System Specification

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1 Grammar

$nat,\ i,\ j,\ n$	::=		natural numbers
	$\mid$	0	
	$\mid$	$1 + i$	
$tm,\ a,\ b,\ c,\ t,\ p,\ A,\ B,\ C$	::=		terms and types
	$\mid$	i	variable
	$\mid$	$\Pi\, A\, B$	dependent function type
	$\mid$	$\lambda A.\, a$	function
	$\mid$	$a\ b$	function application
	$\mid$	$\mathbf{Set}\, i$	universe
	$\mid$	$a\ \sim\ b \in A$	identity type
	$\mid$	$\mathbf{refl}$	reflexivity proof
	$\mid$	$\mathbf{Void}$	empty type
	$\mid$	$\mathbf{J}\ t\ a\ b\ p$	J eliminator
	$\mid$	$\mathbb{B}$	boolean type
	$\mid$	$\mathbf{true}$	
	$\mid$	$\mathbf{false}$	
	$\mid$	$\mathbf{if}\ a\ \mathbf{then}\ b_0\ \mathbf{else}\ b_1$	
$context,\ \Gamma$	::=		contexts
	$\mid$	$\cdot$	
	$\mid$	Γ, A	

2 Dynamics

$\boxed{a \Leftrightarrow b}$ *(Coherence)*

$$\frac{a \Rightarrow^+ c \qquad b \Rightarrow^+ c}{a \Leftrightarrow b}\ \textsc{C-Intro}$$

$\boxed{a \Rightarrow^+ b}$ *(Transitive Closure of Parallel Reduction)*

$$\textsc{PS-One}\ \frac{a \Rightarrow b}{a \Rightarrow^+ b} \qquad \textsc{PS-Step}\ \frac{a \Rightarrow b \qquad b \Rightarrow^+ c}{a \Rightarrow^+ c}$$

$\boxed{a \Rightarrow b}$ *(Parallel Reduction)*

$$\textsc{P-Var}\ \frac{}{i \Rightarrow i} \qquad \textsc{P-Set}\ \frac{}{\mathbf{Set}\ i \Rightarrow \mathbf{Set}\ i} \qquad \textsc{P-Void}\ \frac{}{\mathbf{Void} \Rightarrow \mathbf{Void}} \qquad \textsc{P-Pi}\ \frac{A_0 \Rightarrow A_1 \qquad B_0 \Rightarrow B_1}{\Pi\ A_0\ B_0 \Rightarrow \Pi\ A_1\ B_1}$$

$$\textsc{P-Abs}\ \frac{A_0 \Rightarrow A_1 \qquad a_0 \Rightarrow a_1}{\lambda A_0.\ a_0 \Rightarrow \lambda A_1.\ a_1} \qquad \textsc{P-App}\ \frac{a_0 \Rightarrow a_1 \qquad b_0 \Rightarrow b_1}{a_0\ b_0 \Rightarrow a_1\ b_1} \qquad \textsc{P-AppAbs}\ \frac{a \Rightarrow \lambda A.\ a_0 \qquad b_0 \Rightarrow b_1}{a\ b_0 \Rightarrow a_0[b_1]}$$

$$\textsc{P-True}\ \frac{}{\mathbf{true} \Rightarrow \mathbf{true}} \qquad \textsc{P-False}\ \frac{}{\mathbf{false} \Rightarrow \mathbf{false}}$$

$$\textsc{P-If}\ \frac{a_0 \Rightarrow a_1 \qquad b_0 \Rightarrow b_1 \qquad c_0 \Rightarrow c_1}{\mathbf{if}\ a_0\ \mathbf{then}\ b_0\ \mathbf{else}\ c_0 \Rightarrow \mathbf{if}\ a_1\ \mathbf{then}\ b_1\ \mathbf{else}\ c_1} \qquad \textsc{P-IfTrue}\ \frac{a_0 \Rightarrow \mathbf{true} \qquad b_0 \Rightarrow b_1 \qquad c_0 \Rightarrow c_1}{\mathbf{if}\ a_0\ \mathbf{then}\ b_0\ \mathbf{else}\ c_0 \Rightarrow b_1}$$

$$\textsc{P-IfFalse}\ \frac{a_0 \Rightarrow \mathbf{false} \qquad b_0 \Rightarrow b_1 \qquad c_0 \Rightarrow c_1}{\mathbf{if}\ a_0\ \mathbf{then}\ b_0\ \mathbf{else}\ c_0 \Rightarrow c_1} \qquad \textsc{P-Bool}\ \frac{}{\mathbb{B} \Rightarrow \mathbb{B}} \qquad \textsc{P-Eq}\ \frac{a_0 \Rightarrow a_1 \qquad b_0 \Rightarrow b_1 \qquad A_0 \Rightarrow A_1}{a_0\ \sim\ b_0 \in A_0 \Rightarrow a_1\ \sim\ b_1 \in A_1}$$

