ELECTROMAGNETISM, ELECTRODYNAMICS DUMP; ELECTRICITY AND MAGNETISM DUMP (INCLUDES NOTES AND SOLUTIONS TO PURCELL'S ELECTRICITY AND MAGNETISM

ERNEST YEUNG ERNESTYALUMNI@GMAIL.COM

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gmail : ernestyalumni
linkedin : ernestyalumni
twitter : ernestyalumni

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Abstract. Electricity and Magnetism notes "dump" - Everything about or involving electricity and magnetism, electrodynamics.

Part 1. Math

1. Codifferential, The "Vector Potential" and Laplacian

Keywords: codifferential, vector potential, Laplacian

1.1. Codifferential δ. For smooth manifold M, $\text{dim}M = n$,

$$\boxed{\begin{aligned} &\delta : \Omega^k(M) \to \Omega^{k-1}(M) \\ &\delta = (-1)^{n(k+1)+1} * d* \end{aligned}} \tag{1}$$

For $k = 1, 2$ cases,

$$\begin{aligned} \delta &= (-1)^{n(1+1)+1} * d* = (-1) * d* \\ \delta &= (-1)^{n(2+1)+1} * d* = (-1)^{3n+1} * d* \end{aligned}$$

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For $n = d = 3$,

$$\begin{aligned} &\delta : \Omega^2(M) \to \Omega^1(M) \\ &\delta = *\mathbf{d}* \end{aligned}$$

1.2. the "Vector Potential". If $B = \mathbf{d}A$, $B \in \Omega^2(M)$,

$$\begin{aligned} B = B_{jk} dx^j \wedge dx^k &= \mathbf{d}A = \frac{\partial A_k}{\partial x^j} dx^j \wedge dx^k \\ B_{jk} &= \frac{\partial A_k}{\partial x^j} \end{aligned}$$

So these statements are equivalent:

$$\boxed{B = \mathbf{d}A \Longleftrightarrow \mathbf{B} = \text{curl}\mathbf{A}} \tag{2}$$

Indeed, recall the *deRham cohomology*:

$$H^k_{\text{deRham}}(M) = Z^k(M)/\text{im}d = \frac{Z^k(M)}{d\Omega^{k-1}(M)}$$

And so this form for $B = \mathbf{d}A$ presupposes that

$$B \in [1] = [\mathbf{d}A] \in H^2_{\text{deRham}}(M')$$

But it should be noted on another manifold, this form may not hold; indeed, consider a submanifold or domain $M' \subseteq M$. In this case

$$H^2_{\text{deRham}}(M') \ni B$$

1.3. Laplacian.

Definition 1 (Laplacian)**.**

$$\begin{aligned} &\Delta : \Omega^k(M) \to \Omega^k(M) \\ &\Delta = d\delta + \delta d \end{aligned} \tag{3}$$

Both recognizing the equivalence between these 2 formulations:

$$*\mathbf{d}*B = \delta B \Longleftrightarrow \text{curl}\mathbf{B}$$

and acknowledging that it *has* to be the case that $*\mathbf{d}*B$ is the correct expression, coming from $d * F$ for the electromagnetic 2-form F,

$$\delta B = \delta \mathbf{d}A = \Delta A - \mathbf{d}\delta A$$

Then

$$*\mathbf{d}A = \frac{\sqrt{|\mathbf{g}|}}{(d-2)!}\epsilon_{i_1 i_2 \dots i_{d-2} jk} g^{jj'} g^{kk'} \frac{\partial A_{k'}}{dx^{j'}} dx^{i_1} \wedge \dots \wedge dx^{i_{d-2}} = \sqrt{|\mathbf{g}|}\epsilon_{ijk} g^{jj'} g^{kk'} \frac{\partial A_{k'}}{\partial x^{j'}} dx^i$$

Consider the expression

$$\text{curl}\mathbf{B} = \text{curl}(\text{curl}\mathbf{A})$$

I will generalize it and point out its misgivings.

