THE DIFFERENTIAL GEOMETRY DIFFERENTIAL TOPOLOGY DUMP

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Contents

Abstract. Everything about Differential Geometry, Differential Topology

Part 1.

1. Inverse Function Theorem

Shastri (2011) had a thorough and lucid and explicit explanation of the Inverse Function Theorem [4]. I will recap it here. The following is also a blend of Wienhard's Handout 4 https://web.math.princeton.edu/~wienhard/teaching/M327/handout4.pdf

Definition 1. *Let (X, a) metric space.*
contraction $\phi : X \to X$ *if* $\exists$ *constant* $0 < c < 1$ *s.t.* $\forall\, x, y \in X$

$$d(\phi(x), \phi(y)) \leq cd(x, y)$$

Theorem 1 (Contraction Mapping Principle)**.** *Let (X, d) complete metric space.*
Then $\forall$ *contraction* $\phi : X \to X$, $\exists ! y \in X$ *s.t.* $\phi(y) = y$, y fixed pt.

Proof. Recall def. of complete metric space X, X metric space s.t. $\forall$ Cauchy sequence in X is convergent in X (i.e. has limit in X).

$$\forall\, x_0 \in X, \text{ Define } \begin{aligned} x_1 &= \phi(x_0) \\ x_2 &= \phi(x_1) \\ &\vdots \\ x_j &= \phi(x_{j-1}) \\ &\vdots \\ x_n &= \phi(x_{n-1}) \end{aligned}$$

$$d(x_{n+1}, x_n) = d(\phi(x_n), \phi(x_{n-1})) \leq cd(x_n, x_{n-1}) \leq \dots \leq c^n d(x_1, x_0)$$

for some $0 < c < 1$.

$$d(x_m, x_n) \leq d(x_n, x_{n-1}) + d(x_{n-1}, x_m) \leq d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + \dots + d(x_{m+1}, x_m) \leq \sum_{k=n-1}^m c^k d(x_1, x_0)$$

Thus, $\forall\, \epsilon > 0$, $\exists\, n_0 > 0$, (n_0 large enough) s.t. $\forall\, m, n \in \mathbb{N}$ s.t. $n_0 < n < m$,

$$d(x_m, x_n) \leq \sum_{k=n-1}^m c^k d(x_1, x_0) < \epsilon d(x_1, x_0)$$

Thus, $\{x_n\}$ Cauchy sequence. Since X complete, $\exists$ limit pt. $y \in X$ of $\{x_n\}$.

$$\phi(y) = \phi(\lim_n x_n) = \lim_n \phi(x_n) = \lim_n x_{n+1} = y$$

Since by def. of y limit pt. of $\{x_n\}$, $\forall\, \epsilon > 0$, then $\{n | |x_n - y| \leq \epsilon,\, n \in \mathbb{N}\}$ is infinite.

Consider $\delta > \mathbb{N}$. Consider $\{n | |x_n - y| \leq \delta, n \in \mathbb{N}\}$

$\exists\, N_\delta \in \mathbb{N}$ s.t. $\forall\, n > N_\delta$, $|x_n - y| < \delta$; otherwise, $\forall\, N_\delta$, $\exists\, n > N_\delta$ s.t. $|x_n - y| \geq \delta$. Then $\{n | |x_n - y| \leq \delta, n \in \mathbb{N}\}$ finite. Contradiction.

ϕ cont. so by def. $\forall\, \epsilon > 0$, $\exists\, \delta > 0$ s.t. if $|x_n - y| < \delta$, then $|\phi(x_n) - \phi(y)| < \epsilon$.

Pick N_δ s.t. $\forall\, n > N_\delta$, $|x_n - y| < \delta$, and so $|\phi(x_n) - \phi(y)| < \epsilon$. There are infinitely many $\phi(x_n)$'s that satisfy this, and so $\phi(y)$ is a limit pt.

