n$$

Corollary

Let G be any 3-regular graph on n-vertices and consider the cycle graph on three vertices C_3. Then

$$n(G \circledR C_3) \leq \left(\frac{3!}{|D_3|} \right)^n = \left(\frac{6}{6} \right)^n = 1$$

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Note: The smallest "interesting example" has to be 4-regular

Classification of $K_5 \textcircled{r} C_4$ and $K_5 \textcircled{z} C_4$

The art of doing mathematics consists in finding that special case which contains all the germs of generality.
-**David Hilbert**

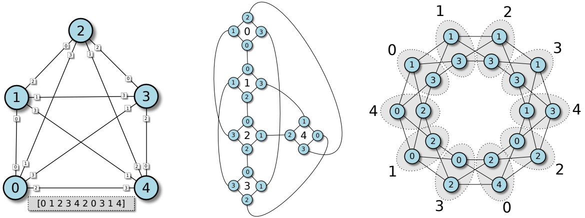

Figure: Example of an enumeration of K_5 and the associated zig-zag and replacement products.

$K_5 \textcircled{z} C_4$

Early Experiments

Led us to graphs which all shared a very similar structure, but with intriguing differences.

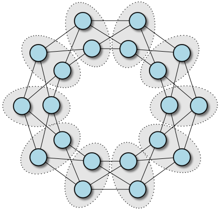

Figure: $K_5 \textcircled{z} C_4$

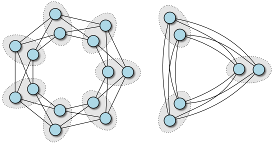

Figure: $K_5 \textcircled{z} C_4$

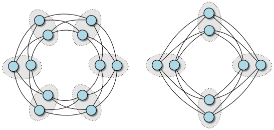

Figure: $K_5 \textcircled{z} C_4$

$K_5 \textcircled{z} C_4$ and $K_5 \textcircled{r} C_4$

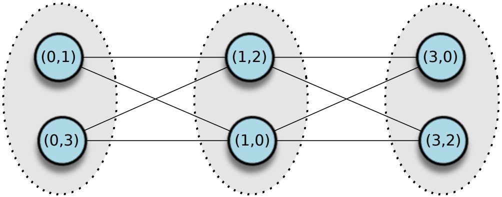

Figure: Example neighborhood in $K_5 \textcircled{z} C_4$

$K_5 \circledz C_4$ and $K_5 \circledr C_4$

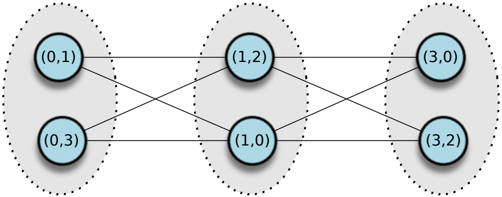

Figure: Example neighborhood in $K_5 \circledz C_4$

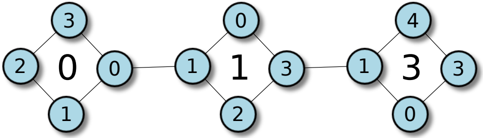

Figure: Corresponding neighborhood in $K_5 \circledr C_4$

$K_5 \circledR C_4$ and K_5

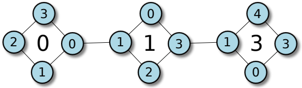

Figure: Neighborhood in $K_5 \circledR C_4$

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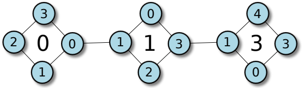

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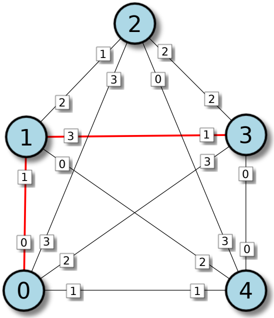


Figure: Corresponding Path in K_5

K_5 and *parity*

Observation:

- Neighborhoods in $K_4 \textcircled{z} C_4$ correspond to paths in K_5
- Paths which *enter* and *leave* by edges labeled with the same parity

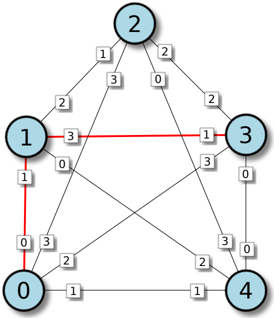


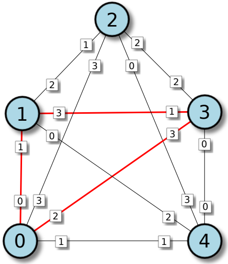
Figure: Corresponding Path in K_5

Parity Circuits in K_5

Definition

Let $p = [u_1 \; u_2 \; \cdots \; u_k]$ be a closed walk on K_5 with enumeration $\mathcal{E}$. p is a **parity circuit** of length k if both

- For each $u_i \in p$, $\eta_{u_i}(u_{i-1}) = \eta_{u_i}(u_{i+1}) \pm 2 \bmod 4$.
- No edge is traversed more than once.

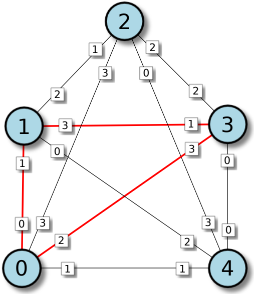


- Parity Circuit [0 1 3]

Figure: Parity circuit of length 3

Parity Circuits in K_5

- The compliment (edgewise) of [0 1 3] leaves edges of the opposite parity.