$$\textsc{P-Refl}\ \frac{}{\mathbf{refl} \Rightarrow \mathbf{refl}} \qquad \textsc{P-J}\ \frac{t_0 \Rightarrow t_1 \qquad a_0 \Rightarrow a_1 \qquad b_0 \Rightarrow b_1 \qquad p_0 \Rightarrow p_1}{\mathbf{J}\ t_0\ a_0\ b_0\ p_0 \Rightarrow \mathbf{J}\ t_1\ a_1\ b_1\ p_1} \qquad \textsc{P-JRefl}\ \frac{t_0 \Rightarrow t_1 \qquad a_0 \Rightarrow a_1 \qquad b_0 \Rightarrow b_1 \qquad p \Rightarrow \mathbf{refl}}{\mathbf{J}\ t_0\ a_0\ b_0\ p \Rightarrow t_1}$$

3 Statics

$\boxed{\vdash \Gamma}$ *(Context Well-Formedness)*

$$\textsc{Ctx-Empty}\ \frac{}{\vdash \cdot} \qquad \textsc{Ctx-Cons}\ \frac{\vdash \Gamma \qquad \Gamma \vdash A : \mathbf{Set}\ i}{\vdash \Gamma, A}$$

$\boxed{\Gamma \vdash a : A}$ *(Typing)*

$$\text{T-Var}\ \frac{\vdash \Gamma \qquad i < |\Gamma|}{\Gamma \vdash i : \uparrow^{1+i} \Gamma_i} \qquad \text{T-Set}\ \frac{\vdash \Gamma \qquad i < j}{\Gamma \vdash \mathbf{Set}\, i : \mathbf{Set}\, j} \qquad \text{T-Pi}\ \frac{\Gamma \vdash A : \mathbf{Set}\, i \qquad \Gamma, A \vdash B : \mathbf{Set}\, i}{\Gamma \vdash \Pi\, A\, B : \mathbf{Set}\, i}$$

$$\text{T-Abs}\ \frac{\Gamma, A \vdash b : B \qquad \Gamma \vdash \Pi\, A\, B : \mathbf{Set}\, i}{\Gamma \vdash \lambda A.\, b : \Pi\, A\, B} \qquad \text{T-App}\ \frac{\Gamma \vdash b : \Pi\, A\, B \qquad \Gamma \vdash a : A}{\Gamma \vdash b\ a : B[a]} \qquad \text{T-Conv}\ \frac{\Gamma \vdash a : A \qquad \Gamma \vdash B : \mathbf{Set}\, i \qquad A \Leftrightarrow B}{\Gamma \vdash a : B}$$

$$\text{T-Refl}\ \frac{\vdash \Gamma \qquad \Gamma \vdash a : A}{\Gamma \vdash \mathbf{refl} : a \sim a \in A} \qquad \text{T-J}\ \frac{\begin{array}{c}\Gamma \vdash a : A \qquad \Gamma \vdash b : A \\ \Gamma \vdash A : \mathbf{Set}\, j \\ \Gamma \vdash p : a \sim b \in A \\ \Gamma, A, 0 \sim \uparrow a \in \uparrow A \vdash C : \mathbf{Set}\, i \\ \Gamma \vdash t : C[a, \mathbf{refl}]\end{array}}{\Gamma \vdash \mathbf{J}\, t\, a\, b\, p : C[b, p]} \qquad \text{T-Bool}\ \frac{\vdash \Gamma}{\Gamma \vdash \mathbb{B} : \mathbf{Set}\, i}$$

$$\text{T-True}\ \frac{\vdash \Gamma}{\Gamma \vdash \mathbf{true} : \mathbb{B}} \qquad \text{T-False}\ \frac{\vdash \Gamma}{\Gamma \vdash \mathbf{false} : \mathbb{B}} \qquad \text{T-If}\ \frac{\begin{array}{c}\Gamma \vdash a : \mathbb{B} \\ \Gamma \vdash b_0 : A \\ \Gamma \vdash b_1 : A\end{array}}{\Gamma \vdash \mathbf{if}\ a\ \mathbf{then}\ b_0\ \mathbf{else}\ b_1 : A}$$

$$\text{T-Void}\ \frac{\vdash \Gamma}{\Gamma \vdash \mathbf{Void} : \mathbf{Set}\, i}$$