System Specification

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1 Grammar

$var,\ x,\ y$	$::=$		variables
$nat,\ i,\ j,\ n$	$::=$		natural numbers
	$\mid$	0	
	$\mid$	$1 + i$	
$set,\ S,\ F$	$::=$		sets
	$\mid$	$\emptyset$	
	$\mid$	$\{a \mid formula\}$	
	$\mid$	$\mathsf{dom}(S)$	
	$\mid$	$F\ a$	
	$\mid$	$Term$	
	$\mid$	$\mathcal{P}(S)$	
	$\mid$	$S_1 \to S_2$	
	$\mid$	S	
	$\mid$	$\mathbf{P}(i)$	
$tm,\ a,\ b,\ c,\ t,\ p,\ A,\ B,\ C$	$::=$		terms and types
	$\mid$	i	variable
	$\mid$	$\Pi\, A\, B$	dependent function type
	$\mid$	$\lambda A.\, a$	function
	$\mid$	$a\ b$	function application
	$\mid$	$\mathbf{Set}\ i$	universe
	$\mid$	$a\ \sim\ b \in A$	identity type
	$\mid$	$\mathbf{refl}$	reflexivity proof
	$\mid$	$\mathbf{Void}$	empty type
	$\mid$	$\mathbf{J}\ t\ a\ b\ p$	J eliminator
	$\mid$	$\mathbb{B}$	boolean type
	$\mid$	$\mathbf{true}$	
	$\mid$	$\mathbf{false}$	
	$\mid$	$\mathbf{if}\ a\ \mathbf{then}\ b_0\ \mathbf{else}\ b_1$	
$context,\ \Gamma$	$::=$		contexts
	$\mid$	$\cdot$	
	$\mid$	Γ, A	

2 Dynamics

$\boxed{a \Leftrightarrow b}$ *(Coherence)*

C-Intro
$$\frac{a \Rightarrow^{+} c \qquad b \Rightarrow^{+} c}{a \Leftrightarrow b}$$

$\boxed{a \Rightarrow^{+} b}$ *(Transitive Closure of Parallel Reduction)*

PS-One
$$\frac{a \Rightarrow b}{a \Rightarrow^{+} b}$$

PS-Step
$$\frac{a \Rightarrow b \qquad b \Rightarrow^{+} c}{a \Rightarrow^{+} c}$$

$\boxed{a \Rightarrow b}$ *(Parallel Reduction)*

P-Var
$$\frac{}{i \Rightarrow i}$$

P-Set
$$\frac{}{\mathbf{Set}\ i \Rightarrow \mathbf{Set}\ i}$$

P-Void
$$\frac{}{\mathbf{Void} \Rightarrow \mathbf{Void}}$$

P-Pi
$$\frac{A_0 \Rightarrow A_1 \qquad B_0 \Rightarrow B_1}{\Pi\ A_0\ B_0 \Rightarrow \Pi\ A_1\ B_1}$$

P-Abs
$$\frac{A_0 \Rightarrow A_1 \qquad a_0 \Rightarrow a_1}{\lambda A_0.\ a_0 \Rightarrow \lambda A_1.\ a_1}$$

P-App
$$\frac{a_0 \Rightarrow a_1 \qquad b_0 \Rightarrow b_1}{a_0\ b_0 \Rightarrow a_1\ b_1}$$

P-AppAbs
$$\frac{\begin{array}{c} a \Rightarrow \lambda A.\ a_0 \\ b_0 \Rightarrow b_1 \end{array}}{a\ b_0 \Rightarrow a_0\langle b_1\rangle}$$

P-True
$$\frac{}{\mathbf{true} \Rightarrow \mathbf{true}}$$

P-False
$$\frac{}{\mathbf{false} \Rightarrow \mathbf{false}}$$

P-If
$$\frac{\begin{array}{c} a_0 \Rightarrow a_1 \\ b_0 \Rightarrow b_1 \qquad c_0 \Rightarrow c_1 \end{array}}{\mathbf{if}\ a_0\ \mathbf{then}\ b_0\ \mathbf{else}\ c_0 \Rightarrow \mathbf{if}\ a_1\ \mathbf{then}\ b_1\ \mathbf{else}\ c_1}$$

P-IfTrue
$$\frac{\begin{array}{c} a_0 \Rightarrow \mathbf{true} \\ b_0 \Rightarrow b_1 \qquad c_0 \Rightarrow c_1 \end{array}}{\mathbf{if}\ a_0\ \mathbf{then}\ b_0\ \mathbf{else}\ c_0 \Rightarrow b_1}$$

P-IfFalse
$$\frac{\begin{array}{c} a_0 \Rightarrow \mathbf{false} \\ b_0 \Rightarrow b_1 \qquad c_0 \Rightarrow c_1 \end{array}}{\mathbf{if}\ a_0\ \mathbf{then}\ b_0\ \mathbf{else}\ c_0 \Rightarrow c_1}$$

P-Bool
$$\frac{}{\mathbb{B} \Rightarrow \mathbb{B}}$$

P-Eq
$$\frac{\begin{array}{c} a_0 \Rightarrow a_1 \\ b_0 \Rightarrow b_1 \qquad A_0 \Rightarrow A_1 \end{array}}{a_0\ \sim\ b_0 \in A_0 \Rightarrow a_1\ \sim\ b_1 \in A_1}$$

P-Refl
$$\frac{}{\mathbf{refl} \Rightarrow \mathbf{refl}}$$

P-J
$$\frac{\begin{array}{c} t_0 \Rightarrow t_1 \qquad a_0 \Rightarrow a_1 \\ b_0 \Rightarrow b_1 \qquad p_0 \Rightarrow p_1 \end{array}}{\mathbf{J}\ t_0\ a_0\ b_0\ p_0 \Rightarrow \mathbf{J}\ t_1\ a_1\ b_1\ p_1}$$

