THE DIFFERENTIAL GEOMETRY DIFFERENTIAL TOPOLOGY DUMP

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Contents

Abstract. Everything about Differential Geometry, Differential Topology

Part 1. Manifolds

1. Inverse Function Theorem

Shastri (2011) had a thorough and lucid and explicit explanation of the Inverse Function Theorem [4]. I will recap it here. The following is also a blend of Wienhard's Handout 4 https://web.math.princeton.edu/~wienhard/teaching/M327/handout4.pdf

Definition 1. *Let (X,a) metric space.*
contraction $\phi : X \to X$ *if* $\exists$ *constant* $0 < c < 1$ *s.t.* $\forall x, y \in X$

$$d(\phi(x), \phi(y)) \leq c d(x,y)$$

Theorem 1 (Contraction Mapping Principle)**.** *Let (X,d) complete metric space.*
Then $\forall$ *contraction* $\phi : X \to X$, $\exists ! y \in X$ *s.t.* $\phi(y) = y$, y fixed pt.

Proof. Recall def. of complete metric space X, X metric space s.t. $\forall$ Cauchy sequence in X is convergent in X (i.e. has limit in X).

$$\forall x_0 \in X, \text{ Define } \begin{aligned} & x_1 = \phi(x_0) \\ & x_2 = \phi(x_1) \\ & \vdots \\ & x_j = \phi(x_{j-1}) \\ & \vdots \\ & x_n = \phi(x_{n-1}) \end{aligned}$$

$$d(x_{n+1}, x_n) = d(\phi(x_n), \phi(x_{n-1})) \leq c d(x_n, x_{n-1}) \leq \dots \leq c^n d(x_1, x_0)$$

for some $0 < c < 1$.

$$d(x_m, x_n) \leq d(x_n, x_{n-1}) + d(x_{n-1}, x_m) \leq d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + \dots + d(x_{m+1}, x_m) \leq \sum_{k=n-1}^m c^k d(x_1, x_0)$$

Thus, $\forall \epsilon > 0$, $\exists n_0 > 0$, (n_0 large enough) s.t. $\forall m, n \in \mathbb{N}$ s.t. $n_0 < n < m$,

$$d(x_m, x_n) \leq \sum_{k=n-1}^m c^k d(x_1, x_0) < \epsilon d(x_1, x_0)$$

Thus, $\{x_n\}$ Cauchy sequence. Since X complete, $\exists$ limit pt. $y \in X$ of $\{x_n\}$.

$$\phi(y) = \phi(\lim_n x_n) = \lim_n \phi(x_n) = \lim_n x_{n+1} = y$$

Since by def. of y limit pt. of $\{x_n\}$, $\forall \epsilon > 0$, then $\{ n | |x_n - y| \leq \epsilon, n \in \mathbb{N} \}$ is infinite.

Consider $\delta > \mathbb{N}$. Consider $\{ n | |x_n - y| \leq \delta, n \in \mathbb{N} \}$

$\exists N_\delta \in \mathbb{N}$ s.t. $\forall n > N_\delta$, $|x_n - y| < \delta$; otherwise, $\forall N_\delta$, $\exists n > N_\delta$ s.t. $|x_n - y| \geq \delta$. Then $\{ n | |x_n - y| \leq \delta, n \in \mathbb{N} \}$ finite. Contradiction.

ϕ cont. so by def. $\forall \epsilon > 0$, $\exists \delta > 0$ s.t. if $|x_n - y| < \delta$, then $|\phi(x_n) - \phi(y)| < \epsilon$.

Pick N_δ s.t. $\forall n > N_\delta$, $|x_n - y| < \delta$, and so $|\phi(x_n) - \phi(y)| < \epsilon$. There are infinitely many $\phi(x_n)$'s that satisfy this, and so $\phi(y)$ is a limit pt.

$$\text{If } \exists y_1, y_2 \in X \text{ s.t. } \begin{aligned} & \phi(y_1) = y_1, \text{ then} \\ & \phi(y_2) = y_2 \end{aligned}$$

$$d(y_1, y_2) = d(\phi(y_1), \phi(y_2)) \leq c d(y_1, y_2) \text{ with } c < 1$$

so $c = 1$ □

Theorem 2 (Inverse Function Theorem)**.** *Suppose open $U \subset \mathbb{R}^n$, let C^1 $f : U \to \mathbb{R}^n$, $x_0 \in U$ s.t. $Df(x_0)$ invertible.*
Then $\exists$ open neighborhoods $V \ni x_0$, $W \ni f(x_0)$ s.t. $V \subseteq U$ and $W \subseteq \mathbb{R}^n$, respectively, and s.t.

(i) *$f : V \to W$ bijection*
(ii) *$g = f^{-1} : V \to U$ differentiable, i.e. $g = f^{-1} : W \to V$ is C^1*

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(iii) $D(f^{-1})$ *cont. on* W.
(iv) $Dg(y) = (Df(g(y)))^{-1} \quad \forall y \in W$

Also, notice that $f(g(y)) = y \, \forall y \in W$.

Proof. Consider $\widetilde{f}(x) = (Df(x_0))^{-1}(f(x+x_0) - f(x_0))$. Then
$\widetilde{f}(0) = 0$ and

$$D\widetilde{f} = (Df(x_0))^{-1}(Df(x+x_0) - 0)$$
$$D\widetilde{f}(0) = (Df(x_0))^{-1}Df(x_0) = 1$$

So let $\widetilde{f} \to f$ (notation) and so assume, without loss of generality, that $U \ni 0$, $f(0) = 0$, $Df(0) = 1$

Choose $0 < \epsilon \leq \frac{1}{2}$. Let $0 < \delta < 1$ s.t. open ball $V = B_\delta(0) \subseteq U$, and $\|Df(x) - 1\| < \epsilon$. $\forall x \in U$, since Df cont. at 0.

Let $W = f(V)$.

$\forall y \in W$, define $\phi_y : V \to \mathbb{R}^n$
$\phi_y(x) = x + (y - f(x))$

$$D(\phi_y)(x) = 1 + -Df(x) \quad \forall x \in V$$
$$\|D(\phi_y)(x)\| = \|1 - Df(x)\| \leq \epsilon < 1$$

$\forall x_1, x_2 \in V$, by mean value Thm. (not the equality that is only valid in 1-dim., but the inequality, that's valid for $\mathbb{R}^d$,

$$\|\phi_y(x_1) - \phi_y(x_2)\| \leq \|D(\phi_y)(x')\| \|x_1 - x_2\|$$

for some $x' = cx_2 + (1-c)x_1$, $c \in [0,1]$. V only needed to be convex set.

$$\Longrightarrow \|\phi_y(x_1) - \phi_y(x_2)\| \leq \epsilon \|x_1 - x_2\|$$

Then ϕ_y contraction mapping.

Suppose $f(x_1) = f(x_2) = y$, $x_1, x_2 \in V$.

$$\phi_y(x_1) = x_1$$
$$\phi_y(x_2) = x_2$$
$$\|\phi_y(x_1) - \phi_y(x_2)\| = \|x_1 - x_2\| \leq \epsilon \|x_1 - x_2\| \quad \forall \epsilon > 0 \Longrightarrow x_1 = x_2$$

$\Longrightarrow f|_U$ injective.

$W = f(V)$, so $f : V \to W$ surjective. f bijective.

Fix $y_0 \in W$, $y_0 = f(x_0)$, $x_0 \in V$.

Let $r > 0$ s.t. $B_r(x_0) \subset V$.

Consider $B_{r\epsilon}(y_0)$. If $y \in B_{r\epsilon}(y_0)$.

$$r\epsilon > \|y - y_0\| = \|y - f(x_0)\| = \|\phi_y(x_0) - x_0\| \text{ with}$$
$$\phi_y(x) = x + (y - f(x))$$

If $x \in B_r(x_0)$,

$$\|\phi_y(x) - x_0\| \leq \|\phi_y(x) - \phi_y(x_0)\| + \|\phi_y(x_0) - x_0\| \leq \epsilon \|x - x_0\| + r\epsilon < 2r\epsilon = r$$

Thus $\phi(B_r(x_0)) = B_r(x_0)$.

By contraction mapping principle, $\exists \, a \in B_r(x_0)$, s.t. $\phi_y(a) = a$. Then $\phi_y(a) = a + (y - f(a)) = a \Longrightarrow f(a) = y$.

$y \in f(V) = W$.

So $B_{r\epsilon}(y_0) \subset W$. W open.

Let $\mathrm{Mat}(n,n) \equiv$ space of all $n \times n$ matrices; $\mathrm{Mat}(n,n) = \mathbb{R}^{n^2}$.

$\square$

There is a proof of the implicit function theorem and its various forms in Shastri (2011) [4], but I found Wienhard's Handout 4 for Math 327 to be clearer.[1]

[1] https://web.math.princeton.edu/~wienhard/teaching/M327/handout4.pdf

Theorem 3 (Implicit Function Theorem)**.** *Let open* $U \subset \mathbb{R}^{m+n} \equiv \mathbb{R}^m \times \mathbb{R}^n$
$C^1 \; f : U \to \mathbb{R}^n$
$(a,b) \in U$ *s.t.* $f(a,b) = 0$ *and* $D_y f|_{(a,b)}$ *invertible.*

Then $\exists$ *open* $V \ni (a,b)$, $V \subset U$
$\exists$ *open neighborhood* $W \ni a$, $W \subseteq \mathbb{R}^m$
$\exists ! \;\; C^1 \; g : W \to \mathbb{R}^n$ *s.t.*

$$\{(x,y) \in V | f(x,y) = 0\} = \{(x, g(x)) | x \in W\}$$

Moreover,

$$dg_x = -(d_y f)^{-1}|_{(x,g(x))} \, d_x f|_{(x,g(x))}$$

and g *smooth if* f.

