Probabilistic Array Programming on Galois Fields

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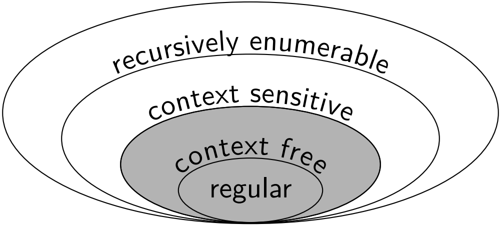
Overview

Recap: Context free grammars

Suppose we have a context free grammar (CFG) $G = \langle V, \Sigma, P, S \rangle$ where V is the set of nonterminals, Σ is the terminals, $P : V \times (V \cup \Sigma)^+$ are the productions, $S \in V$ is the start symbol and $+$ is the Kleene plus.

For example, consider the grammar $S \to SS \mid (S) \mid ()$. This represents the language of balanced parentheses, e.g. $(), ()(), (()), ()(()), (()()), (())() \ldots$

Every CFG has a normal form $P^* : V \times (V^2 \mid \Sigma)$, i.e., every production can be refactored into either $v_0 \to v_1 v_2$ or $v_0 \to \sigma$, where $v_{0\ldots2} : V$ and $\sigma : \Sigma$, e.g., $\{S \to SS \mid (S) \mid ()\} \Leftrightarrow^* \{S \to XR \mid SS \mid LR, L \to (, R \to), X \to LS\}$



Algebraic parsing, distilled

Given a CFG $\mathcal{G} := \langle V, \Sigma, P, S \rangle$ in Chomsky Normal Form, we can construct a recognizer $R_{\mathcal{G}} : \Sigma^n \to \mathbb{B}$ for strings $\sigma : \Sigma^n$ as follows. Let $\mathcal{P}(V)$ be our domain, 0 be $\varnothing$, $\oplus$ be $\cup$, and $\otimes$ be defined as follows:

$$s_1 \otimes s_2 := \{C \mid \langle A, B \rangle \in s_1 \times s_2, (C \to AB) \in P\}$$

E.g., $\{\mathrm{A} \to BC, C \to AD, D \to BA\} \subseteq P \vdash \{A, B, C\} \otimes \{B, C, D\} = \{A, C\}$

By initializing $\mathbf{M}_0[i, j](\mathcal{G}, \sigma) := \{A \mid i + 1 = j, (A \to \sigma_i) \in P\}$ and searching for the least solution to $\mathbf{M} = \mathbf{M} + \mathbf{M}^2$, this will produce a matrix $\mathbf{M}^*$:

$$\mathbf{M}^* = \begin{pmatrix} \varnothing & \{V\}_{\sigma_1} & \dots & \dots & \mathcal{T} \\ \varnothing & \varnothing & \{V\}_{\sigma_2} & \dots & \dots \\ \varnothing & \varnothing & \varnothing & \{V\}_{\sigma_3} & \dots \\ \varnothing & \varnothing & \varnothing & \varnothing & \{V\}_{\sigma_4} \\ \varnothing & \varnothing & \varnothing & \varnothing & \varnothing \end{pmatrix}$$

Valiant (1975) shows that $\sigma \in \mathcal{L}(\mathcal{G})$ iff $S \in \mathcal{T}$, i.e., $\mathbb{1}_{\mathcal{T}}(S) \iff \mathbb{1}_{\mathcal{L}(\mathcal{G})}(\sigma)$.

Kotlin implementation: CFG definition

```
typealias Production = Pair<String, List<String>>
typealias CFG = Set<Production>
val Production.LHS: String get() = first
val Production.RHS: List<String> get() = second
val CFG.nonterminals: Set<String> by cache { map { it.LHS }.toSet() }
val CFG.words: Set<String> by cache { nonterminals + flatMap { it.RHS } }
val CFG.terminals: Set<String> by cache { words - nonterminals }
// Many-to-many mapping of nonterminals to RHS expansions
val CFG.bimap: BidirectionalMap by cache { BidirectionalMap(this) }

fun CFG.makeAlgebra(): Ring<Set<String>> =
  Ring.of(
    // 0 = ∅
    nil = setOf(),
    // x + y = x ∪ y
    plus = { x, y → x union y },
    // x · y = { A0 | A1 ∈ x, A2 ∈ y, (A0 → A1 A2) ∈ P }
    times = { x, y → join(x, y) }
  )

fun CFG.join(ls: Set<String>, rs: Set<String>): Set<String> =
  (ls * rs).flatMap { (l, r) → bimap[listOf(l, r)] }.toSet()
```

