# The Geometry of Physics – Problem Solutions

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These are my solutions to problems given in Theodore Frankel's book "The Geometry of Physics" (second edition). As I could not find any other sources, I do not know whether they are correct or not, so read with care (especially the index battles). If you have a solution that is not in here already, a better way of showing something, or just some useful comment, I'd like to hear about it[1].

20110714 - further solutions begun by Ernest Yeung.

## Conventions

If not mentioned differently, use the following conventions:

- Use Einstein summation. Sometimes I'll typeset a $\sum$ for clarification though.
- The "$\rightharpoondown$" used in the book will be used implicitly, i.e. multiindices are always assumed to be in ascending order.

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# I Manifolds, Tensors and Exterior Forms

# Manifolds and Vector Fields

## Submanifolds of Euclidean Space

**1.1(3)** Consider for $F(A) = \det(A)$, $F : \mathbb{R}^{\ltimes^{\not\models}} \to \mathbb{R}$, that

$$\det(A + tB) = \det\left(A(1 + tA^{-1}B)\right) = \det(A)\det\left(1 + tA^{-1}B\right)$$

How to deal with $\det\left(1 + tA^{-1}B\right)$? Recall that

$$\det\left(1 + tA^{-1}B\right) = \det(A)\left(1 + t\,\mathrm{Tr}\left(A^{-1}B\right)\right)$$

because for $\det((1 + tX))$,

$$\det((1+tX)) = \det\left(\begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} + \begin{pmatrix} tx_{11} & \dots & tx_{1n} \\ \vdots & \ddots & \vdots \\ tx_{n1} & \dots & tx_{nn} \end{pmatrix}\right) = \det\left(\begin{pmatrix} 1 + tx_{11} & \dots & tx_{1n} \\ \vdots & \ddots & \vdots \\ tx_{n1} & \dots & 1 + tx_{nn} \end{pmatrix}\right) =$$
$$= 1 + t\,\mathrm{Tr}(X) + \mathcal{O}(t^2)$$

since recall $\det(A) = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) A_{1\sigma_1} A_{2\sigma_2} \dots A_{n\sigma_n}$, where sum is over all permutations of $\{1, \dots, n\}$, and so only the $A_{11} \dots A_{nn}$ term would have terms of $\mathcal{O}(t)$.

So

$$(DF) \cdot B = \frac{d}{dt} F(x(t)) B = \det(x)\,\mathrm{Tr}\left(x^{-1}B\right)$$

For $x_0 \in Sl(n)$, $\det(x_0) = 1$. Let $B = \frac{r}{n}x$. Then

$$(DF) \cdot B = \mathrm{Tr}\left(x^{-1}\frac{r}{n}x\right) = r$$

$DF = F_*$ is surjective $\forall\, x \in Sl(n)$

## Manifolds

## Tangent Vectors and Mappings

### Tangent or “Contravariant Vectors

### Vectors as Differential Operators

### The Tangent Space to $M^n$ at a Point

### Mappings and Submanifolds of Manifolds

**Definition 1** *$M^m \subset N^n$ (embedded) submanifold of $N^n$. If $M$ locally s.t. $F : N^n \to \mathbb{R}^{n-m}$*

$$\begin{aligned} &F^1(x^1 \dots x^n) = 0 \\ &\vdots \\ &F^{n-m}(x^1 \dots x^n) = 0 \end{aligned}$$

*$n - m$ diff. $F^i$ s.t. $\left|\frac{\partial F^i}{\partial x^j}\right|$ has rank $n - m$*

By implicit function thm., submanifold as graph.

$$\begin{aligned} &(x^1 \dots x^m, y^{m+1} \dots y^n) \\ &y^{m+1} = f^{m=1}(x^1 \dots x^m) \\ &\vdots \\ &y^n = f^n(x^1 \dots x^m) \end{aligned}$$

on $F(x) = 0$

**Theorem 2 (1.12)** *Let $F : M^m \to N^n$, $q \in N^n$ s.t. $F^{-1}(q) \subset M^m$, $F^{-1}(q) \neq \emptyset$*
*If $F_*$ onto, i.e. $F_*$ rank $n$, $\forall\, F^{-1}(q)$,*
*$F^{-1}(q)$ $(n - m)$-dim. submanifold of $M^m$*

# Tensors and Exterior Forms

## 2.1 Covectors and Riemannian Metrics

**2.1(1)** **Want**: $\sum a_i^V v_V^i = \sum a_j^U v_U^j$

$$a_i^U dx_U^i(v) = a_i^U dx_U^i \left( v_U^j \frac{\partial}{\partial x_U^j} \right) = a_i^U v_U^i = a_i^V dx_V^i(v) = a_i^V dx_V^i \left( v_V^j \frac{\partial}{\partial x_V^j} \right) = a_i^V v_V^i$$

or

$$a_j^U v_U^j = a_i^V \frac{\partial x_V^i}{\partial x_U^j} v_U^j = a_i^V \frac{\partial x_V^i}{\partial x_U^j} \frac{\partial x_U^j}{\partial x_V^k}(p_0) v_V^k = a_i^V v_V^i$$

were (1.6) was used.

$$v^i w^i = v_U^i w_U^i = \frac{\partial x_U^i}{\partial x_V^j} v_V^j \frac{\partial x_U^i}{\partial x_V^k} w_V^k = \frac{\partial x_U^i}{\partial x_V^j} \frac{\partial x_U^i}{\partial x_V^k} v_V^j w_V^k$$

transforms as a $(0,2)$ tensor.

**2.1(2)**

(i) Recall

$$g_{ij}^V = \frac{\partial x_U^r}{\partial x_V} \frac{\partial x_U^s}{\partial x_V^j} g_{rs}^U$$

$$\begin{array}{lll} u^1 = r & x = r\cos(\phi)\sin(\theta) \\ u^2 = \theta & y = r\sin(\phi)\sin(\theta) \\ u^3 = \phi & z = r\cos(\theta) \end{array} \quad \begin{array}{lll} \frac{\partial x}{\partial r} = c\phi s\theta & \frac{\partial x}{\partial \theta} = rc\phi c\theta & \frac{\partial x}{\partial \phi} = -rs\phi s\theta \\ \frac{\partial y}{\partial r} = s\phi s\theta & \frac{\partial y}{\partial \theta} = rs\phi c\theta & \frac{\partial y}{\partial \phi} = rc\phi s\theta \\ \frac{\partial z}{\partial r} = c\theta & \frac{\partial z}{\partial \theta} = -rs\theta & \frac{\partial z}{\partial \phi} = 0 \end{array}$$

$$\boxed{\begin{aligned} g_{rr} &= 1 \\ g_{\theta\theta} &= r^2 \\ g_{\phi\phi} &= r^2(\sin(\theta))^2 \end{aligned}}$$

(ii) $\text{grad}(f) = \nabla f$ is contravariant vector, associated to covector $df$, $df(w) = \langle \nabla f, w \rangle$. $(\nabla f)^i = g^{ij} \frac{\partial f}{\partial x^j}$

$g^{ij}$ is **not** $g_{ij}$. What is $g^{ij}$?
Also $(r, \theta, \phi)$ is a non-coordinate bases.

$$ds^2 = (dr)^2 + r^2(d\theta)^2 + r^2(\sin(\theta))^2(d\phi)^2$$

The distance elements are $dr$, $rd\theta$, $r\sin(\theta)\, d\phi$ in this non-coordinate basis $\widehat{r}, \widehat{\theta}, \widehat{\phi}$.
We're using $ds^2 = |d\mathbf{x}|^2 \equiv g(d\mathbf{x}, d\mathbf{x}) = d\mathbf{x} \cdot d\mathbf{x}$ instead of $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$

In the non-coordinate basis $\{\widehat{r}, \widehat{\theta}, \widehat{\phi}\}$. Coordinate basis $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi\}$
Consider

$$g_{\mu'\nu'} \equiv g(\mathbf{e}_{\mu'}, \mathbf{e}_{\nu'}) = g_{\alpha\beta} \widetilde{e}^\alpha(\mathbf{e}_{\mu'}) \widetilde{e}^\beta(\mathbf{e}_{\nu'}) = g_{\alpha\beta} \Lambda^\alpha_{\mu'} \Lambda^\beta_{\nu'}$$

$$g_{\mu\nu} = \mathbf{e}_\mu \cdot \mathbf{e}_\nu$$

$$g_{rr} = 1$$

$$g_{\theta\theta} = r^2$$

$$g_{\phi\phi} = r^2(\sin(\theta))^2$$

So $\mathbf{e}_\theta = \frac{\partial}{\partial \theta}$, $\mathbf{e}_\phi = \frac{\partial}{\partial \phi}$ are **not unit vectors**!

In the coordinate basis $d\mathbf{x} = \mathbf{e}_r dr + \mathbf{e}_\theta d\theta + \mathbf{e}_\phi d\phi = \mathbf{e}_i dx^i$
In the noncoordinate basis, $d\mathbf{x} = \widehat{r}dr + \widehat{\theta}rd\theta + \widehat{\phi}r\sin{(\theta)}\,d\phi$

$$\begin{aligned} \Lambda^r_{\ \widehat{r}} &= 1 \\ \Lambda^\theta_{\ \widehat{\theta}} &= \frac{1}{r} \\ \Lambda^\phi_{\ \widehat{\phi}} &= \frac{1}{r\sin{(\theta)}} \end{aligned}$$

So then, for instance

$$g_{\phi\phi} = r^2(\sin{(\theta)})^2 = g(\partial_\phi, \partial_\phi) = g_{ij}\widetilde{e}^i(\partial_\phi)\widetilde{e}^j(\partial_\phi) = g_{ij}\Lambda^i_{\ \phi}\Lambda^j_{\ \phi} = g_{\widehat{\phi}\widehat{\phi}}r^2(\sin{(\theta)})^2$$

$$\begin{aligned} g_{\widehat{r}\widehat{r}} &= g_{\widehat{\theta}\widehat{\theta}} = g_{\widehat{\phi}\widehat{\phi}} = 1 \\ g_{\widehat{r}\widehat{\theta}} &= g_{\widehat{r}\widehat{\phi}} = g_{\widehat{\theta}\widehat{\phi}} = 0 \end{aligned}$$

in non-coordinate basis, we must give up the following two:

$$\begin{aligned} &d\mathbf{x} \equiv dx^\mu \mathbf{e}_\mu \text{ defines } \mathbf{e}_\mu \text{ coordinate basis} \\ &ds^2 = g_{\mu\nu}dx^\mu dx^\nu \end{aligned}$$

Inverse metric components

$$\begin{aligned} g^{rr} &= 1 \\ g^{\theta\theta} &= \frac{1}{r^2} \\ g^{\phi\phi} &= \frac{1}{r^2(\sin{(\theta)})^2} \end{aligned}$$

The isomorphism of $V$ and $V^*$ (e.g. $T_pM$ and $T_pM^*$) allows us to introduce notation that replaces one-forms with vectors and $(m,n)$ tensors with $(m+n,0)$ tensors.
Replace basis one-forms $\widetilde{e}^\mu \equiv \alpha^\mu$ with set of vectors defined

$$\mathbf{e}^\mu(\cdot) \equiv g^{-1}(\widetilde{e}^\mu, \cdot) = g^{\mu\nu}\mathbf{e}_\mu(\cdot)$$

where $\widetilde{e}^\mu$ basis one form, $\mathbf{e}^\mu$ dual basis vector.
Then

$$\begin{aligned} \mathbf{e}^r &= \mathbf{e}_r = \frac{\partial}{\partial r} = \widehat{r} \\ \mathbf{e}^\theta &= \frac{1}{r^2}\mathbf{e}_\theta = \frac{1}{r}\widehat{\theta} \\ \mathbf{e}^\phi &= \frac{1}{(r\sin{(\theta)})^2}\mathbf{e}_\phi = \frac{1}{r\sin{(\theta)}}\widehat{\phi} \end{aligned}$$

Now

$$\begin{aligned} \widetilde{\nabla} &\equiv \widetilde{e}^\mu\partial_\mu && \text{in coordinate basis} \\ \widetilde{\nabla}x^\mu &= \widetilde{e}^\mu && \text{in a coordinate basis} \\ \boldsymbol{\nabla} &= \mathbf{e}^\mu\partial_\mu = g^{\mu\nu}\mathbf{e}_\mu\partial_\nu \end{aligned}$$

So finally

$$\boxed{\begin{aligned} \boldsymbol{\nabla} &= \widehat{r}\partial_r + \frac{1}{r}\widehat{\theta}\partial_\theta + \frac{1}{r\sin{(\theta)}}\widehat{\phi}\partial_\phi \\ \boldsymbol{\nabla}f &= \widehat{r}\partial_r f + \frac{1}{r}\widehat{\theta}\partial_\theta f + \frac{1}{r\sin{(\theta)}}\widehat{\phi}\partial_\phi f \end{aligned}}$$

Also, in this formulation,

$$\boxed{\begin{aligned} \boldsymbol{\nabla} &= \mathbf{e}_r\partial_r + \frac{1}{r^2}\mathbf{e}_\theta\partial_\theta + \frac{1}{(r\sin{(\theta)})^2}\mathbf{e}_\phi\partial_\phi \\ \boldsymbol{\nabla}f &= \mathbf{e}_r\partial_r f + \frac{1}{r^2}\partial_\theta f\mathbf{e}_\theta + \frac{1}{(r\sin{(\theta)})^2}\partial_\phi f\mathbf{e}_\phi = (\partial_r f)\frac{\partial}{\partial r} + \frac{1}{r^2}\partial_\theta f\frac{\partial}{\partial\theta} + \frac{\partial_\phi f}{(r\sin{(\theta)})^2}\frac{\partial}{\partial\phi} = \\ &= (\nabla f)^i\partial_i = g^{ij}\frac{\partial f}{\partial x^j}\partial_i \end{aligned}}$$

(cf. MIT Physics 8.962 Spring 1999, Edmund Bertschinger. **Introduction to Tensor Calculus for General Relativity** `gr1.pdf`)

(iii) See above. And as before,

$$\begin{aligned} \frac{\partial}{\partial r} &= \widehat{r} \\ \frac{1}{r}\frac{\partial}{\partial \theta} &= \widehat{\theta} \\ \frac{1}{r\sin(\theta)}\frac{\partial}{\partial \phi} &= \widehat{\phi} \end{aligned}$$

### 2.3. The Cotangent Bundle and Phase Space

#### 2.3a. The Cotangent Bundle

#### 2.3b. The Pull-Back of a Covector

**2.3c. The Phase Space in Mechanics** Let $q^1 \dots q^m$ local generalized coordinates, $M^m$ configuration space of a dynamical system.

$$L : TM^m \to \mathbb{R}$$

Consider $(U, q)$, $UV \neq \emptyset$, $q \in UV$
$(V, r)$
$r = r(q)$

$$\dot{r}^j = \frac{\partial r^j}{\partial q^i}\dot{q}^i \qquad (2.27) \qquad \frac{\partial \dot{r}^j}{\partial \dot{q}^i} = \frac{\partial r^j}{\partial q^i}$$

$$\pi_i \equiv \frac{\partial L}{\partial \dot{r}^i} = \frac{\partial L}{\partial q^j}\frac{\partial q^j}{\partial \dot{r}^i} + \frac{\partial L}{\partial \dot{q}^j}\frac{\partial \dot{q}^j}{\partial \dot{r}^i} = \frac{\partial L}{\partial \dot{q}^j}\frac{\partial \dot{q}^j}{\partial \dot{r}^i} = \frac{\partial L}{\partial \dot{q}^j}\frac{\partial q^j}{\partial r^i}$$

$$\Longrightarrow \pi_i = p_j \frac{\partial q^j}{\partial r^i} \qquad (2.29)$$

$p$'s are covector.

$$\Longrightarrow p : TM^m \to T^*M^m$$

cotangent bundle. $T^*M^m$ of covectors to configuration space is phase space.

$$T(q, \dot{q}) = \frac{1}{2}\sum_{jk} g_{jk}(q)\dot{q}^j\dot{q}^k \qquad (2.31)$$

$$p_i = \frac{\partial L}{\partial \dot{q}^i} = \frac{\partial T}{\partial \dot{q}^i} = \sum_j g_{ij}(q)\dot{q}^j \qquad (2.32)$$

think of $2T$ as Riemannian metric on $M^m$.

$$\langle \dot{q}, \dot{q} \rangle = \sum_{ij} g_{ij}(q)\dot{q}^i\dot{q}^j$$

#### 2.3d. The Poincaré 1-Form

### 2.4 Tensors

#### 2.4a. Covariant Tensors

**Definition 3** *covariant tensor of rank $r$*

$$Q : E \times \cdots \times E \to \mathbb{R}$$

$Q(v_1 \dots v_r)$

vector space of covariant $r$th rank tensors $E^* \otimes \cdots \otimes E^* = \otimes^r E^*$

2nd. rank covariant tensor $\alpha \otimes \beta : E \times E \to \mathbb{R}$
$\alpha \otimes \beta(v, w) \equiv \alpha(v)\beta(w)$

**2.4(1)** For any $v, w$ tangent vectors,

$$v = v^i \frac{\partial}{\partial x^i}$$
$$w = w^i \frac{\partial}{\partial x^i}$$

$$(\alpha \otimes \beta)(v,w) = \alpha(v)\beta(w) = a_i dx^i (v^j \frac{\partial}{\partial x^j}) b_k dx^k (w^l \frac{\partial}{\partial x^l}) = a_i b_k v^j w^l \delta^i_j \delta^k_l = a_j v^j b_k w^k =$$
$$= a_j b_k dx^j(v) dx^k(w) = a_j b_k dx^j \otimes dx^k (v,w)$$

For $\alpha, \beta$ in components,

$$\alpha = a_i dx^i$$
$$\beta = b_j dx^j$$

$$a_i b_j dx^i \otimes dx^j (v,w) = a_i b_j v^i w^j = a_j v^j b_k w^k \implies \alpha \otimes \beta = a_i b_j dx^i \otimes dx^j$$

### 2.4(2)(i) Contraction invariant under base transformation

$$A'^i{}_i = A(\mathrm{d}x'^i, \boldsymbol{\partial}'_i) = A\left(\frac{\partial x'^i}{\partial x^j}\mathrm{d}x^j, \frac{\partial x^k}{\partial x'^i}\boldsymbol{\partial}_k\right) = \underbrace{\frac{\partial x'^i}{\partial x^j}\frac{\partial x^k}{\partial x'^i}}_{\frac{\partial x^k}{\partial x^j} = \delta^k_j} \underbrace{A\left(\mathrm{d}x^j, \boldsymbol{\partial}_k\right)}_{=A^j{}_k} = A^j{}_j$$

This is the transformation law of a scalar.

### 2.4(2)(ii) Non-invariant "contraction"

$$\sum_i A'_{ii} = \sum_i A(\boldsymbol{\partial}'_i, \boldsymbol{\partial}'_i) = \sum_i A\left(\frac{\partial x^j}{\partial x'^i}\boldsymbol{\partial}_j, \frac{\partial x^k}{\partial x'^i}\boldsymbol{\partial}_k\right) = \sum_i \frac{\partial x^j}{\partial x'^i}\frac{\partial x^k}{\partial x'^i}\underbrace{A(\boldsymbol{\partial}_j, \boldsymbol{\partial}_k)}_{=A_{jk}}$$
$$= \sum_i \frac{\partial x^j}{\partial x'^i}\frac{\partial x^k}{\partial x'^i} A_{jk} \neq A_{ii}$$

Since the differential quotients do not cancel out, the value of $\sum_i A_{ii}$ is dependant on coordinates; a coordinate-dependant number is neither a scalar nor any other sort of tensor.

### 2.4(3)(i) Transformation behavior of a contraction

$$g'_{ji} v'^i = \frac{\partial x^k}{\partial x'^j}\frac{\partial x^\ell}{\partial x'^i} g_{k\ell} \frac{\partial x'^i}{\partial x^m} v^m = \frac{\partial x^k}{\partial x'^j} \underbrace{\frac{\partial x^\ell}{\partial x'^i}\frac{\partial x'^i}{\partial x^m}}_{=\delta^\ell_m} g_{k\ell} v^m = \frac{\partial x^k}{\partial x'^j} g_{k\ell} v^\ell$$

Thus, $g_{ji}v^i$ transforms like a vector.

### 2.4(3)(ii) Tensor?

$$\partial'_j v'^i = \frac{\partial}{\partial x'^j}\left(\frac{\partial x'^i}{\partial x^k} v^k\right) = \frac{\partial^2 x'^i}{\partial x^\ell \partial x^k}\frac{\partial x^\ell}{\partial x'^j} v^k + \frac{\partial x'^i}{\partial x^k}\frac{\partial v^k}{\partial x'^j} = \frac{\partial^2 x'^i}{\partial x^\ell \partial x^k}\frac{\partial x^\ell}{\partial x'^j} v^k + \frac{\partial x'^i}{\partial x^k}\underbrace{\frac{\partial v^k}{\partial x^\ell}}_{=\partial_\ell v^k}\frac{\partial x^\ell}{\partial x'^j}$$
$$= \underbrace{\frac{\partial^2 x'^i}{\partial x^\ell \partial x^k}\frac{\partial x^\ell}{\partial x'^j} v^k}_{\neq 0} + \frac{\partial x^\ell}{\partial x'^j}\frac{\partial x'^i}{\partial x^k}\partial_\ell v^k$$

Although the second term is the correct tensor transformation law, the first term prevents $\partial_j v^i$ from forming a tensor.

