THE ALGEBRAIC GEOMETRY ALGEBRAIC TOPOLOGY DUMP

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Abstract. Everything about Algebraic Geometry, Algebraic Topology

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1. Geometry, Algebra, and Algorithms

1.1. Polynomials and Affine Space.
fields are important is that linear algebra works over *any* field

Definition 1 (2). *set of all polynomials in $x_1, \dots, x_n$ with coefficients in k, denoted $k[x_1, \dots, x_n]$*

polynomial f *divides* polynomial g provided $g = fh$ for some $h \in k[x_1, \dots, x_n]$

$k[x_1, \dots, x_n]$ satisfies all field axioms except for existence of multiplicative inverses; commutative ring, $k[x_1, \dots, x_n]$ *polynomial ring*

Exercises for 1. **Exercise 1.** $\mathbb{F}_2$ commutative ring since it's an abelian group under addition, commutative in multiplication, and multiplicative identity exists, namely 1. It is a field since for $1 \neq 0$, the multiplicative identity is 1.

Exercise 2.

(a)
(b)
(c)

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Part 2. Reading notes on Cox, Little, O'Shea's *Using Algebraic Geometry*

Using Algebraic Geometry. David A. Cox. John Little. Donal O'Shea. Second Edition. Springer. 2005. ISBN 0-387-20706-6 QA564.C6883 2004

7. Introduction

7.1. **Polynomials and Ideals.** *monomial*

(1)
$$x_1^{\alpha_1} \dots x_n^{\alpha_n} \qquad (1.1)$$

total degree of x^α is $\alpha_1 + \dots + \alpha_n \equiv |\alpha|$

field k, $k[x_1 \dots x_n]$ collection of all polynomials in $x_1 \dots x_n$ with coefficients k.

polynomials in $k[x_1 \dots x_n]$ can be added and multiplied as usual, so $k[x_1 \dots x_n]$ has structure of commutative ring (with identity)
however, only nonzero constant polynomials have multiplicative inverses in $k[x_1 \dots x_n]$, so $k[x_1 \dots x_n]$ not a field
however set of rational functions $\{f/g | f, g \in k[x_1 \dots x_n], \, g \neq 0\}$ is a field, denoted $k(x_1 \dots x_n)$

so
$$f = \sum_\alpha c_\alpha x^\alpha$$
where $c_\alpha \in k$
so
$$f \in k[x_1 \dots x_n] = \{f | f = \sum_\alpha c_\alpha x^\alpha, x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}, c_\alpha \in k\}$$

f homogeneous if all monomials have same total degrees
polynomial f is homogeneous if all monomials have the *same total degree*

Given a collection of polynomials $f_1 \dots f_s \in k[x_1 \dots x_n]$, we can consider all polynomials which can be built up from these by multiplication by arbitrary polynomials and by taking sums

Definition 2 (1.3). *Let $f_1 \dots f_s \in k[x_1 \dots x_n]$*
Let $\langle f_1 \dots f_s \rangle = \{p_1 f_1 + \dots + p_s f_s | p_i \in k[x_1 \dots x_n]$ for $i = 1 \dots s\}$

Exercise 1.

(a) $x^2 = x \cdot (x - y^2) + y \cdot (xy)$

(b)
$$p \cdot (x - y^2) = px - py^2$$
and for $pxy = (py)x$

(c)
$$p(y)(x - y^2) = p(y)x - p(y)y^2 \notin \langle x^2, xy \rangle$$

Exercise 2.
$$\sum_{i=1}^s p_i f_i + \sum_{j=1}^s q_j f_j = \sum_{i=1}^s (p_i + q_i) f_i, \quad p_i + q_i \in k[x_1 \dots x_n]$$
$\langle f_1 \dots f_s \rangle$ closed under sums in $k[x_1 \dots x_n]$

If $f \in \langle f_1 \dots f_s \rangle$,
$p \in k[x_1 \dots x_n]$

$$p \cdot f = p \sum_{i=1}^s q_j f_j = \sum_{i=1}^s p q_j f_j, \quad pq_j \in k[x_1 \dots x_n] \text{ so}$$
$$p \cdot f \in \langle f_1 \dots f_s \rangle$$

Done.

The 2 properties in Ex. 2 are defining properties of ideals in the ring $k[x_1 \dots x_n]$

Definition 3 (1.5). *Let $I \subset k[x_1 \dots x_n]$, $I \neq \emptyset$*
I ideal if

(a) $f + g \in I$, $\forall f, g \in I$
(b) $pf \in I$, $\forall f \in I$, *arbitrary* $p \in k[x_1 \dots x_n]$

Thus $\langle f_1 \dots f_s \rangle$ is an ideal by Ex. 2.

we call it the ideal generated by $f_1 \dots f_s$.

Exercise 3. Suppose $\exists$ ideal J, $f_1 \dots f_s \in J$ s.t. $J \subset \langle f_1 \dots f_s \rangle$
if $f \in \langle f_1 \dots f_s \rangle$, $f = \sum_{i=1}^s p_i f_i$, $p_i \in k[x_1 \dots x_n]$

$\forall i = 1 \dots s$, $p_i f_i \in J$ and so $\sum_{i=1}^s p_i f_i \in J$, by def. of J as an ideal.

$$\langle f_1 \dots f_s \rangle \subseteq J \qquad \Longrightarrow J = \langle f_1 \dots f_s \rangle$$

$\Longrightarrow \langle f_1 \dots f_s \rangle$ is smallest ideal in $k[x_1 \dots x_n]$ containing $f_1 \dots f_s$

Exercise 4. For $I = \langle f_1 \dots f_s \rangle$
$J = \langle g_1 \dots g_t \rangle$

$I = J$ iff $s = t$ and $\forall f \in I$, $f = \sum_{i=1}^t q_i g_i$ and if $0 = \sum_{i=1}^t q_i g_i$, $q_i = 0$, $\forall i = 1 \dots t$, and if $0 = \sum_{i=1}^s p_i f_i$, $p_i = 0$, $\forall i = 1 \dots s$

Definition 4 (1.6).

$$\sqrt{I} = \{ g \in k[x_1 \dots x_n] | g^m \in I \text{ for some } m \geq 1 \}$$

e.g. $x + y \in \sqrt{\langle x^2 + 3xy, 3xy + y^2 \rangle}$
in $\mathbb{Q}[x, y]$ since

$$(x+y)^3 = x(x^2 + 3xy) + y(3xy + y^2) \in \langle x^2 + 3xy, 3xy + y^2 \rangle$$

- (Radical Ideal Property) $\forall$ ideal $I \subset k[x_1 \dots x_n]$, $\sqrt{I}$ ideal, $\sqrt{I} \supset I$
- **(Hilbert basis Thm.)** $\forall$ ideal $I \subset k[x_1 \dots x_n]$
  $\exists$ finite generating set,
  i.e. $\exists \{ f_1 \dots f_2 \} \subset k[x_1 \dots x_n]$ s.t. $I = \langle f_1 \dots f_s \rangle$
- (Division Algorithm in $k[x]$) $\forall f, g \in k[x]$ (EY : in 1 variable)
  $\forall f, g \in k[x]$ (in 1 variable)
  $f = qg + r$, $\exists!$ quotient q, $\exists$ remainder r

7.2.

7.3. Gröbner Bases.

Definition 5 (3.1). *Gröbner basis for $I \equiv G = \{ g_1 \dots g_k \} \subset I$ s.t. $\forall f \in I$, $LT(f)$ divisible by $LT(g_i)$ for some i*

- (Uniqueness of Remainders) let ideal $I \subset k[x_1 \dots x_n]$
  division of $f \in k[x_1 \dots x_n]$ by Grö bner basis for I, produces $f = g + r$, $g \in I$, and no term in r divisible by any element of LT(I)

7.4. Affine Varieties.

affine n-dim. space over k $\quad k^n = \{ (a_1 \dots a_n) | a_1 \dots a_n \in k \}$
$\forall$ polynomial $f \in k[x_1 \dots x_n]$, $(a_1 \dots a_n) \in k^n$
$f : k^n \to k$
$f(a_1 \dots a_n)$ s.t. $x_i = a_i$ i.e.

if $f = \sum_\alpha c_\alpha x^\alpha$ for $c_\alpha \in k$, then
$f(a_1 \dots a_n) = \sum_\alpha c_\alpha a^\alpha \in k$, where $a^\alpha = a_1^{\alpha_1} \dots a_n^{\alpha_n}$

Definition 6 (4.1). *affine variety* $\mathbf{V}(f_1 \dots f_s) = \{ (a_1 \dots a_n) | (a_1 \dots a_n) \in k^n, f_1(x_1 \dots x_n) = \dots = f_s(x_1 \dots x_n) = 0 \}$
subset $V \subset k^n$ is affine variety if $V = V(f_1 \dots f_s)$ for some $\{f_i\}$, polynomial $f_i \in k[x_1 \dots x_n]$

- (Equal Ideals Have Equal Varieties) If $\langle f_1 \dots f_s \rangle = \langle g_1 \dots g_t \rangle$ in $k[x_1 \dots x_n]$, then $\mathbf{V}(f_1 \dots f_s) = \mathbf{V}(g_1 \dots g_t)$

so, recap
if $\langle f_1 \dots f_s \rangle = \langle g_1 \dots g_t \rangle$ in $k[x_1 \dots x_n]$,
then $V(f_1 \dots f_s) = V(g_1 \dots g_t)$

Recall Hilbert basis Thm. $\forall$ ideal $I \subset k[x_1 \dots x_n]$

$$I = \langle f_1 \dots f_s \rangle$$

$\Longrightarrow$ if $I = J$, then $V(I) = V(J)$
think of V defined by I, rather than $f_1 = \dots = f_s = 0$

Exercise 3.
Recall Def. 1.5 Let $I \subset k[x_1 \dots x_n]$

I ideal if $f + g \in I \quad \forall f, g \in I$
$pf \in I$, $\quad \forall f \in I$ arbitrary $p \in k[x_1 \dots x_n]$
Let $f, g \in I(V)$

$$(f+g)(a_1 \dots a_n) = f(a_1 \dots a_n) + g(a_1 \dots a_n) = 0 + 0 = 0 \qquad f + g \in I(V)$$
$$pf(a_1 \dots a_n) = p(a_1 \dots a_n) f(a_1 \dots a_n) = 0 \qquad pf \in I(V)$$

Then $I(V)$ an ideal.
$V = V(x^2)$ in $\mathbb{R}^2$
$I = \langle x^2 \rangle$ in $\mathbb{R}[x, y]$, $I = \{ px^2 | p \in k[x, y] \}$
$I \subset I(V)$, since $px^2 = 0$ for $x^2 = 0$, $(0, b)$, $b \in \mathbb{R}$
But $p(x, y) = x \in I(V)$, as

$$I(V) = \{ f \in k[x_1 \dots x_n] | f(a_1 \dots a_n) = 0, \forall (a_1 \dots a_n) \in V \}$$

$p(0, b) = x = 0$
But $x \notin I$

Exercise 4. $I \subset \sqrt{I}$
Recall Def. 1.6 $\sqrt{I} = \{ g \in k[x_1 \dots x_n] | g^m \in I \text{ for some } m \geq 1 \}$
$\forall f \in I$, $f = f^1$, $m = 1$, so $f \in \sqrt{I}$, $\quad I \subset \sqrt{I}$
Hilbert basis thm., $\forall$ ideal $I \subset k[x_1 \dots x_n]$ s.t. $I = \langle f_1 \dots f_s \rangle$
$\left\{ {}^{V(I) = \{ (a_1 \dots a_n) | (a_1 \dots a_n) \in k^n, f_1(a_1 \dots a_n) = \dots = f_s(a_1 \dots a_n) = 0} \right\}$
$\mathbf{I}(\mathbf{V}(I)) = \{ f \in k[x_1 \dots x_n] | f(a_1 \dots a_n) = 0 \quad \forall (a_1 \dots a_n) \in V(I) \}$
Let $g \in \sqrt{I}$, $g^m \in I$, $g^m = g^{m-1} g$
$g^m(a_1 \dots a_n) = 0 = g^{m-1}(a_1 \dots a_n) g(a_1 \dots a_n) = 0$. Then $g(a_1 \dots a_n) = 0$ or $g^{m-1}(a_1 \dots a_m) = 0$
as $g^m \in I$, and $V(I)$ is s.t. $f_1(a_1 \dots a_n) = \dots = f_s(a_1 \dots a_n) = 0$ for $I = \langle f_1 \dots f_s \rangle$

- (Strong Nullstellensatz) if k algebraically closed (e.g. $\mathbb{C}$), I ideal in $k[x_1 \dots x_n]$, then

$$\mathbf{I}(\mathbf{V}(I) = \sqrt{I}$$

- (Ideal-variety correspondence) Let k arbitrary field

$$I \subset I(V(I))$$
$$V(I(V)) = V \quad \forall V$$

Additional Exercises for Sec.4. Exercise 6.

