THE DIFFERENTIAL GEOMETRY DIFFERENTIAL TOPOLOGY DUMP

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Contents

Abstract. Everything about Differential Geometry, Differential Topology

Part 1. Combinatorics, Probability Theory

Theorem 1 (4.2. of Feller (1968) [1]). *Let $r_1, \dots r_k \in \mathbb{Z}$, s.t. $r_1 + r_2 + \dots + r_k = n;$, $r_i \geq 0$.*
Let

$$\frac{N!}{r_1!r_2!\dots r_k!} = \tag{1}$$

number of ways in which n elemnts can be divided into k ordered parts (partitioned into k subpopulations). cf. Eq. (4.7) of Feller (1968) [1].

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Note that the order of the subpopulations is essential in the sense that $(r_1 = 2, r_2 = 3)$ and $(r_1 = 3, r_2 = 2)$ represent different partitions. However, no attention is paid to the order within the groups.

Proof.

$$\binom{n}{r_1}\binom{n-r_1}{r_2}\binom{n-r_1-r_2}{r_3}\dots\binom{n-r_1-\dots-r_{k-2}}{r_{k-1}} = \frac{n!}{r_1!r_2!\dots r_k!} \tag{2}$$

i.e. in order to effect the desired partition, we have to select r_1 elementsout of n, remaining $n - r_1$ elements select a second group of size r_2, etc. After forming the $(k-1)$st group there remains $n - r_1 - r_2 - \dots - r_{k-1} = r_k$ elements, and these form the last group. □

cf. pp. 37 of Feller (1968) [1] Examples. (g) Bridge. 32 cards are partitioned into 4 equal groups $\to 52!/(13!)^4$.

Probability each player has an ace (?).

The 4 aces can be ordered in $4! = 24$ ways, each order presents 1 possibility of giving 1 ace to each player.
Remaining 48 cards distributed $(48!)/(12!)^4$ ways.

$$\to p = 24\frac{48!}{(12!)^4}/\frac{52!}{(13!)^4}$$

(h) A throw of 12 dice $\to 6^{12}$ different outcomes total. Event each face appears twice can occur in as many ways as 12 dice can be arranged in 6 groups of 2 each.

$$\frac{12!}{(2!)^6}/\frac{52!}{(13!)^4}$$

0.0.1. *Application to Occupancy Problems; binomial coefficients.* cf. Sec. 5 Application to Occupancy Problems of Feller (1968) [1].

Consider randomly placing r balls intos n cells.

Let $r_k =$ occupancy number = number of balls in kth cell.

Every n-tuple of integers satisfying $r_1 + r_2 + \dots + r_n = r$; $r_k \geq 0$. describes a possible configuration of occupancy numbers.

With indistinguishable balls 2 distributions are distinguishable only if the corresponding n-tuples $(r_1, \dots r_n)$ are not identical.

(i) number of distinguishable distributions is

$$A_{r,n} = \binom{n+r-1}{r} = \binom{n+r-1}{n-1} \tag{3}$$

cf. Eq. (5.2) of Feller (1968) [1]

(ii) number of distinguishable distributions in which no cell remains empty is $\binom{r-1}{n-1}$.

Proof. Represent balls by stars, indicate n cells by n spaces between $n+1$ bars. e.g. $\begin{matrix} r = 8 \text{ balls} \\ n = 6 \text{ cells} \end{matrix}$.

$$\begin{array}{cccc} 3 & 10 & 00 & 4 \\ |*** & |*| & || & |****| \end{array}$$

Such a symbol necessarily starts and ends with a bar, but remaining $n-1$ bars and r starts appear in an arbitrary order. In this way, it becomes apparent that the number of distinguishable distributions equals the number of ways of selecting.

r places out of $n+r-1$, $\frac{(n+r-1)!}{(n-1)!r!} = \binom{n-1+r}{r}$

$$|||| \dots || \qquad n+1 \text{ bars}$$
$$*** \cdots ** \qquad r \text{ stars leave } r-1 \text{ spaces}$$

Condition that no cell be empty imposes the restriction that no 2 bars be adjacent. r stars leave $r-1$ spaces of which $n-1$ are to be occupied by bars. Thus $\binom{r-1}{n-1}$ choices.

□

Probability to obtain given occupancy numbers $r_1, \dots r_n = \frac{r!}{r_1!r_2!\dots r_n!}/n^r$, with $\begin{matrix} r \text{ balls} \\ n \text{ cells} \end{matrix}$ given by Thm. 4.2. of Feller (1968) [1], which is the Maxwell-Boltzmann distribution.

(a) Bose-Einstein and Fermi-Dirac statistics. Consider r indistinguishable particles, n cells, each particle assigned to 1 cell.
State of the system - random distribution of r particles in n cells.
If n cells distinguishable, n^r arrangements equiprobable → Maxwell-Boltzmann statistics.
Bose-Einstein statistics: only distinguishable arrangements are considered, and each assigned probability $\frac{1}{A_{r,n}}$

$$A_{r,n} = \binom{n+r-1}{r} = \binom{n-1+r}{n-1} \tag{4}$$

cf. Eq. 5.2 of Feller (1968) [1]
Fermi-Dirac statistics.
(1) impossible for 2 or more particles to be in the same cell. $\to r \leq n$.
(2) all distinguishable arrangements satisfying the first condition have equal probabilities.
→ an arrangement is completely described by stating which of the n cells contain a particle
r particles $\to \binom{n}{r}$ ways r cells chosen.
Fermi-Dirac statistics, there are $\binom{n}{r}$ possible arrangements, prob. $1/\binom{n}{r}$.

pp. 39. Feller (1968) [1]. Consider cells themselves indistinguishable! Disregard order among occupancy numbers.
cf. Feller (1968) [1]

Part 2. Linear Algebra Review

cf. *Change of Basis*, of Appendix B of John Lee (2012) [3].

Exercise B.22. Suppose V, W, X finite-dim. vector spaces
$S: V \to W, \quad T: W \to X$

(a) $\text{rank}S \leq \text{dim}V$ with $\text{rank}S = \text{dim}V$ iff S injective
(b) $\text{rank}S \leq \text{dim}W$ with $\text{rank}S = \text{dim}W$ iff S surjective
(c) if $\text{dim}V = \text{dim}W$ and S either injective or surjective, then S isomorphism
(d) $\text{rank}TS \leq \text{rank}S$ $\text{rank}TS = \text{rank}S$ iff $\text{im}S \bigcap \text{ker}T = 0$
(e) $\text{rank}TS \leq \text{rank}T$ $\text{rank}TS = \text{rank}T$ iff $\text{im}S + \text{ker}T = W$
(f) if S isomorphism, then $\text{rank}TS = \text{rank}T$
(g) if T isomorphism, then $\text{rank}TS = \text{rank}S$

EY : Exercise B.22(d) is useful for showing the chart and atlas of a Grassmannian manifold, found in the More examples, for smooth manifolds.

Proof. (a) Recall the **rank-nullity theorem**:

Theorem 2 (Rank-Nullity Theorem)**.**

$$dim(im(S)) + dim(ker(S)) = dimV \tag{5}$$

Now
$$\text{rank}(S) + \text{dim}(\text{ker}(S)) \equiv \text{dim}(\text{im}(S)) + \text{dim}(\text{ker}(S)) = \text{dim}V$$
$$\implies \boxed{\text{rank}(S) \leq \text{dim}V}$$

If $\text{rank}(S) = \text{dim}V$,
then by rank-nullity theorem, $\text{dim}(\text{ker}(S)) = 0$, implying that $\text{ker}S = \{0\}$.
Suppose $v_1, v_2 \in V$ and that $S(v_1) = S(v_2)$. By linearity of S, $S(v_1) - S(v_2) = S(v_1 - v_2) = 0$, which implies, since $\text{ker}S = \{0\}$, that $v_1 - v_2 = 0$.
$\implies v_1 = v_2$. Then by definition of injectivity, S injective.
If S injective, then $S(v) = 0$ implies $v = 0$. Then $\text{ker}S = \{0\}$. Then by rank-nullity theorem, $\text{rank}(S) = \text{dim}V$.

(b) $\forall\, w \in \text{im}(S)$, $w \in W$. Clearly $\text{rank}S \leq \text{dim}W$.
If S surjective, $\text{im}(S) = W$. Then $\text{dim}(\text{im}(S)) = \text{rank}S = \text{dim}W$.

If $\text{rank}S = \text{dim}W = m$, then $\text{im}(S)$ has basis $\{y_i\}_{i=1}^m$, $y_i \in \text{im}(S)$, so $\exists\, x_i \in V$, $i = 1 \dots m$ s.t. $S(x_i) = y_i$, with $\{S(x_i)\}_{i=1}^m$ linearly independent.
Since $\{S(x_i)\}_{i=1}^m$ linearly independent and $\text{dim}W = m$, $\{S(x_i)\}_{i=1}^m$ basis for W.
$\forall\, w \in W$, $w = \sum_{i=1}^m w^i S(x_i) = S(\sum_{i=1}^m w^i x_i)$. $\sum_{i=1}^m w^i x_i \in V$. S surjective.

(c)

(d) Now
$$\text{dim}V = \text{rank}TS + \text{nullity}TS$$
$$\text{dim}V = \text{rank}S + \text{nullity}S$$
$\text{ker}S \subseteq \text{ker}TS$, clearly, so $\text{nullity}S \leq \text{nullity}TS$
$$\implies \boxed{\text{rank}TS \leq \text{rank}S}$$

If $\text{rank}TS = \text{rank}S$,
then $\text{nullity}S = \text{nullity}TS$
Suppose $w \in \text{Im}S \bigcap \text{ker}T$, $w \neq 0$
Then $\exists\, v \in S$, s.t. $w = S(v)$ and $T(w) = 0$
Then $T(w) = TS(v) = 0$. So $v \in \text{ker}TS$
$v \notin \text{ker}S$ since $w = S(v) \neq 0$
This implies $\text{nullity}TS > \text{nullity}S$. Contradiction.
$\implies \text{Im}S \bigcap \text{ker}T = 0$

If $\text{Im}S \bigcap \text{ker}T = 0$,
Consider $v \in \text{ker}TS$. Then $TS(v) = 0$.
. Then $S(v) \in \text{ker}T$
$S(v) = 0$; otherwise, $S(v) \in \text{Im}S$, contradicting given $\text{Im}S \bigcap \text{ker}T = 0$
$v \in \text{ker}S$

$\text{ker}TS \subseteq \text{ker}S$
$\implies \text{ker}TS = \text{ker}S$
So $\text{nullity}TS = \text{nullity}S$
$\implies \text{rank}TS = \text{rank}S$

(e)

(f)

(g)

□

Part 3. Manifolds

1. Inverse Function Theorem

Shastri (2011) had a thorough and lucid and explicit explanation of the Inverse Function Theorem [5]. I will recap it here. The following is also a blend of Wienhard's Handout 4 https://web.math.princeton.edu/~wienhard/teaching/M327/handout4.pdf

Definition 1. *Let (X,a) metric space.*
contraction $\phi : X \to X$ *if* $\exists$ *constant* $0 < c < 1$ *s.t.* $\forall\, x, y \in X$

$$d(\phi(x),\phi(y)) \leq cd(x,y)$$

Theorem 3 (Contraction Mapping Principle)**.** *Let (X,d) complete metric space.*
Then $\forall$ *contraction* $\phi : X \to X$, $\exists ! y \in X$ *s.t.* $\phi(y) = y$, y fixed pt.

Proof. Recall def. of complete metric space X, X metric space s.t. $\forall$ Cauchy sequence in X is convergent in X (i.e. has limit in X).

$$\forall\, x_0 \in X,\text{ Define } \begin{aligned} & x_1 = \phi(x_0) \\ & x_2 = \phi(x_1) \\ & \vdots \\ & x_j = \phi(x_{j-1}) \\ & \vdots \\ & x_n = \phi(x_{n-1}) \end{aligned}$$

$$d(x_{n+1},x_n) = d(\phi(x_n),\phi(x_{n-1})) \leq cd(x_n,x_{n-1}) \leq \dots \leq c^n d(x_1,x_0)$$

for some $0 < c < 1$.

$$d(x_m,x_n) \leq d(x_n,x_{n-1}) + d(x_{n-1},x_m) \leq d(x_n,x_{n-1}) + d(x_{n-1},x_{n-2}) + \dots + d(x_{m+1},x_m) \leq \sum_{k=n-1}^m c^k d(x_1,x_0)$$

Thus, $\forall\, \epsilon > 0$, $\exists\, n_0 > 0$, (n_0 large enough) s.t. $\forall\, m,n \in \mathbb{N}$ s.t. $n_0 < n < m$,

$$d(x_m,x_n) \leq \sum_{k=n-1}^m c^k d(x_1,x_0) < \epsilon d(x_1,x_0)$$

Thus, $\{x_n\}$ Cauchy sequence. Since X complete, $\exists$ limit pt. $y \in X$ of $\{x_n\}$.

$$\phi(y) = \phi(\lim_n x_n) = \lim_n \phi(x_n) = \lim_n x_{n+1} = y$$

Since by def. of y limit pt. of $\{x_n\}$, $\forall\, \epsilon > 0$, then $\{n | |x_n - y| \leq \epsilon, n \in \mathbb{N}\}$ is infinite.