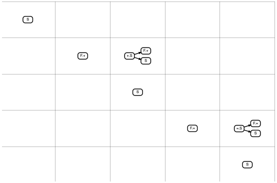
Kotlin implementation: the recognizer

```
// Constructs initial matrix according to: M_{i+1=j} = { A | (A → σ_i) ∈ P }
fun CFG.initialMatrix(str: List<String>): Matrix<Set<String>> =
 Matrix(makeAlgebra(), str.size + 1) { i, j →
   // Aligns nonterminals matching each terminal along superdiagonal
   if (i + 1 ≠ j) emptySet() else bimap[listOf(str[j - 1])].toSet()
 }

// Computes the fixpoint of an abstract matrix function
tailrec fun <T: Matrix<S>, S> T.seekFixpoint(op: (T) → T): T {
 val next = op(this)
 return if (this == next) next else next.seekFixpoint(op)
}

// Checks whether start symbol is contained in the northeasternmost entry
fun CFG.check(s: String): Boolean = START in parse(tokenize(s))[0].last()

// Since matrix is strictly UT, this converges in at most |tokens| steps
fun CFG.parse(tokens: List<String>): Matrix<Set<String>> =
   initialMatrix(tokens).seekFixpoint { it + it * it }
```

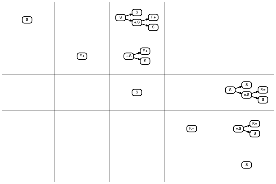
A few observations on algebraic parsing

- The matrix $\mathbf{M}^*$ is strictly upper triangular, i.e., nilpotent of degree n
- Recognizer can be translated into a parser by storing backpointers

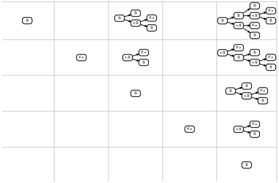
$$\mathbf{M}_1 = \mathbf{M}_0 + \mathbf{M}_0^2$$



$$\mathbf{M}_2 = \mathbf{M}_1 + \mathbf{M}_1^2$$



$$\mathbf{M}_3 = \mathbf{M}_2 + \mathbf{M}_2^2 = \mathbf{M}_4$$



- The $\otimes$ operator is *not* associative: $S \otimes (S \otimes S) \neq (S \otimes S) \otimes S$
- Built-in error recovery: nonempty submatrices $=$ parsable fragments
- `seekFixpoint { it + it * it }` is sufficient but unnecessary
- If we had a way to solve for $\mathbf{M} = \mathbf{M} + \mathbf{M}^2$ directly, power iteration would be unnecessary, could solve for $\mathbf{M} = \mathbf{M}^2$ above superdiagonal

Satisfiability + holes (our contribution)

- Can be lowered onto a Boolean tensor $\mathbb{B}^{n\times n\times|V|}$ (Valiant, 1975)
- Binarized CYK parser can be efficiently compiled to a SAT solver
- Enables sketch-based synthesis in either σ or $\mathcal{G}$: just use variables!
- We simply encode the characteristic function, i.e. $\mathbb{1}_{\subseteq V} : V \to \mathbb{B}^{|V|}$
- $\oplus, \otimes$ are defined as $\boxplus, \boxtimes$, so that the following diagram commutes:

$$\begin{tikzcd} V \times V \arrow[r, "\oplus/\otimes"] \arrow[d, shift left, "\mathbb{1}^2"] & V \arrow[d, shift left, "\mathbb{1}"] \\ \mathbb{B}^{|V|} \times \mathbb{B}^{|V|} \arrow[r, "\boxplus/\boxtimes"'] \arrow[u, shift left, "\mathbb{1}^{-2}"] & \mathbb{B}^{|V|} \arrow[u, shift left, "\mathbb{1}^{-1}"] \end{tikzcd}$$

- These operators can be lifted into matrices/tensors in the usual way
- In most cases, only a few nonterminals are active at any given time
- More sophisticated representations are known for $\binom{n}{0\leq k}$ subsets
- If density is desired, possible to use the Maculay representation
- If you know of a more efficient encoding, please let us know!

