Linear Conjunctive Reachability as Tensor Completion

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Abstract

Brzozowski (1964) defines a regular expression derivative as the suffixes which complete a known prefix. In this work, we establish a Galois connection with Valiant's (1975) fixpoint construction in the context-free setting, and further extend their work into the hierarchy of bounded context-sensitive languages realizable by finite CFL intersection, i.e. conjunctive languages. We illustrate how to lower conjunctive language recognition onto a system of multilinear equations over finite fields. In addition to its theoretical value, this connection has yielded surprisingly useful applications in incremental parsing, code completion and program repair.

1 Introduction

Recall that a CFG is a quadruple consisting of terminals (Σ), nonterminals (V), productions ($P\colon V \to (V \mid \Sigma)^*$), and a start symbol, ($S$). It is a well-known fact that every CFG is reducible to *Chomsky Normal Form*, $P'\colon V \to (V^2 \mid \Sigma)$, in which every production takes one of two forms, either $w \to xz$, or $w \to t$, where $w, x, z : V$ and $t : \Sigma$. For example, the CFG, $P := \{S \to SS \mid (S) \mid ()\}$, corresponds to the CNF:

$$P' = \left\{ S \to QR \mid SS \mid LR, \quad L \to (, \quad R \to), \quad Q \to LS \right\}$$

Given a CFG, $\mathcal{G}' : \langle \Sigma, V, P, S \rangle$ in CNF, we can construct a recognizer $R : \mathcal{G}' \to \Sigma^n \to \mathbb{B}$ for strings $\sigma : \Sigma^n$ as follows. Let 2^V be our domain, 0 be $\varnothing$, $\oplus$ be $\cup$, and $\otimes$ be defined as:

$$X \otimes Z := \left\{ w \mid \langle x, z \rangle \in X \times Z, (w \to xz) \in P \right\} \tag{1}$$

If we define $\sigma_r^{\uparrow} := \{w \mid (w \to \sigma_r) \in P\}$, then initialize $M^0_{r+1=c}(\mathcal{G}', e) := \sigma_r^{\uparrow}$ and solve for the fixpoint $M^* = M + M^2$,

$$M^0 := \begin{pmatrix} \varnothing & \sigma_1^{\rightarrow} & \varnothing & \cdots & \varnothing \\ \vdots & & \ddots & & \vdots \\ & & & & \varnothing \\ & & & & \sigma_n^{\uparrow} \\ \varnothing & \cdots & & & \varnothing \end{pmatrix} \Rightarrow M^* = \begin{pmatrix} \varnothing & \sigma_1^{\rightarrow} & \Lambda & \cdots & \Lambda_\sigma^* \\ \vdots & & \ddots & & \vdots \\ & & & & \Lambda \\ & & & & \sigma_n^{\uparrow} \\ \varnothing & \cdots & & & \varnothing \end{pmatrix}$$

we obtain the recognizer, $R(\mathcal{G}', \sigma) := S \in \Lambda^*_\sigma? \Leftrightarrow \sigma \in \mathcal{L}(\mathcal{G})?$ Full details of the bisimilarity between parsing and matrix multiplication can be found in Valiant [?] and Lee [?], who shows its time complexity to be $\mathcal{O}(n^\omega)$ where ω is the least matrix multiplication upper bound (currently, $\omega < 2.77$).

2 Method

Note that $\bigoplus_{c=1}^{n} M_{r,c} \otimes M_{c,r}$ has cardinality bounded by $|V|$ and is thus representable as a fixed-length vector using the characteristic function, $\mathbb{1}$. In particular, $\oplus, \otimes$ are redefined as $\boxplus, \boxtimes$ over bitvectors so the following diagram commutes,[1]

$$\begin{tikzcd}
\langle \mathcal{G}', \boxed{\Sigma}^{n-1} \rangle \arrow[dr, leftrightarrow] & \text{Set} & \text{Bit} & \langle \mathcal{G}', \Sigma^{n-1} \rangle \arrow[dr] & \text{SAT} \\
\text{Rubix} & M \times M \arrow[r, leftrightarrow, "\mathbb{1}^{2^{n\times n}}", "\mathbb{1}^{-2^{n\times n}}"'] \arrow[d, "\oplus \mid \otimes"] & \mathbb{F}_2^{|V|^{n\times n}} \times \mathbb{F}_2^{|V|^{n\times n}} \arrow[rr, tail, "\varphi^{2^{n\times n}}", "\varphi^{-2^{n\times n}}"'] \arrow[d, "\boxed{+} \mid \boxed{*}"] & & \mathcal{M} \times \mathcal{M} \arrow[d, leftrightarrow, "+ \mid *"] \\
\text{Matrix} & 2^V \times 2^V \arrow[r, leftrightarrow, "\mathbb{1}^2", "\mathbb{1}^{-2}"'] \arrow[d, "\oplus \mid \otimes"] & \mathbb{F}_2^{|V|} \times \mathbb{F}_2^{|V|} \arrow[rr, tail, "\varphi^2", "\varphi^{-2}"'] \arrow[d, "\boxed{\boxplus} \mid \boxed{\boxtimes}"] & & \mathcal{V} \times \mathcal{V} \arrow[d, leftrightarrow, "\boxplus \mid \boxtimes"] \\
\text{Vector} & 2^V \arrow[r, leftrightarrow, "\mathbb{1}", "\mathbb{1}^{-1}"'] & \mathbb{F}_2^{|V|} \arrow[rr, tail, "\varphi", "\varphi^{-1}"'] & & \mathcal{V}
\end{tikzcd}$$

