

## Important Summation Formulas

1.  $\sum_{i=l}^u 1 = \underbrace{1 + 1 + \cdots + 1}_{u-l+1 \text{ times}} = u - l + 1$  ( $l, u$  are integer limits,  $l \leq u$ );  $\sum_{i=1}^n 1 = n$
2.  $\sum_{i=1}^n i = 1 + 2 + \cdots + n = \frac{n(n+1)}{2} \approx \frac{1}{2}n^2$
3.  $\sum_{i=1}^n i^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6} \approx \frac{1}{3}n^3$
4.  $\sum_{i=1}^n i^k = 1^k + 2^k + \cdots + n^k \approx \frac{1}{k+1}n^{k+1}$
5.  $\sum_{i=0}^n a^i = 1 + a + \cdots + a^n = \frac{a^{n+1} - 1}{a - 1}$  ( $a \neq 1$ );  $\sum_{i=0}^n 2^i = 2^{n+1} - 1$
6.  $\sum_{i=1}^n i2^i = 1 \cdot 2 + 2 \cdot 2^2 + \cdots + n2^n = (n-1)2^{n+1} + 2$
7.  $\sum_{i=1}^n \frac{1}{i} = 1 + \frac{1}{2} + \cdots + \frac{1}{n} \approx \ln n + \gamma$ , where  $\gamma \approx 0.5772 \dots$  (Euler's constant)
8.  $\sum_{i=1}^n \lg i \approx n \lg n$

## Sum Manipulation Rules

## Properties of Logarithms

$$1. \sum_{i=l}^u ca_i = c \sum_{i=l}^u a_i$$

$$2. \sum_{i=l}^u (a_i \pm b_i) = \sum_{i=l}^u a_i \pm \sum_{i=l}^u b_i$$

$$3. \sum_{i=l}^u a_i = \sum_{i=l}^m a_i + \sum_{i=m+1}^u a_i, \text{ where } l \leq m < u$$

$$4. \sum_{i=l}^u (a_i - a_{i-1}) = a_u - a_{l-1}$$

1.  $\log_a 1 = 0$
2.  $\log_a a = 1$
3.  $\log_a x^y = y \log_a x$
4.  $\log_a xy = \log_a x + \log_a y$
5.  $\log_a \frac{x}{y} = \log_a x - \log_a y$
6.  $a^{\log_b x} = x^{\log_b a}$
7.  $\log_a x = \frac{\log_b x}{\log_b a} = \log_a b \log_b x$

## Approximation of a Sum by a Definite Integral

$$\int_{l-1}^u f(x)dx \leq \sum_{i=l}^u f(i) \leq \int_l^{u+1} f(x)dx \quad \text{for a nondecreasing } f(x)$$

$$\int_l^{u+1} f(x)dx \leq \sum_{i=l}^u f(i) \leq \int_{l-1}^u f(x)dx \quad \text{for a nonincreasing } f(x)$$

## Floor and Ceiling Formulas

The *floor* of a real number  $x$ , denoted  $\lfloor x \rfloor$ , is defined as the greatest integer not larger than  $x$  (e.g.,  $\lfloor 3.8 \rfloor = 3$ ,  $\lfloor -3.8 \rfloor = -4$ ,  $\lfloor 3 \rfloor = 3$ ). The *ceiling* of a real number  $x$ , denoted  $\lceil x \rceil$ , is defined as the smallest integer not smaller than  $x$  (e.g.,  $\lceil 3.8 \rceil = 4$ ,  $\lceil -3.8 \rceil = -3$ ,  $\lceil 3 \rceil = 3$ ).

1.  $x - 1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x + 1$
2.  $\lfloor x + n \rfloor = \lfloor x \rfloor + n$  and  $\lceil x + n \rceil = \lceil x \rceil + n$  for real  $x$  and integer  $n$
3.  $\lfloor n/2 \rfloor + \lceil n/2 \rceil = n$
4.  $\lceil \lg(n+1) \rceil = \lfloor \lg n \rfloor + 1$

## Miscellaneous

$$1. n! \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \quad \text{as } n \rightarrow \infty \text{ (Stirling's formula)}$$

2. Modular arithmetic ( $n, m$  are integers,  $p$  is a positive integer)

$$(n+m) \bmod p = (n \bmod p + m \bmod p) \bmod p$$

$$(nm) \bmod p = ((n \bmod p)(m \bmod p)) \bmod p$$

**Theorem 4.1 (Master theorem)**

Let  $a \geq 1$  and  $b > 1$  be constants, let  $f(n)$  be a function, and let  $T(n)$  be defined on the nonnegative integers by the recurrence

$$T(n) = aT(n/b) + f(n),$$

where we interpret  $n/b$  to mean either  $\lfloor n/b \rfloor$  or  $\lceil n/b \rceil$ . Then  $T(n)$  has the following asymptotic bounds:

1. If  $f(n) = O(n^{\log_b a - \epsilon})$  for some constant  $\epsilon > 0$ , then  $T(n) = \Theta(n^{\log_b a})$ .
2. If  $f(n) = \Theta(n^{\log_b a})$ , then  $T(n) = \Theta(n^{\log_b a} \lg n)$ .
3. If  $f(n) = \Omega(n^{\log_b a + \epsilon})$  for some constant  $\epsilon > 0$ , and if  $af(n/b) \leq cf(n)$  for some constant  $c < 1$  and all sufficiently large  $n$ , then  $T(n) = \Theta(f(n))$ . ■