SOLUTIONS TO *CALCULUS* VOLUME 2 *MULTI-VARIABLE CALCULUS AND LINEAR ALGEBRA, WITH APPLICATIONS TO DIFFERENTIAL EQUATIONS AND PROBABILITY* BY TOM APOSTOL

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1.5 EXERCISES - INTRODUCTION, THE DEFINITION OF A LINEAR SPACE, EXAMPLES OF LINEAR SPACES, ELEMENTARY CONSEQUENCES OF THE AXIOMS

Recall the following:

Definition 1 (Linear Space.)**.**
Let V be a nonempty set of objects.
Linear space if a set V that satisfies the following ten axioms.

(1) (closure under addition) $\forall x, y \in V, x + y \in V$
(2) (closure under scalar multiplication) $\forall x \in V, \alpha x \in V$
(3) (Additive commutativity) $\forall x, y \in V, x + y = y + x$
(4) (Additive Associativity) $\forall x, y \in V, (x + y) + z = x + (y + z)$
(5) (Additive Identity Existence) $\exists 0 \in V$ such that

$$x + 0 = x, \forall x \in V$$

(6) (Additive Inverse Existence) $\exists (-1)x$ such that

$$x + (-1)x = 0$$

(7) (Scalar Associativity) $\forall x \in V, \forall \alpha, \beta \in \mathbb{R}$ or $\alpha, \beta \in \mathbb{C}$

$$(\alpha\beta)x = \alpha(\beta x)$$

(8) (distributivity for addition in V) $\forall x, y \in V; \forall a \in \mathbb{R}$ or $\forall a \in \mathbb{C}$,

$$a(x + y) = ax + ay$$

(9) (distributivity for addition of numbers) $\forall x \in V, \forall a, b \in \mathbb{R}$ or $\forall a, b \in \mathbb{C}$,

$$(a + b)x = ax + bx$$

(10) (Multiplicative identity existence) $\forall x \in V, 1x = x$

Exercise 1. Consider $x = \frac{p}{q}, y = \frac{r}{s} \in V$ where p, q, r, s are polynomials. $ps + rq, qs$ are polynomials as well.

$$x + y = \frac{p}{q} + \frac{r}{s} = \frac{ps + rq}{qs} \in V$$

$$\alpha \in \mathcal{R}, \quad \alpha p \text{ is a polynomial}$$

$$\alpha x = \frac{\alpha p}{q} \in V$$

$$x + y = \frac{ps + rq}{qs} = \frac{rq + ps}{qs} = r + p$$

$$(x + y) + z = \left(\frac{p}{q} + \frac{r}{s} \right) + \frac{t}{v} = \frac{p}{q} + \left(\frac{r}{s} + \frac{t}{v} \right) = x + (y + z)$$

$$x + 0 = \frac{p}{q} + \frac{0}{q} = \frac{p}{q} = x \text{ so } \frac{0}{q} \in V \text{ if } q \neq 0$$

$$x + (-1)x = \frac{p}{q} + (-1)\frac{p}{q} = \frac{0}{q} = 0$$

$$(\alpha\beta)x = \frac{(\alpha\beta)p}{q} = \frac{\alpha(\beta p)}{q} = \alpha(\beta x) \text{ (follows from associativity of real or complex numbers)}$$

$$\alpha(x + y) = \alpha x + \alpha y \text{ and } (\alpha + \beta)x = \alpha x + \beta x \text{ follows from distributivity for real numbers}$$

$$\text{Consider } x = \frac{p}{q} = \left(\frac{q}{q} \right) \frac{p}{q} = (1)x, \ \frac{q}{q} \in V$$

Exercise 3. All f with $f(0) = f(1)$

$f(0) + g(0) = (f+g)(0) = f(1) + g(1) = (f+g)(1)$
$af(0) = (af)(0) = af(1) = (af)(1)$
$f(x) + g(x) = (f+g)(x) = g(x) + f(x) = (g+f)(x)$
$(f(x) + g(x)) + h(x) = ((f+g)+h)(x) = f(x) + (g(x) + h(x)) = (f + (g+h))(x)$
$0(x) = 0 \quad (f+0)(x) = f(x) + 0(x) = f(x)$
$(-1)f(x) = (-f)(x) \qquad (f + (-1)f)(x) = (f-f)(x) = f(x) + (-1)f(x) = 0 = 0(x)$
$(\alpha\beta)f(x) = \alpha(\beta f(x)) = \alpha(\beta f)(x)$
$a(f+g)(x) = a(f(x) + g(x)) = af(x) + ag(x) = (af + ag)(x)$
$(a+b)f(x) = af(x) + bf(x) = (af)(x) + (bf)(x) = (af + bf)(x)$
$(1f)(x) = f(x)$

Exercise 4. All f with $2f(0) = f(1)$

$2(f+g)(0) = 2f(0) + 2g(0) = f(1) + g(1) = (f+g)(1)$
$2(\alpha f)(0) = 2\alpha f(0) = \alpha f(1) = (\alpha f)(1)$
$f + g = g + f, (f+g) + h = f + (g+h)$ follow from properties of the reals.
$20(0) = 0 = 0(1); \quad (f+0)(x) = f(x) + 0 = f(x)$
$(f + (-f))(x) = f(x) + -f(x) = 0$
$(\alpha\beta)f(x) = \alpha(\beta f(x))$
$a(f+g)(x) = (af)(x) + (ag)(x), (a+b)f(x) = af(x) + bf(x)$ follow from properties of the reals.
$1x = x$

Exercise 5. All f with $f(1) = 1 + f(0)$

$$f(1) + g(1) = (f+g)(1) = 1 + f(0) + 1 + g(0) = 2 + (f+g)(0)$$

So closure under addition is violated.

Exercise 6. All step functions defined on $[0,1]$.

Exercise 8. Even functions, $f(-x) = f(x)$.

$$(f+g)(-x) = f(-x) + g(-x) = f(x) + g(x)$$
$$(\alpha f)(-x) = \alpha f(-x) = \alpha f(x) = (\alpha f)(x)$$
$$(f+g)(x) = f(x) + g(x) = g(x) + f(x) = (g+f)(x) \text{ (follows from commutativity of real numbers)}$$
$$((f+g)+h)(-x) = (f+g)(-x) + h(-x) = f(x) + g(x) + h(x) = f(x) + (g+h)(x) = (f + (g+h))(x)$$
$$\text{(follows from associativity of real numbers)}$$
$$\text{if } 0(-x) = 0(x) = 0 \, \forall x \in D \text{ so } 0 \text{ exists and } (f+0)(x) = f(x) + 0(x) = f(x)$$
$$(-f)(-x) = -f(-x) = -f(x) \text{ so } (f + -f)(x) = f(x) - f(x) = 0(x)$$
$$\alpha(\beta f)(-x) = \alpha(\beta(f(-x))) = \alpha(\beta f(x)) = \alpha(\beta f)(x)$$
$$((\alpha\beta)f)(-x) = (\alpha\beta)f(-x) = (\alpha\beta)f(x) = ((\alpha\beta)f)(x) \text{ (from associativity of the real numbers)}$$
$$\alpha(f+g)(x) = \alpha(f(x) + g(x)) = (\alpha f)(x) + (\alpha g)(x) = (\alpha f + \alpha g)(x)$$
$$(\alpha + \beta)f(x) = \alpha f(x) + \beta f(x) = (\alpha f)(x) + (\beta f)(x) = (\alpha f + \beta f)(x)$$
$$1f(x) = 1(f(x)) = f(x)$$

Exercise 22. All vectors (x, y, z) in V_3 with $z = 0$.

This space is closed under addition and scalar multiplication since

$$(x_1, y_1, 0) + (x_2, y_2, 0) = (x_1 + x_2, y_1 + y_2, 0)$$
$$k(x, y, 0) = (kx, ky, 0)$$

both belong to this space.

Additive commutativity, additive associativity, scalar associativity, distributivity in addition in V, and distributivity in addition of numbers are satisfied automatically, since all vectors in this subset belong to V_3, a linear space.
$(0,0,0)$ belongs to this space, since $z = 0$, so existence of an additive identity is fulfilled.

$(-x, -y, 0) = -(x, y, 0)$ belongs in this space and so the existence of an additive inverse for each element $(x, y, 0)$ in this space is fulfilled.

$1(x, y, 0) = (x, y, 0)$, and so multiplicative identity existence is fulfilled.
This is a linear space. Note that we could've also said that this space is exactly V_2, and V_2 is a linear space.
Exercise 23. All vectors (x, y, z) in V_3 with $x = 0$ or $y = 0$.

Consider $(0, y_1, z_1) + (x_2, 0, z_2) = (x_2, y_1, z_1 + z_2)$. This vector does not belong to this space. This is not a linear space.
Exercise 24. All vectors (x, y, z) in V_3 with $y = 5x$.

This space is closed under addition and scalar multiplication since

$$(x_1, 5x_1, 0) + (x_2, 5x_2, 0) = (x_1 + x_2, 5x_1 + 5x_2, 0) \Longrightarrow 5x_1 + 5x_2 = 5(x_1 + x_2)$$
$$k(x_1, 5x_1, z_1) \Longrightarrow k5x_1 = 5(kx_1)$$

Additive commutativity, additive associativity, scalar associativity, distributivity in addition in V, and distributivity in addition of numbers are satisfied automatically, since all vectors in this subset belong to V_3, a linear space.

$(0, 5(0), 0)$ belongs to this space, so existence of an additive identity is fulfilled.

$(-x, 5(-x), -z) = -(x, 5x, z)$ belongs in this space and so the existence of an additive inverse for each element in this space is fulfilled.

$1(x, 5x, z) = (x, 5x, z)$, and so multiplicative identity existence is fulfilled.
This is a linear space.
Exercise 25. All vectors (x, y, z) in V_3 with $3x + 4y = 1\, z = 0$.

Consider closure:

$$(x_1, y_1, 0) + (x_2, y_2, 0) = (x_1 + x_2, y_1 + y_2, 0)$$
$$z_3 = 0; \quad 3(x_1 + x_2) + 4(y_1 + y_2) = 2$$

Closure under addition is not satisfied. Thus, this is not a linear space.
Exercise 26. All vectors (x, y, z) in V_3 which are scalar multiples of $(1, 2, 3)$.

Closure is fulfilled since $x + y = a(1, 2, 3) + b(1, 2, 3) = (a + b)(1, 2, 3)$, which is a scalar multiple of $(1, 2, 3)$, and $ax = ab(1, 2, 3)$, which is another scalar multiple of $(1, 2, 3)$.

Additive commutativity, additive associativity, scalar associativity, distributivity in addition in V, and distributivity in addition of numbers are satisfied automatically, since all vectors in this subset belong to V_3, a linear space.

$\exists\, 0$ since $0(1, 2, 3) = 0$ is a scalar multiple of $(1, 2, 3)$.

$\exists - 1x \,\forall\, x$, since $(-1)a(1, 2, 3)$ is a scalar multiple of $(1, 2, 3)$.

$1(a(1, 2, 3)) = (a(1, 2, 3))$ is a scalar multiple of $(1, 2, 3)$ and so multiplicative identity existence is satisfied.
This is a linear space.
Exercise 28. All vectors in V_n that are linear combinations of 2 given vectors A and B.

$a_1A + b_1B + a_2A + b_2B = (a_1 + a_2)A + (b_1 + b_2)B$ belongs in this space.
$c(aA + bB) = (ca)A + (cb)B$ belongs in this space.
$x + y = y + x$, $(x + y) + z = x + (y + z)$ follow since the vectors belong in V_n, a linear space.
$0A + 0B = 0; \quad a_1A + b_1B + 0A + 0B = a_1A + b_1B$
$-a_1A + -b_1B$ belongs in this space and $a_1A + b_1B + -a_1A + -b_1B = (a_1 - a_1)A + (b_1 - b_1)B = 0$
$(ab)x = a(bx)$, $a(x + y) = ax + ay$, $(a + b)x = ax + bx$, $1x = x$ follow since the vectors in this space belong in V_n, a linear space.
Exercise 29. Let $V = \mathbb{R}^+$, let $x \text{``}+\text{''}\, y = xy$ and $a \text{``}\cdot\text{''}\, x = x^a$

$x \text{``}+\text{''}\, y = xy \in \mathbb{R}^+$
$a \text{``}\cdot\text{''}\, x = x^a = e^{a \ln x} > 0 \quad a \text{``}\cdot\text{''}\, x \in \mathbb{R}^+$
$x \text{``}+\text{''}\, y = xy = yx = y \text{``}+\text{''}\, x$
$(x \text{``}+\text{''}\, y) \text{``}+\text{''}\, z = xyz = x(yz) = x \text{``}+\text{''}\, (y \text{``}+\text{''}\, z)$
$x \text{``}+\text{''} \text{``}0\text{''} = x1 = x$ so $\text{''}0\text{''} = 1 \in \mathbb{R}^+$

$''(-1)x'' = \frac{1}{x} \in \mathbb{R}^+ \qquad x``+'' ``(-1)x'' = x\frac{1}{x} = 1 = ``0''$
$(ab)x = x^{ab} = (x^b)^a = a(bx)$
$a``\cdot'' (x``+'' y) = a``\cdot'' (xy) = (xy)^a = x^a y^a = (a``\cdot x)``+'' (a``\cdot'' y)$
$(a+b)``\cdot'' x = x^{a+b} = x^a x^b = x^a ``+'' x^b = a``\cdot'' x``+'' b``\cdot'' x$
$1``\cdot'' x = x^1 = x$
This is indeed a linear space.
Exercise 30.

(1) From Axiom 5,6, the Additive Identity Existence and Additive Inverse Existence, that $\exists\, 0 \in V$ s.t. $x + 0 = x$, $\forall\, x \in V$ and $\exists\, (-1)x$ s.t. $x + (-1)x = 0$, then, using associativity, commutativity, and distributivity for addition of numbers,
$$x + 0 = x = x + (x + (-1)x) = 2x + (-1)x = (2 + (-1))x = 1x$$
(2) If Ax.6 is replaced by Ax.6', $\forall\, x \in V, \quad \exists\, y \in V$ s.t. $x + y = 0$,
$$x + 0 = x = x + (x + y) = 2x + y = x$$
So Ax.10 does not hold since $2x + y = x$.

Exercise 31.

(1) $(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2) \qquad a(x_1, x_2) = (ax_1, 0)$.
Not a linear space: violates Additive Inverse existence, which demands $\exists\, (-1)x$ s.t. $x + (-1)x = 0$, not $\exists\, y$ s.t. $x + y = 0$, so that for $(-1)x = (-x_1, 0)$, $(x_1, x_2) + (-1)x = (0, x_2) \neq 0$ and Multiplicative identity existence, since $1x = (x_1, 0) \neq x$.

(2) $(x, 1, x_2) + (y_1, y_2) = (x_1 + y_1, 0) \qquad a(x_1, x_2) = (ax_1, ax_2)$
Not a linear space: violates distributivity in addition of numbers, because
$(a + b)x = ((a + b)x_1, (a + b)x_2)$ but $ax + bx = (ax_1 + bx_1, 0)$

(3) $(x, 1, x_2) + (y_1, y_2) = (x_1, x_2 + y_2) \qquad a(x_1, x_2) = (ax_1, ax_2)$
Not a linear space because it violates Additive Commutativity of elements. $(x_1, x_2) + (y_1, y_2) = (x_1, x_2 + y_2)$ but $(y_1, y_2) + (x_1, x_2) = (y_1, x_2 + y_2)$

(4) $(x, 1, x_2) + (y_1, y_2) = (|x_1 + y_1|, |y_1 + y_2|) \qquad a(x_1, x_2) = (|ax_1|, |ax_2|)$
Not a linear linear space because it violates Distributivity for addition of numbers:
$$(a + b)(x_1, x_2) = (|(a + b)x_1|, |(a + b)x_2|)$$
$$ax + bx = (|ax_1|, |ax_2|) + (|bx_1|, |bx_2|)$$
$$\text{but } |(a + b)x_1| \leq |ax_1| + |bx_1| \text{ in general}$$

Exercise 32. Theorem 1.3.

(1) $0x = 0$
(2) $a0 = 0$
(3) $(-a)x = -(ax) = a(-x)$
(4) If $ax = 0$, then either $a = 0$ or $x = 0$
(5) If $ax = ay$ and $a \neq 0$, then $x = y$
(6) If $ax = bx$ and $x \neq 0$, then $a = b$
(7) $-(x + y) = (-x) + (-y) = -x - y$
(8) $x + x = 2x, x + x + x = 3x, \sum_{j=1}^{n} x = nx$

Part (d), or part 4, is proven by considering this:
If $ax = 0$,
then if $a = 0$ and $x = 0$, done.
If $a \neq 0, \quad ax + a0 = a(x + 0) = 0$
since a is a real number, $\exists \frac{1}{a} \in \mathbb{R}$ s.t. $\left(\frac{1}{a}\right) a = 1$
$$\implies 1(x + 0) = x + 0 = x = \frac{1}{a} 0 = 0$$
If $x \neq 0$, suppose $a \neq 0$.
$$\frac{1}{a} ax = 1x = x = \frac{1}{a} 0 = 0$$
But 0 is unique. Contradiction. So $a = 0$ and that's okay, since by part (a) or part (1) of Thm. 1.3., $0x = 0$.

For part (e), or part 5, if $ax = ay$, $a \neq 0$, then subtract ay from both sides to get $ax - ay = 0 = a(x-y) = 0$. Use distributivity to get $ax - ay = a(x-y) = 0$. Since $a \neq 0$, then from part (d) or part 4, $x - y = 0$ must be true. Then $-y = -x$ or, multiplying both sides by -1, $y = x$.

For part (f), or part 6, if $ax = ay$, subtract bx from both sides and use distributivity to get $ax - bx = (a-b)x = 0$. Since $x \neq 0$, then by part (d), or part 4, $a - b = 0$. Add b to both sides to get $a = b$.

For part (g), or part 7, note that from the existence of an additive inverse, $x + -x = 0$. Consider $x + (-1)x = 0$. $x = 1x$ by the existence of a multiplicative identity, and so using distributivity, $1x + (-1)x = (1 + -1)x = 0x = 0$. Then $(-1)x$ is also an additive inverse for all $x \in V$. But additive inverses are unique, by theorem, so $(-1)x = -x$. Using that and distributivity, we get $(-x) + (-y) = (-1)x + (-1)y = (-1)(x+y) = -(x+y)$. $-x - y = -(x+y)$ because
$x + y + -(x+y) = 0 = x - x + y - y = x + y - x - y$, where we used additive commutativity at the last step. Then subtract $x + y$ from both sides to get $-x - y = -(x+y)$.

For part (h), or part 8, use the existence of a multiplicative identity and distributivity to get $x + x = 1x + 1x = (1+1)x = 2x$. Now, we'll use induction. Assume the nth case, that $\sum_{j=1}^{n} x = nx$.
Consider $\sum_{j=1}^{n+1} x$. $\sum_{j=1}^{n+1} x = \sum_{j=1}^{n} x + x = nx + x = nx + 1x = (n+1)x$. Done.

1.10 Exercises - Subspaces of a linear space, Dependent and independent sets in a linear space, Bases and dimension, Components

Exercise 1. $x = 0$

$$(0, y_1, z_1) + (0, y_2, z_2) = (0, y_1 + y_2, z_1 + z_2) \in S$$
$$k(0, y, z) = (0, ky, kz) \in S$$

Yes, S is a subspace.

$$(0, y, z) = y(0,1,0) + z(0,0,1) \in S$$
$$0 = y(0,1,0) + z(0,0,1) \quad \Longrightarrow z = 0 \quad y = 0$$

$\boxed{dim S = 2}$

Exercise 2. $x + y = 0$

$$(x_1 + x_2, y_1 + y_2, z_1 + z_2) \in S \text{ since} \qquad k(x,y,z) \in S \text{ since}$$
$$x_1 + x_2 + y_1 + y_2 = 0 + 0 = 0 \qquad kx + ky = k(x+y) = 0$$

Yes, S is a subspace.

$$(x, y, z) = (x, -x, z) = x(1, -1, 0) + z(0,0,1)$$
$$0 = x(1,-1,0) + z(0,0,1) \Longrightarrow z = 0, \ x = 0$$

$\boxed{dim S = 2}$

Exercise 3. $x + y + z = 0$

$$(x_1 + x_2, y_1 + y_2, z_1 + z_2) \in S \text{ since} \qquad k(x,y,z) \in S \text{ since}$$
$$x_1 + x_2 + y_1 + y_2 + z_1 + z_2 = 0 + 0 = 0 \qquad k(x+y+z) = kx + ky + kz = 0$$

Yes, S is a subspace.

$$(x, y, -(x+y)) = x(1,0,-1) + y(0,1,-1)$$
$$0 = x(1,0,-1) + y(0,1,-1) \Longrightarrow x = 0, \ y = 0$$

$\boxed{dim S = 2}$

Exercise 4. $x = y$

$$(x_1 + x_2, y_1 + y_2, z_1 + z_2) \in S \text{ since} \qquad k(x,y,z) \in S \text{ since}$$
$$x_1 + x_2 = y + 1 + y + 2 \qquad kx = ky$$

Yes, S is a subspace.

$$(x, y, z) = (x, x, z) = x(1,1,0) + z(0,0,1)$$
$$0 = x(1,1,0) + z(0,0,1) \Longrightarrow x = 0, \ z = 0$$

$\boxed{dim S = 2}$

Exercise 5. $x = y = z$

$$(x_1 + x_2, y_1 + y_2, z_1 + z_2) \in S \text{ since} \qquad k(x,y,z) \in S \text{ since}$$
$$x_1 + x_2 = y + 1 + y + 2 = z_1 + z_2 \qquad kx = ky = kz$$

Yes, S is a subspace.

$$\begin{gathered}(x,y,z)=x(1,1,1)\\ 0=x(1,1,1)\implies x=0,\end{gathered}$$

$\boxed{dimS=1}$

Exercise 6. $x=y$ or $x=z$

If x_1+y_1
If $x_2=y_2$, $x_1+x_2=y_1+y_2$
else if $x_2=z_2$, $\begin{array}{l}x_1+x_2 \text{ may not equal } y_1+y_2\\ x_1+x_2 \text{ may not equal } z_1+z_2\end{array}$ No, S is not a subspace.

Exercise 7. $x^2-y^2=0$

$$(x_1+x_2)^2-(y_1^2+y_2^2)=2x_1x_2-2y_1y_2 \text{ maybe not equal to zero}$$

S not a subspace.

Exercise 8. $x+y=1$

$$\begin{gathered}(x_1+x_2,y_1+y_2,z_1+z_2)\notin S \text{ since}\\ x_1+x_2+y_1+y_2=2\end{gathered}$$

No S is not a subspace.

Exercise 9. $y=2x$ and $z=3x$

$$\begin{gathered}(x_1+x_2,y_1+y_2,z_1+z_2)\in S \text{ since}\\ y_1+y_2=2x_1+2x_2=2(x_1+x_2)\\ z_1+z_2=3x_1+3x_2=3(x_1+x_2)\end{gathered} \qquad \begin{gathered}(kx,ky,kz)\in S \text{ since}\\ ky=k2x=2kx\\ kz=k3x=3kx\end{gathered}$$

S is a subspace.

$$\begin{gathered}(x,y,z)=x(1,2,3)\\ 0=x(1,2,3)\implies x=0\end{gathered}$$

$\boxed{dimS=1}$

Exercise 10.

$$\begin{array}{c}x+y+z=0\\ x-y-z=0\end{array}\iff\begin{array}{c}x=0\\ y=-z\end{array}$$

$$\begin{gathered}(x_1+x_2,y_1+y_2,z_1+z_2)\in S \text{ since}\\ x_1+x_2=0\\ y_1+y_2=-z_1-z_2=-(z_1+z_2)\\ k(x,y,z)\in S \text{ since}\\ kx=0 \qquad ky=-kz\end{gathered} \qquad \begin{gathered}y(0,1,-1)=(x,y,z)\\ 0=y(0,1,-1)\implies y=0\end{gathered}$$

Yes S is a subspace and $\boxed{dimS=1}$

For Exercises 11-20, in the P_n space, we will use the $\{u_k\}$ basis extensively, where $u_k=t^k \quad k=0,1,\dots,n$. It could be shown that this forms a basis, specifically it forms an independent set, by differentiating a degree n polynomial and set it to 0, and then repeating the differentiation.

Exercise 11. $f(0)=0$

$$\begin{gathered}f+g=\sum_{j=1}^n a_jx^j+\sum_{j=1}^n b_jx^j=\sum_{j=1}^n(a_j+b_j)x^j\in S \text{ since}\\ (f+g)(0)=0\end{gathered} \qquad \begin{gathered}kf=\sum_{j=1}^n ka_kx^k\in S \text{ since}\\ kf(0)=0\end{gathered}$$

Yes S is a subspace.

$$f=\sum_{j=1}^n a_jx^j$$

$$0=\sum_{j=1}^n a_jx^j\implies a_j=0$$

$\boxed{dimS=n}$

Exercise 12. $f'(0) = 0$

$$(f+g)' = f' + g' \implies (f+g)'(0) = f'(0) + g'(0) = 0$$
$$(kf)' = kf' \implies kf'(0) = 0$$

Yes S is a subspace.

$$f = \sum_{j=0}^{n} a_j x^j \qquad f'(0) = 0 \implies a_1 = 0$$
$$f' = \sum_{j=1}^{n} j a_j x^{j-1}$$

$$f = a_0 + \sum_{j=2}^{n} a_j x^j$$

$$0 = a_0 + \sum_{j=2}^{n} a_j x^j \implies a_j = 0, \quad j = 0, 2, 3, \dots n$$

$\boxed{dimS = n}$ Note that for the last step, we could've sited the fact that a subset of an independent set, such as the $\{t^k\}$ basis for P_n, is an independent set, by definition, and so if that subset spans S, this subset will be a basis for S.

Exercise 13. $f''(0) = 0$

$$(f+g)'' = f'' + g'' \implies (f+g)''(0) = f''(0) + g''(0) = 0$$
$$(kf)'' = kf'' \implies kf''(0) = 0$$

Yes S is a subspace.

$$f'' = \sum_{j=2}^{n} j(j-1) a_j x^{j-2} \qquad \implies f = a_0 + a_1 x + \sum_{j=3}^{n} a_j x^j$$
$$f''(0) = a_2 = 0$$

Then f is a linear combination of $\{1, x, x^3, x^4, \dots, x^n\}$, $dimS = n$

Exercise 14. $f(0) + f'(0) = 0$

$$f + g + f' + g' = (f+g) + (f+g)' \implies (f+g)(0) + (f+g)'(0) = 0 + 0 = 0$$
$$kf + (kf)' = k(f + f') \implies kf(0) + (kf)'(0) = k(f(0) + f'(0)) = 0$$

Yes S is a subspace.

$$f + f' = \sum_{j=0}^{n} a_j x^j + \sum_{j=1}^{n} j a_j x^{j-1} \qquad f = a_0(1-x) + \sum_{j=2}^{n} a_j x^j$$
$$(f+f')(0) = a_0 + a_1 = 0 \implies a_0 = -a_1$$

If $f = 0$, $a_j = 0$ for $j = 2, 3, \dots n$, by taking $j = 2, 3, \dots n$ derivatives. $a_0 = 0$ for $f(0) = 0$. Thus $\{1 - x, x^2, x^3, \dots, x^n\}$ is independent and span S and thus form a basis.

$\boxed{dimS = n}$

Exercise 15. $f(0) = f(1)$

$$(f+g)(0) = f(0) + g(0) + f(1) + g(1) = (f+g)(1)$$
$$kf(0) = kf(1)$$

Yes S is a subspace.

$$f = \sum_{j=0}^{n} a_j x^j$$

$$f(0) = a_0 = f(1) = a_0 + \sum_{j=1}^{n} a_j$$
$$\implies \sum_{j=1}^{n} a_j = 0 \text{ or } a_1 = -\sum_{j=2}^{n} a_j$$
$$\implies f = a_0 + \sum_{j=2}^{n} a_j (x^j - x)$$

By differentiating

$$f' = (a_0)' + a_2(2x - 1) + a_3(3x^2 - 1)\dots$$
$$f'' = a_2(2) + a_3 x$$
$$f'' = \sum_{j=2}^{n} a_j (j x^{j-2}) = 0 \text{ if } f = 0$$

Then $\{x^{j-2}\}$ is a subset of a basis for P_n.
$a_j = 0$ for $j = 2, \dots n$.
Then $a_0 = 0$.
Thus, $\{1, x^j - x\}$ is independent and spans S. Then $\{1, x^j - x\}$ forms a basis for S.
$\boxed{dim S = n}$
Exercise 16. $f(0) = f(2)$

$$f(0) + g(0) = (f+g)(0) = f(2) + g(2) = (f+g)(2)$$
$$kf(0) = kf(2)$$

Yes S is a subspace.

$$f = \sum_{j=0}^{n} a_j x^j$$
$$f(0) = a_0 = a_0 + \sum_{j=1}^{n} a_j 2^j \Longrightarrow 2a_1 + \sum_{j=2}^{n} 2^j a_j = 0 \text{ or } a_1 = -\sum_{j=2}^{n} 2^{j-1} a_j$$
$$\Longrightarrow f = a_0 + \sum_{j=2}^{n} a_j (x^j - 2^{j-1} x)$$
$$f'' = \sum_{j=2}^{n} a_j j(j-1) x^{j-2} \text{ and } f'' = 0 \text{ if } f = 0$$

$\mathcal{B}_{S_1} = \{1, x, \dots, x^{n-2}\}$ is a subset of the basis $\{1, x, \dots, x^n\} = \mathcal{B}_{P_n}$ for P_n. Then $\mathcal{B}_{S_1}$ is independent, and so $a_j = 0$ for $j = 2, \dots n$. Then for $f = 0$, $a_0 = 0$. Thus $\{1, x^2 - 2x, x^3 - 2^2 x, \dots, x^j - 2^{j-1}x, \dots, x^n - 2^{n-1}x\}$ is independent and spans S and thus forms a basis for S.
$\boxed{dim S = n}$
Exercise 17. f is even. $f(-x) = f(x)$

$$(f+g)(-x) = f(-x) + g(-x) = f(x) + g(x) = (f+g)(x)$$
$$kf(-x) = kf(x)$$

Yes S is a subspace.

$$f(-x) = \sum_{j=0}^{n} a_j x^j (-1)^j = f(x) = \sum_{j=0}^{n} a_j x^j \Longrightarrow \sum_{j=0}^{n} a_j x^j ((-1)^j - 1) = 0$$

$\frac{n}{2} + 1$ if n is even, is the number of possibly nonzero coefficients for f.
$\frac{n-1}{2} + 1 = \frac{n+1}{2}$ if n is odd, is the number of possibly nonzero coefficients for f.

