Set:

George Cantor defined a set as " a collection of well defined objects or things ", objects in a set are called elements or members of a set. Here the term well defined means there must be common property between the elements of a set.

Usually a set is denoted by letters A, B, C ---

for example: $A = \{a, b, c, d\}$

Set Membership: Consider a set A, then x is an element of A, it is written as: $x \in A$.

if x is not an element of A, it is written as: $x \notin A$.

Set Representations:- Sets can be represented by two ways:

(1) Tabular / Roster form: In this form the distinct elements of a set are put within curley braces seperated by comma(,).

Example: $A = \{1, 2, 3, 4, 5\}$

$1 \in A$, $2 \in A$, $3 \in A$, $4 \in A$, $5 \in A$, $7 \notin A$.

(2) Set builder form: In this form the set is represents by a defination or a common property of elements of the given set.

Example: $A = \{x : x < 5 \text{ and } x \in N\}$

$A = \{x : x^2 = 9, x \in Z\}$

Cardinality or cardinal number of a set:

The number of distinct elements in a set is called the cardinality or cardinal number of a set. It is denoted by |Set Name|.

Example: $X = \{a, e, i, o, u\}$

$|X| = 5$.

$A = \{1, 2, 3, 4, 6, 5, 4\}$

$|A| = 6$.

Types of set:

Set have many type as given below.

Finite set: The set in which the number of elements are finite that is called a finite set. Also called countable.

Example:

$Y = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

$|Y| = 10$

Infinite set: The set in which the number of elements are infinite that is called an Infinite set. Example: The Set of Natural Numbers,

$N = \{1, 2, 3, 4, \ldots\}$.

Null Set OR Empty Set: A set is said to be empty set or Null set, if there is no element present in set. It is denoted by ϕ or $\{\ \}$.

Ex:

$$X = \{x : x^2 + 4 = 0, x \in Z\}$$

Here $x = \pm 2i \notin Z$

$X = \phi$

$$A = \{x : x + 4 = 3, x \in N\}$$

$A = \phi$

Singleton Set: A set which contain only the one element is called singleton set.

example: $A = \{2\}$.

$X = \{x : x^2 = 9, x \in N\}$

$X = \{3\}$.

Equal Sets! When the elements of a set A are also the elements of a set B and vice versa then A and B are equal set.

$A = B$.

Example: $X = \{1, 2, 4, 6, 7\}$

$Y = \{1, 4, 2, 7, 6\}$

$X = Y$.

Equivalent Sets: Let A and B are two set, if the number of elements in both the sets are equal then there are equivalent sets. so, $|A| = |B|$

Example: $A = \{a, e, i, o, u\}$

$B = \{1, 2, 3, 4, 5\}$

$|A| = 5$, $|B| = 5$ so $|A| = |B|$.

Subset: Denoted by '$\subseteq$'

Consider two sets A and B. If every element of set A is an element of B, then A is a subset of set B.

$A \subseteq B$.

And the set B is called the super set of set A, $B \supseteq A$.

Proper Subset: Let A and B are two sets. If $A \subseteq B$ but $A \neq B$, mens there is at least one element in B which is not a member of set A, then A is called the proper subset of B. and denoted by $A \subset B$.

Note: (1) Every set is a subset of itself.
(2) Null set ϕ is the subset of every set.
(3) Null set ϕ has no proper subset.
(4) If A is a set with n-elements then it has 2^n subsets.

Power Set: It is denoted by P(SetName) as P(A) or 2^A.
The Power set is the set of all possible subsets of a set is called the power set of it.
example:
$A = \{a, b\}$
$P(A) = \{\phi, \{a\}, \{b\}, \{a, b\}\}$.
$B = \{1, 2, 3\}$
$P(B) = \{\phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}, \{1, 2, 3\}\}$.

Universal Set: It is Denoted by U.
The Universal set is the special kind of set, it is the superset of all sets for a particular domain.
Example: If the domain is english alphabets the Universal set is:
$U = \{a \text{ to } z\}$

"Operations on Sets" (5)

Union of Sets: The operation is denoted by '$\cup$'

Let A and B are two non empty sets, then union of A and B is the set of all elements which belongs to either A or B or in both A and B.

Denoted by $A \cup B$.

Ex: $A = \{1, 2\}$

$B = \{2, 3, 4\}$

$A \cup B = \{1, 2, 3, 4\}$

symbollically,

$$x \in (A \cup B) \Longrightarrow x \in A \text{ or } x \in B.$$

$$A \cup B = \{x : x \in A \text{ or } x \in B\}.$$

Intersection of Sets: It is denoted by '$\cap$'

Let A and B be two non empty sets, the intersection of A and B is the set of all elements which are common in A and B.

Denoted by $A \cap B$.

Ex: $A = \{1, 4, 6\}$

$B = \{2, 4, 6, 7\}$

$A \cap B = \{4, 6\}$.

Symbolically

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$

Disjoint Sets:- Let A and B be the two non non empty sets, if there is no common element in A and B, then A and B are said to be Disjoint sets. $\boxed{A \cap B = \phi}$

Ex: $A = \{4, 6, 7\}$

$B = \{1, 2, 3\}$

$A \cap B = \phi$.

Complement of Set: "Denoted by A', A^c or $\bar{A}$"

Let U be the universal set and A be any subset of it, the complement of A is a set containing elements of universal set those does not belongs to A.

Ex: Let

$$U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$$

$$A = \{2, 4, 6, 7\}$$

then

$$A' = \{1, 3, 5, 8, 9, 10\}$$

symbollicaly:

$$A^c = \{x : x \notin A \text{ and } x \in U\}.$$

OR

$$\boxed{A^c = U - A}$$

Difference of Sets:- "Denoted as A − B"

Let A and B be any two sets, the difference of A and B is a set of all elements which belongs A but not belongs to B, Also Denoted as A/B or $A \sim B$.

$$A - B = \{x : x \in A \text{ and } x \notin B\}$$

Ex!

$$A = \{1, 2, 3, 4, 5, 6, 7\}$$

$$B = \{5, 6\}$$

$$A - B = \{1, 2, 3, 4, 7\}$$

similarly

$$B - A = \{x \mid x \in B \text{ and } x \notin A\}$$

such that

(:)

Symmetric Difference of sets:

Let A and B any two sets, the symmetric difference of A and B is a set containing those elements which are either in A or in B but not in both.