Calculating from definitions for $*$ and d,

$$\mathbf{d}*\mathbf{d}A = \frac{\epsilon_{i_1 i_2 \dots i_{d-2} jk}}{(d-2)!} \frac{\partial}{\partial x^l} \left(\sqrt{|\mathbf{g}|} g^{jj'} g^{kk'} \frac{\partial A_{k'}}{\partial x^{j'}} \right) dx^l \wedge dx^{i_1} \wedge \dots \wedge dx^{i_{d-2}}$$

$$\begin{gathered} *\mathbf{d}*\mathbf{d}A = \frac{\sqrt{|\mathbf{g}|}}{(d-d(-1))!} \frac{\epsilon_{i d' i_1' i_2' \dots i_{d-2}'}}{(d-2)!} g^{l'l} g^{i_1' i_1} g^{i_2' i_2} \dots g^{i_{d-2}' i_{d-2}} \frac{\partial}{\partial x^l} \left(\sqrt{|\mathbf{g}|} g^{jj'} g^{kk'} \frac{\partial A_{k'}}{\partial x^{j'}} \right) \epsilon_{i_1 i_2 \dots i_{d-2} jk} dx^i = \\ \xrightarrow{d=3} \sqrt{|\mathbf{g}|} \epsilon_{i l' m'} g^{l'l} g^{m'm} \frac{\partial}{\partial x^l} \left(\sqrt{|\mathbf{g}|} g^{jj'} g^{kk'} \frac{\partial A_{k'}}{\partial x^{j'}} \right) \epsilon_{mjk} dx^i \end{gathered}$$

In $\mathbb{R}^3$,

$$*\mathbf{d}*\mathbf{d}A \xrightarrow{\mathbb{R}^3} \frac{\partial}{\partial x^j}\left(\frac{\partial A_k}{\partial x^j} \right) - \frac{\partial}{\partial x^k}\left(\frac{\partial A_k}{\partial x^j} \right)$$

At this point, in the "vector calculus" formulation, the partial derivatives in the $-\frac{\partial}{\partial x^k}\left(\frac{\partial A_k}{\partial x^j}\right)$ would be exchanged in order, and then a so-called "choice of gauge" for $\nabla \cdot A \equiv \text{div} A$ would be "made," making this term equal 0. As we clearly see above, this should not be the case. Rather, this choice should be made:

$$\mathbf{d}\delta A = 0 \tag{4}$$

This is because we should directly use the "manifestly covariant" definition of the Laplacian:

$$\begin{aligned} *\mathbf{d} * \mathbf{d}A = (-1)^{d(1-1)+1}\delta\mathbf{d}A = (-1)^{0+1}\delta\mathbf{d}A = (-1)^1(\Delta - \mathbf{d}\delta)A \\ \xrightarrow{d=3} (-1)(\Delta - \mathbf{d}\delta)A \end{aligned} \tag{5}$$

Note that $k=1$, i.e. we're dealing with 1-form A here, in the $(-1)^{d(k-1)+1}$ factor.

So the true expression is this (to reiterate and emphasize the point):

$$\delta B = (-1)(\Delta - \mathbf{d}\delta)A \tag{6}$$

Are there any necessary constraints on A to make it, such that $\mathbf{d}\delta A = 0$? Perhaps we can take a look at how the definition of the codifferential δ necessitates that this diagram commutes:

$$\begin{tikzcd} \Omega^1(M) \arrow[r, "*"] \arrow[d, "\delta"] & \Omega^{d-1}(M) \arrow[d, "\mathbf{d}"] \\ \Omega^0(M) \arrow[r, "*(-1)"] & \Omega^d(M) \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, mapsto, "*"] \arrow[d, mapsto, "\delta"] & *A \arrow[d, mapsto, "\mathbf{d}"] \\ \delta A \arrow[r, mapsto, "*(-1)"] & \mathbf{d}*A \end{tikzcd}$$

Part 2. Maxwell's Equations; My version of Maxwell's Equations

2. My version of Maxwell's equations

2.1. Maxwell's Equations, my version, in "vector calculus" form.

If $\nabla \cdot \mathbf{B} = 0$, then

$$\nabla \times \mathbf{E} = \frac{-1}{c}\left(\frac{\partial \mathbf{B}}{\partial t} \right) \tag{7}$$

