2. $P(y=1|x_i,\omega) = \frac{1}{2}\left(1+\frac{\omega^T x_i}{\sqrt{1+(\omega^T x_i)^2}}\right)$

Hence,

$$P(y=0|x_i,\omega) = 1-\frac{1}{2}\left(1+\frac{\omega^T x_i}{\sqrt{1+(\omega^T x_i)^2}}\right)$$

Assuming Data points are independent, Likelihood function is given by

$$l(\omega) = \prod_{i=1}^{n} P(y_i=1|x_i,\omega)^{y_i}\, P(y_i=0|x_i,\omega)^{1-y_i}$$

$$= \prod_{i=1}^{n}\left[\frac{1}{2}\left(1+\frac{W^T x_i}{\sqrt{1+(\omega^T x_i)^2}}\right)\right]^{y_i}\left[1-\frac{1}{2}\left(1+\frac{W^T x_i}{\sqrt{1+(\omega^T x_i)^2}}\right)\right]^{(1-y_i)}$$

$$= \prod_{i=1}^{n}\left[\frac{1}{2}\left(1+\frac{W^T x_i}{\sqrt{1+(\omega^T x_i)^2}}\right)\right]^{y_i}\left[\frac{1}{2}\left(1-\frac{W^T x_i}{\sqrt{1+(\omega^T x_i)^2}}\right)\right]^{1-y_i}$$

Taking Log

$ll(\omega) =$

$$= \log\left\{\prod_{i=1}^{n}\left[\frac{1}{2}\left(1+\frac{W^T x_i}{\sqrt{1+(\omega^T x_i)^2}}\right)\right]^{y_i}\left[\frac{1}{2}\left(1-\frac{W^T x_i}{\sqrt{1+(\omega^T x_i)^2}}\right)\right]^{1-y_i}\right\}$$

$$= \sum_{i=1}^{n}\log\left\{\left[\frac{1}{2}\left(1+\frac{W^T x_i}{\sqrt{1+(\omega^T x_i)^2}}\right)\right]^{y_i}\right\} + \log\left\{\left[\frac{1}{2}\left(1-\frac{W^T x_i}{\sqrt{1+(\omega^T x_i)^2}}\right)\right]^{1-y_i}\right\}$$

$$= \sum_{i=1}^{n} y_i \log(1/2) + y_i \log\left(1+\frac{\omega^T x_i}{\sqrt{1+(\omega^T x_i)^2}}\right) + (1-y_i)\log(1/2)$$
$$+ (1-y_i)\log\left(1-\frac{\omega^T x_i}{\sqrt{1+(\omega^T x_i)^2}}\right)$$

Scanned by CamScanner

$$= n\log\left(\tfrac{1}{2}\right) + \sum_{i=1}^{n} y_i \log\left(1 + \frac{W^T x_i}{\sqrt{1+(\omega^T x_i)^2}}\right)$$

$$+ (1-y_i)\log\left(1 - \frac{\omega^T x_i}{\sqrt{1+(\omega^T x_i)^2}}\right)$$

Differentiating with respect to ω:

$$\frac{\partial \ell\ell(\omega)}{\partial \omega_j} = \sum_{i=1}^{n} y_i \frac{x_{ij}\left(\sqrt{(\omega^T x_i)^2+1} - \omega^T x_i\right)}{(W^T x_i)^2+1} -$$

$$(1-y_i)\frac{x_{ij}\left(\sqrt{(W^T x_i)^2+1} + W^T x_i\right)}{(W^T x_i)^2+1}$$

$$= \sum_{i=1}^{n} \frac{y_i x_{ij}\left(\sqrt{(\omega^T x_i)^2+1} - W^T x_i\right) - (1-y_i)\left(\sqrt{(\omega^T x_i)^2+1} + W^T x_i\right)}{(W^T x_i)^2+1}$$

Numerator $\rightarrow$

$$= \sum_{i=1}^{n} \frac{y_i x_{ij}\sqrt{(\omega^T x_i)^2+1} - y_i \cancel{x_{ij}\,\omega^T x_i} - x_{ij}\sqrt{(\omega^T x_i)^2+1} - x_{ij}\,\omega^T x_i + y_i x_{ij}\sqrt{(\omega^T x_i)^2+1} + y_i \cancel{x_{ij}\,\omega^T x_i}}{(\omega^T x_i)^2+1}$$

$$= \sum_{i=1}^{n} \frac{y_i x_{ij}\sqrt{(\omega^T x_i)^2+1} - x_{ij}\sqrt{(\omega^T x_i)^2+1} - x_{ij}\,\omega^T x_i + y_i x_{ij}\sqrt{(\omega^T x_i)^2+1}}{(\omega^T x_i)^2+1}$$

$$= \sum_{i=1}^{n} \frac{2 y_i x_{ij}\sqrt{(\omega^T x_i)^2+1} - x_{ij}\sqrt{(\omega^T x_i)^2+1} - x_{ij}\,W^T x_i}{(\omega^T x_i)^2+1}$$

$$= \sum_{i=1}^{n} \frac{x_{ij}\left[(2y_i-1)\sqrt{(W^T x_i)^2+1} - W^T x_i\right]}{(W^T x_i)^2+1}$$