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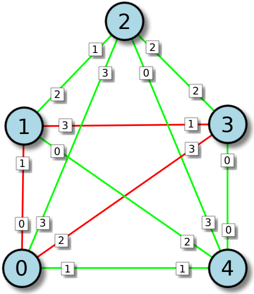
- [0 2 1 4 3 2 4] is a parity circuit of length 7



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Parity Circuits in K_5

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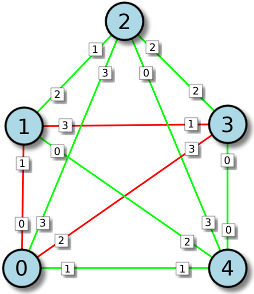


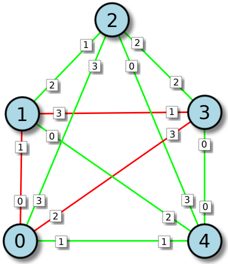
Figure: Parity circuit of length 3

Parity Circuit Decompositions (PCDs)

Definition

Sequence of parity circuits $\mathcal{D} = \left(p_1, p_2, \ldots, p_j\right)$. is a **parity circuit decomposition (PCD)** of K_5 if

- p_i and p_j are edgewise disjoint for $i \neq j$.
- For each $e \in \mathrm{E}(K_5)$ there exists a parity circuit p_i such that e is traversed by p_i.



- $\mathcal{D} = [0\ 1\ 3][0\ 2\ 1\ 4\ 3\ 2\ 4]$

PCDs and $K_5 ⓩ C_4$

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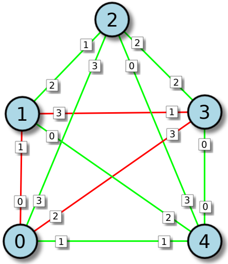

Figure: Parity circuit decomposition of K_5

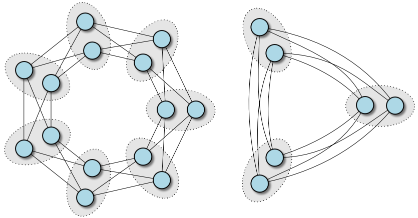

Figure: $K_5 ⓩ C_4$

Equivalence of PCDs and uniqueness

- $p \equiv p'$ if $p = (r \circ c_i) \cdot p'$ where r is a reversal of order and c is a cyclic shift of length i.
- $\mathcal{D} \equiv \mathcal{D}'$ if each parity circuit of $\mathcal{D}$ is equivalent to exactly one parity circuit in $\mathcal{D}'$.

Example

$$[0\ 1\ 3]\,[0\ 2\ 1\ 4\ 3\ 2\ 4] \equiv [3\ 0\ 1]\,[4\ 1\ 2\ 0\ 4\ 2\ 3] \equiv [1\ 0\ 3]\,[1\ 4\ 3\ 2\ 4\ 0\ 2]$$

$$[0\ 1\ 3]\,[0\ 2\ 1\ 4\ 3\ 2\ 4] \not\equiv [3\ 1\ 4]\,[3\ 0\ 4\ 2\ 0\ 1\ 2]$$

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$$[0\ 1\ 3]\,[0\ 2\ 1\ 4\ 3\ 2\ 4] \not\equiv [3\ 1\ 4]\,[3\ 0\ 4\ 2\ 0\ 1\ 2]$$

Note: PCDs are equivalent if the pc's traverse the same edges

Facts about PCDs

- Up to the equivalence discussed, every $\mathcal{E}$ determines a unique $\mathcal{D}_{\mathcal{E}}$

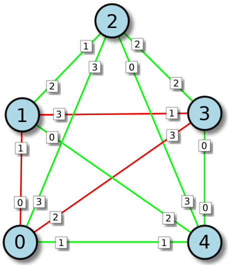


Figure: $\mathcal{D} =$ [0 1 3][0 2 1 4 3 2 4]

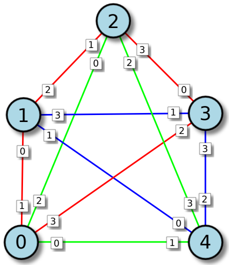


Figure: $\mathcal{D} =$ [0 1 3][0 2 4][1 3 4]

Facts about PCDs

- Up to the equivalence discussed, every $\mathcal{E}$ determines a unique $\mathcal{D}_{\mathcal{E}}$
- Every vertex appears exactly two times in $\mathcal{D}$

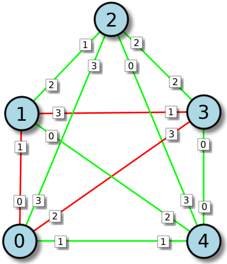


Figure: $\mathcal{D} = [0\ 1\ 3][0\ 2\ 1\ 4\ 3\ 2\ 4]$

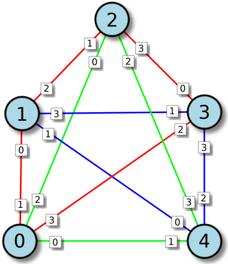
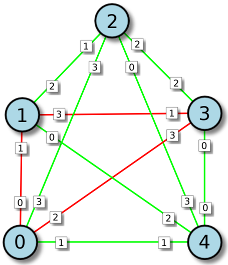