Denoted by $A \oplus B$ or $A \Delta B$.

$$A \oplus B = (A \cup B) - (A \cap B)$$

$A = \{1, 2, 3, 4, 5\}$

$B = \{4, 5, 6\}$

$A \cup B = \{1, 2, 3, 4, 5, 6\}$

$A \cap B = \{4, 5\}$

$A \oplus B = \{1, 2, 3, 6\}$

Laws of Algebra of sets:

(1) Idempotent law:

$A \cup A = A$

$A \cap A = A$

(2) Commutative law:

$A \cup B = B \cup A$

$A \cap B = B \cap A$

(3) Associative law:

$(A \cup B) \cup C = A \cup (B \cup C)$

$(A \cap B) \cap C = A \cap (B \cap C)$

(4) Involution law:

$(A^c)^c = A$

(5) Identity law:

$A \cup \phi = A$, $A \cup U = U$

$A \cap \phi = \phi$, $A \cap U = A$

(6) Distributive Law :-

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

(7) Absorption law :-

$$A \cup (A \cap B) = A$$

$$A \cap (A \cup B) = A$$

(8) complement law :-

$\phi^c = U$, $A \cup A^c = U$

$U^c = \phi$, $A \cap A^c = \phi$

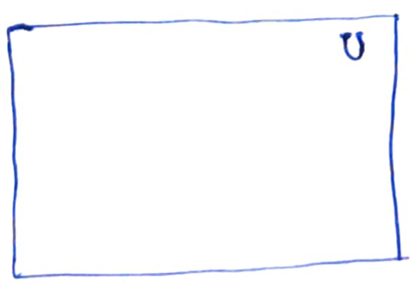
(9) Demorgan's Law :

$$(A \cup B)' = A' \cap B'$$

$$(A \cap B)' = A' \cup B'$$

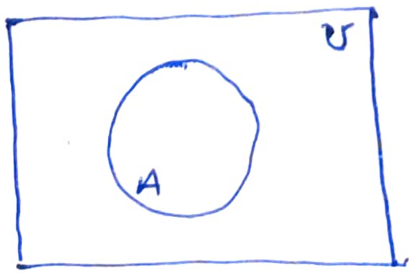
Venn Diagram :- Venn diagram is a pictorial representation of sets and their operations and their relations, in Venn diagrams the universal Set is represented by rectangle base, and other sets are represented by circles.

(1)



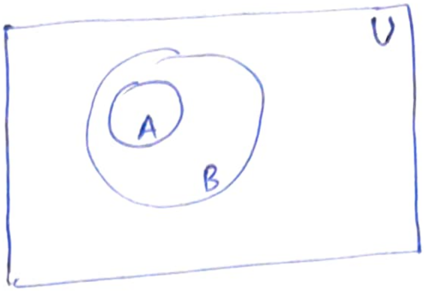
Universal set

(2)



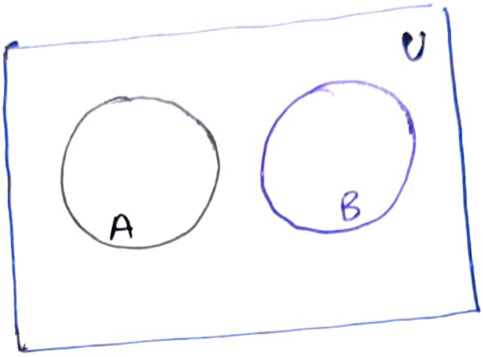
$\leftarrow A \subseteq U$

3



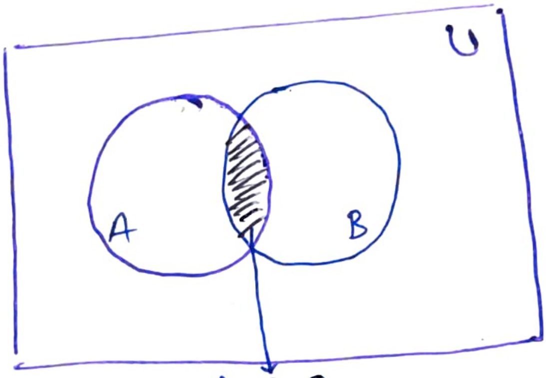
$A \subseteq B$

4



$A \cap B = \phi$

5

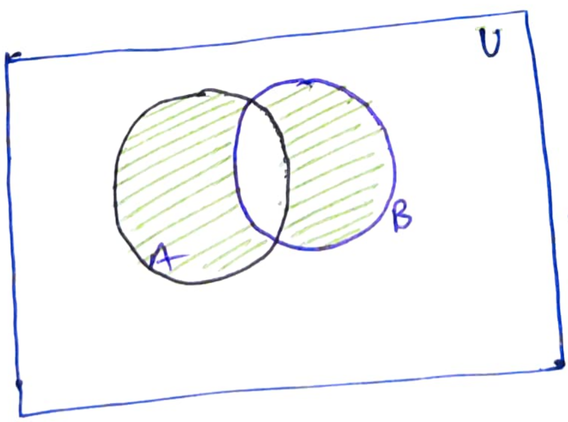


$A \cap B$

$A \cap B \neq \phi$

"shaded part"

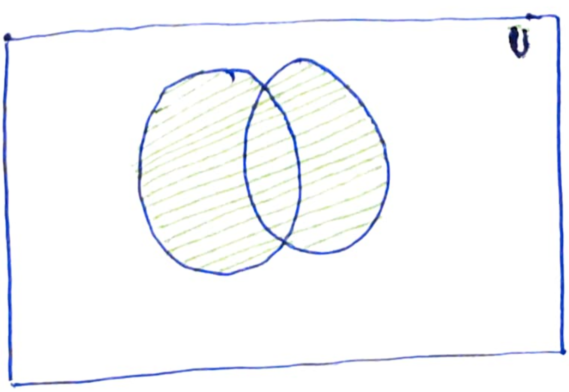
6



$A \oplus B$

"green shaded part"

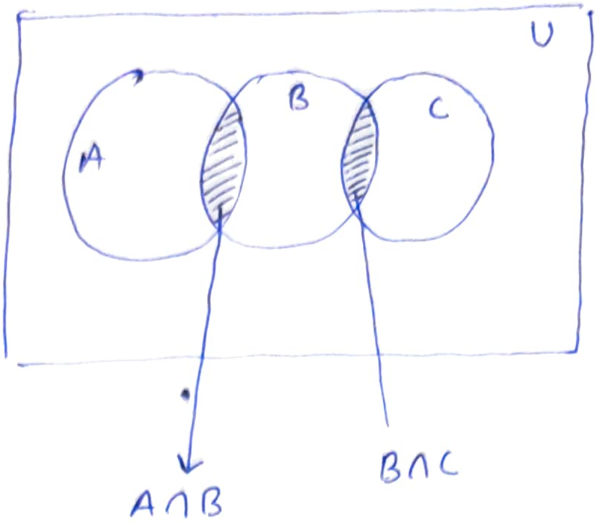
7



$A \cup B$

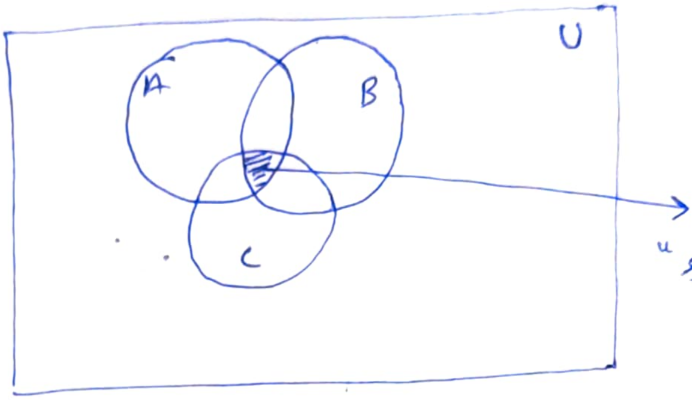
"shaded Part"