Then $\frac{n}{2}$ or $\frac{n-1}{2}$, if n is even, or n is odd, respectively, are the number of needed elements for a subset from the basis $\mathcal{B}_{P_n}$ to span f and form a basis for S.
Exercise 18. f is odd. $f(-x) = f(x)$

$$(f+g)(-x) = f(-x) + g(-x) = -f(x) - g(x) = -(f+g)(x)$$
$$kf(-x) = -kf(x)$$

Yes S is a subspace.

$$f(-x) = \sum_{j=0}^{n} a_j x^j (-1)^j = -f(x) = -\sum_{j=0}^{n} a_j x^j \Longrightarrow \sum_{j=0}^{n} a_j x^j ((-1)^j + 1) = 0$$

$\frac{n}{2}$ if $n = 2K$ is even, is the number of possibly nonzero coefficients for f.
$\frac{n+1}{2} = \frac{n+1}{2}$ if $n = 2K - 1$ is odd, is the number of possibly nonzero coefficients for f.

Then $\frac{n}{2}$ or $\frac{n+1}{2}$, if n is even, or n is odd, respectively, are the number of needed elements for a subset from the basis $\mathcal{B}_{P_n}$ to span f and form a basis for S.
Exercise 19. f has degree $\leq k$, where $k < n$ or $f = 0$

$$f = \sum_{j=0}^{k} a_j x^j; \qquad g = \sum_{j=0}^{k} b_j x^j$$

$$f + g = \sum_{j=0}^{k} (a_j + b_j) x^j \text{ even if } a_j + b_j = 0 \text{ for any or all } j,\ f + g \text{ has degree } \leq k \text{ or } f = 0$$

$$\sum_{j=0}^{k} c_0 a_j x^j \in S$$

f is spanned by $\mathcal{B}_S = \{x^j\}$, $j = 0, 1, \dots, k$ which is a subset of $\mathcal{B}_{P_n}$, which is a basis for P^n. Then $\mathcal{B}_S$ is independent. Then $\mathcal{B}_S$ is a basis for S.
$dimS = k + 1$

Exercise 20. Consider

$$f + g = \left(\sum_{j=0}^{k-1} a_j x^j + a_k x^k \right) + \left(\sum_{j=0}^{k-1} b_j x^j + -a_k x^k \right) = \sum_{j=0}^{k-1} (a_j + b_j) x^j \text{ with } a_{k-1} + b_{k-1} \neq 0$$

$$f + g \notin S$$

Thus S is not a subspace.

Exercise 21.

(1) $\{1, t^2, t^4\}$
$f = a_0 + a_2 t^2 + a_4 t^4$, $\boxed{dimS = 3}$
(2) $\{t, t^3, t^5\}$
$f = a_1 t + a_3 t^3 + a^5 t^5$, $\boxed{dimS = 3}$
(3) $\{t, t^2\}$
$f = a_1 t + a_2 t^2$, $\boxed{dimS = 2}$
(4) $\{1 + t, (1 + t)^2\}$

$$a_1(1 + t) + a_2(1 + t)^2 = a_1(1 + t) + a_2(1 + 2t + t^2) = (a_1 + a_2) + (a_1 + 2a_2)t + a_2 t^2$$

If $a_1(1 + t) + a_2(1 + t)^2 = 0$, $a_2 = 0$, $a_1 = 0$ so $(1 + t), (1 + t)^2$ is independent, $\boxed{dimS = 2}$

Exercise 22. In this exercise, $L(S)$ denotes the subspace spanned by a subset S of a linear space V.

(1) $x \in S, \quad 1x \in L(S) \Longrightarrow S \subseteq L(S)$
(2) If $S = \{x_1, \dots, x_n\}$
$x = \sum a_j x_j \in L(S)$

$S \subseteq T \Longrightarrow x_j \in T$
T is a subspace of V $\Longrightarrow \sum a_j x_j \in T$ (T is closed under addition and scalar multiplication). so $x \in T$
$\boxed{L(S) \subseteq T}$
(3) If S is a subspace of V, S is closed under addition and scalar multiplication.
Repeatedly apply addition and scalar multiplication closure for each $x_j \in S$ and $\forall\, a_j \in \mathbb{R}$, so that $\sum a_j x_j \in S$, $\forall\, a_j \in \mathbb{R}, \forall\, x_j \in S$.

$\Longrightarrow L(S) \subseteq S$

$S \subseteq L(S)$ (as proven in part(a), or part (1), of this exercise).

$L(S) = S$ if S is a subspace of V.

If $L(S) = S$, $\forall \sum a_j x_j \in S$, the S is closed under addition and scalar multiplication. Then S is a subspace of V, by theorem.
(4) Suppose $S = \{x_1, x_2, \dots, x_m\}$
Then since $S \subseteq T$, $T = \{x_1, x_2, \dots, x_m, \dots x_n\}$

$$\sum_{j=1}^{m} a_j x_j = \sum_{j=1}^{n} a_j x_J \in L(T) \text{ with } a_j = 0 \text{ for } j = m + 1, m + 2, \dots, n \Longrightarrow L(S) \subseteq L(T)$$

(5) If $x_1, x_2 \in S\bigcap T$, then $\begin{array}{c} x_1 \in S\bigcap x_2 \in S \\ x_1 \in T\bigcap x_2 \in T \end{array}$. Since S, T are subspaces, $x_1 + x_2, cx_1 \in S\bigcap x_1 + x_2, cx_1 \in T$
Then $x_1 + x_2, cx_1 \in S\bigcap T$. So $S\bigcap T$ is a subspace.

(6) Consider $x \in L(S\bigcap T)$.
$x = \sum a_j x_j$; where $x_j \in S\bigcap T$
Since $\forall\, x_j \in S$, then $x \in L(S)$. Since $\forall\, x_j \in T$, then $x \in L(T)$.
Thus $x \in L(S)\bigcap L(T)$. $\implies L(S\bigcap T) \subseteq L(S)\bigcap L(T)$

(7) Example of when $L(S\bigcap T) \neq L(S)\bigcap L(T)$.
Suppose $S = \{x_1, x_2\}$, $T = \{x_3\}$ and $x_1 + x_2 = x_3$.
$S\bigcap T = \emptyset$. $L(S\bigcap T) = \emptyset$, but $L(S)\bigcap L(T) = \{kx_3 | k \in \mathbb{R}\} = L(T)$

Exercise 23.

(1) $\{1, e^{ax}, e^{bx}\}$, $a \neq b$

$$a_0 + a_1 e^{ax} + a_2 e^{bx} = 0 \xrightarrow{\frac{d}{dx}} a_1 a e^{ax} + a_2 b e^{bx} = 0 \text{ or } a_1 a = -a_2 b e^{(b-a)x}$$

Since x arbitrary, $a_1 = a_2 = 0$.
$\implies \{1, e^{ax}, e^{bx}\}$ independent. $\boxed{dimS = 3}$

(2) $\{e^{ax}, xe^{ax}\}$

$$a_1 e^{ax} + a_2 x e^{ax} = 0 \text{ or } a_1 = -a_2 x$$

x arbitrary, so $a_1 = a_2 = 0$. $\{e^{ax}, xe^{ax}\}$ independent. $\boxed{dimS = 2}$

(3) $\{1, e^{ax}, xe^{ax}\}$

$$a_0 + a_1 e^{ax} + a_2 x e^{ax} = 0 \xrightarrow{\frac{d}{dx}} a a_1 e^{ax} + a_2 e^{ax} + a_2 a x e^{ax} = 0$$
$$a a_1 + a_2 + a_2 a x = 0 \text{ or } a_2 a x = -(a a_1 + a_2)$$
$$x \text{ arbitrary, so } a_2 = 0,\ a_1 = 0$$

Then $a_0 = 0$ and so $\{1, e^{ax}, xe^{ax}\}$ independent. $\boxed{dimS = 3}$

(4) $\{e^{ax}, xe^{ax}, x^2 e^{ax}\}$.

$$a_0 e^{ax} + a_1 x e^{ax} + a_2 x^2 e^{ax} = 0 = a_0 + a_1 x + a_2 x^2$$

$1, x, x^2$ are a subset of independent $\mathcal{B}_{P_n}$ and so $1, x, x^2$ are independent $\implies a_0 = a_1 = a_2 = 0$, and so $\{e^{ax}, xe^{ax}, x^2 e^{ax}\}$ independent. $\boxed{dimS = 3}$.

(5) $\{e^x, e^{-x}, \cosh x\}$
$\cosh x = \frac{e^x + e^{-x}}{2}$ dependent. $\boxed{dimS = 2}$

(6) $\{\cos x, \sin x\}$

$$a \cos x + b \sin x = 0 \text{ or } b \sin x = -a \cos x$$

If $\cos x = 0$, then $\sin x = 1$, so $b = 0$. Otherwise,
$b \tan x = -a$. But x arbitrary $\implies a = 0,\ b = 0$
So $\{\cos x, \sin x\}$ independent. $\boxed{dimS = 2}$

(7) $a \cos^2 x + b \sin^2 x = 0$, so then if $\cos^2 x \neq 0$, we have $b \tan^2 x = -a$. Since x is arbitrary, $a = b = 0$. Then $\{\cos^2 x, \sin^2 x\}$ independent. $\boxed{dimS = 2}$

(8) $\{1, \cos 2x, \sin^2 x\}$

$$\cos 2x = 1 - 2\sin^2 x$$

So the set is dependent. $\boxed{dimS = 2}$, since $\{1, \sin^2 x\}$ independent ($\{\cos^2 x, \sin^2 x\}$ were independent and $1 = \cos^2 x + \sin^2 x$).

(9) $\{\sin x, \sin 2x\}$

$$a \sin x + b \sin 2x = \sin x (a + b2 \cos x) = 0$$

If $\sin x, \cos x \neq 0$, $a + b2\cos x = 0$ or $2b \cos x = -a$. Since x is arbitrary, $a = b = 0$ So then $\{\sin x, \sin 2x\}$ is independent. $\boxed{dimS = 2}$

(10) $\{e^x \cos x, e^{-x} \sin x\}$

$$a e^x \cos x + b e^{-x} \sin x = 0 \text{ or } b \tan x = -a e^{2x}$$

Since x arbitrary, $a = b = 0$. $\boxed{dimS = 2}$

Exercise 24.

(1) Consider $\mathcal{B}_S$, basis for S and $\mathcal{B}_V$ basis for V. $|\mathcal{B}_V| = n$ finite.
If S is infinite-dimensional, then $\exists\, x_{n+1} \in \mathcal{B}_S$ s.t. $x_{n+1} \notin \mathcal{B}_V$ since $\mathcal{B}_V$ finite. Then $\exists\, x_{n+1} \in S$ s.t. $x_{n+1} \notin V$. But $S \subseteq V$
$\Longrightarrow S$ is finite-dimensional.

Consider $\mathcal{B}_S = \{x_1, \dots, x_m\}$ and $\mathcal{B}_V = \{y_1, \dots, y_n\}$.
$\forall\, x_j, x_j \in V$, since $S \subseteq V$.
Suppose $m > n$. Then $\mathcal{B}_S$ linearly dependent, by Thm. 12.10 of Vol.1 (a.k.a. Thm. 1.7 of Vol. 2). This contradicts the fact that $\mathcal{B}_S$ is an independent basis. $\Longrightarrow dimS \leq dimV$

(2) If $S = V$, then by Thm. 12.10 of Vol.1, $\mathcal{B}_S$ must also contain exactly n vectors, since it's a basis for $V = S$.

If $dimS = dimV$, then since $\mathcal{B}_S$ is a set of n linearly independent elements, it forms a basis for V.
Then $\forall\, y \in V, y \in L(\mathcal{B}_S) = S$.
$V \subseteq S \Longrightarrow V = S$.

(3) Use Thm. 12.10 of Vol.1 (a.k.a. Thm. 1.7 of Vol.2): Any set of linear independent elements is a subset of some basis for V.

(4) Consider $\mathcal{B}_V = \{y_1, \dots, y_n\}$
Suppose $\{y_1 + y_2, y_1 - y_2\} = \mathcal{B}_S$
$y_1 + y_2, y_1 - y_2 \notin \mathcal{B}_V$ because if they were, they'd make $\mathcal{B}_V$ dependent.

2.4 Exercises - Linear Transformations and Matrices, Null space and range, Nullity and rank

Exercise 1. $T(x, y) = (y, x)$

$$T(a(x_1,x_2)+b(y_1,y_2)) = T(ax_1+by_1, ax_2+by_2) = (ax_2+by_2, ax_1+by_1) = a(x_2,x_1)+b(y_2,y_1) = aT(x_1,x_2)+bT(y_1,y_2)$$

T is linear.
$T(x, y) = (y, x) = 0$. $nullspaceT = \{0\}$; $\quad kerT = 0$
$T(x, y) = (y, x) = y(1, 0) + x(0, 1)$. $rangeT = V_2$. $rankT = 2$

Exercise 2. $T(x, y) = (x, -y)$

$$T(a(x_1, x_2) + b(y_1, y_2)) = (ax_1 + by_1, -(ax_2 + by_2)) = a(x_1, -x_2) + b(y_1, -y_2) = aT(x_1, x_2) + bT(y_1, y_2)$$

T is linear.
$(x, -y) = 0$. $nullspaceT = \{0\}$; $\quad kerT = 0$
$(x, -y) = x(1, 0) + -y(0, 1)$ $rangeT = V_2$; $\quad rankT = 2$

Exercise 3. $T(x, y) = (x, 0)$.

$$T(a(x_1, x_2) + b(y_1, y_2)) = (ax_1 + by_1, 0) = a(x_1, 0) + b(y_1, 0) = aT(x_1, x_2) + bT(y_1, y_2)$$

T is linear.
$T(x, y) = (x, 0) = 0 \Longrightarrow x = 0, \quad y \in \mathbb{R}$. $nullspaceT = L(\{(0, 1)\}) \quad kerT = 1$
$T(x, y) = (x, 0) = x(1, 0) \quad rangeT = L(\{(1, 0)\})$. $rankT = 1$

Exercise 4. $T(x, y) = (x, x)$

$$T(a(x_1, x_2) + b(y_1, y_2)) = (ax_1 + by_1, ax_1 + by_1) = a(x_1, x_1) + b(y_1, y_1) = aT(x_1, x_2) + bT(y_1, y_2)$$

T is linear.
$T(x, y) = (x, x) = 0 \Longrightarrow x = 0, \quad y \in \mathbb{R}$. $nullspaceT = L(\{(0, 1)\}) \quad kerT = 1$
$T(x, y) = (x, x) = x(1, 1) \quad rangeT = L(\{(1, 1)\})$. $rankT = 1$

Exercise 5. $T(x, y) = (x^2, y^2)$

$$T(a(x_1, x_2) + b(y_1, y_2)) = ((ax_1 + by_1)^2, (ax_2 + by_2)^2) =$$
$$= (a^2x_1^2, a^2x_2^2) + (2abx_1y_1, 2abx_2y_2) + (b^2y_1^2, b^2y_2^2) \neq aT(x_1, x_2) + bT(y_1, y_2)$$

T is not linear.

Exercise 6. $T(x, y) = (e^x, e^y)$

$$T(a(x_1, x_2) + b(y_1, y_2)) = (e^{ax_1+by_1}, e^{ax_2+by_2}) \neq aT(x_1, x_2) + bT(y_1, y_2)$$

T is not linear.

Exercise 7. $T(x, y) = (x, 1)$

$$T(a(x_1, x_2) + b(y_1, y_2)) = (ax_1 + by_1, 1) \neq a(x_1, 1) + b(y_1, 1) = aT(x_1, x_2) + bT(y_1, y_2)$$

T is not linear.
Exercise 8. $T(x,y) = (x+1, y+1)$

$$T(a(x_1,x_2)+b(y_1,y_2)) = (ax_1+by_1+1, ax_2+by_2, 1) \neq aT(x_1,x_2)+bT(y_1,y_2)$$

T is not linear.
Exercise 9. $T(x,y) = (x-y, x+y)$

$$\begin{aligned} T(a(x_1,x_2)+b(y_1,y_2)) &= (ax_1+by_1-ax_2-by_2, ax_1+by_1+ax_2+by_2) = \\ &= a(x_1-x_2, x_1+x_2)+b(y_1-y_2,y_1+y_2) = aT(x_1,x_2)+bT(y_1,y_2) \end{aligned}$$

T is linear.
$T(x,y) = (x-y,x+y) = 0 \Longrightarrow x = y = 0, \quad nullspaceT = \{0\} \quad kerT = 1$
$T(x,y) = (x-y,x+y) = x(1,1)+y(-1,1) \quad rangeT = L(\{(1,1),(-1,1)\}).\ rankT = 2$
Exercise 10. $T(x,y) = (2x-y, x+y)$

$$\begin{aligned} T(a(x_1,x_2)+b(y_1,y_2)) &= (2(ax_1+by_1)-(ax_2+by_2), ax_1+by_1+ax_2+by_2) = \\ &= a(2x_1-x_2, x_1+x_2)+b(2y_1-y_2,y_1+y_2) = aT(x_1,x_2)+bT(y_1,y_2) \end{aligned}$$

T is linear.
$(2x-y,x+y) = 0 \quad nullspaceT = \{0\}. \quad kerT = 0$
$x(2,1)+y(-1,1) = (2x-y,x+y) \quad rangeT = L(\{(2,1),(-1,1)\}).\ rankT = 2$
Exercise 11. T rotates every point through the same angle ϕ about the origin. That is, T maps a point with polar coordinates (r,θ) onto the point with polar coordinates $(r,\theta+\phi)$, where ϕ is fixed. Also, T maps 0 onto itself.
Amazingly, T **is linear** . What's required to show this is persistence.

$$\begin{aligned} x &= (r_1\cos\theta_1, r_1\sin\theta_1) \\ y &= (r_1, \sin\theta_2, r_2\sin\theta_2) \\ ax+by &= (ar_1\cos\theta_1+br_2\cos\theta_2, ar_1\sin\theta_1+br_2\sin\theta_2) \\ |ax+by|^2 &= (ar_1c_1+br_2c_2)^2+(ar_1s_1+br_2s_2)^2 = \\ &= a^2r_1^2c_1^2+2abr_1r_2c_1c_2+b^2r_2^2c_2^2+a^2r_1^2s_1^2+2abr_1s_1r_2s_2+b^2r_2^2s_2^2 = \\ &= a^2r_1^2+b^2r_2^2+2abr_1r_2c(\theta_1-\theta_2) \\ \text{argument of } ax+by &= \arctan\left(\frac{ar_1s\theta_1+br_2s\theta_2}{ar_1c\theta_1+br_2c\theta_2}\right) \end{aligned}$$

So $|T(ax+by)| = |ax+by|$, but the argument of $T(ax+by) = \arctan\left(\frac{ar_1s\theta_1+br_2s\theta_2}{ar_1c\theta_1+br_2c\theta_2}\right)+\phi$.
Consider now $aT(x)+bT(y) = a(r_1,\theta_1+\phi)+b(r_2,\theta_2+\phi)$.

$$\begin{aligned} &\sqrt{(ar_1c(\theta_1+\phi)+br_2c(\theta_2+\phi))^2+(ar_1s(\theta_1+\phi)+br_2s(\theta_2+\phi))^2} = \\ &= \sqrt{(ar_1)^2+(br_2)^2+2abr_1r_2(c(\theta_1+\phi)c(\theta_2+\phi)+s(\theta_1+\phi)s(\theta_2+\phi))} = \sqrt{(ar_1)^2+(br_2)^2+2abr_1r_2c(\theta_1-\theta_2)} \end{aligned}$$

The length is the same for $T(ax+by)$ and $aT(x)+bT(y)$.
The argument of $aT(x)+bT(y)$ is the following:

$$\begin{aligned} \frac{ar_1(s\theta_1c\phi+c\theta_1s\phi)+br_2(s\theta_2c\phi+c\theta_2s\phi)}{ar_1(c\theta_1c\phi-s(\theta_1)s\phi)+br_2(c\theta_2c\phi-s\theta_2s\phi)} &= \frac{ar_1(s\theta_1+c\theta_1\tan\phi)+br_2(s\theta_2+c\theta_2\tan\phi)}{ar_1(c\theta_1-s(\theta_1)\tan\phi)+br_2(c\theta_2-s\theta_2\tan\phi)} = \\ &= \frac{ar_1s\theta_1+br_2s\theta_2+ar_1c\theta_1\tan\phi+br_2c\theta_2\tan\phi}{ar_1c\theta_1+br_2c\theta_1-ar_1s\theta_1\tan\phi-br_2s\theta_2\tan\phi} \end{aligned}$$

Beforehand, recall this trigonometric identity:

$$\tan(x+y) = \frac{\sin(x+y)}{\cos(x+y)} = \frac{\sin x\cos y+\sin y\cos x}{\cos x\cos y-\sin x\sin y} = \frac{\tan x+\tan y}{1-\tan x\tan y}$$

Thus

$$\begin{aligned} \tan\left(\arctan\left(\frac{ar_1s\theta_1+br_2s\theta_2}{ar_1c\theta_1+br_2c\theta_2}\right)+\phi\right) &= \frac{\frac{ar_1s\theta_1+br_2s\theta_2}{ar_1c\theta_1+br_2c\theta_2}+\tan\phi_1}{1-\left(\frac{ar_1s\theta_1+br_2s\theta_2}{ar_1c\theta_1+br_2c\theta_2}\right)\tan\phi} = \\ &= \frac{ar_1s\theta_1+br_2s\theta_2+ar_1c\theta_1\tan\phi+br_2c\theta_2\tan\phi}{ar_1c\theta_1+br_2c\theta_1-ar_1s\theta_1\tan\phi-br_2s\theta_2\tan\phi} \end{aligned}$$

So the arguments for $T(ax+by)$ and $aT(x)+bT(y)$ are, amazingly, the same, modulo some 2π periodicity.

Thus, *rotations are linear transformations.*
$nullspaceT = \{0\}$. $nullT = 0$
$rangeT = \{(r,\theta)\}$. $rankT = 2$
Exercise 12. T maps each point onto its reflection with respect to a fixed line through the origin.

We showed above that rotations are linear transformations. Then without loss of generality, consider reflection about the x-axis.

$$T(a(x_1,x_2)+b(y_1,y_2)) = (ax_1+by_1, -ax_2-by_2) = a(x_1,-x_2)+b(y_1,-y_2)$$
$$aT(x_1,x_2)+bT(y_1,y_2) = a(x_1,-x_2)+b(y_1,y-2)$$

So reflection about the x axis is linear.
Suppose R is the rotation of the fixed line into the x-axis and R is length preserving. Then for $R^{-1}TR$, reflection about any fixed axis, (R^{-1} is linear too, since it's just a rotation in the opposite direction of R)

$$R^{-1}TR(ax+by) = R^{-1}T(aRx+bRy) = R^{-1}(aTRx+bTRy) = aR^{-1}TRx+bR^{-1}TRy$$

T is linear.
$nullT = 0 \quad nullspaceT = \{0\}$
$rankT = 2 \quad rangeT = \{(x,y)\}$
Exercise 13. T maps every point onto the point $(1,1)$.
$T(ax+by) = (1,1) \neq aT(x)+bT(y) = (a+b)(1,1)$
T is nonlinear.
Exercise 14. $T(r,\theta) = (2r,\theta)$

$$T(x_1)+T(x_2) = 2r_1e^{i\theta_1}+2r_2e^{i\theta_2} = 2(r_1e^{i\theta}+r_2e^{i\theta_2})$$
$$T(x_1+y_1) = T(r_1e^{i\theta_1}+r_2e^{i\theta_2}) \text{ so}$$
$$|r_1e^{i\theta_1}+r_2e^{i\theta_2}| = \sqrt{r_1^2+r_2^2+2r_1r_2\cos(\theta_1-\theta_2)} = |T(x_1)+T(x_2)|$$

we see that the argument remains unchanged, while the magnitude is multiplied by 2 in each case

$$nullT = 0 \qquad nullspaceT = \{0\}$$
$$rankT = 2 \qquad rangeT = \{(r,\theta)\}$$

Exercise 15. $T(a(r,\theta)) = (ar,2\theta) = a(r,2\theta) = aT(r,\theta)$.
Consider this counterexample, where $x_1 = 1\vec{e}_x, x_2 = 1\vec{e}_x$. Not linear.
Exercise 16. $T(x,y,z) = (z,y,x)$.

$$T(ax) = aT(x), T(x_1+x_2,y_1+y_2,z_1+z_2) = (z_1,y_1,x_1)+(z_2,y_2,x_2) \implies \text{ linear}$$
$$T(x) = 0 \text{ when } x = 0$$
$$nullspaceT = \{0\} \qquad rangeT = V_3$$
$$nullT = 0 \qquad rankT = 3$$

Exercise 17. $T(x,y,z) = (x,y,0)$.

$$T(a(x,y,z)) = a(x,y,0) = aT(x,y,z); T(x_1+x_2,y_1+y_2,z_1+z_2) =$$
$$= (x_1+x_2,y_1+y_2,0) = T(x_1,y_1,z_1) = T(x_2,y_2,z_2)$$
$$\implies T \text{ linear}$$
$$T(x,y,z) = 0 \text{ if } x = y = 0$$
$$nullspaceT = L(\{(0,0,1)\}) \qquad nullT = 1$$
$$rangeT = L(\{(1,0,0),(0,1,0)\}) \quad rankT = 2$$

Exercise 18. $T(x,y,z) = (x,2y,3z)$.

$$T(ax) = (ax,2ay,3az) = aT(x); T(x_1+x_2) = (x_1+x_2, 2(y_1+y_2), 3(z_1+z_2)) = T(x_1)+T(x_2)$$
$$T(x) = 0 \text{ when } x = y = z = 0$$
$$nullspaceT = \{0\} \qquad nullT = 0$$
$$rangeT = V_3 \qquad rankT = 3$$

Exercise 19. $T(x,y,z) = (x,y,1)$.
$T(x_1)+T(x_2) = (x_1+x_2,y_1+y_2,2) \neq T(x_1+x_2)$ T is not linear.
Exercise 20. $T(x,y,z) = (x+1,y+1,z-1)$

$$T(ax+by) = (ax_1+by_1+1, ax_2+by_2+1, ax_3+by_3-1) \neq aT(x)+bT(y) =$$
$$= a(x_1+1, x_2+1, x_3-1) + b(y_1+1, y_2+1, y_3-1)$$

Exercise 21. $T(x,y,z) = (x+1, y+2, z+3)$

$$T(ax+by) = (ax_1+by_1+1, ax_2+by_2+2, ax_3+by_3+3) \neq aT(x)+bT(y) = a(x_1+1, x_2+2, x_3+3)+b(y_1+1, y_2+2, y_3+3)$$

Exercise 22. $T(x,y,z) = (x, y^2, z^3)$

$$T(ax+by) = (ax_1+by_1, (ax_2+by_2)^2, (ax_3+by_3)^3) = (ax_1+by_1, a^2x_2^2+2abx_2y_2+b^2y_2^2, a^2x_3^2+2abx_3y_3+b^2y_3^2) \neq$$
$$\neq aT(x)+bT(y) = a(x_1, x_2^2, x_3^2) + b(y_1, y_2^2, y_3^2)$$

Exercise 23. $T(x,y,z) = (x+z, 0, x+y)$

$$T(ax+by) = (ax_1+by_1+ax_3+by_3, 0, ax_1+by_1+ax_2+by_2) = a(x_1+x_3, 0, x_1+x_2)+b(y_1+y_3, 0, y_1+y_2) = aT(x)+bT(y)$$

T is linear.

$(x+z, 0, x+y) = 0 \quad \begin{array}{l} x=-z \\ x=-y \end{array} \quad (x,y,z) = x(1,-1,-1) \Longrightarrow nullspaceT = L(\{(1,-1,-1)\}) \quad kerT = 1$

$(x+z, 0, x+y) = (x+z)(1,0,0) + (x+y)(0,0,1) \quad rangeT = L(\{(1,0,0),(0,0,1)\}), \quad rankT = 2$

Exercise 24.

$p(x) = \sum_{j=0}^{n} a_j x^j \qquad (p+q)(x) = \sum_{j=0}^{n} (a_j+b_j)x^j$

$q(x) = \sum_{j=0}^{n} b_j x^j \qquad cp(x) = c\sum_{j=0}^{n} a_j x^j = \sum_{j=0}^{n} ca_j x^j$

$T(p+q) = \sum_{j=0}^{n} (a_j+b_j)(x+1)^j = \sum_{j=0}^{n} a_j(x+1)^j + \sum_{j=0}^{n} b_j(x+1)^j = T(p)+T(q)$

$$T(cp) = \sum_{j=0}^{n} ca_j(x+1)^j = c\sum_{j=0}^{n} a_j(x+1)^j = cT(p)$$

T linear.

Consider $\sum_{j=0}^{n} a_j(x+1)^j = 0$. Apply differentiation repeatedly to get $a_j = 0, \forall\, j = 0, \dots, n$.

$nullspaceT = \{0\}. \quad nullT = 0.$

$\sum_{j=0}^{n} a_j(x+1)^j = \sum_{j=0}^{n} b_j x^j$. $rangeT = L(\{(x+1)^j | j = 0, \dots, n\}); \quad rankT = n+1$

Exercise 25. On $(-1,1)$, $f \in V; \quad g = T(f), g(x) = xf'(x)$

$$T(f+g) = x(f'+g') = xf'+xg' = T(f)+T(g)$$
$$T(af) = x(af)' = axf' = aT(f)$$

$T(f) = xf'(x) = 0$ x is arbitrary, consider $x \neq 0$. $f'(x) = 0 \Longrightarrow f(x) = c_0$.