Proof. Define $F : U \to \mathbb{R}^{m+n}$
$F(x,y) = (x, f(x,y))$

Then $F(a,b) = (a,0)$ (given), and

$$DF = \begin{bmatrix} 1 & \\ \frac{\partial f^i(x,y)}{\partial x^j} & \frac{\partial f^i(x,y)}{\partial y^j} \end{bmatrix} \equiv \begin{bmatrix} 1 & \\ D_x f & D_y f \end{bmatrix}$$

$DF(a,b)$ invertible.

By inverse function theorem, since $DF(a,b)$ invertible at pt. (a,b),

$\exists$ open neighborhoods $V \ni (a,b) \subseteq \mathbb{R}^m \times \mathbb{R}^n$, $\widetilde{W} \ni (a,0) \subseteq \mathbb{R}^m \times \mathbb{R}^n$ s.t. F diffeomorphism with $F^{-1} : \widetilde{W} \to V$.

Set $W = \{x \in \mathbb{R}^m | (x,0) \in \widetilde{W}\}$. Then $\pi_1(\widetilde{W}) = W$ open in $\mathbb{R}^m$.

Define $g : W \to \mathbb{R}^n$,

$$g(x) = \pi_2 \circ F^{-1}(x,0) \text{ or}$$
$$F^{-1}(x,0) = (h(x), g(x))$$

Now $FF^{-1}(x,0) = (x,0) = (h(x), f(h(x), g(x)))$ so $h(x) = x \, \forall x \in W$, $0 = f(x, g(x))$.

Then

$$\{(x,y) \in V | f(x,y) = 0\} = \{(x,y) \in V | F(x,y) = (x,0)\} = \{(x, g(x)) | x \in W, 0 = f(x, g(x))\}$$

Since π smooth and F^{-1} is C^1, g is C^1.

To reiterate, $f(x, g(x)) = 0$ on W.

Using chain rule while differentiating $f(x, g(x)) = 0$,

$$\partial_{x^j} f(x, g(x)) = \frac{\partial f(x,g(x))}{\partial x^k} \frac{\partial x^k}{\partial x^j} + \frac{\partial f(x,g(x))}{\partial y^k} \frac{\partial g^k(x)}{\partial x^j} = D_x f|_{(x,g(x))} + (D_y f)|_{(x,g(x))} \cdot (Dg)_x = 0 \text{ or}$$
$$(Dg)_x = -(D_y f)|_{x,g(x)} \, D_x f|_{(x,g(x))}$$

$\square$

Definition 2. *smooth* $f : M \to N$, *s.t.* $Df(p) : T_p M \to T_{f(p)} N$ *injective. Then* f *immersion at* p.

Shastri (2011) has this as the "Injective Form of Implicit Function Theorem", Thm. 1.4.5, pp. 23 and Guillemin and Pollack (2010) has this as the "Local Immersion Theorem" on pp. 15, Section 3 "The Inverse Function Theorem and Immersions" [3].

Theorem 4 (Local immersion Theorem i.e. Injective Form of Implicit Function Theorem)**.** *Suppose* $f : M \to N$ *immersion at* p, $q = f(p)$.

Then $\exists$ *local coordinates around* p, q, x, y, *respectively s.t.* $f(x_1 \dots x_m) = (x_1 \dots x_m, 0 \dots 0)$.

Proof. Choose local parametrizations

$$\begin{tikzcd} U \subseteq M \arrow[r, "f"] \arrow[d, "\phi"] & N \supseteq V \arrow[d, "\psi"] \\ \phi(U) \arrow[r, "f"] & \psi(V) \end{tikzcd} \qquad \begin{aligned} \phi(p) &= x \\ \psi(q) &= y \end{aligned}$$

$D(\psi f \varphi^{-1}) \equiv Df$. $Df(p)$ injective (given f immersion). $Df(p) \in \text{Mat}(n,m)$

By change of basis in $\mathbb{R}^n$, assume $Df(p) = \begin{pmatrix} I_m \\ 0 \end{pmatrix}$.

Now define $G : \phi(U) \times \mathbb{R}^{n-m} \to \mathbb{R}^n$

$$G(x,z) = f(x) + (0,z)$$

Thus, $DG(x,z) = 1$ and for open $\phi(U) \times U_2$, $G(\phi(U) \times U_2)$ open.
By inverse function theorem, G local diffeomorphism of $\mathbb{R}^n$, at 0.
Now $f = G \circ \mathfrak{i}$, where $\mathfrak{i}$ is canonical immersion.

$$G(x,0) = f(x)$$
$$\implies G^{-1}G(x,0) = (x,0) = G^{-1}f(x)$$

Use $\psi \circ G$ as the local parametrization of N around pt. q. Shrink U, V so that

$$\begin{tikzcd} U \subseteq M \arrow[r, "f"] \arrow[d, "\phi"] & N \supseteq V \arrow[d, "\psi \circ G"] \\ \phi(U) \arrow[r, "\mathfrak{i}"] & \psi \circ G(V) \end{tikzcd}$$

Theorem 5 ((Implicit Function Thm.)). *Let open subset $U \subseteq \mathbb{R}^n \times \mathbb{R}^d$, $(x,y) = (x^1 \dots x^n, y^1 \dots y^k)$ on U. Suppose smooth $\Phi : U \to \mathbb{R}^k$, $(a,b) \in U$, $c = \Phi(a,b)$*

If $k \times k$ matrix $\frac{\partial \Phi^i}{\partial y^j}(a,b)$ nonsingular, then $\exists$ neighborhoods $\begin{matrix} V_0 \subseteq \mathbb{R}^n \text{ of } a \\ W_0 \subseteq \mathbb{R}^k \text{ of } b \end{matrix}$ and smooth $F : V_0 \to W_0$ s.t.

$\Phi^{-1}(c) \bigcap (V_0 \times W_0)$ is graph of F, i.e.
$\Phi(x,y) = c$ for $(x,y) \in V_0 \times W_0$ iff $y = F(x)$.

1.1. **Submersions.** cf. pp. 20, Sec. 4 "Submersions", Ch. 1 of Guillemin and Pollack (2010) [3].
Consider $X, Y \in \mathbf{Man}$, s.t. $\text{dim}X \geq \text{dim}Y$.

Definition 3 (submersion). *If $f : X \to Y$,*
if $Df_x \equiv df_x$ is surjective, *$f \equiv$ **submersion** at x.*

Recall that,

$$Df_x : T_x X \to T_{f(x)} Y$$
$$\text{dim}T_x X \geq \text{dim}T_{f(x)} Y$$
$$\text{rank}Df_x \leq \text{dim}T_{f(x)}Y, \text{ in general, while}$$
$$\text{rank}Df_x = \text{dim}T_{f(x)}Y \text{ iff } Df_x \text{ surjective}$$

Canonical submersion is standard projection:

If $\text{dim}X = k$, $k \geq l$,
$\text{dim}Y = l$

$$(a_1 \dots a_k) \mapsto (a_1 \dots a_l)$$

Theorem 6 (Local Submersion Theorem). *Suppose $f : X \to Y$ submersion at x, and $y = f(x)$, Then $\exists$ local coordinates around x, y s.t.*

$$f(x_1 \dots x_k) = (x_1 \dots x_l)$$

i.e. f locally equivalent to canonical submersion near x

Proof. I'll have a side-by-side comparison of my notation and the 1 used in Guillemin and Pollack (2010) [3] where I can.
For charts $(U,\phi), (V,\psi)$ for X, Y, respectively, $y = f(x)$ for $x \in X$,

$$\begin{tikzcd} U \subseteq X \arrow[r, "f"] \arrow[d, "\phi"] & Y \supseteq V \arrow[d, "\psi \circ G"] \\ \mathbb{R}^k \arrow[r, "\mathfrak{i}"] & \mathbb{R}^l \end{tikzcd} \qquad \begin{tikzcd} x \arrow[r, mapsto, "f"] \arrow[d, mapsto, "\phi"] & f(x) = y \arrow[d, mapsto, "\psi"] \\ \phi(x) = (a^1 \dots a^k) \arrow[r, mapsto, "g"] & g(\phi(x)) = g(a^1 \dots a^k) = \psi(y) \end{tikzcd}$$

Dg_x surjective, so assume it's a $l \times k$ matrix $\begin{bmatrix} \mathbf{1}_l & 0 \end{bmatrix}$.
Define

$\square$

$$\begin{aligned} &G : U \subset \mathbb{R}^k \to \mathbb{R}^k \\ &G(a) \equiv G(a^1 \dots a^k) := (g(a), a_{l+1}, \dots, a_k) \end{aligned} \tag{1}$$

Now

$$DG(a) = \begin{bmatrix} \mathbf{1}_l & 0 \\ & \mathbf{1}_{k-l} \end{bmatrix} = \mathbf{1}_k \tag{2}$$

so G local diffeomorphism (at 0).
So $\exists$ G^{-1} as local diffeomorphism of some U' of a into $U \subset \mathbb{R}^k$.
By construction,

$$g = \mathbb{P}_l \circ G \tag{3}$$

where $\mathbb{P}_l$ is the *canonical submersion*, the projection operator onto $\mathbb{R}^l$.

$$g \circ G^{-1} = \mathbb{P}_l$$

(since G diffeomorphism)

$$\begin{tikzcd} U \subseteq X \arrow[r, "f"] & V \subseteq Y \\ \mathbb{R}^k \arrow[r, "\mathbb{P}_l"] \arrow[u, "\phi^{-1}\circ G^{-1}"] & \mathbb{R}^l \arrow[u, "\psi^{-1}"'] \end{tikzcd}$$

for

$$\begin{tikzcd} \phi^{-1}\circ G^{-1}(a) \equiv \phi^{-1}\circ G^{-1}(a^1 \dots a^k) = x \arrow[r, mapsto, "f"] & f(x) = y = \psi^{-1}(a^1 \dots a^l) \\ (a^1 \dots a^k) \arrow[r, mapsto, "\mathbb{P}_l"] \arrow[u, mapsto, "\phi^{-1}\circ G^{-1}"] & (a^1 \dots a^l) \arrow[u, mapsto, "\psi^{-1}"'] \end{tikzcd}$$

$\Longrightarrow$

□

"An obvious corollary worth noting is that if f is a submersion at x, then it is actually a submersion in a whole neighborhood of x." Guillemin and Pollack (2010) [3]

Suppose f submersion at $x \in f^{-1}(y)$.