Figure: $\mathcal{D} = [0\ 1\ 3][0\ 2\ 4][1\ 3\ 4]$

Facts about PCDs

- Up to the equivalence discussed, every $\mathcal{E}$ determines a unique $\mathcal{D}_{\mathcal{E}}$
- Every vertex appears exactly two times in $\mathcal{D}$
- The smallest parity circuit is of length 3



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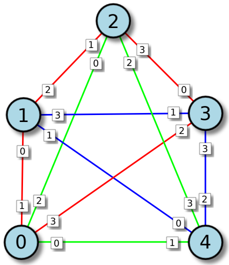

Figure: $\mathcal{D} =$ [0 1 3][0 2 4][1 3 4]

Facts about PCDs

- Up to the equivalence discussed, every $\mathcal{E}$ determines a unique $\mathcal{D}_{\mathcal{E}}$
- Every vertex appears exactly two times in $\mathcal{D}$
- The smallest parity circuit is of length 3
- The lengths of the parity circuits add up to 10.

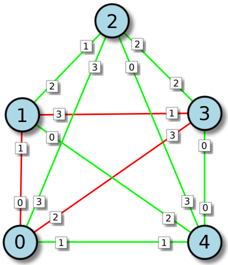

Figure: $\mathcal{D} =$ [0 1 3][0 2 1 4 3 2 4]

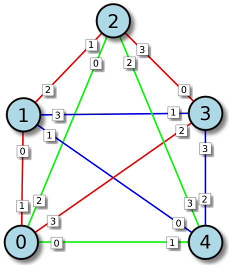

Figure: $\mathcal{D} =$ [0 1 3][0 2 4][1 3 4]

Signature of a PCD

Definition

Let $\mathcal{D} = \{p_1, \ldots, p_k\}$ be a parity circuit decomposition of K_5. The list of lengths of each circuit, ordered by size, is called the **signature** of the PCD.

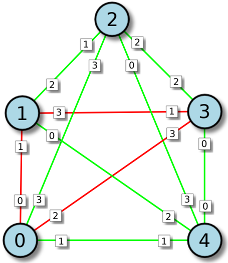

Figure: Example of 3, 7-pcd

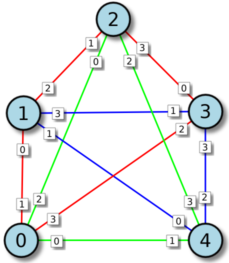

Figure: Example of a 3, 3, 4-pcd

Facts about the signature

Theorem

The signature of a parity circuit decomposition $\mathcal{D}$ is a partition of 10 with terms no smaller than 3. Which implies that the signature of any enumeration is equal to one of the following

- $3, 3, 4$.
- $3, 7$.
- $4, 6$.
- $5, 5$.
- 10.

The number of unique PCD's

- Taken up to the equivalence that we spoke of earlier, we are interested in counting the number of unique PCDs.

Theorem

The number of unique parity circuit decompositions is 3^5

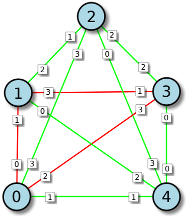

Figure: Example of $3, 7$-pcd

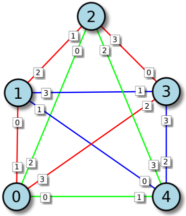

Figure: Example of a $3, 3, 4$-pcd

Sketch of Proof

Let:

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- **E** be the set of all enumerations of K_5

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Sketch of Proof

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- $\mathbf{E}$ be the set of all enumerations of K_5
- $\mathbf{P}$ be the set of all parity circuit decompositions (modulo equivalence)
- $F : \mathbf{E} \mapsto \mathbf{P}$ be the natural mapping that sends an enumeration $\mathcal{E}$ to $\mathcal{D}_{\mathcal{E}}$

Sketch of Proof

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- **P** be the set of all parity circuit decompositions (modulo equivalence)
- $F : \mathbf{E} \mapsto \mathbf{P}$ be the natural mapping that sends an enumeration $\mathcal{E}$ to $\mathcal{D}_{\mathcal{E}}$
- Let $D_4 = \langle (0\ 1\ 2\ 3), (0\ 2)(1\ 3) \rangle$(Dihedral Group)

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- $F : \mathbf{E} \mapsto \mathbf{P}$ be the natural mapping that sends an enumeration $\mathcal{E}$ to $\mathcal{D}_{\mathcal{E}}$
- Let $D_4 = \langle (0\ 1\ 2\ 3)\,, (0\ 2)\,(1\ 3) \rangle$(Dihedral Group)

- We will use F to define a bijection that will allow for us to count the number of unique PCDs.

Lemma

$F(\sigma \cdot \mathcal{E}) = F(\pi \cdot \mathcal{E})$ *if and only if* $\sigma^{-1} \circ \pi \in D_4^5$

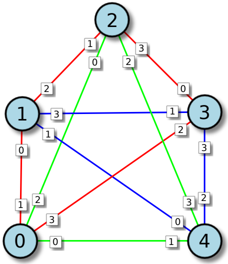

Figure: Example of a $3, 3, 4$-pcd

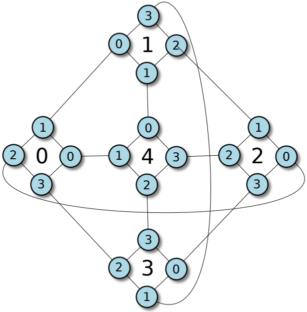

Figure: $K_5 \circledR C_4$

Sketch of Proof

Theorem

Let $\mathcal{E}^0 \in \mathbf{E}$ and let $F^* : S_4^5/D_4^5 \mapsto \mathbf{P}$ given by $\sigma D_4^5 \mapsto F\left(\sigma \cdot \mathcal{E}^0\right)$. Then F^* is a bijection.