8. Solving Polynomial Equations

8.1.

8.2. **Finite-Dimensional Algebras.** Gröbner basis $G = \{g_1 \dots g_t\}$ of ideal $I \subset k[x_1 \dots x_n]$,
recall def.: Gröbner basis $G = \{g_1 \dots g_t\} \subset I$ of ideal I, $\forall f \in I$, $\mathrm{LT}(f)$ divisible by $\mathrm{LT}(g_i)$ for some i
$f \in k[x_1 \dots x_n]$ divide by G produces $f = g + r$, $g \in I$, r not divisible by any $\mathrm{LT}(I)$ uniqueness of r
$f \in k[x_1 \dots x_n]$ divide by G,
Recall from Ch. 1, divide $f \in k[x_1 \dots x_n]$ by G, the division algorithm yields

$$f = h_1 g_1 + \dots + h_t g_t + \overline{f}^G \qquad (2) \quad (2.1)$$

where remainder $\overline{f}^G$ is a linear combination of monomials $x^\alpha \notin \langle \mathrm{LT}(I) \rangle$
since Gröbner basis, $f \in I$ iff $\overline{f}^G = 0$
$\forall f \in k[x_1 \dots x_n]$, we have coset $[f] = f + I = \{f + h | h \in I\}$ s.t. $[f] = [g]$ iff $f - g \in I$
We have a 1-to-1 correspondence

$$\text{remainders} \leftrightarrow \text{cosets}$$
$$\overline{f}^G \leftrightarrow [f]$$

algebraic

$$\overline{f}^G + \overline{g}^G \leftrightarrow [f] + [g]$$
$$\overline{\overline{f}^G \cdot \overline{g}^G} \leftrightarrow [f] \cdot [g]$$

$B = \{x^\alpha | x^\alpha \notin \langle \mathrm{LT}(I) \rangle\}$ is a basis of A, basis monomials, standard monomials
20141023 EY's take
$\forall [f] \in A = k[x_1 \dots x_n]/I$, $[f] = p_i b_i$; $b_i \in B = \{x^\alpha | x^\alpha \notin \langle \mathrm{LT}(I) \rangle\}$
For $I = \langle G \rangle$
e.g. $G = \{x^2 + \frac{3}{2}xy + \frac{1}{2}y^2 - \frac{3}{2}x - \frac{3}{2}y, xy^2 - x, y^3 - y\}$
$\langle \mathrm{LT}(I) \rangle = \langle x^2, xy^2, y^3 \rangle$
e.g. $B = \{1, x, y, xy, y^2\}$
$[f] \cdot [g] = [fg]$
e.g. $f = x$, $g = xy$, $[fg] = [x^2 y]$
now $f = h_1 g_1 + \dots + h_t g_t + \overline{f}^G$

8.3.

8.4. **Solving Equations via Eigenvalues and Eigenvectors.**

9. Resultants

10. Computation in Local Rings

10.1. **Local Rings.**

Definition 7 (1.1)**.**

$$k[x_1 \dots x_n]_{\langle x_1 \dots x_n \rangle} \equiv \{\frac{f}{g} | \text{ rational functions } \frac{f}{g} \text{ of } x_1 \dots x_n \text{ with } g(p) \neq 0 \text{ at } p\}$$

main properties of $k[x_1 \dots x_n]_{\langle x_1 \dots x_n \rangle}$

Proposition 1 (1.2)**.** *Let $R = k[x_1 \dots x_n]_{\langle x_1 \dots x_n \rangle}$. Then*

(a) *R subring of field of rational functions $k(x_1 \dots x_n) \supset k[x_1 \dots x_n]$*
(b) *Let $M = \langle x_1 \dots x_n \rangle \subset R$ (ideal generated by $x_1 \dots X_n$ in R)*
Then $\forall \frac{f}{g} \in R \backslash M$, $\frac{f}{g}$ unit in R (i.e. multiplicative inverse in R)
(c) *M maximal ideal in R*

Exercise 1. if $p = (a_1 \dots a_n) \in k^n$, $R = \{\frac{f}{g} | f, g \in k[x_1 \dots x_n], g(p) \neq 0\}$

(a) R subring of field of rational functions $k(x_1 \dots x_n)$
(b) Let M ideal generated by $x_1 - a_1 \dots x_n - a_n$ in R
Then $\forall \frac{f}{g} \in R \backslash M$, $\frac{f}{g}$ unit in R (i.e. multiplicative inverse in R)
(c) M maximal ideal in R

Proof. let $p = (a_1 \dots a_n) \in k^n$
let $g_1(p) \neq 0$, $g_2(p) \neq 0$

$$\frac{f_1}{g_1} + \frac{f_2}{g_2} = \frac{f_1 g_2 + f_2 g_1}{g_1 g_2} \qquad g_1(p) g_2(p) \neq 0 \text{ so } \frac{f_1}{g_1} + \frac{f_2}{g_2} \in R$$
$$\frac{f_1}{g_1} \cdot \frac{f_2}{g_2} = \frac{f_1 f_2}{g_1 g_2} \qquad g_1(p) g_2(p) \neq 0 \text{ so } \frac{f_1}{g_1} \frac{f_2}{g_2} \in R$$

$f = \frac{f}{1} \in R$, $\quad \forall f \in k[x_1 \dots x_n]$, so $k[x_1 \dots x_n] \subset R$ $\square$

EY : 20141027, to recap,
Let $V = k^n$
Let $p = (a_1 \dots a_n)$
single pt. $\{p\}$ is (an example of) a variety
$I(\{p\}) = \{x_1 - a_1 \dots x_n - a_n\rangle \subset k[x_1 \dots x_n]$

$R \equiv k[x_1 \dots x_n]_{\langle x_1 - a_1 \dots x_n - a_n \rangle}$

$$R = \{\frac{f}{g} | \text{ rational function } \frac{f}{g} \text{ of } x_1 \dots x_n, \, g(p) \neq 0, \, p = (a_1 \dots a_n) \, \}$$

Prop. 1.2. properties

(a) R subring of field of rational functions $k(x_1 \dots x_n)$ $\quad k(x_1 \dots x_n) \subset R$
(b) $M = \langle x_1 \dots a_1 \dots x_n - a_n \rangle \subset R$. ideal generated by $x_1 - a_1 \dots x_n - a_n$
Then $\forall \frac{f}{g} \in R \backslash M$, $\frac{f}{g}$ unit in R ($\exists$ multiplicative inverse in R)
(c) M maximal ideal in R.
in R we allow denominators that are not elements of this ideal $I(\{p\})$

Definition 8 (1.3)**.** *local ring is a ring that has exactly 1 maximal ideal*

Proposition 2 (1.4)**.** *ring R with proper ideal $M \subset R$ is local ring if $\forall \frac{f}{g} \in R \backslash M$ is unit in R*

localization Ex. 8, Ex. 9
parametrization

Exercise 2.

$$x = x(t) = \frac{-2t^2}{1+t^2}$$
$$y = y(t) = \frac{2t}{1+t^2}$$

$k[t]_{\langle t \rangle}$ $\frac{-2t^2}{1+t^2}$ rational function of t. $1+t^2 \neq 0$
if $k = \mathbb{C}$ or $\mathbb{R}$

Consider set of convergent power series in n variables

(3) $$k\{x_1 \dots x_n\} = \{ \sum_{\alpha \in \mathbb{Z}^n_{\geq 0}} c_\alpha x^\alpha | c_\alpha \in k, \text{ series converges in some open } U \ni 0 \in k^n \ \} \quad (1.5)$$

Consider set $k[[x_1 \dots x_n]]$ of formal power series

(4) $$k[[x_1 \dots x_n]] = \{ \sum_{\alpha \in \mathbb{Z}^n_{\geq 0}} c_\alpha x^\alpha | c_\alpha \in k\} \text{ series need not converge} \quad (1.6)$$

variety V

$k[x_1 \dots x_n]/\mathbf{I}(V)$ variety V

10.2. **Multiplicities and Milnor Numbers.** if I ideal in $k[x_1 \dots x_n]$, then denote $Ik[x_1 \dots x_n]_{\langle x_1 \dots x_n \rangle}$ ideal generated by I in larger ring $k[x_1 \dots x_n]_{\langle x_1 \dots x_n \rangle}$

Definition 9 (2.1)**.** *Let I 0-dim. ideal in $k[x_1 \dots x_n]$, so $V(I)$ consists of finitely many pts. in k^n.*
Assume $(0 \dots 0) \in V(I)$
multiplicity of $(0 \dots 0) \in V(I)$ is

$$dim_k k[x_1 \dots x_n]_{\langle x_1 \dots x_n \rangle} / Ik[x_! \dots x_n]_{\langle x_1 \dots x_n \rangle}$$

generally, if $p = (a_1 \dots a_n) \in V(I)$
multiplicity of p, $m(p) = \dim k[x_1 \dots x_n]_M / Ik[x_1 \dots x_n]_M$

$$\dim k[x_1 \dots x_n]_M / Ik[x_1 \dots x_n]_M$$

localizing $k[x_1 \dots x_n]$ at maximal ideal $M = I(\{p\}) = \langle x_1 - a_1 \dots x_n - a_n \rangle$

11.

12.

13. Polytopes, Resultants, and Equations

14. Polyhedral Regions and Polynomials

14.1. **Integer Programming.** Prop. 1.12.

Suppose 2 customers A, B ship to same location
A: ship 400 kg pallet taking up $2\,m^3$ volume
B: ship 500 kg pallet taking up $3\,m^3$ volume

shipping firm trucks carry up to 3700 kg, up to $20\,m^3$

B's product more perishable, paying \$ 15 per pallet

A pays \$ 11 per pallet
How many pallets from A, B each in truck to maximize revenues?

(5) $$\begin{aligned} 4A + 5B &\leq 37 \\ 2A + 3B &\leq 20 \\ A, B &\in \mathbb{Z}^*_{\geq 0} \end{aligned} \quad (1.1)$$

maximize $11A + 15B$

integer programming.
max. or min. value of some linear function

$$l(A_1 \dots A_n) = \sum_{i=1}^n c_i A_i$$

on set $(A_1 \dots A_n) \in \mathbb{Z}^n_{\geq 0}$ s.t.
3. Finally, by introducing additional variables; rewrite linear constraint inequalities as equalities. The new variables are called "slack variables"

(6) $$a_{ij} A_j = b_i, \quad A_j \in \mathbb{Z}_{\geq 0} \quad (1.4)$$

introduce indeterminate z_i, $\forall$ equation in (1.4)

$$z_i^{a_{ij} A_j} = z_i^{b_i}$$

m constraints

$$\prod_{i=1}^m z_i^{a_{ij} A_j} = \prod_{i=1}^m z_i^{b_i} = \left(\prod_{i=1}^m z_i^{a_{ij}} \right)^{A_j}$$

Proposition 3 (1.6)**.** *Let k field, define $\varphi : k[w_1 \dots w_n] \to k[z_1 \dots z_m]$ by*

$$\varphi(w_j) = \prod_{i=1}^m z_i^{a_{ij}} \qquad \forall\, j = 1 \dots n$$

and

$$\varphi(g(w_1 \dots w_n)) = g(\varphi(w_1) \dots \varphi(w_n))$$

$\forall$ general polynomial $g \in k[w_1 \dots w_n]$
Then $(A_1 \dots A_n)$ integer pt. in feasible region iff $\varphi : w_1^{A_1} \dots w_n^{A_n} \mapsto z_1^{b_1} \dots z_m^{b_m}$

Exercise 3.
Now

$$\varphi(w_j) = \prod_{i=1}^m z_i^{a_{ij}}$$
$$z_i^{a_{ij} A_j} = z_i^{b_i}$$

If $(A_1 \dots A_n)$ an integer pt. in feasible region, $a_{ij} A_j = b_i$

$$z_i^{a_{ij}A_j} = z_i^{b_i} = \prod_{j=1}^n z_i^{a_{ij}A_j} \Longrightarrow \prod_{j=1}^n \prod_{i=1}^m (z_i^{a_{ij}})^{A_j} = \prod_{i=1}^m z_i^{b_i} = \prod_{j=1}^n \varphi(w_j)^{A_j} = \prod_{j=1}^n \varphi(w_j)^{A_j} = \varphi\left(\prod_{j=1}^n w_j^{A_j} \right) = \prod_{i=1}^m z_i^{b_i}$$

since $\varphi(g(w_1 \dots w_n)) = g(\varphi(w_1) \dots \varphi(w_n))$

If $\varphi : \prod_{j=1}^n w_j^{A_j} \mapsto \prod_{i=1}^m z_i^{b_i}$

$$\varphi\left(\prod_{j=1}^n w_j^{A_j} \right) = \prod_{j=1}^n (\varphi(w_j))^{A_j} = \prod_{i=1}^m z_i^{b_i} = \prod_{j=1}^n \left(\prod_{i=1}^m z_i^{a_{ij}} \right)^{A_j} \Longrightarrow \prod_{j=1}^n z_i^{a_{ij}A_j} = z_i^{b_i}$$

or $a_{ij}A_j = b_i$. So $(A_1 \dots A_n)$ integer pt.