If $\exists\, y_1, y_2 \in X$ s.t. $\begin{aligned} \phi(y_1) &= y_1 \\ \phi(y_2) &= y_2 \end{aligned}$, then

$$d(y_1, y_2) = d(\phi(y_1), \phi(y_2)) \leq cd(y_1, y_2) \text{ with } c < 1$$

so $c = 1$ □

Theorem 2 (Inverse Function Theorem)**.** *Suppose open $U \subset \mathbb{R}^n$, let C^1 $f : U \to \mathbb{R}^n$, $x_0 \in U$ s.t. $Df(x_0)$ invertible.*
Then $\exists$ open neighborhoods $V \ni x_0$, $W \ni f(x_0)$ s.t. $V \subseteq U$ and $W \subseteq \mathbb{R}^n$, respectively, and s.t.

(i) *$f : V \to W$ bijection*
(ii) *$g = f^{-1} : V \to U$ differentiable, i.e. $g = f^{-1} : W \to V$ is C^1*

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(iii) $D(f^{-1})$ *cont. on* W.
(iv) $Dg(y) = (Df(g(y)))^{-1} \quad \forall\, y \in W$

Also, notice that $f(g(y)) = y \,\forall\, y \in W$.

Proof. Consider $\widetilde{f}(x) = (Df(x_0))^{-1}(f(x+x_0) - f(x_0))$. Then
$\widetilde{f}(0) = 0$ and

$$D\widetilde{f} = (Df(x_0))^{-1}(Df(x+x_0) - 0)$$
$$D\widetilde{f}(0) = (Df(x_0))^{-1}Df(x_0) = 1$$

So let $\widetilde{f} \to f$ (notation) and so assume, without loss of generality, that $U \ni 0$, $f(0) = 0$, $Df(0) = 1$

Choose $0 < \epsilon \leq \frac{1}{2}$. Let $0 < \delta < 1$ s.t. open ball $V = B_\delta(0) \subseteq U$, and $\|Df(x) - 1\| < \epsilon$. $\forall\, x \in U$, since Df cont. at 0.

Let $W = f(V)$.

$\forall\, y \in W$, define $\phi_y : V \to \mathbb{R}^n$
$$\phi_y(x) = x + (y - f(x))$$
$$D(\phi_y)(x) = 1 + -Df(x) \quad \forall\, x \in V$$
$$\|D(\phi_y)(x)\| = \|1 - Df(x)\| \leq \epsilon < 1$$

$\forall\, x_1, x_2 \in V$, by mean value Thm. (not the equality that is only valid in 1-dim., but the inequality, that's valid for $\mathbb{R}^d$,
$$\|\phi_y(x_1) - \phi_y(x_2)\| \leq \|D(\phi_y)(x')\| \|x_1 - x_2\|$$
for some $x' = cx_2 + (1-c)x_1$, $c \in [0,1]$. V only needed to be convex set.
$$\Longrightarrow \|\phi_y(x_1) - \phi_y(x_2)\| \leq \epsilon \|x_1 - x_2\|$$

Then ϕ_y contraction mapping.

Suppose $f(x_1) = f(x_2) = y$, $x_1, x_2 \in V$.
$$\phi_y(x_1) = x_1$$
$$\phi_y(x_2) = x_2$$
$$\|\phi_y(x_1) - \phi_y(x_2)\| = \|x_1 - x_2\| \leq \epsilon \|x_1 - x_2\| \quad \forall\, \epsilon > 0 \Longrightarrow x_1 = x_2$$

$\Longrightarrow f|_U$ injective.

$W = f(V)$, so $f : V \to W$ surjective. f bijective.

Fix $y_0 \in W$, $y_0 = f(x_0)$, $x_0 \in V$.

Let $r > 0$ s.t. $B_r(x_0) \subset V$.

Consider $B_{r\epsilon}(y_0)$. If $y \in B_{r\epsilon}(y_0)$.
$$r\epsilon > \|y - y_0\| = \|y - f(x_0)\| = \|\phi_y(x_0) - x_0\| \text{ with}$$
$$\phi_y(x) = x + (y - f(x))$$

If $x \in B_r(x_0)$,
$$\|\phi_y(x) - x_0\| \leq \|\phi_y(x) - \phi_y(x_0)\| + \|\phi_y(x_0) - x_0\| \leq \epsilon\|x - x_0\| + r\epsilon < 2r\epsilon = r$$

Thus $\phi(B_r(x_0)) = B_r(x_0)$.