Proof.

Since $F(\sigma \cdot \mathcal{E}) = F(\pi \cdot \mathcal{E})$ iff $\sigma^{-1} \circ \pi \in D_4^5$ we have that F^* is an injection of the cosets of S_4^5/D_4^5 and the mapping is surjective by definition. $\square$

Sketch of Proof

Corollary

$|\mathbf{P}| = \left[S_4^5 \ : \ D_4^5\right]$

Lagrange's Theorem gives us that

$$|\mathbf{P}| = \left[S_4^5 \ : \ D_4^5\right] = \frac{(4!)^5}{(8)^5} = 3^5$$

Classifying PCDs

- Next we will seek to classify the 3^5 parity circuit decompositions discussed earlier.

Classifying PCDs

- Next we will seek to classify the 3^5 parity circuit decompositions discussed earlier.
- We will do this by examining how the natural group action provided by S_5 on the vertices of K_5 affects the circuit decomposition.

How S_5 acts on a $\mathcal{D}$

Example

If $\sigma_1 = (0\ 1\ 3\ 4)\,(2)$ then

$$\sigma_1 \cdot [0\ 1\ 3]\,[0\ 2\ 1\ 4\ 3\ 2\ 4] = [1\ 3\ 4]\,[1\ 2\ 3\ 0\ 4\ 2\ 0]$$

Example

If $\sigma_2 = (0\ 3)\,(2\ 4)\,(1)$ then

$$\sigma_1 \cdot [0\ 1\ 3]\,[0\ 2\ 1\ 4\ 3\ 2\ 4] = [3\ 1\ 0]\,[3\ 4\ 1\ 2\ 0\ 4\ 2]$$

How S_5 acts on a $\mathcal{D}$

Example

If $\sigma_1 = (0\ 1\ 3\ 4)\,(2)$ then

$$\sigma_1 \cdot [0\ 1\ 3]\,[0\ 2\ 1\ 4\ 3\ 2\ 4] = [1\ 3\ 4]\,[1\ 2\ 3\ 0\ 4\ 2\ 0]$$

Example

If $\sigma_2 = (0\ 3)\,(2\ 4)\,(1)$ then

$$\sigma_1 \cdot [0\ 1\ 3]\,[0\ 2\ 1\ 4\ 3\ 2\ 4] = [3\ 1\ 0]\,[3\ 4\ 1\ 2\ 0\ 4\ 2]$$

Note: σ_2 fixes the PCD (modulo equivalence).

Counting Using the Orbit Stabilizer Theorem

The orbit-stabilizer theorem relates the cardinality of the orbit of an element under a group action to both the size of the group and it's stabilizer.

$$|\Phi_G(s)| = \frac{|G|}{|\mathrm{stab}_G(s)|}$$

Summary of Classification

Table: Summary of PCDs with given signature

Signature (Type)	Representative PCD	Size of Orbit
3,3,4	[0 1 2 3] [0 2 4] [1 3 4]	$\frac{5!}{8} = 15$
3,7	[0 1 2] [0 3 4 2 3 1 4]	$\frac{5!}{2} = 60$
4,6	[0 1 2 3] [0 4 3 1 4 2]	$\frac{5!}{4} = 30$
5,5	[0 1 2 3 4] [0 2 4 1 3]	$\frac{5!}{20} = 6$
10 (4-distinct)	[0 1 2 3 1 4 3 0 4 2]	$\frac{5!}{10} = 12$
10 (5-distinct/4 start)	[0 1 2 3 4 1 3 0 4 2]	$\frac{5!}{2} = 60$
10 (5-distinct/8 start)	[0 1 2 3 4 2 0 3 1 4]	$\frac{5!}{2} = 60$
Total		$3^5 = 243$

Table: Summary of PCDs with given signature and their stabilizing group

Representative PCD	Stabilizing Group
[0 1 2 3] [0 2 4] [1 3 4]	$\langle$ (0 1 2 3)(4), (1 2)(0 3)(4) $\rangle$
[0 1 2] [0 3 4 2 3 1 4]	$\langle$ (0 2)(3 4)(1) $\rangle$
[0 1 2 3] [0 4 3 1 4 2]	$\langle$ (0 1 2 3)(4) $\rangle$
[0 1 2 3 4] [0 2 4 1 3]	$\langle$ (0 1 2 3 4), (0) (1 2 4 3)$\rangle$
[0 1 2 3 1 4 3 0 4 2]	$\langle$(0 2 1 3 4) , (3) (0 2) (1 4)$\rangle$
[0 1 2 3 4 1 3 0 4 2]	$\langle$(0 2) (1 4) (3)$\rangle$
[0 1 2 3 4 2 0 3 1 4]	$\langle$ (0 3)(1 1)(4) $\rangle$

Justification: PCDs with a 4-cycle

$$\mathrm{stab}_{S_5}([0\ 1\ 2\ 3]\,[0\ 2\ 4]\,[1\ 3\ 4]) = \langle\ (0\ 1\ 2\ 3)(4),\ \ (1\ 2)(0\ 3)(4)\ \rangle$$

and

$$\mathrm{stab}_{S_5}([0\ 1\ 2\ 3]\,[0\ 4\ 3\ 1\ 4\ 2]) = \langle\ (0\ 1\ 2\ 3)(4)\ \rangle$$

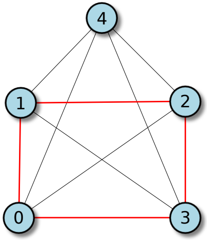

Figure: Both PCD's have 4-cycle.

