Quiz-2 (E): Design and Analysis of Algorithms (Spring-2024)
Solution

Solve the following recurrence using the three methods discussed in the class.

$T(n) = 4T(n/2) + n^2 \quad (T(1)=1)$

Theorem 4.1 (Master theorem)

Let $a \geq 1$ and $b > 1$ be constants, let $f(n)$ be a function, and let $T(n)$ be defined on the nonnegative integers by the recurrence

$$T(n) = aT(n/b) + f(n) ,$$

where we interpret n/b to mean either $\lfloor n/b \rfloor$ or $\lceil n/b \rceil$. Then $T(n)$ has the following asymptotic bounds:

1. If $f(n) = O(n^{\log_b a - \epsilon})$ for some constant $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$.
2. If $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \lg n)$.
3. If $f(n) = \Omega(n^{\log_b a + \epsilon})$ for some constant $\epsilon > 0$, and if $af(n/b) \leq cf(n)$ for some constant $c < 1$ and all sufficiently large n, then $T(n) = \Theta(f(n))$. ■

Master theorem

$a = 4$, $b = 2$, $f(n)=n^2$

Master theorem is applicable as it follows the general form required for the application of Master Theorem.

$n^{\log_b a} = n^{\log_2 4} = n^2 = f(n)$, $k = 0$, so the 2nd case of Master Theorem applies.

So $T(n)=\Theta(n^2 \lg n)$

Iteration method

$$\begin{aligned}
T(n) &= 4T(n/2) + n^2 \\
&= 4(4T(n/2^2) + (n/2)^2) + n^2 \\
&= 4^2T(n/2^2) + (n/2)^2) + n^2 + n^2 \\
&= 4^2(4T(n/2^3) + (n/4)^2) + 2n^2 \\
&= 4^3T(n/2^3) + n^2 + n^2 + n^2 \\
&= 4^3T(n/2^3) + 3n^2
\end{aligned}$$

… after k steps, we have

$$\begin{aligned}
T(n) &= 4^kT(n/2^k) + kn^2 \\
&= n^2T(1) + n^2\log_2 n && \text{because } 2^k=n \\
&= n^2\log_2 n && \text{because } T(1)=1
\end{aligned}$$

Recursion tree method