Consider $\delta > \mathbb{N}$. Consider $\{n | |x_n - y| \leq \delta, n \in \mathbb{N}\}$

$\exists\, N_\delta \in \mathbb{N}$ s.t. $\forall\, n > N_\delta$, $|x_n - y| < \delta$; otherwise, $\forall\, N_\delta$, $\exists\, n > N_\delta$ s.t. $|x_n - y| \geq \delta$. Then $\{n | |x_n - y| \leq \delta, n \in \mathbb{N}\}$ finite. Contradiction.

ϕ cont. so by def. $\forall\, \epsilon > 0$, $\exists\, \delta > 0$ s.t. if $|x_n - y| < \delta$, then $|\phi(x_n) - \phi(y)| < \epsilon$.

Pick N_δ s.t. $\forall\, n > N_\delta$, $|x_n - y| < \delta$, and so $|\phi(x_n) - \phi(y)| < \epsilon$. There are infinitely many $\phi(x_n)$'s that satisfy this, and so $\phi(y)$ is a limit pt.

If $\exists\, y_1, y_2 \in X$ s.t. $\begin{aligned} \phi(y_1) &= y_1 \\ \phi(y_2) &= y_2 \end{aligned}$, then

$$d(y_1,y_2) = d(\phi(y_1),\phi(y_2)) \leq cd(y_1,y_2) \text{ with } c < 1$$

so $c = 1$ □

Theorem 4 (Inverse Function Theorem)**.** *Suppose open $U \subset \mathbb{R}^n$, let C^1 $f : U \to \mathbb{R}^n$, $x_0 \in U$ s.t. $Df(x_0)$ invertible.*
Then $\exists$ *open neighborhoods $V \ni x_0$, $W \ni f(x_0)$ s.t. $V \subseteq U$ and $W \subseteq \mathbb{R}^n$, respectively, and s.t.*

(i) $f : V \to W$ *bijection*
(ii) $g = f^{-1} : V \to U$ *differentiable, i.e.* $g = f^{-1} : W \to V$ *is* C^1
(iii) $D(f^{-1})$ *cont. on* W.
(iv) $Dg(y) = (Df(g(y)))^{-1} \quad \forall\, y \in W$

Also, notice that $f(g(y)) = y \,\forall\, y \in W$.

Proof. Consider $\widetilde{f}(x) = (Df(x_0))^{-1}(f(x+x_0) - f(x_0))$. Then
$\widetilde{f}(0) = 0$ and

$$\begin{aligned} D\widetilde{f} &= (Df(x_0))^{-1}(Df(x+x_0) - 0) \\ D\widetilde{f}(0) &= (Df(x_0))^{-1} Df(x_0) = 1 \end{aligned}$$

So let $\widetilde{f} \to f$ (notation) and so assume, without loss of generality, that $U \ni 0$, $f(0) = 0$, $Df(0) = 1$

Choose $0 < \epsilon \leq \frac{1}{2}$. Let $0 < \delta < 1$ s.t. open ball $V = B_\delta(0) \subseteq U$, and $\|Df(x) - 1\| < \epsilon$. $\forall\, x \in U$, since Df cont. at 0.
Let $W = f(V)$.

$$\forall\, y \in W,\text{ define } \begin{aligned} & \phi_y : V \to \mathbb{R}^n \\ & \phi_y(x) = x + (y - f(x)) \end{aligned}$$

$$\begin{aligned} D(\phi_y)(x) &= 1 + -Df(x) \quad \forall\, x \in V \\ \|D(\phi_y)(x)\| &= \|1 - Df(x)\| \leq \epsilon < 1 \end{aligned}$$

$\forall\, x_1, x_2 \in V$, by mean value Thm. (not the equality that is only valid in 1-dim., but the inequality, that's valid for $\mathbb{R}^d$,

$$\|\phi_y(x_1) - \phi_y(x_2)\| \leq \|D(\phi_y)(x')\| \|x_1 - x_2\|$$

for some $x' = cx_2 + (1-c)x_1$, $c \in [0,1]$. V only needed to be convex set.

$$\Longrightarrow \|\phi_y(x_1) - \phi_y(x_2)\| \leq \epsilon \|x_1 - x_2\|$$

Then ϕ_y contraction mapping.

Suppose $f(x_1) = f(x_2) = y$, $x_1, x_2 \in V$.

$$\begin{gathered} \phi_y(x_1) = x_1 \\ \phi_y(x_2) = x_2 \\ \|\phi_y(x_1) - \phi_y(x_2)\| = \|x_1 - x_2\| \leq \epsilon \|x_1 - x_2\| \quad \forall\, \epsilon > 0 \Longrightarrow x_1 = x_2 \end{gathered}$$

$\Longrightarrow f|_U$ injective.

$W = f(V)$, so $f : V \to W$ surjective. f bijective.

Fix $y_0 \in W$, $y_0 = f(x_0)$, $x_0 \in V$.

Let $r > 0$ s.t. $B_r(x_0) \subset V$.

Consider $B_{r\epsilon}(y_0)$. If $y \in B_{r\epsilon}(y_0)$.

$$\begin{gathered} r\epsilon > \|y - y_0\| = \|y - f(x_0)\| = \|\phi_y(x_0) - x_0\| \text{ with} \\ \phi_y(x) = x + (y - f(x)) \end{gathered}$$

If $x \in B_r(x_0)$,

$$\|\phi_y(x) - x_0\| \leq \|\phi_y(x) - \phi_y(x_0)\| + \|\phi_y(x_0) - x_0\| \leq \epsilon \|x - x_0\| + r\epsilon < 2r\epsilon = r$$

Thus $\phi(B_r(x_0)) = B_r(x_0)$.

By contraction mapping principle, $\exists\, a \in B_r(x_0)$, s.t. $\phi_y(a) = a$. Then $\phi_y(a) = a + (y - f(a)) = a \Longrightarrow f(a) = y$.

$y \in f(V) = W$.

So $B_{r\epsilon}(y_0) \subset W$. W open.

Let $\text{Mat}(n,n) \equiv$ space of all $n \times n$ matrices; $\text{Mat}(n,n) = \mathbb{R}^{n^2}$.

□

There is a proof of the implicit function theorem and its various forms in Shastri (2011) [5], but I found Wienhard's Handout 4 for Math 327 to be clearer.[1]

Theorem 5 (Implicit Function Theorem). *Let open $U \subset \mathbb{R}^{m+n} \equiv \mathbb{R}^m \times \mathbb{R}^n$*

C^1 $f : U \to \mathbb{R}^n$

$(a,b) \in U$ s.t. $f(a,b) = 0$ and $D_y f|_{(a,b)}$ invertible.

Then $\exists$ open $V \ni (a,b)$, $V \subset U$

$\exists$ open neighborhood $W \ni a$, $W \subseteq \mathbb{R}^m$

$\exists!$ C^1 $g : W \to \mathbb{R}^n$ s.t.

$$\{(x,y) \in V | f(x,y) = 0\} = \{(x, g(x)) | x \in W\}$$

Moreover,

$$dg_x = -\left.(d_y f)^{-1}\right|_{(x,g(x))} \left. d_x f \right|_{(x,g(x))}$$

and g smooth if f.

Proof. Define $F : U \to \mathbb{R}^{m+n}$

$$F(x,y) = (x, f(x,y))$$

Then $F(a,b) = (a,0)$ (given), and

$$DF = \begin{bmatrix} 1 & \\ \frac{\partial f^i(x,y)}{\partial x^j} & \frac{\partial f^i(x,y)}{\partial y^j} \end{bmatrix} \equiv \begin{bmatrix} 1 & \\ D_x f & D_y f \end{bmatrix}$$

$DF(a,b)$ invertible.

By inverse function theorem, since $DF(a,b)$ invertible at pt. (a,b),

$\exists$ open neighborhoods $\begin{matrix} V \ni (a,b) \subseteq \mathbb{R}^m \times \mathbb{R}^n \\ \widetilde{W} \ni (a,0) \subseteq \mathbb{R}^m \times \mathbb{R}^n \end{matrix}$ s.t. F diffeomorphism with $F^{-1} : \widetilde{W} \to V$.

Set $W = \{x \in \mathbb{R}^m | (x,0) \in \widetilde{W}\}$. Then $\pi_1(\widetilde{W}) = W$ open in $\mathbb{R}^m$.

Define $g : W \to \mathbb{R}^n$,

$$g(x) = \pi_2 \circ F^{-1}(x,0) \text{ or}$$
$$F^{-1}(x,0) = (h(x), g(x))$$

Now $FF^{-1}(x,0) = (x,0) = (h(x), f(h(x), g(x)))$ so $h(x) = x \,\forall\, x \in W$, $0 = f(x, g(x))$.

Then

$$\{(x,y) \in V | f(x,y) = 0\} = \{(x,y) \in V | F(x,y) = (x,0)\} = \{(x, g(x)) | x \in W, 0 = f(x, g(x))\}$$

Since π smooth and F^{-1} is C^1, g is C^1.

To reiterate, $f(x, g(x)) = 0$ on W.

Using chain rule while differentiating $f(x, g(x)) = 0$,

$$\partial_{x^j} f(x, g(x)) = \frac{\partial f(x,g(x))}{\partial x^k} \frac{\partial x^k}{\partial x^j} + \frac{\partial f(x,g(x))}{\partial y^k} \frac{\partial g^k(x)}{\partial x^j} = \left. D_x f \right|_{(x,g(x))} + \left. (D_y f) \right|_{(x,g(x))} \cdot (Dg)_x = 0 \text{ or}$$
$$(Dg)_x = -\left.(D_y f)\right|_{x,g(x)} \left. D_x f \right|_{(x,g(x))}$$

□

2. Immersions

Definition 2 (Immersion). *smooth $f : M \to N$, s.t. $Df(p) : T_pM \to T_{f(p)}N$ injective. Then f **immersion** at p.*

Absil, Mahony, and Sepulchre [7] pointed out that another definition for a *immersion* can utilize the theorem that rank of $Df \equiv DF = \text{dim} T_pM$. Indeed, recall these facts from linear algebra:

for $T : V \to W$,

It's always true that $\begin{matrix} \text{rank}T \leq V \\ \text{rank}T \leq W \end{matrix}$, and

$\text{rank}T = \text{dim}V$ iff T injective.

$\text{rank}T = \text{dim}W$ iff T surjective.

$$\begin{tikzcd} T_xM \arrow[r, "DF(x)"] & T_{F(x)}N = T_yN \\ x \in M \arrow[u] \arrow[r, mapsto, "F"] & y = F(x) \in N \arrow[u] \end{tikzcd}$$

$$\begin{tikzcd} M \arrow[r, "F"] & N \end{tikzcd}$$

Now

$$\text{dim}T_xM = \text{dim}M$$
$$\text{dim}T_{F(x)}N = \text{dim}N$$

And

$$\text{rank}(DF(x)) \equiv \text{rank of } F$$

I know that the notation above is confusing, but this is what all Differential Geometry books apparently mean when they say "rank of F".

Now

$$\text{rank}(DF(x)) = \text{dim}(\text{im}(DF(x))) = \text{dim}T_xM \text{ iff } DF(x) \text{ injective}$$

If $\forall\, x \in M$, this is the case, then F an **immersion** .

Apply the rank-nullity theorem in this case:

$$\text{rank}(DF(x)) + \text{dimker}(DF(x)) = \text{dim}T_xM = \text{dim}M$$
$$\Longrightarrow \text{rank}(DF(x)) = \text{dim}M \leq \text{dim}T_{F(x)}N = \text{dim}N \text{ or } \text{dim}M \leq \text{dim}N$$

Now

$$\text{rank}(DF(x)) = \text{dim}T_{F(x)}N \text{ iff } DF(x) \text{ surjective}$$

[1] https://web.math.princeton.edu/~wienhard/teaching/M327/handout4.pdf

If $\forall\, x \in M$, this is the case, then F an **submersion** .

$$\text{rank}(DF(x)) = \dim T_{F(x)}N = \dim N \leq \dim M$$

Shastri (2011) has this as the "Injective Form of Implicit Function Theorem", Thm. 1.4.5, pp. 23 and Guillemin and Pollack (2010) has this as the "Local Immersion Theorem" on pp. 15, Section 3 "The Inverse Function Theorem and Immersions" [4].