where $\mathcal{V}$ is a function $\mathbb{F}_2^{|V|} \to \mathbb{F}_2$. Note that while always possible to encode $\mathbb{F}_2^{|V|} \to \mathcal{V}$ using the identity function, φ^{-1} may not exist, as an arbitrary $\mathcal{V}$ might have zero, one, or in general, multiple solutions in $\mathbb{F}_2^{|V|}$. Although holes may occur anywhere, let us consider two cases in which Σ^+ is strictly left- or right-constrained, i.e., $\boxed{x}\,z, x\,\boxed{z} : \Sigma^{|x|+|z|}$.

Valiant's $\otimes$ operator, which yields the set of productions unifying known factors in a binary CFG, naturally implies the existence of a left- and right-quotient, which yield the set of nonterminals that may appear the right or left side of a known factor and its corresponding root. In other words, a known factor not only implicates subsequent expressions that can be derived from it, but also adjacent factors that may be composed with it to form a given derivation.

Left Quotient	Right Quotient
$\frac{\partial}{\partial \overleftarrow{x}} = \left\{ z \mid (w \to xz) \in P \right\}$	$\frac{\partial}{\partial \overrightarrow{z}} = \left\{ x \mid (w \to xz) \in P \right\}$
$\begin{array}{\|c\|c\|} \hline \boxed{x} & \boxed{w} \\ \hline & z \\ \hline \end{array}$	$\begin{array}{\|c\|c\|} \hline x & \boxed{w} \\ \hline & \boxed{z} \\ \hline \end{array}$

The left quotient coincides with the derivative operator first proposed by Brzozowski [?] and Antimirov [?] over regular languages, lifted into the context-free setting (our work). When the root and LHS are fixed, e.g., $\frac{\partial S}{\partial \overleftarrow{x}} : (\overleftarrow{V} \to S) \to \overrightarrow{\mathcal{V}}$ returns the set of admissible nonterminals to the RHS. One may also consider a gradient operator, $\overleftarrow{\nabla} S : (\overleftarrow{\mathcal{V}} \to S) \to \overrightarrow{\mathcal{V}}$, which simultaneously tracks the partials with respect to a set of multiple LHS nonterminals produced by a fixed root.

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[1]Hereinafter, we use gray highlighting to distinguish between expressions containing only $\boxed{\text{constants}}$ from those which may contain free variables.

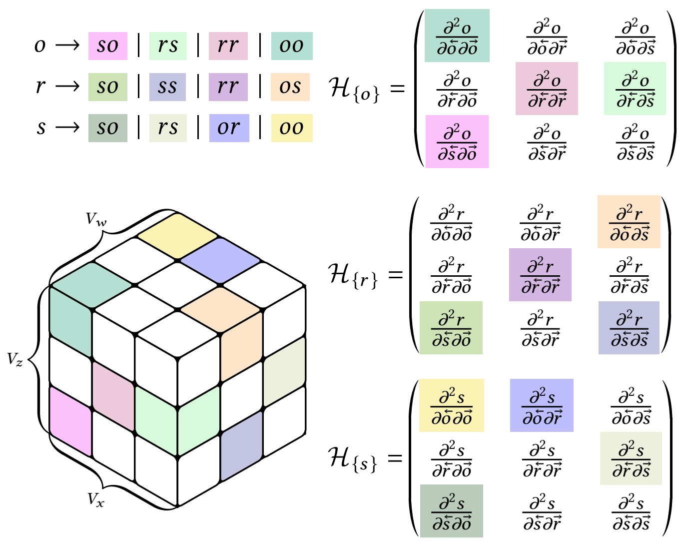


Figure 1. CFGs are witnessed by a rank-3 tensor, whose nonempty inhabitants indicate CNF productions. Gradients in this setting effectively condition the parse tensor M by constraining the superposition of admissible parse forests.