Since, we won't find any analytical solution on equating to zero, ~~so~~
We should find second derivative

$$\frac{\partial^2 ll(\omega)}{\partial \omega_j \partial \omega_k} = \frac{\partial}{\partial w_k} \sum_{i=1}^{n} \frac{x_{ij}\left[(2y_i - 1)\sqrt{(W^T x_i)^2 + 1} - W^T x_i\right]}{(W^T x_i)^2 + 1}$$

$$= \sum_{i=1}^{n} \frac{\partial}{\partial w_k}\left[\frac{\left\{x_{ij}\left[(2y_i - 1)\sqrt{(W^T x_i)^2 + 1} - W^T x_i\right]\right\}}{(W^T x_i)^2 + 1}\right]$$

$$= \sum_{i=1}^{n} \frac{\frac{\partial}{\partial w_k}\left[\left\{x_{ij}\left[(2y_i - 1)\sqrt{(W^T x_i)^2 + 1} - W^T x_i\right]\right\}\right]\left[(W^T x_i)^2 + 1\right] - \left[\left\{x_{ij}\left[(2y_i - 1)\sqrt{(W^T x_i)^2 + 1} - W^T x_i\right]\right\}\right]\frac{\partial}{\partial w}\left[(W^T x_i)^2 + 1\right]}{((W^T x_i)^2 + 1)^2}$$

$$= \sum_{i=1}^{n} \frac{\frac{\partial}{\partial w_k}\left[\left\{x_{ij}\left[(2y_i - 1)\sqrt{(W^T x_i)^2 + 1} - W^T x_i\right]\right\}\right]\left[(W^T x_i)^2 + 1\right]}{((W^T x_i)^2 + 1)^2}$$

$$- \frac{\left[\left\{x_{ij}\left[(2y_i - 1)\sqrt{(W^T x_i)^2 + 1} - W^T x_i\right]\right\}\right]\frac{\partial}{\partial w_k}\left[(W^T x_i)^2 + 1\right]}{((W^T x_i)^2 + 1)^2}$$

Solving two fractions separately:

$$\frac{\partial}{\partial \omega_k}\left[\left\{x_{ij}\left[(2y_i - 1)\sqrt{(\omega^T x_i)^2 + 1} - W^T x_i\right]\right\}\right]$$

$$= x_{ij}\left[\left\{(2y_i - 1)\frac{\partial}{\partial w_k}\left[\sqrt{(W^T x_i)^2 + 1}\right] - \frac{\partial}{\partial w_k}\left[W^T x_i\right]\right\}\right]$$

$$= x_{ij}\left\{(2y_i-1)\frac{1}{2}\frac{2\,W^Tx_i\cdot x_{ik}}{\sqrt{(W^Tx_i)^2+1}} - x_{ik}\right\}$$

$$= x_{ij}\,x_{ik}\left\{\frac{(2y_i-1)\,W^Tx_i}{\sqrt{(W^Tx_i)^2+1}} - 1\right\}$$

$$= x_{ij}\,x_{ik}\left\{\frac{(2y_i-1)\,W^Tx_i - \sqrt{(W^Tx_i)^2+1}}{\sqrt{(W^Tx_i)^2+1}}\right\}$$

$$\frac{\partial}{\partial w_k}\frac{\left\{x_{ij}\left[(2y_i-1)\sqrt{(W^Tx_i)^2+1} - W^Tx_i\right]\right\}\left[((W^Tx_i)^2+1)\right]}{((W^Tx_i)^2+1)^2}$$

$$= \frac{x_{ij}\,x_{ik}\left\{\frac{(2y_i-1)W^Tx_i - \sqrt{(W^Tx_i)^2+1}}{\sqrt{(W^Tx_i)^2+1}}\right\}}{(W^Tx_i)^2+1}$$

$$= x_{ij}\,x_{ik}\left\{\frac{(2y_i-1)W^Tx_i - \sqrt{(W^Tx_i)^2+1}}{((W^Tx_i)^2+1)^{3/2}}\right\}$$

Second Fraction is:

$$\frac{\left\{x_{ij}\left[(2y_i-1)\sqrt{(W^Tx_i)^2+1} - W^Tx_i\right]\right\}\frac{\partial}{\partial w_k}\left[(W^Tx_i)^2+1\right]}{((W^Tx_i)^2+1)^2}$$

$$= \frac{x_{ij}\,x_{ik}\,2W^Tx_i\left[(2y_i-1)\sqrt{(W^Tx_i)^2+1} - W^Tx_i\right]}{((W^Tx_i)^2+1)^2}$$

Combining the two fractions and factoring $x_{ij}\,x_{ik}$, we get:

$$\frac{\partial^2 ll(\omega)}{\partial \omega_j \, \partial \omega_k} = \sum_{i=1}^{n} x_{ij} x_{ik} \left[\frac{(2y_i - 1) W^T x_i - \sqrt{(\omega^T x_i)^2 + 1}}{((W^T x_i)^2 + 1)^{3/2}} \right.$$

$$\left. - \frac{2 W^T x_i \left[(2y_i - 1)\sqrt{(W^T x_i)^2 + 1} - W^T x_i \right.}{((W^T x_i)^2 + 1)^2} \right.$$

$$\boxed{W^{(t+1)} = W^{(0)} - H^{-1} \nabla}$$

H :- Hessian

∇ :- Gradient

Following Expression has been implemented in program.