P-JRefl
$$\frac{\begin{array}{c} t_0 \Rightarrow t_1 \qquad a_0 \Rightarrow a_1 \\ b_0 \Rightarrow b_1 \qquad p \Rightarrow \mathbf{refl} \end{array}}{\mathbf{J}\ t_0\ a_0\ b_0\ p \Rightarrow t_1}$$

3 Statics

$\boxed{\vdash \Gamma}$ *(Context Well-Formedness)*

$$\textsc{Ctx-Empty}\quad \frac{}{\vdash \cdot}$$

$$\textsc{Ctx-Cons}\quad \frac{\vdash \Gamma \qquad \Gamma \vdash A : \textbf{Set}\, i}{\vdash \Gamma, A}$$

$\boxed{\Gamma \vdash a : A}$ *(Typing)*

$$\textsc{T-Var}\quad \frac{\vdash \Gamma \qquad i < |\Gamma|}{\Gamma \vdash i :\uparrow^{1+i} \Gamma_i}$$

$$\textsc{T-Set}\quad \frac{\vdash \Gamma \qquad i < j}{\Gamma \vdash \textbf{Set}\, i : \textbf{Set}\, j}$$

$$\textsc{T-Pi}\quad \frac{\begin{array}{c}\Gamma \vdash A : \textbf{Set}\, i\\ \Gamma, A \vdash B : \textbf{Set}\, i\end{array}}{\Gamma \vdash \Pi\, A\, B : \textbf{Set}\, i}$$

$$\textsc{T-Abs}\quad \frac{\begin{array}{c}\Gamma, A \vdash b : B\\ \Gamma \vdash \Pi\, A\, B : \textbf{Set}\, i\end{array}}{\Gamma \vdash \lambda A.\, b : \Pi\, A\, B}$$

$$\textsc{T-App}\quad \frac{\begin{array}{c}\Gamma \vdash b : \Pi\, A\, B\\ \Gamma \vdash a : A\end{array}}{\Gamma \vdash b\ a : B\langle a\rangle}$$

$$\textsc{T-Conv}\quad \frac{\begin{array}{c}\Gamma \vdash a : A\\ \Gamma \vdash B : \textbf{Set}\, i\\ A \Leftrightarrow B\end{array}}{\Gamma \vdash a : B}$$

$$\textsc{T-Refl}\quad \frac{\vdash \Gamma \qquad \Gamma \vdash a : A}{\Gamma \vdash \textbf{refl} : a\ \sim\ a \in A}$$

$$\textsc{T-J}\quad \frac{\begin{array}{c}\Gamma \vdash a : A \qquad \Gamma \vdash b : A\\ \Gamma \vdash A : \textbf{Set}\, j\\ \Gamma \vdash p : a\ \sim\ b \in A\\ \Gamma, A, 0\ \sim\uparrow a \in \uparrow A \vdash C : \textbf{Set}\, i\\ \Gamma \vdash t : C\langle a, \textbf{refl}\rangle\end{array}}{\Gamma \vdash \textbf{J}\ t\ a\ b\ p : C\langle b, p\rangle}$$

$$\textsc{T-Bool}\quad \frac{\vdash \Gamma}{\Gamma \vdash \mathbb{B} : \textbf{Set}\, i}$$

$$\textsc{T-True}\quad \frac{\vdash \Gamma}{\Gamma \vdash \textbf{true} : \mathbb{B}}$$

$$\textsc{T-False}\quad \frac{\vdash \Gamma}{\Gamma \vdash \textbf{false} : \mathbb{B}}$$

$$\textsc{T-If}\quad \frac{\begin{array}{c}\Gamma \vdash a : \mathbb{B}\\ \Gamma \vdash b_0 : A\\ \Gamma \vdash b_1 : A\end{array}}{\Gamma \vdash \textbf{if}\ a\ \textbf{then}\ b_0\ \textbf{else}\ b_1 : A}$$

$$\textsc{T-Void}\quad \frac{\vdash \Gamma}{\Gamma \vdash \textbf{Void} : \textbf{Set}\, i}$$

4 Semantic Typing

$\boxed{[A]_i \searrow S}$ *(Logical Relation)*

$$\textsc{I-Void}\quad \frac{}{[\textbf{Void}]_i \searrow \emptyset}$$

$$\textsc{I-Bool}\quad \frac{}{[\mathbb{B}]_i \searrow \{a \mid a \Rightarrow^+ \textbf{true} \vee a \Rightarrow^+ \textbf{false}\}}$$

I-Eq

$$\frac{}{[a \ \sim \ b \in A]_i \searrow \{p \mid p \Rightarrow^+ \mathbf{refl}, a \Leftrightarrow b\}}$$

I-Pi

$$\frac{\begin{array}{c}[A]_i \searrow S \\ F \ \in \ S \to \mathcal{P}(\mathit{Term}) \\ \forall a, a \ \in \ S \implies [B\langle a\rangle]_i \searrow F\ a\end{array}}{[\Pi\, A\, B]_i \searrow \{b \mid \forall a, a \ \in \ S \implies b\ a \ \in \ F\ a\}}$$

I-Set

$$\frac{j < i}{[\mathbf{Set}\, j]_i \searrow \{A \mid \exists S, [A]_j \searrow S\}}$$

I-Red

$$\frac{A \Rightarrow B \qquad [B]_i \searrow S}{[A]_i \searrow S}$$