Exercise 4.

$$\prod_{i=1}^m z_i^{b_i} = \prod_{i=1}^m \prod_{j=1}^n z_i^{a_{ij}A_j} = \prod_{j=1}^n \left(\prod_{i=1}^m z_i^{a_{ij}} \right)^{A_j} = \prod_{j=1}^n \varphi(w_j)^{A_j} = \varphi\left(\prod_{j=1}^n w_j^{A_j} \right)$$

So if given $(b_1 \dots b_m) \in \mathbb{Z}^m$, and for a given a_{ij}, $a_{ij}A_j = b_i$

For $m \leq n$, then a_{ij} is surjective, so $\exists\, A_j$ s.t. $\prod_{i=1}^m z_i^{b_i} = \varphi\left(\prod_{j=1}^n w_j^{A_j} \right)$

Proposition 4 (1.8)**.** *Suppose $f_1 \dots f_n \in k[z_1 \dots z_m]$ given*
Fix monomial order in $k[z_1 \dots z_n, w_1 \dots w_n]$ with elimination property:
$\forall$ monomial containing 1 of z_i greater than any monomial containing only w_j

Let $\mathcal{G}$ Gröbner basis for ideal

$$I = \langle f_1 - w_1 \dots f_n - w_n \rangle \subset k[z_1 \dots z_m, w_1 \dots w_n]$$

$\forall\, f \in k[z_1 \dots z_m]$, let $\overline{f}^{\mathcal{G}}$ be remainder on division of f by $\mathcal{G}$
Then

(a) *polynomial f s.t. $f \in k[f_1 \dots f_n]$ iff $g = \overline{f}^{\mathcal{G}} \in k[w_1 \dots w_n]$*

(b) *if $f \in k[f_1 \dots f_n]$ as in part (a),*
$g = \overline{f}^{\mathcal{G}} \in k[w_1 \dots w_n]$
then $f = g(f_1 \dots f_n)$, giving an expression for f as polynomial in f_j

(c) *if $\forall\, f_i, f$ monomials, $f \in k[f_1 \dots f_n]$,*
then g also a monomial.

14.2. **Integer Programming and Combinatorics.**

15. Algebraic Coding Theory

16. The Berlekamp-Massey-Sakata Decoding Algorithm

Gröbner Bases, Martin R. Albrecht of the DTU Crypto Group

Part 3. Algebraic Topology

cf. Bredon (1997) [3]

17. Simplicial Complexes

cf. pp. 245, from Sec. 21 Simplicial Complexes of Ch. 4 Homology Theory in Bredon (1997) [3]
$\mathbf{v}_0, \dots \mathbf{v}_n \in \mathbb{R}^\infty$, "affinely independent" if they span an affine n-plane, i.e.

$$\text{if } \left(\sum_{i=0}^n \lambda_i \mathbf{v}_i = 0, \sum_{i=0}^n \lambda_i = 0 \right), \text{ then } \Longrightarrow \forall\, \lambda_i = 0$$

If not, then, e.g. $\lambda_0 \neq 0$, assume $\lambda_0 = -1$, and solve the equations to get

$$\mathbf{v}_0 = \sum_{i=1}^n \lambda_i \mathbf{v}_i$$

$$\sum_{i=1}^n \lambda_i = 1$$

i.e. $\mathbf{v}_0$ is in affine space spanned by $\mathbf{v}_1 \dots \mathbf{v}_n$.
If $\mathbf{v}_0, \dots \mathbf{v}_n$ affinely independent, then

$$\sigma = (\mathbf{v}_0, \dots \mathbf{v}_n) = \{ \sum_{i=0}^n \lambda_i \mathbf{v}_i | \sum_{i=0}^n \lambda_i = 1,\, \lambda_i \geq 0 \} \quad (7)$$

is "affine simplex" spanned by $\mathbf{v}_i$; also convex hull of $\mathbf{v}_i$.
$\forall\, k \leq n$, k-face of σ is any affine simplex of form $(\mathbf{v}_{i_1}, \dots \mathbf{v}_{i_k})$, where vertices all distinct, so are affinely independent.

Definition 10. *(geometric) simplicial complex $K :=$ collection of affine simplices s.t.*

(1) *$\sigma \in K \Longrightarrow$ any face of $\sigma \in K$; and*
(2) *$\sigma, \tau \in K \Longrightarrow \sigma \bigcap \tau$ is a face of both σ and τ, or $\sigma \bigcap \tau = \emptyset$*

If K simplicial complex, $|K| = \bigcup \{ \sigma | \sigma \in K \} \equiv$ "polyhedron" of K

Definition 11 (Def. 21.2 of Bredon (1997) [3])**.** *polyhedron := space X if $\exists$ homeomorphism $h : |K| \xrightarrow{\approx} X$ for some simplicial complex K. h, K is triangulation of X; (map h, complex K)*

Let K finite simplicial complex.
Choose ordering of vertices $\mathbf{v}_0, \mathbf{v}_1 \dots$ of K.
If $\sigma = (\mathbf{v}_{\sigma_0}, \dots \mathbf{v}_{\sigma_n})$ is simplex of K, where $\sigma_0 < \dots < \sigma_n$, then
let $f_\sigma : \Delta_n \to |K|$ be

$$f_\sigma = [\mathbf{v}_{\sigma_b}, \dots \mathbf{v}_{\sigma_n}]$$

in notation of Def. 1.2. Bredon (1997) [3].
Then this gives CW-complex structure on $|K|$ with f_σ as characteristic maps.

Part 4. Graphs, Finite Graphs

18. Graphs, Finite Graphs, Trees

Serre (1980) [4]
cf. Chapter I. Trees and Amalgams, Section 1 Amalgams, Subsection 1.1 Direct limits of Serre (1980) [4]
Let $(G_i)_{i \in I}$, family of groups.
$\forall$ pair (i, j), let $F_{ij} =$ set of homomorphisms of G_i into G_j
Want: group $G = \varinjlim G_i$ and

$$\{ f_i | f_i : G_i \to G \} \text{s.t. } f_j \circ f = f_i \quad \forall\, f \in F_{ij}$$

group G and family $\{f_i\}$ universal in that
(*) if H group, if $\{ h_i | h_i : G_i \to H; h_j \circ f = h_i \qquad \forall\, f \in F_{ij} \}$,
then $\exists!\, h : G \to H$ s.t. $h_i = h \circ f_i$

i.e. $\text{Hom}(G,H) \simeq \varprojlim \text{Hom}(G_i,H)$, the inverse limit being taken relative to F_{ij}.
i.e. G direct limit of G_i relative to the F_{ij}.

Proposition 5. *$\exists !$ pair G, family $(f_i)_{i\in I}$, i.e. (pair consisting of $G,(f_i)_{i\in I}$, unique up to unique isomorphism.*

Proof. Define G by generators and relations.
Take generating family to be disjoint union of those for G_i.
relations - xyz^{-1} where $x,y,z\in G_i$, $z=xy\in G_i$
xy^{-1} where $x\in G_i$, $y\in G_j$, $y=f(x)$ for at least $f\in F_{ij}$.
Thus, existence of $G,\{f_i\}$.
G represents functor $H\mapsto \varprojlim \text{Hom}(G_i,H)$.
Thus, uniqueness (also from universal property). □

e.g. groups A,G_1,G_2, homomorphisms $f_1 : A\to G_1$.
$f_2 : A\to G_2$
G obtained by amalgamating A in G_1,G_2 by $f_1,f_2 \equiv G_1 *_A G_2$.
1 can have $G=\{1\}$, even though f_1,f_2 non-trivial.
Application: (Van Kampen Thm.)
Let topological space X be covered by open U_1,U_2.
Suppose $U_1,U_2,U_{12}=U_1\bigcap U_2$ arcwise connected.
Let basept. $x\in U_{12}$.
Then $\pi_1(X;x)$ obtained by taking 3 groups

$$\pi_1(U_1;x),\pi_1(U_2;x),\pi_1(U_{12};x)$$

and amalagamating them according to homomorphism

$$\pi_1(U_{12};x)\to\pi_1(U_1;x)$$
$$\pi_1(U_{12};x)\to\pi_1(U_2;x)$$

Exercise 1. Let homomorphisms $f_1 : A\to G_1$ amalgam $G=G_1 *_A G_2$.
$f_2 : A\to G_2$
Define subgroups A^n,G_1^n,G_2^n, of A,G_1,G_2 recursively by

$$A^1=\{1\}$$
$$G_1^1=\{1\}$$
$$G_2^1=\{1\}$$

$A^n=$ subgroup of A generated by $f_1^{-1}(G_1^{n-1})$ and $f_2^{-1}(G_2^{n-1})$

$$G_1^n = \text{subgroup of } G_i \text{ generated by } f_i(A^n)$$

Let A^∞,G_i^∞ be unions of A^n,G_i^n resp.
Show that f_i defines injection $A/A^\infty\to G_i/G_i^\infty$.
So the amalgamation is $G\simeq G_1/G_1^\infty *_{A/A^\infty} G_2/G_2^\infty$.
Take the first induction case (for intuition about the solution).

$$A^2=\langle f_1^{-1}(G_1^1),f_2^{-1}(G_2^1)\rangle = \langle f_1^{-1}(\{1\}),f_2^{-1}(\{1\})\rangle$$
$$G_i^2=f_i(A^2)$$

Let $f_i(a)=f_i(b)\in G_i/G_i^\infty$; $a,b\in A/A^\infty$.

Then since $f_i(a),f_i(b)\in G_i/G_i^\infty$, $f_i(a),f_i(b)\in\{gG_i^\infty|g\in G_i\}$ (quotient is defined to be the set of all left cosets of G_i^∞, which has to be a normal subgroup for G_i/G_i^∞ to be a quotient group).
Since $a,b\in A/A^\infty$, suppose we take $a,b\in A$.
And suppose we take

$$f_i(a)=f_i(a)G_i^\infty = f_i(a)f_i(A^{n_a})=f_i(aA^{n_a})$$
$$f_i(b)=f_i(b)G_i^\infty = f_i(b)f_i(A^{n_b})=f_i(bA^{n_b})$$

Taking f_i^{-1} (recall for group homomorphisms, they map inverse of element of 1st. group to inverse of image of this element).
$aA^{n_a}=bA^{n_b}\in A/A^\infty$ (This is okay as we've "quotiented out A^∞; so indeed, they're equal)
cf. Subsection 1.2 Structure of amalgams of Serre (1980) [4]
Suppose given group A, family of groups $(G_i)_{i\in I}$, and, $\forall\, i\in I$, injective homomorphism $A\to G_i$.
$*_A G_i\equiv$ direct limit (cf. no. 1.1) of family (A,G_i) with respect to these homomorphisms, call it *sum* (in category theory sense, i.e. product) of G_i with A amalgamated.
e.g. $A=\{1\}$,
$*G_i\equiv$ free product of G_i.