If $\nabla \cdot \mathbf{E} = 4\pi\rho_{\text{total}}$, then

$$\begin{aligned} \nabla \times \mathbf{B} = \frac{1}{c}\Big(\frac{\partial \mathbf{E}}{\partial t} + 4\pi \frac{\partial \mathbf{P}}{\partial t} + \\ + 4\pi \mathbf{J}_{\text{free}} + 4\pi c \nabla \times \mathbf{M} \Big) \end{aligned} \tag{8}$$

2.2. Maxwell's Equations, my version, over spacetime manifold M.

For spacetime manifold M, of dimensions $\dim M = d+1$, and for

$$\begin{aligned} E \in \Omega^1(M) \\ B \in \Omega^2(M) \end{aligned}$$

If $\mathbf{d}B = 0$, then

$$\boxed{\mathbf{d}E + \frac{\partial B}{\partial t} = 0} \tag{9}$$

If $\delta E = *\mathbf{d}*E = 4\pi\rho_{\text{total}}$,

$$\boxed{\delta B = *\mathbf{d}*B = \frac{\partial E}{\partial t} + 4\pi\frac{\partial P}{\partial t} + 4\pi J_{\text{free}} + 4\pi c \delta \mathbf{M}} \tag{10}$$

with $\mathbf{M} \in \Omega^2(M)$, magnetization in matter (i.e. matter magnetization) is *necessarily* a 2-form.

2.2.1. *Some of the algebra (scratch) work/explicit calculations, for Maxwell's Equations, my version, over spacetime manifold M.*

$$\mathbf{d}B \Longleftrightarrow \nabla \cdot B$$

since component-wise,

$$\mathbf{d}B = \frac{\partial}{\partial x^k} B_{ij} dx^k \wedge dx^i \wedge dx^j \Longleftrightarrow \nabla \cdot B$$

$$\mathbf{d}E = -\frac{\partial B}{\partial t} \Longleftrightarrow \nabla \times E \equiv \text{curl}E = \frac{-1}{c}\frac{\partial \mathbf{B}}{\partial t}$$

since, component-wise,

$$\mathbf{d}E = \frac{\partial}{\partial x^k} E_i dx^k \wedge dx^i = \frac{\partial}{\partial x^j} E_k dx^j \wedge dx^k = \frac{-\partial}{\partial t} B_{jk} dx^j \wedge dx^k$$

For $\delta E = *\mathbf{d}*E = 4\pi\rho_{\text{total}}$, consider

$$*E = \frac{1}{(d-1)!}\sqrt{\mathbf{g}}\epsilon_{i_1 i_2 \dots i_{d-1} j_1} E_j g^{jj_1} e^{i_1} \wedge e^{i_2} \wedge \dots \wedge e^{i_{d-1}} = \frac{1}{2}\sqrt{\mathbf{g}}\epsilon_{ijk} E_{k'} g^{k'k} dx^i \wedge dx^j$$

Further,

$$\begin{aligned} \mathbf{d}*E &= \frac{1}{(d-1)!}\frac{\partial}{\partial x^k}(\sqrt{\mathbf{g}}E_j g^{jj_1})\epsilon_{i_1 i_2 \dots i_{d-1} j_1} dx^k \wedge dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_{d-1}} = \\ &= \frac{1}{(d-1)!}\frac{\partial}{\partial x^k}(\sqrt{\mathbf{g}}E^{j_1})\epsilon_{i_1 i_2 \dots i_{d-1} j_1}\epsilon^{k i_1 i_2 \dots i_{d-1}}\frac{\text{vol}^d}{\sqrt{|\mathbf{g}|}} = \\ &= \frac{1}{(d-1)!}\frac{\partial}{\partial x^k}(\sqrt{|\mathbf{g}|}E^{j_1})\delta^k_{j_1}(d-1)!\frac{\text{vol}^d}{\sqrt{|\mathbf{g}|}} = \frac{1}{\sqrt{|\mathbf{g}|}}\frac{\partial}{\partial x^k}(\sqrt{|\mathbf{g}|}E^k)\text{vol}^d \end{aligned}$$