Tidyparse IDE plugin

arithmetic_left_recursive/arithmetic.tidy | mini_ocaml/ocaml.tidy | arithmetic_checked/arithmetic.tidy | arithmetic.cfg | simple.tidy | simple.cfg | test.tidy | ocaml/ocaml.tidy | SATValiantTest.kt

```
let rec a = _ _ _ _ _

let rec filter p l = m                                  l -> if p x then x :: _ _ _ els

let curry f = ( fun x
```

```
let rec a = ( <X> , <X> )
let rec a = ( <X> , [] )
let rec a = ( <X> , filter )
let rec a = ( [] , [] )
let rec a = ( [] , filter )
```

Press ↩ to insert, ⇥ to replace

ocaml.cfg

```
S -> X
X -> A | V | ( X , X ) | X X | ( X )
A -> FUN | F | LI | M | L
FUN -> fun V `->` X
F -> if X then X else X
M -> match V with Branch
Branch -> `|` X `->` X | Branch Branch
L -> let V = X
L -> let rec V = X
LI -> L in X
```

ocaml.cfg

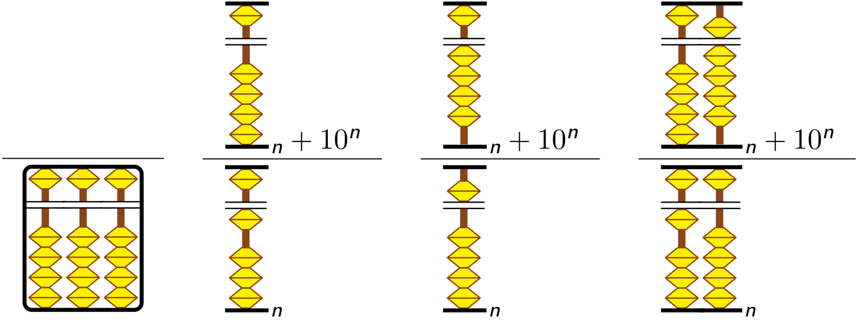
```
V -> Vexp | ( Vexp ) | List | Vexp Vexp
Vexp -> Vname | FunName | Vexp VO Vexp | B
Vexp -> ( Vname , Vname ) | Vexp Vexp | I
List -> [] | V :: V
Vname -> a | b | c | d | e | f | g | h | i
Vname -> j | k | l | m | n | o | p | q | r
Vname -> s | t | u | v | w | x | y | z
FunName -> foldright | map | filter
FunName -> curry | uncurry | ( VO )
VO ->  + | - | * | / | >
VO -> = | < | `||` | `&&`
I -> 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9
B ->  true | false
```

Abbreviated history of algebraic parsing

- Chomsky & Schützenberger (1959) - The algebraic theory of CFLs
- Cocke–Younger–Kasami (1961) - Bottom-up matrix-based parsing
- Brzozowski (1964) - Derivatives of regular expressions
- Earley (1968) - top-down dynamic programming (no CNF needed)
- Valiant (1975) - first realizes the Boolean matrix correspondence
  - Naïvely, has complexity $\mathcal{O}(n^4)$, can be reduced to $\mathcal{O}(n^\omega)$, $\omega < 2.763$
- Lee (1997) - Fast CFG Parsing $\Longleftrightarrow$ Fast BMM, formalizes reduction
- Might et al. (2011) - Parsing with derivatives (Brzozowski $\Rightarrow$ CFL)
- Bakinova, Okhotin et al. (2010) - Formal languages over GF(2)
- Bernady & Jansson (2015) - Certifies Valiant (1975) in Agda
- Cohen & Gildea (2016) - Generalizes Valiant (1975) to parse and recognize mildly context sensitive languages, e.g. LCFRS, TAG, CCG
- **Considine, Guo & Si (2022) - SAT + Valiant (1975) + holes**