8



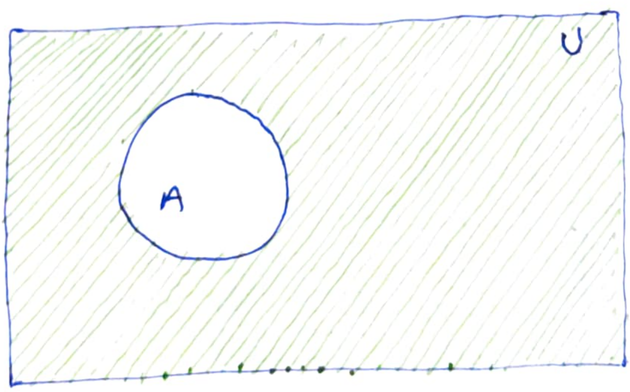
A and C are disjoint sets.

9



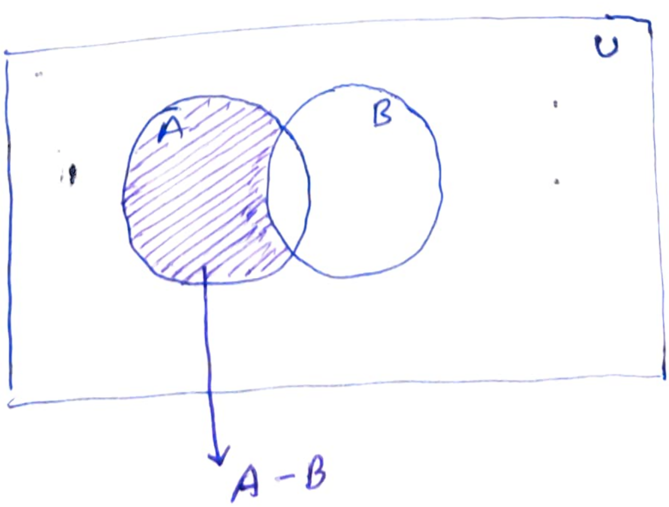
$A \cap B \cap C$
"shaded Part"

10

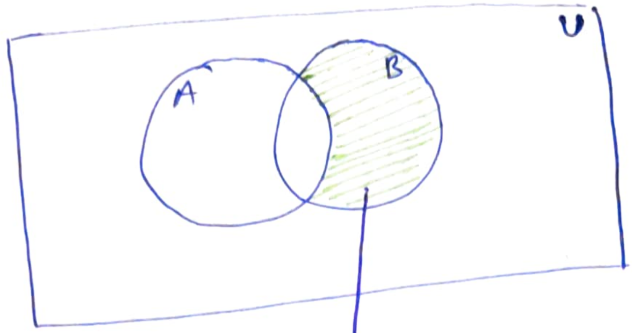


$A' = U - A$ is shaded Part.

11

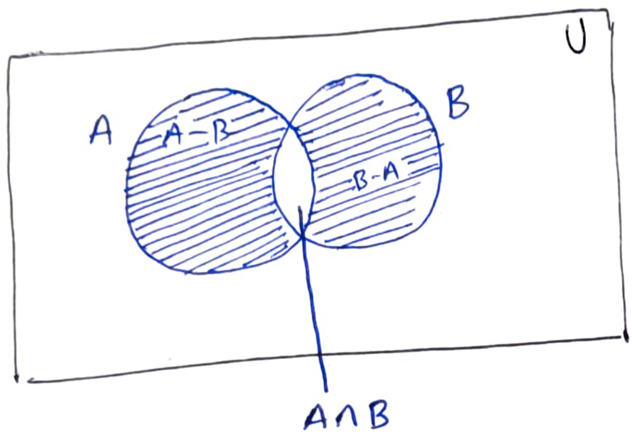


$A - B$



B − A

Set Inclusion Exclusion Principle:-



$n(A) = n(A-B) + n(A \cap B)$ —— (1)

$n(B) = n(B-A) + n(A \cap B)$ —— (2)

and

$n(A \cup B) = n(A-B) + n(B-A) + n(A \cap B)$ —— (3)

from (1), (2) and (3)

$$\boxed{n(A \cup B) = n(A) + n(B) - n(A \cap B)}$$

similarly,

$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) - n(A \cap B \cap C)$

Multiset: Multiset are the sets where an element can occur more than once.
like -

$$A = \{a, a, a, b, b, c\}$$

and

$$B = \{a, a, a, a, b, b, b, d, d\}$$

A and B are multisets, A and B can also be Represented as:

$$A = \{3 \cdot a, 2 \cdot b, 1 \cdot c\}, \quad B = \{4 \cdot a, 3 \cdot b, 2 \cdot d\}$$

The Multiplicity of an element in a multiset is defined to be the number of times the element appears in the multiset.

Ex: Prove $(A \cap B') \cup (A' \cap B) \cup (A' \cap B') = A' \cup B'$

LHS $\Rightarrow$ $(A \cap B') \cup (A' \cap B) \cup (A' \cap B')$

$\Rightarrow$ $(B' \cap A) \cup (B' \cap A') \cup (A' \cap B)$ by commutative law

$\Rightarrow$ $B' \cap (A \cup A') \cup (A' \cap B)$ by distributive law

$\Rightarrow$ $B' \cap (U) \cup (A' \cap B)$ $\quad (A \cup A') = U$

$\Rightarrow$ $B' \cup (A' \cap B)$

$\Rightarrow$ $(B' \cup A') \cap (B' \cup B)$ by distributive law.

$\Rightarrow$ $(B' \cup A') \cap U$

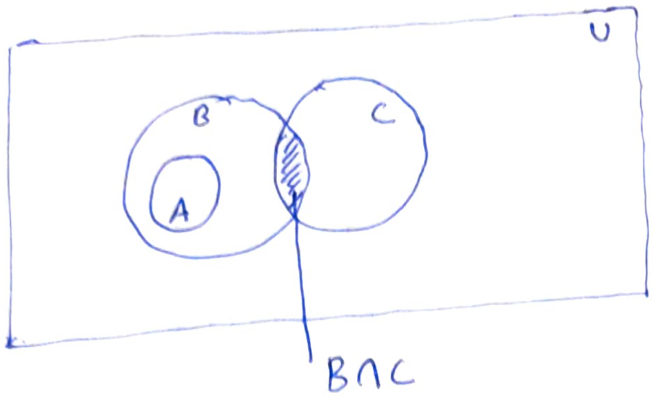
$=$ $(B' \cup A')$

$=$ $A' \cup B'$ $=$ RHS by commutative law.

Ques: Draw the Venn Diagram for set A, B and C where,

(1) A is a subset of B
(2) B and C have elements in common.
(3) A and C are disjoint sets.