By local submersion theorem

$$f(x_1 \dots x_k) = (x_1 \dots x_l)$$

Choose $y = (0, \dots, 0)$.

Then, near x, $f^{-1}(y) = \{(0, \dots 0, x_{l+1} \dots x_k)\}$ i.e. let $V \ni x$ neighborhood of x, define $(x_1 \dots x_k)$ on V.

Then $f^{-1}(y) \bigcap V = \{(0 \dots 0, x_{l+1}, \dots x_k) | x_1 = 0, \dots x_l = 0\}$.

Thus $x_{l+1}, \dots x_k$ form a coordinate system on open set $f^{-1}(y) \bigcap V \subseteq f^{-1}(y)$.

Indeed,

$$\begin{tikzcd} U \subseteq X \arrow[r, "f"] \arrow[d, "\phi"] & V \subseteq Y \arrow[d, "\psi"] \\ \mathbb{R}^k \arrow[r, "\mathbb{P}_l"] & \mathbb{R}^l \end{tikzcd} \qquad \begin{tikzcd} x \arrow[r, mapsto, "f"] \arrow[d, mapsto, "\phi"] & f(x) = y \arrow[d, mapsto, "\psi"] \\ \phi(x) = (x^1 \dots x^k) \arrow[r, mapsto, "\mathbb{P}_l"] & (x^1 \dots x^l) \end{tikzcd}$$

and now

$$\begin{tikzcd} f^{-1}(y) & y \arrow[l, mapsto, "f^{-1}"'] \arrow[d, mapsto, "\psi"] \\ \{(0, \dots 0, x^1 \dots x^k)\} \arrow[u, mapsto, "\phi^{-1}"] & (0 \dots 0) \arrow[l, mapsto, "\mathbb{P}_l^{-1}"] \end{tikzcd}$$

Definition 4 (regular value)**.** *For smooth $f : X \to Y$, $X, Y \in$ **Man**,
$y \in Y$ is a **regular value** for f if $Df_x : T_xX \to T_yY$ surjective $\forall x$ s.t. $f(x) = y$.*

*$y \in Y$ **critical value** if y not a regular value of f.*

Theorem 7 (Preimage theorem)**.** *If y regular value of $f : X \to Y$,
$f^{-1}(y)$ is a submanifold of X, with dim$f^{-1}(y) =$ dim$X -$ dim$Y$*

Proof. Given y is a regular value of $f : X \to Y$,
$\forall x \in f^{-1}(y)$, $Df_x : T_xX \to T_yY$ is surjective. By local submersion theorem,

$$f(x^1 \dots x^k) = (x^1 \dots x^l) = y$$

Since $x \in f^{-1}(y)$, $(x^1 \dots x^k) = (y^1 \dots y^l, x^{l+1} \dots x^k)$.

For this chart for (U, φ), $U \ni x$, consider $(U \cap f^{-1}(y), \psi)$ with $\psi(x) = (x^{l+1} \dots x^k) \quad \forall x \in U \cap f^{-1}(y)$.

$\forall f^{-1}(y)$ submanifold with dim$f^{-1}(y) = k - l =$ dim$X -$ dim$Y$. □

Examples for emphasis

If dim$X >$ dim$Y$,

if $y \in Y$, regular value of $f : X \to Y$,

f submersion, $\forall x \in f^{-1}(y)$

If dim$X =$ dim$Y$,

f local diffeomorphism $\forall x \in f^{-1}(y)$

If dim$X <$ dim$Y$, $\forall y \in f(X)$ is a critical value.

Example: $O(n)$ as a submanifold of Mat(n, n)

Given Mat$(n, n) \equiv M(n) = \{n \times n$ matrices $\}$ is a manifold; in fact Mat$(n, n) \cong \mathbb{R}^{n^2}$,
Consider $O(n) = \{A \in \text{Mat}(n, n) | AA^T = 1\}$.

$$AA^T \in \text{Sym}(n) \equiv S(n) = \{S \in \text{Mat}(n, n) | S^T = S\} = \{\text{ symmetric } n \times n \text{ matrices }\} \tag{4}$$

Sym(n) submanifold of Mat(n, n), Sym(n) diffeomorphic to $\mathbb{R}^k$ (i.e. Sym$(n) \cong \mathbb{R}^k$), $k := \frac{n(n+1)}{2}$.

$$f : \text{Mat}(n, n) \to \text{Sym}(n)$$
$$f(A) = AA^T$$

Notice f is smooth,

$$f^{-1}(1) = O(n)$$
$$Df_A(B) = \lim_{s \to 0} \frac{f(A + sB) - f(A)}{s} = \lim_{s \to 0} \frac{(A + sB)(A^T + sB^T) - AA^T}{s} = AB^T + BA^T$$

If $Df_A : T_A\text{Mat}(n, n) \to T_{f(A)}\text{Sym}(n)$ surjective when $A \in f^{-1}(1) = O(n)$ (???).

Proposition 1. *If smooth $g_1 \dots g_l \in C^\infty(X)$ on X are independent $\forall x \in X$, s.t. $g_i(x) = 0$, $\forall i = 1 \dots l$,
then $Z = \{x \in X | g_1(x) = \dots = g_l(x) = 0\}$ = set of "common zeros" is a* submanifold *of X s.t. dim$Z =$ dim$X - l$.*

Take note that $g_1 \dots g_l$ are independent at x means, really, that $D(g_1)_x \dots D(g_l)_x$ are linearly independent on T_xX.

Proof. Suppose smooth $g_1 \dots g_l \in C^\infty(X)$ on manifold X s.t. dim$X = k \geq l$.

Consider $g = (g_1 \dots g_l) : X \to \mathbb{R}^l$, $Z \equiv g^{-1}(0)$.

Since $\forall g_i$ smooth, $D(g_i)_x : T_xX \to \mathbb{R}$ linear.

Now for

$$Dg_x = (D(g_1)_x \dots D(g_l)_x) : T_xX \to \mathbb{R}^l$$

By rank-nullity theorem (linear algebra), Dg_x surjective iff rank$Dg_x = l$ i.e. l functionals $D(g_1)_x \dots D(g_l)_x$ are linearly independent on T_xX.

"We express this condition by saying the l functions $g_1 \dots g_l$ are independent at x." (Guillemin and Pollack (2010) [3]) □

Jeffrey Lee (2009) [1]

John Lee (2012) [2]

2. Tensors

I'll go through Ch.7 *Tensors* of Jeffrey Lee (2009) [1].

Definition 5 (7.1[1])**.** *Let V, W be modules over commutative ring R, with unity.*
Then, algebraic W-valued tensor on V is multilinear map.

$$\tau : V_1 \times V_2 \times \dots \times V_m \to W \tag{5}$$

where $V_i = \{V, V^*\} \qquad \forall\, i = 1, 2, \dots m$.
If for r, s s.t. $r+s=m$, there are $r\quad V_i = V^*$, $s\, V_i = V$, tensor is r-contravariant, s-covariant; also say tensor of total type $\binom{r}{s}$.

EY : 20170404 Note that

$$(\tau^{i\alpha}_{\beta}\frac{\partial}{\partial x^i} \text{ or } \tau^{i\alpha}_{\beta} e_i)(\omega_j dx^j \text{ or } \omega_j e^j \in V^*)$$
$$(\tau^{\beta}_{i\alpha} dx^i \text{ or } \tau^{\beta}_{i\alpha} e^i)(X^j \frac{\partial}{\partial x^j} \text{ or } X^j e_j \in V)$$

$\exists$ natural map $\begin{aligned} V &\to V^{**}, \\ v &\mapsto \widetilde{v} \end{aligned}$ $\widetilde{v} : \alpha \mapsto \alpha(v)$. If this map is an isomorphism, V is **reflexive** module, and identify V with V^{**}.

Exercise 7.5. Given vector bundle $\pi : E \to M$, open $U \subset M$, consider sections of π on U, i.e. cont. $s : U \to E$, where $(\pi \circ s)(u) = u$, $\forall\, u \in U$.
Consider $E^* \ni \omega = \omega_i e^i$.
$\forall\, s \in \Gamma(E)$, $\omega(s) = \omega_i(s(x))^i$, $\forall\, x \in U \subset M$. So define $\widetilde{s} : \omega, x \mapsto \omega(s(x))$, $\forall\, x \in U$.
If $\widetilde{s} = 0$, $\widetilde{s}(\omega, x) = \omega(s(x)) = 0 \qquad \forall\, \omega \in E^*, \forall\, x \in U$, and so $s = 0$. (Let $\omega_i = \delta_{iJ}$ for some J, and so $s^J(x) = 0 \qquad \forall\, J$).
$s = 0$. So $\ker(s \mapsto \widetilde{s}) = \{0\}$ (so condition for injectivity is fulfilled).
Since $\widetilde{s} : \omega, x \mapsto \omega(s(x))$, $\forall\, \omega \in E^*$, $\forall\, x \in U$, $s \mapsto \widetilde{s}$ is surjective.
$s \mapsto \widetilde{s}$ is an isomorphism so $\Gamma(E)$ is a *reflexive* module.