18.0.1. *reduced word.* $\forall\, i\in I$, choose set S_i of right coset representations of G_i modulo A,
assume $1\in S_i$,
$(a,s)\mapsto as$ is bijection of $A\times S_i$ onto G_i,
$A\times(S_i-\{1\})\to G_i-A$ (onto)
Let $\mathbf{i}=(i_1\dots i_n)$, $n\geq 0$, $i_j\in I$, s.t.

$$i_m\neq i_{m+1} \text{ for } 1\leq m\leq n-1 \tag{8}$$

cf. (T) of Serre (1980) [4].
So *reduced word* m is defined as

$$m=(a;s_1\dots s_n)$$

where $a\in A,s_1\in S_{i_1}\dots s_n\in S_{i_n}$, and $s-j\neq 1\,\forall\, j$.
$f\equiv$ canonical homomorphism of A into group $G=*_AG_i$
$f_i\equiv$ canonical homomorphism of G_i into group $G=*_AG_i$
EY : 20170611 (Further explanations, basic examples, from me):
Given $A,\{G_i\}_{i\in I}$, injective (group) homomorphisms $\{f_i : A\to G_i\}_i$.
$G_i\backslash f_i(A)=\{f_i(A)g|g\in G_i\}$.
Right coset representation of $f_i(A)g\mapsto g$.

e.g. A,G_1,G_2, $f_1 : A\to G_1$.
$f_2 : A\to G_2$

$$G_1\backslash f_1(A)=\{f_1(A)g|g\in G_1\}$$
$$G_2\backslash f_2(A)=\{f_2(A)g|g\in G_2\}$$

$\mathbf{i}=(i_1\dots i_n)$, $i_j\in I$, $i_m\neq i_{m+1}$ for $1\leq m\leq n-1$.
Consider $(1212\dots 12)$
$m=(a;f_1g_2f_3g_4\dots f_{2n-1},g_{2n})$ where f's $\in S_1\subset G_1$, g's $\in S_2\subset G_2$.
and so

Definition 12 (reduced word)**.** ***reduced word*** *of type* $\mathbf{i}$, m,

$$m=(a;s_1\dots s_n) \tag{9}$$

where $a\in A,s_1\in S_{i_1},\dots s_n\in S_{i_n}$, $s_j\neq 1\quad\forall j$,
$\mathbf{i}=(i_1\dots i_n)$, $i_j\in I$, *s.t.* $i_m\neq i_{m+1}$ *for* $1\leq m\leq n-1$,
with $S_i=\{g|g\in f_i(A)g\in f_i(A)G_i\}$

Theorem 1 (1 of Serre (1980) [4]). *$\forall g \in G$, $\exists$ sequence $\mathbf{i}$ s.t. $i_m \neq i_{m+1}$ for $1 \leq m \leq n-1$ and reduced word*

$$m = (a; s_1 \dots s_n)$$

of type $\mathbf{i}$ s.t.

$$g = f(a) f_{i_1}(s_1) \dots f_{i_n}(s_n)$$

Furthermore, $\mathbf{i}$ and m unique.

Remark. Thm. 1 implies $f; f_i$ injective.

Then identify A and G_i with images $f(A), f_i(G_i)$ in G, and reduced decomposition (*) of $g \in G$

$$g = a s_1 \dots s_n, \quad a \in A,\ s_1 \in S_{i_1} - \{1\} \dots s_n \in S_{i_n} - \{1\}$$

Likewise, $G_i \bigcap G_j = A$ if $i \neq j$.

In particular, $S_i - \{1\}$ pairwise disjoint in G.

Proof. Let $X_i \equiv$set of reduced words of type $\mathbf{i}$, $X = \coprod X_i$.

Make G act on X.

In view of universal property of G, sufficient to make $\forall\, i, G_i$ act,

check action induced on A doesn't depend on i

Suppose then that $i \in I$, and let Y_i = set of reduced words of form $(1; s_1 \dots s_n)$, with $i_1 \neq i$.

EY : 20170611

Recall that

$$\begin{aligned} & S_i = \{ g | g \in f_i(A) g \in f_i(A) G_i \} \\ & A \times S_i \to G_i \text{ onto} \\ & A \times (S_i - \{1\}) \to G_i - A \text{ onto} \\ & (a,s) \mapsto as \text{ bijection} \end{aligned}$$

Let Y_i = set of reduced words of form $(1; s_1 \dots s_n) = \{ (1; s_1 \dots s_n) | 1 \in A; s_1 \in S_{i_1} \dots s_n \in S_{i_n}; \mathbf{i} = (i_1 \dots i_n),\ i_j \in I \text{ s.t. } i_m \neq i_{m+1} \text{ for } 1 \leq m \leq n-1 \}$.

$$\begin{gathered} A \times Y_i \to X = \coprod_i X_i \\ (a, (1; s_1 \dots s_n)) \mapsto (a; s_1 \dots s_n) \\ A \times \{ S_i - \{1\} \} \times Y_i \to X \\ ((a,s), (1; s_1 \dots s_n)) \mapsto (a; s, s_1 \dots s_n) \end{gathered}$$

and remember that X_i = set of reduced words of type $\mathbf{i}$.

It's clear that this yields a bijection $A \times Y_i \bigcup A \times (S_i - \{1\}) \times Y_i \to X$.

Let $x \in X$. Then $x \in X_{\mathbf{i}}$ for some $\mathbf{i}$. So x is a reduced word of type $\mathbf{i}$: $x = (a; s_1 \dots s_n)$. Then clearly $x = (a; s_1 \dots s_n) \mapsto (a, (1; s_1 \dots s_n)) \in A \times Y_i$. □

cf. pp. 13, Sec. 2. Trees, 2.1 Graphs of Serre (1980) [4]

Definition 13 (1. of Serre (1980) [4]). ***graph*** *$\Gamma = (X, Y, Y \to X \times X, Y \to Y)$, where* $\begin{aligned} \text{set } X = & \text{ vert } \Gamma \\ \text{set } Y = & \text{ edge } \Gamma \end{aligned}$

$$\begin{aligned} & Y \to X \times X \\ & y \mapsto (o(y), t(y)) \\ & Y \to Y \\ & y \mapsto \overline{y} \end{aligned}$$

s.t. $\forall\, y \in Y$, $\overline{\overline{y}} = y$, $\overline{y} \neq y$, $o(y) = t(\overline{y})$.

vertex $P \in X$ of Γ.

(oriented) edge $y \in Y$, $\overline{y} \equiv$ inverse edge.

origin of $y :=$ vertex $o(y) = t(\overline{y})$.

terminus of $y :=$ vertex $t(y) = o(\overline{y})$

extremities of $y := \{ o(y), t(y) \}$

*If 2 vertices **adjacent**, they're extremities of some edge.*

orientation of graph $\Gamma = Y_+ \subset Y =$ edge Γ s.t. $Y = Y_+ \coprod \overline{Y}_+$. It always exists.

oriented graph defined, up to isomorphism, by giving 2 sets X, Y_+ and $Y+ \to X \times X$.

corresponding set of edges is $Y = Y_+ \coprod \overline{Y}_+$ where $\overline{Y}_+ \equiv$ copy of Y_+

18.0.2. *Realization of a Graph.* cf. Realization of a Graph in Serre (1980) [4].

Let graph Γ, $X = \text{vert}\Gamma$, $Y = \text{edge}\Gamma$.

topological space $T = X \coprod Y \times [0,1]$, where X, Y provided with discrete topology.

Let R be finest equivalence relation on T for which

$$\begin{aligned} & (y,t) \equiv (\overline{y}, 1-t) \\ & (y,0) \equiv o(y) \qquad \forall\, y \in Y, \forall\, t \in [0,1] \\ & (y,1) \equiv t(y) \end{aligned} \tag{10}$$

quotient space $\text{real}(\Gamma) = T/R$ is *realization* of graph Γ. (realization is a functor which commutes with direct limits).

Let $n \in \mathbb{Z}^+$. Consider oriented graph of $n+1$ vertices $0, 1, \dots n$,

Definition 14. *path (of length n) in graph Γ is morphism c of $Path_n$ into Γ*

orientation given by n edges $[i, i+1]$, $0 \leq i < n$, $\begin{aligned} o([i,i+1]) & = i \\ t([i,i+1]) & = i+1 \end{aligned}$

For $n \geq 1$,

$(y_1 \dots y_n)$ sequence of edges $y_i = c([i-1, i])$ s.t.

$$t(y_i) = o(y_{i+1}), \qquad 1 \leq i < n \text{ determine } c$$

If $P_i = c(i)$,

c is a path from P_0 to P_n, and P_0 and P_n are *extremities of the path c*.

pair of form $(y_i, y_{i+1}) = (y_i, \overline{y}_i)$ in path is **backtracking**.

path (of length $n-2$), from P_0 to P_n given (for $n > 2$) by $(y_1 \dots y_{i-1}, y_{i+2} \dots y_n)$

If $\exists$ path from P to Q in Γ, $\exists$ one without backtracking (by induction)

direct limit $\text{Path}_\infty = \varinjlim \text{Path}_n$ provides notion of infinite path.

$\text{Path}_\infty \ni$ infinite sequence $(y_1, y_2, \dots)$ of edges s.t. $t(y_i) = o(y_{i+1}) \quad \forall\, i \geq 1$.

Definition 15 (connected graph; Def. 3 of Serre (1980) [4]). *graph connected if $\forall$ 2 vertices, 2 vertices are extremities of at least 1 path.*

maximal connected subgraphs (under relation of inclusion) are connected components *of graph.*

18.0.3. *Circuits.* Let $n \in \mathbb{Z}^+$, $n \geq 1$.
Consider

set of vertices $\mathbb{Z}/n\mathbb{Z}$, orientation given by n edges $[i, i+1]$, $(i \in \mathbb{Z}/n\mathbb{Z})$ with $\begin{aligned} o([i,i+1]) &= i \\ t([i,i+1]) &= i+1 \end{aligned}$

Definition 16 (circuit; Def. 4 of Serre (1980) [4])**.** *circuit (length n) in graph is subgraph isormorphic to $Circ_n$.*

i.e. subgraph = path $(y_1 \dots y_n)$, without backtracking, s.t. $P_i = t(y_i)$, $(1 \leq i \leq n)$ distinct, s.t. $P_n = o(y_1)$

$n=1$ case: Circ_1, $\mathbb{Z}/\mathbb{Z} = \{0\}$, 1 edge, $[0,1]$, $0 \in \mathbb{Z}/1\mathbb{Z}$, $\begin{aligned} o([0,1]) &= 0 \\ t([0,1]) &= 1 \end{aligned}$

Note Circ_1 has automorphism of order 2, which changes its orientation, i.e.
$\exists$ automorphism $\sigma \in \text{Aut}(\text{Circ}_1)$ s.t. $|\sigma| = 2$, i.e. $\sigma^2 = 1$.
loop := circuit of length 1; so loop $\in \text{Circ}_1$.
path (y_1), $P_1 = t(y_1) = o(y_1)$.
$n=2$ case: Circ_2, $\mathbb{Z}/2\mathbb{Z} = \{0,1\}$, 2 edges $[0,1], [1,2]$,

path (y_1, y_2), $(1 \leq i \leq 2)$, $\begin{aligned} P_1 &= t(y_1) \\ P_2 &= t(y_2) = o(y_1) \end{aligned}$

18.1. **Combinatorial graphs.** Let $(X, S) \equiv$ simplicial complex of dim. ≤ 1, with
$X \equiv$ set
$S \equiv$ set of subsets of X with 1 or 2 elements, containing all the 1-element subsets.
associates with it a graph $\Gamma = (X, \{(P,Q)\})$.
X is its set of vertices.
edges $= \{(P,Q) \in X \times X\}$ s.t. $P \neq Q$, $\{P,Q\} \in S$, with $\overline{(P,Q)} = (Q,P)$

$$\begin{aligned} o(P,Q) &= P \\ t(P,Q) &= Q \end{aligned}$$

In this graph, 2 edges with same origin and same terminus are equal. This is equivalent to (see following Def.)