### 2.4(3)(iii) Tensor? – second attempt

Using the result of (ii), one gets

$$
\begin{aligned}
\partial'_j v'^i - \partial'_i v'^j &= \frac{\partial^2 x'^i}{\partial x^\ell \partial x^k}\frac{\partial x^\ell}{\partial x'^j} v^k + \frac{\partial x^\ell}{\partial x'^j}\frac{\partial x'^i}{\partial x^k}\partial_\ell v^k - \frac{\partial^2 x'^j}{\partial x^\ell \partial x^k}\frac{\partial x^\ell}{\partial x'^i} v^k - \frac{\partial x^\ell}{\partial x'^i}\frac{\partial x'^j}{\partial x^k}\partial_\ell v^k \\
&= \left( \frac{\partial^2 x'^i}{\partial x^\ell \partial x^k}\frac{\partial x^\ell}{\partial x'^j} v^k - \frac{\partial^2 x'^j}{\partial x^\ell \partial x^k}\frac{\partial x^\ell}{\partial x'^i} v^k \right) + \left( \frac{\partial x^\ell}{\partial x'^j}\frac{\partial x'^i}{\partial x^k}\partial_\ell v^k - \frac{\partial x^\ell}{\partial x'^i}\frac{\partial x'^j}{\partial x^k}\partial_\ell v^k \right) \\
&\neq 0 + \frac{\partial x^\ell}{\partial x'^j}\frac{\partial x'^i}{\partial x^k}\left(\partial_\ell v^k - \partial_k v^\ell\right)
\end{aligned}
$$

### 2.4(4)

(i)

$$L = L(q,\dot{q}) = \frac{1}{2} g_{ij}(q)\dot{q}^i \dot{q}^j - V$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^k}\right) = \frac{\partial L}{\partial q^k}$$

$$V = V(q) = V(0) + \frac{\partial V}{\partial q^i} q^i + \frac{1}{2}\frac{\partial^2 V}{\partial q^i \partial q^j} q^i q^j$$

Assume $g$ symmetric in indices.
$\frac{\partial V}{\partial q^k} = 0$ i.e. $q = 0$ nondegenerate minimum for $V$.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) = g_{ij}(0)\dot{q}^j = -\frac{\partial^2 V}{\partial q^i \partial q^j} q^j = -Q_{ij} q^j$$

(ii)

(iii)

## 2.5 The Graßmann or Exterior Algebra

### 2.5a. The Tensor Product of Covariant Tensors

### 2.5b. The Grassmann or Exterior Algebra

$$\alpha = \alpha_{\underline{J}} dx^{\underline{J}} = \frac{1}{p!}\alpha_J dx^J$$

$\alpha_J$ antisymmetric in $J$ and $\alpha_{\underline{J}}$ antisymmetric in $\underline{J}$

$$\alpha_{\underline{J}} = p!\alpha_J$$

**Lemma 1 (2.46)**

$$\delta^{IJ}_M \delta^{KL}_{\underline{J}} = \delta^{IKL}_M$$

**Proof**

$$
\begin{aligned}
&I = (i_1 \dots i_p) \\
&J = (j_1 \dots j_{q+r}) \\
&\underline{J} = (j_1 < \dots < j_{q+r}) \\
&K = (k_1 \dots k_q) \\
&L = (l_1 \dots l_r) \\
&M = (m_1 \dots m_{p+q+r})
\end{aligned}
$$

$KL$ fixed. put $KL$ into (unique) increasing order, by as many transpositions as total number of inversions (cf. Tu, L.W., Introduction to Manifolds, Springer, 2008), Proposition 3.6)
so $\delta^{KL}_{\underline{J}} \neq 0$ for only 1 $\underline{J}$
Suppose $\delta^{KL}_{\underline{J}} = 1$, $KL$ even permutation of $\underline{J}$ (permutation is bijective)
$M$ fixed so suppose $I\underline{J}$ even permutation of $M$
$I\sigma(KL) = f(M)$

$\sigma$ even permutation of $KL$, so put $\sigma(KL)$ into $KL$ by even number of transpositions
This defines even permutation $g$ that's bijective on $I\sigma(KL)$

$$g(I\sigma(KL)) = IKL = gf(M)$$

so $IKL$ even permutation of $M$.
If $I\overline{J}$ odd permutation of $M$, $gf$ odd permutation, $\delta_M^{IKL} = -1$

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### 2.5c. The Geometric Meaning of Forms in $\mathbb{R}^n$

### 2.5(1) Basis expansion of a form

$$\left(a_J \mathrm{d}x^J\right)(\boldsymbol{\partial}_K) = a_J \mathrm{d}x^J (\boldsymbol{\partial}_K) = a_J \delta_K^J = a_K = \alpha(\boldsymbol{\partial}_K)$$

Since this is true for all $\boldsymbol{\partial}_K$, $\alpha = a_J \mathrm{d}x^J$.

### 2.5(2) Components of $\alpha^1 \wedge \beta^2$

$$(\alpha^1 \wedge \beta^2)_{i<j<k} = \sum_{l,m<n} \delta_{ijk}^{lmn} \alpha_l \beta_{mn}$$

All summands where $ijk$ is not a permutation of $lmn$ vanish, so there are 6 possible permutations left:

| (A) | | | (B) | | | (C) | | | (D) | | | (E) | | | (F) | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| i | = | l | i | = | m | i | = | n | i | = | l | i | = | n | i | = | m |
| j | = | m | j | = | n | j | = | l | j | = | n | j | = | m | j | = | l |
| k | = | n | k | = | l | k | = | m | k | = | m | k | = | l | k | = | n |

Of these 6, (C), (D) and (E) contradict $i < j < k$ (given by the problem) with respect to $m < n$ (from the definition of the wedge product), leaving only 3 summands. Thus,

$$\begin{aligned}(\alpha^1 \wedge \beta^2)_{i<j<k} = \sum_{l,m<n} \delta_{ijk}^{lmn} \alpha_l \beta_{mn} &= \underbrace{\delta_{ijk}^{ijk}}_{(A)\to+1} \alpha_i \beta_{jk} + \underbrace{\delta_{ijk}^{kij}}_{(B)\to+1} \alpha_k \beta_{ij} + \underbrace{\delta_{ijk}^{jik}}_{(F)\to-1} \alpha_j \underbrace{\beta_{ik}}_{-\beta_{ki}} \\ &= \alpha_i \beta_{jk} + \alpha_j \beta_{ki} + \alpha_k \beta_{ij} \ .\end{aligned}$$

**2.5(3)** In $\mathbb{R}^3$,
Given

$$\begin{aligned}\alpha^1 &= a_1 dx^1 + \cdots + a_3 dx^3 \\ \beta^1 &= b_1 dx^1 + b_2 dx^2 + b_3 dx^3 \\ \rho^1 &= r_1 dx^1 + r_2 dx^2 + r_3 dx^3 \\ \gamma^2 &= c_1 dx^2 \wedge dx^3 + c_2 dx^3 \wedge dx^1 + c_3 dx^1 \wedge dx^2\end{aligned}$$

$$\alpha^1 \wedge \gamma^2 = (a_1 c_1 + a_2 c_2 + a_3 c_3) dx^1 \wedge dx^2 \wedge dx^3 = a \cdot c dx^1 \wedge dx^2 \wedge dx^3 = a \cdot c \mathrm{vol}(dx)$$

$$\alpha^1 \wedge \beta^1 = (a_1 b_2 - a_2 b_1) dx^1 \wedge dx^2 + (a_1 b_3 - a_3 b_1) dx^1 \wedge dx^3 + (a_2 b_3 - a_3 b_2) dx^2 \wedge dx^3$$

$$\alpha^1 \wedge \beta^1 \wedge \rho^1 = ((a_1 b_2 - a_2 b_1) r_3 + (-r_2)(a_1 b_3 - a_3 b_1) + r_1 (a_2 b_3 - a_3 b_2)) dx^1 \wedge dx^2 \wedge dx^3 = r \cdot (a \times b) \mathrm{vol}(dx)$$

## 2.6 Exterior Differentiation

### 2.6(1) Differential of a 3-Form in $\mathbb{R}^4$

$$\begin{aligned}
\beta^3 &= \beta_J \mathrm{d}x^J = \sum_{i<j<k} \beta_{ijk} \mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k \\
&= \beta_{123} \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 + \beta_{124} \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^4 \\
&\quad + \beta_{134} \mathrm{d}x^1 \wedge \mathrm{d}x^3 \wedge \mathrm{d}x^4 + \beta_{234} \mathrm{d}x^2 \wedge \mathrm{d}x^3 \wedge \mathrm{d}x^4 \\
\Rightarrow \mathrm{d}\beta^3 &= \mathrm{d}\beta_{123} \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 + \mathrm{d}\beta_{124} \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^4 \\
&\quad + \mathrm{d}\beta_{134} \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^3 \wedge \mathrm{d}x^4 + \mathrm{d}\beta_{234} \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \wedge \mathrm{d}x^4 \\
&= \frac{\partial \beta_{123}}{\partial x^i} \mathrm{d}x^i \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 + \frac{\partial \beta_{124}}{\partial x^i} \mathrm{d}x^i \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^4 \\
&\quad + \frac{\partial \beta_{134}}{\partial x^i} \mathrm{d}x^i \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^3 \wedge \mathrm{d}x^4 + \frac{\partial \beta_{234}}{\partial x^i} \mathrm{d}x^i \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \wedge \mathrm{d}x^4 \\
&= \frac{\partial \beta_{123}}{\partial x^4} \mathrm{d}x^4 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 + \frac{\partial \beta_{124}}{\partial x^3} \mathrm{d}x^3 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^4 \\
&\quad + \frac{\partial \beta_{134}}{\partial x^2} \mathrm{d}x^2 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^3 \wedge \mathrm{d}x^4 + \frac{\partial \beta_{234}}{\partial x^1} \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \wedge \mathrm{d}x^4 \\
&= \frac{\partial(-\beta_{123})}{\partial x^4} \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \wedge \mathrm{d}x^4 + \frac{\partial \beta_{124}}{\partial x^3} \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \wedge \mathrm{d}x^4 \\
&\quad + \frac{\partial(-\beta_{134})}{\partial x^2} \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \wedge \mathrm{d}x^4 + \frac{\partial \beta_{234}}{\partial x^1} \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \wedge \mathrm{d}x^4 \\
&\quad \to \text{rename components: } \beta_{234} \to \beta_1,\ -\beta_{134} \to \beta_2,\ \beta_{124} \to \beta_3,\ -\beta_{123} \to \beta_4 \\
&= \left( \frac{\partial \beta_1}{\partial x^1} + \frac{\partial \beta_2}{\partial x^2} + \frac{\partial \beta_3}{\partial x^3} + \frac{\partial \beta_4}{\partial x^4} \right) \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \wedge \mathrm{d}x^4
\end{aligned}$$

In cartesian coordinates, this says something like $d(\mathbf{B} \cdot \mathrm{d}\mathbf{V}) = \operatorname{div}(\mathbf{B})\, \mathrm{d}H$ ("H: Hyperspace volume").

## 2.7 Pull-Backs

### 2.7(1) Proof of homomorphism

Notation: Let $(F_* \mathbf{v}_I) = (F_* \mathbf{v}_{i_1}, F_* \mathbf{v}_{i_2}, \ldots)$ .

$$\begin{aligned}
F^*(\alpha \wedge \beta)(\mathbf{v}_I) &= (\alpha \wedge \beta)(F_* \mathbf{v}_I) = \sum_{J,K} \delta_I^{JK} \alpha(F_* \mathbf{v}_J)\ \beta(F_* \mathbf{v}_K) = \alpha(F_* \mathbf{v}_J) \wedge \beta(F_* \mathbf{v}_K) \\
&= (F^* \alpha(\mathbf{v}_J)) \wedge (F^* \beta(\mathbf{v}_K)) \quad \forall\, \mathbf{v}_I (= \mathbf{v}_{JK}) \\
\Rightarrow F^*(\alpha \wedge \beta) &= (F^* \alpha) \wedge (F^* \beta)
\end{aligned}$$

### 2.7(2) Pull-back onto a surface

Let $(u, v) = (y^1, y^2)$.

$$
\begin{aligned}
\beta^2 &= \beta_{12}\mathrm{d}x^1 \wedge \mathrm{d}x^2 + \beta_{13}\mathrm{d}x^1 \wedge \mathrm{d}x^3 + \beta_{23}\mathrm{d}x^2 \wedge \mathrm{d}x^3 \\
\Rightarrow i^*\beta &= \beta_{12}\frac{\partial x^1}{\partial y^i}\mathrm{d}y^i \wedge \frac{\partial x^2}{\partial y^j}\mathrm{d}y^j + \beta_{13}\frac{\partial x^1}{\partial y^i}\mathrm{d}y^i \wedge \frac{\partial x^3}{\partial y^j}\mathrm{d}y^j + \beta_{23}\frac{\partial x^2}{\partial y^i}\mathrm{d}y^i \wedge \frac{\partial x^3}{\partial y^j}\mathrm{d}y^j \\
&= \beta_{12}\left(\cancel{\frac{\partial x^1}{\partial y^1}\mathrm{d}y^1 \wedge \frac{\partial x^2}{\partial y^1}\mathrm{d}y^1} + \frac{\partial x^1}{\partial y^1}\mathrm{d}y^1 \wedge \frac{\partial x^2}{\partial y^2}\mathrm{d}y^2\right. \\
&\quad \left. + \frac{\partial x^1}{\partial y^2}\mathrm{d}y^2 \wedge \frac{\partial x^2}{\partial y^1}\mathrm{d}y^1 + \cancel{\frac{\partial x^1}{\partial y^2}\mathrm{d}y^2 \wedge \frac{\partial x^2}{\partial y^2}\mathrm{d}y^2}\right) \\
&\quad + \beta_{13}\left(\cancel{\frac{\partial x^1}{\partial y^1}\mathrm{d}y^1 \wedge \frac{\partial x^3}{\partial y^1}\mathrm{d}y^1} + \frac{\partial x^1}{\partial y^1}\mathrm{d}y^1 \wedge \frac{\partial x^3}{\partial y^2}\mathrm{d}y^2\right. \\
&\quad \left. + \frac{\partial x^1}{\partial y^2}\mathrm{d}y^2 \wedge \frac{\partial x^3}{\partial y^1}\mathrm{d}y^1 + \cancel{\frac{\partial x^1}{\partial y^2}\mathrm{d}y^2 \wedge \frac{\partial x^3}{\partial y^2}\mathrm{d}y^2}\right) \\
&\quad + \beta_{23}\left(\cancel{\frac{\partial x^2}{\partial y^1}\mathrm{d}y^1 \wedge \frac{\partial x^3}{\partial y^1}\mathrm{d}y^1} + \frac{\partial x^2}{\partial y^1}\mathrm{d}y^1 \wedge \frac{\partial x^3}{\partial y^2}\mathrm{d}y^2\right. \\
&\quad \left. + \frac{\partial x^2}{\partial y^2}\mathrm{d}y^2 \wedge \frac{\partial x^3}{\partial y^1}\mathrm{d}y^1 + \cancel{\frac{\partial x^2}{\partial y^2}\mathrm{d}y^2 \wedge \frac{\partial x^3}{\partial y^2}\mathrm{d}y^2}\right) \\
&= \left(\beta_{12}\left(\frac{\partial x^1}{\partial y^1}\frac{\partial x^2}{\partial y^2} - \frac{\partial x^1}{\partial y^2}\frac{\partial x^2}{\partial y^1}\right) + \beta_{13}\left(\frac{\partial x^1}{\partial y^1}\frac{\partial x^3}{\partial y^2} - \frac{\partial x^1}{\partial y^2}\frac{\partial x^3}{\partial y^1}\right)\right. \\
&\quad \left. + \beta_{23}\left(\frac{\partial x^2}{\partial y^1}\frac{\partial x^3}{\partial y^2} - \frac{\partial x^2}{\partial y^2}\frac{\partial x^3}{\partial y^1}\right)\right)\mathrm{d}y^1 \wedge \mathrm{d}y^2
\end{aligned}
$$

If one now defines, by renaming the components of $\beta$ again ($\beta_{23} \to \beta_1$, $-\beta_{13} = \beta_{31} \to \beta_2$, $\beta_{12} \to \beta_3$), $\mathbf{b} = (\beta_1, \beta_2, \beta_3)$, the last term can be identified as $\mathbf{b} \cdot \mathbf{n}\,\mathrm{d}y^1 \wedge \mathrm{d}y^2$, and one gets the desired expression

$$i^*\beta = (\mathbf{b}, \mathbf{n})\,\mathrm{d}u \wedge \mathrm{d}v\ .$$

## 2.8

### 2.8c. Orientability and 2-sided Hypersurfaces

If $M$ orientable if $\exists$ orientation $\forall\, TM_x^n$ to $M^n$, cont., or cover $M$ by $(U, \varphi)$, $|J| > 0 \quad \forall$ overlap.
Converse: cont. orientation $\forall\, TM_x^n$, $M$ orientable.

If $M$ orientable, $\forall\, p, q \in M$, curve $C$, $\quad p = C(0)$, $C(t)$, $t \mapsto e_i(t)$ cont.,
$q = C(1)$

Contrapositive! (Mö bius strip)

### 2.8c. Orientability and 2-Sided Hypersurfaces

$M$ submanifold of $W^r$
$N$ transverse to $M$ if $N$ never tangent to $M$, $N \neq 0$ on $M$
hypersurface $M^n$ in $W^{n+1}$ 2-sided in $W$ if $\exists$ (cont.) transverse vector field $N$ along $M$
Möbius band "1-sided", $\nexists$ cont. unit $N$
if $M^n$ 2-sided hypersurface of orientable $W^{n+1}$, then $M^n$ orientable

### 2.9 Interior Products and Vector Analysis

### 2.9a. Interior Products and Contractions

**Definition 4** *interior product*
$i_{\mathbf{v}}\alpha^1 = \alpha(\mathbf{v})$ *if $\alpha$ 1-form*
$i_{\mathbf{v}}\alpha^p(w_2 \ldots w_p) = \alpha^p(v, w_2 \ldots w_p)$ *if $\alpha$ p-form*

*Clearly* $i_{A+B} = i_A + i_B$
$i_{aA} = aA$

**Theorem 5 (2.75)** $i_{\mathbf{v}} : \Lambda^p \to \Lambda^{p-1}$ *antiderivation*

$$i_{\mathbf{v}}(\alpha^p \wedge \beta^q) = [i_{\mathbf{v}}\alpha^p] \wedge \beta^q + (-1)^p \alpha^p \wedge [i_{\mathbf{v}}\beta^q]$$

**Theorem 6 (2.76)** *in components*

$$i_{\mathbf{v}}\alpha = \sum_{i_2 < \dots < i_p} \sum_j v^j a_{ji_2 < \dots < i_p} dx^{i_2} \wedge \dots \wedge dx^{i_p}$$

*i.e.* $(i_{\mathbf{v}}\alpha)_{i_2<\dots<i_p} = \sum_j v^j a_{ji_2<\dots<i_p}$
*or*

$$[i_{\mathbf{v}}\alpha]_k = v^j \alpha_{jk}$$

## 2.9b. Interior Product in $\mathbb{R}^3$

$\mathbf{v} \Longleftrightarrow$ pseudo-2-form $v^2 \equiv i_{\mathbf{v}}\mathrm{vol}^3$

$$i_{\mathbf{v}}\sqrt{g}du^1 \wedge \dots \wedge du^n = \sqrt{g}v^i i_{\partial_i} du^1 \wedge \dots \wedge du^n$$

$$i_{\partial_i} du^1 \wedge \dots \wedge du^n = \sum_{I,j} \delta^j_i \delta^{1\dots n}_{jI} du^I = \sum_I \delta^{1\dots n}_{iI} du^I = ??? = \epsilon^{iI}_{1\dots n} du^1 \wedge \dots \wedge \widehat{du^i} \wedge \dots du^n$$

cf. Nakahara 5.4.3. Interior product and Lie derivative of forms

$$X = X^\mu \frac{\partial}{\partial x^\mu}$$

$$\omega = \frac{1}{p!}\omega_{\mu_1\dots\mu_r} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p}$$

$$i_X\omega = \frac{1}{(p-1)!} X^\nu \omega_{\nu i_2 \dots i_p} dx^{i_2} \wedge \dots \wedge dx^{i_p} = \frac{1}{p!}\sum_{s=1}^p X^{i_s}\omega_{i_1\dots i_s \dots i_p}(-1)^{s-1} dx^{i_1} \wedge \dots \wedge \widehat{dx}^{i_s} \wedge \dots \wedge dx^{i_p}$$

$$\omega^1 = \langle\, , \mathbf{w}\rangle$$

$$i_{\mathbf{v}}\omega^1 = \omega^1(v) = \langle v, w\rangle$$

$$v^1 \wedge \omega^2 = \langle v, w\rangle \mathrm{vol}^3 \qquad (2.82) \tag{1}$$

$$v^1 \wedge \omega^2 = v^1 \wedge i_{\mathbf{w}}\mathrm{vol}^3 = [i_{\mathbf{w}}v^1] \wedge \mathrm{vol}^3 + -i_{\mathbf{w}}(v^1 \wedge \mathrm{vol}^3) = (i_{\mathbf{w}}v^1)\mathrm{vol}^3 = \langle \mathbf{v}, \mathbf{w}\rangle \mathrm{vol}^3$$