where this (generalized) Kronecker delta relation was used:

$$\frac{1}{p!}\delta^{\mu_1 \dots \mu_p}_{\nu_1 \dots \nu_p}\delta^{\nu_1 \dots \nu_p}\rho_1 \dots \rho_p = \delta^{\mu_1 \dots \mu_p}_{\rho_1 \dots \rho_p}$$

where

$$\delta^{\mu_1 \dots \mu_n}_{\nu_1 \dots \nu_n} = \epsilon^{\mu_1 \dots \mu_n}\epsilon_{\nu_1 \dots \nu_n}$$

Note that

$$*1 = \text{vol}$$
$$**1 = (-1)^{0(n-0)}1 = 1 = *\text{vol}$$

and so

$$*\mathbf{d}*E = \delta E = \frac{1}{\sqrt{|\mathbf{g}|}}\frac{\partial}{\partial x^k}(\sqrt{|\mathbf{g}|}E^k)$$

Indeed, we had generalized the divergence, but on a 1-form:

$$\delta : \Omega^1(M) \to C^\infty(M)$$

(11)
$$-\delta E = -\delta(E_k dx^k) = \frac{1}{\sqrt{|\mathbf{g}|}}\frac{\partial}{\partial x^k}(\sqrt{|\mathbf{g}|}E^k) \equiv \frac{1}{\sqrt{|\mathbf{g}|}}\frac{\partial}{\partial x^k}(\sqrt{|\mathbf{g}|}g^{kk_1}E_{k_1})$$

3. Magnetostatics, macroscopic Magnetism, Magnetic permeability, magnetic susceptibility, field H, free currents and field H

Keywords: magnetic permeability, magnetic susceptibility

Suppose we have matter (i.e. the "macroscopic problem", referred to from Jackson (1998), Sec. 5.8 "Macroscopic Equations, Boundary Conditions on B and H", [1]), *not* a vacuum.

Atoms in matter have electrons, e^- in orbit, contributing to (rapidly) fluctuating magnetic moments $\mathbf{m}$, along with e^-'s intrinsic $\mathbf{m}$.

Consider an average macroscopic magnetization or magnetic moment density $\mathbf{M}(\mathbf{x})$ defined in a "vector calculus" manner by Jackson (1998) [1],

$$\mathbf{M}(\mathbf{x}) = \sum_I N_I \langle \mathbf{m}_I \rangle, \qquad I \equiv \text{ index of a particle}$$

Recalling Maxwell's Equations, Eq. 10,

$$\delta B = \frac{\partial E}{\partial t} + 4\pi\frac{\partial P}{\partial t} + 4\pi J_{\text{free}} + 4\pi c\delta\mathbf{M}$$

Consider a time-independent E and negligible P. Then

$$\Longrightarrow \delta B = 4\pi J_{\text{free}} + 4\pi c\delta\mathbf{M}$$

Jackson (1998) [1] considers this magnetization $\mathbf{M}$ as contributing to an *effective current density* by vector calculus arguments of it having a vector potential form, and so he proceeds to write it as (Jackson (1998), Eqn. (5.80) [1])

$$\text{curl}\mathbf{B} = \mu_0(\mathbf{J} + \text{curl}\mathbf{M}) \qquad (SI)$$

Then Jackson *defines* the macroscopic field $\mathbf{H}$, in Jackson (1998), Eqn. (5.81) [1],

$$\mathbf{H} := \frac{1}{\mu_0}\mathbf{B} - \mathbf{M}$$

However, Purcell's treatment is both more lucid, and more grounded in what B field really is physically, less relying upon artificial artifices.

3.1. Free currents $\mathbf{J}_{\text{free}}$ and the field H, magnetic susceptibility.

cf. Purcell (1984) [2], Sec. 11.10 Free Currents, and the Field $\mathbf{H}$

Keywords: $\mathbf{H}$, volume magnetic susceptibility

Bound current $\mathbf{J}_{\text{bound}}$ are current associated with molecular or atomic magnetic moments, including the intrinsic magnetic moment of particles with spin.