$nullspaceT = \{1\}, \quad nullT = 1$

$rangeT = V \quad rankT = dimV - 1 \to \infty$

Exercise 26. $g(x) = \int_a^b f(t)\sin{(x-t)}dt$ for $a \leq x \leq b$

$$T(f+g) = \int_a^b (f(t)+g(t))\sin{(x-t)}dt = \int_a^b f(t)\sin{(x-t)}dt + \int_a^b g(t)\sin{(x-t)}dt$$
$$T(cf) = \int_a^b cf(t)\sin{(x-t)}dt$$

T is linear.

$$g(x) = \int_a^b f(t)(s(x)c(t)-c(x)s(t))dt = s(x)\int_a^b f(t)c(t)dt - c(x)\int_a^b f(t)s(t)dt = k_1s(x)+k_2c(x)$$
$$RangeT = L(\{\sin x, \cos x\}); \quad rankT = 2$$

For the nullspace, now $g(x) = s(x)\int_a^b f(t)c(t)dt - c(x)\int_a^b f(t)s(t)dt$. If we take a look at **Exercise 29** of this section, then we see the **answer: by the orthogonality of** $\sin$**'s and** $\cos$**'s**, $\sin nt$ and $\cos nt$ will be orthogonal to $\cos t$ and $\sin t$ for $n = 2, \dots$. So depending upon a and b, at least the integration over a period of 1 will result in zero for both $\int fc$ and $\int fs$. Then for the "ends" of the integration bound that don't make a full period, make $f(t) = 0$. Since $n = 2, \dots \to \infty$ for $\sin nt$, $\cos nt$ for the choice of $f(t)$,

$nullspaceT = L(\{\sin nt, \cos nt | n = 2, \dots\}), \quad kerT = \infty$

Exercise 27. $T(y) = y'' + Py' + Qy$, P, Q fixed constants.

$$T(ay_1 + by_2) = ay_1'' + by_2'' + P(ay_1' + by_2') + Q(ay_1 + by_2) =$$
$$a(y_1'' + Py_1' + Qy_1) + b(y_2'' + Py_2' + Qy_2) = aT(y_1) + bT(y_2)$$

$$\begin{aligned} &T \text{ is linear} \\ &nullspaceT = L(\{x, 1\}) \qquad nullT = 2 \\ &rangeT = V \qquad rankT = \infty \end{aligned}$$

Exercise 28. If $x = x_k$ is a convergent sequence with limit a, by definition,

$$\forall n \in \mathbb{N}, \exists m = m(n) \in \mathbb{N} \text{ such that } |a - x_k| < \frac{1}{n}\ \forall k \geq m$$

$$T(x) = y_k; cT(x) = cy_k = c(a - x_k) = ca - cx_k$$

(cx_k understood to mean that each x_k term in the sequence is multiplied by c)

Consider $|cx_k - ca| = |c||x_k - a| < \frac{|c|}{n}$ for $k \geq m$

Consider $\frac{n}{|c|} = n_1 . \exists m_1 = m_1(n_1)$ such that $|cx_k - ca| < \frac{1}{n_1}$

Thus cx_k is convergent with limit ca and so $T(cx) = cT(x)$.

Consider two convergent sequences x_k and y_k with limits a and b respectively. Then by definition,

$$\forall n \in \mathbb{N}, \exists m_1 = m_1(n) \in \mathbb{N}, |a - x_k| < \frac{1}{2n}, k \geq m_1$$
$$\forall n \in \mathbb{N}, \exists m_2 = m_2(n) \in \mathbb{N}, |b - y_k| < \frac{1}{2n}, k \geq m_2$$

For $k \geq \max(m_1, m_2)$

$$|a + b - (x_k + y_k)| = |(a - x_k) + (b - y_k)| \leq |a - x_k| + |b - y_k| < \frac{1}{2n} + \frac{1}{2n} = \frac{1}{n}$$

so when we consider $T(x + y)$, $T(x + y) = a + b - (x + y)$,

with $a + b - (x + y)$ convergent sequence defined as above, with limit 0.

so $T(x + y)$ is convergent with limit 0 just like $T(x) + T(y)$. T is linear.

We consider a convergent sequence to be a zero if it is an additive identity to each term in the sequence. Then $nullspaceT =$ space of all sequences consisting of the same term for each term . Also, $rangeT =$ space of all convergent sequences with limit 0.

Exercise 29.

(1)

$$\text{If } \int_{-\pi}^{\pi} f = \int_{-\pi}^{\pi} fc = \int_{-\pi}^{\pi} fs = 0 \text{ and } \int_{-\pi}^{\pi} g = \int_{-\pi}^{\pi} gc = \int_{-\pi}^{\pi} gs = 0$$
$$\int_{-\pi}^{\pi} f + g = \int_{-\pi}^{\pi} (f + g)c = \int_{-\pi}^{\pi} (f + g)s = 0 \text{ and } \int_{-\pi}^{\pi} kf = \int_{-\pi}^{\pi} kfc = \int_{-\pi}^{\pi} kfs = 0$$

(by linearity of integration operation). Then S is closed under addition and scalar multiplication. S is a subspace of V.

(2)

$$\int_{-\pi}^{\pi} c(nt) = \frac{1}{n} s(nt) \Big|_{-\pi}^{\pi} = 0$$
$$\int_{-\pi}^{\pi} s(nt) = \frac{-c(nt)}{n} \Big|_{-\pi}^{\pi} = \frac{-(c(n\pi) - c(-n\pi))}{n} = 0$$

In general,

$$\int_{-\pi}^{\pi} c(nt)c(mt) = \int_{-\pi}^{\pi} \frac{1}{2}(\cos{(n-m)t} + \cos{(n+m)t})dt = 0 + 0 = 0$$
$$\int_{-\pi}^{\pi} s(nt)c(mt) = \int_{-\pi}^{\pi} \frac{1}{2}(\sin{(n+m)t} + \sin{(n-m)t})dt = 0 + 0 = 0$$
$$\int_{-\pi}^{\pi} s(nt)s(mt) = \int_{-\pi}^{\pi} \frac{1}{2}(\cos{(n-m)t} - \cos{(n+m)t})dt = 0 + 0 = 0$$

So S contains the functions $f(x) = \cos nx$ and $f(x) = \sin nx$, since they satisfy the requirements.

(3) As seen above, the set $\mathcal{B}_S = \{\sin nx, \cos nx | n = 2, \dots\}$ consists of orthogonal functions with an inner product defined as $\int$ over a period. Thus, they are independent of each other (orthogonal elements are independent). They belong to S and S, being a subspace, must include all linear combinations of them, and so S is at least infinitely dimensional, since it must contain $\mathcal{B}_S$ in its basis.

(4)
$$T(V) = g(x) = \int_{-\pi}^{\pi} (1 + \cos{(x-t)})f(t)dt = \int_{-\pi}^{\pi} (1 + \cos x \cos t + \sin x \sin t)f(t)dt =$$
$$= \int_{-\pi}^{\pi} f(t)dt + \left(\int_{-\pi}^{\pi} \cos t f(t)dt \right) \cos x + \left(\int_{-\pi}^{\pi} f(t) \sin t dt \right) \sin x$$

$\mathcal{B}_{T(V)} = \{1, \cos x, \sin x\}; \quad rank T(V) = 3$

(5) $T(S) = 0 \Longrightarrow nullspace T = S$

(6) $T(f) = cf$. Note that $cf \in \mathcal{B}_{T(V)}$.

$$\boxed{\begin{array}{l} T(1) = 2\pi(1) \\ T(s) = \pi s \\ T(c) = \pi c \end{array}}$$

Exercise 30. We want the following: Let $T: V \to W$ be a linear transformation of a linear space V into a linear space W. If V is infinite-dimensional, prove that at least one of $T(V)$, or $N(T)$, is infinite-dimensional.

Assume $dim N(T) = k$, $dim T(V) = r$.

Let $e_1, \dots, e_k$ be a basis for $N(T)$.

Let $e_1, \dots, e_k, e_{k+1}, \dots, e_{k+n}$ be independent elements in V, where $n > r$.

Consider $x = \sum_{j=1}^{k+n} a_j e_j$

$$T(x) = a_j \sum_{j=1}^{k+n} T(e_j) = a_j \sum_{j=k+1}^{k+n} T(e_j)(\text{since } e_1, \dots, e_k \in N(T))$$

$x \in V$, so $T(x) \in T(V)$. Since $dim T(V) = r$, and $n > r$, $\{T(e_j) | j = k+1, \dots, k+n\}$ must be dependent (Apostol's Thm.1.5 of Vol.2: any set of $r+1$ elements of a $dim = r$ space is dependent).

Then $\exists \{a_j\}$, a_j's not all zero, s.t.

$$\sum_{j=k+1}^{k+n} a_j T(e_j) = T \sum_{j=k+1}^{k+n} (a_j e_j) = 0$$
$$\Longrightarrow \sum_{j=k+1}^{k+n} a_j e_j \in N(T) \text{ so } \sum_{j=k+1}^{k+n} a_j e_J = \sum_{j=1}^{n} a_j e_j$$

$\Longrightarrow \sum_{j=1}^{k+n} a_j e_j = 0$ is a nontrivial representation of 0. Then $e_1, \dots, e_{k+n}$ are dependent. Contradiction.

2.8 Exercises - Introduction, Motivation for the choice of axioms for a determinant function, A set of axioms for a determinant function, Computation of determinants,

Exercise 1. $V = \{0, 1\}$

$$\begin{array}{l} T_1(0,1) = 0,0 \\ T_2(0,1) = 0,1 \\ T_3(0,1) = 1,0 \\ T_4(0,1) = 1,1 \end{array} \qquad \begin{array}{ccccc} 0,1 & T_1 & T_2 & T_3 & T_4 \\ T_1 & 0,0 & 0,0 & 1,0 & 0,0 \\ T_2 & 0,0 & 0,1 & 1,0 & 1,1 \\ T_3 & 1,1 & 1,0 & 0,1 & 0,0 \\ T_4 & 1,1 & 1,1 & 1,1 & 1,1 \end{array}$$

T_2, T_3 are one-to-one, by inspection. $T_2^{-1} = T_2; T_3^{-1} = T_3$

Exercise 2. $V = \{0, 1, 2\}$. Note, there are obviously $3^3 = 27$ possible ranges and thus 27 possible functions (since for each element in V, there are 3 possible values it could be mapped to).
Consider only the 6 that are one-to-one (choice of 3 values, then 2 values, then 1 value at each subsequent stage).

$$T_1(0,1,2) = 0,1,2$$
$$T_2(0,1,2) = 0,2,1$$
$$T_3(0,1,2) = 1,0,2$$
$$T_4(0,1,2) = 1,2,0$$
$$T_5(0,1,2) = 2,0,1$$
$$T_6(0,1,2) = 2,1,0$$

$0,1,2$	T_1	T_2	T_3	T_4	T_5	T_6
T_1	$0,1,2$	$0,2,1$	$1,0,2$	$1,2,0$	$2,0,1$	$2,1,0$
T_2	$0,2,1$	$0,1,2$	$2,0,1$	$2,1,0$	$1,0,2$	$1,2,0$
T_3	$1,0,2$	$1,2,0$	$0,1,2$	$0,2,1$	$2,1,0$	$2,0,1$
T_4	$1,2,0$	$1,0,2$	$2,1,0$	$2,0,1$	$0,1,2$	$0,2,1$
T_5	$2,0,1$	$2,1,0$	$0,2,1$	$0,1,2$	$1,2,0$	$1,0,2$
T_6	$2,1,0$	$2,0,1$	$1,2,0$	$1,0,2$	$0,2,1$	$0,1,2$

$$T_1^{-1} = T_1$$
$$T_2^{-1} = T_2$$
$$T_3^{-1} = T_3$$
$$T_4^{-1} = T_5$$
$$T_5^{-1} = T_4$$
$$T_6^{-1} = T_6$$

Exercise 3. $T(x, y) = (y, x)$ Suppose $T(x_1, y_1) = (y_1, x_1) = T(x_2, y_2) = (y_2, x_2)$.
Then $y_1 = y_2$, $x_1 = x_2 \to (x_1, y_1) = (x_2, y_2)$
$\boxed{T \text{ is one-to-one on } V\,; \qquad T(V_2) = V_2,\ (u, v) = (y, x)}$
$\boxed{T^{-1} = T}$ (by inspection).
Exercise 4. $T(x, y) = (x, -y)$
Suppose $T(x_1, y_1) = (x_1, -y_1) = T(x_2, y_2) = (x_2, -y_2)$
Then $x_1 = x_2$, $-y_1 = -y_2$ or $y_1 = y_2 \Longrightarrow (x_1, y_1) = (x_2, y_2)$
$\boxed{T \text{ is one-to-one on } V\,, \qquad T(V_2) = V_2,\ (u, v) = (x, -y)}$
$\boxed{T^{-1} = T}$
Exercise 5. $T(x, y) = (x, 0)$.
Note that $T(x, 1) = (x, 0) = T(x, 2)$. T is not one-to-one.
Exercise 6. $T(x, y) = (x, x)$. Note that $T(x, 1) = T(x, 2) = (x, x)$. T is not one-to-one.
Exercise 7. $T(x, y) = (x^2, y^2)$. $T(x, y) = T(x, -y) = (x^2, (-y)^2) = (x^2, y^2)$. T is not one-to-one.
Exercise 8. $T(x, y) = (e^x, e^y)$.
Suppose $T(x_1, y_1) = T(x_2, y_2)$.
Then $e^{x_1} = e^{x_2}$, $e^{y_1} = e^{y_2}$ and since e^x is one-to-one, $\forall\, x \in \mathbb{R}$, $x_1 = x_2$.
$\boxed{T \text{ is one-to-one}}$. $u = e^x$, $v = e^y$, $u, v \in \mathbb{R}^+$. $\boxed{T^{-1}(x, y) = (\ln x, \ln y)}$
Exercise 9. $T(x, y) = (x, 1)$ $T(x, 1) = T(x, 2) = (x, 1)$, so T is not one-to-one.
Exercise 10. $T(x, y) = (x + 1, y + 1)$.
If $T(x_1, y_1) = (x_1 + 1, y_1 + 1) = T(x_2, y_2) = (x_2 + 1, y_2 + 1)$,
then $x_1 = x_2$, $y_1 = y_2$, $(x_1, y_1) = (x_2, y_2)$. T is one-to-one.
$u = x + 1$, $v = y + 1$. $T^{-1}(x, y) = (x - 1, y - 1)$
Exercise 11. $T(x, y) = (x - y, x + y)$.
If $T(x_1, y_1) = (x_1 - y_1, x_1 + y_1) = T(x_2, y_2) = (x_2 - y_2, x_2 + y_2)$

$$\begin{aligned} x_1 - y_1 &= x_2 - y_2 \\ x_1 + y_1 &= x_2 + y_2 \end{aligned} \quad \text{then } x_1 = x_2,\ y_1 = y_2.\ T \text{ is one-to-one.}$$

$$\begin{aligned} u &= x - y \\ v &= x + y \end{aligned} \quad T(V_2) = L(\{(1, 1), (-1, 1)\});\, T^{-1}(x, y) = \left(\tfrac{x+y}{2}, \tfrac{-x+y}{2}\right)$$

Exercise 12. $T(x, y) = (2x - y, x + y)$
If $T(x_1, y_1) = (2x_1 - y_1, x_1 + y_1) = T(x_2, y_2) = (2x_2 - y_2, x_2 + y_2)$

$$\begin{aligned} 2x_1 - y_1 &= 2x_2 - y_2 \\ x_1 + y_1 &= x_2 + y_2 \end{aligned} \quad \text{so} \quad \begin{aligned} x_1 &= x_2 \\ y_1 &= y_2 \end{aligned} \quad T \text{ is one-to-one.}$$

$$\begin{aligned} u &= 2x - y \\ v &= x + y \end{aligned} \quad T(V_2) = L(\{(2, 1), (-1, 1)\}).\ T^{-1}(x, y) = \left(\frac{x+y}{3}, \frac{x-2y}{-3}\right)$$

Exercise 13. $T(x, y, z) = (z, y, x)$
If $T(x_1, y_1, z_1) = (z_1, y_1, x_1) = T(x_2, y_2, z_2) = (z_2, y_2, x_2)$

$$\text{then } \begin{aligned} z_1 &= z_2 \\ y_1 &= y_2 \\ x_1 &= x_2 \end{aligned} \Longrightarrow T \text{ is one-to-one.}$$

$T(V_3) = V_3$, $u = z$, $v = y$, $w = z$. $T^{-1} = T$

Exercise 14. $T(x,y,z) = (x,y,0)$
$T(x,y,1) = T(x,y,2) = (x,y,0)$. T is not one-to-one.
Exercise 15. $T(x,y,z) = (x,2y,3z)$
$T(x_1,y_1,z_1) = (x_1,2y_1,3z_1) = T(x_2,y_2,z_2) = (x_2,2y_2,3z_2)$

$$\begin{aligned} x_1 &= x_2 \\ 2y_1 &= 2y_2 \\ 3z_1 &= 3z_2 \end{aligned} \implies \begin{aligned} x_1 &= x_2 \\ y_1 &= y_2 \\ z_1 &= z_2 \end{aligned} \implies T \text{ is one-to-one}$$

$$\begin{aligned} u &= x \\ v &= 2y \\ w &= 3z \end{aligned} \quad T(V_3) = V_3 \quad T^{-1}(x,y,z) = \left(x, \frac{y}{2}, \frac{z}{3}\right)$$

Exercise 16. $T(x,y,z) = (x,y,x+y+z)$ $T(x_1,y_1,z_1) = (x_1,y_1,x_1+y_1+z_1) = T(x_2,y_2,z_2) = (x_2,y_2,x_2+y_2+z_2)$

$$\begin{aligned} x_1 &= x_2 \\ y_1 &= y_2 \\ x_1+y_1+z_1 &= x_2+y_2+z_2 \end{aligned} \implies z_1 = z_2 \text{ so } T \text{ is one-to-one}$$

Exercise 17. $T(x,y,z) = (x+1,y+1,z-1)$
$T(x_1,y_1,z_1) = (x_1+1,y_1+1,z_1-1) = T(x_2,y_2,z_2) = (x_2+1,y_2+1,z_2-1)$

$$\implies \begin{aligned} x_1 &= x_2 \\ y_1 &= y_2 \\ z_1 &= z_2 \end{aligned} \quad T \text{ is one-to-one}$$

$T(V_3) = V_3 + (1,1,-1)$; $\boxed{T^{-1}(x,y,z) = (x-1,y-1,z+1)}$
Exercise 18. $T(x,y,z) = (x+1,y+2,z+3)$
$T(x_1,y_1,z_1) = (x_1+1,y_1+2,z_1+3) = T(x_2,y_2,z_2) = (x_2+1,y_2+2,z_2+3)$

$$\implies \begin{aligned} x_1 &= x_2 \\ y_1 &= y_2 \\ z_1 &= z_2 \end{aligned} \quad T \text{ is one-to-one}$$

$T(V_3) = V_3 + (1,2,3)$ $\boxed{T^{-1}(x,y,z) = (x-1,y-2,z-3)}$
Exercise 19. $T(x,y,z) = (x,x+y,x+y+z)$
$T(x_1,y_1,z_1) = (x_1,x_1+y_1,x_1+y_1+z_1) = T(x_2,y_2,z_2) = (x_2,x_2+y_2,x_2+y_2+z_2)$

$$\begin{aligned} x_1 &= x_2 \\ y_1 &= y_2 \\ z_1 &= z_2 \end{aligned} \quad T \text{ is one-to-one} \qquad \begin{aligned} T(V_3) &= L(\{(1,1,1),(0,1,1),(0,0,1)\}) \\ T^{-1}(x,y,z) &= (x,y-x,z-y) \end{aligned}$$

Exercise 20. $T(x,y,z) = (x+y,y+z,x+z)$

$$T(x_1,y_1,z_1) = (x_1+y_1,y_1+z_1,x_1+z_1) = T(x_2,y_2,z_2) = (x_2+y_2,y_2+z_2,x_2+z_2)$$

$$\implies \begin{aligned} x_1+y_1 &= x_2+y_2 \\ y_1+z_1 &= y_2+z_2 \\ x_1+z_1 &= x_2+z_2 \end{aligned} \text{ or } \begin{aligned} x_1-z_1 &= x_2-z_2 \\ x_1+z_1 &= x_2+z_2 \end{aligned} \implies x_1 = x_2,\ y_1 = y_2,\ z_1 = z_2 \text{ so that } T \text{ is one-to-one}$$

$$T(V_3) = L(\{(1,0,1),(1,1,0),(0,1,1)\})$$

$$\boxed{T^{-1}(x,y,z) = \left(\frac{x+z-y}{2}, \frac{x+y-z}{2}, \frac{y+z-x}{2}\right)}$$

Exercise 21.

$$T^mT^n = T^mTT^{n-1} = (T^mT)T^{n-1} = T^{m+1}T^{n-1} = \dots = T^{m+n}T^0 = T^{m+n}1 = T^{m+n}$$

$$(T^n)^{-1}T^n = (T^{-1})^nT^n = (T^{-1})(T^{-1})^{n-1}T(T^{n-1}) = \dots =$$

$$= \underbrace{(T^{-1})\dots(T^{-1})}_{n \text{ times}}\underbrace{T\dots(T)}_{n \text{ times}} = \underbrace{(T^{-1})\dots(T^{-1})}_{n-1 \text{ times}}(T^{-1}T)\underbrace{T\dots(T)}_{n-1 \text{ times}} = \underbrace{(T^{-1})\dots(T^{-1})}_{n-1 \text{ times}}1\underbrace{T\dots(T)}_{n-1 \text{ times}} = \dots = T^{-1}T = 1$$

Exercise 22.

$$(ST)^n = (ST)(ST)^{n-1} = \cdots = \underbrace{(ST)\dots(ST)}_{n \text{ times}} =$$
$$= (ST)\dots(ST)(ST) = (ST)\dots(S(TS)T) = (ST)\dots(ST)(SSTT) = \cdots =$$
$$= SST(ST)\dots(ST)T = \cdots = S^nT^n$$

Exercise 23. $(T^{-1}S^{-1})(ST) = T^{-1}S^{-1}ST = T^{-1}1T = 1$, so $(ST)^{-1} = T^{-1}S^{-1}$. Its uniqueness is guaranteed by theorem (left inverses, if they exist, are unique).

Exercise 24. $(ST)^{-1}ST = (ST)^{-1}TS = 1$.

Since left inverses are unique (this theorem is important to use here), then $(TS)^{-1} = (ST)^{-1} \Longrightarrow S^{-1}T^{-1} = T^{-1}S^{-1}$

Exercise 25.

$$(S+T)(S+T) = S^2 + ST + TS + T^2 = S^2 + 2ST + T^2$$
$$(S+T)^3 = (S^2 + ST + TS + T^2)(S+T) = S^3 + STS + TS^2 + T^2S + S^2T + ST^2 + TST + T^3 =$$
$$= S^3 + 3S^2T + 3T^2S + T^3$$

Exercise 26. $S(x,y,z) = (z,y,x)$
$T(x,y,z) = (x, x+y, x+y+z)$

(1)

$$(ST) = (x+y+z, x+y, x)$$
$$(TS) = (z, z+y, x+y+z)$$
$$ST - TS = (x+y, x-z, -y-z)$$
$$S^2 = 1$$
$$T^2 = (x, 2x+y, 3x+2y+z)$$
$$(ST)^2 = (3x+2y+z, 2x+2y+z, x+y+z)$$
$$(TS)^2 = (x+y+z, x+2y+2z, x+2y+3z)$$
$$(ST-TS)^2 = (2x+y-z, x+2y+z, -x+2z+y)$$

(2) If $S(x_1,y_1,z_1) = (z_1,y_1,x_1) = S(x_2,y_2,z_2) = (z_2,y_2,x_2)$, then $z_1 = z_2,\ y_1 = y_2,\ x_1 = x_2$
If $T(x_1,y_1,z_1) = (x_1, x_1+y_1, x_1+y_1+z_1) = T(x_2,y_2,z_2) = (x_2, x_2+y_2, x_2+y_2+z_2)$, then $x_1 = x_2,\ y_1 = y_2,\ z_1 = z_2$. Thus S, T are one-to-one.

$$(S^{-1}) = S$$
$$(T^{-1})(x,y,z) = (x, y-x, z-y)$$
$$(ST)^{-1} = T^{-1}S^{-1} = T^{-1}S \qquad T^{-1}S(x,y,z) = (z, y-z, x-y)$$
$$(TS)^{-1} = S^{-1}T^{-1} \qquad S^{-1}T^{-1}(x,y,z) = S(x, y-x, z-y) = (z-y, y-x, x)$$
$$(T-1)(x,y,z) = (0, x, x+y)$$

(3) $(T-1)^2(x,y,z) = (0,0,x)$
$(T-1)^3(x,y,z) = (0,0,0)$ and for all higher powers

Exercise 27. $T(p) = q(x) = \int_0^x p(t)dt$

$$DT(p) = \frac{d}{dx}\int_0^x p(t)dt = p(x) \quad \text{(by first fundamental thm. of calculus)}$$

$TD(p) = \int_0^x dt p'(t)$
Suppose $p = x+1$, $p' = 1$. $\int_0^x dt 1 = x \neq p$

$$nullspace TD = \{c_0 | \text{ where } c_0 \in \mathbb{R}\}$$
$$range TD = \{ \text{ all polynomials } p \text{ s.t. } p(0) = 0 \}$$

Exercise 28. Let V be linear space of all real polynomials $p(x)$.
$D \equiv$ differential operator
T is a linear map from $p(x)$ onto $xp'(x)$

(1) $p(x) = 2 + 3x - x^2 + 4x^3$
$D, T, DT, TD, DT - TD, T^2D^2 - D^2T^2$.

$$Dp = 3 - 2x + 12x^2$$
$$Tp = 3x - 2x^2 + 12x^3 \qquad DT - TD = 3 - 2x + 12x^2$$
$$DTp = 3 - 4x + 36x^2 \qquad T^2D^2 - D^2T^2 = 24x - (-8 + 216x) = 8 - 192x$$
$$TDp = -2x + 24x^2$$

(2) We want $T(p) = p$. Try $p = \sum_{j=0}^{n} a_j x^j$.

$$T(p) = x\sum_{j=0}^{n} j a_j x^{j-1} = \sum_{j=0}^{n} j a_j x^j = \sum_{j=0}^{n} a_j x^j \Longrightarrow \begin{array}{c} \sum_{j=0}^{n} a_j(j-1)x^j = 0 \\ a_j(j-1)x^j = 0 \end{array} \Longrightarrow j = 1$$

$p = a_1 x$

(3) We want $(DT - 2D)(p) = 0$ or $DT(p) - 2D(p)$

$$DT(p) = \sum_{j=0}^{n} j^2 a_j x^{j-1} = 2\sum_{j=0}^{n} j a_j x^{j-1} \Longrightarrow \sum_{j=0}^{n}(j^2 - 2j)a_j x^{j-1} = \sum_{j=0}^{n} j(j-2)a_j x^{j-1} = 0$$

$$j = 2, 0 \text{ so that } p = a_2 x^2 + a_0$$

(4) We want $(DT - TD)^n(p) = D^n(p)$.

$$D\sum_{j=0}^{n} a_j x^j = \sum_{j=0}^{n} j a_j x^{j-1}$$

$$(DT)\sum_{j=0}^{n} a_j x^j = \sum_{j=0}^{n} j^2 a_j x^{j-1} \qquad (DT - TD)p = \sum_{j=0}^{n} j a_j x^{j-1} = Dp$$

$$TD = \sum_{j=0}^{n} j(j-1)a_j x^{j-1}$$

$$D^n = (DT - TD)^n \quad \forall\, p \in V$$

Exercise 29. $xp(x)$. $T(p) = xp$.

$$\begin{aligned} DT(p) &= D(xp) = p + xp' \\ TD(p) &= Tp' = xp' \end{aligned} \Longrightarrow (DT - TD)(p) = p$$

$$T^n(p) = T^{n-1}(xp) = \cdots = T(x^{n-1}p) = x^n p$$

$$\begin{aligned} DT^n(p) &= nx^{n-1}p + x^n p' \\ T^n D(p) &= T^n(p') = T^{n-1}(xp') = \cdots = T(x^{n-1}p') = x^n p' \end{aligned} \Longrightarrow (DT^n - T^n D)(p) = nx^{n-1}p = nT^{n-1}(p)$$

Exercise 30.

$$n = 1,\ ST - TS = 1$$
$$n = 2,\ ST^2 - T^2 S = ST^2 + T(1 - ST) = T + T = 2T$$
$$\text{Assume the } n\text{th case, } ST^n - T^n S = nT^{n-1}$$
$$ST^{n+1} - T^{n+1}S = ST^n T + T^n(1 - ST) = (nT^{n-1})T + T^n = (n+1)T^n$$

Exercise 31. $p(x) = \sum_{j=0}^{n} c_j x^j$.

$$\begin{aligned} Rp = r &\quad = r(x) = p(0) \\ Sp = s &\quad = s(x) = \sum_{k=1}^{n} c_k x^{k-1} \\ Tp = t &\quad = t(x) = \sum_{k=0}^{n} c_k x^{k+1} \end{aligned}.$$

(1) $p(x) = 2 + 3x - x^2 + x^3$. We want to know $R, S, T, ST, TS, (TS)^2, T^2S^2, S^2T^2, TRS, RST$.

$Rp = p(0) = 2$

$Sp = 3 - x + x^2$

$Tp = 2x + 3x^2 - x^3 + x^4$

$ST(p) = 2 + 3x - x^2 + x^3$

$TS(p) = 3x - x^2 + x^3$

$(TS)^2(p) = 3x - x^2 + x^3$

$T^2S^2 = T^2(-1 + x) = -x^2 + x^3$

$S^2T^2 = S^2(2x^2 + 3x^3 - x^4 + x^5) = 2 + 3x - x^2 + x^3$

$TRSp = 3x$

$RSTp = 2$

(2) R, S, T linear?

$$R(c_1p_1 + c_2p_2) = (c_1p_1 + c_2p_2)(0) = c_1p_1(0) + c_2p_2(0) = c_1R(p_1) + c_2R(p_2)$$

$$c_1S(p_1)+c_2S(p_2)=c_1\sum_{j=1}^{n_1}a_jx^{j-1}+c_2\sum_{j=1}^{n_2}b_jx^{j-1}=\sum_{j=1}^{n_2}h_jx^{j-1}=S(c_1p_1+c_2p_2)$$

where $n_2\geq n_1$, without loss of generality, and $h_j=\begin{cases} c_1a_j+c_2b_j & \text{for } j=0,\dots n_1 \\ c_2b_j & \text{for } j=n_1+1\dots n_2\end{cases}$

and indeed , $c_1p_1+c_2p_2=\sum_{j=0}^{n_2}h_jx^j$

$$c_1p_1+c_2p_2=c_1\sum_{j=0}^{n}a_jx^j+c_2\sum_{j=0}^{n_2}b_jx^j=\sum_{j=0}^{n_1}c_1a_jx^j+\sum_{j=0}^{n_2}c_2b_jx^j=\sum_{j=0}^{n_2}h_jx^j$$

$$h_j=\begin{cases} c_1a_j+c_2b_j & \text{for } j=0,\dots,n_1 \\ c_2b_j & \text{for } j=n_1+1,\dots n_2\end{cases}$$

$$\begin{aligned} T(c_1p_1+c_2p_2) &= \sum_{j=0}^{n}h_jx^{j+1}=\sum_{j=0}^{n_1}(c_1a_j+c_2b_j)x^{j+1}+\sum_{j=n_1+1}^{n_2}(c_2b_j)x^{j+1} \\ &= c_1\sum_{j=0}^{n_1}a_jx^{j+1}+c_2\sum_{j=0}^{n_2}b_jx^{j+1}=c_1T(p_1)+c_2T(p_2)\end{aligned}$$

(3)

$$Rp=p(0)\Longrightarrow \begin{aligned} &nullspace R=\{p|\text{ polynomial } p \text{ of degree }\geq 1\ \} \\ &range R=\{c_0|c_0\in\mathbb{R}\}\end{aligned}$$

$$Sp=\sum_{k=1}^{n}c_kx^{k-1}\quad\Longrightarrow \begin{aligned} &nullspace S=\{c_0|c_0\in\mathbb{R}\} \\ &range S=\{p|\text{ polynomial } p \text{ of degree } n-1\ \}=V\end{aligned}$$

$$Tp=\sum_{j=0}^{n}c_jx^{j+1}\quad \begin{aligned} &nullspace T=0 \\ &range T=\{p|\text{ polynomial of degree }\ \geq 1\}\end{aligned}$$

(4) T is linear. $nullspace T=0$. By thm., T is one-to-one. This thm. for linear transformations is very useful because we simply need to check if the nullspace only contains 0.