$\mathbf{v} \times \mathbf{w}$ 2 form $v^1 \wedge \omega^2$ $i_{v\times w}\mathrm{vol}^3 = v^1 \wedge w^1$
1 form $-i_{\mathbf{v}}\omega^2$

**2.10(1)** Given $T^{\dots i \dots}_{\dots j \dots}$ components of a mixed tensor, $p$ times contravariant and $q$ times covariant, then it transforms as such, by definition,

$$T^{\dots k_i \dots}_{\dots l_j \dots} = \frac{\partial y^{k_1}}{\partial x^{i_1}} \dots \frac{\partial y^{k_i}}{\partial x^{i_i}} \dots \frac{\partial y^{k_q}}{\partial x^{i_q}} \frac{\partial x^{j_1}}{\partial y^{l_1}} \dots \frac{\partial x^{j_j}}{\partial y^{l_j}} \dots \frac{\partial x^{j_p}}{\partial y^{l_p}} T^{\dots i_i \dots}_{\dots j_j \dots}$$

$$T^{\dots k \dots}_{\dots k \dots} = \frac{\partial y^{k_1}}{\partial x^{i_1}} \dots \frac{\partial y^{k}}{\partial x^{i_1}} \dots \frac{\partial y^{k_q}}{\partial x^{i_q}} \frac{\partial x^{j_1}}{\partial y^{l_1}} \dots \frac{\partial x^{j_j}}{\partial y^{k}} \dots \frac{\partial x^{j_p}}{\partial y^{l_p}} T^{\dots i_i \dots}_{\dots j_j \dots} = \frac{\partial y^{k_1}}{\partial x^{i_1}} \dots \widehat{\frac{\partial y^{k}}{\partial x^{i_i}}} \dots \frac{\partial y^{k_q}}{\partial x^{i_q}} \frac{\partial x^{j_1}}{\partial y^{l_1}} \dots \widehat{\frac{\partial x^{j_j}}{\partial y^{l_j}}} \dots \frac{\partial x^{j_p}}{\partial y^{l_p}} T^{\dots i \dots}_{\dots i \dots}$$

$$\frac{\partial y^k}{\partial x^{i_i}}\frac{\partial x^{j_j}}{\partial y^k} = \frac{\partial x^{j_j}}{\partial y^k}\frac{\partial y^k}{\partial x^{i_i}} = \left(\left(\frac{\partial y}{\partial x}\right)^{-1}\right)^{j_j}_k \frac{\partial y^k}{\partial x^{i_i}} = \delta^{j_j}_{i_i}$$

### 2.10(2) Components of the interior product

Let $\alpha = \alpha_J \mathrm{d}x^J$. In general we have the expansion

$$i_{\mathbf{v}}\alpha = i_{v^j\boldsymbol{\partial}_j}\left(\alpha\left(\boldsymbol{\partial}_k, \boldsymbol{\partial}_L\right)\mathrm{d}x^k \wedge \mathrm{d}x^L\right) = v^j \alpha\left(\boldsymbol{\partial}_j, \boldsymbol{\partial}_L\right)\mathrm{d}x^L = v^j \alpha_{jL}\mathrm{d}x^L$$

For a single component this yields

$$\left(i_{\mathbf{v}}\alpha\right)_K = \left(v^j\alpha_{jL}\mathrm{d}x^L\right)\left(\boldsymbol{\partial}_K\right) = v^j\alpha_{jL}\mathrm{d}x^L(\boldsymbol{\partial}_K) = v^j\alpha_{jL}\delta^L_K = v^j\alpha_{jK}$$

**2.10(3)** Recall

$$\nabla^2 f = \Delta f \equiv \operatorname{div}(\operatorname{grad} f) = \frac{1}{\sqrt{g}} \frac{\partial}{\partial u^i} \left[ \sqrt{g} g^{ij} \left( \frac{\partial f}{\partial u^j} \right) \right]$$

Note that $g^{ij}$ is the inverse of $g_{ij}$.
Note that $g = r^4 (\sin(\theta))^2$

$$\begin{aligned} &\partial_r (r^2 \sin(\theta)\, \partial_r f) + \partial_\theta (r^2 \sin(\theta)\, (1/r^2) \partial_\theta f) + \partial_\phi (r^2 \sin(\theta)\, (1/(r^2 (\sin(\theta))^2)) \partial_\phi f) \\ &\implies (1/r) \partial_r (r^2 \partial_r f) + (1/r^2) \partial_\theta (\sin(\theta)\, \partial_\theta f) + (1/(r^2 (\sin(\theta))^2)) \partial_\phi (\partial_\phi f) \end{aligned}$$

**2.10(4) Vector analysis in $\mathbb{R}^3$**

$$\operatorname{grad}(fg) \Leftrightarrow \mathrm{d}(f^0 \wedge g^0) = \mathrm{d}f \wedge g + f \wedge \mathrm{d}g = \overbrace{(\mathrm{d}f)}^{\Leftrightarrow \operatorname{grad}(f)} g + f \overbrace{(\mathrm{d}g)}^{\Leftrightarrow \operatorname{grad}(g)} \Leftrightarrow f \operatorname{grad}(g) + g \operatorname{grad}(f)$$

$$\operatorname{div}(f\,\mathbf{B}) \Leftrightarrow \mathrm{d}(f \wedge \beta^2) = \underbrace{\overbrace{\mathrm{d}f}^{\Leftrightarrow \operatorname{grad}(f)} \wedge \beta^2}_{\Leftrightarrow \operatorname{grad}(f) \cdot \mathbf{B}} + f \wedge \underbrace{\mathrm{d}\beta^2}_{\Leftrightarrow \operatorname{div}(\mathbf{B})} \Leftrightarrow f \operatorname{div}(\mathbf{B}) + \langle \operatorname{grad}(f), \mathbf{B} \rangle$$

**2.10(5) Basis expansion of the cross product**

$$\begin{aligned} \mathbf{v} \times \mathbf{B} &\Leftrightarrow -i_{\mathbf{v}} \beta^2 \\ &= -i_{\mathbf{v}} i_{\mathbf{B}} \operatorname{vol}^3 \\ &= -v^k B^l i_{\boldsymbol{\partial}_k} i_{\boldsymbol{\partial}_l} \operatorname{vol}^3 \\ &= -v^k B^l \operatorname{vol}^3 (\boldsymbol{\partial}_l, \boldsymbol{\partial}_k, \boldsymbol{\partial}_m) \mathrm{d}x^m \\ &= \sqrt{g}\, v^k B^l \varepsilon_{klm} \mathrm{d}x^m \end{aligned}$$

If you're wondering how the "identification stuff" works, read the chapter about the Hodge star operator, it's around page 360. I have no idea why Frankel placed it that late. You might also be interested in the definition of the cross product in 3.1(3)(i).

# Integration of Differential Forms

*one does not integrate vectors; one integrates forms.*
If there is extra structure available, for example, a Riemannian metric, then it is possible to rephrase an integration, say of exterior 1-forms or 2-forms, in terms of a vector interations involving "arc lengths" or "surface areas," but we shall see that even in this case we are *complicating* a basically simple situation.
*If a line integral of a vector occurs in a problem, then usually a deeper look at the situation will show that the vector in question was in fact a covector, that is, a 1-form*!
For example, the strength of the electric field can be determined by the work done in moving a unit charge very slowly along a small path, that is, by a line integral. The electric field strength is a 1-form.
-Theodore Frankel.

### 3.1 Integration over a Parameterized Subset

How does one integrate the Poincaré 2-form $\omega$ over a surface in phase space? -Theodore Frankel.

#### 3.1a. Integration of a $p$-Form in $\mathbb{R}^p$

define integral of a $p$-form over region $(U, o) \subset \mathbb{R}^p$,
orientation $o$; $o(u) = \pm 1$

$$\int_{(U,o)} \alpha = \int a(u) du^1 \wedge \cdots \wedge du^p \equiv = \int_U o(u) a(u) du^1 \dots du^p \qquad (3.1) \qquad (2)$$

$(e_1 \dots e_p) = \left( \frac{\partial}{\partial u^1} \dots \frac{\partial}{\partial u^p} \right)$ has same orientation as $o(u)$

### 3.1b. Integration over Parametrized Subsets

oriented parameterized $p$-subset of manifold $M$ to be pair $(U, o; F)$,
oriented region $(U, o)$ in $\mathbb{R}^p$ and diff. $F : U \to M$

define

$$\int_{(U,o;F)} \alpha^p = \int_{(U,o)} F^* \alpha^p \qquad (3.3) \tag{3}$$

*we make no requirements on the rank of $DF$.*

$$\int_{(U,o;F)} \alpha^p \equiv \int_{(U,o)} (F^*\alpha^p) \left[\frac{\partial}{\partial u^1} \dots \frac{\partial}{\partial u^p}\right] du^1 \wedge \dots \wedge du^p = o(u) \int_U (F^*\alpha^p) \left[\frac{\partial}{\partial u^1} \dots \frac{\partial}{\partial u^p}\right] du^1 \dots du^p \qquad (3.4) \tag{4}$$

$$= o(u) \int_U \alpha^p \left[F_* \frac{\partial}{\partial u^1} \dots F_* \frac{\partial}{\partial u^p}\right] du^1 \dots du^p \qquad (3.5) \tag{5}$$

$$\int_C \alpha^1 = \int_C a_i dx^i = \int_a^b F^*[a_i dx^i] = \int_a^b a_j \frac{dx^j}{dt} dt \qquad (3.6) \tag{6}$$

### 3.1(3)(i) Higher-dimensional cross product

$$\mathbf{A}_i \cdot (\mathbf{A}_1 \times \dots \times \mathbf{A}_i \times \dots \times \mathbf{A}_{n-1}) := \mathrm{vol}^n(\mathbf{A}_i, \mathbf{A}_1, \dots, \mathbf{A}_i, \dots \mathbf{A}_{n-1}) = 0$$

## 3.3 Stokes' Theorem

### 3.3(1) ... in $\mathbb{R}^3$

- $p = 2$

$$\begin{aligned} \omega^1 &= w_1 \mathrm{d}x^1 + w_2 \mathrm{d}x^2 + w_3 \mathrm{d}x^3 \\ \Rightarrow \mathrm{d}\omega^1 &= \left(\frac{\partial w_3}{\partial x^2} + \frac{\partial w_2}{\partial x^3}\right) \mathrm{d}x^2 \wedge \mathrm{d}x^3 + \left(\frac{\partial w_3}{\partial x^1} + \frac{\partial w_1}{\partial x^3}\right) \mathrm{d}x^1 \wedge \mathrm{d}x^3 \\ &\quad + \left(\frac{\partial w_2}{\partial x^1} + \frac{\partial w_1}{\partial x^2}\right) \mathrm{d}x^1 \wedge \mathrm{d}x^2 \end{aligned}$$

This corresponds to the classical Stokes' Theorem

$$\int_A \mathrm{rot}\,(\mathbf{W})\ \mathrm{d}\mathbf{A} = \int_{\partial A} \mathbf{W} \mathrm{d}\mathbf{s}$$

- $p = 3$

$$\begin{aligned} \omega^2 &= w_{12} \mathrm{d}x^1 \wedge \mathrm{d}x^2 + w_{13} \mathrm{d}x^1 \wedge \mathrm{d}x^3 + w_{23} \mathrm{d}x^2 \wedge \mathrm{d}x^3 \\ \Rightarrow \mathrm{d}\omega^2 &= \left(\frac{\partial w_{23}}{\partial x^1} + \frac{\partial w_{31}}{\partial x^2} + \frac{\partial w_{12}}{\partial x^3}\right) \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \end{aligned}$$

This corresponds to Gauß's Law

$$\int_V \mathrm{div}\,(\mathbf{W})\ \mathrm{d}V = \int_{\partial V} \mathbf{W} \mathrm{d}\mathbf{A}$$

### 3.3(2) ... in $\mathbb{R}^4$

- $p = 2$

$$\begin{aligned} \omega^1 &= w_1 \mathrm{d}x^1 + w_2 \mathrm{d}x^2 + w_3 \mathrm{d}x^3 + w_4 \mathrm{d}x^4 \\ \Rightarrow \mathrm{d}\omega^1 &= \left(\frac{\partial w_1}{\partial x^2} - \frac{\partial w_2}{\partial x^1}\right) \mathrm{d}x^1 \wedge \mathrm{d}x^2 + \left(\frac{\partial w_1}{\partial x^3} - \frac{\partial w_3}{\partial x^1}\right) \mathrm{d}x^1 \wedge \mathrm{d}x^3 \\ &\quad + \left(\frac{\partial w_1}{\partial x^4} - \frac{\partial w_4}{\partial x^1}\right) \mathrm{d}x^1 \wedge \mathrm{d}x^4 + \left(\frac{\partial w_2}{\partial x^3} - \frac{\partial w_3}{\partial x^2}\right) \mathrm{d}x^2 \wedge \mathrm{d}x^3 \\ &\quad + \left(\frac{\partial w_2}{\partial x^4} - \frac{\partial w_4}{\partial x^2}\right) \mathrm{d}x^2 \wedge \mathrm{d}x^4 + \left(\frac{\partial w_3}{\partial x^4} - \frac{\partial w_4}{\partial x^3}\right) \mathrm{d}x^3 \wedge \mathrm{d}x^4 \end{aligned}$$

It could be said to be some analogon to the classical Stokes' Theorem in $\mathbb{R}^4$

$$\int_A \mathrm{curl}\,(\mathbf{W})\ \mathrm{d}\mathbf{A} = \int_{\partial A} \mathbf{W}\mathrm{d}\mathbf{s}$$

- $p = 3$

$$\begin{aligned}
\omega^2 &= w_{12}\mathrm{d}x^1 \wedge \mathrm{d}x^2 + w_{13}\mathrm{d}x^1 \wedge \mathrm{d}x^3 + w_{14}\mathrm{d}x^1 \wedge \mathrm{d}x^4 \\
&\quad + w_{23}\mathrm{d}x^2 \wedge \mathrm{d}x^3 + w_{24}\mathrm{d}x^2 \wedge \mathrm{d}x^4 + w_{34}\mathrm{d}x^3 \wedge \mathrm{d}x^4 \\
\Rightarrow \mathrm{d}\omega^2 &= \left(\frac{\partial w_{12}}{\partial x^3} - \frac{\partial w_{13}}{\partial x^2} + \frac{\partial w_{23}}{\partial x^1}\right)\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \\
&\quad + \left(\frac{\partial w_{12}}{\partial x^4} - \frac{\partial w_{14}}{\partial x^2} + \frac{\partial w_{24}}{\partial x^1}\right)\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^4 \\
&\quad + \left(\frac{\partial w_{13}}{\partial x^4} - \frac{\partial w_{14}}{\partial x^3} + \frac{\partial w_{34}}{\partial x^1}\right)\mathrm{d}x^1 \wedge \mathrm{d}x^3 \wedge \mathrm{d}x^4 \\
&\quad + \left(\frac{\partial w_{23}}{\partial x^4} - \frac{\partial w_{24}}{\partial x^3} + \frac{\partial w_{34}}{\partial x^2}\right)\mathrm{d}x^2 \wedge \mathrm{d}x^3 \wedge \mathrm{d}x^4
\end{aligned}$$

The classical analogon is of obviously

$$\int_V \mathrm{wtf}(\mathbf{W})\,\mathrm{d}\mathbf{V} = \int_{\partial V} \mathbf{W}\mathrm{d}\mathbf{A}$$

- $p = 4$
  $\omega^3$ and $\mathrm{d}\omega^3$ have already been calculated in 2.6(1). Using these forms, one gets a 4-dimensional analogon to Gauß's Theorem

$$\int_H \mathrm{div}\,(\mathbf{W})\ \mathrm{d}H = \int_{\partial H} \mathbf{W}\mathrm{d}V$$

# The Lie derivative

## 4.1 The Lie Derivative of a Vector Field

### 4.1a. The Lie Bracket

$X, Y$ - vector fields on $M$

$\phi(t) = \phi_t$ be local flow generated by field $X$

$\phi_t x$ - pt. $t$ seconds along integral curve at $X$, the "orbit" of $x$, starts at time 0 at pt. $x$.

Compare $Y_{\phi_t x}$ at that pt. with results of pushing $Y_x$ to pt. $\phi_t x$ by differential $\phi_{t*}$

Figure 4.1.

Lie derivative of $Y$ with respect to $X$.

$$\begin{aligned}
[\mathcal{L}_X Y]_x &\equiv \lim_{t\to 0} \frac{[Y_{\phi_t x} - \phi_{t*}Y_x]}{t} = \qquad (4.1) \qquad (7)\\
&= \lim_{t\to 0} \phi_{t*}\frac{[\phi_{-t*}Y_{\phi_t x} - Y_x]}{t} = \lim_{t\to 0} \frac{[\phi_{-t*}Y_{\phi_t x} - Y_x]}{t} \qquad (4.2) \qquad (8)
\end{aligned}$$

Hadamard's Lemma. (4.3)

Let $f$ be cont. diff. in neighborhood $U$ of $x_0$

Then for sufficiently small $t$, $\exists\, g = g(t, x) = g_t(x)$ cont. diff. in $t$, pt. $x \in U$ s.t.

$$\begin{aligned}
g_0(x) &= X_x(f) \\
f(\phi_t x) &= f(x) + t g_t(x)
\end{aligned}$$

i.e.

$$f \circ \phi_t = f + t g_t$$

If we accept this for the moment, we many proceed with $\exists$ of limit.