→ Solution:



Here
$A \subseteq B$
$B \cap C \neq \phi$
$A \cap C = \phi$.

Ques: find out the set for following:

(1) $A = \{x : x \in I,\ 6x^2 - 11x + 3 = 0\}$

(2) $B = \{x : x \in N,\ 4 + x = 3\}$

Sol: (i) Given $A = \{x : x \in I,\ 6x^2 - 11x + 3 = 0\}$

After solving $6x^2 - 11x + 3 = 0$, we get $x = 3/2, 1/2$ both are not belongs to I so $x \notin I$, so $A = \{\ \}$, a null set.

(2) $B = \{x : x \in N,\ 4 + x = 3\}$

Here $x \in N$ (Natural numbers), no natural number satisfies the Relation, $4+x$ equals to 3, so B is a Null set,

$B = \phi$.

Note: ① Let A, B and C are three Non empty sets and, if $A \subseteq B$ and $B \subseteq C$ so $A \subseteq C$ also.

② If A and B are disjoint sets so the equation,

$$n(A \cup B) = n(A) + n(B) - n(A \cap B),$$

become

$$n(A \cup B) = n(A) + n(B),$$ because $A \cap B = \phi$ (As A and B are disjoint sets).

Ques: A survey of 1000 customers reading newspaper conducted and reported that 720 customers reads times of India and 450 liked Hindustan times. What is the least number that reads both newspaper?

Sol: Let x be the number of customers reads times of India and y be the number of customers reads Hindustan Times.

Now,

$$n(x) = 720$$

$$n(y) = 450$$

and given that $n(x \cup y) = 1000$

so

$$n(x \cup y) = n(x) + n(y) - n(x \cap y)$$

Here $n(x \cap y)$ is the portion that shows the number of customers read both newspapers, then

$$n(x \cap y) = n(x) + n(y) - n(x \cup y)$$

$$n(x \cap y) = 720 + 450 - 1000$$

$$n(x \cap y) = 170$$

so 170 Customers read both newspapers.

ORDERED PAIRS: Let A and B are two sets and let $a \in A$ and $b \in B$ then the pair of elements as (a, b) is called the ordered pair clearly

$$(a, b) \neq (b, a)$$

if $(a, b) = (b, a)$ then

$a = b$ and $b = a$.

Cartesian Product :

Let A and B are two set the cartesian Product $A \times B$ is a set of ordered pairs of the form (a, b) where $a \in A$ and $b \in B$.

Ex! Let $A = \{1, 2, 3\}$
$B = \{a, b\}$

then
$$A \times B = \{(1,a), (1,b), (2,a), (2,b), (3,a), (3,b)\}$$
$$B \times A = \{(a,1), (a,2), (a,3), (b,1), (b,2), (b,3)\}.$$

symbollacily
$$A \times B = \{(a,b) : a \in A \text{ and } b \in B\}.$$

$$A \times B \neq B \times A$$

Relation :→ Denoted by 'R'

Let A and B be two non empty sets. R is a relation from A to B if R is a subset of $A \times B$. $(R \subseteq A \times B)$.
R is a set of ordered pairs (a, b), where $a \in A$ and $b \in B$, it is also denoted by aRb and read as a related to b by relation R.

so,
$$R = \{(a,b) : a \in A, b \in B, aRb\}$$

Example !→ $A = \{1, 2, 3\}$, $B = \{1, 2\}$
R is from A to B and R is as.
$(a, b) \in R$ iff $a + b$ is even no.

Sol !→ $A \times B = \{(1,1), (1,2), (2,1), (2,2), (3,1), (3,2)\}$

Now in R we only have the pairs, for which a + b = even.

so

$$R = \{(1,1), (2,2), (3,1)\}$$

Ex 1→ from the above example find relation
$S = \{(a,b) : a \in A, b \in A, a+b = \text{odd no.}\}$

Sol !→ $A \times B = \{(1,1), (1,2), (2,1), (2,2), (3,1), (3,2)\}$

then

$$S = \{(1,2), (2,1), (3,1), (3,2)\}$$ Ans.

Domain of Relation :-
Let R be a relation defined from A to B, Domain of Relation R is the set containing of 1st components of all ordered pairs $(a,b) \in R$.
It is denoted by.
D(R) or Dom(R).

$$D(R) = \{a : a \in A \text{ and } aRb\}$$

Range of Relation :-
Let R be a relation defined from A to B, Range of relation R is the set consisting of 2nd component of all ordered pairs $(a,b) \in R$.
Denoted By R(R) or Range(R).

$$R(R) = \{b : b \in B \text{ and } aRb\}$$

Ex:- $R : A \to A$, where $A\{1, 2, 3, 4\}$

R is as: $R = \{(1,1), (2,1), (2,2), (3,1), (3,3), (4,1)$
$(4,2), (4,4)\}$

The Domain of R is as:

$D(R) = \{1, 2, 3, 4\}$

The Range of R is as:

$Range(R) = \{1, 2, 3, 4\}$.

Some Types of Relation:-

<u>Universal Relation:</u>→ A Relation R defined from $A \to B$ is called a Universal relation if $R = A \times B$.

<u>OR</u>

A Relation R defined over set A is called universal relation if $R = A \times A$.

Identity Relation:- The identity relation is denoted by I_A or Δ_A or Δ.

Let A relation R is defined from set A to A, it is called identity relation on set A if

$$R = \{(a,a) : a \in A\}$$

It is also called the diagonal relation.

Empty or Null Relation:- It is also called void relation.

$$R = \phi$$

for Example! $A = \{1, 2, 3, 4\}$, $R: A \to A$

$$R = \{(a,b) : a + b < b\}$$

so $R = \phi$.

Inverse Relation:-

A Relation R^{-1} defined from B to A is called inverse of Relation R defined from A to B if,

$$R^{-1} = \{(b, a) : b \in B, a \in A, aRb\}.$$

Ex 1 → $R : A \to A$ where $A = \{a, b, c\}$

and $R = \{(a, a), (a, b), (a, c), (b, c)\}$

so $R^{-1} = \{(a, a), (b, a), (c, a), (c, b)\}$.

Complement of Relation:-

Denoted by R' or R^{c}

Let R be a relation defined from A to B, The complement of R is R^{c} so,

$$R^{c} = \{(b, a) : a \in A, b \in B, a \not R b\}$$

OR

$$R^{c} = (A \times B) - R$$

Example: $R : A \to B$.

$A = \{1, 2, 3\}$

$B = \{4, 5\}$

$R = \{(1, 4), (3, 4), (3, 5)\}$

$R^{c} = \{(1, 5), (2, 4), (2, 5)\}$.