Definition 17 (combinatorial; Def. 5 of Serre (1980) [4])**.** *graph is combinatorial if it has no circuit of length ≤ 2*

Conversely, it's easy to see that
every combinatorial graph Γ derived (up to isomorphism) by construction above from simplicial complex (X,S), where
$X = \text{vert}\Gamma$
$S =$ set of subset $\{P,Q\}$ of X s.t. P and Q either adjacent or equal.

Part 5. Tensors, Tensor networks; Singular Value Decomposition, QR decomposition, Density Matrix Renormalization Group (DMRG), Matrix Product states (MPS)

19. Introductions to Tensor Networks

https://youtu.be/nsxgAOAEgbgWorkshop introductory overview by José Barbon for the https://www.youtube.com/channel/UC4R1IsRVKs_qlWKTm9pT82QInstitut des Hautes Études Scientifiques (IHÉS) gave me the first impetus to understand tensor networks as I sought to also understand the condensates of entanglement pairs within the black hole.

A Google search for introductions to tensor networks that are on arxiv ("Introduction Tensor Network arxiv") yielded Bridgeman and Chubb's course notes (bf. Bridgeman and Chubb (2017) [7]).

19.1. **List of stuff I want to look at/do/study.** I would like to compare/contrast the following:

- Rotman (2010) [5], Ch. 8, but starting from 8.4 Tensor Products, pp. 574
- Jeffrey Lee (2009) [6], Ch. 7 Tensors

19.2. **Tensor operations; Tensor properties.**

19.2.1. *rank.* r = rank tensor of dim. $d_1 \times \dots \times d_r$ is element of $\mathbb{C}^{d_1 \times \dots \times d_r}$
Tensor product

$$[A \otimes B]_{i_1 \dots i_r, j_1 \dots j_s} := A_{i_1 \dots i_r} \cdot B_{j_1 \dots j_s} \tag{11}$$

19.2.2. *Trace.* Given tensor A, xth, yth indices have identical dims. $(d_x = d_y)$,
partial trace over these 2 dims. is simply joint summation over that index

$$[\text{Tr}_{x,y} A]_{i_1 \dots i_{x-1} i_{x+1} \dots i_{y-1} i_{y+1} \dots i_r} = \sum_{\alpha=1}^{d_x} A_{i_1 \dots i_{x-1} \alpha i_{x+1} \dots i_{y-1} \alpha i_{y+1} \dots i_r} \tag{12}$$

19.2.3. *Contraction.*

19.2.4. *Group and splitting, Bridgeman and Chubb (2017)* [7]. "Rank is a rather fluid concept in the study of tensor networks." Bridgeman and Chubb (2017) [7].
$\mathbb{C}^{a_1 \times \dots \times a_n} \simeq \mathbb{C}^{b_1 \times \dots \times b_m}$ isomorphic as vector spaces if $\prod_i a_i = \prod_i b_i$.
We can "group" or "split" indices to lower or raise rank of given tensor, resp.
Consider contracting 2 arbitrary tensors.
If we group together indices which are and are not involved in contraction,
"It should be noted that not only is this reduction to matrix multiplication pedagogically handy, but this is precisely the manner in which numerical tensor packages perform contraction, allowing them to leverage highly optimised matrix multiplication code." (cf. Bridgeman and Chubb (2017) [7]; check this)
"Owing to freedom in choice of basis, precise details of grouping and splitting aren't unique." (cf. Bridgeman and Chubb (2017) [7]).
1 specific choice of convention:
tensor product basis, defining basis on product space by product of respective bases.
"The canonical use of tensor product bases in quantum information allows for grouping and splitting described above to be - dealt with implicitly."

$$|0\rangle \otimes |1\rangle \equiv |0\rangle \tag{13}$$

and precisely this grouping,

$$\begin{aligned} |0\rangle \otimes |1\rangle &\in \text{Mat}_{\mathbb{C}}(2,2), \text{ whilst} \\ |01\rangle &\in \mathbb{C}^4 \end{aligned} \tag{14}$$

Suppose rank $n+m$ tensor T, group its first n indices, last m indices together.

$$T_{I,J} := T_{i_1 \dots i_n, j_1 \dots j_m}$$

where

$$\begin{aligned} I &:= i_1 + d_1^{(i)} i_2 + d_1^{(i)} d_2^{(i)} i_3 + \dots + d_1^{(i)} \dots d_{n-1}^{(i)} i_n \\ J &:= j_1 + d_1^{(j)} j_2 + d_1^{(j)} d_2^{(j)} j_3 + \dots + d_1^{(j)} \dots d_{m-1}^{j)} j_m \end{aligned}$$

EY : 20170627 to elaborate, consider a functor `flatten` that does what's described above, in the context of category theory (and so this is the generalization):

$$(15)\quad \begin{gathered} \mathbb{K}^{d_1^{(i)}} \times \mathbb{K}^{d_2^{(i)}} \times \dots \times \mathbb{K}^{d_n^{(i)}} \times \mathbb{K}^{d_1^{(j)}} \times \mathbb{K}^{d_2^{(j)}} \times \dots \times \mathbb{K}^{d_m^{(j)}} \xrightarrow{\text{flatten}} \mathbb{K}^{\prod_{p=1}^n d_p^{(i)}} \times \mathbb{K}^{\prod_{q=1}^m d_q^{(j)}} \\ T_{i_1 \dots i_n, j_1 \dots j_m} \xmapsto{\text{flatten}} T_{I,J} \\ \{0,1,\dots d_1^{(i)}\} \times \{0,1,\dots d_2^{(i)}\} \times \dots \times \{0,1,\dots d_n^{(i)}\} \times \{0,1,\dots d_1^{(j)}\} \times \{0,1,\dots d_2^{(j)}\} \times \dots \times \{0,1,\dots d_m^{(j)}\} \xrightarrow{\text{flatten}} \\ \xrightarrow{\text{flatten}} \{0,1,\dots \prod_{p=1}^n d_p^{(i)} - 1\} \times \{0,1,\dots \prod_{q=1}^m d_q^{(j)} - 1\} \\ (i_1, i_2, \dots i_n, j_1, j_2 \dots j_m) \xmapsto{\text{flatten}} (I,J) := (i_1 + d_1^{(i)} i_2 + \dots + d_1^{(i)} \dots d_{n-1}^{(i)} i_n, j_1 + d_1^{(j)} j_2 + \dots + d_1^{(j)} \dots d_{m-1}^{(j)} j_m) \end{gathered}$$

It doesn't make sense to call this "row-major" or "column-major" ordering generalization, because we are not dealing with only 2 indices where we can definitely say the first index indexes the "row" and the second index indexes the "column." At most, possibly, you can alternatively have this:

$$(i_1 \dots i_n, j_1 \dots j_m) \xmapsto{\text{flatten}} (I,J) := (d_2^{(i)} \dots d_n^{(i)} i_1 + d_3^{(i)} \dots d_n^{(i)} i_2 + \dots + i_n, d_2^{(j)} \dots d_m^{(j)} j_1 + \dots + j_m)$$

Note that this is all 0-*based counting* (i.e. we start counting from 0 just like in C,C++,Python, etc.). If you really wanted 1-based counting, you'd have to complicate the above formulas as such:

$$(I,J) := (i_1 + d_1^{(i)}(i_2 - 1) + \dots + d_1^{(i)} \dots d_{n-1}^{(i)}(i_n - 1), j_1 + d_1^{(j)}(j_2 - 1) + \dots + d_1^{(j)} \dots d_{m-1}^{(j)}(j_m - 1))$$

Note that formulas are easily checked by pluggin in the minimum and maximum values for the indices and seeing if they make sense (e.g. plug in $(0,0,\dots,0)$ for all indices for 0-based counting and make sure you get back $I = 0$ or $J = 0$).

19.3. Singular Value Decomposition.

$$T_{I,J} = \sum_\alpha U_{I,\alpha} S_{\alpha,\alpha} \overline{V}_{J,\alpha}$$

$$(16)\quad \begin{gathered} \mathrm{Mat}_{\mathbb{K}}(N,M) \xrightarrow{\text{SVD}} \mathrm{Mat}_{\mathbb{K}}(N,P) \times \mathrm{Mat}_{\mathbb{K}}(P,P) \times \mathrm{Mat}_{\mathbb{K}}(M,P) \\ T_{I,J} \xmapsto{\text{SVD}} U_{I,\alpha}, S_{\alpha,\alpha}, \overline{V}_{I,\alpha} \text{ s.t.} \\ T_{I,J} = \sum_\alpha U_{I,\alpha} S_{\alpha,\alpha} \overline{V}_{J,\alpha} \\ T = USV^\dagger \end{gathered}$$

For the higher-dimensional version of SVD,

$$(17)\quad \begin{gathered} \mathbb{K}^{d_1^{(i)}} \otimes \dots \otimes \mathbb{K}^{d_N^{(i)}} \otimes \mathbb{K}^{d_1^{(j)}} \otimes \dots \otimes \mathbb{K}^{d_M^{(j)}} \xrightarrow{\text{flatten}} \mathrm{Mat}_{\mathbb{K}}(N,M) \xrightarrow{\text{SVD}} \mathrm{Mat}_{\mathbb{K}}(N,P) \times \mathrm{Mat}_{\mathbb{K}}(P,P) \times \mathrm{Mat}_{\mathbb{K}}(M,P) \xrightarrow{\text{splitting}} \\ \xrightarrow{\text{splitting}} \mathbb{K}^{d_1^{(i)}} \otimes \dots \otimes \mathbb{K}^{d_N^{(i)}} \otimes \mathbb{K}^P \times \mathrm{Mat}_{\mathbb{K}}(P,P) \times \mathbb{K}^{d_1^{(j)}} \otimes \dots \otimes \mathbb{K}^{d_M^{(j)}} \otimes \mathbb{K}^P \\ T_{i_1 \dots i_N, j_1 \dots j_M} = \sum_\alpha U_{i_1 \dots i_N, \alpha} S_{\alpha,\alpha} \overline{V}_{j_1 \dots j_M, \alpha} \end{gathered}$$

20. Density Matrix Renormalization Group; Matrix Product States (MPS)

cf. Sec. 4, Matrix Product States (MPS) of Schollwöck [9].

Necessarily, given matrix $M \in \mathrm{Mat}_{\mathbb{K}}(M,N)$ (notation in Bridgeman and Chubb (2017) [7] and CUDA Toolkit Documentation; I will follow the notation in Schollwöck [9] since his A,B denote specific physical meaning).

For

$$U \in \mathrm{Mat}_{\mathbb{K}}(N_A, \min(N_A, N_B)) \text{ s.t. } UU^\dagger = 1$$
$$S \in \mathrm{Mat}_{\mathbb{K}}(\min(N_A, N_B), \min(N_A, N_B))$$

s.t. S diagonal with nonnegative $S_{aa} = s_a$, i.e. $S_{ij} = \delta_{ij} s_i$ s.t. $s_i \geq 0 \quad \forall i = 1, 2, \dots \min(N_A, N_B)$.

$r \equiv$ (Schmidt) rank of $M :=$ number of nonzero singular values.

Assume $s_1 \geq s_2 \geq \dots \geq s_r \geq 0$.

$V^\dagger \in \mathrm{Mat}_{\mathbb{K}}(\min(N_A, N_B), N_B)$ s.t. $V^\dagger V = 1$.

$$\mathrm{Mat}_{\mathbb{K}}(N_A, N_B) \xrightarrow{\text{SVD}} U_{\mathbb{K}}(N_A, \min(N_A, N_B)) \times \mathrm{diag}_{\mathbb{K}}(\min(N_A, N_B)) \times U_{\mathbb{K}}(\min(N_A, N_B), N_B)$$

$$M \xmapsto{\qquad\qquad \text{SVD} \qquad\qquad} USV^\dagger$$

Optimal approximation of M (rank r by matrix M' (rank $r' < r$) property.

In Frobenius norm $\|M\|_F^2 := \sum_{i,j} |M_{ij}|^2$, induced by inner product $\langle M|N\rangle = \mathrm{tr} M^\dagger N$. Indeed,

$$\mathrm{tr} M^\dagger N = (M^\dagger)_{ik} N_{ki} = \overline{M}_{ki} N_{ki}$$

and so for

$$(18)\quad M' = US'V^\dagger, \qquad S' = \mathrm{diag}(s_1, s_2 \dots s_{r'}, 0 \dots)$$

cf. Eq. (19) of Schollwöck [9], i.e. 1 sets all but 1st r' singualr values to 0.