At $x$

$$(f) = \lim_{t\varnothing 0} \frac{[Y_{\phi_t x} - \phi_{t*} Y_x]}{t}(f) \overset{(2.60)}{=} \lim_{t\to 0} \frac{Y_{\phi_t x}(f) - Y_x(f\circ\phi_t)}{t} = \lim_{t\to 0} \frac{Y_{\phi_t x}(f) - Y_x(f + tg_t)}{t} =$$

$$= \lim_{t\to 0} \frac{[Y_{\phi_t x}(f) - Y_x(f)]}{t} - \lim_{t\to 0} Y_x(g_t) \overset{\text{tangent vector def. on integral curve}}{=} X_x[Y(f)] - Y_x(g_0) = X_x\{Y(f)\} - Y_x\{X(f)\}$$

remark (4.2)

$$\mathcal{L}_X Y_x = \{\frac{d}{dt}(\phi_{-t})_* Y_{\phi_t x}\}_{t=0} \qquad (4.7) \tag{9}$$

Proof of Hadamard's Lemma: Define $F(t,x) = (f\circ\phi_t)(x)$
fix $t, x$, put $\mathcal{F}(s) = F(st, x)$
Then $(f\circ\phi_t)(x) - f(x) = \mathcal{F}(1) - \mathcal{F}(0) = \int_0^1 \mathcal{F}'(s)ds = \int_0^1 \frac{d}{ds}F(st,x)ds = \int_0^1 tF_1(st,x)ds$
$F_1$ denotes derivative with respect to 1st. variable.
Thus, define $g_t(x) \equiv \int_0^1 F_1(st,x)ds$
then $(f\circ\phi_t)(x) - f(x) = tg_t(x)$

### 4.1b. Jacobi's Variational Equation

Use the fact that $X^j = \frac{dx^j}{dt}$ along the orbit

$$[\mathcal{L}_X Y]^i = \frac{dY^i}{dt} - \sum_j \left(\frac{\partial X^i}{\partial x^j}\right) Y^j \tag{10}$$

$$\mathcal{L}_V W = [V,W] = \left(V^i \frac{\partial W^i}{\partial x^i} - W^i \frac{\partial V^j}{\partial x^i}\right)\frac{\partial}{\partial x^j}$$

$W^j = W^j(\theta(t))$
If $\dot\theta^i = V^i$

$$V^i \frac{\partial W^j}{\partial x^i} = \frac{dW^j}{dt}$$

$$(\mathcal{L}_V W)^j = \frac{dW^j}{dt} - W^i \frac{\partial V^j}{\partial x^i} \qquad (4.8) \tag{11}$$

$$(\mathcal{L}_V W)_p = \left.\frac{d}{dt}\right|_{t=0} d(\theta_{-t})_{\theta_t(x)}(W_{\theta_t(x)})$$

$$d(\theta_{-t})_{\theta_t(x)}(W_{\theta_t(x)}) = \frac{\partial\theta^i}{\partial x^j}(-t, \theta(t,x))W^j(\theta(t,x))\left.\frac{\partial}{\partial x^i}\right|_x$$

For

$$W = \frac{\partial}{\partial x} = \begin{bmatrix}1\\0\end{bmatrix}$$

$$\theta = \begin{bmatrix}x(t)\\y(t)\end{bmatrix} = \begin{bmatrix}x\cos(t) - y\sin(t)\\ x\sin(t) + y\cos(t)\end{bmatrix}$$

so that

$$\frac{\partial\theta^i}{\partial x^j}(-t,\theta(t,x))W^j(\theta(t,x)) = \begin{bmatrix}\cos((-t)) & -\sin((-t))\\ \sin((-t)) & \cos((-t))\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}\cos(t)\\ -\sin(t)\end{bmatrix} \xrightarrow{\frac{d}{dt}} \begin{bmatrix}-\sin(t)\\ -\cos(t)\end{bmatrix} \xrightarrow{t=0} \begin{bmatrix}0\\-1\end{bmatrix}$$

$$\Longrightarrow (\mathcal{L}_V W)_p = -\frac{\partial}{\partial y} = \mathcal{L}_V \frac{\partial}{\partial x}$$

### 4.1(1) Coordinate expression for $[\mathbf{X}, \mathbf{Y}]$

$$\begin{aligned}[\mathbf{X},\mathbf{Y}] = \mathbf{X}(\mathbf{Y}) - \mathbf{Y}(\mathbf{X}) &= X^i \frac{\boldsymbol\partial}{\boldsymbol\partial u^i}\left(Y^j \frac{\boldsymbol\partial}{\boldsymbol\partial u^j}\right) - Y^j \frac{\boldsymbol\partial}{\boldsymbol\partial u^j}\left(X^i \frac{\boldsymbol\partial}{\boldsymbol\partial u^i}\right)\\
&= X^i \frac{\partial Y^j}{\partial u^i}\frac{\boldsymbol\partial}{\boldsymbol\partial u^i} + \cancel{X^i Y^j \frac{\partial^2}{\partial u^i \partial u^j}} - Y^j \frac{\partial X^i}{\partial u^j}\frac{\boldsymbol\partial}{\boldsymbol\partial u^i} - \cancel{Y^i X^j \frac{\partial^2}{\partial u^j \partial u^j}}\\
&= \left(X^i \frac{\partial Y^j}{\partial u^i} - Y^i \frac{\partial X^j}{\partial u^i}\right)\frac{\boldsymbol\partial}{\boldsymbol\partial u^j}\end{aligned}$$

### 4.2 The Lie Derivative of a Form

If a flow deforms some attribute, say volume, how does one measure the deformation? -Theodore Frankel

#### 4.2a. Lie Derivatives of Forms

#### 4.2b. Formulas Involving the Lie Derivative

**Theorem 7 (4.24)**

$$\mathcal{L}_X i_Y - i_Y \mathcal{L}_X = i_{[X,Y]} \tag{12}$$

**Theorem 8 (4.25)**

$$d\alpha(X,Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X,Y]) \tag{13}$$

**Proof**:

$$\begin{aligned} d\alpha(X,Y) = (i_X d\alpha)(Y) = (\mathcal{L}_X\alpha - di_X\alpha)(Y) = i_Y\mathcal{L}_X\alpha - Y(\alpha(X)) = \mathcal{L}_X i_Y\alpha - i_{[X,Y]}\alpha - Y(\alpha(X)) = \\ = \mathcal{L}_X\alpha(Y) - \alpha([X,Y]) - Y(\alpha(X)) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X,Y]) \end{aligned}$$

Note that

$$\begin{aligned} &di_X\alpha = d(\alpha(X)) \\ &d(\alpha(X))(Y) = \frac{\partial(\alpha(X))}{\partial x^i} Y^i = Y(\alpha(X)) \end{aligned}$$

## Done.

#### 4.2(1) Coordinate expression for $\mathcal{L}_{\mathbf{X}}\alpha^1$

$$\begin{aligned} \mathcal{L}_{\mathbf{X}}\alpha^1 &= i_{\mathbf{X}}\mathrm{d}\alpha + \mathrm{d}i_{\mathbf{X}}\alpha = i_{\mathbf{X}}\mathrm{d}\alpha_i\mathrm{d}u^i + \mathrm{d}i_{\mathbf{X}}\alpha_i\mathrm{d}u^i = i_{\mathbf{X}}\mathrm{d}\alpha_i \wedge \mathrm{d}u^i + \mathrm{d}\alpha_i\mathrm{d}u^i(\mathbf{X}) \\ &= i_{\mathbf{X}}\frac{\partial\alpha_i}{\partial u^j}\mathrm{d}u^j \wedge \mathrm{d}u^i + \frac{\partial\alpha_i}{\partial u^j}\mathrm{d}u^j X^i + \alpha_i\frac{\partial X^i}{\partial u^j}\mathrm{d}u^j \\ &= \frac{\partial\alpha_i}{\partial u^j}\underbrace{\left(i_{\mathbf{X}}\mathrm{d}u^j\right)}_{=X^j}\mathrm{d}u^i - \frac{\partial\alpha_i}{\partial u^j}\mathrm{d}u^j\underbrace{\left(i_{\mathbf{X}}\mathrm{d}u^i\right)}_{=X^i} + \frac{\partial\alpha_i}{\partial u^j}\mathrm{d}u^j X^i + \alpha_i\frac{\partial X^i}{\partial u^j}\mathrm{d}u^j \\ &= \frac{\partial\alpha_i}{\partial u^j}X^j\mathrm{d}u^i \underbrace{- \frac{\partial\alpha_j}{\partial u^i}X^j\mathrm{d}u^i + \frac{\partial\alpha_j}{\partial u^i}X^j\mathrm{d}u^i}_{=0} + \alpha_j\frac{\partial X^j}{\partial u^i}\mathrm{d}u^i \\ &= \left(X^j\frac{\partial\alpha_i}{\partial u^j} + \frac{\partial X^j}{\partial u^i}\alpha_j\right)\mathrm{d}u^i \end{aligned}$$

#### 4.2(2) Compositions of derivations and antiderivations

$$\begin{aligned} (\theta A - A\theta)(\alpha^p \wedge \beta^q) &= \theta(A\alpha \wedge \beta + (-1)^p\alpha \wedge A\beta) - A(\theta\alpha \wedge \beta + \alpha \wedge \theta\beta) \\ &= \theta A\alpha \wedge \beta + A\alpha \wedge \theta\beta + (-1)^p\theta\alpha \wedge A\beta + (-1)^p\alpha \wedge \theta A\beta \\ &\quad - A\theta\alpha \wedge \beta - (-1)^{\deg(\theta\alpha)}\theta\alpha \wedge A\beta - A\alpha \wedge \theta\beta - (-1)^p\alpha \wedge A\theta\beta \\ &\quad \text{(Notice that } \textit{derivations} \text{ alter their argument's degree by} \\ &\quad \text{an } \textit{even} \text{ number; thus the 2nd and 7th, and the 3rd and 6th} \\ &\quad \text{summand cancel each other out)} \\ &= (\theta A - A\theta)\alpha \wedge \beta + (-1)^p\alpha \wedge (\theta A - A\theta)\beta \end{aligned}$$

$$
\begin{aligned}
(AB-BA)(\alpha^p\wedge\beta^q) &= A(B\alpha\wedge\beta+(-1)^p\alpha\wedge B\beta)-B(A\alpha\wedge\beta+(-1)^p\alpha\wedge A\beta)\\
&= AB\alpha\wedge\beta+(-1)^{\deg(B\alpha)}B\alpha\wedge A\beta+(-1)^pA\alpha\wedge B\beta\\
&\quad+\underbrace{(-1)^p(-1)^p}_{=1}\alpha\wedge AB\beta+BA\alpha\wedge\beta+(-1)^{\deg(A\alpha)}A\alpha\wedge B\beta\\
&\quad+(-1)^pB\alpha\wedge A\beta+\underbrace{(-1)^p(-1)^p}_{=1}\alpha\wedge BA\beta\\
&\quad\text{(Again, the 2nd and 7th, 2nd and 6th summands cancel out}\\
&\quad\text{as an \textit{antiderivation} alters its argument's degree by an}\\
&\quad\text{\textit{uneven} number)}\\
&= (AB+BA)\alpha\wedge\beta+\alpha\wedge(AB+BA)\beta
\end{aligned}
$$

**4.2(3)** $i_{[\mathbf{X},\mathbf{Y}]}=\mathcal{L}_\mathbf{X}\circ i_\mathbf{Y}-i_\mathbf{Y}\circ\mathcal{L}_\mathbf{X}$

As stated in the corresponding chapter, it's enough to verify the formula for functions and differentials of functions.

Functions:

$$
\begin{aligned}
i_{[\mathbf{X},\mathbf{Y}]}f&=0\\
\mathcal{L}_\mathbf{X}\underbrace{i_\mathbf{Y}f}_{=0}-\underbrace{i_\mathbf{Y}\mathcal{L}_\mathbf{X}f}_{=0}&=0
\end{aligned}
$$

Differentials:

$$
\begin{aligned}
i_{[\mathbf{X},\mathbf{Y}]}\mathrm{d}f&=\mathrm{d}f([\mathbf{X},\mathbf{Y}])\\
&=[\mathbf{X},\mathbf{Y}](f)\\
\mathcal{L}_\mathbf{X}i_\mathbf{Y}\mathrm{d}f-i_\mathbf{Y}\mathcal{L}_\mathbf{X}\mathrm{d}f&=i_\mathbf{X}\mathrm{d}i_\mathbf{Y}\mathrm{d}f+\mathrm{d}\underbrace{i_\mathbf{X}i_\mathbf{Y}\mathrm{d}f}_{=0}-i_\mathbf{Y}i_\mathbf{X}\underbrace{\mathrm{dd}}_{=0}f-i_\mathbf{Y}\mathrm{d}i_\mathbf{X}\mathrm{d}f\\
&=i_\mathbf{X}\mathrm{d}\mathbf{Y}(f)-i_\mathbf{Y}\mathrm{d}\mathbf{X}(f)=\mathbf{X}(\mathbf{Y}(f))-\mathbf{Y}(\mathbf{X}(f))\\
&=[\mathbf{X},\mathbf{Y}](f)
\end{aligned}
$$

**4.2(3) Fugly proof of** $\mathrm{d}\alpha(\mathbf{X},\mathbf{Y})=\mathbf{X}(\alpha(\mathbf{Y}))-\mathbf{Y}(\alpha(\mathbf{X}))-\alpha([\mathbf{X},\mathbf{Y}])$

Step 1: Calculate single terms.

$$
\begin{aligned}
\mathrm{d}\alpha(\mathbf{X},\mathbf{Y})&=X^kY^l\frac{\partial\alpha_i}{\partial u^j}\mathrm{d}u^j\wedge\mathrm{d}u^i\left(\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}u^k},\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}u^l}\right)=X^kY^l\frac{\partial\alpha_i}{\partial u^j}\left(\delta^j_k\delta^i_l-\delta^j_l\delta^i_k\right)\\
&=X^jY^i\frac{\partial\alpha_i}{\partial u^j}-X^iY^j\frac{\partial\alpha_i}{\partial u^j}\\
\mathbf{X}(\alpha(\mathbf{Y}))&=X^i\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}u^i}\left(\alpha_jY^j\right)=X^iY^j\frac{\partial\alpha_j}{\partial u^i}+\alpha_jX^i\frac{\partial Y^j}{\partial u^i}\\
\mathbf{Y}(\alpha(\mathbf{X}))&=Y^i\frac{\boldsymbol{\partial}}{\boldsymbol{\partial}u^i}\left(\alpha_jX^j\right)=X^jY^i\frac{\partial\alpha_j}{\partial u^i}+\alpha_jY^i\frac{\partial X^j}{\partial u^i}\\
\alpha([\mathbf{X},\mathbf{Y}])&=\alpha([\mathbf{X},\mathbf{Y}]^i\boldsymbol{\partial}_i)=\alpha\left(\left(\frac{\partial Y^i}{\partial u^j}X^j-\frac{\partial X^i}{\partial u^j}Y^j\right)\boldsymbol{\partial}_i\right)=\alpha_iX^j\frac{\partial Y^i}{\partial u^j}-\alpha_iY^j\frac{\partial X^i}{\partial u^j}
\end{aligned}
$$

Step 2: Smash them together.

$$
\begin{aligned}
&\mathbf{X}(\alpha(\mathbf{Y}))-\mathbf{Y}(\alpha(\mathbf{X}))-\alpha([\mathbf{X},\mathbf{Y}])\\
&=X^iY^j\frac{\partial\alpha_j}{\partial u^i}+\alpha_jX^i\frac{\partial Y^j}{\partial u^i}-X^jY^i\frac{\partial\alpha_j}{\partial u^i}-\alpha_jY^i\frac{\partial X^j}{\partial u^i}-\alpha_iX^j\frac{\partial Y^i}{\partial u^j}+\alpha_iY^j\frac{\partial X^i}{\partial u^j}\\
&=\underbrace{X^iY^j\frac{\partial\alpha_j}{\partial u^i}-X^jY^i\frac{\partial\alpha_j}{\partial u^i}}_{=\mathrm{d}\alpha(\mathbf{X},\mathbf{Y})}\underbrace{+\alpha_jX^i\frac{\partial Y^j}{\partial u^i}-\alpha_iX^j\frac{\partial Y^i}{\partial u^j}}_{=0}\underbrace{+\alpha_iY^j\frac{\partial X^i}{\partial u^j}-\alpha_jY^i\frac{\partial X^j}{\partial u^i}}_{=0}\\
&=\mathrm{d}\alpha(\mathbf{X},\mathbf{Y})
\end{aligned}
$$

### 4.3. Differentiation of Integrals

How does one compute the rate of change of an integral when the domain of integration is also changing?

### 4.3a. The Autonomous (Time-Independent) Case

Let $\alpha$ $p$-form

$V$ oriented, compact submanifold of $M$, $\text{dim}V = p$

flow $\phi_t : M \to M$, i.e. 1-parameter "group" of diffeomorphisms $\phi_t$
defined $V(t) \equiv \phi_t V$
Fig. 4.5. EY : 20141031 I don't get this
Let $X = \frac{d}{dt}\phi_t(x)\big|_{t=0}$. $X = \dot{\phi}_t(x)\big|_{t=0}$

$$I(t) = \int_{V(t)} \alpha = \int_V \phi_t^* \alpha$$

$$I'(t) = \lim_{h\to 0} \frac{[I(t+h) - I(t)]}{h} = \lim_{h\to 0} \frac{[\int_V \phi_{t+h}^*\alpha - \int_V \phi_t^*\alpha]}{h} = \lim_{h\to 0}\left[ \int_V \frac{\phi_t^*\{\phi_h^*\alpha - \alpha\}}{h}\right] =$$
$$= \lim_{h\to 0}\left[ \int_{V(t)} \frac{\{\phi_h^*\alpha - \alpha\}}{h}\right] = \int_{V(t)} \lim_{h\to 0} \frac{\{\phi_h^*\alpha - \alpha\}}{h}$$

EY: 20141031 I don't understand the steps in between the lines, equality 4; what happened to the $\phi_t^*$ out in front?

### 4.3b. Time-Dependent Fields

*A time-dependen vector field on a manifold $M$ does not generate a flow!*

$\forall$ time-dependent tensor field $A(t,x)$ on $M$, should be considered a tensor field on product manifold $\mathbb{R}\times M$
$\mathbb{R}\times M$ has local coordinates $(t = x^0, x^1 \dots x^n)$
solve the system of ODE

$$\begin{aligned} &\frac{dx^i}{ds} = v^i(t,x) \qquad x^i(s=0) = x_0^i, \quad i = 1\dots n \\ &\frac{dt}{ds} = 1 \qquad t(s=0) = t_0 \end{aligned} \tag{4.39}$$

get a flow $\phi_s : \mathbb{R}\times M \to \mathbb{R}\times M$

### 4.3c. Differentiating Integrals

Let $\phi_t : M \to M$ 1-parameter family of diffeomorphisms of $M$
don't assume they form a flow, assume $\phi_0 = 1$ and $(t,x) \to \phi_t x$ smooth as a function of $(t,x) \in \mathbb{R}\times M$
Let $\omega_t(x) = \omega(t,x) \in \Omega^p(M)$ be 1 parameter family of forms on $M$
Let $V \subseteq M$ submanifold, $\text{dim}V = p$

### Problems

**4.3(1)** $A, B$ time dependent vector fields on $\mathbb{R}^3$
$\rho(t,\mathbf{x})$ function
Using

$$\frac{d}{dt}\int_{V(t)} \alpha = \frac{d}{dt}\int_{W(t)} \alpha = \int_{W(t)} \mathcal{L}_X \alpha = \int_{W(t)} \mathcal{L}_{v + \frac{\partial}{\partial t}} \alpha$$

if $p = 1$,

$$\frac{d}{dt}\int_{V(t)} \alpha = \frac{d}{dt}\int_{W(t)} \alpha = \int_{W(t)} \mathcal{L}_{v+\frac{\partial}{\partial t}} \alpha = \int_{W(t)} \mathcal{L}_v \alpha + \frac{\partial \alpha}{\partial t} = \int_{W(t)} \frac{\partial \alpha}{\partial t} + i_v \mathbf{d}\alpha + \mathbf{d} i_v \alpha$$

### Additional Problems on Fluid Flow

**4.3(5)**

(i)

(ii) For **circulation** $\oint_{C(t)} u^\flat$,

$$\frac{d}{dt}\oint_{C(t)} u^\flat = \int_{C(t)} \frac{\partial u^\flat}{\partial t} + \mathcal{L}_u u^\flat = \int_{C(t)} d\left(\frac{1}{2}\|u\|^2 - \phi - \int \frac{dp}{\rho}\right) = 0$$

since $C(t)$ is a closed curve. Then the circulation is constant in time.

(iii) **vorticity** $\omega \in \Omega^2(M)$

$\omega := du^\flat$

For some compact submanifold $S \subset M$, $\dim S = 2$,

$$\frac{d}{dt}\int_S \omega = \int_S \mathcal{L}_{\frac{\partial}{\partial t}+u}\omega = \int_S \frac{\partial \omega}{\partial t} + \mathcal{L}_u\omega = \int_S \frac{\partial \omega}{\partial t} + di_u\omega + i_u d\omega = \int_S \frac{\partial du^\flat}{\partial t} + di_u\omega = \int_S d\left(\frac{\partial u^\flat}{\partial t} + i_u\omega\right) =$$
$$= \int_{\partial S}(\frac{\partial u^\flat}{\partial t} + i_u\omega) = \int_{\partial S}(\frac{\partial u^\flat}{\partial t} + \mathcal{L}_u u^\flat - di_u u^\flat) = 0 - \int_{\partial S} du^2 = 0$$

since $\partial S$ is a closed curve.

(iv)

## 4.4 A problem set on Hamiltonian mechanics

### 4.4(1) Symplectic form

$\omega$ is obviously closed as $\omega = \mathrm{d}\lambda$. In order to show non-degeneracy, let

$$\mathbf{X} = Q^i\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} q^i} + P_i\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} p_i}$$
$$\mathbf{X} \neq 0$$

Then

$$i_\mathbf{X}\omega = i_\mathbf{X}\mathrm{d}p_i \wedge \mathrm{d}q^i = (i_\mathbf{X}\mathrm{d}p_i)\,\mathrm{d}q^i - \mathrm{d}p_i\left(i_\mathbf{X}\mathrm{d}q^i\right) = P_i\mathrm{d}q^i - Q^i\mathrm{d}p_i \neq 0$$

i.e. there is no $\mathbf{Y} \neq 0$ so that $i_\mathbf{Y} i_\mathbf{X}\omega = \omega(\mathbf{X},\mathbf{Y}) = 0$ for all $\mathbf{X} \neq 0$, so $\omega$ is a non-degenerate bilinearform.