By contraction mapping principle, $\exists\, a \in B_r(x_0)$, s.t. $\phi_y(a) = a$. Then $\phi_y(a) = a + (y - f(a)) = a \Longrightarrow f(a) = y$.

$y \in f(V) = W$.

So $B_{r\epsilon}(y_0) \subset W$. W open.

Let $\text{Mat}(n,n) \equiv$ space of all $n \times n$ matrices; $\text{Mat}(n,n) = \mathbb{R}^{n^2}$.

□

There is a proof of the implicit function theorem and its various forms in Shastri (2011) [4], but I found Wienhard's Handout 4 for Math 327 to be clearer.[1]

[1]https://web.math.princeton.edu/~wienhard/teaching/M327/handout4.pdf

Theorem 3 (Implicit Function Theorem)**.** *Let open* $U \subset \mathbb{R}^{m+n} \equiv \mathbb{R}^m \times \mathbb{R}^n$
C^1 $f : U \to \mathbb{R}^n$
$(a,b) \in U$ *s.t.* $f(a,b) = 0$ *and* $D_y f|_{(a,b)}$ *invertible.*
Then $\exists$ *open* $V \ni (a,b)$, $V \subset U$
$\exists$ *open neighborhood* $W \ni a$, $W \subseteq \mathbb{R}^m$
$\exists!$ C^1 $g : W \to \mathbb{R}^n$ *s.t.*
$$\{(x,y) \in V | f(x,y) = 0\} = \{(x, g(x)) | x \in W\}$$
Moreover,
$$dg_x = -\left(d_y f\right)^{-1}\big|_{(x,g(x))} d_x f|_{(x,g(x))}$$
and g *smooth if* f.

Proof. Define $F : U \to \mathbb{R}^{m+n}$
$$F(x,y) = (x, f(x,y))$$
Then $F(a,b) = (a,0)$ (given), and
$$DF = \begin{bmatrix} 1 & \\ \frac{\partial f^i(x,y)}{\partial x^j} & \frac{\partial f^i(x,y)}{\partial y^j} \end{bmatrix} \equiv \begin{bmatrix} 1 & \\ D_x f & D_y f \end{bmatrix}$$
$DF(a,b)$ invertible.

By inverse function theorem, since $DF(a,b)$ invertible at pt. (a,b),

$\exists$ open neighborhoods $V \ni (a,b) \subseteq \mathbb{R}^m \times \mathbb{R}^n$ s.t. F diffeomorphism with $F^{-1} : \widetilde{W} \to V$.
$\widetilde{W} \ni (a,0) \subseteq \mathbb{R}^m \times \mathbb{R}^n$

Set $W = \{x \in \mathbb{R}^m | (x,0) \in \widetilde{W}\}$. Then $\pi_1(\widetilde{W}) = W$ open in $\mathbb{R}^m$.

Define $g : W \to \mathbb{R}^n$,
$$g(x) = \pi_2 \circ F^{-1}(x,0) \text{ or}$$
$$F^{-1}(x,0) = (h(x), g(x))$$

Now $FF^{-1}(x,0) = (x,0) = (h(x), f(h(x), g(x)))$ so $h(x) = x \,\forall\, x \in W$, $0 = f(x, g(x))$.

Then
$$\{(x,y) \in V | f(x,y) = 0\} = \{(x,y) \in V | F(x,y) = (x,0)\} = \{(x, g(x)) | x \in W, 0 = f(x, g(x))\}$$

Since π smooth and F^{-1} is C^1, g is C^1.

To reiterate, $f(x, g(x)) = 0$ on W.

Using chain rule while differentiating $f(x, g(x)) = 0$,
$$\partial_{x^j} f(x, g(x)) = \frac{\partial f(x,g(x))}{\partial x^k}\frac{\partial x^k}{\partial x^j} + \frac{\partial f(x,g(x))}{\partial y^k}\frac{\partial g^k(x)}{\partial x^j} = D_x f|_{(x,g(x))} + (D_y f)|_{(x,g(x))} \cdot (Dg)_x = 0 \text{ or}$$
$$(Dg)_x = -\left(D_y f\right)|_{x,g(x)} D_x f|_{(x,g(x))}$$

□

Definition 2. *smooth* $f : M \to N$, *s.t.* $Df(p) : T_pM \to T_{f(p)}N$ *injective. Then* f *immersion at* p.