Abacus arithmetic

- Computational complexity of arithmetic is notation-dependent(!)
- For example, $\pm$ in unary arithmetic is concatenation and decatenation
- Multiplication and division by natural powers of the radix is $\mathcal{O}(1)$
- We can describe the abacus as a kind of abstract rewriting system



Abacus dependent types

```
sealed class B<X, P : B<X, P>>(open val x: X? = null) {
  val T: T<P> get() = T(this as P)
  val F: F<P> get() = F(this as P)
}

class U(val i: Int) : B<Any, U>() // Checked at runtime

object Ø: B<Ø, Ø>(null) // Denotes the end of a bitlist

class T<X>(override val x: X = Ø as X) : B<X, T<X>>(x)
  { companion object: T<Ø>(Ø) }

class F<X>(override val x: X = Ø as X) : B<X, F<X>>(x)
  { companion object: F<Ø>(Ø) }

val b0: F<Ø> = F
val b1: T<Ø> = T
val b2: F<T<Ø>> = T.F // Note the raw type is reversed
val b4: F<F<T<Ø >>> = T.F.F
```

Abacus dependent types

```
typealias B_0<K> = F<K> // Type synonyms for legibility
typealias B_1<K> = T<K>
typealias B_2<K> = F<T<K>>
typealias B_3<K> = T<T<K>>
typealias B_4<K> = F<F<T<K >>>
typealias B_7<K> = T<T<T<K >>>
typealias B_8<K> = F<F<F<T<K>>>>

// Calculates k + 1 for all k == 2ⁿ - 1, 0 <= n < 4
operator fun Ø.plus(t: T<Ø>) = b1
operator fun B_0<Ø>.plus(t: T<Ø>) = b1
operator fun B_1<Ø>.plus(t: T<Ø>): B_2<Ø> = F(x + b1)
operator fun B_3<Ø>.plus(t: T<Ø>): B_4<Ø> = F(x + b1)
operator fun B_7<Ø>.plus(t: T<Ø>): B_8<Ø> = F(x + b1)

// Calculates k + 1 for all k ≡ 2ⁿ - 1 (mod 2ⁿ⁺¹), 1 <= n < 4
operator fun <K: B<*, *>> B_0<K>.plus(t: T<Ø>) = T(x)
operator fun <K: B<*, *>> B_1<F<K>>.plus(t: T<Ø>) = F(x + b1)
operator fun <K: B<*, *>> B_3<F<K>>.plus(t: T<Ø>) = F(x + b1)
operator fun <K: B<*, *>> B_7<F<K>>.plus(t: T<Ø>) = F(x + b1)
```

Abacus dependent types: birds eye view

```
      i\j|  0   1   …  k-1   k  | k+1  k+2   …  k+c  | k+c+1   …  k+c+d-1
         |                      |                    |
    -----+----------------------+--------------------+--------------------┐
        0|                      |                    |                  __|
        1|                      |                    |               __/
        …|        i ± j         |       i + j        |            __/
        …|        j ± i         |       j ± i        |         __/
      k-1|                      |                    |      __/
        k|                      |                    |   __/
   ------+----------------------+--------------------+--┘
      k+1|                      |
      k+2|                      |
        …|        i ± j         |
        …|        j + i         |
    k+c-1|                      |
      k+c|                      |           Run-time
    -----+----------------------|
    k+c+1|                   __/
    k+c+2|                __/          type checked
        …|             __/
        …|          __/
  k+c+d-2|       __/
  k+c+d-1|___/
```

Annotated history of typed eDSLs

- Canning et al. (1989) - F-Bounded Polymorphism is first invented
- Cheney & Hinze (2003) - Phantom types (good for type-safe builders)
- Meijer et al. (2006) - Language integrated querying (LINQ)
- Eder (2011) - Commercial reimplementation LINQ in Java/jOOQ
- Grigore (2016) - Java Generics shown to be Turing Complete
- Erdős (2017) - Encodes Boolean logic into Java type system
- Nakamaru et al. (2017) - Silverchain: a fluent API generator
- **Considine (2019) - Shape-safe matrix multiplication in Kotlin∇**
- Gil & Roth (2019) - Fling, a fluent API parser generator
- Cheng (2020) - Automatic theorem proving in the Scala type system
- Roth (2021) - Encodes CFL into Nominal Subtyping with Variance
- **Considine (2021) - Arithmetic in Kotlin via typelevel abacus**
- We know how to lower parsing onto types, what about vis versa?