Free currents $\mathbf{J}_{\text{free}}$ are ordinary conduction currents.

(12)
$$\mathbf{J}_{\text{bound}} = c\nabla \times \mathbf{M}$$

cf. Purcell (1984), Eq. (44) of Ch. 11 [2]

At a surface, where $\mathbf{M}$ is discontinuous, we have a surface current density $\mathcal{J}$.

By superposition,

(13)
$$\nabla \times \mathbf{B} = \frac{4\pi}{c}(\mathbf{J}_{\text{bound}} + \mathbf{J}_{\text{free}}) = \frac{4\pi}{c}\mathbf{J}_{\text{total}}$$

cf. Purcell (1984), Eq. (50) of Ch. 11 [2]

Thus,

$$\begin{aligned} \nabla \times \mathbf{B} &= \frac{4\pi}{c}(c\nabla \times \mathbf{M}) + \frac{4\pi}{c}\mathbf{J}_{\text{free}} = \\ &= \nabla \times (\mathbf{B} - 4\pi\mathbf{M}) = \frac{4\pi}{c}\mathbf{J}_{\text{free}} \end{aligned}$$

cf. Purcell (1984), Eq. (51) of Ch. 11 [2]

Purcell also defines

$$\mathbf{H} := \mathbf{B} - 4\pi\mathbf{M} \tag{14}$$

cf. Purcell (1984), Eq. (52) of Ch. 11 [2]; and so

$$\nabla \times \mathbf{H} = \frac{4\pi}{c}\mathbf{J}_{\text{free}} \qquad (cgs) \qquad\qquad \nabla \times \mathbf{H} = \mathbf{J}_{\text{free}} \qquad (SI)$$

cf. Purcell (1984), Eq. (53), (53'), respectively, of Ch. 11 [2].

In magnetic systems, it is precisely the free currents that we can control. So **H** is useful:

$$\int_C \mathbf{H}\cdot d\mathbf{l} = \frac{4\pi}{c}\int_S \mathbf{J}_{\text{free}}\cdot d\mathbf{a} = \frac{4\pi}{c}I_{\text{free}} \qquad (cgs) \qquad\qquad \int_C \mathbf{H}\cdot d\mathbf{l} = \int_S \mathbf{J}_{\text{free}}\cdot d\mathbf{a} = I_{\text{free}} \qquad (SI) \tag{15}$$

where in SI, $H \sim \frac{\text{amps}}{\text{meter}}$. cf. Purcell (1984), Eq. (54), (54'), respectively, of Ch. 11 [2].

B is the *fundamental magnetic field vector*; it is **only B** s.t. $\nabla\cdot\mathbf{B} = 0$ or $\mathbf{d}B = 0$

The basic magnetic field inside matter is **B**, *not* **H**. That's not a matter of mere definition, but a *consequence of the absence of magnetic charges*. cf. Purcell (1984)[2].

Now

$$\mathbf{M} = \chi_m\mathbf{H} \tag{16}$$

cf. Purcell (1984), Eq. (56) Ch. 11 [2].

The lines of **H** inside the magnet look just like the lines of **E** inside the polarized cylinder.

- **H** is the fiction of magnetic poles; if there wre magnetic poles then **H** is the macroscopic **B** filed inside the material.

For any material in which **M** is porportional to **H**,

$$\boxed{\begin{gathered}\mathbf{B} = \mathbf{H} + 4\pi\mathbf{M} = (1 + 4\pi\chi_m)\mathbf{H} \\ \mu = 1 + 4\pi\chi_m\end{gathered}} \tag{17}$$