Proposition 2. *For R a ring (special case), $\exists$ module homomorphism:*

tensor product space $\to$ tensor, as a multilinear map, i.e. $\exists$

$$\begin{gathered} (\otimes^r_{i=1} V) \otimes \left(\otimes^s_{j=1} V^*\right) \to T^r{}_s(V; R) \\ u_1 \otimes \dots \otimes u_r \otimes \beta^1 \otimes \dots \otimes \beta^s \in (\otimes^r V) \otimes (\otimes^s V^*) \mapsto (\alpha^1 \dots \alpha^r, v_1 \dots v_s) \mapsto \alpha^1(u_1) \dots \alpha^r(u_r) \beta^1(v_1) \dots \beta^s(v_s) \end{gathered} \tag{6}$$

Indeed, consider

$$(\alpha^1 \dots \alpha^r, v_1 \dots v_s) \in \underbrace{V^* \times \dots \times V^*}_{r} \times \underbrace{V \times \dots \times V}_{s} \mapsto \alpha^1(u_1) \dots \alpha^r(u_r) \beta^1(v_1) \dots \beta^s(v_s)$$

and so for

$$\begin{aligned} \alpha^i &= \alpha^i_\mu e^\mu, \quad i = 1, 2, \dots r,\ \mu = 1, 2, \dots \dim V^* \qquad \alpha^i(u_i) = \alpha^i_\mu u^\mu_i \\ v_i &= v^\mu_i e_\mu, \quad i = 1, 2, \dots s,\ \mu = 1, 2, \dots \dim V \qquad \beta^i(v_i) = \beta^i_\mu v^\mu_i \end{aligned}$$

So that

$$\begin{aligned} \alpha^1(u_1) \dots \alpha^r(u_r) \beta^1(v_1) \dots \beta^s(v_s) &= \alpha^1_{\alpha_1} u^{\alpha_1}_1 \dots \alpha^r_{\alpha_r} u^{\alpha_r}_r \beta^1_{\mu_1} v^{\mu_1}_1 \dots \beta^s_{\mu_s} v^{\mu_s}_s = \\ &= (u^{\alpha_1}_1 \dots u^{\alpha_r}_r \beta^1_{\mu_1} \dots \beta^s_{\mu_s})(\alpha^1_{\alpha_1} \dots \alpha^r_{\alpha_r} v^{\mu_1}_1 \dots v^{\mu_s}_s) \end{aligned}$$

Identify $u_1 \otimes \dots \otimes u_r \otimes \beta^1 \otimes \dots \otimes \beta^s$ with this multiplinear map.

Proposition 3. *If V is finite-dim. vector space, or if $V = \Gamma(E)$, for vector bundle $E \to M$, map*

$$(\otimes^r_{i=1} V) \otimes \left(\otimes^s_{j=1} V^*\right) \to T^r{}_s(V; R) \tag{7}$$

is an isomorphism.

Definition 6. *tensor that can be written as*

$$u_1 \otimes \dots \otimes u_r \otimes \beta^1 \otimes \dots \otimes \beta^s \equiv u_1 \otimes \dots \otimes \beta^s \tag{8}$$

*is **simple** or **decomposable**.*

Now well that not *all* tensors are simple.

Definition 7 (7.7[1], tensor product)**.** $\forall\, S \in T^{r_1}_{s_1}(V)$, $\forall\, T \in T^{r_2}_{s_2}(V)$,
define tensor product

$$\begin{gathered} S \otimes T \in T^{r_1+r_2}_{s_1+s_2}(V) \\ S \otimes T(\theta^1 \dots \theta^{r_1+r_2}, v_1 \dots v_{s_1+s_2}) := S(\theta^1 \dots \theta^{r_1}, v_1 \dots v_{s_1}) T(\theta^{r_1+1} \dots \theta^{r_1+r_2}, v_{s_1+1} \dots v_{s_1+s_2}) \end{gathered} \tag{9}$$

Proposition 4 (7.8[1])**.**

$$\tau^{i_1 \dots i_r}{}_{j_1 \dots j_s} e_{i_1} \otimes \dots \otimes e_{i_r} \otimes e^{j_1} \otimes \dots \otimes e^{j_s} = \tau(e^{i_1} \dots e^{i_r}, e_{j_1} \dots e_{j_s}) e_{i_1} \otimes \dots \otimes e_{i_r} \otimes e^{j_1} \otimes \dots \otimes e^{j_s} = \tau$$

So $\{e_{i_1} \otimes \dots \otimes e_{i_r} \otimes e^{j_1} \otimes \dots \otimes e^{j_s} | i_1 \dots i_r, j_1 \dots j_s \in 1 \dots n\}$ spans $T^r_s(V; R)$

Exercise 7.11. Let basis for $V \quad e_1 \dots e_n$, corresponding dual basis for $V^* \quad e^1 \dots e^n$
Let basis for $V \quad \overline{e}_1 \dots \overline{e}_n$, corresponding dual basis for $V^* \quad \overline{e}^1 \dots \overline{e}^n$
s.t.

$$\begin{aligned} \overline{e}_i &= C^k_{\ i} e_k \\ \overline{e}^i &= (C^{-1})^i_{\ k} e^k \end{aligned}$$

EY:20170404, keep in mind that

$$\begin{gathered} Ax = e_i A^i_{\ k} e^k(x^j e_j) = e_i A^i_{\ j} x^j = A^i_{\ j} x^j e_i \\ Ae_j = e_k A^k_{\ i} e^i(e_j) = A^k_{\ j} e_k = \overline{e}_j \\ \overline{\tau}^i_{\ jk} \overline{e}_i \otimes \overline{e}^j \otimes \overline{e}^k = \overline{\tau}^i_{\ jk} C^l_{\ i} e_l (C^{-1})^j_{\ m} e^m (C^{-1})^k_{\ n} e^n = \overline{\tau}^i_{\ jk} C^l_{\ i} (C^{-1})^j_{\ m} (C^{-1})^k_{\ n} = \tau^l_{\ mn} \\ \overline{\tau}^i_{\ jk} = C^c_{\ k} C^b_{\ j} (C^{-1})^i_{\ a} \tau^a_{\ bc} \end{gathered}$$

On Remark 7.13 of Jeffrey Lee (2009) [1]: first, egregious typo for $L(V, V)$; it shoudl be $L(V, W)$. Onward, for $L(V, W)$,
consider $W \otimes V^* \ni w \otimes \alpha$ s.t.

$$(w \otimes \alpha)(v) = \alpha(v) w \in W, \forall\, v \in V, \text{ so } w \otimes \alpha \in L(V, W)$$

Now consider (category of) left R-module,

$${}_R\mathbf{Mod} \ni {}_{\mathrm{Mat}_{\mathbb{K}}(N,M)}\mathbb{K}^N \tag{10}$$

where

$$\begin{aligned} V &= \mathbb{K}^N \\ W &= \mathbb{K}^M \end{aligned}$$

For $A \in \mathrm{Mat}_{\mathbb{K}}(N, M)$, $x \in \mathbb{K}^N$,

$$\begin{gathered} e_i A^i{}_{,\mu} e^\mu (x^\nu e_\nu) = Ax = e_i A^i{}_\mu x^\mu, \quad i = 1, 2, \dots M,\ \mu = 1, 2, \dots N \\ A \in \mathrm{Mat}_{\mathbb{K}}(N, M) \cong W \otimes V^* \cong L(V, W) \end{gathered}$$

Consider

$$\begin{aligned} \alpha \in (\mathbb{K}^N)^* = V^* &\qquad \alpha = \alpha_\mu e^\mu \\ w \in \mathbb{K}^M = W &\qquad w = w^i e_i \\ \alpha \otimes w = w \otimes \alpha &= w^i \alpha_\mu e_i \otimes e^\mu \end{aligned}$$

(remember, isomoprhism between $\mathrm{Mat}_{\mathbb{K}}(N, M)$ and $W \otimes V^*$ guaranteed, if V, W are free R-modules, $R = \mathbb{K}$).

Let V,W be left R-modules, i.e. $V,W \in {}_R\mathbf{Mod}$.

$$V^* \in \mathbf{Mod}_R$$

For $V^* \otimes W \in \mathbf{Mod}_R \otimes {}_R\mathbf{Mod}$

$$\alpha \in V^*, w \in W$$

$$(\alpha \otimes w)(v) = \alpha(v)w, \text{ for } v \in V \in {}_R\mathbf{Mod}$$

But $(w \otimes \alpha)(v) = w\alpha(v)$.

Note $\alpha(v) \in R$.

Let V,W be right R-modules, i.e. $V,W \in \mathbf{Mod}_R$.

$$V^* \in {}_R\mathbf{Mod}$$

For $W \otimes V^* \in \mathbf{Mod}_R \otimes {}_R\mathbf{Mod}$.

$$\alpha \in V^*, \, w \in W$$

$$(v)(w \otimes \alpha) = w\alpha(v), \text{ with } \alpha(v) \in R, \, v \in V$$

So $W \otimes V^* \cong L(V,W)$, for $V,W \in \mathbf{Mod}_R$

Definition 8 (7.20[1], **contraction**). *Let $(e_1, \dots e_n)$ basis for V, $(e^1 \dots e^n)$ dual basis.*
If $\tau \in T^r_{\ s}(V)$, then for $k \leq r$, $l \leq s$, define

$$\begin{gathered} C^k_l \tau \in T^{r-1}_{\ s-1}(V) \\ C^k_l \tau(\theta^1 \dots \theta^{r-1}, w_1 \dots w_{s-1}) := \\ \sum_{a=1}^n \tau(\theta^1 \dots \underbrace{e^a}_{kth\ position} \dots \theta^{r-1}, w_1 \dots \underbrace{e_a}_{ith\ position} \dots w_{s-1}) \end{gathered} \tag{11}$$

*C^k_l is called **contraction**, for some single $1 \leq k \leq r$, some single $1 \leq l \leq s$,*

$$C^k_l : T^r_s(V) \to T^{r-1}_{s-1}(V)$$

s.t.

$$(C^k_l \tau)^{i_1 \dots \widehat{i}_k \dots i_r}_{\qquad j_1 \dots \widehat{j}_l \dots j_s} := \tau^{i_1 \dots a \dots i_r}_{\qquad j_1 \dots a \dots j_s}$$

Universal mapping properties can be invoked to give a basis free definition of contraction (EY : 20170405???).