Can we lower type checking onto parsing?

First, let us consider the untyped version:

```
Exp → 0 | 1 | ... | T | F
Exp → Exp Op Exp | if ( Exp ) Exp else Exp
Op → and | or | + | *
```

Now, let us consider the GADT/HOAS version:

```
Exp<Bool> → T | F
Op<Bool> → and | or
Exp<Int> → 0 | 1 | ... | 9
Op<Int> → + | *
Exp<E> → Exp<E> Op<E> Exp<E> // Es must be exactly the same!
Exp<E> → if ( Exp<Bool> ) Exp<E> else Exp<E>
```

We can eliminate contextuality by concretizing over `E → Bool | Int`:

```
Exp<Bool> → T | F
Exp<Bool> → Exp<Bool> or Exp<Bool> | Exp<Bool> and Exp<Bool>
Exp<Bool> → if ( Exp<Bool> ) Exp<Bool> else Exp<Bool>
Exp<Int> → 0 | 1 | ... | 9
Exp<Int> → Exp<Int> + Exp<Int> | Exp<Int> * Exp<Int>
Exp<Int> → if ( Exp<Bool> ) Exp<Int> else Exp<Int>
```

Inductive and algberaic graph representations

We can define a graph inductively, using a CFG/ADT:

$$\begin{aligned} \textsc{Vertex} &\rightarrow \textsc{Int} \\ \textsc{Neighbors} &\rightarrow \textsc{Vertex} \mid \textsc{Vertex}\ \ \textsc{Neighbors} \\ \textsc{Context} &\rightarrow ([\textsc{Neighbors}], \textsc{Vertex}, [\textsc{Neighbors}]) \\ \textsc{Graph} &\rightarrow \textsc{Empty} \mid \textsc{Context}\ \ \textsc{Graph} \end{aligned}$$

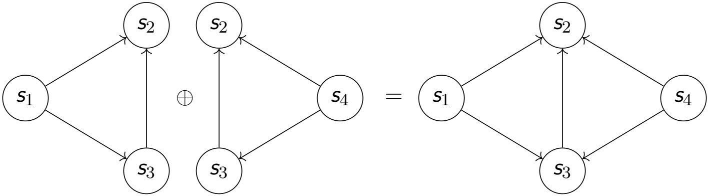
We can also represent graphs algebraically using the graph Laplacian:

$$\mathcal{L}_{i,j} := \begin{cases} \deg(v_i) & \text{if } i = j \\ -1 & \text{if } i \neq j \text{ and } (v_i \rightarrow v_j) \in E \\ 0 & \text{otherwise,} \end{cases}$$

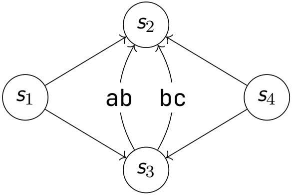
The latter form is preferred for representation learning [Hamilton (2020)].

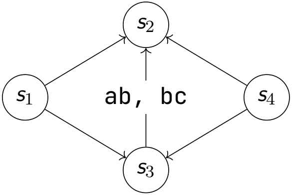
Graph combinator

To merge two unlabeled graphs, apply $G_1 \oplus G_2 = (V_1 \cup V_2) \times (E_1 \cup E_2)$:



Can be specialized to join ADTs, e.g.: $G_1 \oplus G_2 = (V_1 \cup V_2) \times (E_1 \bowtie E_2)$:





A type family for graphs

```
interface IGF<G, E, V> where
  G: IGraph<G, E, V>, E: IEdge<G, E, V>, V: IVertex<G, E, V> {
    val G: (vertices: Set<V>) → G
    val E: (s: V, t: V) → E
    val V: (old: V, edgeMap: (V) → Set<E>) → V

    fun G(vararg graphs: G): G = G(graphs.toList())
    fun G(vararg vertices: V): G = G(vertices.map { it.graph })
    fun G(l: List<Any>): G = when {
      l allAre G → l.fold(G()) { it, acc → it + acc as G }
      l allAre V → list.map { it as V }.toSet()
    }.let { G(it) }

    operator fun G.plus(that: G): G =
      G((this - that) + (this join that) + (that - this))

    operator fun G.minus(that: G): G = G(vertices - that.vertices)

    infix fun G.join(that: G): Set<V> = TODO("Override me!")
}
```

