System Specification

Yiyun Liu

December 22, 2023

1 Grammar

$var,\ x,\ y$	$::=$		variables
$nat,\ i,\ j,\ n$	$::=$		natural numbers
	$\mid$	0	
	$\mid$	$1 + i$	
$set,\ S,\ F,\ I$	$::=$		sets
	$\mid$	$\emptyset$	
	$\mid$	$\{a \mid formula\}$	
	$\mid$	$\mathsf{dom}(S)$	
	$\mid$	$F(a)$	
	$\mid$	$Term$	
	$\mid$	$\mathcal{P}(S)$	
	$\mid$	$S_1 \rightarrow S_2$	
	$\mid$	S	
$tm,\ a,\ b,\ c,\ t,\ p,\ A,\ B,\ C$	$::=$		terms and types
	$\mid$	i	variable
	$\mid$	$\Pi\, A\, B$	dependent function type
	$\mid$	$\lambda A.\, a$	function
	$\mid$	$a\ b$	function application
	$\mid$	$\mathbf{Set}\ i$	universe
	$\mid$	$a\ \sim\ b \in A$	identity type
	$\mid$	$\mathbf{refl}$	reflexivity proof
	$\mid$	$\mathbf{Void}$	empty type
	$\mid$	$\mathbf{J}\ t\ a\ b\ p$	J eliminator
	$\mid$	$\mathbb{B}$	boolean type
	$\mid$	$\mathbf{true}$	
	$\mid$	$\mathbf{false}$	
	$\mid$	$\mathbf{if}\ a\ \mathbf{then}\ b_0\ \mathbf{else}\ b_1$	
$context,\ \Gamma$	$::=$		contexts
	$\mid$	$.$	
	$\mid$	Γ, A	

2 Dynamics

$\boxed{a \Leftrightarrow b}$ *(Coherence)*

$$\textsc{C-Intro}\quad \frac{a \Rightarrow^{+} c \qquad b \Rightarrow^{+} c}{a \Leftrightarrow b}$$

$\boxed{a \Rightarrow^+ b}$ *(Transitive Closure of Parallel Reduction)*

$$\textsc{PS-One}\quad \frac{a \Rightarrow b}{a \Rightarrow^+ b} \qquad \textsc{PS-Step}\quad \frac{a \Rightarrow b \qquad b \Rightarrow^+ c}{a \Rightarrow^+ c}$$

$\boxed{a \Rightarrow b}$ *(Parallel Reduction)*

$$\textsc{P-Var}\quad \frac{}{i \Rightarrow i} \qquad \textsc{P-Set}\quad \frac{}{\mathbf{Set}\ i \Rightarrow \mathbf{Set}\ i} \qquad \textsc{P-Void}\quad \frac{}{\mathbf{Void} \Rightarrow \mathbf{Void}} \qquad \textsc{P-Pi}\quad \frac{A_0 \Rightarrow A_1 \qquad B_0 \Rightarrow B_1}{\Pi\, A_0\, B_0 \Rightarrow \Pi\, A_1\, B_1}$$

$$\textsc{P-Abs}\quad \frac{A_0 \Rightarrow A_1 \qquad a_0 \Rightarrow a_1}{\lambda A_0.\, a_0 \Rightarrow \lambda A_1.\, a_1} \qquad \textsc{P-App}\quad \frac{a_0 \Rightarrow a_1 \qquad b_0 \Rightarrow b_1}{a_0\ b_0 \Rightarrow a_1\ b_1} \qquad \textsc{P-AppAbs}\quad \frac{\begin{array}{c} a \Rightarrow \lambda A.\, a_0 \\ b_0 \Rightarrow b_1 \end{array}}{a\ b_0 \Rightarrow a_0\langle b_1 \rangle}$$

$$\textsc{P-True}\quad \frac{}{\mathbf{true} \Rightarrow \mathbf{true}} \qquad \textsc{P-False}\quad \frac{}{\mathbf{false} \Rightarrow \mathbf{false}}$$

$$\textsc{P-If}\quad \frac{\begin{array}{c} a_0 \Rightarrow a_1 \\ b_0 \Rightarrow b_1 \qquad c_0 \Rightarrow c_1 \end{array}}{\mathbf{if}\ a_0\ \mathbf{then}\ b_0\ \mathbf{else}\ c_0 \Rightarrow \mathbf{if}\ a_1\ \mathbf{then}\ b_1\ \mathbf{else}\ c_1} \qquad \textsc{P-IfTrue}\quad \frac{\begin{array}{c} a_0 \Rightarrow \mathbf{true} \\ b_0 \Rightarrow b_1 \qquad c_0 \Rightarrow c_1 \end{array}}{\mathbf{if}\ a_0\ \mathbf{then}\ b_0\ \mathbf{else}\ c_0 \Rightarrow b_1}$$

$$\textsc{P-IfFalse}\quad \frac{\begin{array}{c} a_0 \Rightarrow \mathbf{false} \\ b_0 \Rightarrow b_1 \qquad c_0 \Rightarrow c_1 \end{array}}{\mathbf{if}\ a_0\ \mathbf{then}\ b_0\ \mathbf{else}\ c_0 \Rightarrow c_1} \qquad \textsc{P-Bool}\quad \frac{}{\mathbb{B} \Rightarrow \mathbb{B}} \qquad \textsc{P-Eq}\quad \frac{\begin{array}{c} a_0 \Rightarrow a_1 \\ b_0 \Rightarrow b_1 \qquad A_0 \Rightarrow A_1 \end{array}}{a_0\ \sim\ b_0 \in A_0 \Rightarrow a_1\ \sim\ b_1 \in A_1}$$