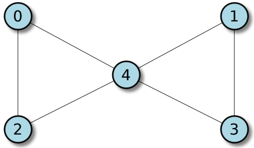

Figure: K_5 with [0 1 2 3] removed

Justification: [0 1 2] [0 3 4 2 3 1 4]

$$\mathrm{stab}_{S_5}([0\ 1\ 2]\,[0\ 3\ 4\ 2\ 3\ 1\ 4]) = \langle\ (0\ 2)(3\ 4)(1)\ \rangle$$

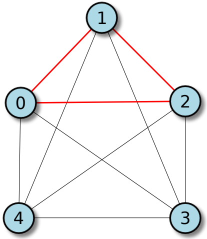

Figure: PCD with 3-cycle.

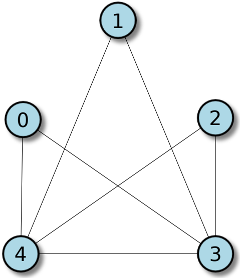

Figure: K_5 with [0 1 2] removed

Justification: [0 1 2 3 4] [0 2 4 1 3]

$$\mathrm{stab}_{S_5}\left([0\ 1\ 2\ 3\ 4]\,[0\ 2\ 4\ 1\ 3]\right) = \langle (0\ 1\ 2\ 3\ 4),\ (0)\,(1\ 2\ 4\ 3)\rangle$$

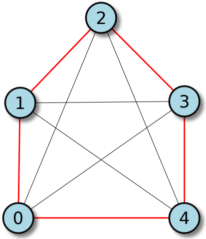

Figure: PCD with 5-cycle.

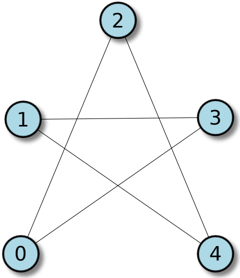

Figure: K_5 with [0 1 2 3 4] removed

PCD's with 10 as a signature

Each of the prior classes have one representative per signature.
PCD's with 10 as the signature have 3 orbit representatives.

- [0 1 2 3 4 2 0 3 1 4] (4 distinct)
- [0 1 2 3 4 1 3 0 4 2] (5 distinct, 4 start)
- [0 1 2 3 4 2 0 3 1 4] (5 distinct, 8 start)

PCD's with 10 as a signature: Terminology

Definition

We will say that a PCD is n-**distinct** if the maximum number of unique vertices traversed before duplicates is equal to n

Definition

We will call the starting vertex of a distinct sequence when reading from left to right a **forward start**, denoted $\underline{i}$, and if it is the beginning of a distinct sequence reading from right to left a **backward start**, which we denote $\bar{i}$.

PCD's with 10 as a signature

The 3 orbit-representatives for PCDs with signature 10.

- $[\underline{0}\ \overline{1}\ \underline{2}\ \overline{3}\ \underline{1}\ \overline{4}\ \underline{3}\ \overline{0}\ \underline{4}\ \overline{2}]$ (4-distinct)
- $[\underline{0}\ 1\ 2\ 3\ \overline{4}\ \underline{1}\ 3\ 0\ 4\ \overline{2}]$ (5 distinct, 4 start)
- $[\underline{0}\ 1\ 2\ \overline{3}\ \underline{\overline{4}}\ \underline{2}\ 0\ 3\ \overline{1}\ \underline{\overline{4}}]$ (5 distinct, 8 start)

All of the stabilizing groups are calculated by observing that S_5 must send n-distinct elements to n-distinct elements.

Table: Summary of PCDs with given signature and their stabilizing group

Representative PCD	Stabilizing Group
[0 1 2 3] [0 2 4] [1 3 4]	$\langle(0\ 1\ 2\ 3)(4), (1\ 2)(0\ 3)(4)\rangle$
[0 1 2] [0 3 4 2 3 1 4]	$\langle(0\ 2)(3\ 4)(1)\rangle$
[0 1 2 3] [0 4 3 1 4 2]	$\langle(0\ 1\ 2\ 3)(4)\rangle$
[0 1 2 3 4] [0 2 4 1 3]	$\langle(0\ 1\ 2\ 3\ 4), (0)\,(1\ 2\ 4\ 3)\rangle$
[0 1 2 3 1 4 3 0 4 2]	$\langle(0\ 2\ 1\ 3\ 4)\,,\ (3)\,(0\ 2)\,(1\ 4)\rangle$
[0 1 2 3 4 1 3 0 4 2]	$\langle(0\ 2)\,(1\ 4)\,(3)\rangle$
[0 1 2 3 4 2 0 3 1 4]	$\langle\ (0\ 3)(1\ 1)(4)\ \rangle$

PCDs, $K_5 \textcircled{r}\ C_4$ and $K_5 \textcircled{z}\ C_4$

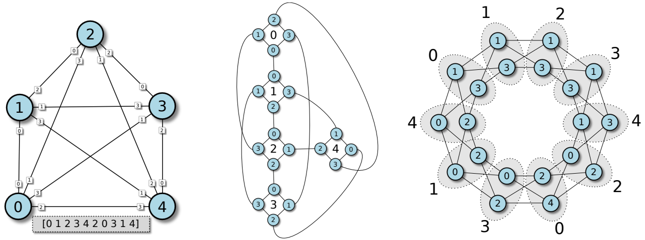

Figure: An enumeration on K_5 with associated replacement and zig-zag products.