Shastri (2011) has this as the "Injective Form of Implicit Function Theorem", Thm. 1.4.5, pp. 23 and Guillemin and Pollack (2010) has this as the "Local Immersion Theorem" on pp. 15, Section 3 "The Inverse Function Theorem and Immersions" [3].

Theorem 4 (Local immersion Theorem i.e. Injective Form of Implicit Function Theorem)**.** *Suppose* $f : M \to N$ *immersion at* p, $q = f(p)$.
Then $\exists$ *local coordinates around* p, q, x, y, *respectively s.t.* $f(x_1 \dots x_m) = (x_1 \dots x_m, 0 \dots 0)$.

Proof. Choose local parametrizations

$$\begin{tikzcd} U \subseteq M \arrow[r, "f"] \arrow[d, "\phi"] & N \supseteq V \arrow[d, "\psi"] \\ \phi(U) \arrow[r, "f"] & \psi(V) \end{tikzcd} \qquad \begin{aligned} \phi(p) &= x \\ \psi(q) &= y \end{aligned}$$

$D(\psi f \varphi^{-1}) \equiv Df$. $Df(p)$ injective (given f immersion). $Df(p) \in \text{Mat}(n,m)$

By change of basis in $\mathbb{R}^n$, assume $Df(p) = \begin{pmatrix} I_m \\ 0 \end{pmatrix}$.

Now define $G : \phi(U) \times \mathbb{R}^{n-m} \to \mathbb{R}^n$

$G(x,z) = f(x) + (0,z)$

Thus, $DG(x,z) = 1$ and for open $\phi(U) \times U_2$, $G(\phi(U) \times U_2)$ open.
By inverse function theorem, G local diffeomorphism of $\mathbb{R}^n$, at 0.
Now $f = G \circ \mathfrak{i}$, where $\mathfrak{i}$ is canonical immersion.

$$G(x,0) = f(x)$$
$$\implies G^{-1}G(x,0) = (x,0) = G^{-1}f(x)$$

Use $\psi \circ G$ as the local parametrization of N around pt. q. Shrink U, V so that

$$\begin{tikzcd} U \subseteq M \arrow[r, "f"] \arrow[d, "\phi"] & N \supseteq V \arrow[d, "\psi \circ G"] \\ \phi(U) \arrow[r, "\mathfrak{i}"] & \psi \circ G(V) \end{tikzcd}$$

□

Theorem 5 ((Implicit Function Thm.)). *Let open subset $U \subseteq \mathbb{R}^n \times \mathbb{R}^d$, $(x,y) = (x^1 \dots x^n, y^1 \dots y^k)$ on U. Suppose smooth $\Phi : U \to \mathbb{R}^k$, $(a,b) \in U$, $c = \Phi(a,b)$*

If $k \times k$ matrix $\frac{\partial \Phi^i}{\partial y^j}(a,b)$ nonsingular, then $\exists$ neighborhoods $\begin{aligned} V_0 \subseteq \mathbb{R}^n &\text{ of } a \\ W_0 \subseteq \mathbb{R}^k &\text{ of } b \end{aligned}$ and smooth $F : V_0 \to W_0$ s.t.

$\Phi^{-1}(c) \bigcap (V_0 \times W_0)$ is graph of F, i.e.
$\Phi(x,y) = c$ for $(x,y) \in V_0 \times W_0$ iff $y = F(x)$.

Jeffrey Lee (2009) [1]
John Lee (2012) [2]

Part 2. Prástaro

Prástaro (1996) [6]

1.0.1. *Affine Spaces.* cf. Sec. 1.2 - *Affine Spaces* of Prástaro (1996) [6]

Definition 3 (affine space)**.**

$$\begin{aligned} &\textit{affine space} \qquad (M, \mathbf{M}, \alpha) \\ &\qquad \textit{with} \\ &M \equiv \textit{ set (set of pts.)} \\ &\mathbf{M} \equiv \textit{ vector space (space of free vectors)} \\ &\alpha \equiv \mathbf{M} \times M \to M \equiv \textit{ translation operator} \\ &\alpha : (v,p) \mapsto p' \equiv p + v \end{aligned} \tag{1}$$