(5) If $n\geq 1$, $(TS)^n=\boxed{(1-R)^n}$ since $TS(p)=p-R(p)=(1-R)(p)$.
$S^nT^n=1$

Exercise 32. If $x=\{x_j\}$ is a convergent sequence, $\lim_{j\to\infty}x_j=a$, let $T(x)=\{y_n\}$, $y_n=a-x_n$ for $n\geq 1$.
V = linear space of all real convergent sequences $\{x_j\}$.
T is linear, since

$$\begin{aligned} T(c_1x_1+c_2x_2) &= \{c_1a_1+c_2a_2-(c_1x_{1j}+c_2x_{2j})\}=\{c_1(a_1-x_{1j})+c_2(a_2-x_{2j})\}=c_1\{(a_1-x_{1j})\}+c_2\{(a_2-x_{2j})\}= \\ &= c_1T(x_1)+c_2T(x_2)\end{aligned}$$

where $\lim_{j\to\infty}(c_1x_{1j}+c_2x_{2j})=c_1a_1+c_2a_2$.
Note that all sequences of a constant number, constant sequences, get mapped to the same sequence of zeroes. Thus, T is not one-to-one.

2.12 Exercises - Linear transformations with prescribed values, Matrix representations of linear transformations, Construction of a matrix representation in diagonal form

Exercise 1.

(1) $a_{ij}=\delta_{ij}$
(2) $a_{ij}=0$
(3) $a_{ij}=c\delta_{ij}$

Exercise 2.

(1) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0\end{bmatrix}$

(2) $\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0\end{bmatrix}$

(3) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1\end{bmatrix}$

Exercise 3.

(1) $$\begin{aligned} T(3i-4j) &= -5i+7j \\ T^2(3i-4j) &= 9i-12j \end{aligned}$$

(2) $T = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} \qquad T^2 = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

(3)

$$e_1 = i - j$$
$$e_2 = 3i + j$$
$$T(e_1) = T(i-j) = i+j-2i+j = -i+2j = -\left(\frac{e_1+e_2}{4}\right) + 2\left(\frac{e_2-3e_1}{4}\right) \qquad = \frac{-7e_1+e_2}{4}$$
$$T(e_2) = T(3i+j) = 3i+3j+2i-j = 5i+2j = 5\left(\frac{e_1+e_2}{4}\right) + 2\left(\frac{e_2-3e_1}{4}\right) \qquad = \frac{-e_1+7e_2}{4}$$
$$T = \frac{1}{4}\begin{bmatrix} -7 & -1 \\ 1 & 7 \end{bmatrix}, \qquad T^2 = 12\begin{bmatrix} 1 & \\ & 1 \end{bmatrix}$$

Exercise 4.

$$T = 2\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \qquad T^2 = 4\begin{bmatrix} 1 & \\ & 1 \end{bmatrix}$$

Exercise 5. $T : V_3 \to V_3$ be a linear transformation s.t.

(1) $$\begin{aligned} T(k) &= 2i+3j+5k \\ T(j+k) &= i \\ T(i+j+k) &= j-k \end{aligned} \qquad \Longrightarrow T(i+2j+3k) = \boxed{3i+4j+4k}$$

$$\begin{aligned} T(i) &= -i+j-k \\ T(j) &= -i-3j-5k \\ T(k) &= 2i+3j+5k \end{aligned} \qquad T(c_1 i + c_2 j + c_3 k) = (-c_1-c_2+2c_3)i + (c_1-3c_2+3c_3)j + (-c_1-5c_2+5c_3)k = 0$$

So we have a system of linear equations,

$$\begin{aligned} -c_1-c_2+2c_3 &= 0 \\ c_1-3c_2+3c_3 &= 0 \\ -c_1-5c_2+5c_3 &= 0 \end{aligned}$$

to solve, which could be done by Gauss-Jordan or simply to add them up cleverly to get $c_3 = 0$ first, and then $c_2, c_1 = 0$.

$\Longrightarrow nullspaceT = 0, \quad nullT = 0. \ rangeT = V_3, \quad rankT = 3$

(2) $T = \begin{bmatrix} -1 & -1 & 2 \\ 1 & -3 & 3 \\ -1 & -5 & 5 \end{bmatrix}$

Exercise 6.

$$\begin{aligned} e_1 &= (2,3,5) & T(e_1) &= 2(-1,1,-1)+3(-1,-3,5)+5(2,3,5) = (5,8,8) = 2e_1+2e_3+e_2 \\ e_2 &= (1,0,0) & T(e_2) &= (-1,1,-1) = -e_2+e_3 \\ e_3 &= (0,1,-1) & T(e_3) &= (-1,-3,-5)+(-2,-3,-5) = (-3,-6,-10) = -2e_1+2e_2 \end{aligned}$$

$$T = \begin{bmatrix} 2 & 0 & -2 \\ 1 & -1 & 1 \\ 2 & 1 & 0 \end{bmatrix}$$

Exercise 7. Given $T(0) = (0,0)$, $T(j) = (1,1)$, $T(k) = (1,-1)$

(1) $T(4i-j+k) = (-1,-1)+(1,-1) = (0,-2)$

Determine the nullspace of T by considering

$$T(x) = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = 0 \Longrightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So then $nullspaceT = L(\{(1,0,0)\})$.

$nullT = 1$
$rangeT = 2$ (by nullity-rank thm.)

(2) $T = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$

(3) $\begin{matrix} w_1 = (1,1) \\ w_2 = (1,2) \end{matrix}$ $\quad \begin{matrix} T(i) = 0 \\ T(j) = w_1 \\ T(k) = 3w_1 - 2w_2 \end{matrix}$ $\quad T = \begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & -2 \end{bmatrix}$

(4)

$$\boxed{\begin{aligned} e_1 &= i \\ e_2 &= j \\ e_3 &= \frac{3j-k}{2} \end{aligned}}$$

Exercise 8. Given $\begin{matrix} T(i) = (1,0,1) \\ T(j) = (-1,0,1) \end{matrix}$,

(1) $T(2i-3j) = (2,0,2) + (3,0,-3) = (5,0,-1)$. $\begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} \xrightarrow{\text{Gauss-Jordan elimination}} \begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \Longrightarrow c_1 = c_2 = 0.$

So then $\begin{matrix} nullT = 0 \\ rankT = 2 \end{matrix}$

(2) Again, $T = \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$

(3)

$$\begin{aligned} e_1 &= \frac{i-j}{2} & T(e_1) = T\left(\frac{i-j}{2}\right) &= (1,0,0) & w_1 &= (1,0,0) \\ & & & & w_2 &= (0,0,1) = k \\ e_2 &= \frac{i+j}{2} & T\left(\frac{i+j}{2}\right) &= (0,0,1) = w_2 = k & w_3 &= (0,1,0) \end{aligned}$$

Exercise 9. Given
$T(i) = (1,0,1)$
$T(j) = (1,1,1)$

(1) $T(2i-3j) = (2,0,2) - (3,3,3) = (-1,-3,-1)$. By inspection of the matrix for T, $\begin{matrix} nullT = 0 \\ rankT = 2 \end{matrix}$.

(2) $T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$

(3) Note that

$$\begin{matrix} T(j-i) = (0,1,0) \\ T(i) = (1,0,1) \end{matrix} \qquad \begin{matrix} w_1 = (1,0,1) \\ w_2 = (0,1,0) \\ w_3 = (0,0,1) \end{matrix} \qquad \Longrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Exercise 10. Let V and W be linear spaces, each with dimension 2.

(1) Given $\begin{matrix} T(e_1+e_2) = 3e_1 + 9e_2 \\ T(3e_1+2e_2) = 7e_1 + 23e_2 \end{matrix}$, then $\begin{matrix} T(-e_2) = -2e_1 + -4e_2 = -(2e_1+4e_2) \\ T(e_1) = e_1 + 5e_2 \end{matrix}$, so that $\boxed{T(e_2 - e_1) = e_1 - e_2}$.

By inspection of matrix T, $\begin{matrix} nullT = 0 \\ rankT = 2 \end{matrix}$

(2) $T = \begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix}$

(3) With a basis of (e_1, e_2) for V and a desired basis of the form $(e_1 + ae_2, 2e_1 + be_2)$ for W,

$$\begin{aligned} T(e_1) &= e_1 + 5e_2 & &\Longrightarrow a = 5 \\ T(e_2) &= 2e_1 + 4e_2 & &\Longrightarrow b = 4 \end{aligned}$$

Exercise 11. $(\sin x, \cos x)$

$$D(s,c) = (c,-s) \Longrightarrow D = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad D^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Exercise 12. $(1, x, e^x)$

$$D(1,x,e^x)=(0,1,e^x) \qquad D=\begin{bmatrix}0&1&0\\0&0&0\\0&0&1\end{bmatrix} \qquad D^2=\begin{bmatrix}&1&\\&&\\&&1\end{bmatrix}\begin{bmatrix}&1&\\&&\\&&1\end{bmatrix}=\begin{bmatrix}&&\\&&\\&&1\end{bmatrix}$$

Exercise 13. $(1,1+x,1+x+e^x)$

$$D(1,1+x,1+x+e^x) \qquad D=\begin{bmatrix}0&1&1\\0&0&-1\\0&0&1\end{bmatrix} \qquad D^2=\begin{bmatrix}&1&1\\&&-1\\&&1\end{bmatrix}\begin{bmatrix}&1&1\\&&-1\\&&1\end{bmatrix}=\begin{bmatrix}&&\\&&-1\\&&1\end{bmatrix}$$

Exercise 14. (e^x,xe^x)

$$D(e^x,xe^x)=(e^x,e^x+xe^x) \qquad D=\begin{bmatrix}1&1\\0&1\end{bmatrix} \qquad D^2=\begin{bmatrix}1&1\\&1\end{bmatrix}\begin{bmatrix}1&1\\&1\end{bmatrix}=\begin{bmatrix}1&2\\&1\end{bmatrix}$$

Exercise 15. $(-c,s)$.

$$D(-c,s)=(s,c) \qquad D=\begin{bmatrix}0&-1\\1&0\end{bmatrix} \qquad D^2=\begin{bmatrix}&-1\\1&\end{bmatrix}\begin{bmatrix}&-1\\1&\end{bmatrix}=\begin{bmatrix}-1&\\&-1\end{bmatrix}$$

Exercise 16. $(\sin x,\cos x,x\sin x,x\cos x)$

$$D(s,c,xs,xc)=(c,-s,s+xc,c+-xs)$$

$$D=\begin{bmatrix}0&-1&1&0\\1&0&0&1\\0&0&0&-1\\0&0&1&0\end{bmatrix} \qquad D^2=\begin{bmatrix}&-1&1&\\1&&&1\\&&&-1\\&&1&\end{bmatrix}\begin{bmatrix}&-1&1&\\1&&&1\\&&&-1\\&&1&\end{bmatrix}=\begin{bmatrix}-1&&-2&\\&-1&2&\\&&-1&\\&&&-1\end{bmatrix}$$

Exercise 17. $(e^x\sin x,e^x\cos x)$

$$D(e^xs,e^xc)=(e^xs+e^xc,e^xc-e^xs) \qquad D=\begin{bmatrix}1&-1\\1&1\end{bmatrix} \qquad D^2=\begin{bmatrix}1&-1\\1&1\end{bmatrix}\begin{bmatrix}1&-1\\1&1\end{bmatrix}=\begin{bmatrix}&-2\\2&\end{bmatrix}$$

Exercise 18. $(e^{2x}\sin 3x,e^{2x}\cos 3x)$

$$D(e^{2x}s,e^{2x}c)=(2e^{2x}s+3e^{2x}c,2e^{2x}c+-3e^{2x}s) \qquad D=\begin{bmatrix}2&-3\\3&2\end{bmatrix} \qquad D^2=\begin{bmatrix}2&-3\\3&2\end{bmatrix}\begin{bmatrix}2&-3\\3&2\end{bmatrix}=\begin{bmatrix}-5&-12\\12&-5\end{bmatrix}$$

Exercise 19. $(1,x,x^2,x^3)$. $T(p)=xp'$.

$D(1,x,x^2,x^3)=(0,1,2x,3x^2)$

$T(1,x,x^2,x^3)=(0,x,2x^2,3x^3)$

(1) $T=\begin{bmatrix}0&0&0&0\\0&1&0&0\\0&0&2&0\\0&0&0&3\end{bmatrix}$

(2) $DT=\begin{bmatrix}0&1&0&0\\0&0&4&0\\0&0&0&9\\0&0&0&0\end{bmatrix}$

(3) $TD=\begin{bmatrix}0&0&0&0\\0&0&2&0\\0&0&0&6\\0&0&0&0\end{bmatrix}$

(4) $TD-DT=\begin{bmatrix}0&-1&0&0\\0&0&-2&0\\0&0&0&-3\\0&0&0&0\end{bmatrix}$

(5) $T^2=\begin{bmatrix}&1&&\\&&2&\\&&&3\end{bmatrix}\begin{bmatrix}&1&&\\&&2&\\&&&3\end{bmatrix}=\begin{bmatrix}&1&&\\&&4&\\&&&9\end{bmatrix}$

(6)

$$T^2D^2 = \begin{bmatrix} & & & \\ & 1 & & \\ & & 4 & \\ & & & 9 \end{bmatrix}\begin{bmatrix} & & 2 & \\ & & & 6 \\ & & & \\ & & & \end{bmatrix} = \begin{bmatrix} & & & \\ & & & 6 \\ & & & \\ & & & \end{bmatrix}$$

$$D^2T^2 = \begin{bmatrix} & & 2 & \\ & & & 6 \\ & & & \\ & & & \end{bmatrix}\begin{bmatrix} & & & \\ & 1 & & \\ & & 4 & \\ & & & 9 \end{bmatrix} = \begin{bmatrix} & & 8 & \\ & & & 54 \\ & & & \\ & & & \end{bmatrix}$$

$$T^2D^2 - D^2T^2 = \begin{bmatrix} & & -8 & \\ & & & -48 \\ & & & \\ & & & \end{bmatrix}$$

Exercise 20. $TD = \begin{bmatrix} & & & \\ & & 2 & \\ & & & 6 \\ & & & \end{bmatrix}$.

Note that $(TD)(x^3, x^2, x, 1) = (6x^2, 2x, 0, 0)$, so if we let $\begin{matrix} w_1 = x^2 \\ w_2 = x \end{matrix}$, then $(TD) = \begin{bmatrix} 6 & & \\ & 2 & \end{bmatrix}$

2.16 Exercises - Linear spaces of matrices, Isomorphism between linear transformations and matrices, Multiplication of matrices

Exercise 1. $A = \begin{bmatrix} 1 & -4 & 2 \\ -1 & 4 & -2 \end{bmatrix}$, $\quad B = \begin{bmatrix} 1 & 2 \\ -1 & 3 \\ 5 & -2 \end{bmatrix}$, $\quad C = \begin{bmatrix} 2 & 2 \\ 1 & -1 \\ 1 & -3 \end{bmatrix}$

$$B + C = \begin{bmatrix} 3 & 4 & & \\ 0 & 2 & 6 & -5 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & -4 & 2 \\ -1 & 4 & -2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & 3 \\ 5 & -2 \end{bmatrix} = \begin{bmatrix} 15 & -14 \\ -15 & 14 \end{bmatrix} \qquad AC = \begin{bmatrix} 1 & -4 & 2 \\ -1 & 4 & -2 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ 1 & -1 \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 2 \\ -1 & 3 \\ 5 & -2 \end{bmatrix}\begin{bmatrix} 1 & -4 & 2 \\ -1 & 4 & -2 \end{bmatrix} = \begin{bmatrix} -1 & 4 & -2 \\ -4 & 16 & -8 \\ 7 & -28 & 14 \end{bmatrix} \qquad CA = \begin{bmatrix} 2 & 2 \\ 1 & -1 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} 1 & -4 & 2 \\ -1 & 4 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 2 & -8 & 4 \\ 4 & -16 & 8 \end{bmatrix}$$

$$A(2B - 3C) = 2AB - 3AC = \begin{bmatrix} 30 & -28 \\ -30 & 28 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 30 & -28 \\ -30 & 28 \end{bmatrix}$$

Exercise 2. $A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$

(1) $AB = \begin{bmatrix} & 1 \\ & 2 \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} b_{21} & b_{22} \\ 2b_{21} & 2b_{22} \end{bmatrix} = 0$

$\implies b_{21} = b_{22} = 0$ or $\begin{bmatrix} b_{11} & 0 \\ b_{21} & 0 \end{bmatrix}$

(2) $BA = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}\begin{bmatrix} & 1 \\ & 2 \end{bmatrix} = \begin{bmatrix} & b_{11} + 2b_{12} \\ & b_{21} + 2b_{22} \end{bmatrix} \implies \begin{matrix} b_{11} = -2b_{12} \\ b_{21} = -2b_{22} \end{matrix}$

$$\begin{bmatrix} b_{11} & b_{11}/-2 \\ b_{21} & b_{21}/-2 \end{bmatrix} = b_{11}\begin{bmatrix} 1 & -1/2 \\ 0 & 0 \end{bmatrix} + b_{21}\begin{bmatrix} 0 & 0 \\ 1 & -1/2 \end{bmatrix}$$

Exercise 3.

(1) $\begin{bmatrix} & & 1 & \\ 1 & & & \\ & 1 & & \\ & & & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ 9 \\ 6 \\ 5 \end{bmatrix} \implies \begin{matrix} c = 1 \\ a = 9 \\ b = 6 \\ d = 5 \end{matrix}$

(2)

$$\begin{bmatrix} a & b & c & d \\ 1 & 4 & 9 & 2 \end{bmatrix}\begin{bmatrix} 1 & & 2 & \\ & & 1 & \\ & 1 & & \\ & & 1 & \end{bmatrix} = \begin{bmatrix} 1 & & 6 & 6 \\ 1 & 9 & 8 & 4 \end{bmatrix} \qquad \begin{matrix} a = 1 \\ c = 0 \\ b = 6 \\ d = -2 \end{matrix}$$

Exercise 4. $AB - BA$

(1)

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

$$B = \begin{bmatrix} 4 & 1 & 1 \\ -4 & 2 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

$$BA = \begin{bmatrix} 4 & 1 & 1 \\ -4 & 2 & 0 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 7 & 11 & 13 \\ & -6 & -4 \\ 6 & 6 & 9 \end{bmatrix}$$

$$AB = \begin{bmatrix} -2 & 9 & 3 \\ 6 & 10 & 4 \\ -1 & 11 & 4 \end{bmatrix}$$

$$AB - BA = \boxed{\begin{bmatrix} -9 & -2 & -10 \\ 6 & 16 & 8 \\ -7 & 5 & -5 \end{bmatrix}}$$

(2)

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 2 \\ -1 & 2 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 3 & 1 & -2 \\ 3 & -2 & 4 \\ -3 & 5 & 11 \end{bmatrix}$$

$$AB = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 2 \\ -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & -2 \\ 3 & -2 & 4 \\ -3 & 5 & 11 \end{bmatrix} = \begin{bmatrix} 6 & 2 & -4 \\ 0 & 9 & 24 \\ 0 & 0 & 21 \end{bmatrix}$$

$$BA = \begin{bmatrix} 3 & 1 & -2 \\ 3 & -2 & 4 \\ -3 & 5 & 11 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 2 \\ -1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 9 & -3 & 0 \\ 0 & 6 & 0 \\ -12 & 27 & 21 \end{bmatrix}$$

$$AB - BA = \begin{bmatrix} 6 & 2 & -4 \\ 0 & 9 & 24 \\ 0 & 0 & 21 \end{bmatrix} - \begin{bmatrix} 9 & -3 & 0 \\ 0 & 6 & 0 \\ -12 & 27 & 21 \end{bmatrix} = \begin{bmatrix} -3 & 5 & -4 \\ 0 & 3 & 24 \\ 12 & -27 & 0 \end{bmatrix}$$

Exercise 5. $A^n A^m = A^{m+n}$.

Matrix multiplication is associative; $A^n A^m = A^{n-1}(AA^m) = A^{n-1}A^{m+1} = \cdots = A^0 A^{m+n} = A^{m+n}$

Exercise 6. $A = \begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix} \qquad A^2 = \begin{bmatrix} 1 & 2 \\ & 1 \end{bmatrix}$

$$A^3 = \begin{bmatrix} 1 & 2 \\ & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ & 1 \end{bmatrix}$$

Assume nth case is true $A^n = \begin{bmatrix} 1 & n \\ & 1 \end{bmatrix} \qquad A^{n+1} = AA^n = \begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix} \begin{bmatrix} 1 & n \\ & 1 \end{bmatrix} = \begin{bmatrix} 1 & n+1 \\ & 1 \end{bmatrix}$

Exercise 7. $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$$A^2 = \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \begin{bmatrix} c & -s \\ s & c \end{bmatrix} = \begin{bmatrix} c^2 - s^2 & -2sc \\ 2sc & -s^2 + c^2 \end{bmatrix} = \begin{bmatrix} \cos(2\theta) & -\sin(2\theta) \\ \sin(2\theta) & \cos(2\theta) \end{bmatrix}$$

Assume nth case is true : $A^n = \begin{bmatrix} \cos(n\theta) & -\sin(n\theta) \\ \sin(n\theta) & \cos(n\theta) \end{bmatrix}$

$$A^{n+1} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos(n\theta) & -\sin(n\theta) \\ \sin(n\theta) & \cos(n\theta) \end{bmatrix} = \begin{bmatrix} \cos(n\theta)\cos\theta - \sin(n\theta)\sin\theta & -\cos\theta\sin(n\theta) - \sin\theta\cos(n\theta) \\ \cos(n\theta)\sin\theta + \cos\theta\sin(n\theta) & -\sin\theta\sin(n\theta) + \cos(n\theta)\cos\theta \end{bmatrix}$$
$$= \begin{bmatrix} \cos(n+1)\theta & -\sin(n+1)\theta \\ \sin(n+1)\theta & \cos(n+1)\theta \end{bmatrix}$$

Exercise 8. Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$
$$A^3 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 6 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$
$$A^4 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 6 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 10 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

(Assume nth case is true)

$$A^n = \begin{bmatrix} 1 & n & \frac{n(n+1)}{2} \\ & 1 & n \\ & & 1 \end{bmatrix}$$
$$A^{n+1} = \begin{bmatrix} 1 & 1 & 1 \\ & 1 & 1 \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & n & \frac{n(n+1)}{2} \\ & 1 & n \\ & & 1 \end{bmatrix} = \begin{bmatrix} 1 & n+1 & \frac{n^2+n}{2} + \frac{2n}{2} + \frac{2}{2} \\ & 1 & n+1 \\ & & 1 \end{bmatrix} =$$
$$= \begin{bmatrix} 1 & n+1 & \frac{(n+2)(n+1)}{2} \\ & 1 & n+1 \\ & & 1 \end{bmatrix}$$

Exercise 9. Given $\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ -2 & 2 \end{bmatrix} - \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}$$

Consider that

$$A^3 = 2A^2 - A = 2(2A - 1) - A = 3A - 2$$
$$A^4 = (2A - 1)(2A - 1) = 4A^2 - 4A + 1 = 4(2A - 1) - 4A + 1 = 4A - 3$$

Then assume the nth case, that $A^n = nA - (n - 1)$.

$$A^{n+1} = nA^2 - (n - 1)A = n(2A - 1) - (n - 1)A = 2nA - n - nA + A =$$
$$= (n + 1)A - n$$

So for $n = 100$, we have $A^{100} = \begin{bmatrix} 1 & \\ 100 & 1 \end{bmatrix}$

Exercise 10. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$,

$$A^2 = \begin{bmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{bmatrix} = \begin{bmatrix} a^2 + bc & b(a + d) \\ (a + d)c & bc + d^2 \end{bmatrix}$$

If $b = 0, d = 0, a = 0$, so $c = 0$. So the only other way for $A^2 = 0$ is for $a = -d \neq 0$. $a^2 + bc = 0$ or $a^2 = -bc$. For instance,

$$\begin{bmatrix} 1 & \pm 1 \\ \mp 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & \pm 1 \\ \mp 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Exercise 11. Let

$$A_{ij} = a_{ij} \qquad \begin{aligned} (E_{11})_{ij} &= \delta_{1i}\delta_{1j} \\ (E_{12})_{ij} &= \delta_{1i}\delta_{2j} \\ (E_{21})_{ij} &= \delta_{2i}\delta_{1j} \\ (E_{22})_{ij} &= \delta_{2i}\delta_{2j} \end{aligned}$$

(1) If $AB - BA = [A, B] = 0 \quad \forall B \in M_{22}$, then since $E_{ij} \in M_{22}$, $[A, E_{ij}] = 0 \quad \forall i = 1, 2, 3, 4$.

If $[A, E_{ij}] = 0$, then since $\forall B \in M_{22}$, $B = \sum b_{ij}E_{ij}$, so that

$$[A, B] = [A, \sum b_{ij}E_{ij}] = \sum b_{ij}[A, E_{ij}] = 0$$

(2) Given $[A, E_{ij}] = 0$,

$$\begin{aligned} (AE_{lm})_{ij} &= \sum_{k=1}^{2} a_{ik}(E_{lm})_{kj} = \sum_{k=1}^{2} a_{ik}\delta_{lk}\delta_{mj} = \\ &= a_{il}\delta_{mj} \\ (E_{lm}A)_{ij} &= \sum_{k=1}^{2} (E_{lm})_{ik}a_{kj} = \sum_{k=1}^{2} \delta_{li}\delta_{mk}a_{kj} = \\ &= \delta_{li}a_{mj} \end{aligned} \implies (AE_{lm} - E_{lm}A)_{ij} = a_{il}\delta_{mj} - \delta_{li}a_{mj} = 0 \text{ or } a_{il}\delta_{mj} = \delta_{li}a_{mj}$$

$$\text{if } m = j,\ a_{il} = \delta_{li} a_{jj}, \qquad \begin{matrix} \text{if } l = i,\ a_{ii} = a_{jj} \\ \text{if } l \neq i,\ a_{il} = 0 \end{matrix}$$

$$\text{if } m \neq j,\ \delta_{li} a_{mj} = 0, \qquad \begin{matrix} \text{if } l = i,\ a_{mj} = 0 \\ \text{if } l \neq i,\ a_{mj} \text{ unknown} \end{matrix}$$

i, j is completely arbitrary, and $(AE_{lm} - E_{lm}A)_{ij} = 0$ must be true $\forall\, i = 1, 2,\ j = 1, 2$, then $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} = aI$ is the A.

Exercise 12. Suppose A s.t. $A^2 = I$.

$(A^2)_{ij} = \sum_{k=1}^{2} a_{ik} a_{kj} = \delta_{ij}$

$$\text{if } i = j, \quad \sum_{k=1}^{2} a_{ik} a_{ki} = 1 \implies a_{i1} a_{1i} + a_{i2} a_{2i} = 1$$

$$\text{if } i \neq j, \quad \sum_{k=1}^{2} a_{ik} a_{kj} = 0 \implies a_{i1} a_{1j} = -a_{i2} a_{2j}$$

If $a_{11} = 0$, $a_{12} a_{21} = 1$ but $0 = -a_{12} a_{21}$. Similar if $a_{22} = 0$. Then $a_{11}, a_{22} \neq 0$

If $a_{12} = 0$,
$a_{11}^2 = 1$
$a_{22}^2 = 1$

if $a_{21} = 0$, then $A = \pm \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \pm \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
if $a_{21} \neq 0$, $a_{21} a_{11} = -a_{22} a_{21} \implies a_{11} = -a_{22}$
If $a_{21} = 0$, $a_{11}^2 = a_{22}^2 = 1$
if $a_{21} \neq 0$, then $a_{11} a_{12} = -a_{12} a_{22} \implies a_{11} = -a_{22}$
If $a_{12}, a_{21} \neq 0$, then
$a_{11}^2 + a_{12} a_{21} = 1 a_{22}^2 + a_{12} a_{21} \quad = 1$

$$\implies \begin{bmatrix} \sqrt{1 - bc} & b \\ c & -\sqrt{1 - bc} \end{bmatrix}$$

Exercise 13. Given
$A = \begin{bmatrix} 2 & -1 \\ -2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 7 & 6 \\ 9 & 8 \end{bmatrix}$. Find 2×2 matrices C and D s.t. $\begin{matrix} AC = B \\ DA = B \end{matrix}$

$$A^{-1} = \frac{1}{4} \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix} \qquad \begin{matrix} A^{-1}AC = C = A^{-1}B \\ DAA^{-1} = D = BA^{-1} \end{matrix} \qquad \begin{matrix} \frac{1}{4} \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 7 & 6 \\ 9 & 8 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 30 & 26 \\ 32 & 28 \end{bmatrix} & = C \\ \begin{bmatrix} 7 & 6 \\ 9 & 8 \end{bmatrix} \frac{1}{4} \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 33 & 19 \\ 43 & 25 \end{bmatrix} & = D \end{matrix}$$

Exercise 14.

(1)

$$\begin{matrix} AB = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} & = \begin{bmatrix} 0 & -2 \\ 2 & 4 \end{bmatrix} \\ BA = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} & = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \end{matrix} \qquad AB \neq BA$$

(2)

$$(A + B)^2 = A^2 + AB + BA + B^2$$
$$(A + B)(A - B) = A^2 - AB + BA - B^2$$

(3) $[A, B] = 0$

2.20 Exercises - Systems of linear equations, Computation techniques, Inverses of square matrices

Exercise 1.

$$\begin{bmatrix} 1 & 1 & 3 \\ 2 & -1 & 4 \\ & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \\ 3 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 5 \\ 2 & -1 & 4 & 11 \\ & -1 & 1 & 3 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 4 & 8 \\ 2 & 0 & 3 & 8 \\ & -1 & 1 & 3 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 8/5 \\ 0 & 0 & 1 & 8/5 \\ 0 & 1 & 0 & -7/5 \end{array}\right]$$

Exercise 2. Solution doesn't exist since

$$\begin{aligned} 5x + 3y + 3z &= 2 \\ 3x + 2y + z &= 1 \end{aligned} \qquad \Longrightarrow x + y - z = 0 \text{ but } x + y - z = 1$$

Exercise 3.