IN general,

$$\forall \, v_1 \dots v_s \in V, \forall \, \alpha^1 \dots \alpha^r \in V^*$$

so that

$$\begin{aligned} v_j &= v^\mu_j e_\mu \qquad j = 1 \dots s, \quad \mu = 1, \dots \mathrm{dim} V \\ \alpha^i &= \alpha^i_\mu e^\mu \qquad i = 1 \dots r, \quad \mu = 1 \dots \mathrm{dim} V^* \end{aligned}$$

then $\forall \, \tau \in T^r_{\ s}(V)$,

$$\begin{gathered} \tau(\alpha^1 \dots \alpha^r, v_1 \dots v_s) = \tau(\alpha^1_{\mu_1} e^{\mu_1} \dots \alpha^r_{\mu_r} e^{\mu_r}, v_1^{\nu_1} e_{\nu_1} \dots v_s^{\nu_s} e_{\nu_s}) = \\ = \alpha^1_{\mu_1} \dots \alpha^r_{\mu_r} v_1^{\nu_1} \dots v_s^{\nu_s} \tau(e^{\mu_1} \dots e^{\mu_r}, e_{\nu_1} \dots e_{\nu_s}) = \alpha^1_{\mu_1} \dots \alpha^r_{\mu_r} v_1^{\nu_1} \dots v_s^{\nu_s} \tau^{\mu_1 \dots \mu_r}_{\qquad \nu_1 \dots \nu_s} \end{gathered}$$

which is equivalent to

$$\begin{tikzcd} \tau \in T^r_{\ s}(V) \arrow[r, mapsto, "\alpha^1 \otimes \dots \otimes \alpha^r \otimes v_1 \otimes \dots \otimes v_s \otimes"] & \alpha^1 \otimes \dots \otimes \alpha^r \otimes v_1 \otimes \dots \otimes v_s \otimes \tau \arrow[d, mapsto, "C^1_{s+1} C^2_{s+2} \dots C^r_{r+s} C^r_1 C^{r+1}_2 \dots C^{r+s}_s"'] \\ & \tau(\alpha^1 \dots \alpha^r, v_1 \dots v_s) \in R \end{tikzcd}$$

where I've tried to express the right-R-module, "right action" on $\alpha^1 \otimes \dots \otimes \alpha^r \otimes v_1 \otimes \dots \otimes v_s \in V^* \otimes \dots \otimes V$.

Conlon (2008) [10]

Part 2. Prástaro

Prástaro (1996) [6]

2.0.1. *Affine Spaces.* cf. Sec. 1.2 - *Affine Spaces* of Prástaro (1996) [6]

Definition 9 (affine space)**.**

$$\begin{gathered} \textit{affine space} \qquad (M, \mathbf{M}, \alpha) \\ \textit{with} \\ M \equiv \textit{ set (set of pts.)} \\ \mathbf{M} \equiv \textit{ vector space (space of free vectors)} \\ \alpha \equiv \mathbf{M} \times M \to M \equiv \textit{ translation operator} \\ \alpha : (v,p) \mapsto p' \equiv p + v \end{gathered} \tag{12}$$

*Note: α is a **transitive** action and without fixed pts. (free).*
i.e. $\forall \, p \in M$,

$\forall$ pt. $O \in M$, $\alpha : (v,O) \mapsto O' \equiv O + v$, $\alpha(\cdot, O) \equiv \alpha_O \equiv \alpha(O)$. $\alpha_O(v) = O' = O + \mathbf{v}$ $\quad \forall \, O' \in M$, $\exists \, \mathbf{v} \in \mathbf{M}$ s.t. $O' = O + \mathbf{v}$ $\implies M \equiv \mathbf{M}$.

$\forall \, (O, \{e_i\})_{1 \leq i \leq n}$, where $\{e_i\}$ basis of $\mathbf{M}$, $M \equiv \mathbf{M} = \mathbb{R}^n$ so isomorphism $M \simeq \mathbb{R}^n$

Definition 10. *$(O, \{e_i\}) \equiv$ affine frame.*
$\forall$ affine frame $(O, \{e_i\})$, $\exists$ coordinate system $x^\alpha : M \to \mathbb{R}$,
where $x^\alpha(p)$ is αth component, in basis $\{e_i\}$, of vector $p - O$

Theorem 8 (1.4 Prástaro (1996) [6])**.** *Let $(x^\alpha), (\overline{x}^\alpha)$ 2 coordinate systems correspond to affine frames $(O, \{e_i\})$, $(\overline{O}, \{\overline{e}_i\})$, respectively.*

$$\overline{x}^\alpha = A^\alpha_{\ \beta} x^\beta + y^\alpha \tag{13}$$

where

$$y^\alpha \in \mathbb{R}^n, \qquad A^\alpha_{\ \beta} \in GL(n;\mathbb{R})$$

Definition 11 (1.10 Prástaro (1996) [6])**.**

$$A(n) \equiv Gl(n, \mathbb{R}) \times \mathbb{R}^n \tag{14}$$

affine group of dim. n

Theorem 9 (1.5)**.** *symmetry group of n-dim. affine space, called affine group $A(M)$ of M. $\exists$ isomoprhism,*

$$A(M) \simeq A(n), \qquad f \mapsto (f^\alpha_{\ \beta}, y^\alpha)\,; \qquad f^\alpha \equiv x^\alpha \circ f = f^\alpha_{\ \beta} x^\beta + y^\alpha \tag{15}$$

cf. Eq. 1.4 Prástaro (1996) [6]

Definition 12 (Conlon, 10.1.2)**.** *If $X, Y \in \mathfrak{X}(M)$, $M \subset \mathbb{R}^m$, **Levi-Civita connection** on $M \subset \mathbb{R}^m$*

$$\begin{gathered} \nabla : \mathfrak{X}(M) : \mathfrak{X}(M) \to \mathfrak{X}(M) \\ \nabla_X Y := p(D_X Y) \end{gathered} \tag{16}$$

with

$$D_X Y := \sum_{j=1}^m X(Y^j) \frac{\partial}{\partial x^j} = \sum_{i,j=1}^m X^i \frac{\partial Y^j}{\partial x^i} \frac{\partial}{\partial x^j} \qquad \begin{gathered} \forall\, X = \sum_{i=1}^m X^i \frac{\partial}{\partial x^i}, \\ \forall\, Y = \sum_{i=1}^m Y^i \frac{\partial}{\partial x^i} \end{gathered}$$

$$\begin{aligned} &\nabla_{fX} Y = f(D_{fX} Y) = p(f D_X Y) = f p D_X Y = f \nabla_X Y \\ &\nabla_X fY = p(D_X fY) = p \left(\sum_{i,j=1}^m \left(X^i f \frac{\partial Y^j}{dx^i} + X^i Y^j \frac{\partial f}{\partial x^i} \right) \frac{\partial}{\partial x^j} \right) = f \nabla_X Y + p \sum_{j=1}^m X(f) Y^j \frac{\partial}{\partial x^j} = f \nabla_X Y + X(f) p(Y) \end{aligned}$$

Definition 13 (Conlon, 10.1.4; Christoffel symbols)**.**

$$\begin{gathered} \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma^k_{ij} \frac{\partial}{\partial x^k} \qquad \textit{(Conlon's notation)} \\ \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i} = \Gamma^k_{ij} \frac{\partial}{\partial x^k} \qquad \textit{(F. Schuller's notation)} \end{gathered} \tag{17}$$

Definition 14 (torsion)**.**

$$\begin{aligned} &T : \mathfrak{X}(M) \in \mathfrak{X}(M) \to \mathfrak{X}(M) \\ &T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] \end{aligned} \tag{18}$$

If $T = 0$, ∇ torsion-free or symmetric.

$$\begin{gathered} T(fX, Y) = f \nabla_X Y - (f \nabla_Y X + Y(f) X) - \{ (fXY - (Y(f)X + fYX) \} = fT(X,Y) \\ T(X, fY) = f \nabla_X Y + X(f) Y - f \nabla_Y X - \{ ((X(f)Y + fXY) - fYX \} = fT(X,Y) \end{gathered}$$

Thus, $T(X,Y)$ $C^\infty(M)$-bilinear.
$T \in \tau^2_1(M)$.
$T(v,w) \in T_x M$ defined, $\forall\, v, w \in T_x M$, $\forall\, x \in M$.
Thus, torsion is a **tensor**.

Exercise 10.1.7 Conlon (2008)[10] **.** .
If $T(X,Y) = 0$,

$$T(e_i, e_j) = \Gamma^k_{ji} e_k - \Gamma^k_{ij} e_k - 0 = 0 \Longrightarrow \Gamma^k_{ji} = \Gamma^k_{ij}$$

If $\Gamma^k_{ij} = \Gamma^k_{ji}$, $T(e_i, e_j) = 0$.