3 Context-sensitive reachability

It is well-known that the family of CFLs is not closed under intersection. For example, consider $\mathcal{L}_\cap := \mathcal{L}(\mathcal{G}_1) \cap \mathcal{L}(\mathcal{G}_2)$:

$$P_1 := \left\{ S \to LR, \quad L \to ab \mid aLb, \quad R \to c \mid cR \right\}$$
$$P_2 := \left\{ S \to LR, \quad R \to bc \mid bRc, \quad L \to a \mid aL \right\}$$

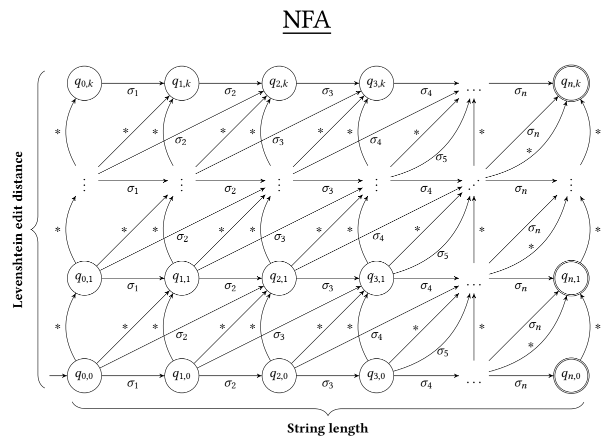
Note that $\mathcal{L}_\cap$ generates the language $\left\{ a^d b^d c^d \mid d > 0 \right\}$, which according to the pumping lemma is not context-free. We can encode $\bigcap_{i=1}^{c} \mathcal{L}(\mathcal{G}_i)$ as a polygonal prism with upper-triangular matrices adjoined to each rectangular face. More precisely, we intersect all terminals $\Sigma_\cap := \bigcap_{i=1}^{c} \Sigma_i$, then for each $t_\cap \in \Sigma_\cap$ and CFG, construct an equivalence class $E(t_\cap, \mathcal{G}_i) = \{w_i \mid (w_i \to t_\cap) \in P_i\}$ and bind them together:

$$\bigwedge_{t \in \Sigma_\cap} \bigwedge_{j=1}^{c-1} \bigwedge_{i=1}^{|\sigma|} E(t_\cap, \mathcal{G}_j) \equiv_{\sigma_i} E(t_\cap, \mathcal{G}_{j+1}) \tag{2}$$

Figure 2. Orientations of a $\bigcap_{i=1}^{4} \mathcal{L}(\mathcal{G}_i) \cap \Sigma^6$ configuration. As $c \to \infty$, this shape approximates a circular cone whose symmetric axis joins σ_i with orthonormal unit productions $w_i \to t_\cap$, and $S_i \in \Lambda_\sigma^*$? represented by the outermost bitvector inhabitants. Equations of this form are equiexpressive with the family of CSLs realizable by finite CFL intersection.

4 Levenshtein Reachability

Levenshtein reachability is recognized by the nondeterministic infinite automaton (NIA) whose topology $\mathcal{L}$ = can be factored into a product of (a) the monotone Chebyshev topology, equipped with horizontal transitions accepting σ_i and vertical transitions accepting Kleene stars, and (b) the monotone knight's topology, equipped with transitions accepting σ_{i+2}. The structure of this space is representable as an acyclic NFA [**?**], populated by accept states within radius k of $q_{n,0}$, or equivalently, a left-linear CFG whose productions finitely instantiate the transition dynamics:



By intersection with a conjunctive language, we obtain a language $\mathcal{L}(\mathcal{G})$ that is both a subset of Σ^n and accepted by $\mathcal{G}$. We can then define the Levenshtein reachability problem as follows: given a string $\sigma \in \Sigma^n$, find the smallest k such that $\sigma \in \mathcal{L}(\mathcal{G})$.

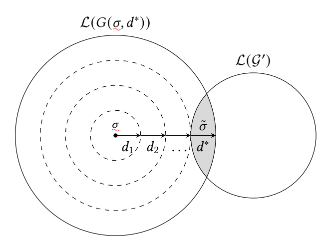


Figure 3. LED is computed gradually by incrementing d until $\mathcal{L}_d^\cap \neq \varnothing$.

5 Conclusion

Not only is linear algebra over finite fields an expressive language for inference, but also an efficient framework for inference on languages themselves. We illustrate a few of its applications for parsing incomplete strings and repairing syntax errors in context- free and sensitive languages. In contrast with LL and LR-style parsers, our technique can recover partial forests from invalid strings by examining the structure of M^*. In future work, we hope to extend our method to more natural grammars like PCFG and LCFRS.