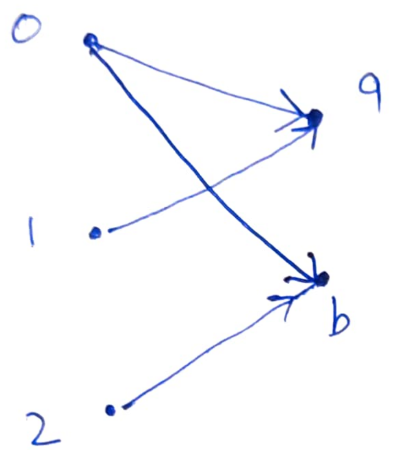
Representation of Relation :-

Let $A = \{0, 1, 2\}$, $B = \{a, b\}$

$R : A \longrightarrow B$

and $R = \{(0, a), (0, b), (1, a), (2, b)\}$

Representation by directed Graph :-



Representation by Matrix :-

R	a	b
0	1	1
1	1	0
2	0	1

Properties of Relation:-

Reflexive Relation: A Relation R on a set A is called reflexive if $(a, a) \in R$ for every element $a \in A$.

Symmetric Relation: A Relation R on a set A is called symmetric if $(a, b) \in R$ then $(b, a) \in R$, for all $a, b \in A$.

means, $(a, b) \in R \Rightarrow (b, a) \in R$, $a, b \in A$.

Antisymmetric Relation: A Relation R on a set A is antisymmetric such that for all $a, b \in A$, if $(a, b) \in R$ and $(b, a) \in R$ then $a = b$.

[* If a Relation is R is symmetric then $R = R^{-1}$]

Asymmetric Relation: A Relation R is defined on set A is said to be Asymmetric relation, if $(a, b) \in R \Rightarrow (b, a) \notin R$, $a, b \in A$

Ex:-> $A = \{1, 2, 3\}$ $R: A \to A$

$R = \{(1, 2), (2, 3), (4, 2), (4, 1)\}$

Since $(1, 2) \in R \Rightarrow (2, 1) \notin R$

$(2, 3) \in R \Rightarrow (3, 2) \notin R$

$(4,1) \in R \Rightarrow (1,4) \notin R$

$(4,2) \in R \Rightarrow (2,4) \notin R$.

so R is Asymmetric.

Irreflexive Relation: A Relation R on set A is said to be irreflexive if no element in set A is related with itself.

$$[\forall a \in A, (a,a) \notin R].$$

Transitive Relation: A Relation R on a set A is called transitive if $(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \in R$, $\forall a,b,c \in A$.

OR

$$[(a,b) \in R \text{ and } (b,c) \in R \Rightarrow (a,c) \in R, a,b,c \in A]$$

Equivalence Relation: A Relation R is on a set A is Equivalence Relation if R is:

1) Transitive: if

$$[(a,b) \in R \text{ and } (b,c) \in R \text{ then } (a,c) \in R, a,b,c \in A]$$

2) Reflexive: if

$$[\forall a \in A, (a,a) \in R]$$

3) Symmetric: if

$$[(a,b) \in R \xRightarrow{\text{then}} (b,a) \in R, a,b \in A]$$

Partial Ordering (POSET) :-

A Relation 'R' on a set S is called partial ordering or partial order relation if R is reflexive, antisymmetric (should be Asymmetric) and transitive. So the set S is a Poset with partially ordered Relation 'R'.

Denoted by (S, R)

OR

$(S, \preceq)$.

In R the pair (a, b) means a R b or $a \preceq b$ is called comparable.

Hasse Diagram :-

Hasse Diagram is a graphically representation of partial order Relation on a set, it is a reduced form of Relational graph.