Exercise 10.1.8, Conlon (2008)[10]**.**
If $M \subset \mathbb{R}^m$ smoothly embedded submanifold, $\forall\, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^i} \in T_x M$, spanning $T_x M$, consider $\frac{\partial}{\partial x^j} = X^k_j \frac{\partial}{\partial \widetilde{x}^k}$, $\frac{\partial}{\partial x^i} = X^k_i(\widetilde{x}) \frac{\partial}{\partial \widetilde{x}^k}$

$$\begin{gathered} \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i} = p D_{X^k_j \frac{\partial}{\partial \widetilde{x}^k}} X^l_i \frac{\partial}{\partial \widetilde{x}^l} = p \left(X^k_j \frac{\partial X^l_i}{\partial \widetilde{x}^k} \frac{\partial}{\partial \widetilde{x}^l} \right) = X^k_j p \left(\frac{\partial X^l_i}{\partial \widetilde{x}^k} \frac{\partial}{\partial \widetilde{x}^l} \right) \\ \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = X^k_i p \left(\frac{\partial X^l_j}{\partial \widetilde{x}^k} \frac{\partial}{\partial \widetilde{x}^l} \right) \end{gathered}$$

If $X \in \mathfrak{X}(M)$, smooth $s : [a,b] \to M$,
then $\forall\, s(t)$,

$$X'_{s(t)} = \nabla_{\dot{s}(t)} X \in T_{s(t)} M$$

In fact, it's often natural to consider fields $X_{s(t)}$ along s, parametrized by parameter t, allowing

$$X_{s(t_1)} \neq X_{s(t_2)}$$

each of $s(t_1) = s(t_2)$.

Definition 15 (10.1.9)**.** *Let smooth $s : [a,b] \to M$.*
Vector field along s is smooth $v : [a,b] \to TM$ s.t.

$$\begin{tikzcd} & TM \arrow[d, mapsto, "\pi"] \\ {[a,b]} \arrow[ur, mapsto, "v"] \arrow[r, mapsto, "s"] & M \end{tikzcd}$$

commutes.
Note that $v \in \mathfrak{X}(s) \subset \mathfrak{X}(M)$

e.g. $(Y|s)(t) = Y_{s(t)}$, restriction of $Y \in \mathfrak{X}(M)$ to s.
e.g. $\dot{s}(t) \in \mathfrak{X}(M)$.
$\forall\, v, w \in \mathfrak{X}(s)$, $v + w \in \mathfrak{X}(s)$,

$$(fv + gv)(t) := (f(s(t)) + g(s(t)))v(t) = f(s(t))v(t) + g(s(t))v(t) = (f+g)v(t)$$

Likewise,

$$f(v + w) = fv + fw$$

$\mathfrak{X}(s)$ is a real vector space and $C^\infty[a,b]$-module.

Definition 16 (10.1.10)**.** *Let conection ∇ on M.*
***Associated covariant derivative** is operator*

$$\frac{\nabla}{dt} \mathfrak{X}(s) \to \mathfrak{X}(s)$$

$\forall$ smooth s on M, s.t.
(1) $\frac{\nabla}{dt}$ *$\mathbb{R}$-linear*
(2) $\left(\frac{\nabla}{dt} \right) (fv) = \frac{df}{dt} v + f \frac{\nabla}{dt} v$, $\forall\, f \in C^\infty[a,b]$, $\forall\, v \in \mathfrak{X}(s)$
(3) *If $Y \in \mathfrak{X}(M)$, then*

$$\frac{\nabla}{dt} (Y|s)(t) = \nabla_{\dot{s}(t)} Y \in T_{s(t)} M, \quad a \leq t \leq b$$

Theorem 10 (Conlon Thm. 10.1.11[10])**.** *$\forall$ connection ∇ on M, $\exists!$ associated covariant derivative $\frac{\nabla}{dt}$*

Proof. Consider arbitrary coordinate chart $(U, x^1 \dots x^n)$.
Consider smooth curve $s : [a,b] \to U$.
Let $v \in \mathfrak{X}(s)$, $v(t) = v^i(t) \frac{\partial}{\partial x^i}$; $\dot{s}(t) = s^j \frac{\partial}{\partial x^j}$.

$$\frac{\nabla v}{dt} = \frac{dv^i(t)}{dt} \frac{\partial}{\partial x^i} + v^i(t) \frac{\nabla}{dt} \frac{\partial}{\partial x^i} = \frac{dv^i}{dt} \frac{\partial}{\partial x^i} + v^i \nabla_{\dot{s}(t)} \frac{\partial}{\partial x^i} = \dot{v}^i \frac{\partial}{\partial x^i} + v^i \dot{s}^j \Gamma^k_{ij} \frac{\partial}{\partial x^k} = \left(\dot{v}^k + v^i \dot{s}^j \Gamma^k_{ij} \right) \frac{\partial}{\partial x^k}$$

This is an explicit, local formula in terms of connection, proving uniqueness.
Existence: $\forall$ coordinate chart $(U, x^1 \dots x^n)$, $\left(\dot{v}^k + v^i \dot{s}^j \Gamma^k_{ij} \right) \frac{\partial}{\partial x^k} =: \frac{\nabla v}{dt}$.

$$\frac{\nabla}{dt}(fv) = \dot{f} v^k + f \dot{v}^k + f v^i \dot{s}^j = \dot{f} v + f \frac{\nabla v}{dt}$$

If f constant, then $\frac{\nabla}{dt}$ is $\mathbb{R}$-linear. ∎

Definition 17 (10.1.12 Conlon (2008)[10])**.** *Let (M, ∇). Let $v \in \mathfrak{X}(s)$ for smooth $s : [a,b] \to M$.*
If $\frac{\nabla v}{dt} \equiv 0$ on s, then v is ***parallel*** *along s.*

Theorem 11 (10.1.13)**.** *Let (M, ∇), smooth $s : [a,b] \to M$, $c \in [a,b]$, $v_0 \in T_{s(c)} M$.*
Then $\exists !$ parallel field $v \in \mathfrak{X}(s)$ s.t. $v(c) = v_0$.
v parallel transport along s.

Proof.

$$\begin{aligned} \dot{s}(t) &= \dot{s}^j(t) e_j \\ v(t) &= v^i(t) e_i \\ v_0 &= a^i e_i \end{aligned}$$

$$0 = \left(\frac{dv^k}{dt}(t) + v^i(t) \dot{s}^j(t) \Gamma^k_{ij}(s(t)) \right) e_k$$

or equivalently

$$\frac{dv^k}{dt} = -v^i \dot{s}^j \Gamma^k_{ij}, \qquad 1 \leq k \leq n \qquad (10.1) \tag{19}$$

with initial conditions $v^k(c) = a^k$, $1 \leq k \leq n$.
By existence and uniqueness of solutions of O.D.E.
$\exists \, \epsilon > 0$ s.t. $\exists !$ solutions $v^k(t)$. For $c - \epsilon < t < c + \epsilon$.
In fact, these ODEs being linear in v^k, by ODE theory (Appendix C, Thm. C.4.1).
$\nexists$ restriction on ϵ, so $\exists ! \, v^k(t) \quad \forall \, t \in [a,b]$, $1 \leq k \leq n$ ∎

2.0.2. *Principal bundle, vector bundle case for parallel transport.* Recall the 2 different forms or viewpoints for Lie-algebra valued 1-forms, or vector-valued 1-forms, or sections of 1-form-valued endomorphisms:

$$\omega^k_{\ i\mu} dx^\mu \equiv \omega^k_{\ i} \in \Omega^1(M, \mathfrak{gl}(n, \mathbb{F})) = \Gamma(\mathfrak{gl}(n, \mathbb{R} \otimes T^*M|_U))$$

for $i, k = 1 \dots n = \dim E$.
$\mu = 1 \dots d = \dim E$
Now

$$D_X \mu = X^\mu D_{\frac{\partial}{\partial x^\mu}} \mu = X^\mu \left[\left(\frac{\partial}{\partial x^\mu} \mu^k \right) e_k + \mu^i \omega^k_{\ i\mu} e_k \right] = \left(X(\mu^k) + \mu^i \omega^k_{\ i}(X) \right) e_k = \left(d\mu^k(X) + \mu^i \omega^k_{\ i}(X) \right) e_k$$

So then define

$$\begin{aligned} &D : \Gamma(E) \to \Gamma(E) \otimes \Gamma(T^*M) \\ &D\mu = D(\mu^i e_i) = e_k (d\mu^k + \mu^i \omega^k_{\ i}) \equiv (d + A)\mu \end{aligned} \tag{20}$$

Also, D can be defined for this case:

$$D : \Gamma(\text{End}(E)) \to \Gamma(\text{End} E) \otimes \Gamma(T^*M)$$

Let $\sigma = \sigma^i_{\ j} e_i \otimes e^j \in \Gamma(\text{End}(E))$

$$\begin{aligned} D\sigma &= D(\sigma^i_{\ j} e_i) \otimes e^j + \sigma^i_{\ j} e_i \otimes D^* e^j = \left(d\sigma^k_{\ j} + \sigma^i A^k_{\ i} \right) e_k \otimes e^j + \sigma^i_{\ j} e_i \otimes (A^*)^{\ j}_k e^k = \\ &= (d\sigma^k_{\ j} + \sigma^i_{\ j} A^k_{\ i}) e_k \otimes e^j + \sigma^k_{\ i} e_j \otimes (-A^i_{\ j}) e^j = (d\sigma^k_{\ j} + [A, \sigma]^k_{\ j}) e_k \otimes e^j \end{aligned} \tag{21}$$

cf. Def. 4.1.4 of Jost (2011), pp. 138.
For $\mu \in \Gamma(E)$, smooth $s : [a,b] \to M$, $X(t) = \dot{s}(t)$,

$$D_{\dot{s}(t)} \mu = \dot{s}^\mu D_{\frac{\partial}{\partial x^\mu}} \mu = \dot{s}^\mu \left[\frac{\partial \mu^k}{\partial x^\mu} e_k + \mu^i \omega^k_{\ i\mu} e_k \right] = \left[\dot{s}^\mu \frac{\partial \mu^k}{\partial x^\mu} + \dot{s}^\mu \mu^i \omega^k_{\ i\mu} \right] e_k = \frac{d}{dt} \mu(s(t)) + \mu^i \dot{s}^\mu \omega^k_{\ i\mu} e_k \tag{22}$$