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 1 \\ 5 & 3 & 3 & 2 \\ 7 & 4 & 5 & 3 \end{array}\right] = \left[\begin{array}{ccc|c} 0 & 2 & -8 & -2 \\ 0 & 3 & -12 & -3 \\ 1 & 0 & 3 & 1 \end{array}\right] = \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 1 & -4 & -1 \\ 1 & 0 & 3 & 1 \end{array}\right]$$

$$\begin{aligned} x &= 1 - 3z \\ y &= -1 + 4z \end{aligned} \quad \Longrightarrow \boxed{\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + z \begin{pmatrix} -3 \\ 4 \\ 1 \end{pmatrix}}$$

Exercise 4.

$$\left[\begin{array}{ccc|c} 7 & 4 & 5 & 3 \\ 1 & 1 & -1 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 0 & -3 & 12 & 3 \\ 1 & 1 & -1 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 3 & 1 \\ 0 & 1 & -4 & -1 \end{array}\right]$$

$$\begin{aligned} x + 3z &= 1 \\ y - 4z &= -1 \end{aligned} \quad \boxed{\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + z \begin{pmatrix} -3 \\ 4 \\ 1 \end{pmatrix}}$$

Exercise 5.

$$\left[\begin{array}{cccc|c} 3 & -2 & 5 & 1 & 1 \\ 1 & 1 & -3 & 2 & 2\ 7 \\ 6 & 1 & -4 & 3 & \end{array}\right] = \left[\begin{array}{cccc|c} 0 & -5 & 14 & -5 & -5 \\ 1 & 1 & -3 & 2 & 2\ -5 \\ 0 & -5 & 14 & -9 & \end{array}\right] = \left[\begin{array}{cccc|c} 0 & 0 & 0 & 4 & 0 \\ 1 & 1 & -3 & 0 & 2\ -5 \\ 0 & -5 & 14 & 0 & \end{array}\right] \Longrightarrow \boxed{\begin{pmatrix} x \\ y \\ z \\ u \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 1/5 \\ 14/5 \\ 1 \\ 0 \end{pmatrix}}$$

Exercise 6.

$$\left[\begin{array}{cccc|c} 1 & 1 & -3 & 1 & 5 \\ 2 & -1 & 1 & -2 & 2 \\ 7 & 1 & -7 & 3 & 3 \end{array}\right] = \left[\begin{array}{cccc|c} 1 & 1 & -3 & 1 & 5 \\ 0 & -3 & 7 & -4 & -8 \\ 0 & -6 & 14 & -4 & -32 \end{array}\right] = \left[\begin{array}{cccc|c} 1 & 1 & -3 & 1 & 5 \\ 0 & 0 & 0 & -2 & 8 \\ 0 & -1 & 7/3 & -2/3 & -16/3 \end{array}\right] = \left[\begin{array}{cccc|c} 1 & 0 & -2/3 & 0 & 1 \\ 0 & 0 & 0 & 1 & -4 \\ 0 & 1 & -7/3 & 0 & 8 \end{array}\right]$$

$$\begin{aligned} x + \frac{-2}{3} z &= 1 \\ u &= -4 \\ y - \frac{7}{3} z &= 8 \end{aligned} \quad \boxed{\begin{pmatrix} x \\ y \\ z \\ u \end{pmatrix} = \begin{pmatrix} 1 \\ 8\ 0 \\ -4 \end{pmatrix} + z \begin{pmatrix} 2/3 \\ 7/3 \\ 1 \\ 0 \end{pmatrix}}$$

Exercise 7.

$$\left[\begin{array}{ccccc|c} 1 & 1 & 2 & 3 & 4 & 0 \\ 2 & 2 & 7 & 11 & 14 & 0 \\ 3 & 3 & 6 & 10 & 15 & 0 \end{array}\right] = \left[\begin{array}{ccccc|c} 1 & 1 & 2 & 3 & 4 & 0 \\ 0 & 0 & 3 & 5 & 6 & 0 \\ 0 & 0 & 0 & 1 & 3 & 0 \end{array}\right] \Longrightarrow \begin{aligned} x + y &= -v \\ z &= 3v \\ u &= -3v \end{aligned} \Longrightarrow \boxed{\begin{pmatrix} x \\ y \\ z \\ u \\ v \end{pmatrix} = x \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + v \begin{pmatrix} 0 \\ -1 \\ 3 \\ -3 \\ 1 \end{pmatrix}}$$

Exercise 8.

$$\left[\begin{array}{cccc|c} 1 & -2 & 1 & 2 & -2 \\ 2 & 3 & -1 & -5 & 9 \\ 4 & -1 & 1 & -1 & 5 \\ 5 & -3 & 2 & 1 & 3 \end{array}\right] = \left[\begin{array}{cccc|c} 1 & -2 & 1 & 2 & -2 \\ 0 & 7 & -3 & -9 & 13 \\ 0 & 7 & -3 & -9 & 13 \\ 0 & 7 & -3 & -9 & 13 \end{array}\right] \Longrightarrow \begin{aligned} x + \frac{1}{7} z - \frac{4}{7} u &= \frac{12}{7} \\ y + \frac{-3}{7} z - \frac{9}{7} u &= \frac{13}{7} \end{aligned}$$

$$\Longrightarrow \boxed{\begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix} = \begin{pmatrix} 12/7 \\ 13/7 \\ 0 \\ 0 \end{pmatrix} + z \begin{pmatrix} -1/7 \\ 3/7 \\ 1 \\ 0 \end{pmatrix} + u \begin{pmatrix} 4/7 \\ 9/7 \\ 0 \\ 1 \end{pmatrix}}$$

Exercise 9.

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 2 & -1 & 3 & 2 \\ 5 & -1 & a & 6 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & -3 & & -2 \\ & -1 & & -4 \\ 0 & -6 & a-10 & \end{array}\right] \quad \text{if } a-8 \neq 0, \qquad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4/3 \\ 2/3 \\ 0 \end{pmatrix}$$

$$\text{if } a = 8, \quad \left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ & -3 & -1 & -2 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 5/3 & 4/3 \\ & 1 & 1/3 & 2/3 \end{array}\right] \Longrightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = z \begin{pmatrix} -5/3 \\ -1/3 \\ 1 \end{pmatrix} + \begin{pmatrix} 4/3 \\ 2/3 \\ 0 \end{pmatrix}$$

Exercise 10.

(1)

$$\begin{pmatrix} x \\ y \\ z \\ u \end{pmatrix} = \begin{pmatrix} -5/7 \\ 9/7 \\ 0 \\ 0 \end{pmatrix} + z \begin{pmatrix} 4/7 \\ 11/7 \\ 1 \\ 0 \end{pmatrix} + u \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

(2)

$$\left[\begin{array}{cccc|c} 5 & 2 & -62 & & -1 \\ 1 & -1 & 1 & -1 & -2 \\ 1 & 1 & 1 & 0 & 6 \end{array}\right] = \left[\begin{array}{cccc|c} 0 & -3 & -11 & 2 & -31 \\ 0 & -2 & 0 & -1 & -8 \\ 1 & 1 & 1 & 0 & 6 \end{array}\right] = \left[\begin{array}{cccc|c} 0 & 0 & -11 & 7/2 & -19 \\ 0 & 1 & 0 & 1/2 & 4 \\ 1 & 0 & 1 & -1/2 & 2 \end{array}\right] = \left[\begin{array}{cccc|c} 1 & 0 & 0 & -2/11 & 3/11 \\ 0 & 1 & 0 & 1/2 & 4 \\ 0 & 0 & -1 & 7/22 & -19/11 \end{array}\right]$$

$$\boxed{\begin{pmatrix} x \\ y \\ z \\ u \end{pmatrix} = \begin{pmatrix} 3/11 \\ 4 \\ 19/11 \\ 0 \end{pmatrix} + u \begin{pmatrix} 2/11 \\ -1/2 \\ 7/22 \\ 1 \end{pmatrix}}$$

Exercise 11.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad-bc & -ba+ba \\ cd-cd & -bc+ad \end{bmatrix} = (ad-bc)I$$

If $ad - bc \neq 0$, then for $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$,

$$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

otherwise, if $ad - bc = 0$, $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = 0$

Use thm. from determinants.

$$det(AA^{-1}) = detAdetA^{-1} = detI = 1$$
$$detA,\ detA^{-1} \neq 0$$

Exercise 12. $\begin{bmatrix} 2 & 3 & 4 \\ 2 & 1 & 1 \\ -1 & 1 & 2 \end{bmatrix}$.

$$\left[\begin{array}{ccc|ccc} 2 & 3 & 4 & 1 & & \\ 2 & 1 & 1 & & 1 & \\ -1 & 1 & 2 & & & 1 \end{array}\right] = \left[\begin{array}{ccc|ccc} 0 & 5 & 8 & 1 & & 2 \\ 0 & 3 & 5 & & 1 & 1 \\ -1 & 1 & 2 & & & 1 \end{array}\right] = \left[\begin{array}{ccc|ccc} 0 & 1 & 8/5 & 1/5 & 0 & 2/5 \\ 0 & 0 & 1/5 & -3/5 & 1 & 4/5 \\ 1 & -1 & 2 & & & -1 \end{array}\right] =$$

$$= \left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 5 & -8 & -6 \\ 0 & 0 & 1 & -3 & 5 & 4 \\ 1 & -1 & 0 & -6 & 10 & 7 \end{array}\right] = \left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 5 & -8 & -6 \\ 0 & 0 & 1 & -3 & 5 & 4 \\ 1 & 0 & 0 & -3 & 5 & 4 \end{array}\right] =$$

$$\Longrightarrow \boxed{\begin{bmatrix} -1 & 2 & 1 \\ 5 & -8 & -6 \\ -3 & 5 & 4 \end{bmatrix}}$$

Exercise 13.

$$\left[\begin{array}{ccc|ccc}1 & 2 & 2 & 1 & & \\ 2 & -1 & 1 & & 1 & \\ 1 & 3 & 2 & & & 1\end{array}\right] = \left[\begin{array}{ccc|ccc}1 & 0 & 2 & 3 & & -2 \\ 0 & -1 & -3 & -6 & 1 & 4 \\ 0 & 1 & 0 & -1 & 0 & 1\end{array}\right] = \left[\begin{array}{ccc|ccc}1 & 0 & 2 & 3 & & 2 \\ 0 & 0 & -3 & -7 & 1 & 5 \\ 0 & 1 & 0 & -1 & & 1\end{array}\right] = \left[\begin{array}{ccc|ccc}1 & & & -5/3 & 2/3 & 4/3 \\ & & 1 & 7/3 & -1/3 & -5/3 \\ & 1 & & -1 & & 1\end{array}\right]$$

$$\boxed{\begin{bmatrix}\frac{-5}{3} & \frac{2}{3} & \frac{4}{3} \\ -1 & 0 & 1 \\ \frac{7}{3} & \frac{-1}{3} & \frac{-5}{3}\end{bmatrix}}$$

Exercise 14.

$$\left[\begin{array}{ccc|ccc}1 & -2 & 1 & 1 & & \\ -2 & 5 & -4 & & 1 & \\ 1 & -4 & 6 & & & 1\end{array}\right] = \left[\begin{array}{ccc|ccc}1 & 0 & -3 & 5 & 2 & \\ 0 & 1 & -2 & 2 & 1 & \\ 0 & -2 & 5 & -1 & & 1\end{array}\right] = \left[\begin{array}{ccc|ccc}1 & & & 14 & 8 & 3 \\ & 1 & & 8 & 5 & 2 \\ & & 1 & 3 & 2 & 1\end{array}\right]$$

$$\boxed{\begin{bmatrix}14 & 8 & 3 \\ 8 & 5 & 2 \\ 3 & 2 & 1\end{bmatrix}}$$

Exercise 15.

$$\left[\begin{array}{cccc|cccc}1 & 2 & 3 & 4 & 1 & & & \\ & 1 & 2 & 3 & & 1 & & \\ & & 1 & 2 & & & 1 & \\ & & & 1 & & & & 1\end{array}\right] = \left[\begin{array}{cccc|cccc}1 & & -1 & -2 & 1 & -2 & & \\ & 1 & & -1 & & 1 & -2 & \\ & & 1 & & & & 1 & -2 \\ & & & 1 & & & & 1\end{array}\right] = \left[\begin{array}{cccc|cccc}1 & & & -2 & 1 & -2 & 1 & -2 \\ & 1 & & & & 1 & -2 & 1 \\ & & 1 & & & & 1 & -2 \\ & & & 1 & & & & 1\end{array}\right] =$$

$$\boxed{\begin{bmatrix}1 & -2 & 1 & 0 \\ & 1 & -2 & 1 \\ & & 1 & -2 \\ 1 & & & \end{bmatrix}}$$

Exercise 16. $$\begin{bmatrix} & 1 & & & & \\ 2 & 0 & 2 & & & \\ & 3 & & 1 & & \\ & & 1 & & 2 & \\ & & & 3 & & 1 \\ & & & & 2 & \end{bmatrix}^{-1} = \begin{bmatrix} & 1/2 & & -1 & & 1 \\ 1 & & & & & \\ & & & 1 & & -1 \\ -3 & & 1 & & & \\ & & & & & 1/2 \\ 9 & & -3 & & 1 & \end{bmatrix}$$

2.21 Miscellaneous exercises on matrices

Exercise 3. Use the eigenvalue method.

$$\begin{bmatrix}1 & 2 \\ 5 & 4\end{bmatrix}\begin{bmatrix}x \\ y\end{bmatrix} = -\begin{bmatrix}x \\ y\end{bmatrix} \Longrightarrow \begin{bmatrix}x \\ y\end{bmatrix} = \begin{bmatrix}1 \\ -1\end{bmatrix}$$

$$\begin{bmatrix}1 & 2 \\ 5 & 4\end{bmatrix}\begin{bmatrix}x \\ y\end{bmatrix} = 6\begin{bmatrix}x \\ y\end{bmatrix} \Longrightarrow \begin{aligned} x + 2y &= 6x \\ 2y &= 5x\end{aligned} \quad \begin{bmatrix}x \\ y\end{bmatrix} = \begin{bmatrix}2 \\ 5\end{bmatrix}$$

Indeed, we obtain P since

$$\begin{bmatrix}1 & 2 \\ 5 & 4\end{bmatrix}\begin{bmatrix}\frac{2}{\sqrt{29}} & \frac{1}{\sqrt{2}} \\ \frac{5}{\sqrt{29}} & \frac{-1}{\sqrt{2}}\end{bmatrix} = \begin{bmatrix}\frac{12}{\sqrt{29}} & \frac{-1}{\sqrt{2}} \\ \frac{30}{\sqrt{29}} & \frac{-1}{\sqrt{2}}\end{bmatrix}$$

$$\frac{1}{\frac{-7}{\sqrt{58}}}\begin{bmatrix}\frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{-5}{\sqrt{29}} & \frac{2}{\sqrt{29}}\end{bmatrix}\begin{bmatrix}1 & 2 \\ 5 & 4\end{bmatrix}\begin{bmatrix}\frac{2}{\sqrt{29}} & \frac{1}{\sqrt{2}} \\ \frac{5}{\sqrt{29}} & \frac{-1}{\sqrt{2}}\end{bmatrix} = \begin{bmatrix}6 & \\ & -1\end{bmatrix}$$

$$\boxed{P = \begin{bmatrix}\frac{2}{\sqrt{29}} & \frac{1}{\sqrt{2}} \\ \frac{5}{\sqrt{29}} & \frac{-1}{\sqrt{2}}\end{bmatrix}}$$

Exercise 4. $(A^2)_{ij} = \sum_{k=1}^{2} a_{ik}a_{kj} = a_{il}a_{1j} + a_{i2}a_{2j} = a_{ij}$
If $i = j$, $a_{i1}a_{1i} + a_{i2}a_{2i} = a_{ii}$
it must be that $i = 1$ or $i = 2$. Then rewrite as $a_{ii}^2 + a_{ij}a_{ji} = a_{ii}$
If $i \neq j$. $a_{i1}a_{1j} + a_{i2}a_{2j} = a_{ij}$
it must be that $i = 1$ or $i = 2$ and $j = 2$ or $j = 1$, respectively. then

$$a_{ii}a_{ij} + a_{ij}a_{jj} = a_{ij}$$

If $a_{ij}=0$, $a_{ii}^2=a_{ii}$. $\quad a_{ii}=1$

$$\begin{bmatrix}1 & a\\0 & 1\end{bmatrix}\begin{bmatrix}1 & a\\0 & 1\end{bmatrix}=\begin{bmatrix}1 & 2a\\0 & 1\end{bmatrix}=A\implies a=2a,\ \text{so } a=0$$

If $a_{ij}\neq 0$, $a_{ii}+a_{jj}=1$
Note that a_{ij},a_{ji} must be both nonzero for the following:

$$\begin{aligned}a_{ii}^2-a_{ii}+a_{ij}a_{ji}&=0\\ a_{ii}&=\frac{1\pm\sqrt{1-4a_{ij}a_{ji}}}{2}\\ a_{jj}=1-a_{ii}&=\frac{1\mp\sqrt{1-4a_{ij}a_{ji}}}{2}\end{aligned}$$

Exercise 5. $A^2=A$

$$\begin{aligned}(A+I)^2&=3A+I\\ (A+I)^3&=(3A+I)(A+I)=7A+I\\ (A+I)^{k+1}&=(I+(2^k-1)A)(A+I)=A(1+2^k-1)+I+(2^k-1)A=(2^{k+1}-1)A+I\end{aligned}$$

Exercise 6.

$$\begin{aligned}x'&=a(x-vt)\\ y'&=y\\ z'&=z\\ t'&=a(t-vx/c^2)\end{aligned}$$

$$\begin{aligned}L(v)&=a\begin{bmatrix}1 & -v\\-vc^{-2} & 1\end{bmatrix}\\ L(v)L(u)&=a\begin{bmatrix}1 & -v\\-vc^{-2} & 1\end{bmatrix}b\begin{bmatrix}1 & -u\\-uc^{-2} & 1\end{bmatrix}=ab\begin{bmatrix}1+uv/c^2 & -u-v\\ \frac{-v-u}{c^2} & \frac{uv}{c^2}+1\end{bmatrix}\\ &=\frac{c}{\sqrt{c^2-v^2}}\frac{c}{\sqrt{c^2-u^2}}\left(1+\frac{uv}{c^2}\right)\begin{bmatrix}1 & \frac{-(u+v)}{1+uv/c^2}\\ \frac{-(u+v)}{1+\frac{uv}{c^2}} & 1\end{bmatrix}\end{aligned}$$

$$\gamma=\frac{1}{\sqrt{1-\left(\frac{w}{c}\right)^2}}=\left(1-\left(\frac{-(u+v)/c}{(1+\frac{uv}{c^2})}\right)^2\right)^{-1/2}=\left(\left(1+\frac{2uv}{c^2}+\frac{u^2v^2}{c^4}-\left(\frac{u^2+2uv+v^2}{c^2}\right)\right)/\left(1+\frac{uv}{c^2}\right)^2\right)^{-1/2}=$$
$$=\left(\left(1+\frac{u^2v^2}{c^2}-\frac{u^2}{c^2}-\frac{v^2}{c^2}\right)/\left(1+\frac{uv}{c^2}\right)^2\right)^{-1/2}$$

Exercise 7.

(1) $(A^T)_{ij}=A_{ji}$
$(A^T)^T_{ij}=(A^T)_{ji}=A_{ij}\implies(A^T)^T=A$
(2) $(A+B)^T_{ij}=(A+B)_{ji}=A_{ji}+B_{ji}=A^T_{ij}+B^T_{ij}\implies(A+B)^T=A^T+B^T$
(3) $(cA)^T_{ij}=(cA)_{ji}=c_Aji=c(A^T)_{ij}$
(4) $(AB)^T_{ij}=(AB)_{ji}=\sum_k a_{jk}b_{ki}=\sum_k b_{ki}a_{jk}=\sum_k(B^T)_{ik}(A^T)_{kj}=(B^TA^T)_{ij}$
(5) $A^{-1}A=1\implies(A^{-1}A)^T=A^T(A^{-1})^T=1$ then $(A^{-1})^T=(A^T)^{-1}$ (recall that a right inverse is also a left inverse).

Exercise 8. $\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$

$$A_iA_j=\sum_{k=1}^n a_{ik}a_{jk}=\sum_{k=1}^n a_{ik}a^T_{kj}=(AA^T)_{ij}=\delta_{ij}$$

Exercise 9.

(1) If $\begin{aligned}AA^T&=1\\BB^T&=1\end{aligned}$, $(A+B)(A+B)^T=2+BA^T+AB^T$
(2) $(AB)(AB)^T=(AB)B^TA^T=1$
(3) B is given to be orthogonal.

Exercise 10.

(1)

$$\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}\begin{bmatrix}1 & -1\\ 1 & -1\end{bmatrix}\begin{bmatrix}-1 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}-1 & 1\\ -1 & -1\end{bmatrix}\begin{bmatrix}-1 & -1\\ 1 & -1\end{bmatrix}\begin{bmatrix}-1 & -1\\ -1 & -1\end{bmatrix}$$

(2) $(X+Y)\cdot(X+Z) = X^2 + X\cdot Z + Y\cdot X + Y\cdot Z = X^2$ Lemma 1 is true.
Lemma 2 $\quad (x_i+y_i)(x_i+z_i) = x_i^2 + x_i(z_i+y_i) + y_i z_i$

x_i	y_i	z_i	
1	1	1	4
1	1	-1	0
1	-1	-1	0
1	-1	1	0
-1	1	1	0
-1	-1	1	0
-1	1	-1	0
-1	-1	-1	4

Assume A is Hadamard.
Then by Lemma 1, $(A_i+A_j)\cdot(A_i+A_k) = A_i^2 = n, \quad i,j,k$ distinct.

$$(A_i+A_j)\cdot(A_i+A_k) = A_i^2 + A_i\cdot A_k + A_j\cdot A_i + A_j\cdot A_k = \sum_{l=1}^n a_{il}^2 + \sum_{l=1}^n a_{il}a_{kl} + \sum_{l=1}^n a_{jl}a_{il} + \sum_{l=1}^n a_{jl}a_{kl} =$$

$$= \sum_{l=1}^n (a_{il}+a_{jl})(a_{il}+a_{kl})$$

By Lemma 2, $(a_{il}+ajl)(a_{il}+a_{kl}) = 0$ or 4
then $(A_i+A_j)\cdot(A_i+A_k) = \sum_{l=1}^n (a_{il}+a_{jl})(a_{il}+a_{kl}) = 4m$, where $m \leq n$
$\implies \boxed{n = 4m}$

3.6 Exercises - Introduction, Motivation for the choice of axioms for a determinant function, A set of axioms for a determinant function, Computation of determinants,

Exercise 1.

(1) $\begin{vmatrix}2 & 1 & 1\\ 1 & 4 & -4\\ 1 & 0 & 2\end{vmatrix}$

$$\begin{vmatrix}0 & 1 & -3\\ 0 & 4 & -6\\ 1 & 0 & 2\end{vmatrix} = \begin{vmatrix}0 & 1 & -3\\ 0 & 0 & -6\\ 1 & 0 & 2\end{vmatrix} = \begin{vmatrix}1 & 0 & 2\\ 0 & 1 & -3\\ 0 & 0 & -6\end{vmatrix} = \boxed{6}$$

(2) $\begin{vmatrix}3 & 0 & 8\\ 5 & 0 & 7\\ -1 & 4 & 2\end{vmatrix} = 3(-28) + 8(20) = 4(-3(7)+2(20)) = \boxed{76}$

(3) $\begin{vmatrix}a & 1 & 0\\ 2 & a & 2\\ 0 & 1 & a\end{vmatrix} = a(a^2-2) - 1(2a) = a(a^2-4) = a(a-2)(a+2)$

Exercise 2. Given $det\begin{bmatrix}x & y & z\\ 3 & 0 & 2\\ 1 & 1 & 1\end{bmatrix} = 1$,

(1) $\begin{bmatrix}2x & 2y & 2z\\ \frac{3}{2} & 0 & 1\\ 1 & 1 & 1\end{bmatrix}$

$$\begin{vmatrix}2x & 2y & 2z\\ \frac{3}{2} & 0 & 1\\ 1 & 1 & 1\end{vmatrix} = 2\left(\frac{1}{2}\right) detA = 1$$

(2) $\begin{bmatrix} x & y & z \\ 3x+3 & 3y & 3z+2 \\ x+1 & y+1 & z+1 \end{bmatrix}$

$$\begin{vmatrix} x & y & z \\ 3x+3 & 3y & 3z+2 \\ x+1 & y+1 & z+1 \end{vmatrix} = \begin{vmatrix} x & y & z \\ 3x+3 & 3y & 3z+2 \\ x & y & z \end{vmatrix} + \begin{vmatrix} x & y & z \\ 3x+3 & 3y & 3z+2 \\ 1 & 1 & 1 \end{vmatrix} = 0 + \begin{vmatrix} x & y & z \\ 3x & 3y & 3z \\ 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} x & y & z \\ 3 & 0 & 2 \\ 1 & 1 & 1 \end{vmatrix} = \boxed{1}$$

(3) $\begin{bmatrix} x-1 & y-1 & z-1 \\ 4 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix}$

$$\begin{vmatrix} x-1 & y-1 & z-1 \\ 4 & 1 & 3 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} x & y & z \\ 4 & 1 & 3 \\ 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} -1 & -1 & -1 \\ 4 & 1 & 3 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} x & y & z \\ 3+1 & 0+1 & 2+1 \\ 1 & 1 & 1 \end{vmatrix} = \boxed{1}$$

Exercise 3.

(1)

$$\begin{vmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & b^2-a^2 & c^2-a^2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & 0 & (c^2-a^2)-(c-a)(b+a) \end{vmatrix} = (b-a)((c-a)(c+a)-(c-a)(b+a)) =$$
$$= \boxed{(b-a)(c-a)(c-b)}$$

(2)

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & b^3-a^3 & c^3-a^3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & (b-a) & c-a \\ 0 & (b-a)(b^2+ba+a^2) & (c-a)(c^2+ca+a^2) \end{vmatrix}$$

subtract the second column off the third column modulo a factor

$$\begin{pmatrix} 0-0 \\ c-a-(c-a) \\ (c-a)(c^2+ca+a^2-(b^2+ba+a^2)) \end{pmatrix}$$
$$= \begin{vmatrix} 1 & 0 & 0 \\ 0 & (b-a) & 0 \\ 0 & (b-a)(b^2+ba+a^2) & (c-a)(c^2-b^2+ca-ab) \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & (b-a) & 0 \\ 0 & 0 & (c-a)(c-b)(c+b+a) \end{vmatrix}$$
$$= (a+b+c)(c-a)(c-b)(b-a)$$

$$\begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & b^2-a^2 & c^2-a^2 \\ 0 & b^3-a^3 & c^3-a^3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & (b-a)(b+a) & (c-a)(c+a) \\ 0 & (b-a)(b^2+ba+a^2) & (c-a)(c^2+ca+a^2) \end{vmatrix}$$

subtract the second column off the third column modulo a factor

$$\begin{pmatrix} 0 \\ (c-a)(c+a) \\ (c-a)(c^2+ca+a^2) \end{pmatrix} - \frac{(c-a)(c+a)}{(b-a)(b+a)} \begin{pmatrix} 0 \\ (b-a)(b+a) \\ (b-a)(b^2+ba+a^2) \end{pmatrix} =$$
$$= \begin{pmatrix} 0 \\ 0 \\ (c-a)(c^2+ac+a^2-\frac{(c+a)}{(b+a)}(b^2+ba+a^2)) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \frac{(c-a)}{(b+a)}(c-b)(ac+ab+bc) \end{pmatrix}$$
$$= (b-a)(c-a)(c-b)(ac+ab+bc)$$

Exercise 4.

(1)

$$\begin{vmatrix} 1 & -1 & 1 & 1 \\ 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 2 & 0 & 0 & 2 \\ 2 & 0 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 2 \\ 2 & 0 & 0 & 0 \end{vmatrix} = (-1)8$$

(2)

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & b-a & c-a & d-a \\ 0 & b^2-a^2 & c^2-a^2 & d^2-a^2 \\ 0 & b^3-a^3 & c^3-a^3 & d^3-a^3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & b-a & b^2-a^2 & b^3-a^3 \\ 0 & c-a & c^2-a^2 & c^3-a^2 \\ 0 & d-a & d^2-a^2 & d^3-a^3 \end{vmatrix}$$

$$= (b-a)(c-a)(d-a) \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & (b+a) & b^2+ba+a^2 \\ 0 & 1 & (c+a) & c^2+ac+a^2 \\ 0 & 1 & (d+a) & d^2+ad+a^2 \end{vmatrix} =$$

(Now I use the addition of column $\begin{matrix} 0 \\ 1 \\ 1 \\ 1 \end{matrix}$, which doesn't change the determinant)

$$= (b-a)(c-a)(d-a) \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & b & (b+a)b \\ 0 & 1 & c & (c+a)c \\ 0 & 1 & d & (d+a)d \end{vmatrix} =$$

$$= (b-a)(c-a)(d-a) \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c-b & (c+a)c-(b+a)b \\ 0 & 0 & d-b & (d+a)d-(b+a)b \end{vmatrix}$$

$$(b-a)(c-a)(d-a) \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c-b & c^2-b^2 \\ 0 & 0 & d-b & d^2-b^2 \end{vmatrix} =$$

$$= (b-a)(c-a)(d-a)(c-b)(d-b) \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & c+b \\ 0 & 0 & 1 & d+b \end{vmatrix} =$$

$$\boxed{(b-a)(c-a)(d-a)(c-b)(d-b)(d-c)}$$

(3)
(4)

$$\begin{vmatrix} a & 1 & 0 & 0 & 0 \\ 4 & a & 2 & 0 & 0 \\ 0 & 3 & a & 3 & 0 \\ 0 & 0 & 2 & a & 4 \\ 0 & 0 & 0 & 1 & a \end{vmatrix} = \begin{vmatrix} a & 1 & & & \\ & a-\frac{4}{a} & 2 & & \\ & 3 & a & 3 & \\ & & 2 & a & 4 \\ & & & 1 & a \end{vmatrix} = \begin{vmatrix} a & 0 & & & \\ & a-\frac{4}{a} & 2 & & \\ & 3 & a & 3 & \\ & & 2 & a-\frac{4}{a} & 4 \\ & & & 0 & a \end{vmatrix}$$

$$= \begin{vmatrix} a & & & & \\ & a-\frac{4}{a} & & & \\ & 0 & a-\frac{12}{a-\frac{4}{a}} & 0 & \\ & & 0 & a-\frac{4}{a} & 0 \\ & & & 0 & a \end{vmatrix} =$$

$$= a^2(a-\frac{4}{a})^2\left(a-\frac{12}{a-\frac{4}{a}}\right) = a^2(a-\frac{4}{a})(a^2-4-12)$$

$$= a(a^2-4)(a^2-16)$$

(5)

$$\begin{vmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 \end{vmatrix} = \begin{vmatrix} 1 & & & & & \\ 0 & 0 & 0 & -2 & -2 & -2 \\ & & -2 & -2 & & \\ & -2 & -2 & & -2 & \\ & -2 & & -2 & & \\ & -2 & -2 & & & -2 \end{vmatrix} = \begin{vmatrix} 1 & & & & & \\ & 2 & 2 & -2 & -2 & 0 \\ & & -2 & -2 & & \\ & -2 & -2 & & -2 & \\ & -2 & & -2 & & \\ & 0 & 0 & & & -2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & & & & & \\ & 2 & & & & \\ & & -2 & -2 & & \\ & & & -2 & -4 & \\ & & 0 & -6 & -2 & \\ 0 & & 0 & & & -2 \end{vmatrix} = (1)(2)(-2)(-2)(10)(-2) = -160$$

Exercise 5. Consider $A = (a_{ij})$ s.t. $a_{ij} = 0$ whenever $i < j$.