Linear Finite State Registers

Let $\mathbf{M} : \mathrm{GF}(2^{n\times n})$ be a square matrix $\mathbf{M}^0_{r,c} = P_c$ if $r = 0$ else $\mathbb{1}[c = r - 1]$, where P is a feedback polynomial over $GF(2^n)$ with coefficients $P_{1\ldots n}$ and semiring operators $\oplus := \veebar, \otimes := \wedge$:

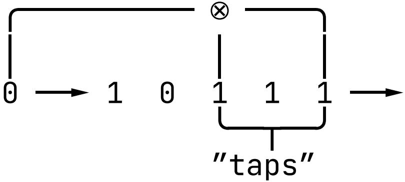
$$\mathbf{M}^t V = \begin{pmatrix} P_1 & P_2 & P_3 & P_4 & P_5 \\ \top & \circ & \circ & \circ & \circ \\ \circ & \top & \circ & \circ & \circ \\ \circ & \circ & \top & \circ & \circ \\ \circ & \circ & \circ & \top & \circ \end{pmatrix}^t \begin{pmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \end{pmatrix}$$

Selecting any $V \neq \mathbf{0}$ and coefficients P_j from a known *primitive polynomial* then powering the matrix $\mathbf{M}$ generates an ergodic sequence over $\mathrm{GF}(2^n)$:

$$\mathbf{S} = \begin{pmatrix} V & \mathbf{M}V & \mathbf{M}^2 V & \mathbf{M}^3 V & \cdots & \mathbf{M}^{2^n-1} V \end{pmatrix}$$

This sequence has *full periodicity*, i.e., for all $i, j \in [0, 2^n), \mathbf{S}_i = \mathbf{S}_j \Rightarrow i = j$.

	1	+	x^3	+	x^5
P = 20 =	0	0	1	0	1
	‖	‖	‖	‖	‖
	a	b	c	d	e



$$
\begin{array}{ccccc}
a & b & c & d & e \\
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0
\end{array}
\quad * \quad
\begin{array}{c}
V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5
\end{array}
$$

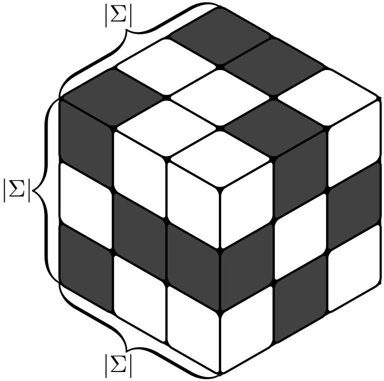
$$M * V$$

S_0 =	1	0	1	1	1
S_1 =	0	1	0	1	1
S_2 =	1	0	1	0	1
S_3 =	0	1	0	1	0
S_4 =	0	0	1	0	1

$S_4 = M \ldots S_1 = M * V$

S_4	S_1	V
‖	‖	‖
0	0	1
0	1	0
1	0	1
0	1	1
1	0	1

Multidimensional sampling: the hasty pudding trick



To uniformly sample $\boldsymbol{\sigma} \sim \Sigma^n$ without replacement, we could track historical samples, or, we can form an injection $GF(2^n) \rightharpoonup \Sigma^d$, cycle a primitive polynomial over $GF(2^n)$, then discard samples that do not identify an element in any indexed dimension. This procedure rejects $(1 - |\Sigma| 2^{-\lceil \log_2 |\Sigma| \rceil})^d$ samples on average and requires $\sim \mathcal{O}(1)$.

```
e.g., Σ² = {A, B, C}², x⁴ + x³ + 1

  S₀     S₁     S₂     S₃     S₄     S₅     S₆     S₇
  ||     ||     ||     ||     ||     ||     ||     ||
[1000] [0100] [0010] [1001] [1100] [0110] [1011] [0101]
 C A    B A    A C    C B           B C           B B
```