Use singular value decomposition (SVD) to derive Schmidt decomposition of general quantum state.

$\forall$ pure state $|\psi\rangle$ on AB,

$$|\psi\rangle = \sum_{i,j} \Psi_{ij} |i\rangle_A |j\rangle_B$$

where $\{|i\rangle_A\}, \{|j\rangle_B\}$ orthonormal bases of A, B ((complex) Hilbert spaces), with dim. N_A, N_B, respectively.

Let $\Psi_{i,j} \in \mathrm{Mat}_{\mathbb{K}}(N_A, N_B)$.

Then **reduced density operators** $\widehat{\rho}_A, \widehat{\rho}_B$ are such that

$$\widehat{\rho}_A = \mathrm{tr}_B |\psi\rangle\langle\psi|$$
$$\widehat{\rho}_B = \mathrm{tr}_A |\psi\rangle\langle\psi|$$

In matrix form,

$$\rho_A = \Psi\Psi^\dagger$$
$$\rho_B = \Psi^\dagger\Psi$$

Indeed,

$$(\rho_A)_{ij} = \Psi_{ik} \overline{\Psi}_{jk}$$
$$(\rho_B)_{ij} = \overline{\Psi}_{ki} \Psi_{kj}$$
$$|\psi\rangle\langle\psi| = \sum_{i,j} \Psi_{ij} |i\rangle_A |j\rangle_B \sum_{l,m} \overline{\Psi}_{lm} \langle l|_A \langle m|_B$$
$$\mathrm{tr}_B |\psi\rangle\langle\psi| = \sum_{i,j} \Psi_{ik} \overline{\Psi}_{jk} |i\rangle_A \rangle j|_A$$

In matrix form,

$$\rho_A = \Psi\Psi^\dagger$$
$$\rho_B = \Psi^\dagger\Psi$$

Carry out SVD on Ψ in Eq. (20) of Schollwöck [9],

$$|\psi\rangle = \sum_{i,j} \Psi_{ij} |i\rangle_A |j\rangle_B$$

$$|\psi\rangle = \sum_{ij} \Psi_{ij} |i\rangle_A |j\rangle_B = \sum_{ij} \sum_{a=1}^{\min(N_A,N_B)} U_{ia} S_{aa} \overline{V}_{ja} |i\rangle_A |j\rangle_B = \sum_{a=1}^{\min(N_A,N_B)} \sum_i U_{ia} |i\rangle_A s_a \sum_j \overline{V}_{ja} |j\rangle_B = \sum_{a=1}^{\min(N_A,N_B)} s_a |a\rangle_A |a\rangle_B$$

Due to orthogonality of $U, V^\dagger$, $\{|a\rangle_A\}, \{|a\rangle_B\}$ orthonormal, and can be extended to be orthonormal bases of A, B.

If we restrict the sum to run only over the $r \leq \min(N_A, N_B)$ positive nonzero singular values (i.e., for $\sum_{a=1}^{\min(N_A,N_B)}$, $_a > 0$ $\forall a \leq r$, and so

$$|\psi\rangle = \sum_{a=1}^r s_a |a\rangle_A |a\rangle_B$$

$r = 1$ (classical) product states. $|\psi\rangle = s_1 |1\rangle_A |1\rangle_B$.
$r > 1$ entangled (quantum) states.

Schmidt decomposition on reduced density operators for A and B:

$$\widehat{\rho}_A = \sum_{a=1}^r s_a^2 |a\rangle_A \langle a|_A$$
$$\widehat{\rho}_B = \sum_{a=1}^r s_a^2 |a\rangle_B \langle a|_B$$

Respective eigenvectors are left and right singular vectors.

Von Neumann entropy can be read off:

$$S_{A|B}(|\psi\rangle) = -\mathrm{tr}\widehat{\rho}_A \log_2 \widehat{\rho}_A = -\sum_{a=1}^r s_a^2 \log_2 s_a^2$$

In view of large size of Hilbert spaces, approximate $|\psi\rangle$ by some $|\widetilde{\psi}\rangle$ spanned over state spaces A, B that have dims. r' only.

Since 2-norm of $|\psi\rangle$,

$$\| |\psi\rangle \|_2^2 = \sum_{ij} |\Psi_{ij}|^2 = \|\Psi\|_F^2$$

since

$$\| |\psi\rangle \|_2^2 = \sum_{a=1}^r s_a^2 = \sum_{ij} |\Psi_{ij}|^2$$

iff $\{|i\rangle\}, \{|j\rangle\}$ orthonormal. Optimal approx. of 2-norm given by optimal approx. of Ψ by $\overline{\Psi}$ in Frobenius norm, where $\overline{\Psi}$ is matrix of rank r'.

$\overline{\Psi} = US'V^\dagger$, $S' = \mathrm{diag}(s_1, \dots s_{r'}, 0 \dots)$ from above.

$\Longrightarrow$ Schmidt decomposition of approximate state

$$|\overline{\Psi}\rangle = \sum_{a=1}^{r'} s_a |a\rangle_A |a\rangle_B \tag{19}$$

cf. Eq. (27) of Schollwöck [9], where s_a must be rescaled if normalization desired.

20.1. **QR decomposition.** cf. 4.1.2. of Schollwöck [9].

If actual value of singular values not used explicitly, then use *QR decomposition*.

QR decomposition: $\forall M \in \mathrm{Mat}_{\mathbb{K}}(N_A, N_B)$,

$$M = QR,\ Q \in U_{\mathbb{K}}(N_A), \text{ i.e. } Q^\dagger Q = 1 = QQ^\dagger,\ R \in \mathrm{Mat}_{\mathbb{K}}(N_A, N_B) \text{ s.t. upper triangular, i.e. } R_{ij} = 0 \text{ if } i > j \tag{20}$$

thin QR decomposition: assume $N_A > N_B$. Then bottom $N_A - N_B$ rows of R are 0, so

$$M = Q \begin{bmatrix} R_1 \\ 0 \end{bmatrix} = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \begin{bmatrix} R_1 \\ 0 \end{bmatrix} = Q_1 R_1$$
$$Q_1 \in \mathrm{Mat}_{\mathbb{K}}(N_A, N_B)$$
$$R_1 \in \mathrm{Mat}_{\mathbb{K}}(N_B, N_B)$$

While $Q_1^\dagger Q_1 = 1$ in general $Q_1 Q_1^\dagger \neq 1$

21. Matrix Product States (MPS)

cf. Section 4.13 Decomposition of arbitrary quantum states into MPS of Schollwöck [9].

Consider lattice of L sites, d-dim. local state spaces $\{\sigma_i\}_{i=1,\dots L}$.

Most general pure quantum state on lattice (assume normalized)

$$|\psi\rangle = \sum_{\sigma_1 \dots \sigma_L} c_{\sigma_1 \dots \sigma_L} |\sigma_1 \dots \sigma_L\rangle \tag{21}$$

cf. Eq. (30) of Schollwöck [9],

21.1. **Left-canonical matrix product state.** cf. Schollwöck [9],

Consider the process of refactoring or "flattening", which I claim to be a functor *flatten*:

$$|\psi\rangle \in \mathcal{H} \text{ s.t. } \dim\mathcal{H} = d^L \mapsto \Psi \in \mathrm{Mat}_{\mathbb{K}}(d, d^{L-1})$$

$$\Psi_{\sigma_1,(\sigma_2 \dots \sigma_L)} = c_{\sigma_1 \dots \sigma_L}$$

$$\xrightarrow{\text{SVD}} c_{\sigma_1 \dots \sigma_L} = \Psi_{\sigma_1,(\sigma_2 \dots \sigma_L)} = \sum_{a}^{r_1} U_{\sigma_1, a_1} S_{a_1, a_1} (V^\dagger)_{a_1,(\sigma_2 \dots \sigma_L)} \equiv \sum_{a_1}^{r_1} U_{\sigma_1, a_1} c_{a_1, \sigma_2 \dots \sigma_L} \tag{22}$$

i.e.

$$(\mathbb{K}^d)^L \to \mathrm{Mat}_{\mathbb{K}}(1, r) \times \mathrm{Mat}_{\mathbb{K}}(r_1 d, d^{L-2})$$
$$c_{\sigma_1 \dots \sigma_L} \mapsto A_{a_1}^{\sigma_1}, \Psi_{(a_1 \sigma_2),(\sigma_3 \dots \sigma_L)}$$

s.t.

$$c_{\sigma_1 \dots \sigma_L} = \sum_{a_1}^{r_1} A_{a_1}^{\sigma_1} \Psi_{(a_1 \sigma_2),(\sigma_3 \dots \sigma_L)}$$

where rank $r_1 \leq d$.

$$U \in \mathrm{Mat}_{\mathbb{K}}(d, \min(d, r)) = \mathrm{Mat}_{\mathbb{K}}(d, r)$$

Consider d row vectors A^{σ_1}, $A_{a_1}^{\sigma_1} = U_{\sigma_1, a_1}$.

$$c_{a_1 \sigma_2 \dots \sigma_L} = \sum_{a_1}^{r_1} A_{a_1}^{\sigma_1} \Psi_{(a_1,\sigma_2),(\sigma_3 \dots \sigma_L)} \text{ with}$$
$$\Psi_{(a_1 \sigma_2),(\sigma_3 \dots \sigma_L)} \in \mathrm{Mat}_{\mathbb{K}}(r_1 d, d^{L-2})$$

So from Eq. (34) of Schollwöck [9],

$$c_{\sigma_1 \dots \sigma_L} = \sum_{a_1}^{r_1} \sum_{a_2}^{r_2} A_{a_1}^{\sigma_1} U_{(a_1 \sigma_2), a_2} S_{a_2, a_2} (V^\dagger)_{a_2,(\sigma_3 \dots \sigma_L)} = \sum_{a_1}^{r_1} \sum_{a_2}^{r_2} A_{a_1}^{\sigma_1} A_{a_1, a_2}^{\sigma_2} \Psi_{(a_2 \sigma_3),(\sigma_4 \dots \sigma_L)} \tag{23}$$

So for

$$U \in \mathrm{Mat}_{\mathbb{K}}(d, r_1 \times r_2) \mapsto \{A^{\sigma_2}\}_{\sigma_2}, \qquad |\{A^{\sigma_2}\}_{\sigma_2}| = d, \qquad A^{\sigma_2} \in \mathrm{Mat}_{\mathbb{K}}(r_1, r_2)$$

$A_{a_1, a_2}^{\sigma_2} = U_{(a_1, \sigma_2), a_2}$ and multiplied S and $V^\dagger$,

$$SV^\dagger \mapsto \Psi \in \mathrm{Mat}_{\mathbb{K}}(r_2 d, d^{L-3}); \qquad r_2 \leq r_1 d \leq d^2$$

and so continuing the application of SVD and refactoring (what I call applying the *flatten* functor)

$$\xrightarrow{\text{SVD}} c_{\sigma_1 \dots \sigma_L} = \sum_{a_1 \dots a_{L-1}} A_{a_1}^{\sigma_1} A_{a_1 a_2}^{\sigma_2} \dots A_{a_{L-2}, a_{L-1}}^{\sigma_{L-1}} A_{a_L - 1}^{\sigma_L} \equiv A^{\sigma_1} A^{\sigma_2} \dots A^{\sigma_{L-1}} A^{\sigma_L}$$

21.1.1. *Matrix Product State (definition).*

Definition 18 (Matrix Product State)**.**

$$|\psi\rangle = \sum_{\sigma_1 \dots \sigma_L} A^{\sigma_1} A^{\sigma_2} \dots A^{\sigma_{L-1}} A^{\sigma_L} |\sigma_1 \dots \sigma_L\rangle \tag{24}$$

Maximally, the dims. are

$$(1 \times d), (d \times d^2) \dots (d^{L/2-1} \times d^{L/2}), (d^{L/2} \times d^{L/2-1}) \dots (d^2 \times d), (d \times 1)$$

Since $\forall$ SVD, $U^\dagger U = 1$,

$$\delta_{a_l, a_l'} = \sum_{a_{l-1} a_l} (U^\dagger)_{a_l, (a_{l-1}\sigma_l)} U_{(a_{l-1}\sigma_l), a_l'} = \sum_{a_{l-1}\sigma_l} (A^{\sigma_l})^\dagger_{a_l, a_{l-1}} A^{\sigma_l}_{a_{l-1}, a_l'} = \sum_{\sigma_l} ((A^{\sigma_2})^\dagger A^{\sigma_l})_{a_l, a_l'}$$

or

$$\sum_{\sigma_l} (A^{\sigma_l})^\dagger A^{\sigma_l} = 1 \tag{25}$$

cf. Eq. (38) of Schollwöck [9],

If for $\{A^{\sigma_l}\}_{\sigma_l}$, $\sum_{\sigma_l} (A^{\sigma_l})^\dagger A = 1$, $\{A^{\sigma_l}\}_{\sigma_l}$ are **left-normalized**; matrix product states that consist of only left-normalized matrices are **left-canonical**.