Suppose $a_{11} = 0$. Then $a_{1j} = 0, \forall\, j \leq n$, since a row of A is entirely zero, by homogeneity property of determinants, $detA = 0$.

Suppose $a_{ii} = 0$ for some $1 < i \leq n$.

then i rows have $n - (i-1)$ components equal to zero. Therefore, these i rows can span a psace of at most $i - 1$ dimensions. then the i rows are dependent. Then $detA = 0$ by Thm. (the determinant vanishes if its rows are dependent).

Then assume a_{ii} nonzero $\forall\, i \leq n$

Let $A_n = B_n + C_n$, where $B_n = \begin{bmatrix} 0 & 0 & \dots & 0 & a_{nn} \end{bmatrix}$ and $C_n = \begin{bmatrix} a_{n1} & a_{n2} & \dots & a_{n,\,n-1} & 0 \end{bmatrix}$

$$\begin{aligned} det(A) &= det(A_1, A_2, \dots, A_n) = det(A_1,\, A_2, \dots, B_n + C_n) = det(A_1, A_2, \dots, B_n) + det(A_1, A_2, \dots, C_n) = \\ &= det(A_1, A_2, \dots, B_n) \end{aligned}$$

Also, $A_{n-1} = B_{n-1} + C_{n-1}$, where $B_{n-1} = \begin{bmatrix} 0 & 0 & \dots & a_{n-1,n-1} & 0 \end{bmatrix}$ and $C_{n-1} = \begin{bmatrix} a_{n-1,1} & a_{n-1,2} & \dots & a_{n-1,\,n-2} & 0 & 0 \end{bmatrix}$

$$\begin{aligned} det(A) &= det(A_1, A_2, \dots, A_{n-1}, B_n) = det(A_1,\, A_2, \dots, B_{n-1} + C_{n-1}, B_n) = \\ &= det(A_1, A_2, \dots, B_{n-1}, B_n) + det(A_1, A_2, \dots, C_{n-1}, B_n) = \\ &= det(A_1, A_2, \dots, B_{n-1}, B_n) \end{aligned}$$

Then $detA = det(B_1, B_2, \dots, B_n)$.

By homogeneity of determinants, $detA = \prod i = 1^n a_{ii} detI = \prod_{i=1}^n a_{ii}$

Exercise 6.

$$\begin{aligned} F &= f_1 g_2 - f_2 g_1 \\ F' &= f_1' g_2 + f_1 g_2' - f_2' g_1 - f_2 g_1' = f_1' g_2 - f_2' g_1 + f_1 g_2' - f_2 g_1' = \\ &= \begin{vmatrix} f_1' & f_2' \\ g_1 & g_2 \end{vmatrix} + \begin{vmatrix} f_1 & f_2 \\ g_1' & g_2' \end{vmatrix} \end{aligned}$$

Exercise 7.

$$\begin{aligned} F &= f_1 \begin{vmatrix} g_2 & g_3 \\ h_2 & h_3 \end{vmatrix} - f_2 \begin{vmatrix} g_1 & g_3 \\ h_1 & h_3 \end{vmatrix} + f_3 \begin{vmatrix} g_1 & g_2 \\ h_1 & h_2 \end{vmatrix} \\ F' &= \begin{vmatrix} f_1' & f_2' & f_3' \\ g_1 & g_2 & g_3 \\ h_1 & h_2 & h_3 \end{vmatrix} + f_1 \left(\begin{vmatrix} g_2' & g_3' \\ h_2 & h_3 \end{vmatrix} + \begin{vmatrix} g_2 & g_3 \\ h_2' & h_3' \end{vmatrix} \right) - f_2 \left(\begin{vmatrix} g_1' & g_3' \\ h_1 & h_3 \end{vmatrix} + \begin{vmatrix} g_1 & g_3 \\ h_1' & h_3' \end{vmatrix} \right) + f_3 \left(\begin{vmatrix} g_1' & g_2' \\ h_1 & h_2 \end{vmatrix} + \begin{vmatrix} g_1 & g_2 \\ h_1' & h_2' \end{vmatrix} \right) = \\ &= \begin{vmatrix} f_1' & f_2' & f_3' \\ g_1 & g_2 & g_3 \\ h_1 & h_2 & h_3 \end{vmatrix} + \begin{vmatrix} f_1 & f_2 & f_3 \\ g_1' & g_2' & g_3' \\ h_1 & h_2 & h_3 \end{vmatrix} + \begin{vmatrix} f_1 & f_2 & f_3 \\ g_1 & g_2 & g_3 \\ h_1' & h_2' & h_3' \end{vmatrix} \end{aligned}$$

Exercise 8. Using the previous results:

(1)

$$F' = \begin{vmatrix} f_1' & f_2' \\ f_1' & f_2' \end{vmatrix} + \begin{vmatrix} f_1 & g_1 \\ f_2'' & g_2'' \end{vmatrix} = \boxed{\begin{vmatrix} f_1 & g_1 \\ f_2'' & g_2'' \end{vmatrix}}$$

(2)

$$F = \begin{vmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{vmatrix}] \Longrightarrow \begin{vmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{vmatrix}$$

Exercise 9.

(1)

$$(U+V)_{ij} = u_{ij} + v_{ij} = \begin{cases} u_{ij} & \text{if } i<j \\ 0 & \text{otherwise} \end{cases} + \begin{cases} v_{ij} & \text{if } i<j \\ 0 & \text{otherwise} \end{cases} = \begin{cases} u_{ij}+v_{ij} & \text{if } i<j \\ 0 & \text{otherwise} \end{cases}$$

$$(UV)_{ij} = \sum_{k=1}^{n} u_{ik}v_{kj} \sum_{i<k} u_{ik}v_{kj} = \sum_{i<k,k<j} u_{ik}v_{kj} = \begin{cases} \sum_{i<k,k<j} u_{ik}v_{kj} & \text{if } i \leq j \\ 0 & \text{otherwise} \end{cases}$$

(2)

$$det(UV) = \prod_{i=1}^{n} \left(\sum_{i<k,k<j}^{n} u_{ik}v_{ki} \right) = \prod_{i=1}^{n} u_{ii}v_{ii} = \left(\prod_{i=1}^{n} u_{ii} \right) \left(\prod_{i=1}^{n} v_{ii} \right) = detUdetV$$

(3) Suppose $UU^{-1} = 1$.

$$xdet1 = 1 = (detU)(detU^{-1}), \ U \text{ and } 1 \text{ are } 2 \ n \times n \text{ triangular matrices}$$

U^{-1} exists since $detU \neq 0$.

$$detU^{-1} = 1/detU$$

Exercise 10. Use the cofactor matrix to get the inverse, that $\frac{(cofA)^T}{detA} = A^{-1}$.

$$detA = 16, \quad detA^{-1} = \frac{1}{16}$$

$$\begin{pmatrix} \frac{1}{2} & \frac{-3}{4} & \frac{1}{8} & \frac{1}{16} \\ 0 & \frac{1}{2} & \frac{-3}{4} & \frac{1}{8} \\ 0 & 0 & \frac{1}{2} & \frac{-3}{4} \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 2 & 3 & 4 & 5 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

3.11 Exercises - The product formula for determinants, The determinant of the inverse of a nonsingular matrix, Determinants and independence of vectors, The determinant of a block-diagonal matrix

Exercise 1.

(1) If $A = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}; B = \begin{bmatrix} -4 & 2 \\ 3 & 6 \end{bmatrix}; A+B = \begin{bmatrix} -3 & 5 \\ 5 & 11 \end{bmatrix} \begin{matrix} detA = -1 \\ detB = -30 \end{matrix} \ det(A+B) = -58$

(2) $det(A+B)^2 = det(A+B)(A+B) = det(A+B)det(A+B) = (det(A+B))^2$

(3) If $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}; B = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}, A+B = \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$$

$$B^2 = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix}$$

$$A^2 + 2AB + B^2 = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} + 2 \begin{bmatrix} 0 & 1 \\ 2 & -3 \end{bmatrix} + \begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

$$\Longrightarrow det(A^2 + 2AB + B^2) = -9$$

(4) likewise, $det(A^2 + B^2) = det \begin{bmatrix} 3 & -2 \\ -4 & 3 \end{bmatrix} = 1$

Exercise 2.

(1) Assume A is $n \times n$, B is $m \times m$, and C is $p \times p$.

$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ is a $n+m \times n+m$ matrix, D.

$\begin{bmatrix} A & & \\ & B & \\ & & C \end{bmatrix} = \begin{bmatrix} D & \\ & C \end{bmatrix}$. Then by Thm. 3.7, $det \begin{bmatrix} D & 0 \\ 0 & C \end{bmatrix} = detDdetC$

$$detD = det\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = detAdetB \text{ (by Thm. 3.7). Then } det\begin{bmatrix} A & & \\ & B & \\ & & C \end{bmatrix} = detAdetBdetC$$

(2) Assume $det\begin{bmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_n \end{bmatrix} = \prod_{i=1}^{n} detA_i$.

Consider $det\begin{bmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_{n+1} \end{bmatrix}$

Now $\begin{bmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_n \end{bmatrix} = D_n$, a square matrix of size $\sum_{i=1}^{n} N_i \times \sum_{i=1}^{n} N_i$ where $N_i =$ size of matrix A_i.

$det\begin{bmatrix} D_n & \\ & A_{n+1} \end{bmatrix} = detD_n detA_{n+1}$ by Thm. 3.7. $detD_n$, by induction assumption, is $detD_n = \prod_{i=1}^{n} detA_i$.

$\implies det\begin{bmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_{n+1} \end{bmatrix} = \prod_{i=1}^{n+1} detA_i$

Exercise 3.

$$detA = det\begin{bmatrix} 1 & & & \\ & 1 & & \\ a & b & c & d \\ e & f & g & h \end{bmatrix} = det\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & c & d \\ & & g & h \end{bmatrix} = det\begin{bmatrix} c & d \\ g & h \end{bmatrix}$$
$$detB = det\begin{bmatrix} a & b & c & d \\ e & f & g & h \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{vmatrix} a & b & & \\ e & f & & \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = det\begin{bmatrix} a & b \\ e & f \end{bmatrix}$$

Exercise 4. If $X = \begin{bmatrix} & A & \\ & & I_m \end{bmatrix}$ where A is $(n-m) \times n$ and I is $m \times m$, then $\forall\, a_{ij}$ entry, $(n-m)+1 \leq i \leq n$, $(n-m)+1 \leq j \leq n$.
$[0\,0\,\dots\,0,\, -a_{ij},\, 0, \dots\, 0]$ could be added to the ith row since Gauss-Jordan row operations do not change the determinant, by determinant properties. Then $detX = \begin{bmatrix} A_{n-m} & \\ & I_m \end{bmatrix}$. By Thm. 3.7, $detX = detA_{n-m}$.

Similarly for $Y = \begin{bmatrix} I_m & \\ & A \end{bmatrix}$.

Exercise 5. $A = \begin{bmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ e & f & g & h \\ x & y & z & w \end{bmatrix}$ $\qquad detA = det\begin{bmatrix} a & b \\ c & d \end{bmatrix} det\begin{bmatrix} g & h \\ z & w \end{bmatrix}$

Exercise 6. $A = \begin{bmatrix} B & 0 \\ C & D \end{bmatrix}$ where B is $m \times m$, C, D are $(n-m) \times (n-m)$.

$$detA = f(A_1, A_2, \dots, A_n), \quad A_i = C_i + D_i, \quad m+1 \leq i \leq n$$
$$detA = f(A_1, A_2, \dots, A_n) = f(A_1, A_2, \dots, C_{m+1} + D_{m+1}, \dots, A_n) =$$
$$= f(A_1, A_2, \dots, C_{m+1}, \dots, A_n) + f(A_1, A_2, \dots, D_{m+1}, \dots, A_n)$$

Consider $A_1, A_2, \dots, C_{m+1}$, $m+1$ rows with m possibly nonzero components. Then $A_1, \dots, C_{m+1}$ span at most a $dimm$ subspace. Then $A_1, \dots, C_{m+1}$ dependent. By Thm., $f(A_1, A_2, \dots, C_{m+1}, \dots, A_n) = 0$

$$detA = f(A_1, A_2, \dots, D_{m+1}, \dots, A_n)$$

Likewise for $i = m+2, \dots, n$

$$\Longrightarrow detA = f(A_1, A_2, \dots, D_{m+1}, \dots, D_n) = detBdetD$$

(By Thm. for det of block-diagonal matrices)

Exercise 7.

(1) $\begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 2 & 3 & -1 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 5 & -1 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 4 \end{vmatrix} = 4$

(2) $\begin{vmatrix} 1 & -1 & 2 & 1 \\ -1 & 2 & -1 & 0 \\ 3 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 1 \\ 0 & -1 & 1 & -3 \\ 1 & 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 0 & -1 & 2 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & -1 & -3 \\ 1 & 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & -8 \\ 0 & 0 & -1 & -3 \\ 1 & 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & -3 \\ 0 & 0 & 0 & -8 \end{vmatrix} = -8$

(3) $\begin{vmatrix} 1 & 0 & 0 & 01 & \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{vmatrix} = 0$

3.17 Exercises - Expansion formulas for determinants. Minors and cofactors. 3.13 Existence of the determinant function, The determinant of a transpose, The cofactor matrix, Cramer's rule

Exercise 1.

(1) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad cofA = \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}$

(2) $\begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & 1 \\ -1 & -2 & 0 \end{bmatrix} \qquad cofA = \begin{bmatrix} 2 & -1 & 1 \\ -6 & 3 & 5 \\ -4 & -2 & 2 \end{bmatrix}$

(3) $\begin{bmatrix} 3 & 1 & 2 & 4 \\ 2 & 0 & 5 & 1 \\ 1 & -1 & -2 & 6 \\ -2 & 3 & 2 & 3 \end{bmatrix} \qquad cofA = \begin{bmatrix} 109 & 113 & -41 & -13 \\ -40 & -92 & 74 & 16 \\ -41 & -79 & 7 & -47 \\ -50 & 38 & 16 & 20 \end{bmatrix}$

Exercise 2.

(1) $\frac{1}{-2}\begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}$

(2) $\frac{1}{8}\begin{bmatrix} 2 & -6 & -4 \\ -1 & 3 & -2 \\ 1 & 5 & 2 \end{bmatrix}$

(3) $\frac{1}{184}\begin{bmatrix} 109 & -40 & -41 & -50 \\ 113 & -92 & 79 & 38 \\ -41 & 74 & 7 & 16 \\ -13 & 16 & -47 & 20 \end{bmatrix}$

Exercise 3. Note that for $\lambda I - A$, $det(\lambda I - A) = 0 = det(A - \lambda I)$

(1)

$$\begin{vmatrix} -\lambda & 3 \\ 2 & -1-\lambda \end{vmatrix} = 0 \Longrightarrow \lambda + \lambda^2 - 6 = 0 = (\lambda+3)(\lambda-2) = 0$$

(2)

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & -2 \\ 2 & -2 & 0 \end{bmatrix}$$

$$\begin{vmatrix} 1-\lambda & & 2 \\ & -1-\lambda & -2 \\ 2 & -2 & -\lambda \end{vmatrix} = (1-\lambda)(\lambda(1+\lambda) - 4) + 2(2(1+\lambda)) =$$

$$= (1-\lambda)(\lambda^2 + \lambda - 4) + 4(1+\lambda) = (1-\lambda)(\lambda^2 + \lambda - 4) + 4(1+\lambda) =$$

$$= -\lambda^3 + 9\lambda = \lambda(-\lambda^2 + 9) \Longrightarrow \boxed{\lambda = \pm 3, 0}$$

(3)

$$\begin{vmatrix} 11-\lambda & -2 & 8 \\ 19 & -3-\lambda & 14 \\ -8 & 2 & -5-\lambda \end{vmatrix} = \begin{vmatrix} 3-\lambda & 0 & 3-\lambda \\ 19 & -3-\lambda & 14 \\ -8 & 2 & -5-\lambda \end{vmatrix} = \begin{vmatrix} 0 & 0 & 3-\lambda \\ 5 & -3-\lambda & 14 \\ \lambda-3 & 2 & -5-\lambda \end{vmatrix}$$

$$= (3-\lambda)(10+\lambda^2-9) = (3-\lambda)(1+\lambda^2)$$

$$\boxed{\lambda = 3, \pm i}$$

Exercise 4.

(1) $((cofA)^T)_{ij} = (cofA)_{ji} = (-1)^{i+j}detA_{ji} = (-1)^{i+j}det(A_{ji})^T = (-1)^{i+j}det(A^T)_{ij} = cof(A^T)_{ij}$

(2) See Part (c), and then use $A(cofA)^T = (detA)I$, Thm. 3.12.

(3) $((cofA)^T A)_{ij} = \sum_k (cofA)^T_{ik} a_{kj} = \sum_k a_{kj}(cofA)_{ki}$

Recall that column expansions can be done on determinants, and that $detA = detA^T$.

Consider B matrix whose jth column is equal to the ith column for some $j \neq i$, but remaining rows are the same as A.
then $detB = 0$

$$detB = \sum_k^n b_{kj} cofb_{kj} \qquad (j\text{th column expansion of } B)$$

$$b_{kj} = a_{ij}$$

$$cofb_{kj} = cofa_{kj} \qquad (\text{since } B \text{ differs from } A \text{ only in the } j\text{th column})$$

$$\Longrightarrow \sum_k^n a_{ij} cofa_{kj} = 0$$

If $i = j$, $\sum_k a_{ki}(cofA)_{ki} = detA$ (by ith column expansion of $detA$)

Exercise 5.

(1)
$$\begin{aligned} x+2y+3z &= 8 \\ 2x-y+4z &= 7 \\ -y+z &= 1 \end{aligned} \Longrightarrow \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 7 \\ 1 \end{bmatrix}$$

$$x = \frac{1}{-7}\begin{vmatrix} 8 & 2 & 3 \\ 7 & -1 & 4 \\ 1 & -1 & 1 \end{vmatrix} = \frac{1}{-7}\begin{vmatrix} 0 & 10 & -5 \\ 0 & 6 & -3 \\ 1 & -1 & 1 \end{vmatrix} = \frac{1}{-7}\begin{vmatrix} 1 & -1 & 1 \\ 0 & 10 & -5 \\ 0 & 6 & -3 \end{vmatrix} = 0$$

$$y = \frac{1}{-7}\begin{vmatrix} 1 & 8 & 3 \\ 2 & 7 & 4 \\ & 1 & 1 \end{vmatrix} = \frac{1}{-7}\begin{vmatrix} 1 & 0 & -5 \\ 2 & 0 & -3 \\ & 1 & 0 \end{vmatrix} = -7/-7 = 1$$

$$z = \frac{1}{-7}\begin{vmatrix} 1 & 2 & 8 \\ 2 & -1 & 7 \\ & -1 & 1 \end{vmatrix} = \frac{-1}{7}\begin{vmatrix} 1 & 0 & 10 \\ 0 & -1 & -13 \\ & -1 & 1 \end{vmatrix} = \frac{-14}{-7} = 2$$

(2)
$$\begin{aligned} x+y+2z &= 0 \\ 3x-y-z &= 3 \\ 2x+5y+3z &= 4 \end{aligned} \Longrightarrow \begin{bmatrix} 1 & 1 & 2 \\ 3 & -1 & -1 \\ 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix}$$

$$\begin{vmatrix} 1 & 1 & 2 \\ 3 & -1 & -1 \\ 2 & 5 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 2 \\ 0 & -4 & -7 \\ 0 & 3 & -1 \end{vmatrix} = 25$$

$$\begin{vmatrix} 0 & 1 & 2 \\ 3 & -1 & -1 \\ 4 & 5 & 3 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 2 \\ 3 & 0 & 1 \\ 4 & 0 & -7 \end{vmatrix} = -(-21-4) = 25 \qquad \Longrightarrow \boxed{x=1}$$

$$\begin{vmatrix} 1 & 0 & 2 \\ 3 & 3 & -1 \\ 2 & 4 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 2 \\ 0 & 3 & -7 \\ 0 & 4 & -1 \end{vmatrix} = 25 \qquad \Longrightarrow \boxed{y=1}$$

$$\boxed{z=-1}$$

Exercise 6.

(1) Vector form of lines: $tA + P_1 = X$; $A = P_2 - P_1$.

$$t\begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \end{pmatrix} + \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$
$$t\begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \end{pmatrix} + - \begin{pmatrix} x - x_1 \\ y - y_1 \end{pmatrix} = 0$$

Then $A, X - P_1$ are linearly dependent.
Then if A, $X - P_1$ form rows of a matrix,

$$\begin{vmatrix} x - x_1 & y - y_1 \\ x_2 - x_1 & y_2 - y_1 \end{vmatrix} = 0$$

Also

$$t\begin{pmatrix} x_2 \\ t_2 \end{pmatrix} + (1-t)\begin{pmatrix} x_1 \\ t_1 \end{pmatrix} - \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

we can extend this to say

$$t\begin{pmatrix} x_2 \\ y_2 \\ 1 \end{pmatrix} + (1-t)\begin{pmatrix} x_1 \\ y_1 \\ 1 \end{pmatrix} - \begin{pmatrix} x_1 \\ y_1 \\ 1 \end{pmatrix} = 0 =$$
$$= tX_2 + (1-t)X_1 - X = 0$$

X, X_1, X_2 are dependent, and so if X, X_1, X_2 form rows of a matrix, then

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

(2) Recall the vector form for planes: $P = \{X | X = P + sA + tB\}$, A, B are independent.

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + s\begin{pmatrix} x_1 - x_0 \\ y_1 - y_0 \\ z_1 - z_0 \end{pmatrix} + t\begin{pmatrix} x_2 - x_0 \\ y_2 - y_0 \\ z_2 - z_0 \end{pmatrix} = P + sA + tB$$
$$0 = P - X + sA + tB$$

$P - X, A, B$ are dependent. Then consider $P - X, A, B$ to be rows of a matrix. Then

$$\begin{vmatrix} x_0 - x & y_0 - y & z_0 - z \\ x_1 - x_0 & y_1 - y_0 & z_1 - z_0 \\ x_2 - x_0 & y_2 - y_0 & z_2 - z_0 \end{vmatrix} = 0$$

We could also rewrite this equation like this:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} - s\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + s\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} - t\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} + t\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = 0$$
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} + (t+s-1)\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} - t\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} - s\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = 0$$

Extend by 1 for a new row.

$$\begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} + (t+s-1)\begin{pmatrix} x_0 \\ y_0 \\ z_0 \\ 1 \end{pmatrix} - t\begin{pmatrix} x_2 \\ y_2 \\ z_2 \\ 1 \end{pmatrix} - s\begin{pmatrix} x_1 \\ y_1 \\ z_1 \\ 1 \end{pmatrix} = 0$$

This shows that these 4 vectors are linearly dependent. Consider the vectors as rows of matrix to obtain:

$$\begin{vmatrix} x & y & z & 1 \\ x_0 & y_0 & z_0 & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \end{vmatrix} = 0$$

(3) We have 3 noncollinear points that satisfy some specific equation for a circle in the $x-y$ plane.

$(x-x_0)^2+(y-y_0)^2=\rho^2=(x^2-2xx_0+x_0^2)+(y^2-2yy_0+y_0^2)$ or $x^2-2x_0x+y^2-2y_0y-(\rho^2-x_0^2-y_0^2)=0$

So x_0, y_0, the coordinates for the origin, and ρ, the radius of the circle, are 3 unknowns and 3 equations are needed. To fix the "scale" of the coordinates, we need a 4th equation.

$$\Longrightarrow \begin{aligned} x_1^2-2x_0x_1+y_1^2-2y_0y_1-(\rho^2-x_0^2-y_0^2)&=0 \\ x_2^2-2x_0x_2+y_2^2-2y_0y_2-(\rho^2-x_0^2-y_0^2)&=0 \\ x_3^2-2x_0x_3+y_3^2-2y_0y_3-(\rho^2-x_0^2-y_0^2)&=0 \\ x^2-2x_0x+y^2-2y_0y-(\rho^2-x_0^2-y_0^2)&=0 \end{aligned} \qquad \Longrightarrow \begin{pmatrix} x^2+y^2 \\ x_1^2+y_1^2 \\ x_2^2+y_2^2 \\ x_3^2+y_3^2 \end{pmatrix} -2x_0 \begin{pmatrix} x \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} -2y_0 \begin{pmatrix} y \\ y_1 \\ y_2 \\ y_3 \end{pmatrix} -(\rho^2-x_0^2-y_0^2) \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} =0$$

Notice how x_j^2 and y_j^2 must be "*correlated*" in that their relative values are not independent, but must be 1 to 1.

We can also consider "getting rid" of the ρ^2 unknown by taking an equation minus the previous equation:

$$\begin{aligned} x_1^2-2x_1x_0+x_0^2+y_1^2-2y_1y_0+y_0^2&=\rho^2 \\ -(x^2-2xx_0+x_0^2+y^2-2yy_0+y_0^2&=\rho^2) \\ \Longrightarrow x_1^2-x^2+-2x_0(x_1-x)+y_1^2-y^2+-2y_0(y_1-y)&=0 \end{aligned}$$

So that we get

$$\begin{pmatrix} x_1^2-x^2 \\ x_2^2-x_1^2 \\ x_3^2-x_2^2 \\ x^2-x_3^2 \end{pmatrix} + -2x_0 \begin{pmatrix} x_1-x \\ x_2-x_1 \\ x_3-x_2 \\ x-x_3 \end{pmatrix} + \begin{pmatrix} y_1^2-y^2 \\ y_2^2-y_1^2 \\ y_3^2-y_2^2 \\ y^2-y_3^2 \end{pmatrix} + -2y_0 \begin{pmatrix} y_1-y \\ y_2-y_1 \\ y_3-y_2 \\ y-y_3 \end{pmatrix} =0$$

Thus, these 4 vectors above are linearly dependent, which implies

$$\boxed{\begin{vmatrix} x_1^2-x^2 & x_2^2-x_1^2 & x_3^2-x_2^2 & x^2-x_3^2 \\ x_1-x & x_2-x_1 & x_3-x_2 & x-x_3 \\ y_1^2-y^2 & y_2^2-y_1^2 & y_3^2-y_2^2 & y^2-y_3^2 \\ y_1-y & y_2-y_1 & y_3-y_2 & y-y_3 \end{vmatrix}=0}$$

Exercise 7. $F(x)=det[f_{ij}(x)]$

$i=1 \qquad f_{11} \Longrightarrow |f_{11}|=F(x) \qquad \Longrightarrow F'(x)=f'_{11}=detA_1$

$i=2 \qquad \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix} = f_{11}f_{22}-f_{12}f_{21}=F(x)$

$$F'(x)=f'_{11}f_{22}+f_{11}f'_{22}-f'_{12}f_{21}-f_{12}f'_{21}$$

$$\begin{vmatrix} f'_{11} & f'_{12} \\ f_{21} & f_{22} \end{vmatrix} + \begin{vmatrix} f_{11} & f_{12} \\ f'_{21} & f'_{22} \end{vmatrix} = |A_1|+|A_2| = f'_{11}f_{22}-f_{21}f'_{12}+f_{11}f'_{22}-f_{12}f'_{21}=F'(x)$$

Assume n case is true.

$$F(x)=det(f_{ij}(x))=\sum_{k=1}^{n+1} f_{n+1,\,k}(-1)^{n+1+k}det(f)_{n+1,\,k}$$

$$F'(x)=\sum_{k=1}^{n+1} f'_{n+1,k}cof(f)_{n+1,k}+\sum_{k=1}^{n+1} f_{n+1,k}(-1)^{n+1+k}(det(f)_{n+1,k})'$$

$\sum_{k=1}^{n+1} f'_{n+1,k}cof(f)_{n+1,k}=detA_{n+1}$, matrix obtained by differentiating the $n+1$ row of $[f_{ij}]$
$(det(f)_{n+1,k})'=\sum_{l=1}^{n} detB_l$ where B_l is the matrix obtained by differentiating the lth row of $(f)_{n+1,k}$, $l=1,\dots,n$

$$\sum_{k=1}^{n+1} f_{n+1,k}(-1)^{n+1+k}(det(f)_{n+1,k})' = \sum_{k=1}^{n+1} f_{n+1,k}(-1)^{n+1+k}\sum_{l=1}^{n} detB_l = \sum_{l=1}^{n}\sum_{k=1}^{n+1} f_{n+1,k}(-1)^{n+k+1}detB_l =$$
$$=\sum_{l=1}^{n} detA_l$$

$$\Longrightarrow F'(x)=detA_{n+1}+\sum_{l=1}^{n} detA_l = \boxed{\sum_{l=1}^{n+1} detA_l}$$

Exercise 8. Consider $W(x) = [u_j^{(i-1)}(x)]$.

$|W(x)| = |[u_j^{(i-1)}(x)]|$ Use Ex.7: $F'(x) = \sum_{i=1}^n detA_i(x)$, where $A_i(x)$ is the matrix obtained by differentiating the functions in the ith row of $[f_{ij}(x)]$, then

$|W(x)|' = \sum_{i=1}^n detA_i(x)$, where $A_i(x)$ is the matrix obtained by differentiating the functions in the ith row of $[u_j^{(i-1)}(x)]$.