Multidimensional No-Replacement Sampler

```
fun List<Int>.bitLens() = map { ceil(log2(it.toDouble())).toInt() }

// Splits a bitvector into designated chunks and returns indices
// (10101011, [3, 2, 3]) -> [101, 01, 011] -> [4, 1, 3]
fun List<Boolean>.toIndexes(bitLens: List<Int>): List<Int> =
  bitLens.fold(listOf<List<Boolean>>() to this) { (a, b), i ->
    (a + listOf(b.take(i))) to b.drop(i)
  }.first.map { it.toInt() }

fun Sequence<List<Boolean>>.hastyPudding(lengths: List<Int>) =
  map { it.toIndexes(lengths.bitLens()) }
   .filter { it.zip(lengths).all { (a, b) -> a < b } }

fun <T> List<Set<T>>.sampleWithoutReplacement(
  lengths: List<Int> = map { it.size },
  bitLens: List<Int> = map(Set<T>::size).bitLens(),
  degree: Int = bitLens.sum().also { println("LFSR(GF(2^$it))") }
): Sequence<List<T>> =
  LFSR(degree).hastyPudding(lengths)
    .map { zip(it).map { (dims, idx) -> dims[idx] } }
```

Recap: classical logic in a nutshell

$$\frac{a \veebar b}{(p \vee q) \wedge \neg(p \wedge q)}\,\text{XOR} \qquad \frac{a \rightarrow b}{\neg a \vee b}\,\text{Impl} \qquad \frac{a \leftrightarrow b}{(\neg a \vee b) \wedge (\neg b \vee a)}\,\text{Iff}$$

$$\frac{\neg\neg a}{a}\,\text{2Neg} \qquad \frac{a \bullet (b \bullet c)}{(a \bullet b) \bullet c}\,\text{Assoc}_{\wedge\vee} \qquad \frac{a \bullet b}{b \bullet a}\,\text{Comm}_{\wedge\vee}$$

$$\frac{a \wedge (b \vee c)}{(a \wedge b) \vee (a \wedge c)}\,\text{Dist}_{\wedge} \qquad \frac{a \vee (b \wedge c)}{(a \vee b) \wedge (a \vee c)}\,\text{Dist}_{\vee}$$

$$\frac{\neg(a \vee b)}{\neg a \wedge \neg b}\,\text{DeMorgan}_{\vee} \qquad \frac{\neg(a \wedge b)}{\neg a \vee \neg b}\,\text{DeMorgan}_{\wedge}$$

Normalization in classical logic

Conjunctive Normal Form

$$\textsc{Conj} \to (\textsc{Disj}) \mid \textsc{Conj} \land (\textsc{Disj})$$
$$\textsc{Unit} \to \mathrm{VAR} \mid \neg\textsc{Var} \mid \bot \mid \top$$
$$\textsc{Disj} \to \textsc{Unit} \mid \textsc{Disj} \lor \textsc{Disj}$$

$$\cfrac{\cfrac{\cfrac{\cfrac{\neg(p \lor \neg q) \lor \neg\neg r}{\neg(p \lor \neg q) \lor r}\,\text{2Neg}}{(\neg p \land \neg\neg q) \lor r}\,\text{DeMorgan}}{(\neg p \land q) \lor r}\,\text{2Neg}}{(\neg p \lor r) \land (q \lor r)}\,\text{Dist}$$

Zhegalkin Normal Form

$$f(x_1, \ldots x_n) = \bigoplus_{i \subseteq \{1,\ldots,n\}} a_i x^i$$

i.e., a_i's filter the powerset.

$$\cfrac{\cfrac{\cfrac{\cfrac{x + (y \land \neg z)}{x + y(1 \oplus z)}}{x + (y \oplus yz)}}{x \oplus (y \oplus yz) \oplus x(y \oplus yz)}}{x \oplus y \oplus xy \oplus yz \oplus xyz}$$