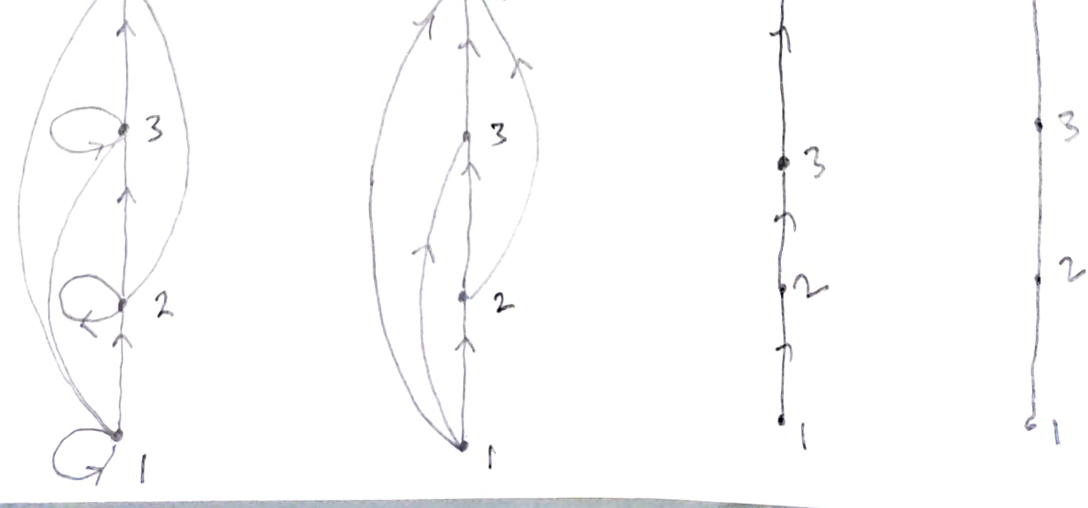
for example :-

$(S, \preceq)$ is a Poset, and $R = \{(a, b) : a \preceq b\}$

$S = \{1, 2, 3, 4\}$

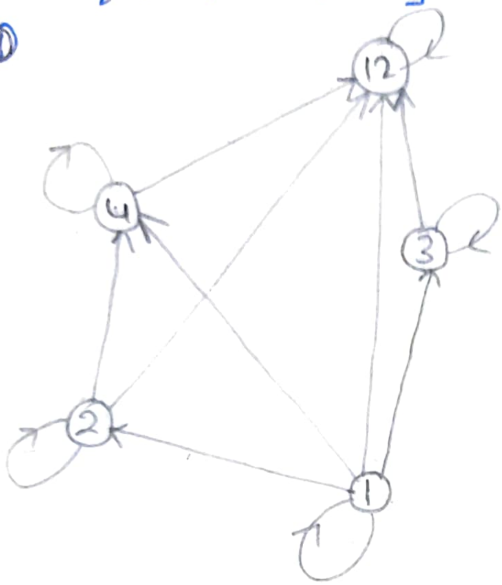
Construction of Hasse diagram :-

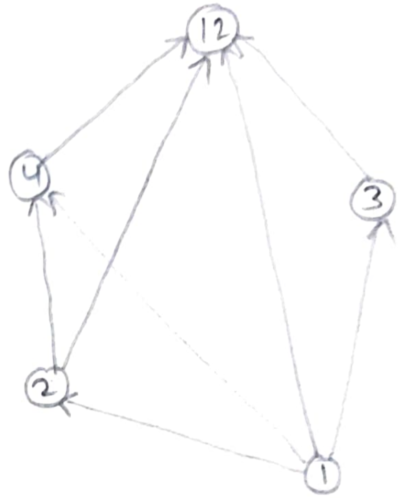
① ② ③ ④

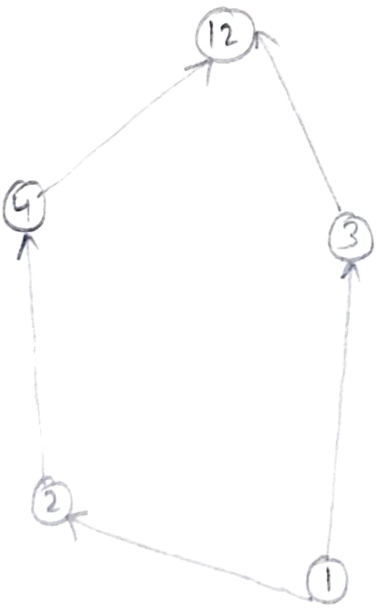


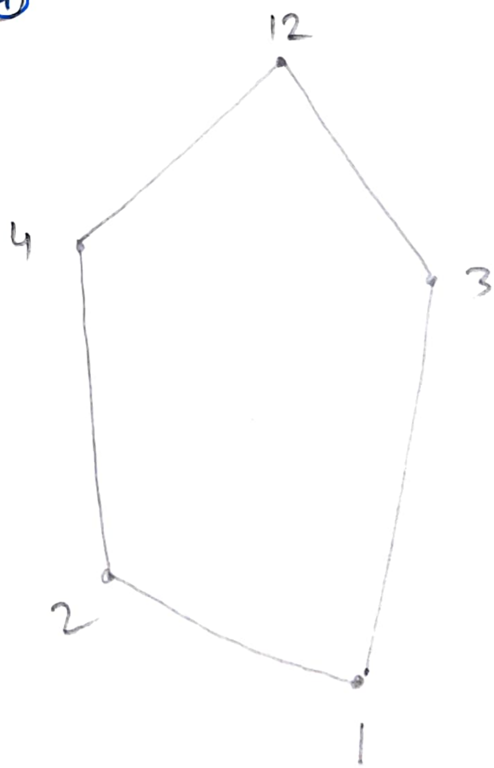
Example! → Let $S = \{1, 2, 3, 4, 12\}$, consider the partial order is divisibility on set S means a/b, the Hasse diagram for $(S, \leq)$ is as! (23)

$S = \{1, 2, 3, 4, 12\}$

① 

② 

③ 

④ 

Hasse Diagram.

Minimal element:

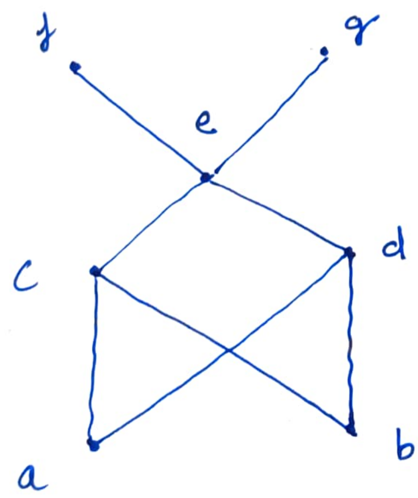
An element is called minimal element if or other element strictly preceds it. In an Hasse diagram 'a' is called minimal if no edge incedent 'a' from below.

Maximal element:

An element is called maximal element if no other element strictly succed it. In an Hasse diagram 'b' is called maximal element if no edge goes out from b in upward direction.

Example:

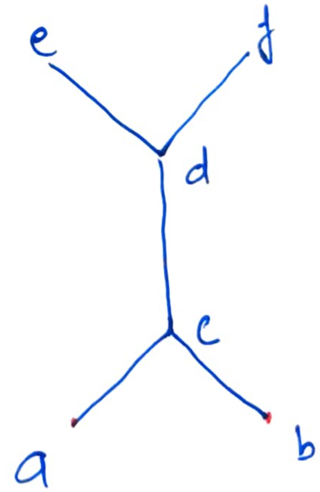
①



minimal = {a, b}.

maximal = {f, g}.

②



Minimal = {a, b}

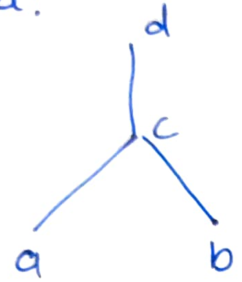
maximal = {e, f}

Least and greatest Element :→ (unique element)

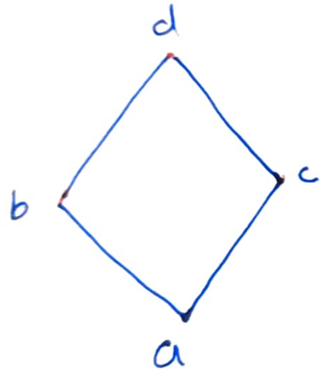
Let $(A, \leq)$ be a poset. Then an element $b \in A$ is the least element of A if for every element $a \in A$, $b \leq a$.

Similarly an element $b \in A$ is called greatest if for every $a \in A$, $a \leq b$.

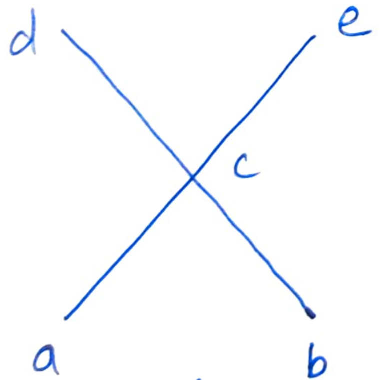
Example!



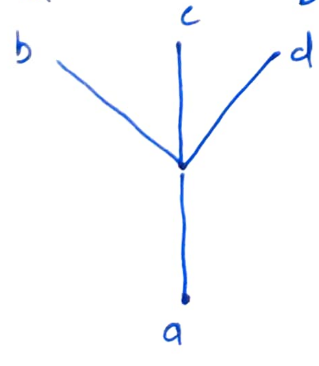
Least = ? not exists.
greatest = d.



Least = a.
greatest = d.



Least = not exists
greatest = not exists.



Least = a
greatest = not exists.

Upper Bound :→

Let $(P, \leq)$ be any poset and A is a subset of P, An element $m \in P$ is called upper bound of A, if m succeds every element of A, means –

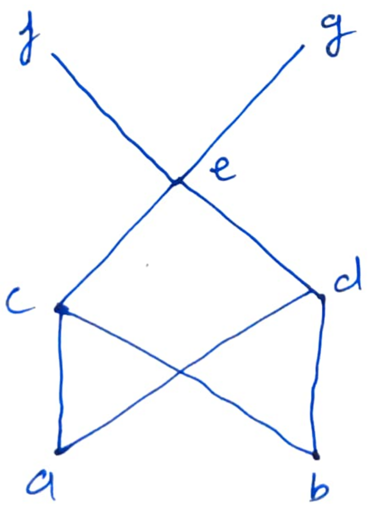
$$\boxed{\forall\ x \in A,\ x \leq m}.$$

Lower Bound :→

Let $(P, \leq)$ be any Poset and A is a subset of P, An element $m \in P$ is called lower bound of A if m Presceeds every element of A, means

$$\boxed{\forall\ x \in A,\ m \leq x}.$$

Example :→



Upper Bound of $\{c\} = \{c, e, f, g\}$

upper Bound of $\{e\} = \{e, f, g\}$

upper Bound of $\{f\} = \{f\}$

lower Bound of $c = \{a, b, c\}$

lower Bound of $\{c, d, e\} =$

Least upper Bound :→ (Supremum)

Let $(P, \leq)$ be a Poset and A is a subset of P. If an upper bound of A precedes all other upperbounds of A then it is called least upper bound of A.