### 4.4(1) Symplectic volume form

$$\omega^n := \bigwedge_{k=1}^{n}\omega = \bigwedge_{k=1}^{n}\mathrm{d}p_{i_k}\wedge\mathrm{d}q^{i_k} = \mathrm{d}p_{i_1}\wedge\mathrm{d}q^{i_1}\wedge\mathrm{d}p_{i_2}\wedge\mathrm{d}q^{i_2}\wedge\ldots\wedge\mathrm{d}p_{i_n}\wedge\mathrm{d}q^{i_n}$$

All summands with equal indices vanish, only distinct $i_k$ indices yield a term, thus there are $(n-k+1)$ choices for $i_k$. Combine them all to get a total of

$$\prod_{k=1}^{n}(n-k+1) = (n-1+1)(n-2+1)\cdots(n-n+1) = n(n-1)\cdots 1 = n!$$

So $n!$ choices exist. Next, rearrange the “wedge factors” so the indices are in ascending order, yielding a factor of $\pm 1$. Now

$$\omega^n = \pm n!\,\mathrm{d}p_1\wedge\mathrm{d}q^1\wedge\mathrm{d}p_2\wedge\mathrm{d}q^2\wedge\ldots\wedge\mathrm{d}p_n\wedge\mathrm{d}q^n$$

**P. 147: Derivation of Hamilton's equations** The paragraph below (4.49) says “comparing these two expressions” and doesn't explain it any further. This is what's happening.
Let

$$\mathcal{H} = \mathcal{H}(q,p,t) = p_i\dot{q}^i - L(q,\dot{q},t)$$

Then

$$\mathrm{d}\mathcal{H} = \mathrm{d}\mathcal{H}(q,p,t) = \frac{\partial\mathcal{H}}{\partial q^i}\mathrm{d}q^i + \frac{\partial\mathcal{H}}{\partial p_i}\mathrm{d}p_i + \frac{\partial\mathcal{H}}{\partial t}\mathrm{d}t$$

but also

$$\mathrm{d}\mathcal{H} = \mathrm{d}(p_i\dot{q}^i - L(q,\dot{q},t)) = \mathrm{d}p_i\dot{q}^i + \cancel{p_i\mathrm{d}\dot{q}^i} - \underbrace{\frac{\partial L}{\partial q^i}}_{=\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^i}=\dot{p}_i}\mathrm{d}q^i - \underbrace{\cancel{\frac{\partial L}{\partial \dot{q}^i}\mathrm{d}\dot{q}^i}}_{=-p_i\mathrm{d}\dot{q}^i} - \frac{\partial L}{\partial t}\mathrm{d}t$$

$$= -\dot{p}_i\mathrm{d}q^i + \dot{q}^i\mathrm{d}p_i - \frac{\partial L}{\partial t}\mathrm{d}t$$

Comparing these two results for $\mathrm{d}\mathcal{H}$ yields Hamilton's equations

$$\dot{q}^i = \frac{\partial \mathcal{H}}{\partial p_i} \qquad \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q^i} \qquad \frac{\partial L}{\partial t} = -\frac{\partial \mathcal{H}}{\partial t}$$

### 4.4(4) Hamilton in shrt

$$\begin{aligned}
i_{\mathbf{X}}\omega &= \mathrm{d}p_i \wedge \mathrm{d}q^i \left(X^j \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} q^j} + X^{n+j}\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} p_j}\right) \\
&= \mathrm{d}p_i \wedge \mathrm{d}q^i \left(X^j \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} q^j}\right) + \mathrm{d}p_i \wedge \mathrm{d}q^i \left(X^{n+j}\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} p_j}\right) \\
&= \underbrace{\mathrm{d}p_i\left(X^j \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} q_j}\right)}_{=0}\mathrm{d}q^i - \underbrace{\mathrm{d}q^i\left(X^j \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} q^j}\right)}_{=X^i}\mathrm{d}p_i + \underbrace{\mathrm{d}p_i\left(X^{n+j} \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} p_j}\right)}_{=X^{n+i}}\mathrm{d}q^i - \underbrace{\mathrm{d}q^i\left(X^{n+j} \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} p_j}\right)}_{=0}\mathrm{d}p_i \\
&= -X^i\mathrm{d}p_i + \sum_i X^{n+i}\mathrm{d}q^i = -\frac{\mathrm{d}q^i}{\mathrm{d}t}\mathrm{d}p_i + \frac{\mathrm{d}p_i}{\mathrm{d}t}\mathrm{d}q^i = -\frac{\partial \mathcal{H}}{\partial p_i}\mathrm{d}q^i - \frac{\partial \mathcal{H}}{\partial q^i}\mathrm{d}p_i = -\mathrm{d}\mathcal{H}(q,p)
\end{aligned}$$

### 4.4(5) Lie derivative of the symplectic Poincaré 2-form

$$\mathcal{L}_{\mathbf{X}}\omega = i_{\mathbf{X}}\mathrm{d}\omega + \mathrm{d}i_{\mathbf{X}}\omega = i_{\mathbf{X}}\mathrm{d}^2\lambda - \mathrm{d}^2\mathcal{H} = 0$$

Since $\mathcal{L}$ is a derivation on the exterior algebra, $\mathcal{L}_{\mathbf{X}}\omega^n$ vanishes as well.

### 4.4(8) Hmltn n shrtr

This is basically the same procedure as in 4.4(4).

$$\mathbf{X} = \frac{\partial q^i}{\partial t}\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} q^i} + \frac{\partial p_i}{\partial t}\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} p_i} + \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} t}$$

$$\begin{aligned}
0 = i_{\mathbf{X}}\Omega &= i_{\mathbf{X}}(\mathrm{d}p_i \wedge \mathrm{d}q^i - \mathrm{d}\mathcal{H}\wedge t) = (i_{\mathbf{X}}\mathrm{d}p_i)\mathrm{d}q^i - \mathrm{d}p_i(i_{\mathbf{X}}\mathrm{d}q^i) - (i_{\mathbf{X}}\mathrm{d}\mathcal{H})\mathrm{d}t + \mathrm{d}\mathcal{H}\underbrace{(i_{\mathbf{X}}\mathrm{d}t)}_{=1} \\
&= \frac{\partial p_i}{\partial t}\mathrm{d}q^i - \frac{\partial q^i}{\partial \mathrm{d}}p_i - i_{\mathbf{X}}\left(\frac{\partial \mathcal{H}}{\partial q^i}\mathrm{d}q^i + \frac{\partial \mathcal{H}}{\partial p_i}\mathrm{d}p_i + \frac{\partial \mathcal{H}}{\partial t}\mathrm{d}t\right)\mathrm{d}t + \frac{\partial \mathcal{H}}{\partial q^i}\mathrm{d}q^i + \frac{\partial \mathcal{H}}{\partial p_i}\mathrm{d}p_i + \frac{\partial \mathcal{H}}{\partial t}\mathrm{d}t \\
&= \frac{\partial p_i}{\partial t}\mathrm{d}q^i - \frac{\partial q^i}{\partial \mathrm{d}}p_i \underbrace{- \frac{\partial \mathcal{H}}{\partial q^i}\frac{\partial q^i}{\partial t}\mathrm{d}t - \frac{\partial \mathcal{H}}{\partial p_i}\frac{\partial p_i}{\partial t}\mathrm{d}t - \frac{\partial \mathcal{H}}{\partial t}\mathrm{d}t}_{=-\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t}\mathrm{d}t=-\frac{\partial \mathcal{H}}{\partial t}\mathrm{d}t} + \frac{\partial \mathcal{H}}{\partial q^i}\mathrm{d}q^i + \frac{\partial \mathcal{H}}{\partial p_i}\mathrm{d}p_i + \frac{\partial \mathcal{H}}{\partial t}\mathrm{d}t \\
&= \left(\frac{\partial p_i}{\partial t} + \frac{\partial \mathcal{H}}{\partial q^i}\right)\mathrm{d}q^i + \left(-\frac{\partial q^i}{\partial t} + \frac{\partial \mathcal{H}}{\partial p_i}\right)\mathrm{d}p_i + \left(-\frac{\partial \mathcal{H}}{\partial t} + \frac{\partial \mathcal{H}}{\partial t}\right)\mathrm{d}t \\
&\Rightarrow \quad \dot{q}^i = \frac{\partial \mathcal{H}}{\partial p_i}\ ; \quad \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q^i}
\end{aligned}$$

I don't think it can still become any shorter.

### 4.4(9) Lie derivative of the pre-symplectic Poincaré 2-form

$$\mathcal{L}_{\mathbf{X}}\Omega = i_{\mathbf{X}}\mathrm{d}\Omega + \mathrm{d}\underbrace{i_{\mathbf{X}}\Omega}_{=0} = i_{\mathbf{X}}\mathrm{d}^2\Lambda = 0$$

# The Poincare Lemma and Potentials

## 5.1. A More General Stokes's Theorem

Let $V$ compact oriented submanifold of $M^n$
smooth $F : M^n \to W^m$
$F(V) \subset W$ need not be a submanifold, might have self-interactions, pathologies.

$$\int_{F(V)} \beta^p = \int_V F^* \beta^p \qquad (5.1) \tag{14}$$

generalizes (3.17), def.

$$\int_{F(V)} d\beta^{p-1} = \int_V F^* d\beta^{p-1} = \int_V dF^* \beta^{p-1} = \int_{\partial V} F^* \beta^{p-1} = \int_{F(\partial V)} \beta^{p-1}$$

(*question*, 2nd., 3rd. equality)

Answer: recall Naturality (cf. wikipedia exterior derivative)
$\Omega^k$ contravariant smooth functor $M \mapsto^{\Omega^k} \Lambda^k(M)$

$$\begin{array}{ccc} \Omega^k(N) & \xrightarrow{F^*} & \Omega^k(M) \\ \downarrow d & & \downarrow d \\ \Omega^{k+1}(N) & \xrightarrow{F^*} & \Omega^{k+1}(M) \end{array}$$

So that

$$dF^* = F^* d$$

---

define $\partial F(V) = F(\partial V)$

generalized Stoke's thm.

$$\int_{F(V)} d\beta^{p-1} = \int_{\partial F(V)} \beta^{p-1} \qquad (5.2) \tag{15}$$

manifold needs only "piecewise smooth" boundaries.

## 5.2. Closed Forms and Exact Forms

$\beta^p$ closed if $d\beta = 0$
$\beta^p$ exact if $\beta^p = d\alpha^{p-1}$, some $\alpha^{p-1}$

**Theorem 9 (5.3)** *Let $M^n$ with 1st Betti number 0, $b_1 = 0$, i.e. $\forall$ closed oriented piecewise smooth curve $C$ is the boundary of some compact oriented "surface". Then $\forall$ closed 1-form $\beta^1$ on $M^n$ is exact.*

---

Let $x, y \in M$, $y$ fixed.
oriented $C(y,x)$ starts at $y$, ends at $x$
define

$$f(x) \equiv \int_{C(y,x)} \beta^1$$

If $\exists$ another $C^1(y,x)$, then $C - C'$ closed oriented curve.
By given, $\exists$ oriented compact surface $F(V)$ s.t. $\partial F(V) = C$.

$$\int_C \beta - \int_{C'} \beta = \int_{C-C'} \beta = \oint_{\partial F(V)} \beta = \int_{F(V)} d\beta = 0$$

$\int_C \beta = \int_{C'} \beta$. $f$ independent of curve.
Let $\mathbf{v}_x$ vector at $x$.
Let vector field $\mathbf{v}$ coincide with $\mathbf{v}_x$ at $x$, defined in neighborhood of curve $C(y,x)$, $v = 0$ at $y$.

$\phi_t$ flow generated by $v$, $\phi_t C(y,x)$ curve joining $y$ to $\phi_i x$

$$\left[\frac{d\phi_i x}{dt}\right]_{t=0} = v_f$$

$$df(v) = \frac{d}{dt} f\{\phi_t x\}_{t=0} = \left[\frac{d}{dt}\int_{\phi_t C(y,x)} \beta\right]_{t=0} = \int_{C(y,x)} \mathcal{L}_v \beta =$$

---

## 5.3. Complex Analysis

### 5.5 Finding potentials

#### 5.5(1) Product of a closed and an exact form

Let $\kappa$ be a closed k-form and $\varepsilon$ be an exact form with $\mathrm{d}\tilde{\varepsilon} = \varepsilon$. Then

$$\kappa \wedge \varepsilon = \kappa \wedge \mathrm{d}\tilde{\varepsilon} = (-1)^k (\mathrm{d}(\kappa \wedge \tilde{\varepsilon}) - \underbrace{\mathrm{d}\kappa}_{=0} \wedge \tilde{\varepsilon}) = \mathrm{d}\left((-1)^k \kappa \wedge \tilde{\varepsilon}\right)$$

# 6 Holonomic and Nonholonomic Constraints

## 6.1. The Robenius Integrability Condition

Can one always find a surface orthogonal to a family of curves in $\mathbb{R}^3$?

## 6.2. Integrability and Constraints

## 6.3. Heuristic Thermodynamics via Caratheodory

Can one go adiabatically from some state to any nearby state?

### 6.3a. Introduction

### 6.3b. The First Law of Thermodynamics

Consider system of regions of fluids separated by "diathermous" membranes
allow only passage of heat, not fluids

assume system connected
assume each state in thermal equilibrium
Let $p_i, v_i$ (uniform) pressure and volume of $i$th region
at thermal equilibrium, "equations of state" $p_i v_i = n_i R T_i$
$p_i v_i = n_i R T_i$ eliminate all but 1 pressure

$$\Longrightarrow p_1, v_1, v_2, \ldots v_n$$

assume globally defined energy function $U$
path in $M^{n+1}$ represents sequence of states each in equilibrium, i.e. assume very slow changes in time, quasi-static irreversible processes, e.g. "stirring"
on $M$, dim$M = n+1$, assume $\exists$ work 1-form $W$, work done by system

$$W = p_i dv_i = p_i(U, v_1 \ldots v_n) dv_i \qquad i = 1 \ldots n$$

heat 1-form, heat added or removed from system, assume $Q \neq 0$

$$Q = \sum_{i=0}^{n} Q_i(U, v_1 \ldots v_n) dv_i \quad (v_0 = U)$$

1st. law of thermodynamics

$$dU = Q - W$$

energy conservation

### 6.3c. Some Elementary Changes of State

1. Heating at constant volume.

path $\gamma_I \in M$, $\dim M = n+1$ s.t. $dv_1 = \cdots = dv_n = 0$. $W = 0$ $\quad dU = Q_0 dU$. $\dot{\gamma}_I = c_0 \frac{\partial}{\partial U}$

2. Quasi-static adiabatic process. No heat exchanged,

$$Q(\dot{\gamma}_{\text{II}}) = 0 \qquad \text{so } dU = -W$$

3. Stirring at constant volume adiabatic, but not quasistatic

$Q, W$ makes no sense but

work is being done by (or on) system, $U(y') - U(x)$, difference of internal energy

assume connected mechanical manifold $V$, $\dim V = n$

diff. $\pi : M \to V$

$\pi$ onto

$\pi_*$ onto

$\pi$ submersion

By main thm. on submanifolds of Sec. 1.3d,

if $v \in V$, then $\pi^{-1}(v)$ 1-dim. embedded submanifold of $M$

assume $\forall \pi^{-1}(v)$ connected, we're assuming given any pair of states

lying on $\pi^{-1}(v)$, 1 of them can be obtained by other by "heating at constant volume"

assume $W$ on $M$ is 0 when $W|_{\pi^{-1}(v)} = 0$

on the other hand, $Q \neq 0$ on $\pi^{-1}(v)$; $dU = Q \neq 0$ (first law)

$(U, v^1 \dots v^n)$ local coordinate system for $M$ ($U$ global coordinate)

### 6.3d. The Second Law of Thermodynamics

cyclic process starts and ends at the same state

Kelvin 2nd. law of thermodynamics

$\nexists$ quasistatic cyclic process can $Q$ converted entirely into $W$

Caratheodory (1909) 2nd. law of thermodynamics

$\forall$ neighborhood $N \ni$ state $x$, $\exists y$ not accessible from $x$ via quasistatic adiabatic paths, i.e. paths s.t. $Q = 0$

# II Geometry and Topology

# $\mathbb{R}^3$ and Minkowski Space

## 7.1 Curvature and Special Relativity

### 7.1.a. Curvature of a Space Curve in $\mathbb{R}^3$

$\mathbf{x} = \mathbf{x}(t) \qquad \left(\frac{ds}{dt}\right)^2 = v^2 \qquad s(t) = \int_0^t \|\mathbf{x}(u)\| du$

$\|\mathbf{v}\| = v$

$$\begin{aligned}
\dot{\mathbf{x}} &= \frac{d\mathbf{x}}{dt} = \mathbf{v} = \frac{d\mathbf{x}}{ds}\frac{ds}{dt} = \mathbf{T}v \\
\mathbf{a} &= \ddot{x} = \mathbf{v} = \dot{v}\mathbf{T} + v\dot{\mathbf{T}} = \frac{d^2 s}{dt^2}\mathbf{T} + v\frac{d\mathbf{T}}{ds}\frac{ds}{dt} = \dot{v}\mathbf{T} + v^2\frac{d\mathbf{T}}{ds} \\
\mathbf{v} \times \mathbf{a} &= v^3 \mathbf{T} \times \frac{d\mathbf{T}}{ds} = v^3 \kappa(s) \mathbf{T} \times \mathbf{n}
\end{aligned}$$

so

$$\text{unit tangent vector } \mathbf{T} = \frac{d\mathbf{x}}{ds} = \dot{\mathbf{x}}\left(\frac{dt}{ds}\right) = \frac{\mathbf{v}}{v}$$

Note that

$$\frac{d\mathbf{T}}{ds} \cdot \mathbf{T} = \frac{1}{2}\frac{d}{ds}(\mathbf{T} \cdot \mathbf{T}) = \frac{1}{2}\frac{d}{ds}(1) = 0$$

Now

$$\frac{d\mathbf{T}}{ds} = \kappa(s)\mathbf{n}(s) \qquad (7.1)$$

where $\mathbf{n}$ principal normal, $\kappa(s) \geq 0$ curvature of $C$.

$$\implies \kappa = \frac{\|\mathbf{v} \times \mathbf{a}}{v^3}$$

### 7.1(1)

$$\begin{aligned}
x &= \cos(\omega t) \\
y &= \sin(\omega t) \\
z &= kt
\end{aligned}$$

$$\mathbf{v}\times\mathbf{a} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ -\omega s\omega t & \omega c\omega t & k \\ -\omega^2 c\omega t & -\omega^2 s\omega t & 0 \end{vmatrix} = \begin{pmatrix} k\omega^2 s(\omega t) \\ -k\omega^2 c(\omega t) \\ \omega^3 \end{pmatrix}$$

$$\implies \kappa = \frac{\|\mathbf{v}\times\mathbf{a}\|}{v^3} = \frac{\sqrt{k^2\omega^4+\omega^6}}{\sqrt{(\omega^2+k^2)^3}} = \frac{\omega^2}{\omega^2+k^2}$$

### 7.1(2)

Given $\mathbf{B} = \mathbf{T}\times\mathbf{n}$,

$$\frac{d\mathbf{B}}{ds} = \frac{d\mathbf{T}}{ds}\times\mathbf{n} + \mathbf{T}\times\frac{d\mathbf{n}}{ds} = \mathbf{T}\times\frac{d\mathbf{n}}{ds}$$

so

$$\mathbf{n}\times\frac{d\mathbf{B}}{ds} = \mathbf{n}\times(\mathbf{T}\times\frac{d\mathbf{n}}{ds}) = \left(\mathbf{n}\times\frac{d\mathbf{n}}{ds}\right)\mathbf{T} - (\mathbf{n}\times\mathbf{T})\frac{d\mathbf{n}}{ds} = 0$$

Indeed

$$\begin{aligned}
\mathbf{T}\cdot\frac{d\mathbf{B}}{ds} &= 0 \\
\mathbf{B}\cdot\frac{d\mathbf{B}}{ds} &= \frac{d}{ds}(\mathbf{B}\cdot\mathbf{B}) = \frac{d}{ds}(1) = 0
\end{aligned}$$

Then $\frac{d\mathbf{B}}{ds} \parallel \mathbf{n}$.