Corollary

If F is a replacement product isomorphism. $F \cdot \mathcal{D}$ is a $\mathcal{E}'$-parity circuit decomposition whenever $\mathcal{D}$ is a $\mathcal{E}$-parity circuit decomposition

PCDs and $K_5 \textcircled{r} C_4$

Corollary

If F is a replacement product isomorphism. $F \cdot \mathcal{D}$ is a $\mathcal{E}'$-parity circuit decomposition whenever $\mathcal{D}$ is a $\mathcal{E}$-parity circuit decomposition

- Shows that two enumerations generate rp-isomorphic replacement products iff they generate the same PCD on K_5

PCDs and $K_5 \circledR\, C_4$

Corollary

If F is a replacement product isomorphism. $F \cdot \mathcal{D}$ is a $\mathcal{E}'$-parity circuit decomposition whenever $\mathcal{D}$ is a $\mathcal{E}$-parity circuit decomposition

- Shows that two enumerations generate rp-isomorphic replacement products iff they generate the same PCD on K_5
- The characterization of equivalent PCD's also characterize replacement products.

PCDs and the Replacement Product

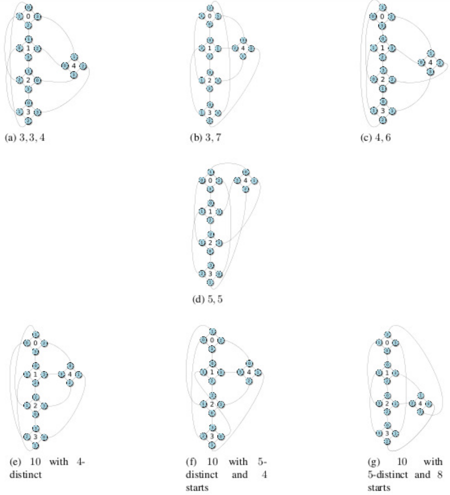

Figure: The 7 classes of $K_5 \textcircled{r} C_4$

PCD's and the Zig-Zag Product: 2-trellis

Definition

A t-**trellis** over a cycle C_n is a graph with $t \cdot n$ vertices, partitioned into $S_1, \ldots, S_n$, such that each S_i has t elements and each element of S_i is connected to each element of S_{i+1} and S_{i-1} where i is taken modulo n.

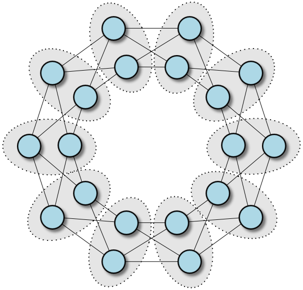

Figure: A 2-trellis over C_{10}

PCD's and the Zig-Zag Product

Theorem

$[u_1 \ \cdots \ u_k]$ is a parity circuit with parity a_i at u_i if and only if $K_5 \, ⓩ \, C_4$ has a subgraph that is a 2-trellis over C_k with $S_i = \{(u_i, 1 - a_i), (u_i, 3 - a_i)\}$ for $i = 1, \ldots, k$.

Theorem

Each connected component of $K_5 \, ⓩ \, C_4$ is a 2-trellis over a cycle C_j.

Theorems demonstrate that there is a one-to-one correspondence between the PCDs of K_5 and the components of $K_5 \textcircled{z} C_4$.

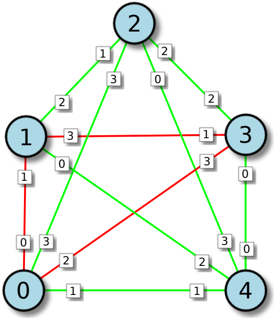

Figure: K_5 with $3,7$-pcd

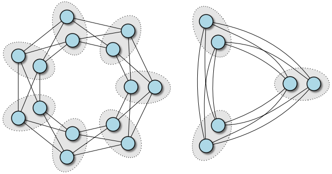

Figure: $K_5 \textcircled{z} C_4$ with two components

PCDs and the Zig-Zag Product

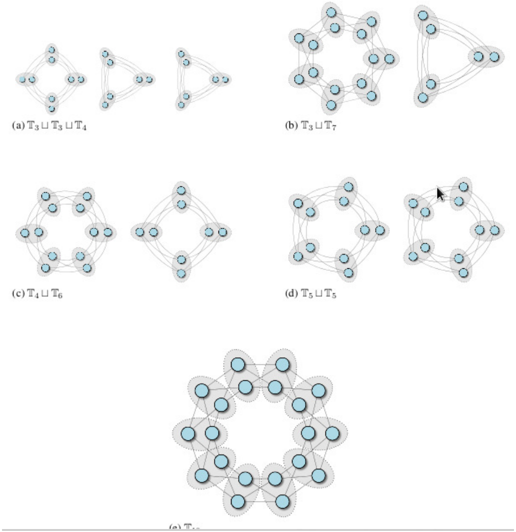

Figure: The 5 classes of $K_5 \circledR C_4$

Generalization of the Zig-Zag Product

- Finally we would like to define a zig-zag like product in a more general setting.