So *if* there was a linear response between magnetization **M** and the measured macroscopic field **H**, related through the volume magnetic susceptibility, χ_m, ($\mathbf{M} = \chi_m\mathbf{H}$), then for $\delta B = 4\pi J_{\text{free}} + 4\pi c\delta\mathbf{M}$,

$$\delta(B - 4\pi cM) = \delta H = \delta\frac{B}{\mu} = 4\pi J_{\text{free}} \Longrightarrow \mathbf{B} = \mu 4\pi J_{\text{free}}$$

and so we have the usual expression (make the comparison)

$$\text{curl}\mathbf{B} = \mu 4\pi\mathbf{J}_{\text{free}}$$

Obtaining an integral form,

$$*\delta B = **\mathbf{d}*B = (-1)^{2(d-2)}\mathbf{d}*B = \mu 4\pi *J_{\text{free}} \xrightarrow{\int_S} \int_S \mathbf{d}*B = \int_{\partial S} *B = \mu 4\pi\int_S *J_{\text{free}}$$

Jackson seems to imply to treat macroscopic field **H** as what you measure, since $\mathbf{J}_{\text{free}}$ is what one can measure and control, pointed out sagely by Purcell. So consider this, as I write down an integral form,

$$*\delta H = \mathbf{d}*H = 4\pi *J_{\text{free}} \xrightarrow{\int_S} \int_S \mathbf{d}*H = \int_{\partial S} *H = 4\pi\int_S *J_{\text{free}}$$

If, over S, $\mathbf{J}_{\text{free}}$ is uniform, $\frac{4\pi}{c}\int_S *\mathbf{J}_{\text{free}} = \frac{4\pi}{c}I_{\text{free}}$. If we can measure the current, we can obtain the line integral of **H**. But we should really be aware that what we're *really* measuring is $B - 4\pi c\mathbf{M}$ - would it be possible to measure the macroscopic **M** itself?

4. Eddy Currents

I build upon the physical setup proposed by Jackson (1998) [1] in Section 5.18 "Quasi-Static Magnetic Fields in Conductors; Eddy Currents; Magnetic Diffusion."

For a system (with characteristic) length L, L being small,
compared to electromagnetic wavelength associated with dominant time scale of problem T,

$$f := \frac{1}{T}; \quad \omega = 2\pi f; \quad \omega\lambda = c \Longrightarrow \lambda = \frac{c}{\omega} = \frac{c}{2\pi f} = \frac{Tc}{2\pi}$$

$$\frac{L}{\lambda} = \frac{LTc}{2\pi} \gg 1$$

From Maxwell's equations, in particular, Faraday's Law, and in its integral form (over 2-dim. *closed* surface S),

$$\mathbf{d}E + \frac{\partial}{\partial t}B = 0 \text{ or } -\mathbf{d}E = \frac{\partial B}{\partial t} \xrightarrow{\int_S} \int_S \frac{\partial B}{\partial t} = -\int_S \mathbf{d}E = -\int_{\partial S} E \tag{18}$$

So on S, changing magnetic flux $\int\frac{\partial B}{\partial t}$ results in E field, circulating around boundary of S, ∂S.

We know that in a conductor, free conducting electrons get pushed around by E fields, result in a current density J.

J is related to E, *empirically* (by Ohm's Law)

$$J = \sigma E$$

where σ is the resistivity.

Then use the force law on this induced current J from the B field set up:

$$F_{\text{net}} = \frac{1}{c}\int_S J \times B dA$$

By working through the right-hand rule, F_{net} the force on those currents induced in the conductor due to the B that's there, is in the direction to help oppose changing (increasing or decreasing $\frac{\partial B}{\partial t}$).

To find B, suppose $B = dA$, i.e. $B \in H^2_{\text{deRham}}(M)$, i.e. $B = \text{curl}A$.

For sure,

$$\delta(B - 4\pi c\mathbf{M}) = 4\pi J \Longleftrightarrow \text{curl}(B - 4\pi c\mathbf{M}) = \text{curl}H = 4\pi J$$

Be warned now that the relation $B = \mu H$ may not be valid on all domains of interest; μ could even be a tensor! (e.g. $B_{ij} = \mu_{ij}^{kl}H_{kl}$). However, both Jackson (1998) [1] in Sec. 5.18 Quasi-Static Magnetic Fields in Conductors; Eddy Currents; Magnetic Diffusion, pp. 219, and Smythe (1968), Ch. X (his Ch. 10), pp. 368 [5], continues on *as if* this relation is linear: $B = \mu H$.