Note: α is a ***transitive*** *action and without fixed pts. (free).*
i.e. $\forall\, p \in M$,

$\forall$ pt. $O \in M$, $\alpha : (v,O) \mapsto O' \equiv O + v$, $\alpha(\cdot, O) \equiv \alpha_O \equiv \alpha(O)$. $\alpha_O(v) = O' = O + \mathbf{v}$ $\quad \forall O' \in M, \exists\, \mathbf{v} \in \mathbf{M}$ s.t. $O' = O + \mathbf{v}$
$\implies M \equiv \mathbf{M}$.
$\forall (O, \{e_i\})_{1 \leq i \leq n}$, where $\{e_i\}$ basis of $\mathbf{M}$, $M \equiv \mathbf{M} = \mathbb{R}^n$ so isomorphism $M \simeq \mathbb{R}^n$

Definition 4. $(O, \{e_i\}) \equiv$ *affine frame.*
$\forall$ *affine frame $(O, \{e_i\})$, $\exists$ coordinate system $x^\alpha : M \to \mathbb{R}$,*
where $x^\alpha(p)$ is αth component, in basis $\{e_i\}$, of vector $p - O$

Theorem 6 (1.4 Prástaro (1996) [6])**.** *Let $(x^\alpha), (\overline{a}^\alpha)$ 2 coordinate systems correspond to affine frames $(O, \{e_i\})$, $(\overline{O}, \{\overline{e}_i\})$, respectively.*

$$\overline{x}^\alpha = A^\alpha_{\ \beta} x^\beta + y^\alpha \tag{2}$$

where

$$y^\alpha \in \mathbb{R}^n, \qquad A^\alpha_{\ \beta} \in GL(n;\mathbb{R})$$

Definition 5 (1.10 Prástaro (1996) [6])**.**

$$A(n) \equiv Gl(n, \mathbb{R}) \times \mathbb{R}^n \tag{3}$$

affine group of dim. n

Theorem 7 (1.5)**.** *symmetry group of n-dim. affine space, called affine group $A(M)$ of M. $\exists$ isomoprhism,*

$$A(M) \simeq A(n), \qquad f \mapsto (f^\alpha_{\ \beta}, y^\alpha)\,; \qquad f^\alpha \equiv x^\alpha \circ f = f^\alpha_{\ \beta} x^\beta + y^\alpha \tag{4}$$

cf. Eq. 1.4 Prástaro (1996) [6]

Part 3. Complex Manifolds

EY : 20170123 I don't see many good books on Complex Manifolds for physicists other than Nakahara's. I will supplement this section on Complex Manifolds with external links to the notes of other courses that I found useful to myself.
Complex Manifolds - Lecture Notes Koppensteiner (2010) [7]
Lectures on Riemannian Geometry, Part II: Complex Manifolds by Stefan Vandoren
Vandoren (2008) [8]

Part 4. Morse Theory

2. Morse Theory introduction from a physicist

I needed some physical motivation to understand Morse theory, and so I looked at Hori, et. al. [9].

cf. pp. 43, Sec. 3.4 Morse Theory, from Ch. 3. Differential and Algebraic Topology of Hori, et. al. [9].

Consider smooth $f : M \to \mathbb{R}$, with non-degenerate critical points.

If no critical values of f between a and b ($a < b$), then subspace on which f takes values less than a is deformation retract of subspace where f less than b, i.e.

$$\{x \in M | f(x) < b\} \times [0,1] \xrightarrow{F} \{x \in M | f(x) < b\}$$

$\forall\, x \in M$ s.t. $f(x) < b$,

$$\begin{aligned} & F(x,0) = x \\ & F(x,1) \in \{x \in M | f(x) < a\} \end{aligned} \qquad \text{and } F(a',1) = a' \qquad \forall\, a' \in M \text{ s.t. } f(a') < a$$

To show this, consider $-\nabla f / |\nabla f|^2$

Morse lemma: $\forall$ critical pt. p s.t. $\exists$ choice of coordinates s.t.

$$f = -(x_1^2 + x_2^2 + \dots + x_\mu^2) + x_{\mu+1}^2 + \dots + x_n^2 \qquad (5)$$

where $f(p) = 0$ and p is at origin of these coordinates.