$$
\begin{tikzcd}
\mathrm{E}(G) \arrow[d, shift right, "\varsigma_G"'] \arrow[d, shift left, "\tau_G"] \\
\mathrm{V}(G)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{E}(G) \arrow[loop above, "\varrho_G"] \arrow[d, "\varsigma_G"] \\
\mathrm{V}(G)
\end{tikzcd}
$$

Generalization of the Zig-Zag Product

- We use a very general definition of graph and digraph.(Similar to Harary [3], Mac Lane [4], and Serre [9])

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- Introduce the *concatenation* and the *sandwich product* of graphs.

Generalization of the Zig-Zag Product

- We use a very general definition of graph and digraph.(Similar to Harary [3], Mac Lane [4], and Serre [9])
- Introduce the *concatenation* and the *sandwich product* of graphs.
- Demonstrate that the zig-zag product of graphs may be concisely presented as the sandwich product of two relatively simple graphs.

A New (and old) way to define a graph

Definition

A **directed graph** (or **digraph**), denoted by the letter G, is a collection of two sets E and V together with two functions ς_G and τ_G from E to V where

- E and V are known as the **edge** and **vertex** sets of the digraph G.
- ς_G and τ_G are known as the **source** and **terminus** functions of the digraph G.

$$\begin{tikzcd} \mathrm{E}(G) \arrow[d, shift right, "\varsigma_G"'] \arrow[d, shift left, "\tau_G"] \\ \mathrm{V}(G) \end{tikzcd}$$

A New (and old) way to define a graph

Definition

An **undirected graph**, or just **graph**, is a digraph G in which there exists a unique involution $\varrho_G : E \mapsto E$ such that $\tau_G = \sigma_G \circ \varrho_G$. We call the function ϱ_G the **rotation mapping** of the graph G.

$$\begin{tikzcd}
\mathrm{E}(G) \arrow[loop above, "\varrho_G"] \arrow[d, "\varsigma_G"] \\
\mathrm{V}(G)
\end{tikzcd}$$

Definition

Let A, B, and C be sets and let $f : A \mapsto C$ and $g : B \mapsto C$ be functions. Then the **pullback** of set functions f and g is the set

$$A \times_C B = \{(a, b) \in A \times B \mid f(a) = g(b)\}$$

together with the standard coordinate projections π_1 and π_2.

$$\begin{tikzcd}
A \times_C B \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] & B \arrow[d, "g"] \\
A \arrow[r, "f"'] & C
\end{tikzcd}$$

Definition

Let G and H be directed graphs with a common vertex set V and let $E(G) \times_V E(H)$ be the pullback of the mappings τ_G and ς_H. Then the **concatenation** of G with H, denoted by $G \circledcirc H$, is the directed graph with vertex set V, edge set $E(G) \times_V E(H)$, source map $\varsigma = \varsigma_G \circ \pi_1$ and terminus map $\tau = \tau_H \circ \pi_2$ as depicted in the diagram

$$\begin{tikzcd}
 & E(G) \times_V E(H) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] \arrow[dd, shift right, "\varsigma"'] \arrow[dd, shift left, "\tau"] & \\
E(G) \arrow[dr, "\varsigma_G"'] & & E(H) \arrow[dl, "\tau_H"] \\
 & V &
\end{tikzcd}$$

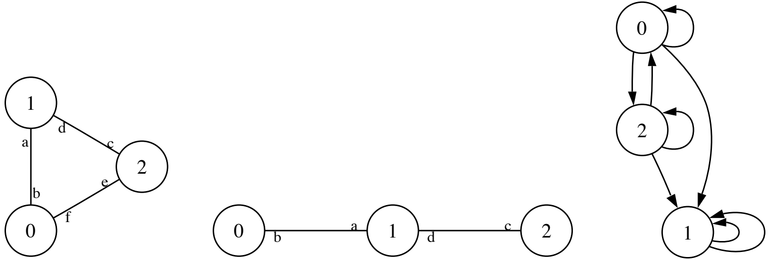

Figure: Three cycle, three chain (with indicated source and terminus labeling) and their concatenation.

Definition

Let G and H be graphs with common vertex set V. Then the **sandwich product** of G with H, denoted $G \circledS H$, is the graph with vertex set V, edge set

$$\left\{(f, e, f') \in F \times E \times F \mid \tau_H(f) = \varsigma_G(e) \text{ and } \tau_G(e) = \varsigma_H\left(f'\right)\right\}$$

with source map which takes $(e, f, e') \mapsto (\varsigma_G(e), \varsigma_H(f))$ and rotation map

$$\varrho((f, e, f')) = \left(\varrho_H(f), \varrho_G(e), \varrho_H(f')\right)$$

Definition

Let G and H be graphs. The the **zig product** of G with H, denoted $G \textcircled{1} H$, is the graph depicted in the diagram

$$\begin{tikzcd}
V(G) \times E(H) \arrow[loop above, "id_{V(G)} \times \varrho_H"] \arrow[d, "id_{V(G)} \times \varsigma_H"'] \\
V(G) \times V(H)
\end{tikzcd}$$

