UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS
APAI4012 High-performance computing: algorithms and applications
Computer programming project

Please select **Two** from the following **Five** problems. Write a report about ways to solve the two problems you selected. You should also submit the scripts. Scripts should be runable independently: after running the code, indicated input and output will be printed in the Command Window. Functions should be detailed and illustrated with 'scripts' showing each feature.
Each problem will have 7.5% credits. These two problems count for 15% credits in the Final Grade.
Due Date: 6 Dec 2024 (Friday.)

1. (Gaussian elimination method)
   (a) Write `function x = gauss(A,b)` that uses Gaussian elimination without pivoting to solve the $n \times n$ system $Ax = b$. Note that the function does not perform the LU decomposition but just applies straightforward row operations. If a pivot element is zero, print an error message and exit.
   (b) Use your function to solve three random systems of size 3×3, 4×4, and 10×10. For example, you can use $A = rand(10, 10)$ to generate matrix A and $b = rand(10, 1)$ to generate vector b, which have uniformly distributed random numbers. You can also try $A = randn(10, 10)$ to generate matrix A and $b = randn(10, 1)$ to generate vector b, which have normal distributed random numbers.

2. (Image compression with low-rank SVD) We know that a $m \times n$ pixels gray image can be expressed by a $m \times n$ matrix. Here we have two gray images. (.jpg format)
   (a) Write a function `[U, S, V] = SVD("xxxx.jpg")` that transform a gray image into a matrix A and do singular value decomposition of A. Here $A = USV^T$.
   (b) Compute the summation of singular values of A, i.e. the trace of S. We do low-rank approximations of A, according to the rate of singular values summation, restore the low-rank approximation matrix of A to a gray image, and compare them with the original image. (The rate of singular values summation means if A has r singular value that $\sigma_1 \geq \sigma_2 \geq ... \geq \sigma_r$, then the rate of k-rank approximation is $(\sum_{i=1}^{k} \sigma_i)/(\sum_{i=1}^{r} \sigma_i)$)

3. (Finite difference method for boundary value problem) This problem takes an approach to the column buckling application. Apply a compressive axial force, or load, P, to the top of a vertical thin elastic column of uniform cross-section having length L. See [Moodle\Related Learning Materials\ Matlab–The Algebraic Eigenvalue Problem\Section 18.1.3.] for more details.
   (a) If the beam is hinged at both ends, we obtain the boundary value problem for deflection $y(x)$,
   $$EI\frac{d^2y}{dx^2} + Py = 0,\ y(0) = 0,\ y(L) = 0,$$
   where E is Young's modulus of elasticity and I is the area moment of inertia of column cross-section. Approach the problem using finite differences with n intervals of length $h = \frac{L}{n}$ and central finite difference approximation. Show that the result is a matrix eigenvalue problem of the form
   $$Ay = \lambda y,$$
   where A is a symmetric tridiagonal matrix. What is the relationship between λ and the critical load?
   (b) For copper, $E = 117 \times 10^9 N/m^2$. Assume $I = .0052m^4$ and that $L = 1.52m$. Using $h = \frac{1.52}{25}$ find the smallest three values of λ, compute critical loads, and graph (x_i, y_i) for each value of λ on the same axes. Relate the results to the discussion in Section 18.1.3.

Note: Suppose that EV is a 26×3 matrix containing the eigenvectors corresponding to λ_1, λ_2, and λ_3 as well as the zero values at the endpoints. Each row has the format $[0\ y_2\ y_3\ \cdots\ y_{24}\ y_{25}\ 0]^T$. To create a good-looking graph place these statements in your program.

```
minval = min(min(EV));
maxval = max(max(EV));
axis equal;
axis([0 L minval maxval]);
hold on;
graph each displacement using different line styles
add a legend
hold off;
```

4. (Finite difference methods for Parabolic equations) Consider the following problem

$$\begin{aligned} u_t &= u_{xx} + u(1-u), \quad (x,t) \in (0,1) \times (0,t_F) \\ u_x(0,t) &= u_x(1,t) = 0, \quad t \in [0,t_F] \\ u(x,0) &= f(x), \quad x \in [0,1] \end{aligned}$$

where $u = u(x,t)$. Note that this problem comes from the population growth model, which is also called Fisher's equation.

(a) Develop an explicit scheme to solve this problem with $f(x) = \sin^2(\pi x)$, $\Delta x = 0.02$, $\Delta t = 0.001$, $t_F = 5$.

(b) What do you observe for $t_F \to \infty$? (Hint: in practice, you cannot approach $t_F = \infty$. But you can compute the solution at $t_F = 10$, $t_F = 20$, $t_F = 40$, etc and observe the behaviors of the solutions.)

5. (Finite difference methods for Hyperbolic equations) Consider the following 2D hyperbolic equation

$$\begin{aligned} u_{tt} = u_{xx} + u_{yy}, &\qquad (x,y,t) \in (0,1) \times (0,1) \times (0,t_F) \\ u(x,y,0) = f(x,y), u_t(x,y,0) = g(x,y), &\qquad (x,y) \in (0,1) \times (0,1) \\ u(0,y,t) = u_L(y,t), u(1,y,t) = u_R(y,t) &\qquad t \in [0,f_F] \\ u(x,0,t) = u_B(x,t), u(x,1,t) = u_T(x,t) &\qquad t \in [0,f_F] \end{aligned}$$

Implement the explicit scheme for solving the above problem with the corresponding initial and boundary conditions such that the analytic solution is

$$u(x,y,t) = \sin(\pi x)\sin(\pi y)\cos(\sqrt{2}\pi t)$$

For example, the initial conditions are $u(x,y,0) = \sin(\pi x)\sin(\pi y)$ and $u_t(x,y,0) = 0$. The boundary conditions can be obtained similarly.

(a) Let $h = 1/10$ and $t_F = 0.5$. Solve the problem with $\Delta t = 0.5h$ and $\Delta t = 0.8h$. Plot the numerical and analytic solutions at t_F. What do you observe?

(b) What happens if you try a smaller mesh size, say $h = 1/20$ and $\Delta t = 0.5h$ and $\Delta t = 0.8h$?