- difference between
$$f^{-1}(\{x \leq -\epsilon\}),\ f^{-1}(\{x \leq +\epsilon\})$$
can be determined by local analysis and only depends on μ, $\mu \equiv$ "Morse index" = number of negative eigenvalues of Hessian of f at critical pt.

  Answer:

  $f^{-1}(\{x \leq +\epsilon\})$ can be obtained from $f^{-1}(\{x \leq -\epsilon\})$ by "attaching μ-cell" along boundary $f^{-1}(0)$
- "attaching μ-cell to X mean, take μ-ball $B_\mu = \{|x| \leq 1\}$ in μ-dim. space, identity pts. on boundary $S^{\mu-1}$ with pts. in the space X, through cont. $f : S^{\mu-1} \to X$, i.e. take
$$X \coprod B_\mu$$
with $x \sim f(x) \quad \forall\, x \in \partial B_\mu = S^{\mu-1}$.
- find homology of M,

  f defines chain complex C_f^*, kth graded piece C^{α_k}, α_k is number of critical pts. with index k.
$$\begin{aligned} & \partial : C_p^k \to C_p^{k-1} \\ & \partial x_a = \sum_b \Delta_{a,b} x_b \end{aligned} \qquad (6)$$
where $\Delta_{a,b} :=$ signed number of lines of gradient flow from x_a to x_b, b labels pts. of index $k-1$.

  Gradient flow line is path $x(t)$ s.t. $\dot{x} = \nabla(f)$, with $\begin{aligned} x(-\infty) &= x_a \\ x(+\infty) &= x_b \end{aligned}$
- To define this number ($\Delta_{a,b}$?), construct moduli space of such lines of flow (???) by intersecting outward and inward flowing path spaces from each critical point, and then show this moduli space is oriented, 0-dim. manifold (pts. with signs)
- $\partial^2 = 0$ proof

  ∂, boundary of space of paths connecting critical points, whose index differs by 2 = union over compositions of paths between critical pts. whose index differs by 1.

  $\Longrightarrow$ coefficients of ∂^2 are sums of signs of pts. in 0-dim. space, which is boundary of 1-dim. space.

  These signs must therefore add to 0, so $\partial^2 = 0$.

Hori, et. al. [9] is good for physics, but there isn't much thorough, step-by-step explanations of the math. I will look at Hirsch (1997) [5] and Shastri (2011) [4] at the same time.

2.1. Introduction, definitions of Morse Functions, for Morse Theory.

cf. Ch. 6, Morse Theory of Hirsch (1997) [5], Section 1. Morse Functions, pp. 143-

Recall for TM, $T_xM \xrightarrow{\varphi} \mathbb{R}^n$.

Cotangent bundle T^*M defined likewise:

$$T_x^*M \xrightarrow{\varphi} \text{ dual vector space } (\mathbb{R}^n)^* = L(\mathbb{R}^n, \mathbb{R})$$

i.e.

$$T^*M = \bigcup_{x \in M} (M_x^*) \qquad M_x^* = L(M_x, \mathbb{R})$$

If chart (φ, U) on M, natural chart on T^*M is

$$\begin{aligned} & T^*U \to \varphi(U) \times (\mathbb{R}^n)^* \\ & \lambda \in M_x^* \mapsto (\varphi(x), \lambda \varphi_x^{-1}) \end{aligned}$$

Projection map

$$\begin{aligned} & p : T^* \to M \\ & M_x^* \mapsto x \end{aligned}$$

Let C^{r+1} map, $1 \leq r \leq \omega$, $f : M \to \mathbb{R}$, $\forall\, x \in M$, linear map $T_xf : M_x \to \mathbb{R}$ belongs to M_x^*

$$T_xf = Df_x \in M_x^*$$

Then

$$\begin{aligned} & Df : M \to T^*M \\ & x \mapsto Df_x = Df(x) \end{aligned}$$

is C^r section of T^*M.

Definition 6. ***critical point*** *x of f is zero of Df, i.e.*

$$Df(x) = 0 \qquad (7)$$

of vector space M_x^*.