Denoted by LUB(A) or Sup(A).

OR $(P, \leq)$ and $A \subseteq P$.

An element $a \in A$ called a least upper bound of A if a is an upper bound of A and $a \leq a_1$, for all upper bounds of A.

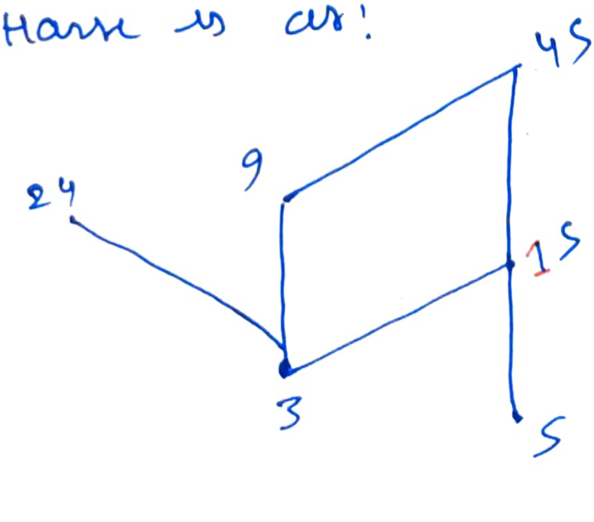
Greatest lower Bound :→ (Infimum)

Let $(P, \leq)$ be a Poset and A is a subset of P. If a lower bound of A succeeds all other lower bounds of A, then that is called the Greatest lower Bound of A.

Denoted by GLB(A) or Inf(A).

Ex:- $(P, \leq)$, $P = (\{3, 5, 9, 15, 24, 45\}, \text{"|"})$

the Hasse is as:



minimal = $\{3, 5\}$

Maximal = $\{24, 45\}$

Now LUB of $\{3, 5\}$

UB of 3 = $\{3, 9, 15, 24, 45\}$

UB of 5 = $\{5, 15, 45\}$

UB of $\{3, 5\}$ = $\{15, 45\}$

lub $\{3, 5\}$ = 15 ✓

Now GLB $\{15, 45\}$

LB of $\{15\}$ = $\{3, 5, 15\}$

LB of $\{45\}$ = $\{5, 15, 9, 3, 45\}$

LB of $\{15, 45\}$ = $\{3, 5, 15\}$

GLB of $\{15, 45\}$ = 15 ✓

"Lattice":→ Let $(L, \leq)$ be any Poset.

A Lattice is a poset $(L, \leq)$ in which each of its subset containing two elements has a LUB and GLB.

Let any subset of L is $\{a, b\}$ so

lub $\{a, b\}$ = $a \vee b$ or (a join b)

Join

glb $\{a, b\}$ = $a \wedge b$ or (a meet b)

meet.

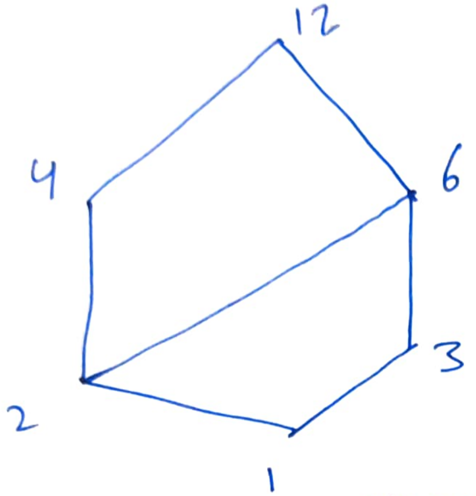
Note! If for every pair of elements in set L there exists a meet and join then $(L, \leq)$ is a lattice. (for all possible pairs)

Ex1→ $(D_{12}, |)$ is a lattice.

$D_{12} = \{1, 2, 3, 4, 6, 12\}$

In case of D_m, it is the set of divisores of m, and
$a \vee b = lcm(a, b)$
$a \wedge b = gcd(a, b)$
$a, b \in D_m$

Sol: Hasse diagram is as:



for jain or LUB or supremum :-

$\vee$	1	2	3	4	6	12
1	1	2	3	4	6	12
2	2	2	6	4	6	12
3	3	6	3	12	6	12
4	4	4	12	4	12	12
6	6	6	6	12	6	12
12	12	12	12	12	12	12

→

for meet or GLB or infimum :-
($\wedge$)

$\wedge$	1	2	3	4	6	12
1	1	1	1	1	1	1
2	1	2	1	2	2	2
3	1	1	3	1	3	3
4	1	2	1	4	2	4
6	1	2	3	2	6	6
12	1	2	3	4	6	12

Here meet and join for every pair of element exists. so $(D_{12}, '|')$ is a lattice.

Properties of lattice :-

(a) Let L be a lattice, for every $a, b \in L$

$a \vee b = b$ iff $a \leq b$

$a \wedge b = a$ iff $a \leq b$

$a \wedge b = a$ iff $a \vee b = b$

(b) Idempotent Property :

$a \vee a = a$

$a \wedge a = a$

(c) Commutative law :→

$$a \vee b = b \vee a$$
$$a \wedge b = b \wedge a$$

(d) Associative law :-

$$a \vee (b \vee c) = (a \vee b) \vee c$$
$$a \wedge (b \wedge c) = (a \wedge b) \wedge c$$

(e) Absortion Property :-

$$a \vee (a \wedge b) = a$$
$$a \wedge (a \vee b) = a$$

(f) Isatonicity :→

Let $a, b, c \in L$

If $a \leq b$

then $a \vee c \leq b \vee c$

$a \wedge c \leq b \wedge c$

(g) Stability :- Let $a, b, c \in L$

If $a \leq b$ and $a \leq c$

then $a \leq b \vee c$.

$a \leq b \wedge c$.

(h) Distributive inequality :-

let $a, b, c \in L$

$$a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$$
$$a \wedge (b \vee c) \leq (a \wedge b) \vee (a \wedge c)$$

(c) Commutative law:→

$$a \vee b = b \vee a$$
$$a \wedge b = b \wedge a$$

(d) Associative law:-

$$a \vee (b \vee c) = (a \vee b) \vee c$$
$$a \wedge (b \wedge c) = (a \wedge b) \wedge c$$

(e) Absortion Property:-

$$a \vee (a \wedge b) = a$$
$$a \wedge (a \vee b) = a$$

(f) Isotonicity:→

Let $a, b, c \in L$

If $a \leq b$

then $a \vee c \leq b \vee c$

$a \wedge c \leq b \wedge c$

(g) Stability:- Let $a, b, c \in L$

If $a \leq b$ and $a \leq c$

then $a \leq b \vee c$.