$$\textsc{P-Refl}\quad \frac{}{\mathbf{refl} \Rightarrow \mathbf{refl}} \qquad \textsc{P-J}\quad \frac{\begin{array}{c} t_0 \Rightarrow t_1 \qquad a_0 \Rightarrow a_1 \\ b_0 \Rightarrow b_1 \qquad p_0 \Rightarrow p_1 \end{array}}{\mathbf{J}\ t_0\ a_0\ b_0\ p_0 \Rightarrow \mathbf{J}\ t_1\ a_1\ b_1\ p_1} \qquad \textsc{P-JRefl}\quad \frac{\begin{array}{c} t_0 \Rightarrow t_1 \qquad a_0 \Rightarrow a_1 \\ b_0 \Rightarrow b_1 \qquad p \Rightarrow \mathbf{refl} \end{array}}{\mathbf{J}\ t_0\ a_0\ b_0\ p \Rightarrow t_1}$$

3 Statics

$\boxed{\vdash \Gamma}$ *(Context Well-Formedness)*

$$\textsc{Ctx-Empty}\quad \frac{}{\vdash \cdot} \qquad \textsc{Ctx-Cons}\quad \frac{\vdash \Gamma \qquad \Gamma \vdash A : \mathbf{Set}\ i}{\vdash \Gamma, A}$$

$\boxed{\Gamma \vdash a : A}$ *(Typing)*

$$\textsc{T-Var}\ \frac{\vdash \Gamma \qquad i < |\Gamma|}{\Gamma \vdash i : \uparrow^{1+i} \Gamma_i}$$

$$\textsc{T-Set}\ \frac{\vdash \Gamma \qquad i < j}{\Gamma \vdash \textbf{Set}\, i : \textbf{Set}\, j}$$

$$\textsc{T-Pi}\ \frac{\Gamma \vdash A : \textbf{Set}\, i \qquad \Gamma, A \vdash B : \textbf{Set}\, i}{\Gamma \vdash \Pi\, A\, B : \textbf{Set}\, i}$$

$$\textsc{T-Abs}\ \frac{\Gamma, A \vdash b : B \qquad \Gamma \vdash \Pi\, A\, B : \textbf{Set}\, i}{\Gamma \vdash \lambda A.\, b : \Pi\, A\, B}$$

$$\textsc{T-App}\ \frac{\Gamma \vdash b : \Pi\, A\, B \qquad \Gamma \vdash a : A}{\Gamma \vdash b\ a : B\langle a \rangle}$$

$$\textsc{T-Conv}\ \frac{\Gamma \vdash a : A \qquad \Gamma \vdash B : \textbf{Set}\, i \qquad A \Leftrightarrow B}{\Gamma \vdash a : B}$$

$$\textsc{T-Refl}\ \frac{\vdash \Gamma \qquad \Gamma \vdash a : A}{\Gamma \vdash \textbf{refl} : a\ \sim\ a \in A}$$

$$\textsc{T-J}\ \frac{\Gamma \vdash a : A \qquad \Gamma \vdash b : A \qquad \Gamma \vdash A : \textbf{Set}\, j \qquad \Gamma \vdash p : a\ \sim\ b \in A \qquad \Gamma, A, 0\ \sim \uparrow a \in \uparrow A \vdash C : \textbf{Set}\, i \qquad \Gamma \vdash t : C\langle a, \textbf{refl} \rangle}{\Gamma \vdash \textbf{J}\, t\, a\, b\, p : C\langle b, p \rangle}$$

$$\textsc{T-Bool}\ \frac{\vdash \Gamma}{\Gamma \vdash \mathbb{B} : \textbf{Set}\, i}$$

$$\textsc{T-True}\ \frac{\vdash \Gamma}{\Gamma \vdash \textbf{true} : \mathbb{B}}$$

$$\textsc{T-False}\ \frac{\vdash \Gamma}{\Gamma \vdash \textbf{false} : \mathbb{B}}$$

$$\textsc{T-If}\ \frac{\Gamma \vdash a : \mathbb{B} \qquad \Gamma \vdash b_0 : A \qquad \Gamma \vdash b_1 : A}{\Gamma \vdash \textbf{if}\ a\ \textbf{then}\ b_0\ \textbf{else}\ b_1 : A}$$

$$\textsc{T-Void}\ \frac{\vdash \Gamma}{\Gamma \vdash \textbf{Void} : \textbf{Set}\, i}$$

4 Semantic Typing

$\boxed{[A]_i^I \searrow S}$ *(Logical Relation)*

$$\textsc{I-Void}\ \frac{}{[\textbf{Void}]_i^I \searrow \emptyset}$$

$$\textsc{I-Bool}\ \frac{}{[\mathbb{B}]_i^I \searrow \{a \mid a \Rightarrow^+ \textbf{true} \vee a \Rightarrow^+ \textbf{false}\}}$$

$$\textsc{I-Eq}\ \frac{}{[a\ \sim\ b \in A]_i^I \searrow \{p \mid p \Rightarrow^+ \textbf{refl}, a \Leftrightarrow b\}}$$

I-Pi

$$\frac{[A]^I_i \searrow S \qquad F \in S \to \mathcal{P}(\mathit{Term}) \qquad \forall a, a \in S \implies [B\langle a\rangle]^I_i \searrow F(a)}{[\Pi\, A\, B]^I_i \searrow \{b \mid \forall a, a \in S \implies b\ a \in F(a)\}}$$

I-Set

$$\frac{j < i}{[\mathbf{Set}\, j]^I_i \searrow I(i)}$$

I-Red

$$\frac{A \Rightarrow B \qquad [B]^I_i \searrow S}{[A]^I_i \searrow I(j)}$$