Thus, set of critical pts. of f is counter-image of submanifold $Z^* \subset T^*M$ of zeros.

Note $Z^* \approx M$, codim. of Z^* is $n = \dim M$.

Definition 7. ***Morse function*** *f if $\forall$ critical pts. of f are nondegenerate.*

Note set of critical pts. closed discrete subset of M.

Let open $U \subset \mathbb{R}^n$, let C^2 map $g : U \to \mathbb{R}$,

critical pt. $p \in U$ nondegenerate iff

- linear $D(Dg)(p) : \mathbb{R}^n \to (\mathbb{R}^n)^*$ bijective
- identify $L(\mathbb{R}^n, (\mathbb{R}^n)^*)$ with space of bilinear maps $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $\Longrightarrow$ equivalent to condition that symmetric bilinear $D^2g(p) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ non-degenerate
- *$n \times n$ Hessian matrix*
$$\left[\frac{\partial^2 g}{\partial x^i \partial x^j}(p) \right]$$
has rank n

Hessian of g at critical pt. p is quadratic form H_pf associated to bilinear form $D^2g(p)$

$$\Longrightarrow H_pf(y) = D^2g(p)(y,y) = \sum_{i,j} \frac{\partial^2 g}{\partial x^i \partial x^j}(p) y^i y^j$$

Let open $V \subset \mathbb{R}^n$, suppose C^2 diffeomorphism $h : V \to U$.
Let $q = h^{-1}(p)$, so q is critical pt. of $gh : V \to \mathbb{R}$.

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "H_q(gh)"] \arrow[d, "Dh(q)"] & \mathbb{R} \\ \mathbb{R}^n \arrow[ur, "H_pg"] & \end{tikzcd}$$

(quadratic) form (H_pf) invariant under diffeomorphisms.
Let C^2 $f : M \to \mathbb{R}$.
$\forall$ critical pt. x of f, define
Hessian quadratic form

$$H_xf : M_x \to \mathbb{R}$$

$$H_xf : M_x \xrightarrow{D\varphi_x} \mathbb{R}^n \xrightarrow{H_{\varphi(x)}(f\varphi^{-1})} \mathbb{R}$$

where φ is any chart at x.
Thus, critical pt. of a C^2 real-valued function nondegenerate iff associated Hessian quadratic form is nondegenerate.
Let Q nondegenerate quadratic form on vector space E.
Q negative definite on subspace $F \subset E$ if $Q(x) < 0$ whenever $x \in F$ nonzero.
Index of $Q \equiv \text{Ind}Q$, is largest possible dim. of subspace on which Q is negative definite.
cf. 1.1. Morse's Lemma of Ch. 6, pp. 145, Morse Theory of Hirsch (1997) [5]

Lemma 1 (Morse's Lemma). *Let $p \in M$ be nondegenerate critical pt. of index k of C^{r+2} map $f : M \to \mathbb{R}$, $1 \leq r \leq \omega$.*
Then $\exists\, C^r$ chart (φ, U) at p s.t.

(8)
$$f\varphi^{-1}(u_1 \dots u_n) = f(p) - \sum_{i=1}^k u_i^2 + \sum_{i=k+1}^n u_i^2$$

Let ${}^TQ \equiv Q^T$ denote tranpose of matrix Q.

Lemma 2. *Let $A = diag\{a_1, \dots, a_n\}$ diagonal $n \times n$ matrix, with diagonal entries ± 1.*
Then $\exists$ neighborhood N of A in vector space of symmetric $n \times n$ matrices, C^∞ map

(9)
$$P : N \to GL(n, \mathbb{R})$$

s.t. $P(A) = I$, and if $P(B) = Q$, then $Q^TBQ = A$

Proof. Let $B = [b_{ij}]$ be symmetri matrix near A s.t. $b11 \neq 0$ and b_{11} has same sign as a_1.
Consider $x = Ty$ where

$$x_1 = \left[y_1 - \frac{b_{12}}{b_{11}} y_2 - \dots - \frac{b_{1n}}{b_{11}} y_n \right] / \sqrt{|b_n|}$$
$$x_k = y_k \text{ for } k = 2, \dots n$$

$\square$

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