For $i = 1, \dots, n-1$, $i+1 = 2, \dots, n$ and there's a $k = i+1$ row s.t. $k = 2, \dots, n$ so that $detA_i = 0$,

For $i = n$, $[u_j^{(n-1)}(x)]' = [u_j^{(n)}(x)]$ and for $k = 1, \dots, n-1$, $[u_j^{(k-1)}(x)]$ is different from $[u_j^{(n)}(x)]$.
$\Longrightarrow |W(x)|' = detA_n(x)$, where $A_n(x)$ is the matrix obtained by differentiating the functions in the nth row of $[u_j^{(i-1)}(x)]$

4.4 Exercises - Linear transformations with diagonal matrix representations, Eigenvectors and eigenvalues of a linear transformations, Linear independence of eigenvectors corresponding to distinct eigenvalues

Exercise 1.

(1) $$\begin{aligned} T(x) &= \lambda x \\ aT(x) &= (a\lambda)x \end{aligned}$$

(2) $$\begin{aligned} T_1(x) &= \lambda_1 x \\ T_2(x) &= \lambda_2 x \end{aligned} \quad (aT_1 + bT_2)(x) = a\lambda_1 x + b\lambda_2 x = (a\lambda_1 + b\lambda_2)x$$

Exercise 2.

$$\begin{aligned} T(x) &= \lambda x \\ T^2(x) &= T(T(x)) = T(\lambda x) = \lambda T(x) = \lambda^2 x \\ T^n(x) &= \lambda^n x \\ T^{n+1}(x) &= T(T^n(x)) = T(\lambda^n x) = \lambda^n \lambda x = \lambda^{n+1} x \end{aligned}$$

Let $P(x) = \sum_{j=0}^N a_j x^j$
$P(T)(x) = \sum_{j=0}^N a_j T^j(x)$.

If x is an eigenvector, $P(T)(x) = \sum_{j=0}^N a_j T^j(x) = \sum_{j=0}^N a_j \lambda^j(x) = P(\lambda)x$.
Exercise 3. $V = V_2(\mathbb{R})$, plane as a real linear space.

$T =$ rotation of V through an angle of $\frac{\pi}{2}$ radians.

$$T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \text{ or } \begin{aligned} T(e_1) &= e_2 \\ T(e_2) &= -e_1 \end{aligned} \qquad \Longrightarrow \begin{aligned} T^2(x) &= T^2(x_1e_1 + x_2e_2) = T(x_1e_2 + x_2(-e_1)) = \\ &= -x_1e_1 + x_2(-e_2) = -x \end{aligned}$$

Exercise 4.

$$\begin{aligned} T^2 x_\lambda &= \lambda^2 x_\lambda \\ T^2 - \lambda^2 I &= (T + \lambda I)(T - \lambda I) \\ det(T^2 - \lambda^2 I) &= 0 = det(T + \lambda I) det(T - \lambda I) = 0 \end{aligned}$$

$det(T + \lambda I)$ or $det(T - \lambda I)$ is zero, so λ or $-\lambda$ is an eigenvalue of T.
Exercise 5. Let V be the linear space of all real functions differentiable on $(0, 1)$.

Let $f \in V$.
Define $g = T(f)$ s.t. $g(t) = tf'(t)\ \forall t \in (0, 1)$
Suppose f is an eigenfunction of T.

$$g(t) = T(f)(t) = tf'(t) = \lambda f(t) \Longrightarrow \boxed{f(t) = c_0 t^\lambda}$$

In solving this ordinary differential equation, $\lambda \in \mathbb{R}$
Exercise 6. $V = p(x)$ of degree $\leq n$.

$p \in V$, $q = T(p)$ s.t. $q(t) = p(t+1)\ \forall t$

$$p(t) = \sum_{j=0}^{N} a_j t^j$$

$$T(p(t)) = q(t) = p(x+1) = \sum_{j=0}^{N} a_j (t+1)^j = \lambda \sum_{j=0}^{N} a_j t^j$$

$$N = 0 \qquad a_0 = \lambda a_0 \Longrightarrow \lambda = 1$$

$$N = 1 \qquad \begin{aligned} a_1(t+1) + a_0 &= \lambda(a_1 t + a_0) \\ t(a_1(1-\lambda)) + a_1 + a_0(1-\lambda) &= 0 \\ \text{if } a_1 = 0, \text{ then } a_0 = 0 \text{ or } \lambda = 1 \\ \text{if } \lambda = 1,\ a_1 = 0 \end{aligned}$$

$$N = 2 \qquad \begin{aligned} a_2(t+1)^2 + a_1(t+1) + a_0 &= \lambda(a_2 t^2 + a_1 t + a_0) \\ a_2(t^2 + 2t + 1) + a_1(t+1) + a_0 &= \lambda(a_2 t^2 + a_1 t + a_0) \\ t^2(a_2(1-\lambda)) + t(2a_2 + a_1(1-\lambda)) + a_2 + a_1 + (1-\lambda)a_0 &= 0 \\ \text{if } a_2 = 0, \text{ we're left with } N = 1 \text{ case} \\ \text{if } \lambda = 1,\ a_2 = 0; \text{ we're left with } N = 1 \text{ case.} \end{aligned}$$

Assume $\sum_{j=0}^{N} a_j(t+1)^j = \lambda \sum_{j=0}^{N} a_j t^j$, $a_j = 0 \quad \forall j = 1, \dots N$

$$\begin{aligned} \sum_{j=0}^{N+1} a_j (t+1)^j &= \lambda \sum_{j=0}^{N+1} a_j t^j \\ \sum_{j=0}^{N+1} a_j \sum_{k=0}^{j} \binom{j}{k} t^k &= \lambda \sum_{j=0}^{N+1} a_j t^j \\ \sum_{j=0}^{N+1} a_j \sum_{k=0}^{j} \binom{j}{k} t^k - \lambda a_j t^j &= 0 \end{aligned}$$

$a_{N+1}(1-\lambda) = 0$.
If $\lambda = 1$, $t^N : \ a_{N+1}(N+1) + a_N - \lambda a_N = 0; \quad a_{N+1} = 0$, and then we could rewrite the equation, and coefficients, as Nth case, which we've shown to yield only a_0 to be nonzero.

Exercise 7. Let $V =$ linear space of functions continuous on $(-\infty, \infty)$ s.t. $\exists \int_{-\infty}^{x} f(t)dt \quad \forall x \in \mathbb{R}$
If $f \in V$, let $g = T(f)$ s.t. $g(x) = \int_{-\infty}^{x} f(t)dt$

$$T(f)(x) = g(x) = \int_{-\infty}^{x} f(t)dt = \lambda f(x)$$

$$\Longrightarrow f(x) = \lambda f'(x) \Longrightarrow \boxed{f(x) = c_0 e^{\lambda x}}$$

$$\int_{-\infty}^{x} f(t)dt = \left(\frac{c_0 e^{\lambda t}}{\lambda} \right)\Bigg|_{-\infty}^{x} = \frac{c_0 e^{\lambda x}}{\lambda} - \lim_{t \to -\infty} \frac{c_0 e^{\lambda t}}{\lambda}$$

A limit only exists if $\lambda > 0$.

Exercise 8.

$$g(x) = T(f)(x) = \int_{-\infty}^{x} t f(t) dt = \lambda f(x)$$

$$x f(x) = \lambda f'(x)$$

$$\text{if } \lambda \neq 0,\ \ln\left(\frac{f(x)}{f(0)}\right) = \frac{\frac{1}{2}x^2}{\lambda} \Longrightarrow \boxed{f(x) = c_0 e^{\frac{x^2}{2\lambda}}}$$

$$\int_{-\infty}^{x} t c_0 e^{t^2/2\lambda} dt = c_0 e^{x^2/2\lambda} - \lim_{t \in -\infty} c_0 e^{t^2/2\lambda}$$

Limit only exists if $\lambda < 0$.

Exercise 9.

$$T(f) = f'' = \lambda f$$
$$f(t) = c_n \sin nt \quad f(0) = f(\pi) = 0$$
$$f''(t) = -n^2 c_n \sin nt = \lambda f \quad \implies \lambda_n = -n^2$$

Exercise 10. $T(x) = (y_n)$. $y_n = a - x_n; \quad n \geq 1$
$T((x_n)) = (a - x_n) = \lambda(x_n)$

The sequences are equal, so $a - x_n = \lambda x_n$ or $a = (\lambda + 1)x_n$.
(x_n) is a convergent sequence, so

$$x_n - a = x_n - (\lambda+1)x_n = -\lambda x_n \text{ must go to zero}$$

So $\lim_{n\to\infty} x_n = 0$, $a = 0$. Then by $a = (\lambda+1)x_n$, $\lambda = -1$ for nonzero x_n.

$$\boxed{\lambda = -1,\ (x_n) \text{ s.t. } \lim_{n\to\infty} x_n = 0 \text{ and } x_n \text{ nonconstant}}$$

If x_n is constant, $a = (\lambda+1)a \implies \lambda = 0$.

$$\boxed{\lambda = 0, \quad x_n = a,\ (x_n) \text{ is a constant sequence}}$$

Exercise 11. $T(x) = \lambda x$; $T(y) = \mu y$.

$$T(ax + by) = \beta(ax + by) = a\lambda x + b\mu y$$
$$a(\beta - \lambda)x + b(\beta - \mu)y = 0$$

Use Thm., Thm. 4.2: Let $u_1, u_2, \dots, u_k$ be eigenvectors of linear linear transformation $T : S \to V$.
Assume corresponding eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_k$ distinct.
Then $u_1, u_2, \dots, u_k$ are independent.

$$\implies x, y \text{ independent} \quad \beta - \lambda = \beta - \mu = 0$$

If $\lambda = \beta$, $\mu \neq \beta$ or $\mu = \beta$, $\lambda \neq \beta$. (given that λ, μ distinct).

Exercise 12. Suppose $x, y \in S$, so $\begin{array}{l} T(x) = \lambda x \\ T(y) = \mu y \end{array}$

Suppose $\lambda \neq \mu$.
Suppose $ax + by \in S$ for some $a, b \in \mathbb{R}$. Then by definition of S, $ax + by$ is an eigenvector of T.
Then by Exercise 11, a or b is zero. Suppose $b = 0$.

$$T(ax + by) = \beta(ax + by) = aT(x) + bT(y) = a\lambda x + b\mu y$$
$$T(ax) = \beta(ax) = a\lambda x = \lambda(ax)$$

ax nonzero, so $\beta = \lambda$.

So if $x \in S$, so is $ax \in S$, $a \neq 0$ and $T(x) = cx = T(ax) = c(ax)$
This must be true $\forall\, x \in S \implies T(x) = cx \quad \forall\, x \in S$.

4.8 Exercises - The finite-dimensional case. Characteristic polynomials. Calculation of eigenvalues and eigenvectors in the finite-dimensional case. Trace of a matrix

Exercise 1.

(1) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\lambda_{1,2} = 1 \qquad \zeta_{\lambda=1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}; \qquad E(1) = 2$

(2) $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$$\begin{vmatrix} \lambda - 1 & -1 \\ 0 & \lambda - 1 \end{vmatrix} = (\lambda - 1)^2 = 0$$
$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \implies \zeta_{\lambda=1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \quad E(1) = 1$$

(3) $\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}$

$$\begin{vmatrix}\lambda-1 & 0\\ -1 & \lambda-1\end{vmatrix} = (\lambda-1)^2 = 0$$

$$\boxed{\lambda = 1, \quad \zeta_{\lambda=1} = \begin{bmatrix}0\\ 1\end{bmatrix}; \quad E(1) = 1}$$

(4) $\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$

$$\begin{vmatrix}\lambda-1 & -1\\ -1 & \lambda-1\end{vmatrix} = (\lambda-1)^2 - 1 = (\lambda-2)\lambda = 0; \quad \lambda = 0, 2$$

$$\lambda = 2, \, \zeta_{\lambda=2} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ 1\end{bmatrix}; \qquad E(2) = 1$$

$$\lambda = 0, \, \zeta_{\lambda=0} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ -1\end{bmatrix}; \qquad E(0) = 1$$

Exercise 2. $\begin{bmatrix}1 & a\\ b & 1\end{bmatrix}$; $a > 0, \, b > 0$

$$\begin{vmatrix}\lambda-1 & -a\\ -b & \lambda-1\end{vmatrix} = (\lambda-1)^2 - ab = \lambda^2 - 2\lambda + 1 - ab = 0 \qquad \lambda_\pm = \frac{2 \pm \sqrt{4 - 4(1)(1-ab)}}{2} = 1 \pm \sqrt{ab}$$

$$\begin{bmatrix}1 & a\\ b & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = (1 \pm \sqrt{ab})\begin{bmatrix}x_1\\ x_2\end{bmatrix} \Longrightarrow \begin{array}{l} x_1 + ax_2 = (1 \pm \sqrt{ab})x_1 \\ bx_1 + x_2 = (1 \pm \sqrt{ab})x_2 \end{array}$$

$$\boxed{\lambda_\pm = 1 \pm \sqrt{ab}; \quad \zeta_{\lambda=1\pm\sqrt{ab}} = \begin{pmatrix}\sqrt{a}\\ \pm\sqrt{b}\end{pmatrix} \quad E(1 \pm \sqrt{ab}) = 1}$$

Exercise 3. $\begin{bmatrix}c\theta & -s\theta\\ s\theta & c\theta\end{bmatrix}$

$$\begin{vmatrix}\lambda - c\theta & s\theta\\ -s\theta & \lambda - c\theta\end{vmatrix} = \lambda^2 - 2\lambda c\theta + 1 = 0 \Longrightarrow \lambda = c\theta \pm is\theta$$

if $\theta = 2\pi n$, $\qquad \lambda = 1, \, \xi_{\lambda=1} = \begin{bmatrix}1\\ 0\end{bmatrix}, \begin{bmatrix}0\\ 1\end{bmatrix}$

if $\theta \neq 2\pi n$, $\qquad \lambda = e^{-\pm i\theta} \quad \begin{bmatrix}c\theta & -s\theta\\ s\theta & c\theta\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = e^{\pm i\theta}\begin{bmatrix}x\\ y\end{bmatrix} \Longrightarrow \xi_{\lambda=e^{\pm i\theta}} = 1/\sqrt{2}\begin{bmatrix}1\\ \pm i\end{bmatrix}$

Exercise 4. $A = \begin{bmatrix}a & b\\ c & d\end{bmatrix}$

$$P_1 = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} \qquad |\lambda I - P_1| = \lambda^2 - 1 = 0$$

$$P_2 = \begin{bmatrix}0 & -i\\ i & 0\end{bmatrix} \qquad |\lambda I - P_2| = \lambda^2 - 1 = 0$$

$$P_3 = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} \qquad |\lambda I - P_3| = (\lambda-1)(\lambda+1) = 0$$

$$\begin{vmatrix}\lambda - a & -b\\ -c & \lambda - a\end{vmatrix} = \lambda^2 - (a+d)\lambda + (ad - bc) = 0 = (\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2$$

$$\Delta = ad - bc = -1 \Longrightarrow a = -d$$

$$\boxed{A = \begin{bmatrix}a & b\\ c & -a\end{bmatrix}} \text{ where } a^2 + bc = 1$$

Exercise 5.

$$det(A-\lambda I)=\lambda^2-(a+d)\lambda+(ad-bc)=0 \Longrightarrow \lambda=\frac{(a+d)\pm\sqrt{(a+d)^2-4(1)(ad-bc)}}{2}$$
$$\text{if } \sqrt{(a+d)^2-4(ad-bc)}>0,\ \lambda \text{ real and distinct}$$
$$\text{if } \sqrt{(a+d)^2-4(ad-bc)}=0,\ \lambda \text{ real and equal}$$
$$\text{if } \sqrt{(a+d)^2-4(ad-bc)}<0,\ \lambda \text{ complex conjugates}$$

Exercise 6.

$$\begin{bmatrix}1&1&1\\a&b&c\\d&e&f\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}=3\begin{bmatrix}1\\1\\1\end{bmatrix} \qquad \begin{aligned}a+b+c&=3\\d+e+f&=3\end{aligned}$$
$$\begin{bmatrix}1&1&1\\a&b&c\\d&e&f\end{bmatrix}\begin{bmatrix}1\\0\\-1\end{bmatrix}=0\begin{bmatrix}1\\0\\-1\end{bmatrix} \qquad \begin{aligned}a-c&=0 \quad a=c\\d-f&=0 \quad d=f\end{aligned}$$
$$\begin{bmatrix}1&1&1\\a&b&c\\d&e&f\end{bmatrix}\begin{bmatrix}1\\-1\\0\end{bmatrix}=0\begin{bmatrix}1\\-1\\0\end{bmatrix} \qquad \begin{aligned}a&=b\\d&=e\end{aligned}$$

$$a=b=c=d=e=f=1$$

Exercise 7.

(1) $\begin{bmatrix}1&0&0\\-3&1&0\\4&-7&1\end{bmatrix}$

$$\begin{vmatrix}1-\lambda&&\\-3&1-\lambda&\\4&-7&1-\lambda\end{vmatrix}=\begin{vmatrix}1-\lambda&&\\&1-\lambda&\\&&1-\lambda\end{vmatrix}=0 \quad \Longrightarrow \lambda=1$$

$$\begin{bmatrix}1&0&0\\-3&1&0\\4&-7&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} \quad \Longrightarrow \begin{aligned}x_1&=0\\x_2&=0\\x_3&=1\end{aligned} \quad \Longrightarrow \zeta_{\lambda=1}=\begin{bmatrix}0\\0\\1\end{bmatrix}$$

(2) $\begin{bmatrix}2&1&3\\1&2&3\\3&3&20\end{bmatrix}$

$$\begin{vmatrix}2-\lambda&1&3\\1&2-\lambda&3\\3&3&20-\lambda\end{vmatrix}=\begin{vmatrix}1-\lambda&1&3\\-1+\lambda&2-\lambda&3\\0&3&20-\lambda\end{vmatrix}=\begin{vmatrix}1-\lambda&1&3\\0&3-\lambda&6\\0&3&20-\lambda\end{vmatrix}=\begin{vmatrix}1-\lambda&&\\&3-\lambda&6\\&3&20-\lambda\end{vmatrix}$$

$$(1-\lambda)((3-\lambda)(20-\lambda)-18)=(1-\lambda)(60-23\lambda+\lambda^2-18)=(1-\lambda)(42-23\lambda+\lambda^2)=(1-\lambda)(\lambda-21)(\lambda-2)=0$$
$$\Longrightarrow \lambda=1,\ 2,\ 21$$

(3) $\begin{bmatrix}5&-6&-6\\-1&4&2\\3&-6&-4\end{bmatrix}$

$$\begin{vmatrix}\lambda-5&6&6\\1&\lambda-4&-2\\-3&6&\lambda+4\end{vmatrix}=\begin{vmatrix}\lambda-2&0&-\lambda+2\\0&\lambda-4&-1\\0&6&\lambda+1\end{vmatrix}=(\lambda-2)^2(\lambda-1)$$
$$\Longrightarrow \xi_{\lambda=1}=\frac{1}{\sqrt{19}}\begin{bmatrix}3\\-1\\3\end{bmatrix},\ \xi_{\lambda=2}=\frac{1}{\sqrt{18}}\begin{bmatrix}4\\1\\1\end{bmatrix},\ \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\-1\end{bmatrix}$$

Exercise 8.

(1) $\begin{bmatrix} & & 1 & \\ & & & 1 \\ 1 & & & \\ & 1 & & \end{bmatrix}$

$$\begin{vmatrix} -\lambda & & 1 & \\ & -\lambda & & 1 \\ 1 & & -\lambda & \\ & 1 & & -\lambda \end{vmatrix} = \begin{vmatrix} -\lambda & & 1-\lambda^2 & \\ & -\lambda & & 1-\lambda^2 \\ 1 & & 0 & \\ & 1 & & 0 \end{vmatrix} = (1-\lambda^2)^2 \Longrightarrow \lambda = \pm 1$$

(2) $\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix} \Longrightarrow (\lambda-1)^2(\lambda+1)^2 = 0$ so $\lambda = \pm 1$

(3) $\begin{bmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{bmatrix}$

$$\begin{vmatrix} \lambda & -1 & & \\ -1 & \lambda & & \\ & & \lambda & -1 \\ & & -1 & \lambda \end{vmatrix} = (\lambda^2-1)(\lambda^2-1) \Longrightarrow \lambda = \pm 1$$

(4) $\begin{bmatrix} & -i & & \\ i & & & \\ & & & -i \\ & & i & \end{bmatrix}$

$$\begin{vmatrix} -\lambda & -i & & \\ - & -\lambda & & \\ & & -\lambda & -i \\ & & i & -\lambda \end{vmatrix} = \begin{vmatrix} -\lambda & -i \\ i & -\lambda \end{vmatrix}^2 = (\lambda^2-1)^2$$

(5) $\begin{bmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{bmatrix} \Longrightarrow ((\lambda+1)(\lambda-1))^2 = 0$

Exercise 10.

Exercise 11. Let $(AB)x = \lambda x$

$$A^{-1}(AB)x = \lambda(A^{-1}x) = BX$$

Let $x = Ay$

$$\lambda(A^{-1}Ay) = BAy = \lambda y$$

So if λ is eigenvalue of AB, λ is also an eigenvalue of BA (A is invertible).

Exercise 13.

Exercise 14.

(1)

$$tr(A+B) = \sum_{i=1}^{N}(A+B)_{ii} = \sum_{i=1}^{N}(a_{ii}+b_{ii}) = \sum_{i=1}^{N} a_{ii} + \sum_{i=1}^{N} b_{ii} = trA + trB$$

(2)

$$tr(cA) = \sum_{i=1}^{N}(cA)_{ii} = \sum_{i=1}^{N} ca_{ii} = c\sum_{i=1}^{N} a_{ii} = ctrA$$

(3) $tr(AB) = \sum_{j=1}^{N}(AB)_{jj} = \sum_{j=1}^{N}\sum_{k=1}^{N} a_{jk}b_{kj} = \sum_{k=1}^{N}\sum_{j=1}^{N} b_{kj}a_{jk} = \sum_{j=1}^{N}(BA)_{jj} = tr(BA)$

(4) $trA^T = \sum_{j=1}^{N}(A^T)_{jj} = \sum_{j=1}^{N} a_{jj} = trA$

Exercise 1. Given

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad f(\lambda) = \lambda^2 - 2\lambda + 1 \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix} = x_{\lambda=1}$$

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad g(\lambda) = (\lambda - 1)^2 \quad \zeta_{\lambda=1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Suppose $C^{-1}BC = A$.
$C^{-1}C = I = A$. But $A \neq I$.
Contradiction. So $\nexists\, C$ invertible s.t. $C^{-1}BC = A$

Exercise 2.

(1) $A = \begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix}$

$$\begin{vmatrix} \lambda - 1 & \\ -1 & \lambda - 3 \end{vmatrix} = (\lambda - 1)(\lambda - 3) \quad \Longrightarrow \quad \begin{aligned} \xi_{\lambda=1} &= \frac{2}{\sqrt{5}} \begin{bmatrix} 1 \\ -1/2 \end{bmatrix} \\ \xi_{\lambda=3} &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{aligned}$$

$$\boxed{C = \begin{bmatrix} 2/\sqrt{5} & 0 \\ -1/\sqrt{5} & 1 \end{bmatrix}}$$

$$\text{indeed, } \frac{1}{2/\sqrt{5}} \begin{bmatrix} 1 & 0 \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} \begin{bmatrix} 1 & \\ 1 & 3 \end{bmatrix} C = I$$

(2)

$$A = \begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix}$$

$$\begin{vmatrix} \lambda - 1 & -2 \\ -5 & \lambda - 4 \end{vmatrix} = \lambda^2 - 5\lambda + 4 - 10 = (\lambda - 6)(\lambda + 1)$$

$$\begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -1, 6 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \Longrightarrow \quad \begin{aligned} \xi_{\lambda=-1} &= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} \\ \xi_{\lambda=6} &= \frac{1}{\sqrt{29}} \begin{bmatrix} 2 \\ 5 \end{bmatrix} \end{aligned}$$

$$C = \begin{bmatrix} 1/\sqrt{2} & 2/\sqrt{29} \\ -1/\sqrt{2} & 5/\sqrt{29} \end{bmatrix}$$

$$\text{indeed, } 1/(7/\sqrt{2}\sqrt{29}) \begin{bmatrix} 5/\sqrt{29} & -2/\sqrt{29} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix} C = \begin{bmatrix} -1 & \\ & 6 \end{bmatrix}$$

(3) $A = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}$

$$\begin{vmatrix} \lambda - 2 & -1 \\ 1 & \lambda - 4 \end{vmatrix} = (\lambda - 3)^2$$

Suppose nonsingular C exists, s.t. $C^{-1}AC = 3I \Longrightarrow A = 3I$. Contradiction.

(4)

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} \quad \Longrightarrow \begin{vmatrix} \lambda - 2 & -1 \\ 1 & \lambda \end{vmatrix} = \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2$$

Suppose nonsingular C exists s.t. $C^{-1}AC = I \Longrightarrow A = I$. Contradiction.

Exercise 3.

$$\begin{aligned} [y_1, y_2] &= [x_1, x_2]A \\ [z_1, z_2] &= [x_1, x_2]B \\ [z_1, z_2] &= [y_1, y_2]C = [x_1, x_2]B = [x_1, x_2]AC \end{aligned} \qquad \begin{aligned} B &= AC \\ A^{-1}B &= C \end{aligned}$$

Exercise 4.

(1)

$$A=\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix} \qquad \Longrightarrow \begin{vmatrix}\lambda&0&-1\\0&\lambda-1&0\\-1&0&\lambda\end{vmatrix}=\lambda^3-\lambda^2-\lambda+1$$

$$\lambda^3-\lambda^2-\lambda+1=(\lambda-1)^2(\lambda+1)$$

Note that we could still obtain the following independent eigenvectors: $\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\-11\end{bmatrix}$

(2)

$$A=\begin{bmatrix}1&-1&-1\\1&3&1\\-1&-1&1\end{bmatrix} \qquad \Longrightarrow f(\lambda)=\lambda^3-5\lambda^2+8\lambda-4=$$

$$=(\lambda^2-4\lambda+4)(\lambda-1)=(\lambda-2)^2(\lambda-1)$$

$$\Longrightarrow x_{\lambda=2}=\begin{bmatrix}1\\-1\\0\end{bmatrix}, \quad x_{\lambda=2}=\begin{bmatrix}1\\0\\-1\end{bmatrix}, \quad x_{\lambda=1}=\begin{bmatrix}-1\\1\\-1\end{bmatrix}$$

Exercise 5. Generally, we'll have

$$C^{-1}AC=\lambda I+\begin{bmatrix}&\\1&\end{bmatrix}\Longrightarrow A=\lambda I+C\begin{bmatrix}&\\1&\end{bmatrix}C^{-1}$$

$$\begin{bmatrix}a&b\\c&d\end{bmatrix}\begin{bmatrix}&\\1&\end{bmatrix}C^{-1}=\begin{bmatrix}b&0\\d&0\end{bmatrix}\frac{1}{detC}\begin{bmatrix}d&-b\\-c&a\end{bmatrix}=$$

$$=\frac{1}{detC}\begin{bmatrix}bd&-d^2\\d^2&-bd\end{bmatrix}$$

(1) So for $A=\begin{bmatrix}2&-1\\0&2\end{bmatrix}$. Then $d=0,\ b=1,\ c=-1$ for C.

(2) For $A=\begin{bmatrix}2&1\\-1&4\end{bmatrix}\Longrightarrow\lambda=3$. So $bd=-1,\ \ ad-bc=-1$.

$$\text{if } b=1,\ \ d=-1,\ \ -a-c=-1 \quad a+c=1$$
$$\text{if } b=-1,\ \ d=1,\ \ a+c=-1$$

Exercise 6.

$$\begin{bmatrix}0&-1&\\&&1\\-1&-3&5\end{bmatrix}\Longrightarrow\begin{vmatrix}\lambda&1&\\&\lambda&-1\\1&3&\lambda-3\end{vmatrix}=(\lambda-1)^3$$

$$\Longrightarrow\frac{1}{\sqrt{3}}\begin{bmatrix}1\\-1\\-1\end{bmatrix}$$

Suppose $C^{-1}AC=I$. $A=CC^{-1}=1$ so diagonalizing matrix cannot exist for this A.
CHECK this result.

Exercise 7.

(1) $\forall$ matrix A, we can always consider the characteristic polynomial $|\lambda I-A|=f(\lambda)$. So $\exists\, n$ roots, $\lambda_j\in\mathbb{C}$ If $\lambda_j\neq 0$, $\forall\, j=1,\dots,n$, then $det(C^{-1}AC)=detA=det(\Lambda)=\prod_{j=1}\lambda_j\neq 0$. So A nonsingular.
If A nonsingular, $detA\neq 0$, so

$$detA=det(C^{-1}AC)=det\Lambda\neq 0 \qquad \Longrightarrow \lambda_j\neq\ \forall\, j$$

(2) A nonsingular,

$$det(AA^{-1})=detAdetA^{-1}=detC^{-1}ACdetD^{-1}AD=det\Lambda_A det\Lambda_A=\prod_{j=1}^{n}\lambda_j\prod_{k=1}^{n}b_k=1$$

λ_j, b_k distinct, so $b_k=\frac{1}{\lambda_j}$

Exercise 8.

(1) $A^2=-1$ so $A^{-1}=-A$, so A nonsingular.
(2) $detA^2=(detA)^2=(-1)^n$. $(detA)^2>0$, so $(-1)^n=1$; n even.

(3) $Ax = \lambda x$
$A^2 x = -x = \lambda^2 x \quad \lambda^2 = -1$

(4) $detA = 1$ From fundamental theorem of algebra, roots of the characteristic polynomial must come in complex conjugate pairs. We already showed that the eigenvalues are purely imaginary. So there must be a whole number of pairs of complex conjugate eigenvalues that multiply together to get -1. **CHECK** this result.

5.5 Exercises - Eigenvalues and inner products, Hermitian and skew-Hermitian transformations, Eigenvalues and eigenvectors of Hermitian and skew-Hermitian operators, Orthogonality of eigenvectors corresponding to distinct eigenvalues

Exercise 1.

$$\begin{aligned} &\text{if } T(x) = \lambda x, (T(x), y) = (\lambda x, y) = \lambda(x, y)\, \forall y \in E \\ &\text{if } (T(x), y) = (\lambda x, y)\, \forall y \in E \\ &\quad \implies (T(x) - \lambda x, y) = 0\, \forall y \in E \\ &\quad\quad \implies T(x) = \lambda x \end{aligned}$$

Exercise 2.

$$\begin{aligned} &(T(x), y) = (cx, y) = c(x, y) \\ &(x, T(y)) = (x, cy) = \bar{c}(x, y) \\ &\text{since } V \text{ is a real Euclidean space } c \in \mathbb{R} \text{ for } (T(x), y), (x, T(y)) \in \mathbb{R} \end{aligned}$$

Exercise 3.