Theorem 6 (Local immersion Theorem i.e. Injective Form of Implicit Function Theorem)**.** *Suppose $f : M \to N$ immersion at p, $q = f(p)$.*

Then $\exists$ local coordinates around p, q, x, y, respectively s.t. $f(x_1 \dots x_m) = (x_1 \dots x_m, 0 \dots 0)$.

Proof. Choose local parametrizations

$$\begin{tikzcd} U \subseteq M \arrow[r, "f"] \arrow[d, "\phi"] & N \supseteq V \arrow[d, "\psi"] \\ \phi(U) \arrow[r, "f"] & \psi(V) \end{tikzcd} \qquad \begin{aligned} \phi(p) &= x \\ \psi(q) &= y \end{aligned}$$

$D(\psi f \varphi^{-1}) \equiv Df$. $Df(p)$ injective (given f immersion). $Df(p) \in \text{Mat}(n, m)$

By change of basis in $\mathbb{R}^n$, assume $Df(p) = \begin{pmatrix} I_m \\ 0 \end{pmatrix}$.

Now define $G : \phi(U) \times \mathbb{R}^{n-m} \to \mathbb{R}^n$

$$G(x, z) = f(x) + (0, z)$$

Thus, $DG(x, z) = 1$ and for open $\phi(U) \times U_2$, $G(\phi(U) \times U_2)$ open.

By inverse function theorem, G local diffeomorphism of $\mathbb{R}^n$, at 0.

Now $f = G \circ \mathfrak{i}$, where $\mathfrak{i}$ is canonical immersion.

$$\begin{gathered} G(x, 0) = f(x) \\ \implies G^{-1}G(x, 0) = (x, 0) = G^{-1}f(x) \end{gathered}$$

Use $\psi \circ G$ as the local parametrization of N around pt. q. Shrink U, V so that

$$\begin{tikzcd} U \subseteq M \arrow[r, "f"] \arrow[d, "\phi"] & N \supseteq V \arrow[d, "\psi \circ G"] \\ \phi(U) \arrow[r, "\mathfrak{i}"] & \psi \circ G(V) \end{tikzcd}$$

$\square$

Theorem 7 (Implicit Function Thm.)**.** *Let open subset $U \subseteq \mathbb{R}^n \times \mathbb{R}^d$, $(x, y) = (x^1 \dots x^n, y^1 \dots y^k)$ on U. Suppose smooth $\Phi : U \to \mathbb{R}^k$, $(a, b) \in U$, $c = \Phi(a, b)$*

If $k \times k$ matrix $\frac{\partial \Phi^i}{\partial y^j}(a, b)$ nonsingular, then $\exists$ neighborhoods $\begin{matrix} V_0 \subseteq \mathbb{R}^n \text{ of } a \\ W_0 \subseteq \mathbb{R}^k \text{ of } b \end{matrix}$ and smooth $F : V_0 \to W_0$ s.t.

$\Phi^{-1}(c) \bigcap (V_0 \times W_0)$ is graph of F, i.e.
$\Phi(x, y) = c$ for $(x, y) \in V_0 \times W_0$ iff $y = F(x)$.

3. Submersions

cf. pp. 20, Sec. 4 "Submersions", Ch. 1 of Guillemin and Pollack (2010) [4].

Consider $X, Y \in \mathbf{Man}$, s.t. $\dim X \geq \dim Y$.

Definition 3 (submersion)**.** *If $f : X \to Y$,*
if $Df_x \equiv df_x$ is surjective, *$f \equiv$ **submersion** at x.*

Recall that,

$$\begin{aligned} Df_x &: T_x X \to T_{f(x)}Y \\ \dim T_x X &\geq \dim T_{f(x)}Y \\ \text{rank} Df_x &\leq \dim T_{f(x)}Y, \text{ in general, while} \\ \text{rank} Df_x &= \dim T_{f(x)}Y \text{ iff } Df_x \text{ surjective} \end{aligned}$$

Canonical submersion is standard projection:

If $\begin{aligned} \dim X &= k,\ k \geq l, \\ \dim Y &= l \end{aligned}$

$$(a_1 \dots a_k) \mapsto (a_1 \dots a_l)$$

Theorem 8 (Local Submersion Theorem)**.** *Suppose $f : X \to Y$ submersion at x, and $y = f(x)$, Then $\exists$ local coordinates around x, y s.t.*

$$f(x_1 \dots x_k) = (x_1 \dots x_l)$$

i.e. f locally equivalent to canonical submersion near x

Proof. I'll have a side-by-side comparison of my notation and the 1 used in Guillemin and Pollack (2010) [4] where I can.

For charts $(U, \phi), (V, \psi)$ for X, Y, respectively, $y = f(x)$ for $x \in X$,

$$\begin{tikzcd} U \subseteq X \arrow[r, "f"] \arrow[d, "\phi"] & Y \supseteq V \arrow[d, "\psi \circ G"] \\ \mathbb{R}^k \arrow[r, "\mathfrak{i}"] & \mathbb{R}^l \end{tikzcd} \qquad \begin{tikzcd} x \arrow[r, mapsto, "f"] \arrow[d, mapsto, "\phi"] & f(x) = y \arrow[d, mapsto, "\psi"] \\ \phi(x) = (a^1 \dots a^k) \arrow[r, mapsto, "g"] & g(\phi(x)) = g(a^1 \dots a^k) = \psi(y) \end{tikzcd}$$

Dg_x surjective, so assume it's a $l \times k$ matrix $\begin{bmatrix} \mathbf{1}_l & 0 \end{bmatrix}$.

Define

$$\begin{aligned} & G : U \subset \mathbb{R}^k \to \mathbb{R}^k \\ & G(a) \equiv G(a^1 \dots a^k) := (g(a), a_{l+1}, \dots, a_k) \end{aligned} \tag{6}$$

Now

$$DG(a) = \begin{bmatrix} \mathbf{1}_l & 0 \\ & \mathbf{1}_{k-l} \end{bmatrix} = \mathbf{1}_k \tag{7}$$

so G local diffeomorphism (at 0).

So $\exists$ G^{-1} as local diffeomorphism of some U' of a into $U \subset \mathbb{R}^k$.

By construction,

$$g = \mathbb{P}_l \circ G \tag{8}$$

where $\mathbb{P}_l$ is the *canonical submersion*, the projection operator onto $\mathbb{R}^l$.

$$g \circ G^{-1} = \mathbb{P}_l$$

(since G diffeomorphism)

$$\begin{tikzcd} U \subseteq X \arrow[r, "f"] & V \subseteq Y \\ \mathbb{R}^k \arrow[u, "\phi^{-1}\circ G^{-1}"] \arrow[r, "\mathbb{P}_l"] & \mathbb{R}^l \arrow[u, "\psi^{-1}"'] \end{tikzcd}$$

for

$$\begin{tikzcd} \phi^{-1}\circ G^{-1}(a) \equiv \phi^{-1}\circ G^{-1}(a^1 \dots a^k) = x \arrow[r, mapsto, "f"] & f(x) = y = \psi^{-1}(a^1 \dots a^l) \\ (a^1 \dots a^k) \arrow[u, mapsto, "\phi^{-1}\circ G^{-1}"] \arrow[r, mapsto, "\mathbb{P}_l"] & (a^1 \dots a^l) \arrow[u, mapsto, "\psi^{-1}"'] \end{tikzcd}$$

$\Longrightarrow$

$\square$

"An obvious corollary worth noting is that if f is a submersion at x, then it is actually a submersion in a whole neighborhood of x." Guillemin and Pollack (2010) [4]

Suppose f submersion at $x \in f^{-1}(y)$.

By local submersion theorem

$$f(x_1 \dots x_k) = (x_1 \dots x_l)$$

Choose $y = (0, \dots, 0)$.

Then, near x, $f^{-1}(y) = \{(0, \dots 0, x_{l+1} \dots x_k)\}$ i.e. let $V \ni x$ neighborhood of x, define $(x_1 \dots x_k)$ on V.

Then $f^{-1}(y) \bigcap V = \{(0 \dots 0, x_{l+1}, \dots x_k) | x_1 = 0, \dots x_l = 0\}$.

Thus $x_{l+1}, \dots x_k$ form a coordinate system on open set $f^{-1}(y) \bigcap V \subseteq f^{-1}(y)$.

Indeed,

$$\begin{tikzcd} U \subseteq X \arrow[r, "f"] \arrow[d, "\phi"] & V \subseteq Y \arrow[d, "\psi"] \\ \mathbb{R}^k \arrow[r, "\mathbb{P}_l"] & \mathbb{R}^l \end{tikzcd} \qquad \begin{tikzcd} x \arrow[r, mapsto, "f"] \arrow[d, mapsto, "\phi"] & f(x) = y \arrow[d, mapsto, "\psi"] \\ \phi(x) = (x^1 \dots x^k) \arrow[r, mapsto, "\mathbb{P}_l"] & (x^1 \dots x^l) \end{tikzcd}$$

and now

$$\begin{tikzcd} f^{-1}(y) & y \arrow[l, mapsto, "f^{-1}"'] \arrow[d, mapsto, "\psi"] \\ \{(0, \dots 0, x^1 \dots x^k)\} \arrow[u, mapsto, "\phi^{-1}"] & (0 \dots 0) \arrow[l, mapsto, "\mathbb{P}_l^{-1}"] \end{tikzcd}$$

Definition 4 (regular value)**.** *For smooth $f : X \to Y$, $X, Y \in$ **Man**,*
*$y \in Y$ is a **regular value** for f if $Df_x : T_xX \to T_yY$ surjective $\forall\, x$ s.t. $f(x) = y$.*
*$y \in Y$ **critical value** if y not a regular value of f.*

Absil, Mahony, and Sepulchre [7] pointed out that another definition for a *regular value* can utilize the theorem that rank of $Df \equiv DF = \dim T_pN = \dim N$, iff $DF(p)$ surjective, for $p \in M$, $F : M \to N$. Then

regular value $y \in N$, of F, if rank of $F \equiv \text{rank}(DF(x)) = \dim N$, $\forall\, x \in F^{-1}(y)$, for $F : M \to N$.

Theorem 9 (Preimage theorem)**.** *If y regular value of $f : X \to Y$,*
$f^{-1}(y)$ is a submanifold of X, with $\dim f^{-1}(y) = \dim X - \dim Y$

Proof. Given y is a regular value of $f : X \to Y$,
$\forall\, x \in f^{-1}(y)$, $Df_x : T_xX \to T_yY$ is surjective. By local submersion theorem,

$$f(x^1 \dots x^k) = (x^1 \dots x^l) = y$$

Since $x \in f^{-1}(y)$, $(x^1 \dots x^k) = (y^1 \dots y^l, x^{l+1} \dots x^k)$.

For this chart for (U, φ), $U \ni x$, consider $(U \cap f^{-1}(y), \psi)$ with $\psi(x) = (x^{l+1} \dots x^k) \quad \forall\, x \in U \cap f^{-1}(y)$.

$\forall\, f^{-1}(y)$ submanifold with $\dim f^{-1}(y) = k - l = \dim X - \dim Y$. $\square$

Examples for emphasis

If $\dim X > \dim Y$,

if $y \in Y$, regular value of $f : X \to Y$,

f submersion, $\forall\, x \in f^{-1}(y)$

If $\dim X = \dim Y$,

f local diffeomorphism $\forall\, x \in f^{-1}(y)$

If $\dim X < \dim Y$, $\forall\, y \in f(X)$ is a critical value.

Example: $O(n)$ as a submanifold of Mat(n,n)

Given $\text{Mat}(n,n) \equiv M(n) = \{n \times n \text{ matrices }\}$ is a manifold; in fact $\text{Mat}(n,n) \cong \mathbb{R}^{n^2}$,
Consider $O(n) = \{A \in \text{Mat}(n,n) | AA^T = 1\}$.

$$AA^T \in \text{Sym}(n) \equiv S(n) = \{S \in \text{Mat}(n,n) | S^T = S\} = \{\text{ symmetric } n \times n \text{ matrices }\} \tag{9}$$

$\text{Sym}(n)$ submanifold of $\text{Mat}(n,n)$, $\text{Sym}(n)$ diffeomorphic to $\mathbb{R}^k$ (i.e. $\text{Sym}(n) \cong \mathbb{R}^k$), $k := \frac{n(n+1)}{2}$.

$$f : \text{Mat}(n,n) \to \text{Sym}(n)$$
$$f(A) = AA^T$$

Notice f is smooth,

$$f^{-1}(1) = O(n)$$

$$Df_A(B) = \lim_{s \to 0} \frac{f(A + sB) - f(A)}{s} = \lim_{s \to 0} \frac{(A + sB)(A^T + sB^T) - AA^T}{s} = AB^T + BA^T$$

If $Df_A : T_A\text{Mat}(n,n) \to T_{f(A)}\text{Sym}(n)$ surjective when $A \in f^{-1}(1) = O(n)$ (???).