Definition

Let G and H be graphs and let $\phi : \mathrm{E}(G) \mapsto \mathrm{V}(H)$ be an onto mapping. The **zag product** of G and H, denoted $G ⓐ H$, is the graph with vertex set $V(G) \times V(H)$, edge set $\mathrm{E}(G ⓐ H) = \mathrm{E}(G)$, source mapping $\varsigma_{\mathrm{zag}} = \varsigma_g \times \phi$ and rotation mapping $\varrho_{\mathrm{zag}} = \varrho_G$ as depicted the diagram

$$\begin{tikzcd} E(G) \arrow[loop above, "\varrho_G"] \arrow[d, "\varsigma_G \times \phi"'] \\ V(G) \times V(H) \end{tikzcd}$$

Definition

Let G and H be graphs with $\phi : \mathrm{E}(G) \mapsto \mathrm{V}(H)$. Then the **zig-zag** product of G with H, denoted $G \text{ⓩ} H$, is the graph with vertex set $V(G) \times V(H)$, edge set

$$E(G \text{ⓩ} H) = \left\{ \left(f, e, f'\right) \mid \tau_H(f) = \phi(e) \text{ and } \phi(\varrho(e)) = \varsigma_H\left(f'\right) \right\},$$

source mapping given by $\varsigma((f, e, f')) = (\varsigma_G(e), \varsigma_H(f))$ and rotation mapping ϱ given by

$$\varrho(e, f, e') = \left(\varrho_H\left(f'\right), \varrho_G(e), \varrho_H(f)\right)$$

Theorem

$$G \text{ⓩ} H = (G \text{ⓐ} H) \text{ⓢ} (G \text{ⓘ} H)$$

Further Topics of Study

- The classification of $G \textcircled{r} C_4$ and $G \textcircled{z} C_4$ for any 4-regular graph

Further Topics of Study

- The classification of $G \textcircled{r} C_4$ and $G \textcircled{z} C_4$ for any 4-regular graph
- The classification of $G \textcircled{r} C_n$ and $G \textcircled{z} C_n$.

Further Topics of Study

- The classification of $G \text{ⓡ} C_4$ and $G \text{ⓩ} C_4$ for any 4-regular graph
- The classification of $G \text{ⓡ} C_n$ and $G \text{ⓩ} C_n$.
- The Sandwich Product and the generalization to non-regular graphs.

Further Topics of Study

- The classification of $G \textcircled{r} C_4$ and $G \textcircled{z} C_4$ for any 4-regular graph
- The classification of $G \textcircled{r} C_n$ and $G \textcircled{z} C_n$.
- The Sandwich Product and the generalization to non-regular graphs.
- The Zig-Zag Product and its connection with the semi-direct product of groups.

Further Topics of Study

- The classification of $G \circledr C_4$ and $G \circledz C_4$ for any 4-regular graph
- The classification of $G \circledr C_n$ and $G \circledz C_n$.
- The Sandwich Product and the generalization to non-regular graphs.
- The Zig-Zag Product and its connection with the semi-direct product of groups.
- The Zig-Zag Product and its connection with the semi-direct product of groups

The classification of $G Ⓡ C_4$ and $G Ⓩ C_4$ for any 4-regular graph

We would like to extend the definitions and theorems contained in the thesis to

- $G Ⓡ C_4$ and $G Ⓩ C_4$ for any 4-regular graph.
- little work needs to be done to extend the concepts as they almost depend completely on the properties of C_4
- Though much is not known about how $\mathrm{Aut}(G)$ being more general affects the classes of PCDs
- it is suspected that when G has a small automorphism group there will likely be many classes of replacement and zig-zag products.

The classification of $G \textcircled{r} C_n$ and $G \textcircled{z} C_n$

- The zig-zag product when taken over C_4 is always a 2-trellis. (which has bad expansion)
- It is suspected that the zig-zag product over larger cycles will yield a richer class of graphs with possibly better expansion properties.
- Still simple enough that a full classification is practical.

The Sandwich Product and the generalization to non-regular graphs

- We considered a graph just a collection of functions.
- have opened up an intriguing and, some would say, elegant method for describing the zig-zag product in a more general setting.
- Today the definitions are too broad to be useful for analysis, but by placing additional restrictions on the function involved there is hope for further development.

The Zig-Zag Product and its connection with the semi-direct product of groups

- Alon et al. [1] presented an interesting connection between the semi-direct product and the zig-zag product of graphs.
- That with suitably chosen generating sets, the Cayley graphs of the semi-direct product of two groups can be seen as the zig-zag product of the constituent Cayley graphs.
- let $G = \mathbb{Z}_5$ with Cayley generating set $S_G = \mathbb{Z}_5 \setminus \{0\}$ and $H = \mathbb{Z}_4$ with generating set $S_H = \{\pm 1\}$, then we have Cayley graphs which mirror K_5 and C_4
- Obvious questions are how does the structure of the groups effect the enumeration which would be generated by these Cayley graphs.

So in conclusion, as with most mathematical work, we are left with as many questions as we have answers. The hope of the author is that this thesis will be seen as a good first step.

Any Questions?

Copy of Presentation:
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Copy of Thesis:
http://dl.dropbox.com/u/1768136/thesis.pdf

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