View Density Matrix Renormalization Group (DMRG) decomposition of universe into blocks A and B, split lattice into parts A,B, where A compries sites 1 through l and B sites $l+1$ through L.

$$\begin{aligned} |a_l\rangle_A &= \sum_{\sigma_1 \dots \sigma_l} (A^{\sigma_1} A^{\sigma_2} \dots A^{\sigma_l})_{a_l, 1} |\sigma_1 \dots \sigma_l\rangle \\ |a_l\rangle_B &= \sum_{\sigma_{l+1} \dots \sigma_L} (A^{\sigma_{l+1}} A^{\sigma_{l+2}} \dots A^{\sigma_L})_{a_l, 1} |\sigma_{l+1} \dots \sigma_L\rangle \end{aligned}$$

s.t. matrix product state (MPS) is

$$|\psi\rangle = \sum_{a_l} |a_l\rangle_A |a_l\rangle_B$$

21.1.2. *Summarize this procedure of constructing, from a pure state, the matrix product state (version) by successive application Singular Value Decomposition (SVD) from the Category Theory point of view.* Consider all applications of SVD to get to a matrix

$$(\mathbb{K}^d)^L \xrightarrow{\text{SVD}} (\text{Mat}_{\mathbb{K}}(1, r_1))^d \times (\text{Mat}_{\mathbb{K}}(r_1, r_2))^d \times \dots \times (\text{Mat}_{\mathbb{K}}(r_{L-2}, r_{L-1}))^d \times (\text{Mat}_{\mathbb{K}}(r_{L-1}, 1))^d$$

$$c_{\sigma_1 \dots \sigma_L} \xmapsto{\quad\text{SVD}\quad} c_{\sigma_1 \dots \sigma_L} = \sum_{a_1 \dots a_{L-1}} A^{\sigma_1}_{a_1} A^{\sigma_2}_{a_1 a_2} \dots A^{\sigma_{L-1}}_{a_{L-2}, a_{L-1}} A^{\sigma_L}_{a_{L-1}}$$

product state (MPS):
and remember the maximal values that the r_i's can take:

$$\begin{array}{lll} r_1 \leq d & r_{L/2} \leq d^{L/2} & r_{L-2} \leq d^2 \\ r_2 \leq d^2 & r_{L/2+1} \leq d^{L/2-1} & r_{L-1} \leq d \end{array}$$

Let us explicitly note the functors (that were applied) flatten (and its inverse), and the application of SVD, explicitly:

$$\begin{array}{ccccccccc} (\mathbb{K}^d)^L & \xrightarrow{\text{flatten}^{-1}} & \text{Mat}_{\mathbb{K}}(d, d^{L-1}) & \xrightarrow{\text{SVD}} & U_{\mathbb{K}}(d, r_1) \times \text{diag}_{\mathbb{K}}(r_1) \times U_{\mathbb{K}}(r_1, d^{L-1}) & \xrightarrow{\cong} & (\text{Mat}_{\mathbb{K}}(1, r_1))^d \times \text{Mat}_{\mathbb{K}}(r_1 d, d^{L-2}) & \xrightarrow{\text{flatten}} & (\text{Mat}_{\mathbb{K}}(1, r_1))^d \times (\mathbb{K}^{r_1}) \times (\mathbb{K}^d)^{L-1} \\ c_{\sigma_1 \dots \sigma_L} & \xmapsto{\text{flatten}^{-1}} & c_{\sigma_1 \dots \sigma_L} = \Psi_{\sigma_1, (\sigma_2 \dots \sigma_L)} & \xmapsto{\text{SVD}} & \Psi_{\sigma_1, (\sigma_2 \dots \sigma_L)} = \sum_{a_1}^{r_1} U_{\sigma_1 a_1} S_{a_1, a_1} (V^\dagger)_{a_1, (\sigma_2 \dots \sigma_L)} & \xmapsto{\cong} & c_{a_1 \sigma_2 \dots \sigma_L} = \sum_{a_1}^{r_1} A_{a_1}^{\sigma_1} \Psi_{(a_1, a_2), (\sigma_3 \dots \sigma_L)} & \xmapsto{\text{flatten}} & c_{a_1 \sigma_2 \dots \sigma_L} = \sum_{a_1}^{r_1} A_{a_1}^{\sigma_1} c_{a_1 \sigma_2 \dots \sigma_L} \end{array}$$

with $\cong$ in this case denoting an isomorphism (clearly).

In considering some kind of recursive algorithm, so to repeat some series of steps until a matrix product state is obtained, consider this:

$$\begin{array}{ccc} (\mathbb{K}^d)^L & \longrightarrow & (\text{Mat}_{\mathbb{K}}(1, r_1))^d \times \mathbb{K}^{r_1} \times (\mathbb{K}^d)^{L-1} \\ c_{\sigma_1 \dots \sigma_L} & \longmapsto & c_{\sigma_1 \dots \sigma_L} = \sum_{a_1}^{r_1} A_{a_1}^{\sigma_1} c_{a_1 \sigma_2 \dots \sigma_L} \end{array}$$

So in summary, to obtain matrix product states, starting from a matrix,

$$\boxed{\begin{array}{ccccccc} \text{Mat}_{\mathbb{K}}(d, d^{L-1}) & \longrightarrow & (\text{Mat}_{\mathbb{K}}(1, r_1))^d \times \text{Mat}_{\mathbb{K}}(r_1 d, d^{L-2}) & \longrightarrow & \dots & \longrightarrow & (\text{Mat}_{\mathbb{K}}(1, r_1))^d \times (\text{Mat}_{\mathbb{K}}(r_1, r_2))^d \times \dots \times (\text{Mat}_{\mathbb{K}}(r_{n-1}, r_n))^d \times (\text{Mat}_{\mathbb{K}}(r_n d, d^{L-(n+1)}))^d \\ \Psi_{\sigma_1, (\sigma_2 \dots \sigma_L)} & \longmapsto & \sum_{a_1}^{r_1} A_{a_1}^{\sigma_1} \Psi_{(a_1, \sigma_2), (\sigma_3 \dots \sigma_L)} & \longmapsto & \dots & \longmapsto & \sum_{a_1, a_2, \dots a_n}^{r_1, r_2, \dots r_n} A_{a_1}^{\sigma_1} A_{a_1 a_2}^{\sigma_2} \dots A_{a_{n-1} a_n}^{\sigma_n} \Psi_{(a_n \sigma_{n+1}), (\sigma_{n+2} \dots \sigma_L)} \end{array}} \tag{26}$$

21.2. **Right-canonical matrix product state.** cf. Schollwöck [9],

We can start from right in order to obtain

$$c_{\sigma_1 \dots \sigma_L} = \Psi_{(\sigma_1 \dots \sigma_{L-1}), \sigma_L} = \sum_{a_{L-1}} U_{(\sigma_1 \dots \sigma_{L-1}), a_{L-1}} S_{a_{L-1}, a_{L-1}} (V^\dagger)_{a_{L-1}, \sigma_L} = \sum_{a_{L-1}} \Psi_{(\sigma_1 \dots \sigma_{L-2}), (\sigma_{L-1} a_{L-1})} B_{a_{L-1}}^{\sigma_L} =$$
$$= \sum_{a_{L-1}, a_{L-2}} U_{(\sigma_1 \dots \sigma_{L-2}), a_{L-2}} S_{a_{L-2}, a_{L-2}} (V^\dagger)_{a_{L-2}, (\sigma_{L-1} a_{L-1})} B_{a_{L-1}}^{\sigma_L} = \sum_{a_{L-2}, a_{L-1}} \Psi_{(\sigma_1 \dots \sigma_{L-3}), (\sigma_{L-2} a_{L-2})} B_{a_{L-2}, a_{L-1}}^{\sigma_{L-1}} B_{a_{L-1}}^{\sigma_L} = \dots$$

or consider

$$\begin{array}{ccccccccc} (\mathbb{K}^d)^L & \xrightarrow{\text{flatten}^{-1}} & \text{Mat}_{\mathbb{K}}(d^{L-1}, d) & \xrightarrow{\text{SVD}} & U_{\mathbb{K}}(d^{L-1}, r_{L-1}) \times \text{diag}_{\mathbb{K}}(r_{L-1}) \times U_{\mathbb{K}}(r_{L-1}, d) & \xrightarrow{\cong} & \text{Mat}_{\mathbb{K}}(d^{L-2}, d r_{L-1}) \times (\text{Mat}_{\mathbb{K}}(r_{L-1}, 1))^d & \xrightarrow{\text{SVD}} & \\ c_{\sigma_1 \dots \sigma_L} & \xmapsto{\text{flatten}^{-1}} & c_{\sigma_1 \dots \sigma_L} = \Psi_{(\sigma_1 \dots \sigma_{L-1}), \sigma_L} & \xmapsto{\text{SVD}} & c_{\sigma_1 \dots \sigma_L} = \sum_{a_{L-1}}^{r_{L-1}} U_{(\sigma_1 \dots \sigma_{L-1}), a_{L-1}} S_{a_{L-1}, a_{L-1}} (V^\dagger)_{a_{L-1}, \sigma_L} & \xmapsto{\cong} & \begin{gathered} U_{(\sigma_1 \dots \sigma_{L-1}), a_{L-1}} S_{a_{L-1}, a_{L-1}} = \Psi_{(\sigma_1 \dots \sigma_{L-2}), (\sigma_{L-1} a_{L-1})} \\ (V^\dagger)_{a_{L-1}, \sigma_L} = B_{a_{L-1}}^{\sigma_L} \\ c_{\sigma_1 \dots \sigma_L} = \sum_{a_{L-1}} \Psi_{(\sigma_1 \dots \sigma_{L-2}), (\sigma_{L-1}, a_{L-1})} B_{a_{L-1}}^{\sigma_L} \end{gathered} & \xmapsto{\text{SVD}} & \end{array}$$

$$\begin{array}{ccccc} \xrightarrow{\text{SVD}} & U_{\mathbb{K}}(d^{L-2}, r_{L-2}) \times \text{diag}_{\mathbb{K}}(r_{L-2}) \times U_{\mathbb{K}}(r_{L-2}, d r_{L-1}) \times (\text{Mat}_{\mathbb{K}}(r_{L-1}, 1))^d & \xrightarrow{\cong} & \text{Mat}_{\mathbb{K}}(d^{L-3}, d r_{L-2}) \times (\text{Mat}_{\mathbb{K}}(r_{L-2}, r_{L-1}))^d \times (\text{Mat}_{\mathbb{K}}(r_{L-1}, 1))^d \\ \xmapsto{\text{SVD}} & c_{\sigma_1 \dots \sigma_L} = \sum_{a_{L-1}, a_{L-2}} U_{(\sigma_1 \dots \sigma_{L-2}), a_{L-2}} S_{a_{L-2}, a_{L-2}} (V^\dagger)_{a_{L-2}, (\sigma_{L-1} a_{L-1})} B_{a_{L-1}}^{\sigma_L} & \xmapsto{\cong} & \begin{gathered} U_{(\sigma_1 \dots \sigma_{L-2}), a_{L-2}} S_{a_{L-2}, a_{L-2}} = \Psi_{(\sigma_1 \dots \sigma_{L-3}), (\sigma_{L-2} a_{L-2})} \\ (V^\dagger)_{a_{L-2}, (\sigma_{L-1} a_{L-1})} = B_{a_{L-2} a_{L-1}}^{\sigma_{L-1}} \\ c_{\sigma_1 \dots \sigma_L} = \sum_{a_{L-1}, a_{L-2}} \Psi_{(\sigma_1 \dots \sigma_{L-3}), (\sigma_{L-2}, a_{L-2})} B_{a_{L-2}, a_{L-1}}^{\sigma_{L-1}} B_{a_{L-1}}^{\sigma_L} \end{gathered} \end{array}$$

with $\cong$ in this case denoting an isomorphism (clearly).