Proposition 1. *If smooth $g_1 \dots g_l \in C^\infty(X)$ on X are independent $\forall\, x \in X$, s.t. $g_i(x) = 0$, $\forall\, i = 1 \dots l$,*
then $Z = \{x \in X | g_1(x) = \dots = g_l(x) = 0\}$ = set of "common zeros" is a submanifold *of X s.t. $\dim Z = \dim X - l$.*

Take note that $g_1 \dots g_l$ are independent at x means, really, that $D(g_1)_x \dots D(g_l)_x$ are linearly independent on T_xX.

Proof. Suppose smooth $g_1 \dots g_l \in C^\infty(X)$ on manifold X s.t. $\dim X = k \geq l$.

Consider $g = (g_1 \dots g_l) : X \to \mathbb{R}^l$, $Z \equiv g^{-1}(0)$.

Since $\forall\, g_i$ smooth, $D(g_i)_x : T_xX \to \mathbb{R}$ linear.

Now for

$$Dg_x = (D(g_1)_x \dots D(g_l)_x) : T_xX \to \mathbb{R}^l$$

By rank-nullity theorem (linear algebra), Dg_x surjective iff $\text{rank} Dg_x = l$ i.e. l functionals $D(g_1)_x \dots D(g_l)_x$ are linearly independent on T_xX.

"We express this condition by saying the l functions $g_1 \dots g_l$ are independent at x." (Guillemin and Pollack (2010) [4]) $\square$

4. Submanifolds; immersed submanifold, embedded submanifolds, regular submanifolds

Recall immersion:
$F: M \to N$ immersion iff DF injective iff $\text{rank}DF = \text{dim}M$.
Consider manifolds $M \subseteq N$.

Consider inclusion map $\begin{aligned} i &: M \to N. \\ i &: x \mapsto x \end{aligned}$

If i immersion, $Di(x) = \frac{\partial y^i}{\partial x^j} = \delta_j{}^i$ if $y^i = x^i$, $\forall\, i = 1, \dots \text{dim}M$.

Definition 5 (immersed submanifold)**.** ***immersed submanifold*** *$M \subseteq N$ if inclusion $i: M \to N$ is an immersion.*

cf. 3.3 Embedded Submanifolds of Absil, Mahony, and Sepulchre [7], also Ch. 5 Submanifolds, pp. 108, **Immersed Submanifolds** of John Lee (2012) [3].
Immersed submanifolds often arise as images of immersions.

Proposition 2 (Images of Immersions as submanifolds)**.** *Suppose smooth manifold M,*
smooth manifold with or without boundaries N,
injective, smooth immersion $F: M \to N$ (F injective itself, not just immersion)
Let $S = F(M)$.
Then S has unique topology and smooth structure of smooth submanifolds of N s.t. $F: M \to S$ diffeomorphism.

cf. Prop. 5.18 of John Lee (2012) [3].

Proof. Define topology of S: set $U \subseteq S$ open iff $F^{-1}(U) \subseteq M$ open $(F^{-1}(U \cap V) = F^{-1}(U) \cap F^{-1}(V), F^{-1}(U \cup V) = F^{-1}(U) \cup F^{-1}(V))$.
Define smooth structure of S: $\{F(U), \varphi \circ F^{-1} | (U, \varphi) \in \text{atlas for } M$, i.e. (U, φ) any smooth chart of $M\}$.
"smooth compatibility condition":

$$(\varphi_2 \circ F^{-1})(\varphi_i F^{-1})^{-1} = \varphi_2 \circ F^{-1} F \varphi_1^{-1} = \varphi_2 \varphi_1^{-1}$$

since $\varphi_2\varphi_1^{-1}$ diffeomorphism ($\varphi_2\varphi_1^{-1}$ bijection and it and inverse is differentiable)
F diffeomorphism onto $F(M)$.
and these are the only topology and smooth structure on S with this property:

$$S \xrightarrow{F^{-1}} M \xrightarrow{F} N \qquad = \qquad S \hookrightarrow M$$

and F^{-1} diffeomorphism, F smooth immersion, so $i: S \to M$ smooth immersion. □

Jeffrey Lee (2009) [2]

5. Tensors

I'll go through Ch.7 *Tensors* of Jeffrey Lee (2009) [2].

Definition 6 (7.1[2])**.** *Let V, W be modules over commutative ring R, with unity.*
Then, algebraic W-valued tensor on V is multilinear map.

$$\tau: V_1 \times V_2 \times \dots \times V_m \to W \tag{10}$$

where $V_i = \{V, V^*\} \qquad \forall\, i = 1, 2, \dots m$.
If for r, s s.t. $r + s = m$, there are $r \quad V_i = V^*$, $s\, V_i = V$, tensor is r-contravariant, s-covariant; also say tensor of total type $\binom{r}{s}$.

EY : 20170404 Note that

$$(\tau^{i\alpha}_\beta \frac{\partial}{\partial x^i} \text{ or } \tau^{i\alpha}_\beta e_i)(\omega_j dx^j \text{ or } \omega_j e^j \in V^*)$$
$$(\tau^\beta_{i\alpha} dx^i \text{ or } \tau^\beta_{i\alpha} e^i)(X^j \frac{\partial}{\partial x^j} \text{ or } X^j e_j \in V)$$

$\exists$ natural map $\begin{aligned} V &\to V^{**}, \\ v &\mapsto \widetilde{v} \end{aligned}$ $\widetilde{v}: \alpha \mapsto \alpha(v)$. If this map is an isomorphism, V is **reflexive** module, and identify V with V^{**}.

Exercise 7.5. Given vector bundle $\pi: E \to M$, open $U \subset M$, consider sections of π on U, i.e. cont. $s: U \to E$, where $(\pi \circ s)(u) = u, \quad \forall\, u \in U$.
Consider $E^* \ni \omega = \omega_i e^i$.
$\forall\, s \in \Gamma(E)$, $\omega(s) = \omega_i(s(x))^i, \quad \forall\, x \in U \subset M$. So define $\widetilde{s}: \omega, x \mapsto \omega(s(x)), \quad \forall\, x \in U$.
If $\widetilde{s} = 0$, $\widetilde{s}(\omega, x) = \omega(s(x)) = 0 \qquad \forall \omega \in E^*, \forall\, x \in U$, and so $s = 0$. (Let $\omega_i = \delta_{iJ}$ for some J, and so $s^J(x) = 0 \qquad \forall\, J$).
$s = 0$. So $\ker(s \mapsto \widetilde{s}) = \{0\}$ (so condition for injectivity is fulfilled).
Since $\widetilde{s}: \omega, x \mapsto \omega(s(x))$, $\forall\, \omega \in E^*, \forall\, x \in U$, $s \mapsto \widetilde{s}$ is surjective.
$s \mapsto \widetilde{s}$ is an isomorphism so $\Gamma(E)$ is a *reflexive* module.

Proposition 3. *For R a ring (special case), $\exists$ module homomorphism:*

tensor product space $\to$ tensor, as a multilinear map, i.e. $\exists$

$$\begin{gathered} (\otimes^r_{i=1} V) \otimes \left(\otimes^s_{j=1} V^*\right) \to T^r_s(V; R) \\ u_1 \otimes \dots \otimes u_r \otimes \beta^1 \otimes \dots \otimes \beta^s \in (\otimes^r V) \otimes (\otimes^s V^*) \mapsto (\alpha^1 \dots \alpha^r, v_1 \dots v_s) \mapsto \alpha^1(u_1) \dots \alpha^r(u_r)\beta^1(v_1) \dots \beta^s(v_s) \end{gathered} \tag{11}$$

Indeed, consider

$$(\alpha^1 \dots \alpha^r, v_1 \dots v_s) \in \underbrace{V^* \times \dots \times V^*}_{r} \times \underbrace{V \times \dots \times V}_{s} \mapsto \alpha^1(u_1) \dots \alpha^r(u_r)\beta^1(v_1) \dots \beta^s(v_s)$$

and so for

$$\begin{aligned} \alpha^i &= \alpha^i_\mu e^\mu, \quad i = 1, 2, \dots r,\ \mu = 1, 2, \dots \text{dim}V^* \qquad & \alpha^i(u_i) &= \alpha^i_\mu u^\mu_i \\ v_i &= v^\mu_i e_\mu, \quad i = 1, 2, \dots s,\ \mu = 1, 2, \dots \text{dim}V \qquad & \beta^i(v_i) &= \beta^i_\mu v^\mu_i \end{aligned}$$

So that

$$\begin{aligned} \alpha^1(u_1) \dots \alpha^r(u_r)\beta^1(v_1) \dots \beta^s(v_s) &= \alpha^1_{\alpha_1} u_1^{\alpha_1} \dots \alpha^r_{\alpha_r} u_r^{\alpha_r} \beta^1_{\mu_1} v_1^{\mu_1} \dots \beta^s_{\mu_s} v_s^{\mu_s} = \\ &= (u_1^{\alpha_1} \dots u_r^{\alpha_r} \beta^1_{\mu_1} \dots \beta^s_{\mu_s})(\alpha^1_{\alpha_1} \dots \alpha^r_{\alpha_r} v_1^{\mu_1} \dots v_s^{\mu_s}) \end{aligned}$$

Identify $u_1 \otimes \dots \otimes u_r \otimes \beta^1 \otimes \dots \otimes \beta^s$ with this multiplinear map.

Proposition 4. *If V is finite-dim. vector space, or if $V = \Gamma(E)$, for vector bundle $E \to M$, map*

$$(\otimes^r_{i=1} V) \otimes \left(\otimes^s_{j=1} V^*\right) \to T^r_s(V; R) \tag{12}$$

is an isomorphism.

Definition 7. *tensor that can be written as*

$$u_1 \otimes \dots \otimes u_r \otimes \beta^1 \otimes \dots \otimes \beta^s \equiv u_1 \otimes \dots \otimes \beta^s \tag{13}$$

*is **simple** or **decomposable**.*

Now well that not *all* tensors are simple.

Definition 8 (7.7[2], tensor product)**.** $\forall\, S \in T^{r_1}_{s_1}(V), \forall\, T \in T^{r_2}_{s_2}(V)$,
define tensor product

$$\begin{gathered} S \otimes T \in T^{r_1 + r_2}_{s_1 + s_2}(V) \\ S \otimes T(\theta^1 \dots \theta^{r_1 + r_2}, v_1 \dots v_{s_1 + s_2}) := S(\theta^1 \dots \theta^{r_1}, v_1 \dots v_{s_1}) T(\theta^{r_1 + 1} \dots \theta^{r_1 + r_2}, v_{s_1 + 1} \dots v_{s_1 + s_2}) \end{gathered} \tag{14}$$

Proposition 5 (7.8[2])**.**

$$\tau^{i_1 \dots i_r}{}_{j_1 \dots j_s} e_{i_1} \otimes \dots \otimes e_{i_r} \otimes e^{j_1} \otimes \dots \otimes e^{j_s} = \tau(e^{i_1} \dots e^{i_r}, e_{j_1} \dots e_{j_s}) e_{i_1} \otimes \dots \otimes e_{i_r} \otimes e^{j_1} \otimes \dots \otimes e^{j_s} = \tau$$

So $\{ e_{i_1} \otimes \dots \otimes e_{i_r} \otimes e^{j_1} \otimes \dots \otimes e^{j_s} | i_1 \dots i_r, j_1 \dots j_s \in 1 \dots n \}$ spans $T^r_s(V;R)$

Exercise 7.11. Let basis for V $e_1 \dots e_n$, corresponding dual basis for V^* $e^1 \dots e^n$
Let basis for V $\overline{e}_1 \dots \overline{e}_n$, corresponding dual basis for V^* $\overline{e}^1 \dots \overline{e}^n$
s.t.

$$\begin{aligned} \overline{e}_i &= C^k{}_i e_k \\ \overline{e}^i &= (C^{-1})^i{}_k e^k \end{aligned}$$