$a \leq b \wedge c$.

(h) Distributive inequality:-

let $a, b, c \in L$

$$a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$$
$$a \wedge (b \vee c) \leq (a \wedge b) \vee (a \wedge c)$$

Isomorphic Lattice :->

Let L_1 and L_2 are two lattices they are said to be isomorphic to each other if there is bijective (one to one) mapping

$$F : L_1 \rightarrow L_2 \text{ such that.}$$

(a) $f(a \vee b) = f(a) \vee f(b)$

(b) $f(a \wedge b) = f(a) \wedge f(b)$

$[a, b \in L_1]$

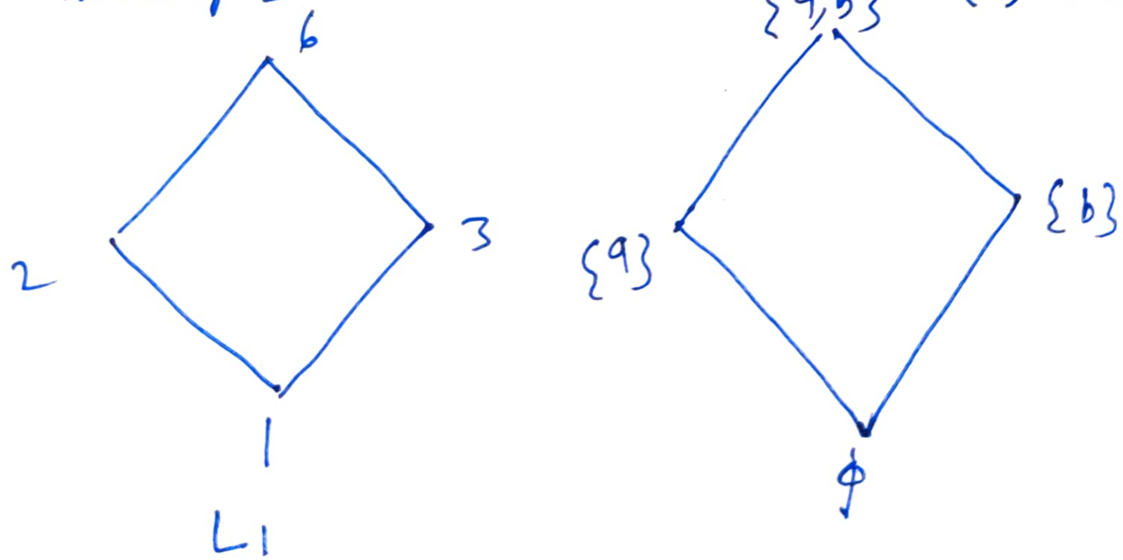
If both (a) and (b) are satisfied then L_1 and L_2 are Isomorphic.

Example :-> $(\underset{L_1}{D_6}, /)$ and $(\underset{L_2}{P(A)}, \underset{\text{'means subset'}}{\leq})$

- where $A = \{a, b\}$ show Both are somorphic.

sol :- $D_6 = \{1, 2, 3, 4\}$, $P(A) = \{\phi, \{a\}, \{b\}, \{a, b\}\}$

Hasse of L_1



Let $f: L_1 \Rightarrow L_2$ is Bijective such that

$f(1) = \phi$
$f(2) = \{a\}$
$f(3) = \{b\}$
$f(4) = \{a, b\}$

for join 'v' :— [by $f(a \vee b) = f(a) \vee f(b)$]

$f(1 \vee 2) = f(2) = \{a\}$
$f(1 \vee 3) = f(3) = \{b\}$
$f(1 \vee 6) = f(6) = \{a, b\}$
$f(2 \vee 3) = f(6) = \{a, b\}$.
$f(2 \vee 6) = f(6) = \{a, b\}$

LHS

and

$f(1) \vee f(2) = \phi \vee \{a\} = \{a\}$
$f(1) \vee f(3) = \phi \vee \{b\} = \{b\}$
$f(1) \vee f(6) = \phi \vee \{a, b\} = \{a, b\}$
$f(2) \vee f(3) = \{a\} \vee \{b\} = \{a, b\}$
$f(2) \vee f(6) = \{a\} \vee \{a, b\} = \{a, b\}$
$f(3) \vee f(6) = \{b\} \vee \{a, b\} = \{a, b\}$.

RHS

for meet '$\wedge$':

By $f(a \wedge b) = F(a) \wedge f(b)$.

$\rightarrow$

$f(1 \wedge 2) = f(1) = \phi$
$f(1 \wedge 3) = f(1) = \phi$
$f(1 \wedge 6) = f(1) = \phi$
$f(2 \wedge 3) = f(1) = \phi$
$f(2 \wedge 6) = f(2) = \{a\}$
$f(3 \wedge 6) = f(3) = \{b\}$

LHS

$f(1) \wedge f(2) = \phi \wedge \{a\} = \phi$
$f(1) \wedge f(3) = \phi \wedge \{b\} = \phi$
$f(1) \wedge f(6) = \phi \wedge \{a, b\} = \phi$
$f(2) \wedge f(3) = \{a\} \wedge \{b\} = \phi$
$f(2) \wedge f(6) = \{a\} \wedge \{a, b\} = \{a\}$.
$f(3) \wedge f(6) = \{b\} \wedge \{a, b\} = \{b\}$.

RHS

So both conditions are satisfied L_1 and L_2 are "Isomorphic".

—x—

* Distributed Lattices :->

A Lattice L is said to be distributive if for any elements a, b and c of L, the following distributive law holds good / satisfied.

(i) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$.

(ii) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$.

* Bounded Lattice :->

" A Lattice $(L, \leq)$ is said to be a bounded lattice if it has a greatest element 'I' (say I) and a lest element say 'O', such that:

$$a \vee I = I \ , \ a \wedge I = I$$
$$a \vee O = a \ , \ a \wedge O = a$$

for any $a \in L$, and $O \leq a \leq I$.

Complement in a Lattice :->

Let $(L, \leq)$ be a lattice, and the greatest element is "I" and the least element is 'O'. Now the complement of any $a \in L$ is a' if it satisfies the following.

$$\begin{bmatrix} a \vee a' = I \\ a \wedge a' = O \end{bmatrix}$$

"Here I is the greatest element in Lattice in L and O is the lest element in L.

Modular Lattices :→

A Lattice $(L, \leq)$ is called Modular if for any elements a, b and c in L the following property is satisfied

$$a \leq c \text{ implies } a \vee (c \wedge b) = (a \vee c) \wedge b.$$

Consider an Example a and 1, so $a \leq 1$, By taking b as a 3rd element, we have

$$a \leq 1 \text{ implies } a \vee (b \wedge 1) = (a \vee b) \wedge 1.$$

—x—