Let $D_{\dot{s}(t)} \mu = 0$. Then,

$$\frac{d}{dt} \mu(s(t)) = -\mu^i \dot{s}^\mu \omega^k_{\ i\mu} e_k \tag{23}$$

Recall, given vector bundle $E \xrightarrow{\pi} N$, given $\varphi : M \to N$, then pullback

$$\varphi^* E \to M \tag{24}$$

i.e.

$$\begin{tikzcd} \varphi^* E \arrow[d, "\psi"] & E \arrow[l, "\varphi^*"'] \arrow[d, "\pi"] & (\varphi^* E)_x = E_{\varphi(x)} \\ M \arrow[r, "\varphi"] & N & x \in M \arrow[u, mapsto] \end{tikzcd}$$

i.e. if $s \in \Gamma(E)$,

$$\varphi^* s = s \circ \varphi \in \Gamma(\varphi^* E)$$

Thus,

$$\begin{tikzcd} \gamma^* E \arrow[d] & E \arrow[l, "\gamma^*"'] \arrow[d, "\pi"] & (\varphi^* E)_c = E_{\gamma(c)} \\ {[a,b]} \arrow[r, "\gamma"] & M & c \in {[a,b]} \arrow[u, mapsto] \end{tikzcd}$$

For

$$\begin{aligned} \dot{v}^k &= -v^i \dot{s}^j \Gamma^k_{\ ij} \\ v^k(c) &= v^k_0 \qquad 1 \leq k \leq m \\ \dot{v} &= -v^i \dot{s}^j \Gamma_{ij} \\ (v \dot{+} w) &= -(v^i + w^i) \dot{s}^j \Gamma_{ij} (v + w)(c) = v(c) + w(c) = v_0 + w_0 \end{aligned}$$

so $v + w \in \mathfrak{X}(s)$ is parallel transport of $v_0 + w_0$.
Likewise, $\forall \, a \in \mathbb{F}$, $av \in \mathfrak{X}(s)$ is the parallel transport of av_0.

$$\dot{\mu}^k = -\mu^i \dot{s}^\mu \omega^k_{\ i\mu} = -\mu^i \omega^k_{\ i}(\dot{s}^\mu)$$

Suppose $\gamma^* E$ trivialized over $[a,b]$.
Closed interval is contractible, so this is always possible.
For chart (U,φ),

$$\begin{tikzcd} \gamma^* E \arrow[d] & E \arrow[l, "\gamma^*"] \arrow[d, "\pi"] \\ {[a,b]} \arrow[r, "\gamma"] & M \end{tikzcd} \qquad \begin{tikzcd} E|_U \arrow[r, "\psi"] & U \times V \\ U \subset M \arrow[u, "\pi^{-1}"] \arrow[ur] & \end{tikzcd}$$

Consider

$$\begin{aligned} \varphi &: [a,b] \times V \to \gamma^* E \\ \varphi(t,\cdot) &= \gamma^* \circ \psi^{-1}(\gamma(t),\cdot) \end{aligned}$$

$\forall\, \mu \in \Gamma(E|_{x\in M})$,
$\mu = \mu^i e_i$.
$\varphi(t,e_i) = \epsilon_i$ is a basis for $\gamma^* E$.
$\forall\, \sigma \in \Gamma(\gamma^* E)$,

$$\sigma = \sigma^i \epsilon_i, \quad \sigma^i : [a,b] \to \mathbb{F}$$

$$\nabla_{\frac{\partial}{\partial x^\mu}} \sigma = \frac{\partial \sigma^k}{\partial x^\mu}\epsilon_k + \omega^k_{\ j\mu}\sigma^j \epsilon_k = \left(\frac{\partial \sigma^k}{\partial x^\mu} + \omega^k_{\ ,j\mu}\sigma^j \right)\epsilon_k$$

$$\nabla \sigma = \epsilon_k \otimes (d\sigma^k + \omega^k_{\ j\mu} dx^\mu \sigma^j) = \epsilon_k \otimes (d\sigma^k + \omega^k_{\ j}\sigma^j)$$

$$\nabla_{\frac{d}{dt}} \sigma = \epsilon_k \otimes \left(\frac{d\sigma^k}{dt} + \omega^k_{\ j\mu}\dot{x}^\mu \sigma^j \right)$$

Now

$$\frac{d}{dt} = \dot{x}^\nu \frac{\partial}{\partial x^\nu}$$

Then σ parallel along γ if

$$\frac{d\sigma^k}{dt} + \omega^k_{\ j\mu}\dot{x}^\mu \sigma^j = 0$$

Definition 18 (3.1.4 [11]). *Parallel transport along γ is*

(25)
$$\begin{aligned} P_\gamma &: E_{\gamma(a)} \to E_{\gamma(b)} \\ P_\gamma(v) &\mapsto \sigma(b) \end{aligned}$$

where $\sigma \in \Gamma(\gamma^* E)$, σ unique and s.t. $\sigma(a) = v$.

Lemma 1 (10.1.16[10]). *holonomy*

$$h_s : T_x M \to T_{x_0} M$$

if ∇ around piecewise smooth loop s is a linear transformation.

Lemma 2 (10.1.18 Conlon (2008)[10]). *Let piecewise smooth loop $s : [a,b] \to M$ at x_0.*
Let weak reparametrization $\widetilde{s} = s \circ r : [c,d] \to M$.
If reparametrization is orientation-preserving, then $h_{\widetilde{s}} = h_s$,
If reparametrization is orientation-reversing, then $h_{\widetilde{s}} = h_s^{-1}$,

Proof. Without loss of generality, assume smooth s, r

$$\begin{aligned} \widetilde{s}(\tau) &= s(r(\tau)) \\ \widetilde{v}(\tau) &= v(r(\tau)) \end{aligned}$$

$$\begin{aligned} \widetilde{u}^j(\tau) &= \frac{dt}{d\tau}(\tau) u^j(r(\tau)) \\ \frac{d\widetilde{v}^k}{d\tau}(\tau) &= \frac{dr}{d\tau}(\tau)\frac{dv^k}{dt}(r(\tau)) \\ \frac{d\widetilde{v}^k}{d\tau} &= -\widetilde{v}^i \widetilde{u}^j \Gamma^k_{ij} \end{aligned}$$

since

$$\begin{aligned} \frac{dv^k}{dt} &= -v^i u^j \Gamma^k_{ij}; \qquad 1 \leq k \leq n \\ v^k(c) &= a^k; \qquad 1 \leq k \leq a \end{aligned}$$

$$\frac{dr}{d\tau}\frac{dv^k}{dt} = -v^i \frac{dr}{d\tau} u^j \Gamma^k_{\ ij} = \frac{d\widetilde{v}^k}{d\tau} = -\widetilde{v}^i \widetilde{u}^j \Gamma^k_{\ ij}$$

Thus, if $r(c) = a$, $r(d) = b$

$$h_{\widetilde{s}}(v_0) = \widetilde{v}(d) = v(b) = h_s(v_0)$$

If $r(c) = a$, $r(d) = b$, then

$$\widetilde{v}(c) = v(b) = h_s(v_0)$$

and

$$h_{\widetilde{s}}(h_s(v_0)) = h_{\widetilde{s}}(v(b)) = \widetilde{v}(d) = v(a) = v_0$$

At this point, I will switch to my notation because it clarified to me, at least, what was going on, in that a holonomy h_s is *invariant* under orientation-preserving reparametrization, and its inverse is well-defined.
For $\widetilde{s} = s \circ t : [c,d] \to M$,
piecewise smooth t is reparametrized, i.e.

(26)
$$t : [c,d] \to [a,b]$$

Now,

$$\begin{aligned} \frac{d}{d\tau}\widetilde{s}(\tau) &= \frac{d}{d\tau}\widetilde{s}(t(\tau)) = \dot{s}(t)\frac{dt}{d\tau}(\tau) \equiv \dot{s}\frac{dt}{d\tau} \\ v^k(t) &= v^k(t(\tau)) = v^k(\tau) \\ \frac{dv^k}{d\tau}(t(\tau)) &= \frac{dv^k}{dt}\frac{dt}{d\tau} = \frac{dt}{d\tau}(-v^i(\tau)\dot{s}^j(t)\Gamma^k_{\ ij}) = -v^i(\tau)\frac{d\widetilde{s}^j}{d\tau}\Gamma^k_{\ ij} \end{aligned}$$

Consider

$$h_s(v_0) = v(b)$$

If $t(c) = a$,
$t(d) = b$

$$h_{\widetilde{s}}(v_0) = \widetilde{v}(d) = v(t(d)) = v(b) = h_s(v_0)$$

If $t(c) = b$,
$t(d) = a$

$$\begin{aligned} h_{\widetilde{s}}(h_s(v_0)) &= h_{\widetilde{s}}(v(b)) = h_{\widetilde{s}}(v(t(c))) = h_{\widetilde{s}}(\widetilde{v}(c)) = \\ &= \widetilde{v}(d) = v(t(d)) = v(a) = v_0 \end{aligned}$$

Thus,

$$\boxed{h_{\tilde{s}} = h_s^{-1}}$$

□

I am working through Conlon (2008) [10] , Clarke and Santoro (2012) [11], and Schreiber and Waldorf (2007) , concurrently, for holonomy.

Part 3. Complex Manifolds

EY : 20170123 I don't see many good books on Complex Manifolds for physicists other than Nakahara's. I will supplement this section on Complex Manifolds with external links to the notes of other courses that I found useful to myself.

Complex Manifolds - Lecture Notes Koppensteiner (2010) [7]

Lectures on Riemannian Geometry, Part II: Complex Manifolds by Stefan Vandoren

Vandoren (2008) [8]

Part 4. Morse Theory

3. Morse Theory introduction from a physicist

I needed some physical motivation to understand Morse theory, and so I looked at Hori, et. al. [9].

cf. pp. 43, Sec. 3.4 Morse Theory, from Ch. 3. Differential and Algebraic Topology of Hori, et. al. [9].

Consider smooth $f : M \to \mathbb{R}$, with non-degenerate critical points.