EY:20170404, keep in mind that

$$\begin{gathered} Ax = e_i A^i{}_k e^k(x^j e_j) = e_i A^i{}_j x^j = A^i{}_j x^j e_i \\ A e_j = e_k A^k{}_i e^i(e_j) = A^k{}_j e_k = \overline{e}_j \\ \overline{\tau}^i{}_{jk} \overline{e}_i \otimes \overline{e}^j \otimes \overline{e}^k = \overline{\tau}^i{}_{jk} C^l{}_i e_l (C^{-1})^j{}_m e^m (C^{-1})^k{}_n e^n = \overline{\tau}^i{}_{jk} C^l{}_i (C^{-1})^j{}_m (C^{-1})^k{}_n = \tau^l{}_{mn} \\ \overline{\tau}^i{}_{jk} = C^c{}_k C^b{}_j (C^{-1})^i{}_a \tau^a{}_{bc} \end{gathered}$$

On Remark 7.13 of Jeffrey Lee (2009) [2]: first, egregious typo for $L(V,V)$; it shoudl be $L(V,W)$. Onward, for $L(V,W)$,
consider $W \otimes V^* \ni w \otimes \alpha$ s.t.

$$(w \otimes \alpha)(v) = \alpha(v) w \in W, \forall v \in V, \text{ so } w \otimes \alpha \in L(V,W)$$

Now consider (category of) left R-module,

$$_R\mathbf{Mod} \ni {}_{\mathrm{Mat}_{\mathbb{K}}(N,M)}\mathbb{K}^N \tag{15}$$

where

$$\begin{aligned} V &= \mathbb{K}^N \\ W &= \mathbb{K}^M \end{aligned}$$

For $A \in \mathrm{Mat}_{\mathbb{K}}(N,M)$, $x \in \mathbb{K}^N$,

$$\begin{gathered} e_i A^i{}_{,\mu} e^\mu (x^\nu e_\nu) = Ax = e_i A^i{}_\mu x^\mu, \quad i = 1, 2, \dots M, \, \mu = 1, 2, \dots N \\ A \in \mathrm{Mat}_{\mathbb{K}}(N,M) \cong W \otimes V^* \cong L(V,W) \end{gathered}$$

Consider

$$\begin{aligned} \alpha &\in (\mathbb{K}^N)^* = V^* & \alpha &= \alpha_\mu e^\mu \\ w &\in \mathbb{K}^M = W & w &= w^i e_i \\ \alpha \otimes w &= w \otimes \alpha = w^i \alpha_\mu e_i \otimes e^\mu \end{aligned}$$

(remember, isomoprhism between $\mathrm{Mat}_{\mathbb{K}}(N,M)$ and $W \otimes V^*$ guaranteed, if V, W are free R-modules, $R = \mathbb{K}$).
Let V, W be left R-modules, i.e. $V, W \in {}_R\mathbf{Mod}$.

$$V^* \in \mathbf{Mod}_R$$

For $V^* \otimes W \in \mathbf{Mod}_R \otimes {}_R\mathbf{Mod}$

$$\begin{gathered} \alpha \in V^*, w \in W \\ (\alpha \otimes w)(v) = \alpha(v) w, \text{ for } v \in V \in {}_R\mathbf{Mod} \end{gathered}$$

But $(w \otimes \alpha)(v) = w\alpha(v)$.
Note $\alpha(v) \in R$.
Let V, W be right R-modules, i.e. $V, W \in \mathbf{Mod}_R$.

$$V^* \in {}_R\mathbf{Mod}$$

For $W \otimes V^* \in \mathbf{Mod}_R \otimes {}_R\mathbf{Mod}$.

$$\begin{gathered} \alpha \in V^*, \, w \in W \\ (v)(w \otimes \alpha) = w\alpha(v), \text{ with } \alpha(v) \in R, \, v \in V \end{gathered}$$

So $W \otimes V^* \cong L(V,W)$, for $V, W \in \mathbf{Mod}_R$

Definition 9 (7.20[2], **contraction**)**.** *Let $(e_1, \dots e_n)$ basis for V, $(e^1 \dots e^n)$ dual basis.*
If $\tau \in T^r_s(V)$, then for $k \leq r$, $l \leq s$, define

$$\begin{gathered} C^k_l \tau \in T^{r-1}_{s-1}(V) \\ C^k_l \tau(\theta^1 \dots \theta^{r-1}, w_1 \dots w_{s-1}) := \\ \sum_{a=1}^n \tau(\theta^1 \dots \underbrace{e^a}_{kth\ position} \dots \theta^{r-1}, w_1 \dots \underbrace{e_a}_{ith\ position} \dots w_{s-1}) \end{gathered} \tag{16}$$

*C^k_l is called **contraction**, for some single $1 \leq k \leq r$, some single $1 \leq l \leq s$,*

$$C^k_l : T^r_s(V) \to T^{r-1}_{s-1}(V)$$

s.t.

$$(C^k_l \tau)^{i_1 \dots \widehat{i}_k \dots i_r}{}_{j_1 \dots \widehat{j}_l \dots j_s} := \tau^{i_1 \dots a \dots i_r}{}_{j_1 \dots a \dots j_s}$$

Universal mapping properties can be invoked to give a basis free definition of contraction (EY : 20170405???).
IN general,

$$\forall v_1 \dots v_s \in V, \forall \alpha^1 \dots \alpha^r \in V^*$$

so that

$$\begin{aligned} v_j &= v^\mu_j e_\mu & j &= 1 \dots s, & \mu &= 1, \dots \mathrm{dim} V \\ \alpha^i &= \alpha^i_\mu e^\mu & i &= 1 \dots r, & \mu &= 1 \dots \mathrm{dim} V^* \end{aligned}$$

then $\forall \tau \in T^r_s(V)$,

$$\begin{gathered} \tau(\alpha^1 \dots \alpha^r, v_1 \dots v_s) = \tau(\alpha^1_{\mu_1} e^{\mu_1} \dots \alpha^r_{\mu_r} e^{\mu_r}, v^{\nu_1}_1 e_{\nu_1} \dots v^{\nu_s}_s e_{\nu_s}) = \\ = \alpha^1_{\mu_1} \dots \alpha^r_{\mu_r} v^{\nu_1}_1 \dots v^{\nu_s}_s \tau(e^{\mu_1} \dots e^{\mu_r}, e_{\nu_1} \dots e_{\nu_s}) = \alpha^1_{\mu_1} \dots \alpha^r_{\mu_r} v^{\nu_1}_1 \dots v^{\nu_s}_s \tau^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s} \end{gathered}$$

which is equivalent to

$$\begin{tikzcd} \tau \in T^r_s(V) \arrow[r, mapsto, "\alpha^1 \otimes \dots \otimes \alpha^r \otimes v_1 \otimes \dots \otimes v_s \otimes"] & \alpha^1 \otimes \dots \otimes \alpha^r \otimes v_1 \otimes \dots \otimes v_s \otimes \tau \arrow[d, mapsto, "C^1_{s+1} C^2_{s+2} \dots C^r_{r+s} C^r_1 C^{r+1}_2 \dots C^{r+s}_s"'] \\ & \tau(\alpha^1 \dots \alpha^r, v_1 \dots v_s) \in R \end{tikzcd}$$

where I've tried to express the right-R-module, "right action" on $\alpha^1 \otimes \dots \otimes \alpha^r \otimes v_1 \otimes \dots \otimes v_s \in V^* \otimes \dots \otimes V$.
Conlon (2008) [12]

Part 4. Prástaro

Prástaro (1996) [8]

5.0.1. *Affine Spaces.* cf. Sec. 1.2 - *Affine Spaces* of Prástaro (1996) [8]

Definition 10 (affine space).

$$\begin{gathered} \textit{affine space} \qquad (M, \mathbf{M}, \alpha) \\ \textit{with} \\ M \equiv \textit{ set (set of pts.)} \\ \mathbf{M} \equiv \textit{ vector space (space of free vectors)} \\ \alpha \equiv \mathbf{M} \times M \to M \equiv \textit{ translation operator} \\ \alpha : (v,p) \mapsto p' \equiv p + v \end{gathered} \tag{17}$$

*Note: α is a **transitive** action and without fixed pts. (free).*
i.e. $\forall p \in M$,

$\forall$ pt. $O \in M$, $\alpha : (v, O) \mapsto O' \equiv O + v$, $\alpha(\cdot, O) \equiv \alpha_O \equiv \alpha(O)$. $\alpha_O(v) = O' = O + \mathbf{v}$ $\quad \forall O' \in M, \exists \mathbf{v} \in \mathbf{M}$ s.t. $O' = O + \mathbf{v}$
$\Longrightarrow M \equiv \mathbf{M}$.
$\forall (O, \{e_i\})_{1 \leq i \leq n}$, where $\{e_i\}$ basis of $\mathbf{M}$, $M \equiv \mathbf{M} = \mathbb{R}^n$ so isomorphism $M \simeq \mathbb{R}^n$

Definition 11. $(O, \{e_i\}) \equiv$ *affine frame.*
$\forall$ *affine frame* $(O, \{e_i\})$, $\exists$ *coordinate system* $x^\alpha : M \to \mathbb{R}$,
where $x^\alpha(p)$ *is* α*th component, in basis* $\{e_i\}$*, of vector* $p - O$

Theorem 10 (1.4 Prástaro (1996) [8]). *Let* $(x^\alpha), (\overline{a}^\alpha)$ *2 coordinate systems correspond to affine frames* $(O, \{e_i\})$, $(\overline{O}, \{\overline{e}_i\})$, *respectively.*

$$\overline{x}^\alpha = A^\alpha_{\ \beta} x^\beta + y^\alpha \tag{18}$$

where

$$y^\alpha \in \mathbb{R}^n, \qquad A^\alpha_{\ \beta} \in GL(n;\mathbb{R})$$

Definition 12 (1.10 Prástaro (1996) [8]).

$$A(n) \equiv Gl(n,\mathbb{R}) \times \mathbb{R}^n \tag{19}$$

affine group of dim. n

Theorem 11 (1.5). *symmetry group of n-dim. affine space, called affine group $A(M)$ of M. $\exists$ isomoprhism,*

$$A(M) \simeq A(n), \qquad f \mapsto (f^\alpha_{\ \beta}, y^\alpha)\,; \qquad f^\alpha \equiv x^\alpha \circ f = f^\alpha_{\ \beta} x^\beta + y^\alpha \tag{20}$$

cf. Eq. 1.4 Prástaro (1996) [8]

Part 5. Holonomy

Definition 13 (Conlon, 10.1.2). *If $X, Y \in \mathfrak{X}(M)$, $M \subset \mathbb{R}^m$, **Levi-Civita connection** on $M \subset \mathbb{R}^m$*

$$\begin{gathered} \nabla : \mathfrak{X}(M) : \mathfrak{X}(M) \to \mathfrak{X}(M) \\ \nabla_X Y := p(D_X Y) \end{gathered} \tag{21}$$

with

$$D_X Y := \sum_{j=1}^m X(Y^j) \frac{\partial}{\partial x^j} = \sum_{i,j=1}^m X^i \frac{\partial Y^j}{\partial x^i} \frac{\partial}{\partial x^j} \qquad \begin{aligned} \forall X &= \sum_{i=1}^m X^i \frac{\partial}{\partial x^i}, \\ \forall Y &= \sum_{i=1}^m Y^i \frac{\partial}{\partial x^i} \end{aligned}$$

$$\nabla_{fX} Y = f(D_{fX} Y) = p(fD_X Y) = fpD_X Y = f\nabla_X Y$$

$$\nabla_X fY = p(D_X fY) = p\left(\sum_{i,j=1}^m \left(X^i f \frac{\partial Y^j}{dx^i} + X^i Y^j \frac{\partial f}{\partial x^i} \right) \frac{\partial}{\partial x^j} \right) = f\nabla_X Y + p\sum_{j=1}^m X(f) Y^j \frac{\partial}{\partial x^j} = f\nabla_X Y + X(f)p(Y)$$

Definition 14 (Conlon, 10.1.4; Christoffel symbols).

$$\begin{aligned} &\nabla_{\frac{\partial}{\partial x^i}} \tfrac{\partial}{\partial x^j} = \Gamma^k_{ij} \tfrac{\partial}{\partial x^k} \qquad \textit{(Conlon's notation)} \\ &\nabla_{\frac{\partial}{\partial x^j}} \tfrac{\partial}{\partial x^i} = \Gamma^k_{ij} \tfrac{\partial}{\partial x^k} \qquad \textit{(F. Schuller's notation)} \end{aligned} \tag{22}$$

Definition 15 (torsion).

$$\begin{aligned} &T : \mathfrak{X}(M) \in \mathfrak{X}(M) \to \mathfrak{X}(M) \\ &T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] \end{aligned} \tag{23}$$

If $T = 0$, ∇ torsion-free or symmetric.

$$\begin{gathered} T(fX,Y) = f\nabla_X Y - (f\nabla_Y X + Y(f)X) - \{(fXY - (Y(f)X + fYX)\} = fT(X,Y) \\ T(X,fY) = f\nabla_X Y + X(f)Y - f\nabla_Y X - \{((X(f)Y + fXY) - fYX\} = fT(X,Y) \end{gathered}$$

Thus, $T(X,Y)$ $C^\infty(M)$-bilinear.
$T \in \tau^2_1(M)$.
$T(v,w) \in T_x M$ defined, $\forall v, w \in T_x M$, $\forall x \in M$.
Thus, torsion is a **tensor**.