Some common algebraic and logical forms

a_1	a_2	a_3	a_4	ZNF	Logical	CNF
0	0	0	0	0	$\bot$	$x \wedge \neg x$
1	0	0	0	1	$\top$	$x \vee \neg x$
0	1	0	0	x	x	x
1	1	0	0	1 + x		$\neg x$
0	0	1	0	y	y	y
1	0	1	0	1 + y		$\neg y$
0	1	1	0	x + y	$x \oplus y$	$(x \vee y) \wedge (\neg x \vee \neg y)$
1	1	1	0	1 + x + y	$x \iff y$	$(x \vee \neg y) \wedge (\neg x \vee y)$
0	0	0	1	xy	$x \wedge y$	$x \wedge y$
1	0	0	1	1 + xy	$\neg(x \wedge y)$	$(\neg x) \vee (\neg y)$
0	1	0	1	x + xy	$x \wedge ()$	$x \wedge (\neg y)$
1	1	0	1	1 + x + xy	$x \implies y$	$(\neg x) \vee y$
0	0	1	1	y + xy	$() \wedge y$	$(\neg x) \wedge y$
1	0	1	1	1 + y + xy	$x \impliedby y$	$x \vee (\neg y)$
0	1	1	1	x + y + xy	$x \vee y$	$x \vee y$
1	1	1	1	1 + x + y + xy	$\neg(x \vee y)$	$(\neg x) \wedge (\neg y)$

Facts about finite fields

- For every prime number p and positive integer n, there exists a finite field with p^n elements, denoted $GF(p^n)$, $\mathbb{Z}/p^n$ or $\mathbb{F}_p^n$.
- The following instruction sets have identical expressivity:
  - Pairs: $\{\vee, \neg\}, \{\wedge, \neg\}, \{\rightarrow, \neg\}, \{\rightarrow, \bot\}, \{\rightarrow, \veebar\}, \{\wedge, \veebar\}, \ldots$
  - Triples: $\{\vee, =, \veebar\}, \{\vee, \veebar, \top\}, \{\wedge, =, \bot\}, \{\wedge, =, \veebar\}, \{\wedge, \veebar, \top\}, \ldots$
- In other words, we can compute any Boolean function $\mathbb{B}^n \rightarrow \mathbb{B}$ by composing any one of the above operator sets in an orderly fashion.
- $\mathbb{F}_2$ corresponds to arithmetic modulo 2, i.e., $\oplus := \veebar, \otimes := \wedge$.
- There are (at least) two schools of thought about Boolean circuits:
  - Logical: Conjunctive Normal Form (CNF). Not necessarily unique.
  - Algebra: Zhegalkin Normal Form (ZNF). It is necessarily unique.
- The type $\mathbb{F}_2^n \rightarrow \mathbb{F}_2$ possesses 2^{2^n} inhabitants.

Preface to "Two Memoirs on Pure Analysis"

"*Long algebraic calculations were at first hardly necessary for mathematical progress... It was only since Euler that concision has become indispensable to continuing the work this great geometer has given to science. Since Euler, calculation has become more and more necessary and... the algorithms so complicated that progress would be nearly impossible without the elegance that modern geometers have brought to bear on their research, and by which means the mind can promptly and with a glance grasp a large number of operations.*

...

It is clear that elegance, so admirably and aptly named, has no other purpose.

...

Jump headlong into the calculations! Group the operations, classify them by their difficulties and not their appearances. This, I believe, is the mission of future geometers. This is the road on which I am embarking in this work."

Évariste Galois, 1811-1832

What's the point?

- Algebraists have developed a powerful language for rootfinding
- Tradition handed down from Euler, Galois, Borel, Kleene, Chomsky
- We know closed forms for exponentials of structured matrices
- Solving these forms can be much faster than power iteration
- Unifies many problems in PL, probability and graph theory
- Context free parsing is just rootfinding on a semiring algebra
- Type checking sans recursive types is just graph reachability
- Unification/simplification is lazy hypergraph search
- Bounded program synthesis is matrix factorization/completion
- By doing so, we can leverage well-known algebraic techniques

Future work

Parsing

- The line between parsing and computation is blurry
- Investigate connection between dynamical and term rewrite systems
- Extend Valiant's parser to tensors/context-sensitive languages
- Recover the original parse tree or eliminate Chomsky Normal Form
- What is the connection to Leibnizian differentiability?

Probability

- Look into Markov chains (detailed balance, stationarity, reversibility)
- Fuse Valiant parser and probabilistic context free grammar
- Message passing and graph diffusion processes
- Look into constrained optimization (e.g., L/QP) to rank feasible set

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