If no critical values of f between a and b ($a < b$), then subspace on which f takes values less than a is deformation retract of subspace where f less than b, i.e.

$$\{x \in M | f(x) < b\} \times [0,1] \xrightarrow{F} \{x \in M | f(x) < b\}$$

$\forall\, x \in M$ s.t. $f(x) < b$,

$$\begin{aligned} & F(x,0) = x \\ & F(x,1) \in \{x \in M | f(x) < a\} \end{aligned} \qquad \text{and } F(a',1) = a' \qquad \forall\, a' \in M \text{ s.t. } f(a') < a$$

To show this, consider $-\nabla f / |\nabla f|^2$

Morse lemma: $\forall$ critical pt. p s.t. $\exists$ choice of coordinates s.t.

$$f = -(x_1^2 + x_2^2 + \dots + x_\mu^2) + x_{\mu+1}^2 + \dots + x_n^2 \qquad (27)$$

where $f(p) = 0$ and p is at origin of these coordinates.

- difference between
$$f^{-1}(\{x \leq -\epsilon\}),\ f^{-1}(\{x \leq +\epsilon\})$$
can be determined by local analysis and only depends on μ, $\mu \equiv$ "Morse index" = number of negative eigenvalues of Hessian of f at critical pt.

  Answer:

  $f^{-1}(\{x \leq +\epsilon\})$ can be obtained from $f^{-1}(\{x \leq -\epsilon\})$ by "attaching μ-cell" along boundary $f^{-1}(0)$
- "attaching μ-cell to X mean, take μ-ball $B_\mu = \{|x| \leq 1\}$ in μ-dim. space, identity pts. on boundary $S^{\mu-1}$ with pts. in the space X, through cont. $f : S^{\mu-1} \to X$, i.e. take
$$X \coprod B_\mu$$
with $x \sim f(x) \quad \forall\, x \in \partial B_\mu = S^{\mu-1}$.
- find homology of M,

  f defines chain complex C_f^*, kth graded piece C^{α_k}, α_k is number of critical pts. with index k.
$$\begin{aligned} & \partial : C_p^k \to C_p^{k-1} \\ & \partial x_a = \sum_b \Delta_{a,b} x_b \end{aligned} \qquad (28)$$
where $\Delta_{a,b} :=$ signed number of lines of gradient flow from x_a to x_b, b labels pts. of index $k-1$.

  Gradient flow line is path $x(t)$ s.t. $\dot{x} = \nabla(f)$, with $\begin{aligned} x(-\infty) = x_a \\ x(+\infty) = x_b \end{aligned}$
- To define this number ($\Delta_{a,b}$?), construct moduli space of such lines of flow (???) by intersecting outward and inward flowing path spaces from each critical point, and then show this moduli space is oriented, 0-dim. manifold (pts. with signs)
- $\partial^2 = 0$ proof

  ∂, boundary of space of paths connecting critical points, whose index differs by 2 = union over compositions of paths between critical pts. whose index differs by 1.

  $\Longrightarrow$ coefficients of ∂^2 are sums of signs of pts. in 0-dim. space, which is boundary of 1-dim. space.

  These signs must therefore add to 0, so $\partial^2 = 0$.

Hori, et. al. [9] is good for physics, but there isn't much thorough, step-by-step explanations of the math. I will look at Hirsch (1997) [5] and Shastri (2011) [4] at the same time.

3.1. Introduction, definitions of Morse Functions, for Morse Theory.

cf. Ch. 6, Morse Theory of Hirsch (1997) [5], Section 1. Morse Functions, pp. 143-

Recall for TM, $T_xM \xrightarrow{\varphi} \mathbb{R}^n$.

Cotangent bundle T^*M defined likewise:

$$T_x^*M \xrightarrow{\varphi} \text{ dual vector space } (\mathbb{R}^n)^* = L(\mathbb{R}^n, \mathbb{R})$$

i.e.

$$T^*M = \bigcup_{x \in M} (M_x^*) \qquad M_x^* = L(M_x, \mathbb{R})$$

If chart (φ, U) on M, natural chart on T^*M is

$$\begin{aligned} & T^*U \to \varphi(U) \times (\mathbb{R}^n)^* \\ & \lambda \in M_x^* \mapsto (\varphi(x), \lambda\varphi_x^{-1}) \end{aligned}$$

Projection map

$$\begin{aligned} & p : T^* \to M \\ & M_x^* \mapsto x \end{aligned}$$

Let C^{r+1} map, $1 \leq r \leq \omega$, $f : M \to \mathbb{R}$, $\forall\, x \in M$, linear map $T_x f : M_x \to \mathbb{R}$ belongs to M_x^*

$$T_x f = Df_x \in M_x^*$$

Then

$$\begin{aligned} & Df : M \to T^*M \\ & x \mapsto Df_x = Df(x) \end{aligned}$$

is C^r section of T^*M.

Definition 19. ***critical point*** *x of f is zero of Df, i.e.*

$$Df(x) = 0 \qquad (29)$$

of vector space M_x^*.

Thus, set of critical pts. of f is counter-image of submanifold $Z^* \subset T^*M$ of zeros.
Note $Z^* \approx M$, codim. of Z^* is $n = \dim M$.

Definition 20. ***Morse function*** *f if $\forall$ critical pts. of f are nondegenerate.*

Note set of critical pts. closed discrete subset of M.
Let open $U \subset \mathbb{R}^n$, let C^2 map $g : U \to \mathbb{R}$,
critical pt. $p \in U$ nondegenerate iff

- linear $D(Dg)(p) : \mathbb{R}^n \to (\mathbb{R}^n)^*$ bijective
- identify $L(\mathbb{R}^n, (\mathbb{R}^n)^*)$ with space of bilinear maps $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $\Longrightarrow$ equivalent to condition that symmetric bilinear $D^2g(p) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ non-degenerate
- $n \times n$ *Hessian matrix*
$$\left[\frac{\partial^2 g}{\partial x^i \partial x^j}(p) \right]$$
has rank n

Hessian of g at critical pt. p is quadratic form $H_p f$ associated to bilinear form $D^2 g(p)$

$$\Longrightarrow H_p f(y) = D^2 g(p)(y, y) = \sum_{i,j} \frac{\partial^2 g}{\partial x^i \partial x^j}(p) y^i y^j$$

Let open $V \subset \mathbb{R}^n$, suppose C^2 diffeomorphism $h : V \to U$.
Let $q = h^{-1}(p)$, so q is critical pt. of $gh : V \to \mathbb{R}$.

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "H_q(gh)"] \arrow[d, "Dh(q)"'] & \mathbb{R} \\ \mathbb{R}^n \arrow[ur, "H_p g"'] & \end{tikzcd}$$

(quadratic) form $(H_p f)$ invariant under diffeomorphisms.
Let C^2 $f : M \to \mathbb{R}$.
$\forall$ critical pt. x of f, define
Hessian quadratic form

$$H_x f : M_x \to \mathbb{R}$$

$$H_x f : M_x \xrightarrow{D\varphi_x} \mathbb{R}^n \xrightarrow{H_{\varphi(x)}(f\varphi^{-1})} \mathbb{R}$$

where φ is any chart at x.
Thus, critical pt. of a C^2 real-valued function nondegenerate iff associated Hessian quadratic form is nondegenerate.
Let Q nondegenerate quadratic form on vector space E.
Q negative definite on subspace $F \subset E$ if $Q(x) < 0$ whenever $x \in F$ nonzero.
Index of $Q \equiv \mathrm{Ind}Q$, is largest possible dim. of subspace on which Q is negative definite.
cf. 1.1. Morse's Lemma of Ch. 6, pp. 145, Morse Theory of Hirsch (1997) [5]

Lemma 3 (Morse's Lemma)**.** *Let $p \in M$ be nondegenerate critical pt. of index k of C^{r+2} map $f : M \to \mathbb{R}$, $1 \leq r \leq \omega$.*
Then $\exists\, C^r$ chart (φ, U) at p s.t.

$$f\varphi^{-1}(u_1 \dots u_n) = f(p) - \sum_{i=1}^{k} u_i^2 + \sum_{i=k+1}^{n} u_i^2 \tag{30}$$

Let ${}^TQ \equiv Q^T$ denote tranpose of matrix Q.

Lemma 4. *Let $A = diag\{a_1, \dots, a_n\}$ diagonal $n \times n$ matrix, with diagonal entries ± 1.*
Then $\exists$ neighborhood N of A in vector space of symmetric $n \times n$ matrices, C^∞ map

$$P : N \to GL(n, \mathbb{R}) \tag{31}$$

s.t. $P(A) = I$, and if $P(B) = Q$, then $Q^T B Q = A$

Proof. Let $B = [b_{ij}]$ be symmetri matrix near A s.t. $b11 \neq 0$ and b_{11} has same sign as a_1.
Consider $x = Ty$ where

$$x_1 = \left[y_1 - \frac{b_{12}}{b_{11}} y_2 - \cdots - \frac{b_{1n}}{b_{11}} y_n \right] / \sqrt{|b_n|}$$
$$x_k = y_k \text{ for } k = 2, \dots n$$

□

4. Lagrange multipliers

From *wikipedia:Lagrange multiplier*, `https://en.wikipedia.org/wiki/Lagrange_multiplier`, find local minima (maxima), pt. $a \in N$, s.t. $\exists$ neighborhood U s.t. $f(x) \geq f(a)$ $(f(x) \leq f(a))$ $\forall x \in U$.
For $f : U \to \mathbb{R}$, open $U \subset \mathbb{R}^n$, find $x \in U$ s.t. $D_x f \equiv Df(x) = 0$, check if Hessian $H_x f < 0$.
Maxima may not exit since U open.
References:
Relative Extrema and Lagrange Multipliers
Other interesting links:
The Lagrange Multiplier Rule on Manifolds and Optimal Control of nonlinear systems

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