Exercise 10.1.7 Conlon (2008)[12] . .
If $T(X,Y) = 0$,

$$T(e_i, e_j) = \Gamma^k_{ji} e_k - \Gamma^k_{ij} e_k - 0 = 0 \Longrightarrow \Gamma^k_{ji} = \Gamma^k_{ij}$$

If $\Gamma^k_{ij} = \Gamma^k_{ji}$, $T(e_i, e_j) = 0$.

Exercise 10.1.8, Conlon (2008)[12].
If $M \subset \mathbb{R}^m$ smoothly embedded submanifold, $\forall \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^i} \in T_x M$, spanning $T_x M$, consider $\frac{\partial}{\partial x^j} = X^k_j \frac{\partial}{\partial \widetilde{x}^k}$, $\frac{\partial}{\partial x^i} = X^k_i(\widetilde{x}) \frac{\partial}{\partial \widetilde{x}^k}$

$$\begin{gathered} \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i} = pD_{X^k_j \frac{\partial}{\partial \widetilde{x}^k}} X^l_i \frac{\partial}{\partial \widetilde{x}^l} = p\left(X^k_j \frac{\partial X^l_i}{\partial \widetilde{x}^k} \frac{\partial}{\partial \widetilde{x}^l} \right) = X^k_j p\left(\frac{\partial X^l_i}{\partial \widetilde{x}^k} \frac{\partial}{\partial \widetilde{x}^l} \right) \\ \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = X^k_i p\left(\frac{\partial X^l_j}{\partial \widetilde{x}^k} \frac{\partial}{\partial \widetilde{x}^l} \right) \end{gathered}$$

If $X \in \mathfrak{X}(M)$, smooth $s : [a,b] \to M$,
then $\forall s(t)$,

$$X'_{s(t)} = \nabla_{\dot{s}(t)} X \in T_{s(t)} M$$

In fact, it's often natural to consider fields $X_{s(t)}$ along s, parametrized by parameter t, allowing

$$X_{s(t_1)} \neq X_{s(t_2)}$$

each of $s(t_1) = s(t_2)$.

Definition 16 (10.1.9)**.** *Let smooth $s : [a,b] \to M$.*
Vector field along s is smooth $v : [a,b] \to TM$ s.t.

$$\begin{tikzcd} & & TM \arrow[d, "\pi"] \\ {[a,b]} \arrow[urr, mapsto, "v"] \arrow[rr, mapsto, "s"] & & M \end{tikzcd}$$

commutes.
Note that $v \in \mathfrak{X}(s) \subset \mathfrak{X}(M)$

e.g. $(Y|s)(t) = Y_{s(t)}$, restriction of $Y \in \mathfrak{X}(M)$ to s.
e.g. $\dot{s}(t) \in \mathfrak{X}(M)$.
$\forall \, v, w \in \mathfrak{X}(s)$, $v + w \in \mathfrak{X}(s)$,

$$(fv + gv)(t) := (f(s(t)) + g(s(t)))v(t) = f(s(t))v(t) + g(s(t))v(t) = (f+g)v(t)$$

Likewise,

$$f(v+w) = fv + fw$$

$\mathfrak{X}(s)$ is a real vector space and $C^\infty[a,b]$-module.

Definition 17 (10.1.10)**.** *Let conection ∇ on M.*
Associated covariant derivative *is operator*

$$\frac{\nabla}{dt}\mathfrak{X}(s) \to \mathfrak{X}(s)$$

$\forall$ *smooth s on M, s.t.*

(1) $\frac{\nabla}{dt}$ $\mathbb{R}$-*linear*
(2) $\left(\frac{\nabla}{dt}\right)(fv) = \frac{df}{dt}v + f\frac{\nabla}{dt}v$, $\forall \, f \in C^\infty[a,b]$, $\forall \, v \in \mathfrak{X}(s)$
(3) *If $Y \in \mathfrak{X}(M)$, then*

$$\frac{\nabla}{dt}(Y|s)(t) = \nabla_{\dot{s}(t)} Y \in T_{s(t)}M, \quad a \leq t \leq b$$

Theorem 12 (Conlon Thm. 10.1.11[12])**.** $\forall$ *connection ∇ on M, $\exists!$ associated covariant derivative $\frac{\nabla}{dt}$*

Proof. Consider arbitrary coordinate chart $(U, x^1 \dots x^n)$.
Consider smooth curve $s : [a,b] \to U$.
Let $v \in \mathfrak{X}(s)$, $v(t) = v^i(t)\frac{\partial}{\partial x^i}$; $\dot{s}(t) = s^j \frac{\partial}{\partial x^j}$.

$$\frac{\nabla v}{dt} = \frac{dv^i(t)}{dt}\frac{\partial}{\partial x^i} + v^i(t)\frac{\nabla}{dt}\frac{\partial}{\partial x^i} = \frac{dv^i}{dt}\frac{\partial}{\partial x^i} + v^i \nabla_{\dot{s}(t)}\frac{\partial}{\partial x^i} = \dot{v}^i \frac{\partial}{\partial x^i} + v^i \dot{s}^j \Gamma^k_{ij}\frac{\partial}{\partial x^k} = \left(\dot{v}^k + v^i \dot{s}^j \Gamma^k_{ij}\right)\frac{\partial}{\partial x^k}$$

This is an explicit, local formula in terms of connection, proving uniqueness.
Existence: $\forall$ coordinate chart $(U, x^1 \dots x^n)$, $\left(\dot{v}^k + v^i \dot{s}^j \Gamma^k_{ij}\right)\frac{\partial}{\partial x^k} =: \frac{\nabla v}{dt}$.

$$\frac{\nabla}{dt}(fv) = \dot{f}v^k + f\dot{v}^k + fv^i\dot{s}^j = \dot{f}v + f\frac{\nabla v}{dt}$$

If f constant, then $\frac{\nabla}{dt}$ is $\mathbb{R}$-linear. $\square$

Definition 18 (10.1.12 Conlon (2008)[12])**.** *Let (M, ∇). Let $v \in \mathfrak{X}(s)$ for smooth $s : [a,b] \to M$.*
If $\frac{\nabla v}{dt} \equiv 0$ on s, then v is ***parallel*** *along s.*

Theorem 13 (10.1.13)**.** *Let (M, ∇), smooth $s : [a,b] \to M$, $c \in [a,b]$, $v_0 \in T_{s(c)}M$.*
Then $\exists!$ parallel field $v \in \mathfrak{X}(s)$ s.t. $v(c) = v_0$.
v parallel transport along s.

Proof.

$$\begin{aligned} \dot{s}(t) &= \dot{s}^j(t)e_j \\ v(t) &= v^i(t)e_i \\ v_0 &= a^i e_i \\ 0 &= \left(\frac{dv^k}{dt}(t) + v^i(t)\dot{s}^j(t)\Gamma^k_{ij}(s(t))\right)e_k \end{aligned}$$

or equivalently

$$\frac{dv^k}{dt} = -v^i \dot{s}^j \Gamma^k_{ij}, \qquad 1 \leq k \leq n \qquad (10.1) \tag{24}$$

with initial conditions $v^k(c) = a^k$, $1 \leq k \leq n$.
By existence and uniqueness of solutions of O.D.E.
$\exists \, \epsilon > 0$ s.t. $\exists!$ solutions $v^k(t)$. For $c - \epsilon < t < c + \epsilon$.
In fact, these ODEs being linear in v^k, by ODE theory (Appendix C, Thm. C.4.1).
$\nexists$ restriction on ϵ, so $\exists! \, v^k(t) \quad \forall \, t \in [a,b]$, $1 \leq k \leq n$

$\square$

5.0.2. *Principal bundle, vector bundle case for parallel transport.* Recall the 2 different forms or viewpoints for Lie-algebra valued 1-forms, or vector-valued 1-forms, or sections of 1-form-valued endomorphisms:

$$\omega^k_{\ i\mu}dx^\mu \equiv \omega^k_{\ i} \in \Omega^1(M, \mathfrak{gl}(n,\mathbb{F})) = \Gamma(\mathfrak{gl}(n, \mathbb{R} \otimes T^*M|_U))$$

for $i, k = 1 \dots n = \dim E$.
$\mu = 1 \dots d = \dim E$
Now

$$D_X\mu = X^\mu D_{\frac{\partial}{\partial x^\mu}}\mu = X^\mu\left[\left(\frac{\partial}{\partial x^\mu}\mu^k\right)e_k + \mu^i \omega^k_{\ i\mu}e_k\right] = \left(X(\mu^k) + \mu^i\omega^k_{\ i}(X)\right)e_k = \left(d\mu^k(X) + \mu^i\omega^k_{\ i}(X)\right)e_k$$

So then define

$$\begin{aligned} &D : \Gamma(E) \to \Gamma(E) \otimes \Gamma(T^*M) \\ &D\mu = D(\mu^i e_i) = e_k(d\mu^k + \mu^i\omega^k_{\ i}) \equiv (d + A)\mu \end{aligned} \tag{25}$$

Also, D can be defined for this case:

$$D : \Gamma(\mathrm{End}(E)) \to \Gamma(\mathrm{End}E) \otimes \Gamma(T^*M)$$

Let $\sigma = \sigma^i_{\ j} e_i \otimes e^j \in \Gamma(\mathrm{End}(E))$

$$\begin{aligned} D\sigma = D(\sigma^i_{\ j}e_i) \otimes e^j + \sigma^i_{\ j}e_i \otimes D^*e^j = \left(d\sigma^k_{\ j} + \sigma^i A^k_{\ i}\right)e_k \otimes e^j + \sigma^i_{\ j}e_i \otimes (A^*)^j_{\ k}e^k = \\ = (d\sigma^k_{\ j} + \sigma^i_{\ j}A^k_{\ i})e_k \otimes e^j + \sigma^k_{\ i}e_j \otimes (-A^i_{\ j})e^j = (d\sigma^k_{\ j} + [A,\sigma]^k_{\ j})e_k \otimes e^j \end{aligned} \tag{26}$$

cf. Def. 4.1.4 of Jost (2011), pp. 138.
For $\mu \in \Gamma(E)$, smooth $s : [a,b] \to M$, $X(t) = \dot{s}(t)$,

$$D_{\dot{s}(t)}\mu = \dot{s}^\mu D_{\frac{\partial}{\partial x^\mu}}\mu = \dot{s}^\mu\left[\frac{\partial \mu^k}{\partial x^\mu}e_k + \mu^i\omega^k_{\ i\mu}e_k\right] = \left[\dot{s}^\mu\frac{\partial\mu^k}{\partial x^\mu} + \dot{s}^\mu\mu^i\omega^k_{\ i\mu}\right]e_k = \frac{d}{dt}\mu(s(t)) + \mu^i\dot{s}^\mu\omega^k_{\ i\mu}e_k \tag{27}$$

Let $D_{\dot{s}(t)}\mu = 0$. Then,

(28)
$$\frac{d}{dt}\mu(s(t)) = -\mu^i \dot{s}^\mu \omega^k_{\ i\mu} e_k$$

Recall, given vector bundle $E \xrightarrow{\pi} N$, given $\varphi : M \to N$, then pullback

(29)
$$\varphi^* E \to M$$

i.e.

$$\begin{tikzcd}
\varphi^*E \arrow[d, "\psi"] & E \arrow[l, "\varphi^*"] \arrow[d, "\pi"] & (\varphi^*E)_x = E_{\varphi(x)} \\
M \arrow[r, "\varphi"] & N & x \in M \arrow[u, mapsto]
\end{tikzcd}$$

i.e. if $s \in \Gamma(E)$,

$$\varphi^* s = s \circ \varphi \in \Gamma(\varphi^* E)$$

Thus,

$$\begin{tikzcd}
\gamma^*E \arrow[d] & E \arrow[l, "\gamma^*"] \arrow[d, "\pi"] & (\varphi^*E)_c = E_{\gamma(c)} \\
{[a,b]} \arrow[r, "\gamma"] & M & c \in [a,b] \arrow[u, mapsto]
\end{tikzcd}$$

For

$$\dot{v}^k = -v^i \dot{s}^j \Gamma^k_{\ ij}$$
$$v^k(c) = v_0^k \qquad 1 \leq k \leq m$$
$$\dot{v} = -v^i \dot{s}^j \Gamma_{ij}$$
$$(v \dot{+} w) = -(v^i + w^i)\dot{s}^j \Gamma_{ij} (v+w)(c) = v(c) + w(c) = v_0 + w_0$$

so $v + w \in \mathfrak{X}(s)$ is parallel transport of $v_0 + w_0$.