Nevertheless, as we want to find B by finding its "vector potential" A, we obtain a diffusion equation:

$$\begin{aligned} -\delta B = *\mathbf{d}*\mathbf{d}A = (-1)\delta\mathbf{d}A = (-1)(\Delta - \mathbf{d}\delta)A \xrightarrow{\mathbf{d}\delta A = 0} -\Delta A = \\ = 4\pi\mu J = 4\pi\mu\sigma E = 4\pi\mu\sigma\left(-\frac{\partial A}{\partial t}\right) \\ \Longrightarrow \boxed{\Delta A = 4\pi\mu\sigma\frac{\partial A}{\partial t}} \end{aligned} \tag{19}$$

where in the first 2 steps (equalities), $-\delta B = *\mathbf{d}*\mathbf{d}A = (-1)\delta\mathbf{d}A$ it's interesting to note that the codifferential δ for the 2 form B had to be written out explicitly, and then the codifferential for the 1-form A is *different* from the δ for B by a(n important) factor of (-1); where $\mathbf{d}\delta A = 0$ must be satisfied by the form A takes; and where, since $B = \mathbf{d}A$,

$$\mathbf{d}E + \frac{\partial B}{\partial t} = \mathbf{d}E + \frac{\partial}{\partial t}\mathbf{d}A = \mathbf{d}\left(E + \frac{\partial A}{\partial t}\right) = 0 \Longrightarrow E = -\frac{\partial A}{\partial t} + \text{grad}\Phi \xrightarrow{\Phi = \text{ constant}} E = -\frac{\partial A}{\partial t} \tag{20}$$

whereas a choice of gauge for E was chosen so that $\Phi = $ constant (and so a particular form for E was chosen, amongst those in the *same* equivalence class of $H^1_{\text{deRham}}(M)$.

To ensure that the differential geometry formulation is in agreement with the practical vector calculus formulation, compare Eq. 19 with Eq. (5.160) of Jackson (1998) [1] and Eq. (10) in Sec. 10.00 of Smythe (1968) [5].

To summarize what's going on, I think one should at least understand in one's head how Maxwell's Equations apply, (and I will try to write in SI here)

$$\boxed{\begin{gathered} \int_S \frac{\partial \mathbf{B}}{\partial t} dA = -\oint \mathbf{E}\cdot d\mathbf{s} \Longrightarrow \mathbf{J} = \sigma \mathbf{E} \Longrightarrow \mathbf{F}_{\text{net}} = \int_S \mathbf{J}\times \mathbf{B} dA \\ \text{to find } \mathbf{B} = ? \qquad \text{using form } \mathbf{B} = \nabla \times \mathbf{A}, \\ \nabla^2 \mathbf{A} = \mu\sigma \frac{\partial \mathbf{A}}{\partial t} \qquad (SI) \end{gathered}} \tag{21}$$

where, a change in magnetic flux over a surface S over the conductor, $\int_S \frac{\partial \mathbf{B}}{\partial t} dA$ induces a circulation of E field around S, $-\oint \mathbf{E}\cdot d\mathbf{s}$, and this E field is pushing around *free conducting charges* according to Ohm's law, $\mathbf{J} = \sigma\mathbf{E}$, with σ being the conductivity of the conducting material, and this current density $\mathbf{J}$ is then acted upon by the prevailing B field, according to the usual force law. To find $\mathbf{B}$, one can find $\mathbf{A}$ and *try* to find $\mathbf{A}$ analytically.

Keep in mind that for $\nabla^2\mathbf{A} = \mu\sigma\frac{\partial \mathbf{A}}{\partial t}$, we had used, critically, the Maxwell equation $\nabla\times\mathbf{H} = \mathbf{J}$, with $\mathbf{J}$ being the *induced current of free conducting charges on the conductor*. This $\mathbf{H}$ will contribute (through linear superposition) to the $\mathbf{B}$ that could already be there due to the permanent magnet.

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