Define torsion $\frac{d\mathbf{B}}{ds} = \tau(s)\mathbf{n}$

Using $CAB - BAC$,

$$\begin{aligned}
\mathbf{n}\times\mathbf{B} = \mathbf{n}\times(\mathbf{T}\times\mathbf{n}) &= \mathbf{T} \\
\mathbf{T}\times\mathbf{B} = \mathbf{T}\times(\mathbf{T}\times\mathbf{n}) = (\mathbf{n}\cdot\mathbf{T})\mathbf{T} - (\mathbf{T}\cdot\mathbf{T})\mathbf{n} &= -\mathbf{n} \\
\implies \mathbf{n} &= \mathbf{B}\times\mathbf{T}
\end{aligned}$$

So

$$\frac{d\mathbf{n}}{ds} = \frac{d\mathbf{B}}{ds}\times\mathbf{T} + \mathbf{B}\times\frac{d\mathbf{T}}{ds} = \tau\mathbf{n}\times\mathbf{T} + \mathbf{B}\times\kappa\mathbf{n} = \boxed{-\tau\mathbf{B} + -\kappa\mathbf{T} = \frac{d\mathbf{n}}{ds}}$$

## 7.2 Electromagnetism in Minkowski Space

### 7.2(3) Field strength 2-Form

Notation: $\mathrm{d}x^0 = \mathrm{d}t$; $\mathrm{d}x^{ij\cdots} = \mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \cdots$. The expansion for $*F$ was taken from (14.20) combined with

(3.41).

$$\begin{aligned}
F \wedge F &= (E_i \mathrm{d}x^{i0} + B_{J=\{1,2,3\}} \mathrm{d}x^J) \wedge (E_k \mathrm{d}x^{k0} + B_{L=\{1,2,3\}} \mathrm{d}x^L) \\
&= -E_i E_k \cancel{\mathrm{d}x^{i0k0}} + B_J B_L \cancel{\mathrm{d}x^{JL}} + E_i B_L \mathrm{d}x^{i0L} + B_J E_k \mathrm{d}x^{Jk0} \\
&= -2 E_i B_J \mathrm{d}x^{0iJ} \\
&= -2(E_1 B_{23} \mathrm{d}x^{0123} + E_2 B_{13} \mathrm{d}x^{0213} + E_3 B_{12} \mathrm{d}x^{0312}) \\
&= -2 \underbrace{(E_1 B_{23} + E_2 B_{31} + E_3 B_{12})}_{=\langle \mathbf{E}, \mathbf{B} \rangle} \underbrace{\mathrm{d}x^{0123}}_{=\mathrm{vol}^4} \\
&= -2 \langle \mathbf{E}, \mathbf{B} \rangle \mathrm{vol}^4 \\
F \wedge *F &= (E_i \mathrm{d}x^{i0} + B_{J=\{1,2,3\}} \mathrm{d}x^J) \wedge (-(*B)_k \mathrm{d}x^{k0} + (*E)_{L=\{1,2,3\}} \mathrm{d}x^L) \\
&= -E_i B_j^* \cancel{\mathrm{d}x^{i0k0}} + B_J E_L^* \cancel{\mathrm{d}x^{JL}} + E_i E_L^* \mathrm{d}x^{0iL} - B_J B_k^* \mathrm{d}x^{0Jk} \\
&= B_k^* B_J \mathrm{d}x^{0kJ} - E_i E_L^* \mathrm{d}x^{0iL} \\
&= B_k^* B_J \mathrm{d}x^{0kJ} - E_k E_J^* \mathrm{d}x^{0kJ} \\
&= (B_k^* B_J - E_k E_J^*) \mathrm{d}x^{0kJ} \\
&\qquad \text{(permute k, J so their combination is in increasing order;} \\
&\qquad \text{permuting the double indices of B, E cancels out the minus)} \\
&= (\|\mathbf{B}\|^2 - \|\mathbf{E}\|^2) \mathrm{vol}^4
\end{aligned}$$

# The Geometry of Surfaces in $\mathbb{R}^3$

# 9 Covariant Differentiation and Curvature

### 9.1 Covariant Differentiation

**Definition 10** *affine connection or covariant differentiation is operator* $\nabla(X, v) \mapsto \nabla_X v$ *vector* $X$ *at* $p$ *vector field* $v$ *at* $p$

$$\begin{aligned} \nabla_X(av + bw) &= a\nabla_X v + b\nabla_X w \\ \nabla_{aX+bY} v &= a\nabla_X v + b\nabla_Y v \end{aligned} \qquad (9.2) \qquad (16)$$

$$\nabla_X(fv) = X(f)v + f\nabla_X v \qquad \text{(“Leibniz rule”)}$$

*demand if* $X$ *smooth,* $\nabla_X v$ *smooth vector field.*

in our work up until now, we have always used local coordinates $x$ to yield a basis $\frac{\partial}{\partial x^i}$ for tangent vectors in a patch $U$.

For many purposes, however, it is advantageous to use a more general basis.

frame of vector fields in $U$ - $n$ linearly independent smooth vector fields

$$\mathbf{e} = (e_1 \dots e_n)$$

coordinate frame = special case, $e_i = \frac{\partial}{\partial x^i}$ for some coordinate system $x$ in $U$

frame $\mathbf{e}$ usually not coordinate frame, since $[e_i, e_j]$ usually not 0 while $[\partial_i, \partial_j] = 0$

**Theorem 11 (9.3)** *frame* $\mathbf{e}$ *is locally a coordinate frame iff*

$$[e_i, e_j] = 0 \quad \forall\, i, j$$

Proof:

We need only show that $[e_i, e_j] = 0$ implies $\exists$ functions $(x^i)$ such taht

$$e_i = \frac{\partial}{\partial x^i}$$

Let $\sigma$ be the dual form basis. From (4.25)

$$d\sigma^i(e_j, e_k) = -\sigma^i([e_j, e_k]) \qquad (9.4) \qquad (17)$$

Let $e = (e_1 \dots e_n)$ frame in $U$. Then $X = e_j X^j$

$$\xrightarrow{(9.2)} \nabla_X(e_k v^k) = X(v^k)e_k + v^k \nabla_X e_k = X^j e_j v^k e_k + v^k X^j \nabla_{e_j} e_k = X^j e_j (v^k) e_k + X^j e_i \omega^i_{jk} v^k = \\ = X^j e_i \omega^i_{jk} v^k + X^j e_j (v^k) e_k \quad (9.5) \tag{18}$$

where $\omega^i_{jk}$ defined

$$\nabla_{e_j} e_k = e_i \omega^i_{jk} \qquad (9.6) \tag{19}$$

when $e_j = \partial_j$ coordinate frame, $\omega^i_{jk} = \Gamma^i_{jk}$
since $X(v^k) = dv^k(X)$

$$\nabla_X v = e_i \{ dv^i(X) + X^j \omega^i_{jk} v^k \}$$

$\omega^i_{jk}$ coefficients of affine connection.

using dual basis $\sigma$ of 1-forms,

$$\nabla_X v = e_i \{ dv^i(X) + \omega^i_{jk} \sigma^j(X) v^k \} = e_i \{ dv^i + \omega^i_{jk} \sigma^j v^k \}(X) \qquad (9.7) \tag{20}$$

i.e.

$$\boxed{\nabla_X v = dv(X) + \omega^i_{jk} + \omega^i_{jk} \sigma^j(X) v^k e_i} \tag{21}$$

when frame $e$ is coordinate frame $e_i = \partial_i = \frac{\partial}{\partial x^i}$, $\quad \sigma^i = dx^i$

$$\nabla_X v = \partial_i \{ \frac{\partial v^i}{\partial x^j} + \omega^i_{jk} v^k \} dx^j(X)$$

i.e.

$$(\nabla_X v)^i = \left[ \frac{\partial v^i}{\partial x^j} + \omega^i_{jk} v^k \right] X^j \qquad (9.8) \tag{22}$$

since $\nabla_X v$ assumed to be vector, conclude

$$\nabla_j v^i = v^i \big|_j \equiv \frac{\partial v^i}{\partial x^j} + \omega^i_{jk} v^k \qquad (9.9) \tag{23}$$

form the components of a mixed tensor, covariant derivative of vector $v$.

### 9.3 Cartan's Exterior Covariant Differential

### 9.3c. Cartan's Structural Equations

denote
row matrix $e \equiv (e_1 \dots e_n)$
column $\sigma \equiv (\sigma^1 \dots \sigma^n)^T$

$n \times n$ matrix of connection 1-forms $\omega = (\omega^i{}_j)$

column vector of torsion 2-forms $\tau = (\tau^1 \dots \tau^n)^T$

### 9.3d. The Exterior Covariant Differential of a Vector-Valued Form

$\alpha$ vector-valued $p$-form
locally, $\alpha = e_i \otimes \alpha^i$, $\forall \alpha^i = a^i{}_{\underline{J}}(x) \sigma^J$ locally defined $p$-form

**exterior covariant differential**, vector-valued $(p+1)$ form $\nabla \alpha$
defined by Leibniz rule

$$\nabla \alpha = \nabla(e_i \otimes \alpha^i) = (\nabla e_i) \otimes_\wedge \alpha^i + e_i \otimes d\alpha^i$$

where

$$(\nabla e_i) \otimes_\wedge \alpha^i = (e_k \otimes \omega^k_i) \otimes_\wedge \alpha^i \equiv e_k \otimes (\omega^k_i \wedge \alpha^i)$$

column of $p$ forms $\alpha = (\alpha^1 \dots \alpha^n)^T$

$$\nabla \alpha = e \otimes (d\alpha + \omega \wedge \alpha) \qquad (9.31) \tag{24}$$

### 9.3(1) Basis expansion of the curvature form

$$
\begin{aligned}
\theta^i{}_j &= \mathrm{d}\omega^i{}_j + \omega^i{}_r \wedge \omega^r{}_j = \mathrm{d}\left(\omega^i{}_{\ell j}\mathrm{d}u^\ell\right) + \left(\omega^i{}_{kr}\mathrm{d}u^k\right) \wedge \left(\omega^r{}_{\ell j}\mathrm{d}u^\ell\right) \\
&= \underbrace{\left(\partial_k\omega^i{}_{\ell j} + \omega^i{}_{kr}\omega^r{}_{\ell j}\right)}_{\equiv\text{“}(k\ell)\text{”}} \underbrace{\mathrm{d}u^k \wedge \mathrm{d}u^\ell}_{\equiv \mathrm{d}u^{k\ell}} = \frac{1}{2}\left((k\ell)\mathrm{d}u^{k\ell} + (k\ell)\mathrm{d}u^{k\ell}\right) \\
&\qquad \text{(In the second summand, commute the wedge product,} \\
&\qquad \text{afterwards rename } k \leftrightarrow \ell\text{)} \\
&= \frac{1}{2}\left((k\ell)\mathrm{d}u^{k\ell} - (\ell k)\mathrm{d}u^{k\ell}\right) \\
&= \frac{1}{2}\underbrace{\left(\partial_k\omega^i{}_{\ell j} - \partial_\ell\omega^i{}_{kj} + \omega^i{}_{kr}\omega^r{}_{\ell j} - \omega^i{}_{\ell r}\omega^r{}_{kj}\right)}_{=R^i{}_{jk\ell}} \mathrm{d}u^k \wedge \mathrm{d}u^\ell \\
&= \frac{1}{2}R^i{}_{jk\ell}\mathrm{d}u^k \wedge \mathrm{d}u^\ell
\end{aligned}
$$

### 9.3(2) Covariant derivative of the identity form

$$
\boldsymbol{\nabla}\text{“}\mathrm{d}\mathbf{r}\text{”} = \boldsymbol{\nabla}\left(\mathbf{e}_i \otimes \sigma^i\right) = \mathbf{e}_i \otimes \underbrace{\left(\mathrm{d}\sigma^i + \omega^i{}_j \wedge \sigma^j\right)}_{=\tau^i} = \mathbf{e}_i \otimes \tau^i
$$

*Remark:* The reason for calling $\mathbf{e}_i \otimes \sigma^i$ the identity form is because

$$
\mathbf{e}_i \otimes \sigma^i(\mathbf{v}) = \mathbf{e}_i \otimes \sigma^i(v^j\mathbf{e}_j) = \mathbf{e}_i v^j \underbrace{\sigma^i(\mathbf{e}_j)}_{=\delta^i_j} = \mathbf{e}_i v^i = \mathbf{v}
$$

## 9.4 Change of Basis and Gauge Transformations

### 9.4(1) Transformation of the curvature form

For readability, let $\bar{P} \equiv P^{-1}$.

$$
\begin{aligned}
\theta' &= \mathrm{d}\omega' + \omega' \wedge \omega' \\
&= \mathrm{d}(\bar{P}\omega P + \bar{P}\mathrm{d}P) \\
&\quad + (\bar{P}\omega P + \bar{P}\mathrm{d}P) \wedge (\bar{P}\omega P + \bar{P}\mathrm{d}P) \\
&= \mathrm{d}(\bar{P}\omega P) + \mathrm{d}(\bar{P}\mathrm{d}P) \\
&\quad + \bar{P}\omega P \wedge \bar{P}\omega P + \bar{P}\omega P \wedge \bar{P}\mathrm{d}P + \bar{P}\mathrm{d}P \wedge \bar{P}\omega P + \bar{P}\mathrm{d}P \wedge \bar{P}\mathrm{d}P \\
&= \mathrm{d}\bar{P} \wedge \omega P + \bar{P}\mathrm{d}\omega P - \bar{P}\omega \wedge \mathrm{d}P + \mathrm{d}\bar{P} \wedge \mathrm{d}P + \cancel{\bar{P}\mathrm{d}^2 P} \\
&\quad + \bar{P}\omega P \wedge \bar{P}\omega P + \bar{P}\omega P \wedge \bar{P}\mathrm{d}P + \bar{P}\mathrm{d}P \wedge \bar{P}\omega P + \bar{P}\mathrm{d}P \wedge \bar{P}\mathrm{d}P \\
&\quad \text{(Use } 0 = \mathrm{d}\mathbb{1} = \mathrm{d}(\bar{P}P) = \mathrm{d}\bar{P}P + \bar{P}\mathrm{d}P \Leftrightarrow \mathrm{d}\bar{P} = -\bar{P}\mathrm{d}P\bar{P}; \\
&\quad \text{Also, the matrices “commute” with the wedge product, i.e. “} A \wedge B = AB\, \wedge \text{”)} \\
&= -\bar{P}\mathrm{d}P \wedge \bar{P}\omega P + \bar{P}\mathrm{d}\omega P - \bar{P}\omega \wedge \mathrm{d}P - \bar{P}\mathrm{d}P \wedge \bar{P}\mathrm{d}P \\
&\quad + \bar{P}\omega \wedge \omega P + \bar{P}\omega \wedge \mathrm{d}P + \bar{P}\mathrm{d}P \wedge \bar{P}\omega P + \bar{P}\mathrm{d}P \wedge \bar{P}\mathrm{d}P \\
&= \bar{P}\mathrm{d}\omega P + \bar{P}\omega \wedge \omega P \\
&= \bar{P}(\mathrm{d}\omega + \omega \wedge \omega)P \\
&= \bar{P}\theta P
\end{aligned}
$$

And this dear children is why indices should be left away. (Yes, it's the same exercise.)

$$
\begin{aligned}
\theta'^i{}_j &= \mathrm{d}\omega'^i{}_j + \omega'^i{}_k \wedge \omega'^k{}_j \\
&= \mathrm{d}(\bar{P}^i{}_l \omega^l{}_m P^m{}_j + \bar{P}^i{}_l \mathrm{d}P^l{}_j) \\
&\qquad + (\bar{P}^i{}_l \omega^l{}_m P^m{}_k + \bar{P}^i{}_l \mathrm{d}P^l{}_k) \wedge (\bar{P}^k{}_n \omega^n{}_o P^o{}_j + \bar{P}^k{}_n \mathrm{d}P^n{}_j) \\
&= \mathrm{d}(\bar{P}^i{}_l \omega^l{}_m P^m{}_j) + \mathrm{d}(\bar{P}^i{}_l \mathrm{d}P^l{}_j) \\
&\qquad + \bar{P}^i{}_l \omega^l{}_m P^m{}_k \wedge \bar{P}^k{}_n \omega^n{}_o P^o{}_j + \bar{P}^i{}_l \omega^l{}_m P^m{}_k \wedge \bar{P}^k{}_n \mathrm{d}P^n{}_j \\
&\qquad + \bar{P}^i{}_l \mathrm{d}P^l{}_k \wedge \bar{P}^k{}_n \omega^n{}_o P^o{}_j + \bar{P}^i{}_l \mathrm{d}P^l{}_k \wedge \bar{P}^k{}_n \mathrm{d}P^n{}_j \\
&= \mathrm{d}\bar{P}^i{}_l \wedge \omega^l{}_m P^m{}_j + \bar{P}^i{}_l \mathrm{d}\omega^l{}_m P^m{}_j - \bar{P}^i{}_l \omega^l{}_m \wedge \mathrm{d}P^m{}_j + \mathrm{d}\bar{P}^i{}_l \wedge \mathrm{d}P^l{}_j + \cancel{\bar{P}^i{}_l \mathrm{d}^2 P^l{}_j} \\
&\qquad + \bar{P}^i{}_l \omega^l{}_m P^m{}_k \wedge \bar{P}^k{}_n \omega^n{}_o P^o{}_j + \bar{P}^i{}_l \omega^l{}_m P^m{}_k \wedge \bar{P}^k{}_n \mathrm{d}P^n{}_j \\
&\qquad + \bar{P}^i{}_l \mathrm{d}P^l{}_k \wedge \bar{P}^k{}_n \omega^n{}_o P^o{}_j + \bar{P}^i{}_l \mathrm{d}P^l{}_k \wedge \bar{P}^k{}_n \mathrm{d}P^n{}_j \\
&\qquad (\text{Use } 0 = \mathrm{d}\delta^i{}_j = \mathrm{d}(\bar{P}^i{}_k P^k{}_j) = \mathrm{d}\bar{P}^i{}_k P^k{}_j + \bar{P}^i{}_k \mathrm{d}P^k{}_j \Leftrightarrow \mathrm{d}\bar{P}^i{}_j = -\bar{P}^i{}_k \mathrm{d}P^k{}_l \bar{P}^l{}_j; \\
&\qquad \text{Also, the matrices “commute” with the wedge product, i.e. “} A^i{}_k \wedge B^k{}_j = A^i{}_k B^k{}_j \wedge \text{”}) \\
&= -\bar{P}^i{}_r \mathrm{d}P^r{}_s \wedge \bar{P}^s{}_l \omega^l{}_m P^m{}_j + \bar{P}^i{}_l \mathrm{d}\omega^l{}_m P^m{}_j - \bar{P}^i{}_l \omega^l{}_m \wedge \mathrm{d}P^m{}_j - \bar{P}^i{}_r \mathrm{d}P^r{}_s \wedge \bar{P}^s{}_l \mathrm{d}P^l{}_j \\
&\qquad + \bar{P}^i{}_l \omega^l{}_m \wedge \omega^m{}_o P^o{}_j + \bar{P}^i{}_l \omega^l{}_m \wedge \mathrm{d}P^m{}_j \\
&\qquad + \bar{P}^i{}_l \mathrm{d}P^l{}_m \wedge \bar{P}^m{}_n \omega^n{}_o P^o{}_j + \bar{P}^i{}_l \mathrm{d}P^l{}_k \wedge \bar{P}^k{}_n \mathrm{d}P^n{}_j \\
&= \bar{P}^i{}_l \mathrm{d}\omega^l{}_m P^m{}_j + \bar{P}^i{}_l \omega^l{}_m \wedge \omega^m{}_n P^n{}_j \\
&= \bar{P}^i{}_l (\mathrm{d}\omega^l{}_m + \omega^l{}_n \wedge \omega^n{}_m) P^m{}_j \\
&= \bar{P}^i{}_l \theta^l{}_m P^m{}_j
\end{aligned}
$$

### 9.4(2) Transformation of the curvature form

The Transformation rule for basis vectors is

$$
\mathbf{e}' = \mathbf{e}P \Leftrightarrow e'_i = e_j P^j{}_i
$$

The transformation from cartesian to polar coordinates is given by

$$
\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x(r,\varphi) \\ y(r,\varphi) \end{pmatrix} = \begin{pmatrix} r\cos(\varphi) \\ r\sin(\varphi) \end{pmatrix}
$$

$$
P = \frac{\partial(x,y)}{\partial(r,\varphi)} = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi} \end{pmatrix} = \begin{pmatrix} \cos(\varphi) & -r\sin(\varphi) \\ \sin(\varphi) & r\cos(\varphi) \end{pmatrix}
$$

Using Mathematica to skip the annoying 2nd semester homework assignment parts of finding the inverse and calculating derivatives,

$$
\mathrm{d}P = \begin{pmatrix} -\sin(\varphi)\,\mathrm{d}\varphi & -\sin(\varphi)\,\mathrm{d}r - r\cos(\varphi)\,\mathrm{d}\varphi \\ \cos(\varphi)\,\mathrm{d}\varphi & \cos(\varphi)\,\mathrm{d}r - r\sin(\varphi)\,\mathrm{d}\varphi \end{pmatrix}
$$

$$
P^{-1} = \begin{pmatrix} \cos(\varphi) & \sin(\varphi) \\ -\frac{1}{r}\sin(\varphi) & \frac{1}{r}\cos(\varphi) \end{pmatrix}
$$

Multiplying these two expressions yields, as desired,

$$
\omega' = \cancel{P^{-1}\omega P} + P^{-1}\mathrm{d}P = \begin{pmatrix} 0 & -r\mathrm{d}\varphi \\ \frac{1}{r}\mathrm{d}\varphi & \frac{1}{r}\mathrm{d}r \end{pmatrix}
$$

Since $\theta = 0$, $\theta'$ vanishes as well. This is obvious from the transformation behavior of $\theta$; direct computation confirms this, as

$$
\begin{aligned}
\theta' &= \mathrm{d}\omega' + \omega' \wedge \omega' \\
&= \mathrm{d}\begin{pmatrix} 0 & -r\mathrm{d}\varphi \\ \frac{1}{r}\mathrm{d}\varphi & \frac{1}{r}\mathrm{d}r \end{pmatrix} + \begin{pmatrix} 0 & -r\mathrm{d}\varphi \\ \frac{1}{r}\mathrm{d}\varphi & \frac{1}{r}\mathrm{d}r \end{pmatrix} \wedge \begin{pmatrix} 0 & -r\mathrm{d}\varphi \\ \frac{1}{r}\mathrm{d}\varphi & \frac{1}{r}\mathrm{d}r \end{pmatrix} \\
&= \begin{pmatrix} \cancel{\mathrm{d}0} & \mathrm{d}(-r\mathrm{d}\varphi) \\ \mathrm{d}\left(\frac{1}{r}\mathrm{d}\varphi\right) & \cancel{\mathrm{d}\left(\frac{1}{r}\mathrm{d}r\right)} \end{pmatrix} + \begin{pmatrix} \cancel{0 \wedge 0} - \cancel{r\mathrm{d}\varphi \wedge \frac{1}{r}\mathrm{d}\varphi} & \cancel{0 \wedge (-r\mathrm{d}\varphi)} - r\mathrm{d}\varphi \wedge \frac{1}{r}\mathrm{d}r \\ \cancel{\frac{1}{r}\mathrm{d}\varphi \wedge 0} + \frac{1}{r}\mathrm{d}r \wedge \frac{1}{r}\mathrm{d}\varphi & \cancel{\frac{1}{r}\mathrm{d}\varphi \wedge (-r\mathrm{d}\varphi)} + \cancel{\frac{1}{r}\mathrm{d}r \wedge \frac{1}{r}\mathrm{d}r} \end{pmatrix} \\
&= \begin{pmatrix} 0 & -\mathrm{d}r \wedge \mathrm{d}\varphi \\ -\frac{1}{r^2}\mathrm{d}r \wedge \mathrm{d}\varphi & 0 \end{pmatrix} + \begin{pmatrix} 0 & \mathrm{d}r \wedge \mathrm{d}\varphi \\ \frac{1}{r^2}\mathrm{d}r \wedge \mathrm{d}\varphi & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}
\end{aligned}
$$

This exercise made the advantage of the matrix notation clear: use the connection coefficients like normal matrices, only that you put a wedge in between their components' differential form "factors".