Likewise, $\forall\, a \in \mathbb{F}$, $av \in \mathfrak{X}(s)$ is the parallel transport of av_0.

$$\dot{\mu}^k = -\mu^i \dot{s}^\mu \omega^k_{\ i\mu} = -\mu^i \omega^k_{\ i}(\dot{s}^\mu)$$

Suppose $\gamma^* E$ trivialized over $[a,b]$.

Closed interval is contractible, so this is always possible.

For chart (U, φ),

$$\begin{tikzcd}
\gamma^*E \arrow[d] & E \arrow[l, "\gamma^*"] \arrow[d, "\pi"] & E|_U \arrow[r, "\psi"] & U \times V \\
{[a,b]} \arrow[r, "\gamma"] & M & U \subset M \arrow[u, "\pi^{-1}"] \arrow[ur] &
\end{tikzcd}$$

Consider

$$\varphi : [a,b] \times V \to \gamma^* E$$
$$\varphi(t, \cdot) = \gamma^* \circ \psi^{-1}(\gamma(t), \cdot)$$

$\forall\, \mu \in \Gamma(E|_{x \in M})$,
$\mu = \mu^i e_i$.

$\varphi(t, e_i) = \epsilon_i$ is a basis for $\gamma^* E$.

$\forall\, \sigma \in \Gamma(\gamma^* E)$,

$$\sigma = \sigma^i \epsilon_i, \quad \sigma^i : [a,b] \to \mathbb{F}$$
$$\nabla_{\frac{\partial}{\partial x^\mu}} \sigma = \frac{\partial \sigma^k}{\partial x^\mu}\epsilon_k + \omega^k_{\ j\mu}\sigma^j \epsilon_k = \left(\frac{\partial \sigma^k}{\partial x^\mu} + \omega^k_{\ ,j\mu}\sigma^j \right)\epsilon_k$$
$$\nabla \sigma = \epsilon_k \otimes (d\sigma^k + \omega^k_{\ j\mu} dx^\mu \sigma^j) = \epsilon_k \otimes (d\sigma^k + \omega^k_{\ j}\sigma^j)$$
$$\nabla_{\frac{d}{dt}} \sigma = \epsilon_k \otimes \left(\frac{d\sigma^k}{dt} + \omega^k_{\ j\mu}\dot{x}^\mu \sigma^j \right)$$

Now

$$\frac{d}{dt} = \dot{x}^\nu \frac{\partial}{\partial x^\nu}$$

Then σ parallel along γ if

$$\frac{d\sigma^k}{dt} + \omega^k_{\ j\mu}\dot{x}^\mu \sigma^j = 0$$

Definition 19 (3.1.4 [13])**.** *Parallel transport along γ is*

(30)
$$P_\gamma : E_{\gamma(a)} \to E_{\gamma(b)}$$
$$P_\gamma(v) \mapsto \sigma(b)$$

where $\sigma \in \Gamma(\gamma^* E)$, σ unique and s.t. $\sigma(a) = v$.

Lemma 1 (10.1.16[12])**.** *holonomy*

$$h_s : T_x M \to T_{x_0} M$$

if ∇ around piecewise smooth loop s is a linear transformation.

Lemma 2 (10.1.18 Conlon (2008)[12])**.** *Let piecewise smooth loop $s : [a,b] \to M$ at x_0.*

Let weak reparametrization $\widetilde{s} = s \circ r : [c,d] \to M$.

If reparametrization is orientation-preserving, then $h_{\widetilde{s}} = h_s$,
If reparametrization is orientation-reversing, then $h_{\widetilde{s}} = h_s^{-1}$,

Proof. Without loss of generality, assume smooth s, r

$$\widetilde{s}(\tau) = s(r(\tau))$$
$$\widetilde{v}(\tau) = v(r(\tau))$$
$$\widetilde{w}^j(\tau) = \frac{dt}{d\tau}(\tau) u^j(r(\tau))$$
$$\frac{d\widetilde{v}^k}{d\tau}(\tau) = \frac{dr}{d\tau}(\tau)\frac{dv^k}{dt}(r(\tau))$$
$$\frac{d\widetilde{v}^k}{d\tau} = -\widetilde{v}^i \widetilde{w}^j \Gamma^k_{ij}$$

since

$$\frac{dv^k}{dt} = -v^i u^j \Gamma^k_{ij}; \qquad 1 \leq k \leq n$$
$$v^k(c) = a^k; \qquad 1 \leq k \leq a$$
$$\frac{dr}{d\tau}\frac{dv^k}{dt} = -v^i \frac{dr}{d\tau} u^j \Gamma^k_{\ ij} = \frac{d\widetilde{v}^k}{d\tau} = -\widetilde{v}^i \widetilde{w}^j \Gamma^k_{\ ij}$$

Thus, if $r(c) = a$, $r(d) = b$

$$h_{\widetilde{s}}(v_0) = \widetilde{v}(d) = v(b) = h_s(v_0)$$

If $r(c) = a$, $r(d) = b$, then

$$\widetilde{v}(c) = v(b) = h_s(v_0)$$

and

$$h_{\widetilde{s}}(h_s(v_0)) = h_{\widetilde{s}}(v(b)) = \widetilde{v}(d) = v(a) = v_0$$

At this point, I will switch to my notation because it clarified to me, at least, what was going on, in that a holonomy h_s is *invariant* under orientation-preserving reparametrization, and its inverse is well-defined.

For $\widetilde{s} = s \circ t : [c,d] \to M$,
piecewise smooth t is reparametrized, i.e.

$$t : [c,d] \to [a,b] \tag{31}$$

Now,

$$\begin{gathered} \frac{d}{d\tau}\widetilde{s}(\tau) = \frac{d}{d\tau}\widetilde{s}(t(\tau)) = \dot{s}(t)\frac{dt}{d\tau}(\tau) \equiv \dot{s}\frac{dt}{d\tau} \\ v^k(t) = v^k(t(\tau)) = v^k(\tau) \\ \frac{dv^k}{d\tau}(t(\tau)) = \frac{dv^k}{dt}\frac{dt}{d\tau} = \frac{dt}{d\tau}(-v^i(\tau)\dot{s}^j(t)\Gamma^k_{\ ij}) = -v^i(\tau)\frac{d\widetilde{s}^j}{d\tau}\Gamma^k_{\ ij} \end{gathered}$$

Consider

$$h_s(v_0) = v(b)$$

If $t(c) = a$,
$t(d) = b$

$$h_{\widetilde{s}}(v_0) = \widetilde{v}(d) = v(t(d)) = v(b) = h_s(v_0)$$

If $t(c) = b$,
$t(d) = a$

$$\begin{gathered} h_{\widetilde{s}}(h_s(v_0)) = h_{\widetilde{s}}(v(b)) = h_{\widetilde{s}}(v(t(c))) = h_{\widetilde{s}}(\widetilde{v}(c)) = \\ = \widetilde{v}(d) = v(t(d)) = v(a) = v_0 \end{gathered}$$

Thus,

$$\boxed{h_{\widetilde{s}} = h_s^{-1}}$$

□

I am working through Conlon (2008) [12] , Clarke and Santoro (2012) [13], and Schreiber and Waldorf (2007) , concurrently, for holonomy.

Part 6. Complex Manifolds

EY : 20170123 I don't see many good books on Complex Manifolds for physicists other than Nakahara's. I will supplement this section on Complex Manifolds with external links to the notes of other courses that I found useful to myself.

Complex Manifolds - Lecture Notes Koppensteiner (2010) [9]

Lectures on Riemannian Geometry, Part II: Complex Manifolds by Stefan Vandoren

Vandoren (2008) [10]

Part 7. Jets, Jet bundles, h-principle, h-Prinzipien

cf. Eliashberg and Misahchev (2002) [15]

cf. Ch. 1 Jets and Holonomy, Sec. 1.1 Maps and sections of Eliashberg and Misahchev (2002) [15].

Visualize $f : \mathbb{R}^n \to \mathbb{R}^q$ as graph $\Gamma_f \subset \mathbb{R}^n \times \mathbb{R}^q$.

Consider this graph as image of $\mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^q$, i.e.

$$x \mapsto (x, f(x))$$

$\mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^q$ is called section (by mathematicians),
$x \mapsto (x, f(x))$
is called *field* or $\mathbb{R}^q$-valued field (by physicists).

cf. Ch. 1 Jets and Holonomy, Sec. 1.2 Coordinate definition of jets of Eliashberg and Misahchev (2002) [15].

Definition 20 (r-jet)**.** *Given (smooth) $f : \mathbb{R}^n \to \mathbb{R}^q$, given $x \in \mathbb{R}^n$.*

r-jet of f at x - sequence of derivatives of f, up to order r, $\equiv$

$$J^r_f(x) = (f(x), f'(x) \dots f^{(r)}(x)) \tag{32}$$

$f^{(q)}$ consists of all partial derivatives $D^\alpha f$, $\alpha = (\alpha_1 \dots \alpha_n)$, $|\alpha| = \alpha_1 + \dots + \alpha_n = s$, ordered lexicographically.

e.g. $q = 1$,
$f : \mathbb{R}^n \to \mathbb{R}$.
1-jet of f at $x = J^1_f(x) = (f(x), f^{(1)}(x))$.

$$f^{(1)}(x) = \{D^\alpha f | \alpha = (\alpha_1 \dots \alpha_n), |\alpha| = \alpha_1 + \dots + \alpha_n = 1\} = \left(\frac{\partial f}{\partial x^1}, \frac{\partial f}{\partial x^2}, \dots \frac{\partial f}{\partial x^n}\right)$$

Let $d_r = d(n,r) =$ number of all partial derivatives D^α of order r of function $\mathbb{R}^n \to \mathbb{R}$.

Consider r-jet $J^r_f(x)$ of map $f : \mathbb{R}^n \to \mathbb{R}^q$ as pt. of space $\mathbb{R}^q \times \mathbb{R}^{qd_1} \times \mathbb{R}^{qd_2} \times \dots \times \mathbb{R}^{qd_r} = \mathbb{R}^{qN_r}$, where $N_r = N(n,r) = 1 + d_1 + d_2 + \dots + d_r$, i.e.

$$J^r_f(x) = (f(x), f^{(1)}(x), \dots f^{(r)}(x)) \in \mathbb{R}^q \times \mathbb{R}^{qd_1} \times \dots \times \mathbb{R}^{qd_r} = \mathbb{R}^{qN_r}$$

Exercise 1.

Given order r, consider n-tuple of (positive) integers $(r_1, r_2 \dots r_n)$ s.t. $r_1 + r_2 + \dots + r_n = r$, and $r_k \geq 0$.
Imagine $r_k =$ occupancy number, num ber of balls in kth cell. $(r_1 \dots r_n)$ describes a positive ocnfiguration of occupancy numbers, with indistinguishable balls; 2 distributions are distinguishable only if corresponding n-tuples $(r_1 \dots r_n)$ not identical.

Represent balls by stars, and indicate n cells by n spaces between $n+1$ bars.