And so we can explicitly state the recursion step, for the purpose of writing numerical implementations/algorithms: $\forall\, l = 1, 2 \dots L$,

$$\begin{gathered} \text{Mat}_{\mathbb{K}}(d^{L-l}, dr_{L-(l-1)}) \longrightarrow \text{Mat}_{\mathbb{K}}(d^{L-(l+1)}, dr_{L-l}) \times (\text{Mat}_{\mathbb{K}}(r_{L-l}, r_{L-(l-1)}))^d \\ \Psi_{(\sigma_1 \dots \sigma_{L-l}),(\sigma_{L-(l-1)} a_{L-(l-1)})} \longmapsto \Psi_{(\sigma_1 \dots \sigma_{L-l}),(\sigma_{L-(l-1)} a_{L-(l-1)})} = \sum_{a_{L-l}} \Psi_{(\sigma_1 \dots \sigma_{L-(l+1)}),(\sigma_{L-l} a_{L-l})} B^{\sigma_{L-(l-1)}}_{a_{L-l}, a_{L-(l-1)}} \end{gathered}$$

and we finally obtained, after successive applications SVD, the matrix product state:

$$\begin{gathered} (\mathbb{K}^d)^L \longrightarrow \text{Mat}_{\mathbb{K}}(d^{L-1}, d) \longrightarrow (\text{Mat}_{\mathbb{K}}(1, r_1))^d \times (\text{Mat}_{\mathbb{K}}(r_1, r_2))^d \times \dots \times (\text{Mat}_{\mathbb{K}}(r_{L-2}, r_{L-1}))^d \times (\text{Mat}_{\mathbb{K}}(r_{L-1}, 1))^d \\ c_{\sigma_1 \dots \sigma_L} \longmapsto \Psi_{(\sigma_1 \dots \sigma_{L-l}),\sigma_L} \longmapsto c_{\sigma_1 \dots \sigma_L} = \sum_{a_1 \dots a_{L-1}} B^{\sigma_1}_{a_1} B^{\sigma_2}_{a_1 a_2} \dots B^{\sigma_{L-1}}_{a_{L-2} a_{L-1}} B^{\sigma_L}_{a_{L-1}} \end{gathered}$$

Since

(27) $$V^\dagger V = 1$$

, then

(28) $$\delta_{a_l a_l'} = \sum_{\sigma_m a_m} (V^\dagger)_{a_l (\sigma_m a_m)} V_{(\sigma_m a_m) a_l'} = \sum_{\sigma_m a_m} B^{\sigma_m}_{a_l a_m} \overline{B}^{\sigma_m}_{a_l' a_m} \Longrightarrow \sum_{\sigma_m} \boxed{B^{\sigma_m} (B^{\sigma_m})^\dagger = 1}$$

The B-matrices that obey this condition are referred to as **right-normalized** matrices. A matrix product state (MPS) entirely consisting of a product of these right-normalized matrices is called **right-canonical**.

21.2.1. *Numerical implementation; both in BLAS and cuBLAS.* As stated in the CUDA Toolkit Documentation v8.0 for cuSOLVER, under section 5.3.6. `cusolverDn<t>gesvd()` and Remark 1, `gesvd` "only supports" `m>=n`, for matrix you want to decompose $A \in \text{Mat}_{\mathbb{K}}(m, n)$. So number of rows must be greater than or equal to number of columns. And so we can only consider right-normalized matrices in a practical implementation.

I suspect it's the same in BLAS.

Consider the very first step, $l = 1$, in a procedure to calculate the matrix product state.

$$\begin{gathered} \text{Mat}_{\mathbb{K}}(d^{L-1}, d) \xrightarrow{\text{SVD}} U_{\mathbb{K}}(d^{L-1}, r_{L-1}) \times \text{diag}_{\mathbb{K}}(r_{L-1}) \times U_{\mathbb{K}}(r_{L-1}, d) \xrightarrow{\cong} \text{Mat}_{\mathbb{K}}(d^{L-2}, dr_{L-1}) \times (\text{Mat}_{\mathbb{K}}(r_{L-1}, 1))^d \\ \Psi_{(\sigma_1 \dots \sigma_{L-1}),\sigma_L} \xmapsto{\text{SVD}} = \sum_{a_{L-1}}^{r_{L-1}} U_{(\sigma_1 \dots \sigma_{L-1}),a_{L-1}} S_{a_{L-1},a_{L-1}} (V^\dagger)_{a_{L-1},\sigma_L} \xmapsto{\cong} \begin{matrix} U_{(\sigma_1 \dots \sigma_{L-1}),a_{L-1}} S_{a_{L-1},a_{L-1}} = \Psi_{(\sigma_1 \dots \sigma_{L-2}),(\sigma_{L-1} a_{L-1})} \\ (V^\dagger)_{a_{L-1},\sigma_L} = B^{\sigma_L}_{a_{L-1}} \end{matrix} \end{gathered}$$

with $\cong$ in this case denoting an isomorphism, the *reshaping* of a matrix into different matrix size dimensions, which should be the inverse of a "flatten" functor, which I'll denote as flatten^{-1} as well (and this is this same isomorphism we're talking about).

Let's deal with the specific procedure of flatten^{-1}, how it reshapes indices in accordance with different matrix size dimensions, and with the so-called "stride" when going from, say, 2-dimensional indices to a "flattened" 1-dimensional index.

Note also as a practical numerical implementation design point, LAPACK's linear algebra BLAS library package and CUBLAS assumes *column*-major ordering.

Consider $i = 1, 2, \dots L-1$ (for site i) (or for 0-based counting, starting to count from 0, $i = 0, 1, \dots L-2$; be aware of this difference as in practical numerical implementation, in C, C++, Python, it assumes 0-based counting).

For a state space of dimension d, we can consider the specific example of $d = 2$, representing say a spin-1/2 system. Then index σ_i can be 0 or 1: $\sigma_i \in \{0, 1\}$. In general, $\sigma_i \in \{0, 1, \dots d-1\}$. I may use d or 2 in the context of the number of states (basis vectors) of the spin system (state vector space).

Consider site i. Suppose the spin system there interacts most with sites $i-1$, $i+1$, and then next sites $i-2$, $i+2$, etc. So the values at $\sigma_{i-1}, \sigma_{i+1}$, etc. are most important in calculating interactions with spin system at site i.

Then we seek this reshaping of the matrix index - assuming 0-based counting/ordering, for $l = 1$:

$$\begin{gathered} \{0,1\}^{L-1} \xrightarrow{(\text{flatten})^{-1}} \{0, 1, \dots 2^{L-1} - 1\} \\ (\sigma_0, \sigma_1, \dots \sigma_{L-2}) \xmapsto{(\text{flatten})^{-1}} I_{L-1} := \sigma_0 + 2\sigma_1 + \dots + 2^i \sigma_i + \dots + 2^{L-2} \sigma_{L-2} = \sum_{i=0}^{L-2} 2^i \sigma_i \end{gathered}$$

In this way, states of a site i are closest in memory addresses in the allocation of a 1-dim. array, on CPU or GPU memory, so that memory access operations should be efficient.

Assuming SVD doesn't change the striding, and defining the result of matrix multiplication:

$$U_{(\sigma_0, \sigma_1 \dots \sigma_{L-2}),a_{L-1}} S_{a_{L-1},a_{L-1}} =: (US)_{(\sigma_0 \dots \sigma_{L-2}),a_{L-1}} \in \text{Mat}_{\mathbb{K}}(d^{L-1}, r_{L-1})$$

We can reshape (i.e. $(\text{flatten})^{-1}$) in such a manner:

$$\begin{gathered} \text{Mat}_{\mathbb{K}}(d^{L-1}, r_{L-1}) \xrightarrow{(\text{flatten})^{-1}} \text{Mat}_{\mathbb{K}}(d^{L-2}, dr_{L-1}) \\ (US)_{(\sigma_0 \dots \sigma_{L-2}),a_{L-1}} \xmapsto{(\text{flatten})^{-1}} \Psi_{(\sigma_0, \sigma_1, \dots \sigma_{L-3}),(\sigma_{L-2} a_{L-1})} \\ \{0, 1, \dots 2^{L-1} - 1\} \times \{0, 1, \dots r_{L-1} - 1\} \xrightarrow{(\text{flatten})^{-1}} \{0, 1, \dots 2^{L-2} - 1\} \times \{0, 1, \dots dr_{L-1} - 1\} \\ I_{L-1}, a_{L-1} \xmapsto{(\text{flatten})^{-1}} I_{L-1} \mod 2^{L-2}, \frac{I_{L-1}}{2^{L-2}} + d a_{L-1} \end{gathered}$$

Reshaping $V^\dagger$ at iteration $l=1$ can be done as follows:

$$U_{\mathbb{K}}(r_{L-1},d) \xrightarrow{(\text{flatten})^{-1}} (\text{Mat}_{\mathbb{K}}(r_{L-1},1))^d$$

$$(V^\dagger)_{a_{L-1},\sigma_{L-1}} \xmapsto{(\text{flatten})^{-1}} (V^\dagger)_{a_{L-1},\sigma_{L-1}} = B^{\sigma_{L-1}}_{a_{L-1}}$$

$$\{0,1,\dots r_{L-1}-1\}\times\{0,1,\dots d-1\} \xrightarrow{(\text{flatten})^{-1}} (\{0,1,\dots r_{L-1}-1\})^d$$

$$a_{L-1},\sigma_{L-1} \xmapsto{(\text{flatten})^{-1}} a_{L-1}$$

Let's do this same procedure, reshaping or $(\text{flatten})^{-1}$, for a general l iteration.

$$\text{Mat}_{\mathbb{K}}(d^{L-l},r_{L-l}) \xrightarrow{(\text{flatten})^{-1}} \text{Mat}_{\mathbb{K}}(d^{L-(l+1)},dr_{L-l})$$

$$(US)_{(\sigma_0\dots\sigma_{L-(l+1)}),a_{L-l}} \xmapsto{(\text{flatten})^{-1}} \Psi_{(\sigma_0,\sigma_1,\dots\sigma_{L-(l+2)}),(\sigma_{L-(l+1)}a_{L-l})}$$

$$\{0,1,\dots d^{L-l}-1\}\times\{0,1,\dots r_{L-l}-1\} \xrightarrow{(\text{flatten})^{-1}} \{0,1,\dots d^{L-(l+1)}-1\}\times\{0,1,\dots dr_{L-l}-1\}$$

$$I_{L-l},a_{L-l} \xmapsto{(\text{flatten})^{-1}} I_{L-l} \mod d^{L-(l+1)}, \frac{I_{L-l}}{d^{L-(l+1)}}+da_{L-l}$$

$$U_{\mathbb{K}}(r_{L-l},dr_{L-(l-1)}) \xrightarrow{(\text{flatten})^{-1}} (\text{Mat}_{\mathbb{K}}(r_{L-l},r_{L-(l-1)}))^d$$

$$(V^\dagger)_{a_{L-l},(\sigma_{L-l}a_{L-(l-1)})} \xmapsto{(\text{flatten})^{-1}} (V^\dagger)_{a_{L-l},(\sigma_{L-l}a_{L-(l-1)})} = B^{\sigma_{L-l}}_{a_{L-l},a_{L-(l-1)}}$$

$$\{0,1,\dots r_{L-l}-1\}\times\{0,1,\dots dr_{L-(l-1)}-1\} \xrightarrow{(\text{flatten})^{-1}} (\{0,1,\dots r_L-1\}\times\{0,1,\dots r_{L-(l-1)}-1\})^d$$

$$a_{L-l},(\sigma_{L-l}a_{L-(l-1)}) := a_{L-l},\sigma_{L-1}+da_{L-(l-1)} \xmapsto{(\text{flatten})^{-1}} a_{L-l},\frac{(\sigma_{L-1}a_{L-(l-1)})}{d}; \sigma_{L-l} = (\sigma_{L-l}a_{L-(l-1)}) \mod d$$

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