### Parallel Displacement and Curvature on a Surface

When is parallel displacement independent of path?

We saw in Section 8.7 that parallel displacement of a vector between 2 pts. of a surface is path-dependent; This phenomenon is referred to as holonomy.

**Theorem 12 (9.61)** *Let $U \subset M^2$ compact in Riemannian surface with piecewise smooth boundary $\partial U$*
*Assume $U$ covered by single orthonormal frame field e (e.g. $U$ contained in coordinate patch)*
*Let unit vector $\mathbf{v}$ parallel translated around $\partial U$*
$\mathbf{e}$ *defined orientation.*

*Then angle $\Delta\alpha$ between $\mathbf{v}_0, \mathbf{v}_f$ is*

$$\Delta\alpha = \iint K dS = \iint_U K\sigma^1 \wedge \sigma^2$$

Proof. parametrize $\partial U$, let $T$ tangent, let $\alpha = \arccos\langle e_1, v\rangle$
Although $\alpha$ (like $\mathbf{v}$) is not single-valued on $\partial U$, $d\alpha = \left(\frac{d\alpha}{ds}\right) ds$ well-defined.

$$\Delta\alpha = \arccos\langle v_0, v_f\rangle = \oint_{\partial U} d\alpha$$

For

$$\mathbf{v} = \mathbf{e}_1 \cos(\alpha) + \mathbf{e}_2 \sin(\alpha)$$

then

$$\nabla v = e(dv + \omega v) = e_1(dv^1 + \omega_{12}v^2) + e_2(dv^2 + \omega_{21}v^1) = e_1(-\sin(\alpha)\, d\alpha + \omega_{12}\sin(\alpha)) + e_2(\cos(\alpha)\, d\alpha + \omega_{21}\cos(\alpha)) = \\ = (-e_1 \sin(\alpha) + e_2\cos(\alpha))(d\alpha - \omega_{12})$$

To say that $v$ parallel displaced around $\partial U$ is to say
$\nabla v(T) = 0$, i.e. $d\alpha - \omega_{12} = 0$ along $\partial U$ (9.62)

$$d\alpha(T) = \omega_{12}(T)$$

Then

$$\Delta\alpha = \oint_{\partial U} d\alpha = \oint_{\partial U} \omega_{12} = \iint_U d\omega_{12} = \\ = \iint_U \theta_{12} = \iint_U K\sigma^1 \wedge \sigma^2$$

# 10 Geodesics

# 11 Relativity, Tensors, and Curvature

# 12 Curvature and Topology: Synge's Theorem

# 13 Betti Numbers and De Rham's Theorem

# 14 Harmonic Forms

# III Lie Groups, Bundles, and Chern Forms

# 15. Lie groups

## 15.1 Lie Groups, Invariant Vector Fields and Forms

### 15.1a Lie Groups

Topological $GL(n,\mathbb{R})$ is an open subset of $\mathbb{R}^{n^2}$ and as such is a $n^2$-dim. manifold.
(cf. pp. 392)

#### Examples

4. $G = Sl(n,\mathbb{R})$. $Sl(n,\mathbb{R})$ subgroup of $Gl(n,\mathbb{R})$, $\det x = 1$. Prob. 1.1(3). Submanifold of $\dim Sl(n,\mathbb{R}) = n^2 - 1$
5. $G = O(n)$, Sec. 1.1. Submanifold of $\dim \frac{n(n-1)}{2}$.
6. $G = U(n)$. Sec. 1. submanifold of complex $n^2$ space or real $2n^2$ space.
8. $G = T^n$ abelian group of diagonal matrices of form $z = \mathrm{diag}[e^{i\theta_1} \dots e^{i\theta_n}]$ (15.2)

This group is topologically $S^1 \times \cdots \times S^1$, $n$-torus. Since circle connected, $T^n$ connected. From this, $U(n)$ also connected!

#### 15.1b. Invariant Vector Fields and Forms

## 15.2 One-parameter subgroups

### 15.2(1) Generator of rotations

$$\begin{aligned} e^{\vartheta J} &= \sum_{k=0}^{\infty} \frac{\vartheta^k J^k}{k!} = \sum_{k=0}^{\infty} \frac{\vartheta^{2k} J^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{\vartheta^{2k+1} J^{2k+1}}{(2k+1)!} = \sum_{k=0}^{\infty} \frac{\vartheta^{2k} (J^2)^k}{(2k)!} + \sum_{k=0}^{\infty} \frac{\vartheta^{2k+1} J (J^2)^k}{(2k+1)!} \\ &= I \sum_{k=0}^{\infty} \frac{\vartheta^{2k} (-1)^k}{(2k)!} + J \sum_{k=0}^{\infty} \frac{\vartheta^{2k+1} (-1)^k}{(2k+1)!} = I \cos(\vartheta) + J \sin(\vartheta) \end{aligned}$$

### 15.2(2) Generator of A(1)

$$\begin{aligned} X &= \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \\ X^2 &= \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} aa & ab \\ 0 & 0 \end{pmatrix} = a \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = aX \\ \Rightarrow X^n &= \begin{cases} I & n = 0 \\ a^{n-1} X & n > 0 \end{cases} \end{aligned}$$

$$\begin{aligned} \Rightarrow e^{tX} &= \sum_{k=0}^{\infty} \frac{1}{k!} t^k X^k = I + \sum_{k=1}^{\infty} \frac{1}{k!} t^k a^{k-1} X = I + \frac{1}{a} X \sum_{k=1}^{\infty} \frac{1}{k!} t^k a^k \\ &= I + \frac{1}{a} X \left( \sum_{k=0}^{\infty} \frac{1}{k!} t^k a^k - \frac{t^0 a^0}{0!} \right) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & \frac{b}{a} \\ 0 & 0 \end{pmatrix} e^{ta} - \begin{pmatrix} 1 & \frac{b}{a} \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} e^{ta} & \frac{b}{a} e^{ta} - \frac{b}{a} \\ 0 & 0 \end{pmatrix} \end{aligned}$$

### 15.3(1) Maurer-Cartan equations

$$
\begin{aligned}
\mathrm{d}\sigma^U(\mathbf{X}_R,\mathbf{X}_S) &= \cancel{\mathbf{X}_R(\sigma^U(\mathbf{X}_S))} - \cancel{\mathbf{X}_S(\sigma^U(\mathbf{X}_R))} - \sigma^U([\mathbf{X}_R,\mathbf{X}_S]) \\
&= -\sigma^U(C^T_{RS}\mathbf{X}_T) = -C^U_{RS}
\end{aligned} \tag{25}
$$

$$
\Rightarrow \mathrm{d}\sigma^U = \frac{1}{2}\mathrm{d}\sigma^U(\mathbf{X}_R,\mathbf{X}_S)\sigma^R\wedge\sigma^S \overset{(25)}{=} -\frac{1}{2}C^U_{RS}\sigma^R\wedge\sigma^S
$$

$$
\begin{aligned}
\Rightarrow 0 &= \mathrm{d}(\mathrm{d}\sigma^U)(\mathbf{X}_L,\mathbf{X}_M,\mathbf{X}_S) \\
&\overset{(4.27)}{=} \mathbf{X}_L(\mathrm{d}\sigma^U(\mathbf{X}_M,\mathbf{X}_S)) - \mathbf{X}_M(\mathrm{d}\sigma^U(\mathbf{X}_L,\mathbf{X}_S)) + \mathbf{X}_S(\mathrm{d}\sigma^U(\mathbf{X}_L,\mathbf{X}_M)) \\
&\qquad - \mathrm{d}\sigma^U([\mathbf{X}_L,\mathbf{X}_M],\mathbf{X}_S) + \mathrm{d}\sigma^U([\mathbf{X}_L,\mathbf{X}_S],\mathbf{X}_M) - \mathrm{d}\sigma^U([\mathbf{X}_M,\mathbf{X}_S],\mathbf{X}_L) \\
&= \cancel{\mathbf{X}_L(-C^U_{MS})} - \cancel{\mathbf{X}_M(-C^U_{LS})} + \cancel{\mathbf{X}_S(-C^U_{LM})} \\
&\qquad - \mathrm{d}\sigma^U(C^R_{LM}\mathbf{X}_R,\mathbf{X}_S) + \mathrm{d}\sigma^U(C^R_{LS}\mathbf{X}_R,\mathbf{X}_M) - \mathrm{d}\sigma^U(C^R_{MS}\mathbf{X}_R,\mathbf{X}_L) \\
&= C^U_{RS}C^R_{LM} + C^U_{RM}C^R_{SL} + C^U_{RL}C^R_{MS}
\end{aligned}
$$

## The Lie Algebra of a Lie Group

### 15.3a. The Lie Algebra

### 15.3b. The Exponential Map

**Theorem 13 (15.27)** *map* $\exp: g \to G$ *sending* $A \mapsto e^A$ *diffeomorphism of some neighborhood of* $0 \in g$ *onto neighborhood of* $e \in G$.

Pf.
vector $X \in g$

$$
\exp_*(X) = \frac{d}{dt}(\exp tX)\bigg|_{t=0} = \frac{d}{dt}\left(1 + tX + \frac{1}{2}t^2X^2 + \dots\right)\bigg|_{t=0} = X
$$

$\exp_*: g \to g$ is the identity, exp local diffeomorphism by inverse function thm. (The Jacobian is nonsingular).

If $G$ not a matrix group,
given $X$ at $e$, $e^{tX} = \exp(tX)$ curve through $e$ whose tangent vector at $t = 0$ is vector $X$ (recall $e^{tX}$ is the integral curve through $e$ of left invariant vector field $X$).
Thus $\exp_*(X) = X$

# 16. Vector Bundles in Geometry and Physics

### 16.3(1) Connection on a tensor product space

$$
\begin{aligned}
\nabla''_{\mathbf{X}}\Lambda &= \nabla''_{\mathbf{X}}(\mathbf{e}_a\otimes\mathbf{e}'_R\lambda^{aR}) = \nabla_{\mathbf{X}}\mathbf{e}_a\otimes\mathbf{e}'_R\lambda^{aR} + \nabla_{\mathbf{X}}(\mathbf{e}_a\otimes\mathbf{e}'_R\lambda^{aR}) \\
&= \underbrace{X^i\mathbf{e}_b\omega^b_{ia}\otimes\mathbf{e}'_R\lambda^{aR}}_{b\leftrightarrow a} + \underbrace{\mathbf{e}_a\otimes\mathbf{e}'_S\omega^S_{iR}\lambda^{aR}X^i}_{S\leftrightarrow R} + \mathbf{e}_a\otimes\mathbf{e}'_R\underbrace{\mathrm{d}\lambda^{aR}X^i}_{=X^i\partial_i\lambda^{aR}} \\
&= X^i\mathbf{e}_a\otimes\mathbf{e}'_R(\partial_i\lambda^{aR} + \omega^a_{ib}\lambda^{bR} + \omega^R_{iS}\lambda^{aS})
\end{aligned}
$$

# 17. Fiber Bundles, Gauss-Bonnet, and Topological Quantization

A vector bundle is a family of vector spaces parameterized by points in the base space. How do we parameterize a family of manifolds, say Lie groups?

## 17.1. Fiber Bundles and Principal Bundles

### 17.1a. Fiber Bundles

### 17.1b. Principal Bundles and Frame Bundles

frame $\mathbf{e}$ at $p$ chosen

$$f(p) = e_\alpha(p) g_\alpha(p) \qquad (17.4) \tag{26}$$

$$\begin{aligned} \Phi_\alpha &: U_\alpha \times G \to \pi^{-1}(U_\alpha) \\ \Phi_\alpha(p,g) &= e_\alpha(p) g = (e_\alpha)_i g^i{}_j = f_j \end{aligned} \tag{27}$$

in an overlap, the same frame (17.4) will have another representation

$$\mathbf{f}(p) = \mathbf{e}_\beta(p) g_\beta(p) \qquad (17.5) \tag{28}$$

$$\begin{aligned} e_\beta(p) &= e_\alpha(p) \tau_{\alpha\beta}(p) \\ \tau_{\alpha\beta}(p) &\equiv \tau_{\alpha\beta} \\ g_\alpha(p) &= \tau_{\alpha\beta}(p) g_\beta(p) \end{aligned}$$

diffeomorphism

$$\tau_{\alpha\beta}(p) : G \to G$$

left translation of $G$ by (transition) matrix $\tau_{\alpha\beta}(p)$

### 17.1c. Action of the Structure Group on a Principal Bundle

Let $\mathbf{f} = (\mathbf{f}_1 \dots \mathbf{f}_n)$ frame at $p$, $\mathbf{f} \in P$

**Theorem 14 (17.8)**

$$(f \in P, g \in G) \to (fg) \in P$$

*freely when $g \neq e$ and*

$$\pi(fg) = \pi(f)$$

*i.e. preserves fibers*

Proof: $\pi(\mathbf{f}) = p$

$\Phi_\alpha : U_\alpha \times G \to \pi^{-1}(U_\alpha)$ local trivialization
$\Phi_\alpha(p, g_\alpha) = \mathbf{f} \Longrightarrow \Phi_\alpha^{-1}(\mathbf{f}) = (p, g_\alpha)$ $\quad \exists!\, g_\alpha$ for $\mathbf{f}$
Let $g \in G$,
right action of $g$ on $\pi^{-1}(U_\alpha)$ is (locally action)

$$\Phi_\alpha(p, g_\alpha g) = fg$$

if $p \in U_\alpha \bigcap U_\beta$

$$fg = \Phi_\beta(p, g_\beta g) = \Phi_\beta(p, \tau_{\beta\alpha}(p) g_\alpha g) = \Phi_\alpha(p, g_\alpha g)$$

$\tau_{\beta\alpha} = \Phi_\beta^{-1} \Phi_\alpha$

---

We see in this proof that the essential point is that *left translations* in $G$ (say by $\tau_{\beta\alpha}$) *commute with right translations* (say by $g$).

## 17.2.

## 17.3. Chern's Proof of the Gauss-Bonnet-Poincaré Theorem

### 17.3a. A Connection in the Frame Bundle of a Surface

$$\omega\left(\frac{d\mathbf{x}}{dt}\right) \in \mathfrak{g} = \mathfrak{u}(1) \qquad (17.14) \tag{29}$$

# 18. Connections and Associated Bundles

## 18.1. Forms with Values in a Lie Algebra

What do we mean by $g^{-1}dg$?

### 18.1.a. The Maurer-Cartan Form

If we think of $\omega$ as being a form that takes its values in the fixed vector space $\mathfrak{g}$, rather than as a matrix of 1-forms, we shall have an equivalent picture that is in many ways more closely related to the terminology used in physics.

exterior form is differential form

**Maurer-Cartan 1-form on $G$**

Let $\{E_R\}$ basis for $\mathfrak{g}$

$\{X_R\}$ left invariant fields on $G$ obtained by left translating $E$'s

$\{\sigma^R\}$ left invariant 1-forms on $G$ forming, $\forall\, g \in G$, basis dual to $\{X_R\}$

$\sigma^R(X_S) = \delta^R_S$

Then

$$\Omega \equiv E_R \otimes \sigma^R \qquad (18.1) \tag{30}$$

$$\Omega(Y_g) = E_R \sigma^R(Y_g) = E_R Y^R$$

$Y = X_R Y^R$ at $g \in G$, left translates back to 1

$\Omega : T_g G \to T_e G$

cf. Nakahara

$$\Omega : Y \mapsto (L_{g^{-1}})_* Y = (L_g)^{-1}_* Y,\, Y \in T_g G$$

Classically, Cartan wrote $\forall\, p \in M$, vector valued 1 form taking each $Y$ vector at $p$ into itself

$$dp = \partial_i \otimes dx^i = \partial_i \otimes \delta^i{}_j dx^j$$

$$\Omega = g^{-1} dg \qquad (18.2) \tag{31}$$

$dg$ takes $Y$ at $g$ into $Y$, $g^{-1}$ left transltates $Y$ back to $e$

# 19. The Dirac Equation

Spin is what makes the world go 'round. -Theodore Frankel

## 19.1a. The Rotation Group $SO(3)$ of $\mathbb{R}^3$

$$E_1 = \begin{bmatrix} & & \\ & & -1 \\ & 1 & \end{bmatrix}$$

$$E_2 = \begin{bmatrix} & & 1 \\ & & \\ -1 & & \end{bmatrix}$$

$$E_3 = \begin{bmatrix} & -1 & \\ 1 & & \\ & & \end{bmatrix}$$

$(E_i)$'s are basis for $\mathfrak{so}(3)$

$$[E_i, E_j] = \epsilon_{ijk} E_k$$

$c^k_{ij} = \epsilon_{ijk}$

Consider 1-parameter group of rotations with angular velocity $\omega$, $\omega = \frac{d\theta}{dt}$

$$\left.\frac{d\mathbf{r}}{dt}\right|_{t=0} = \omega \times \mathbf{r}(0)$$

On the other hand, 1-parameter subgroup is of form $R(t) = e^{tS}$, $S$ skew-symmetric matrix

$$r(t) = R(t) r(0) = e^{tS} r(0)$$

$$\left.\frac{dr}{dt}\right|_{t=0} = Sr(0) \text{ so } S(r) = \omega \times r$$

$$E_j(r) = e_j \times r$$

$$R(t) = \exp\left(E_j \omega^j t\right) \equiv \exp\left(E \cdot \omega t\right) \qquad (19.3) \tag{32}$$

$$R(\theta) = \exp\left(\theta E \cdot n\right) \qquad (19.4) \tag{33}$$

## 19.1b. $SU(2)$: The Lie Algebra $\mathfrak{su}(2)$

$\mathfrak{su}(2) = \mathfrak{g} = \{X | X = -X^\dagger; \mathrm{tr}(X) = 0\}$
$i\mathfrak{g} = \{X | X = X^\dagger; \mathrm{tr}(X) = 0\}$,
basis

$$\begin{aligned} \sigma_1 &= \begin{bmatrix} & 1 \\ 1 & \end{bmatrix} \\ \sigma_2 &= \begin{bmatrix} & -i \\ i & \end{bmatrix} \\ \sigma_3 &= \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} \end{aligned} \qquad (19.5)$$

$$[\sigma_j, \sigma_k] = 2i\epsilon_{ijk}\sigma_i \qquad (19.6)$$

We shall see that $SU(2)$ simply connected.
Lie group theory states that $\exists$ homomorphism from $SU(2)$ onto $SO(3)$ Pf. : Frobenius thm.
$\mathrm{Ad} : SU(2) \to SO(3)$
Claim: adjoint representation $\mathrm{Ad}(g) = gYg^{-1}$ of $SU(2)$ on 3-dim. Lie algebra $\mathfrak{su}(2)$ yields (Thm. 19.2) standard representation of $SO(3)$ on $\mathbb{R}^3$

$$\mathbf{x} = \mathbf{x} \cdot \sigma = x^R \sigma_R = \begin{bmatrix} z & x - iy \\ x + iy & -z \end{bmatrix} = x_*$$

inverse

$$\begin{aligned} x &= \frac{1}{2}\mathrm{tr}(x_*\sigma_1) \\ y &= \frac{1}{2}\mathrm{tr}(x_*\sigma_2) \\ z &= \frac{1}{2}\mathrm{tr}(x_*\sigma_3) \end{aligned} \qquad (19.8)$$