With $n+1$ bars, r stars, 2 bars are fixed. $n-1$ bars and r stars to arrange linearly, so a total of $n-1+r$ objects to arrange. r stars indistinguishable amongst themselves, so choose r out of $n-1+r$ to be stars.

$$\Longrightarrow d_r = d(n,r) = \binom{n-1+r}{r} \tag{33}$$

Use *induction* (cf. Ch. 4 Binomial Coefficients).

$$N_0 = N(n,0) = \binom{n-1+0}{0} = 1$$

$$N_1 = N(n,1) = 1 + \binom{n-1+1}{1} = 1 + n = \frac{(n+1)!}{n!1!}$$

Induction step:

$$N_{r-1} = N(n, r-1) = \sum_{k=1}^{r-1} d_k + 1 = \binom{n+r-1}{r-1}$$

and so

$$N_r = N(n,r) = \sum_{k=1}^r d_k + 1 = \sum_{k=1}^r \binom{n-1+k}{k} + 1 = \sum_{k=1}^{r-1} \binom{n-1+k}{k} + \binom{n-1+r}{r} + 1 =$$
$$= \binom{n+r-1}{r-1} + \binom{n-1+r}{r} = \frac{(n+r-1)!}{(r-1)!n!} + \frac{(n-1+r)!}{r!(n-1)!} = \frac{(n+r)!}{n!r!} = \binom{n+r}{r}$$

$$\begin{tikzcd} \mathbb{R}^{qN_r} & & J^r_f(x) & \\ \mathbb{R}^n \arrow[u, "J^r_f"] \arrow[r, "f"] & \mathbb{R}^q & x \arrow[u, "J^r_f"] \arrow[r, mapsto, "f"] & f(x) \end{tikzcd}$$

Definition 21 (space of r-jets)**.** *space of r-jects of maps $\mathbb{R}^n \to \mathbb{R}^q$ or space of r-jets of sections $\mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^q \equiv$*

$$J^r(\mathbb{R}^n, \mathbb{R}^q) = \mathbb{R}^n \times \mathbb{R}^{qN_r} \equiv \mathbb{R}^n \times \mathbb{R}^q \times \mathbb{R}^{qd_1} \times \mathbb{R}^{qd_2} \times \dots \times \mathbb{R}^{qd_r} \quad (34)$$

e.g. $J^1(\mathbb{R}^n, \mathbb{R}^q) = \mathbb{R}^n \times \mathbb{R}^q \times M_{q\times n}$, where $M_{q\times n} = \mathbb{R}^{qn}$ is the space of $(q\times n)$-matrices.

Part 8. Morse Theory

6. Morse Theory introduction from a physicist

I needed some physical motivation to understand Morse theory, and so I looked at Hori, et. al. [11].

cf. pp. 43, Sec. 3.4 Morse Theory, from Ch. 3. Differential and Algebraic Topology of Hori, et. al. [11].

Consider smooth $f: M \to \mathbb{R}$, with non-degenerate critical points.

If no critical values of f between a and b ($a < b$), then subspace on which f takes values less than a is deformation retract of subspace where f less than b, i.e.

$$\{x \in M | f(x) < b\} \times [0,1] \xrightarrow{F} \{x \in M | f(x) < b\}$$

$\forall\, x \in M$ s.t. $f(x) < b$,

$$\begin{aligned} &F(x,0) = x \\ &F(x,1) \in \{x \in M | f(x) < a\} \end{aligned} \qquad \text{and } F(a',1) = a' \qquad \forall\, a' \in M \text{ s.t. } f(a') < a$$

To show this, consider $-\nabla f / |\nabla f|^2$

Morse lemma: $\forall$ critical pt. p s.t. $\exists$ choice of coordinates s.t.

$$f = -(x_1^2 + x_2^2 + \dots + x_\mu^2) + x_{\mu+1}^2 + \dots + x_n^2 \quad (35)$$

where $f(p) = 0$ and p is at origin of these coordinates.

- difference between
$$f^{-1}(\{x \leq -\epsilon\}),\ f^{-1}(\{x \leq +\epsilon\})$$
can be determined by local analysis and only depends on μ, $\mu \equiv$ "Morse index" = number of negative eigenvalues of Hessian of f at critical pt.
  Answer:
$$f^{-1}(\{x \leq +\epsilon\}) \text{ can be obtained from } f^{-1}(\{x \leq -\epsilon\}) \text{ by "attaching } \mu\text{-cell" along boundary } f^{-1}(0)$$
- "attaching μ-cell to X mean, take μ-ball $B_\mu = \{|x| \leq 1\}$ in μ-dim. space, identity pts. on boundary $S^{\mu-1}$ with pts. in the space X, through cont. $f: S^{\mu-1} \to X$, i.e. take
$$X \coprod B_\mu$$
with $x \sim f(x)$ $\forall\, x \in \partial B_\mu = S^{\mu-1}$.
- find homology of M,
  f defines chain complex C_f^*, kth graded piece C^{α_k}, α_k is number of critical pts. with index k.
$$\begin{aligned} &\partial : C_p^k \to C_p^{k-1} \\ &\partial x_a = \sum_b \Delta_{a,b} x_b \end{aligned} \quad (36)$$
where $\Delta_{a,b} :=$ signed number of lines of gradient flow from x_a to x_b, b labels pts. of index $k-1$.

  Gradient flow line is path $x(t)$ s.t. $\dot{x} = \nabla(f)$, with $\begin{aligned} x(-\infty) = x_a \\ x(+\infty) = x_b \end{aligned}$
- To define this number ($\Delta_{a,b}$?), construct moduli space of such lines of flow (???) by intersecting outward and inward flowing path spaces from each critical point, and then show this moduli space is oriented, 0-dim. manifold (pts. with signs)
- $\partial^2 = 0$ proof
  ∂, boundary of space of paths connecting critical points, whose index differs by 2 = union over compositions of paths between critical pts. whose index differs by 1.
  $\Longrightarrow$ coefficients of ∂^2 are sums of signs of pts. in 0-dim. space, which is boundary of 1-dim. space.
  These signs must therefore add to 0, so $\partial^2 = 0$.

Hori, et. al. [11] is good for physics, but there isn't much thorough, step-by-step explanations of the math. I will look at Hirsch (1997) [6] and Shastri (2011) [5] at the same time.

6.1. Introduction, definitions of Morse Functions, for Morse Theory.

cf. Ch. 6, Morse Theory of Hirsch (1997) [6], Section 1. Morse Functions, pp. 143-

Recall for TM, $T_xM \xrightarrow{\varphi} \mathbb{R}^n$.

Cotangent bundle T^*M defined likewise:

$$T_x^*M \xrightarrow{\varphi} \text{ dual vector space } (\mathbb{R}^n)^* = L(\mathbb{R}^n, \mathbb{R})$$

i.e.

$$T^*M = \bigcup_{x\in M} (M_x^*) \qquad M_x^* = L(M_x, \mathbb{R})$$

If chart (φ, U) on M, natural chart on T^*M is

$$\begin{aligned} &T^*U \to \varphi(U) \times (\mathbb{R}^n)^* \\ &\lambda \in M_x^* \mapsto (\varphi(x), \lambda\varphi_x^{-1}) \end{aligned}$$

Projection map

$$\begin{aligned} &p: T^* \to M \\ &M_x^* \mapsto x \end{aligned}$$

Let C^{r+1} map, $1 \leq r \leq \omega$, $f: M \to \mathbb{R}$, $\forall\, x \in M$, linear map $T_x f : M_x \to \mathbb{R}$ belongs to M_x^*

$$T_x f = Df_x \in M_x^*$$

Then

$$\begin{aligned} &Df : M \to T^*M \\ &x \mapsto Df_x = Df(x) \end{aligned}$$

is C^r section of T^*M.

Definition 22. ***critical point*** *x of f is zero of Df, i.e.*

$$Df(x) = 0 \tag{37}$$

of vector space M_x^*.

Thus, set of critical pts. of f is counter-image of submanifold $Z^* \subset T^*M$ of zeros.
Note $Z^* \approx M$, codim. of Z^* is $n = \dim M$.

Definition 23. ***Morse function*** *f if $\forall$ critical pts. of f are nondegenerate.*

Note set of critical pts. closed discrete subset of M.
Let open $U \subset \mathbb{R}^n$, let C^2 map $g : U \to \mathbb{R}$,
critical pt. $p \in U$ nondegenerate iff

- linear $D(Dg)(p) : \mathbb{R}^n \to (\mathbb{R}^n)^*$ bijective
- identify $L(\mathbb{R}^n, (\mathbb{R}^n)^*)$ with space of bilinear maps $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $\Longrightarrow$ equivalent to condition that symmetric bilinear $D^2 g(p) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ non-degenerate
- $n \times n$ *Hessian matrix*
$$\left[\frac{\partial^2 g}{\partial x^i \partial x^j}(p) \right]$$
has rank n

Hessian of g at critical pt. p is quadratic form $H_p f$ associated to bilinear form $D^2 g(p)$

$$\Longrightarrow H_p f(y) = D^2 g(p)(y,y) = \sum_{i,j} \frac{\partial^2 g}{\partial x^i \partial x^j}(p) y^i y^j$$

Let open $V \subset \mathbb{R}^n$, suppose C^2 diffeomorphism $h : V \to U$.
Let $q = h^{-1}(p)$, so q is critical pt. of $gh : V \to \mathbb{R}$.

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "H_q(gh)"] \arrow[d, "Dh(q)"'] & \mathbb{R} \\ \mathbb{R}^n \arrow[ur, "H_p g"'] & \end{tikzcd}$$

(quadratic) form $(H_p f)$ invariant under diffeomorphisms.
Let C^2 $f : M \to \mathbb{R}$.
$\forall$ critical pt. x of f, define
Hessian quadratic form

$$H_x f : M_x \to \mathbb{R}$$

$$H_x f : M_x \xrightarrow{D\varphi_x} \mathbb{R}^n \xrightarrow{H_{\varphi(x)}(f\varphi^{-1})} \mathbb{R}$$

where φ is any chart at x.
Thus, critical pt. of a C^2 real-valued function nondegenerate iff associated Hessian quadratic form is nondegenerate.
Let Q nondegenerate quadratic form on vector space E.
Q negative definite on subspace $F \subset E$ if $Q(x) < 0$ whenever $x \in F$ nonzero.
Index of $Q \equiv \text{Ind}Q$, is largest possible dim. of subspace on which Q is negative definite.
cf. 1.1. Morse's Lemma of Ch. 6, pp. 145, Morse Theory of Hirsch (1997) [6]

Lemma 3 (Morse's Lemma)**.** *Let $p \in M$ be nondegenerate critical pt. of index k of C^{r+2} map $f : M \to \mathbb{R}$, $1 \leq r \leq \omega$.*
Then $\exists C^r$ chart (φ, U) at p s.t.

$$f\varphi^{-1}(u_1 \dots u_n) = f(p) - \sum_{i=1}^k u_i^2 + \sum_{i=k+1}^n u_i^2 \tag{38}$$

Let ${}^T Q \equiv Q^T$ denote tranpose of matrix Q.

Lemma 4. *Let $A = diag\{a_1, \dots, a_n\}$ diagonal $n \times n$ matrix, with diagonal entries ± 1.*
Then $\exists$ neighborhood N of A in vector space of symmetric $n \times n$ matrices, C^∞ map

$$P : N \to GL(n, \mathbb{R}) \tag{39}$$

s.t. $P(A) = I$, and if $P(B) = Q$, then $Q^T B Q = A$

Proof. Let $B = [b_{ij}]$ be symmetri matrix near A s.t. $b11 \neq 0$ and b_{11} has same sign as a_1.
Consider $x = Ty$ where

$$\begin{gathered} x_1 = \left[y_1 - \frac{b_{12}}{b_{11}} y_2 - \dots - \frac{b_{1n}}{b_{11}} y_n \right] / \sqrt{|b_n|} \\ x_k = y_k \text{ for } k = 2, \dots n \end{gathered}$$

□

7. Lagrange multipliers

From *wikipedia:Lagrange multiplier*, `https://en.wikipedia.org/wiki/Lagrange_multiplier`, find local minima (maxima), pt. $a \in N$, s.t. $\exists$ neighborhood U s.t. $f(x) \geq f(a)$ $(f(x) \leq f(a))$ $\forall x \in U$.
For $f : U \to \mathbb{R}$, open $U \subset \mathbb{R}^n$, find $x \in U$ s.t. $D_x f \equiv Df(x) = 0$, check if Hessian $H_x f < 0$.
Maxima may not exit since U open.
References:
Relative Extrema and Lagrange Multipliers
Other interesting links:
The Lagrange Multiplier Rule on Manifolds and Optimal Control of nonlinear systems

Part 9. Classical Mechanics applications

cf. Arnold, Kozlov, Neishtadt (2006) [16].
If known forces $\mathbf{F}_1 \dots \mathbf{F}_n$ acts on points, then

$$\sum_{i=1}^n \langle m_i \ddot{\mathbf{r}}_i - \mathbf{F}_i, \xi_i \rangle = 0 \tag{40}$$

cf. Eq. (1.26) of Arnold, Kozlov, Neishtadt (2006) [16], where $\xi_1, \dots \xi_n$ are arbitrary tangent vectors to M, $\xi_i, \dots \xi_n \in TM$.
$\sum_{i=1}^n \langle m_i \ddot{\mathbf{r}}_i - \mathbf{F}_i, \xi_i \rangle$ called "general equation of dynamics" or d'Alembert-Lagrange principle.

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