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \mapsto \sigma_1$$

$$\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \mapsto \sigma_2$$

$$\mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \mapsto \sigma_3$$

$\mathbf{e}_i \cdot \sigma = \sigma_i$
real scalar product in $i\mathfrak{g}$

$$\langle h, h' \rangle = \mathrm{tr}(hh')$$

as $\mathrm{tr}(\sigma_j\sigma_k) = 2\delta_{jk}$
Recall, $\forall$ Lie group $G$ acts on $\mathfrak{g}$ by adjoint action

$$\begin{aligned} &\mathrm{Ad} : G \to Gl(\mathfrak{g}) \\ &\mathrm{Ad}(g)(X) = gXg^{-1} \qquad \forall\, X \in \mathfrak{g} \end{aligned}$$

In this case, $SU(2) = \{u | u^\dagger u = 1\}$, $\mathfrak{su}(2) = \{X | X^\dagger = -X, \mathrm{tr}X\}$, $\sigma_i$, $i = 1, 2, 3$ basis for $\mathfrak{su}(2)$

$$\begin{aligned} &\mathrm{Ad} : G \to Gl(\mathfrak{g}) \\ &\mathrm{Ad}(g)(X) = gXg^{-1} \qquad \forall\, X \in \mathfrak{g} \\ &\mathrm{Ad} : SU(2) \to \mathfrak{su}(2) \\ &\mathrm{Ad}(u)(X) = uXu^{-1} \qquad \forall\, X \in \mathfrak{su}(2) \end{aligned}$$

Consider action of $SU(2)$ on $i\mathfrak{su}(2) = i\mathfrak{g}$ hermitian traceless matrices $X$. We'll still call this Ad
$\forall\, u \in SU(2)$

$$\begin{aligned} &\mathrm{Ad}(u) : i\mathfrak{g} \to i\mathfrak{g} \\ &\mathrm{Ad}(u) : i\mathfrak{su}(2) \to i\mathfrak{su}(2) \quad x_* \mapsto ux_*u^{-1} \quad \forall\, x_* \in i\mathfrak{su}(2) \end{aligned}$$

$\forall\, u \in SU(2)$, we're associated a $3\times 3$ matrix

$$\mathrm{Ad}(u) : \mathbb{R}^3 \to \mathbb{R}^3 \text{ using (19.7)} \begin{array}{l} \mathbb{R}^3 \to i\mathfrak{g} \\ x \mapsto x\cdot\sigma = x^R\sigma_R = x_* \end{array}$$

Note Ad is a representation of $SU(2)$ by $3\times 3$ matrices

$$\mathrm{Ad}(uu')(x_*) = uu'x_*(uu')^{-1} = \mathrm{Ad}(u)\mathrm{Ad}(u')x_*$$

Note

$$\langle \mathrm{Ad}(u)x_*, \mathrm{Ad}(u)x_*\rangle = \mathrm{tr}(ux_*u^{-1}ux_*u^{-1})\mathrm{tr}(x_*x_*) = \langle x_*x_*\rangle$$

so $\mathrm{Ad}(u) \in O(3)$, Ad representation of $SU(2)$ by orthogonal $3\times 3$ matrices.

## 19.1c. $SU(2)$ is Topologically the 3-Sphere

fundamental representation of $SU(2)$ by $2\times 2$ complex unitary matrices $uu^\dagger = 1$
$\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}$
Recall that the general form of $SU(2)$ matrices is the following: (cf. wikipedia)

$$SU(2) = \{ \begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} | \alpha, \beta \in \mathbb{C},\, |\alpha|^2 + |\beta|^2 = 1\}$$

$S^3 \subset \mathbb{C}^2 \approx \mathbb{R}^4$

$$S^3 = \{(z_1, z_2)^T || z_1|^2 + |z_2|^2 = 1\}$$

Note $SU(2) : S^3 \to S^3$ as $UU^\dagger = 1 \qquad \forall U \in SU(2)$ i.e. (unitary)
Note $SU(2)$ acts transitively on $S^3$

Pf:

$$(1,0)^T \in S^3, \qquad u = \begin{bmatrix} z_1 & -\overline{z}_2 \\ z_2 & \overline{z}_1 \end{bmatrix} \in SU(2)$$

$$u\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \qquad \forall \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in S^3 \text{ i.e. (arbitrary)}$$

So any $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in S^3$ can be "reached" from $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \in S^3$ by some $u \in SU(2)$
From (17.10), topologically

$$S^3 \approx \frac{SU(2)}{H}$$

where $H$ is stability subgroup of pt. $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$
But (19.11), $H = \{1\}$

$$\Longrightarrow SU(2) \approx S^3$$

In fact,

$$\begin{aligned} & SU(2) \to S^3 \\ & u = \begin{pmatrix} z_1 & -\overline{z}_2 \\ z_2 & \overline{z}_1 \end{pmatrix} \mapsto \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \end{aligned}$$

In particular $SU(2) = S^3$ connected.
Since $\mathrm{Ad}(u) \in O(3)$ orthogonal matrix, $\det\mathrm{Ad}(u) = \pm 1$
since $u$ cont., and connected $S^3$, $\det\mathrm{Ad}(u) = +1$. Thus $\mathrm{Ad}(u) \in SO(3)$

$$\mathrm{Ad} : SU(2) \to SO(3)$$

## Ad : $SU(2) \to SO(3)$ in More Detail

**Theorem 15 (19.12)** *representation $Ad : SU(2) \to SO(3)$ given in (19.10)*

$$\begin{aligned} u &\in SU(2) \\ x_* &\in i\mathfrak{su}(2) \qquad (19.10) \\ x_* &\mapsto u x_* u^{-1} \end{aligned} \tag{34}$$

*is onto, i.e. $\forall$ rotation in $\mathbb{R}^3$, of form (19.10)*

*Furthermore, this representation is $2:1$, i.e. $\forall$ rotation $R$, $\exists$ exactly 2 $\pm u \in SU(2)$, s.t. $Ad(\pm u) = R$*

EY : 20150217 What we have is this:

$$\begin{aligned} \mathbb{R}^3 &\doteq^* \mathfrak{su}(2) \\ (x,y,z) &\overset{*}{\mapsto} x^R \sigma_R \end{aligned} \qquad x_*^{-1}(X) = \frac{1}{2}(\mathrm{tr}(X\sigma_1), \mathrm{tr}(X\sigma_2), \mathrm{tr}(X\sigma_3))^T$$

$$\begin{aligned} SU(2) &\xrightarrow{\mathrm{Ad}} \mathrm{Gl}(\mathfrak{su}(2)) \\ u &\mapsto \mathrm{Ad}(u) \subset SO(3) \end{aligned}$$

$$\begin{aligned} i\mathfrak{su}(2) &\xrightarrow{\mathrm{Ad}(u)} i\mathfrak{su}(2) \\ x_* &\mapsto u x_* u^{-1} \end{aligned}$$

Pf: Let $u(t)$ 1-parameter subgroup of $SU(2)$

$u(t) = \exp\left(\frac{t}{i}h\right)$, $h$ $2\times 2$ hermitian matrix (i.e. $h = h^\dagger$), $u(t) \in SU(2)$

$u(t) \to$ 1-parameter subgroup of $SO(3)$ under $\begin{aligned} i\mathfrak{su}(2) &\to \mathbb{R}^{\not{k}} \\ x_* &\overset{-1}{\overset{*}{\mapsto}} \mathbf{x} \end{aligned}$

$$\mathrm{Ad}u(t)\mathbf{x} \sim \mathrm{Ad}u(t)x_* = e^{-ith}x_* e^{ith}$$

$$\begin{aligned} \left.\frac{d}{dt}\right|_0 \mathrm{Ad}(u(t))x_* = \left.\frac{d}{dt}\right|_0 e^{-ith}x_*e^{ith} &= -i[h, x_*] = -i[h^j\sigma_j, x^k\sigma_k] = -ih^jx^k[\sigma_j,\sigma_k] = -ih^jx^k\epsilon_{jki}\sigma^i(2i) = \\ &= 2\epsilon_{jki}h^jx^k\sigma^i = 2(h\times x)^i\sigma_i \end{aligned}$$

EY : 20150217 Keep in mind

$$\begin{tikzcd} \mathbb{R}^3 \arrow[r, "\mathrm{Ad}(u)"] \arrow[d, dash, "\wr"'] & \mathbb{R}^3 \arrow[d, dash, "\wr"'] \\ i\mathfrak{su}(2) \arrow[r, "\mathrm{Ad}(u)"] & i\mathfrak{su}(2) \end{tikzcd}$$

angular velocity vector of 1-parameter group $\mathrm{Ad}u(t)x \in \mathbb{R}^3$

$$\omega = 2h$$

From (19.3)

$$R(t) = \exp(E_j\omega^jt) \equiv \exp(E\cdot\omega t) \qquad (19.3) \tag{35}$$

$$\mathrm{Ad}\exp\left(\frac{\sigma}{i}\cdot\mathbf{h}t\right)x_* \sim R(t)\mathbf{x} = \exp(\mathbf{E}\cdot 2\mathbf{h}t)\mathbf{x} \qquad (19.13) \tag{36}$$

or

$$i\mathfrak{su}(2) \xrightarrow{\mathrm{Ad}\exp\left(\frac{\sigma}{i}ht\right)} i\mathfrak{su}(2)$$

$$i\mathfrak{su}(2) \to \mathbb{R}^3$$

$$\boxed{\mathrm{Ad}\exp\left(\frac{\sigma}{i}ht\right)x_* \mapsto R(t)x = \exp(E\cdot 2ht)x}$$

$$\mathrm{Ad}_*\left(\frac{\sigma_\alpha}{2i}\right) = E_\alpha \qquad (19.14) \tag{37}$$

e.g. $h \in i\mathfrak{su}(2)$

$$h = \sigma_3,\ h = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$t = \theta$

$u(t) \in SU(2)$

$$u(t) = \exp\left(\frac{t}{i}h\right) = \exp\left(\frac{t}{i}\sigma_3\right) = \begin{bmatrix} e^{-i\theta} & \\ & e^{i\theta} \end{bmatrix} = \exp\left(\frac{\theta}{i}\sigma_3\right)$$

with $\sigma_3 = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}$

$\exp(E \cdot 2ht) \in SO(3)$

$$\overset{\mathrm{Ad}_*\left(\frac{\sigma_\alpha}{2i}\right)}{\mapsto} \exp(E \cdot 2h\theta) = \exp(2\theta E_3) = \exp\begin{bmatrix} & -2\theta & \\ 2\theta & & \\ & & \end{bmatrix} = \begin{bmatrix} \cos(2\theta) & -\sin(2\theta) & \\ \sin(2\theta) & \cos(2\theta) & \\ & & 1 \end{bmatrix}$$

with $\cdot h = E \cdot \sigma_3 = E_3 = \begin{bmatrix} & -1 & \\ 1 & & \\ & & \end{bmatrix}$

For $SU(2)$, for $0 \leq \theta 2\pi$,

$\exp\left(\frac{\theta\sigma_3}{i}\right)$ is a simple closed curve, $\begin{bmatrix} e^{-i\theta} & \\ & e^{i\theta} \end{bmatrix}$

$\exp(2\theta E_3)$ yields 2 full rotations

$\forall$ rotation of $\mathbb{R}^3$ is a rotation about some size, i.e. $R = \exp(E \cdot \omega\theta) \in SO(3)$

By (19.13),

$$\mathrm{Ad}\exp\left(\frac{\sigma}{i} \cdot ht\right)x_* \mapsto R(t)x = \exp(E \cdot 2ht) = \exp(E \cdot \omega\theta)$$

So that

$$\mathrm{Ad}\exp\left(\frac{\sigma}{2i} \cdot \omega\theta\right) = R$$

for

$$\begin{aligned} \omega &= 2h \\ E &= \frac{\sigma}{2i} \end{aligned}$$

So Ad onto. $\mathrm{Ad} : SU(2) \to SO(3)$

If $\mathrm{Ad}(u) = R$, $u = u(t) = \exp\left(\frac{t}{i}h\right) \mapsto R = \exp(E \cdot \omega t)$ $\quad \mathrm{Ad}(u)x_* = ux_*u^{-1} \mapsto Rx$

$\mathrm{Ad}(-u)x_* = ux_*u^{-1} \mapsto Rx$

So Ad representation is at least $2:1$ i.e. not faithful

It's an elementary result of group theory that

if $\phi : G \to G'$ homomorphism of $G$ onto $G'$, then $G'$ isomorphic to coset $G/H$, where $H = \phi^{-1}(e')$ is kernel

(17.10) fundamental principle, $\forall G$ that acts on $G'$ by

$$(g, g') \mapsto \phi(g)g'$$

and stability subgroup of $e' \in G'$ is kernel $H = \phi^{-1}(e')$ $\qquad \ker\phi = H = \phi^{-1}(e')$

$\mathrm{Ad} : SU(2) \to SO(3)$ $\qquad \ker\mathrm{Ad} = \{\pm 1\}$

(17.11) $\to SU(2)$ is fiber bundle over $SO(3)$;

$$\begin{aligned} &p^{-1} : SO(3) \to SU(2) \\ &p^{-1}(R) \mapsto \{\pm u\} \text{ exactly 2 pts.} \end{aligned}$$

$$\begin{array}{ccc} S^3 & \overset{\simeq}{\text{———}} & SU(2) \\ p\downarrow & & p\downarrow \\ \mathbb{R}P^3 & \overset{\simeq}{\text{———}} & SO(3) \end{array}$$

$$
\begin{array}{ll}
S^3/p = \mathbb{R}P^3 & \\
x \in S^3 & \\
x \sim -x & [x] = \{x, -x\}
\end{array}
$$

# 20. Yang-Mills Fields

## 20.1. Noether's Theorem for Internal Symmetries

*How do symmetries yield conservation laws?*

$\phi$ $N$-tuple $\phi^a(t, \mathbf{x}) = \phi^a(x)$, local representation of a section of some vector bundle $E$,

$$
\begin{tikzcd}
E \arrow[d, "\pi"] \\
M
\end{tikzcd}
$$

In the case of a Dirac electorn, we have seen that $E$ is the bundle of complex 4-component Dirac spinors over a perhaps curved spacetime. If $E$ is not a trivial bundle (or if we insist on using curvilinear coordinates) we shall have to deal with the fact that $\partial_j \phi^a$ do not form a tensor.

### 20.1a. The Tensorial Nature of Lagrange's Equations

Let $M^{n+1}$ (pseudo-) Riemannian manifold, let $E$ vector bundle over $M$; for definiteness, let fiber be $\mathbb{R}^N$.
section of this bundle over $U \subset M$ is described by $N$ real-valued functions $\{\phi^a_U\}$,
where $\phi_V = c_{VU} \phi_U$ and
$c_{VU}(x)$ is $N \times N$ transition matrix function, $c^a_{VUb}$.

notation $\begin{array}{l} \{\Phi^a\} \\ \{\Phi^a_\alpha\} \end{array}$

$\Phi^a_\alpha = \tau_{\alpha\beta} \Phi_\beta$

Lagrangian $L_0(x, \phi, \phi_x) \equiv L_0(x, \Phi, \partial_j \Phi^a)$

## 20.2. Weyl's Gauge Invariance Revisited

### 20.2a. The Dirac Lagrangian

### 20.2b. Weyl's Gauge Invariance Revisited

### 20.2c. The Electromagnetic Lagrangian

Instead of considering a change of (spacetime) coordinates $x$, we shall look at a change of the *field* (fiber) coordinate $\psi$, i.e. *a gauge transformation.*

Since the phase of $\psi$ is not measurable, we *should* be able to have invariance under a *local* gauge transformation, where $\alpha = \alpha(x)$ varies with the spacetime point $x$!
Clearly the Dirac equation and Lagrangian are *not* invariant under such a substitution because of the appearance of terms involving $d\alpha$.
It must be that *there is some background field that is interacting with the electron.* This background field will manifest itself through the appearance of the connection.

## The Yang-Mills Nucleon

How did the groups $SU(2)$ and $SU(3)$ appear in particle physics?

### 20.3a. The Heisenberg Nucleon

### 20.3b. The Yang-Mills Nucleon

### 20.3c. A Remark on Terminology

We have related the connection matrices $\omega$ to the gauge potentials $A$ by

$$\omega = -iqA$$

$q$ is called a generalized **charge**.

# 21.

# A. Elasticity

## A.a. The Classical Cauchy Stress Tensor and Equations of Motion

$B(t)$ compact body, might be portion of larger body in motion
mass 3 form

$$m^3 \equiv \rho \text{ vol}$$

mass conservation

$$\frac{d}{dt}\int_{B(t)} m^3 = \int_{B(t)} \mathcal{L}_{\mathbf{v}+\frac{\partial}{\partial t}} m^3 = 0$$

**b** external force density (per unit mass)

$$\frac{d}{dt}\int_{B(t)} v^i m^3 = \int_{B(t)} b^i m^3 + \int_{\partial B(t)} t^{ij} n_j da$$

## A.b. Stresses in Terms of Exterior Forms

$t$ pseudo $(n-1)$ form on $M^n$ with values in tangent bundle $TM$ (vector bundle language)

$$\mathbf{t} = \mathbf{e}_r \otimes \mathfrak{t}^r \equiv \mathbf{e}_r \otimes \mathfrak{t}^r{}_{\underline{J}} \sigma^J \qquad (38)$$

$$\int_{\partial B} \mathbf{t} = \int_{\partial B} \mathbf{e}_r \otimes \mathfrak{t}^r = \mathbf{e}_r \int_{\partial B} \mathfrak{t}^r = \mathbf{e}_r \int_{\partial B} \mathfrak{t}^r_{\underline{J}} dx^J$$

as total traction that part of body outside $\partial B$ exerts on $B$
(2.73) $\forall$ Riemannian $M$, write $(n-1)$ form $\mathfrak{t}^r$ in terms of vector $t^{(r)}$

$$\mathfrak{t}^r = i(\mathbf{t}^r)\text{vol} = i(\mathbf{t}^{(r)})\sqrt{g}\epsilon_{\underline{I}} dx^I = \sqrt{g}\mathbf{t}^{(r)i}\epsilon_{i\underline{J}} dx^J$$

$$\begin{gathered} t^r{}_{\underline{J}} = \sqrt{g} t^{(r)i} \epsilon_{i\underline{J}} \qquad (A.6) \\ t^{ri} \equiv t^{(r)i} = \frac{1}{\sqrt{g}} t^r{}_{\underline{J}} \epsilon^{iJ} \end{gathered} \qquad (39)$$

relation between stress form $t^r$ and Cauchy's stress tensor $t^{ri}$
assuming $\mathbf{t} = \mathbf{e}_r \otimes \mathfrak{t}^r$ is $(n-1)$ form section of the tangent bundle, thus from (9.31), we have
20141102 EY recall

$$\nabla \alpha = e \otimes (d\alpha + \omega \wedge \alpha) \qquad (9.31)$$

and

$$\nabla \alpha = \nabla(e_i \otimes \alpha^i) = (\nabla e_i) \otimes_\wedge \alpha^i + e_i \otimes d\alpha^i$$

where

$$(\nabla e_i) \otimes_\wedge \alpha^i = (e_k \otimes \omega^k_{\ i} \otimes_\wedge \alpha^i \equiv e_k \otimes (\omega^k_{\ i} \wedge \alpha^i)$$

$$\nabla t = \nabla(\mathbf{e}_r \otimes \mathfrak{t}^r) = \nabla \mathbf{e}_r \otimes_\wedge \mathfrak{t}^r + \mathbf{e}_r \otimes d\mathfrak{t}^r = \mathbf{e}_r \otimes (d\mathfrak{t}^r + \omega^r_{\ s} \wedge \mathfrak{t}^s) = \mathbf{e}_r \otimes \nabla \mathfrak{t}^r \qquad (A.7) \qquad (40)$$

## A.f. Hamilton's Principle in Elasticity

$$\delta U = \int_B \delta E_{RC} dX^R \wedge S^C \qquad (A.24) \qquad (41)$$

$$\delta U = \int_B S^{CR} \delta E_{RC} \text{VOL}^n \qquad (A.25) \qquad (42)$$

Hamilton's principle

$$\delta \int T dt - \int \delta U dt + \int \delta W dt = 0 \qquad (A.26) \qquad (43)$$

We don't write $\delta \int U dt$ because we don't assume

We don't assume that $\exists$ stored energy function $U$ so we don't write $\delta \int U dt$, $\exists$ only differential $\delta U$