(1) Assume $T : V \to V$ is a Hermitian transformation.

Use induction:

$$\begin{aligned} (Tx, y) &= (x, Ty) \\ (T^2 x, y) &= (Tx, Ty) = (x, T^2 y) \\ (T^{n+1} x, y) &= (Tx, T^n y) = (x, T^{n+1} y) \end{aligned}$$

T^{-1} is Hermitian since

$$(T^{-1}x, y) = (T^{-1}x, TT^{-1}y) = (TT^{-1}x, T^{-1}y) = (x, T^{-1}y)$$

Neat trick, no?

(2) $(T(x), y) = -(x, T(y))$. Now $(T^2(x), y) = -(T(x), T(y)) = (-1)^2(x, T^2(y))$.
Assume the nth case, $(T^n(x), y) = (-1)^n(x, T^n(y))$, i.e. T^n is Hermitian (skew-Hermitian) if n is even (odd). Then consider that

$$(T^{n+1}(x), y) = -(T^n(x), T(y)) = -(-1)^n(x, T^{n+1}(y)) = (-1)^{n+1}(x, T^{n+1}(y))$$

$$(T^{-1}(x), y) = (T^{-1}(x), TT^{-1}y) = -(TT^{-1}(x), T^{-1}(y)) = -(x, T^{-1}(y))$$

So T^{-1} is skew-Hermitian.

Exercise 4.

(1)

$$\begin{aligned} ((aT_1 + bT_2)(x), y) = (aT_1(x) + bT_2(x), y) = a(T_1(x), y) + b(T_2(x), y) = (x, (aT_1)(y)) + (x, (bT_2)y) = \\ = (x, (aT)(y) + (bT_2)(y)) = (x, (aT_1 + bT_2)y) \end{aligned}$$

(2)

$$\begin{gathered} (T_1 T_2(x), y) = (T_2(x), T_1(y)) = (x, T_2 T_1(y)) \\ \text{if } T_1 T_2 = T_2 T_1; T_1 T_2 \text{ is Hermitian} \end{gathered}$$

Exercise 5. Let $V = V_3(\mathbb{R})$.

$$\begin{aligned} (T(x), y) &= \sum_{j=1}^{3} (T(x))_j y_j = x_1 y_1 + x_2 y_2 - x_3 y_3 \\ &= x_1 y_1 + x_2 y_2 + x_3(-y_3) = (x, T(y)) \end{aligned}$$

Exercise 6. $\int_0^1 f(t)dt = F(1) - F(0) = 0, F(1) = F(0)$, likewise, for $g \in V$, $\int_0^1 g(t)dt = G(1) - G(0) = 0, G(1) = G(0)$.

The trick is to use integration by parts

$$(Tf,g)=\int_0^1 (Tf)(t)g(t)dt=\int_0^1\int_0^t f(x)dxg(t)dt=$$
$$=\int_0^1 F(t)g(t)dt-F(0)\int_0^1 g(t)dt=F(t)G(t)|_0^1-\int_0^1 f(t)G(t)dt=-(f,Tg)$$

Exercise 7.

(1)
$$(Tf,g)=\int_{-1}^1 Tf(t)g(t)dt=\int_{-1}^1 f(-t)g(t)dt=\int_1^{-1} f(t)g(-t)(-dt)=\int_{-1}^1 f(t)g(-t)dt=$$
$$=(f,Tg)$$

(2)
$$(Tf,g)=\int_{-1}^1 f(t)f(-t)g(t)dt \text{ but } (f,Tg)=\int_{-1}^1 f(t)g(t)g(-t)dt$$
$\implies$ Neither symmetric nor skew-symmetric (choose different coefficients for f and g)

(3)
$$(Tf,g)=\int_{-1}^1 Tf(t)g(t)dt=\int_{-1}^1 (f(t)+f(-t))g(t)dt=\int_{-1}^1 fg+-\int_1^{-1} f(t)g(-t)dt=$$
$$=\int_{-1}^1 f(t)(g(t)+g(-t))dt=(f,Tg)$$
Hermitian.

(4)
$$(Tf,g)=\int_{-1}^1 (f(t)-f(-t))g(t)dt=\int_{-1}^1 fg-\int_{-1}^1 f(-t)g(t)dt=$$
$$=\int_{-1}^1 fg-\int_1^{-1} f(t)g(-t)(-dt)=\int_{-1}^1 f(g(t)-g(-t))dt=(f,Tg)$$
Hermitian.

Exercise 8. Given $(f,g)=\int_a^b f(t)g(t)w(t)dt$, $T(f)=\frac{(pf')'+qf}{w}$

$$(Tf,g)=\int_a^b (Tf)(t)g(t)w(t)dt=\int_a^b \frac{(pf')'(t)+q(t)f(t)}{w(t)}g(t)w(t)dt=\int_a^b ((pf')'+qf)g$$
$$\int_a^b (pf')'g=(pf')g|_a^b-\int_a^b (pf')g'=-\int_a^b (pf')g'$$

$$\text{since } f,g \text{ satisfy } \begin{array}{c} p(a)f(a)=0 \\ p(b)f(b)=0 \end{array}$$

$$\int_a^b (pf')g'=pg'f|_a^b-\int_a^b (pg')'f=0-\int_a^b (pg')'f$$
$$\implies (Tf,g)=\int_a^b (pg')'f+qfg=\int_a^b wf\frac{((pg')'+qg)}{w}=(f,Tg)$$

Exercise 9. Let V be a subspace of a complex Euclidean space E.
Let $T:V\to E$ be a linear transformation and define a scalar-valued function Q on V as follows:
$$Q(x)=(T(x),x)\quad \forall\, x\in V$$

(1) T Hermitian.
$$(Tx,x)=(x,Tx)=\overline{(Tx,x)}=\overline{Q(x)}=Q(x)\implies Q(x)\in\mathbb{R}$$

(2)
$$(Tx,x)=-(x,Tx)=-\overline{(Tx,x)}=-\overline{Q(x)}=Q(x)\implies Q(x) \text{ pure imaginary}$$

(3)
$$Q(tx)=(T(tx),tx)=(tTx,tx)\quad \text{(since } T \text{ is linear)}$$
$$Q(tx)=t(Tx,tx)=t\overline{(tx,Tx)}=t\bar{t}(Tx,x)=t\bar{t}Q(x)$$

(4)

$$Q(x+y) = (T(x+y), x+y) = (Tx+Ty, x+y) = (Tx, x+y) + (Ty, x+y) =$$
$$= \overline{(x+y, Tx)} + \overline{(x+y, Ty)} = \overline{(x, Tx)} + \overline{(y, Tx)} + \overline{(x, Ty)} + \overline{(y, Ty)} =$$
$$= (Tx, x) + (Tx, y) + (Ty, x) + (Ty, y) = Q(x) + Q(y) + (T(x), y) + (T(y), x)$$
$$Q(x+ty) = Q(x) + Q(ty) + (T(x), ty) + (T(ty), x) = Q(x) + t\bar{t}Q(y) + \bar{t}(T(x), y) + t(T(y), x)$$

(5) Suppose $T(x) = y \neq 0$ for some $x \in V, \;\; y \in E, \;\; x \neq 0$

$$(T(x), x) = (y, x) = 0$$

$y \neq x$, otherwise $(x, x) = 0; \;\; x = 0$. Contradiction. Done.

$$Q(ax+by) = (T(ax+by), ax+by) = |a|^2(T(x), x) + \bar{b}a(T(x), y) + b\bar{a}(T(y), x) + |b|^2(T(y), y)$$
$$= a\bar{b}(T(x), y) + b\bar{a}(T(y), x) = 0$$

Let $a = 1, \, b = -1$

$$-(T(x), y) = (T(y), x) = (y, y) > 0$$

Let $a\bar{b} = i$

$$(T(x), y) = (T(y), x) = (y, y) > 0$$

Contradiction. Thus $y = 0$.

(6)

$$Q(x+ty) = Q(x) + t\bar{t}Q(y) + \bar{t}(T(x), y) + t(T(y), x)$$
$$Q \in \mathbb{R} \Longrightarrow \overline{Q(x+ty)} = Q(x+ty) \Longrightarrow t(y, T(x)) + \bar{t}(x, T(y)) = \bar{t}(T(x), y) + t(T(y), x)$$
$$\Longrightarrow t((y, T(x)) - (T(y), x)) + \bar{t}((x, T(y)) - (T(x), y)) = 0$$

Suppose $t = a + bi$, a, b arbitrary

$$\Longrightarrow \begin{array}{l} (y, T(x)) - (T(y), x) + ((x, T(y)) - (T(x), y)) = 0 \\ (y, T(x)) - (T(y), x) - ((x, T(y)) - (T(x), y)) = 0 \end{array} \Longrightarrow (y, T(x)) - (T(y), x) = 0 \text{ so } T \text{ is Hermitian.}$$

Exercise 10. Legendre polynomials:

$$P_n(t) = \frac{1}{2^n n!} f_n^{(n)}(t) \text{ where } f_n(t) = (t^2 - 1)^n$$

(1)

$$(t^2 - 1)f_n'(t) = (t^2 - 1)(n)(t^2 - 1)^{n-1}(2t) = 2nt(t^2 - 1)^n = 2ntf_n(t)$$

(2) Leibniz's formula. If $h(x) = f(x)g(x)$, prove that the nth derivative of n is given by the formula.

$$h^{(n)}(x) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(x) g^{(n-k)}(x)$$

so then

$$((t^2-1)f_n'(t))^{(n+1)} = \sum_{k=0}^{n+1} \binom{n+1}{k} (t^2 - 1)^{(k)} (f_n'(t))^{(n+1-k)} =$$
$$= (t^2 - 1)f_n^{(n+2)}(t) + (n+1)2tf_n^{(n+1)}(t) + \frac{(n+1)n}{2} 2f_n^{(n)}(t)$$

$$(t^2 - 1)' = 2t$$
$$(t^2 - 1)'' = 2$$
$$(t^2 - 1)'' = 0$$

$$(2ntf_n(t))^{(n+1)} = (2n)(t(f_n(t))^{(n+1)} + (n+1)(f_n(t))^{(n)})$$
$$\Longrightarrow (t^2 - 1)f_n^{(n+2)}(t) + 2t(n+1)f_n^{(n+1)}(t) + (n+1)nf_n^{(n)}(t) = (2n)(tf_n^{(n+1)}(t) + (n+1)(f_n(t))^{(n)})$$

(3) $P_n(t) = \frac{1}{2^n n!} f_n^{(n)}(t)$

$$\Longrightarrow (t^2 - 1)P_n'' + 2t(n+1)P_n' + (n+1)nP_n = (2n)(tP_n' + (n+1)P_n)$$
$$\Longrightarrow (t^2 - 1)P_n'' + 2tP_n' - (n+1)nP_n = 0 \text{ or } \boxed{((t^2 - 1)P_n')' = n(n+1)P_n}$$

5.11 Exercises - Existence of an orthonormal set of eigenvectors for Hermitian and skew-Hermitian operators acting on finite-dimensional spaces; Matrix representations for Hermitian and skew-Hermitian operators; Hermitian and skew-Hermitian matrices. The adjoint of a matrix; Diagonalization of a Hermitian or skew-Hermitian matrix; Unitary matrices. Orthogonal matrices

Exercise 1.

(1) $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 2 & 3 & 4 \end{bmatrix}$. Symmetric. Hermitian.

(2) $\begin{bmatrix} 0 & i & 2 \\ i & 0 & 3 \\ -2 & -3 & 4i \end{bmatrix}$. Skew-Hermitian.

(3) $\begin{bmatrix} 0 & i & 2 \\ -i & 0 & 3 \\ -2 & -3 & 0 \end{bmatrix}$. Skew-symmetric.

(4) $\begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & 0 \end{bmatrix}$. Skew-Hermitian. Skew-symmetric.

Exercise 2.

(1)

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(2)

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} r\cos\alpha \\ r\sin\alpha \end{bmatrix} = \begin{bmatrix} r\cos\alpha\cos\theta - r\sin\alpha\sin\theta \\ r\cos\alpha\sin\theta + r\sin\alpha\cos\theta \end{bmatrix} = r \begin{bmatrix} \cos(\alpha+\theta) \\ \sin(\alpha+\theta) \end{bmatrix}$$

Exercise 3.

(1) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ (reflection in the xy-plane).

$$\begin{bmatrix} 1 & & \\ & 1 & \\ & & -1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & & -1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \qquad \begin{aligned} \begin{bmatrix} 1 & & \\ & 1 & \\ & & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} &= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 1 & & \\ & 1 & \\ & & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} &= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 1 & & \\ & 1 & \\ & & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \end{aligned}$$

(2) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ (reflection through the x-plane).

$$\begin{bmatrix} 1 & & \\ & -1 & \\ & & -1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & -1 & \\ & & -1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \qquad \begin{aligned} \begin{bmatrix} 1 & & \\ & -1 & \\ & & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} &= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 1 & & \\ & -1 & \\ & & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} &= -\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 1 & & \\ & -1 & \\ & & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \end{aligned}$$

(3) $\begin{bmatrix}-1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{bmatrix}$ (reflection through the origin).

$$\begin{bmatrix}-1 & & \\ & -1 & \\ & & -1\end{bmatrix}\begin{bmatrix}-1 & & \\ & -1 & \\ & & -1\end{bmatrix} = \begin{bmatrix}1 & & \\ & 1 & \\ & & 1\end{bmatrix} \qquad \begin{bmatrix}-1 & & \\ & -1 & \\ & & -1\end{bmatrix} i, j, k = -i, j, k)$$

(4) $\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\theta & -\sin\theta\\ 0 & \sin\theta & \cos\theta\end{bmatrix}$ (rotation about the x-axis).

$$\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\theta & -\sin\theta\\ 0 & \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\theta & +\sin\theta\\ 0 & -\sin\theta & \cos\theta\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\theta & -\sin\theta\\ 0 & \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\theta & -\sin\theta\\ 0 & \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}0\\ \cos\theta\\ \sin\theta\end{bmatrix}$$

$$\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\theta & -\sin\theta\\ 0 & \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\ -\sin\theta\\ \cos\theta\end{bmatrix}$$

(5) $\begin{bmatrix}-1 & 0 & 0\\ 0 & \cos\theta & -\sin\theta\\ 0 & \sin\theta & \cos\theta\end{bmatrix}$ (rotation about x-axis followed by reflection in the yz-plane).

$$\begin{bmatrix}-1 & 0 & 0\\ 0 & \cos\theta & -\sin\theta\\ 0 & \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}-1 & 0 & 0\\ 0 & \cos\theta & \sin\theta\\ 0 & -\sin\theta & \cos\theta\end{bmatrix} = \begin{bmatrix}1 & & \\ & 1 & \\ & & 1\end{bmatrix}\begin{bmatrix}-1 & & \\ & 1 & \\ & & 1\end{bmatrix}\begin{bmatrix}1 & & \\ & c_\theta & -s_\theta\\ & s_\theta & c_\theta\end{bmatrix} = \begin{bmatrix}-1 & 0 & 0\\ 0 & \cos\theta & -\sin\theta\\ 0 & \sin\theta & \cos\theta\end{bmatrix} =$$

$= $ (reflection in the yz-plane)(rotation about x-axis)

Exercise 4. A real orthogonal matrix A is called *proper* if $detA = 1$, and *improper* if $detA = -1$.

(1)

$$\begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}a & c\\ b & d\end{bmatrix} = \begin{bmatrix}a^2+b^2 & ac+bd\\ ac+bd & c^2+d^2\end{bmatrix} \implies \begin{aligned} ac+bd &= 0\\ a^2+b^2 &= 1\\ ad-bc &= 1\end{aligned}$$

$$\implies \cos^2\theta\left(\frac{-c}{\sin\theta}\right) - \sin\theta c = 1 \quad \implies \begin{aligned} a &= \cos\theta\\ b &= \sin\theta\\ c &= -\sin\theta\\ d &= \cos\theta\end{aligned}$$

(2)

$$\begin{bmatrix}1 & \\ & -1\end{bmatrix}\begin{bmatrix}1 & \\ & -1\end{bmatrix} = \begin{bmatrix}1 & \\ & 1\end{bmatrix} \qquad det\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} = -1$$

$$\begin{bmatrix}-1 & \\ & 1\end{bmatrix}\begin{bmatrix}-1 & \\ & 1\end{bmatrix} = \begin{bmatrix}1 & \\ & 1\end{bmatrix} \qquad det\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix} = -1$$

From previous part, all improper 2×2 matrices:

$$\begin{bmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{bmatrix}$$

Exercise 13. If A is a real skew-symmetric matrix, prove that both $I - A$ and $I + A$ are nonsingular and that $(I - A)(I + A)^{-1}$ is orthogonal.

Note: Notice the difference between skew-Hermitian and skew symmetric and use eignevalue eqn. and one-to-one.

Skew-Hermitian matrices must be square matrices (from how Skew-Hermitian operators, T, are defined as $T : V \to V$). For skew-symmetric matrices, eigenvalues must equal zero, since $-\lambda = \overline{\lambda}$

$$\begin{gathered} C^{-1}AC = \Lambda = 0 \\ det(1 \pm A) = det(C^{-1}C)det(1 \pm A) = det(C^{-1})det(1 \pm A)detC = \\ = det(1 \pm C^{-1}AC) = det(1 \pm 0) = \boxed{1} \end{gathered}$$

To prove orthogonality of $(I - A)(I + A)^{-1}$, use $(AB)^T = B^T A^T$ extensively.
We know that A is real skew-symmetric, so that $A = -A^T$.

$$\begin{aligned} (1 + A)(1 + A)^{-1} &= 1 \\ ((1 + A)(1 + A)^{-1})^T = ((1 + A)^{-1})^T (1 + A^T) &= 1^T = 1 \end{aligned}$$

Thus,

$$\begin{gathered} (1 - A)(1 + A)^{-1}((1 + A)^{-1}(1 - A))^T = \\ = (1 - A)(1 + A)^{-1}(1 - A^T)((1 + A)^{-1})^T = (1 - A)\left((1 + A)^{-1}(1 + A)\right)((1 + A)^{-1})^T = \\ = (1 - A)((1 + A)^{-1})^T = (1 + A^T)((1 + A)^{-1})^T = (1 + A)^T((1 + A)^{-1})^T = ((1 + A)^{-1}(1 + A))^T = 1^T = 1 \end{gathered}$$

Note that we have $((1 - A)(1 + A)^{-1})^T = ((1 + A)^{-1}(1 - A))^T$ because if $(1 + A)^{-1} = B$,

$$\begin{aligned} &(1 + A)B = B + AB = 1 \\ &B(1 + A) = B + BA = 1 \quad \text{(since a left inverse is a right inverse)} \end{aligned} \quad \Longrightarrow BA = AB$$

$$(1 - A)(1 + A)^{-1} = (1 - A)B = B - AB = B - BA = B(1 - A) = (1 + A)^{-1}(1 - A)$$

Exercise 14.

(1) Counterexample:

$$A = \begin{bmatrix} 1 & \\ & e^{-i2\pi/3} \end{bmatrix} \qquad B = \begin{bmatrix} e^{i2\pi/3} & \\ & 1 \end{bmatrix} \qquad (A + B) = \begin{bmatrix} 1 + e^{i2\pi/3} & \\ & 1 + e^{-i2\pi/3} \end{bmatrix}$$

$$A^* = \begin{bmatrix} 1 & \\ & e^{i2\pi/3} \end{bmatrix} \qquad B^* = \begin{bmatrix} e^{-i2\pi/3} & \\ & 1 \end{bmatrix} \qquad A^* + B^* = (A + B)^* = \begin{bmatrix} 1 + e^{-i2\pi/3} & \\ & 1 + e^{i2\pi/3} \end{bmatrix}$$

$$(A + B)(A^* + B^*) = \begin{bmatrix} 2 + e^{i2\pi/3} + e^{-i2\pi/3} & \\ & 2 + e^{i2\pi/3} + e^{-i2\pi/3} \end{bmatrix}$$

(2) If A and B are unitary, then AB is unitary.

$$\begin{gathered} AA^* = BB^* = 1 \\ (AB)(AB)^* = (AB)B^*A^* = A(BB^*)A^* = A1A^* = 1 \end{gathered}$$

(3)
(4)

5.15 Exercises - Quadratic forms, Reduction of a real quadratic form to a diagonal form, Applications to analytic geometry

Exercise 1. $4x_1^2 + 4x_1x_2 + x_2^2$. $\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} = A$

$$\begin{vmatrix} 4 - \lambda & 2 \\ 2 & 1 - \lambda \end{vmatrix} = 4 - 5\lambda + \lambda^2 - 4 = \lambda(\lambda - 5)$$

$$\xi_{\lambda=5} = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}; \quad \xi_{\lambda=0} = \begin{bmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{bmatrix}$$

$$C = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}$$

Exercise 2. $\quad x_1x_2 \qquad A = \begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix}$. $\lambda = \frac{1}{2}, -\frac{1}{2}$.

$$\xi_{\lambda=1/2} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \quad \xi_{\lambda=-1/2} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

Exercise 3. $x_1^2 + 2x_1x_2 - x_2^2$. $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

Exercise 4. $34x_1^2 - 24x_1x_2 + 41x_2^2$. $\quad A = \begin{bmatrix} 34 & -12 \\ -12 & 41 \end{bmatrix}$.

$\begin{vmatrix} \lambda - 34 & 12 \\ 12 & \lambda - 41 \end{vmatrix} = \lambda^2 - 75\lambda + 34(41) - 144 = \lambda^2 - 75\lambda + 1250.$ $\boxed{\lambda = 50, 25}$

$$\boxed{\begin{array}{ll} \xi_{\lambda=50} = \frac{1}{5}\begin{bmatrix} 3 \\ -4 \end{bmatrix} & \\ & C = \frac{1}{5}\begin{bmatrix} 3 & 4 \\ -4 & 3 \end{bmatrix} \\ \xi_{\lambda=25} = \frac{1}{5}\begin{bmatrix} 4 \\ 3 \end{bmatrix} & \end{array}}$$

Exercise 5. $x_1^2 + x_1x_2 + x_1x_3 + x_2x_3$.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad \Longrightarrow \begin{vmatrix} \lambda - 1 & -1 & -1 \\ -1 & \lambda & -1 \\ -1 & -1 & \lambda \end{vmatrix} = \begin{vmatrix} \lambda & -1 & 0 \\ -1 & \lambda - 2 & 0 \\ & 0 & \lambda + 1 \end{vmatrix} = (\lambda + 1)(\lambda^2 - 2\lambda - 1)$$

$$\xi_{\lambda=-1} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & \sqrt{2}/2 & -\sqrt{2}/2 \\ 1/\sqrt{2} & 1/2 & 1/2 \\ -1/\sqrt{2} & 1/2 & 1/2 \end{bmatrix}$$

$$\xi_{\lambda=1\pm\sqrt{2}} = \frac{1}{2}\begin{bmatrix} \pm\sqrt{2} \\ 1 \\ 1 \end{bmatrix}$$

Exercise 6. $2x_1^2 + 4x_1x_3 + x_2^2 - x_3^2$

$$A = \begin{bmatrix} 2 & & 2 \\ & 1 & \\ 2 & & -1 \end{bmatrix} \quad \Longrightarrow \begin{vmatrix} \lambda - 2 & & -2 \\ & \lambda - 1 & \\ -2 & & \lambda + 1 \end{vmatrix} = (\lambda - 1)(\lambda - 3)(\lambda + 2)$$

$$\begin{aligned} \xi_{\lambda=1} &= (0, 1, 0) \\ \xi_{\lambda=3} &= \frac{1}{\sqrt{5}}(2, 0, 1) \\ \xi_{\lambda=-2} &= \frac{1}{\sqrt{5}}(1, 0, -2) \end{aligned} \quad \Longrightarrow C = \begin{bmatrix} 0 & 2/\sqrt{5} & 1/\sqrt{5} \\ 1 & 0 & 0 \\ 0 & 1/\sqrt{5} & -2/\sqrt{5} \end{bmatrix}$$

Exercise 7. $3x_1^2 + 4x_1x_@ + 8x_1x_3 + 4x_2x_3 + 3x_3^2$.

$$A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$$

$$det(\lambda I - A) = \begin{vmatrix} \lambda - 3 & -2 & -4 \\ -2 & \lambda & -2 \\ -4 & -2 & \lambda - 3 \end{vmatrix} = \begin{vmatrix} \lambda + 1 & -2 & 0 \\ -2\lambda - 2 & \lambda & -2\lambda - 2 \\ 0 & -2 & \lambda + 1 \end{vmatrix} = \begin{vmatrix} \lambda + 1 & -2 & 0 \\ -2\lambda - 2 & \lambda - 4 & 0 \\ 0 & -2 & \lambda + 1 \end{vmatrix} =$$

$$= (\lambda + 1)(\lambda - 4)(\lambda + 1) + 2(\lambda + 1)(-2\lambda - 2) = (\lambda + 1)(\lambda^2 - 3\lambda - 4 - 4\lambda - 4) = (\lambda + 1)^2(\lambda - 8)$$

$$\begin{bmatrix}3&2&4\\2&0&2\\4&2&3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=-1\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\implies \begin{aligned}4x_1+2x_2+4x_3=0\\2x_1+x_2+2x_3=0\end{aligned}$$

$$\implies \xi_{\lambda=-1}=\frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\-1\end{bmatrix}$$

$$\begin{bmatrix}3&2&4\\2&0&2\\4&2&3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=9\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\implies\begin{vmatrix}-5&2&4\\2&-8&2\\4&2&-5\end{vmatrix}=\begin{vmatrix}0&-18&9\\1&-4&1\\0&18&-9\end{vmatrix}=\begin{vmatrix}&2&-1\\1&0&-1\\&&\end{vmatrix}$$

$$\implies \xi_{\lambda=8}=\frac{1}{3}\begin{bmatrix}2\\1\\2\end{bmatrix}$$

$$\begin{vmatrix}e_1&e_2&e_3\\1&0&-1\\2&1&2\end{vmatrix}=(1,-4,1)\implies \xi_{\lambda=-1}=\frac{1}{3\sqrt{2}}\begin{bmatrix}1\\-4\\1\end{bmatrix}$$

$$\boxed{C=\begin{bmatrix}1/\sqrt{2}&1/3\sqrt{2}&2/3\\0&-4/3\sqrt{2}&1/3\\-1/\sqrt{2}&1/3\sqrt{2}&2/3\end{bmatrix}}$$

Exercise 8. $y^2-2xy+2x^2-5=0$.

$$\begin{bmatrix}2&-1\\-1&1\end{bmatrix}\implies\begin{vmatrix}\lambda-2&1\\1&\lambda-1\end{vmatrix}=\lambda^2-3\lambda+1=0$$

$$\implies \lambda=\frac{3\pm\sqrt{5}}{2}$$

$$\begin{bmatrix}2&-1\\-1&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\frac{3\pm\sqrt{5}}{2}\begin{bmatrix}x_1\\x_2\end{bmatrix}\implies \begin{aligned}\xi_{\lambda=\frac{3+\sqrt{5}}{2}}&=\frac{1}{\sqrt{\frac{5-\sqrt{5}}{2}}}\begin{bmatrix}1\\\frac{1-\sqrt{5}}{2}\end{bmatrix}\\ \xi_{\lambda=\frac{3-\sqrt{3}}{2}}&=\frac{1}{\sqrt{5+\sqrt{5}2}}\begin{bmatrix}1\\\frac{1+\sqrt{5}}{2}\end{bmatrix}\end{aligned}$$

$$\implies C=\begin{bmatrix}\frac{1}{\sqrt{\frac{5-\sqrt{5}}{2}}}&\frac{1}{\sqrt{\frac{5+\sqrt{5}}{2}}}\\\sqrt{\frac{2}{5-\sqrt{5}}}\left(\frac{1-\sqrt{5}}{2}\right)&\sqrt{\frac{2}{5+\sqrt{5}}}\frac{1+\sqrt{5}}{2}\end{bmatrix}$$

$$\implies \frac{3+\sqrt{5}}{2}x^2+\frac{3-\sqrt{5}}{2}y^2=5$$

Ellipse centered about $(0,0)$.

Exercise 9. $y^2-2xy+5x=0$

$$\begin{bmatrix}0&-1\\-1&1\end{bmatrix}=A$$

$$|\lambda I-A|=\begin{vmatrix}\lambda&1\\1&\lambda-1\end{vmatrix}=\lambda^2-\lambda-1\quad\implies\lambda=\frac{1\pm\sqrt{1-4(1)(-1)}}{2}=\frac{1\pm\sqrt{5}}{2}$$

$$\begin{bmatrix}0&-1\\-1&1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\frac{1+\sqrt{5}}{2}\begin{bmatrix}x\\y\end{bmatrix}\implies \xi_{\lambda=\frac{1+\sqrt{5}}{2}}=\frac{1}{\sqrt{\frac{5+\sqrt{5}}{2}}}\begin{bmatrix}1\\\frac{1+\sqrt{5}}{-2}\end{bmatrix}$$

$$\begin{bmatrix}0&-1\\-1&1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\frac{1-\sqrt{5}}{2}\begin{bmatrix}x\\y\end{bmatrix}\implies \xi_{\lambda=\frac{1-\sqrt{5}}{2}}=\frac{1}{\sqrt{\frac{5-\sqrt{5}}{2}}}\begin{bmatrix}1\\\frac{-1+\sqrt{5}}{2}\end{bmatrix}$$

$$\implies C=\begin{bmatrix}\sqrt{\frac{2}{5+\sqrt{5}}}&\sqrt{\frac{2}{5-\sqrt{5}}}\\\sqrt{\frac{2}{5+\sqrt{5}}}\left(\frac{1+\sqrt{5}}{-2}\right)&\sqrt{\frac{2}{5-\sqrt{5}}}\left(\frac{-1+\sqrt{5}}{2}\right)\end{bmatrix}$$

$$y^2 - 2xy + 5x = 0 \Longrightarrow$$

$$\frac{1+\sqrt{5}}{2}x^2 + \frac{1-\sqrt{5}}{2}y^2 + 5\left(\sqrt{\frac{2}{5+\sqrt{5}}}x + \sqrt{\frac{2}{5-\sqrt{5}}}y\right) = \frac{1+\sqrt{5}}{2}x^2 + 5\sqrt{\frac{2}{5+\sqrt{5}}}x + \frac{1-\sqrt{5}}{2}y^2 + 5\sqrt{\frac{2}{5-\sqrt{5}}}y = 0 =$$

$$= \frac{1+\sqrt{5}}{2}\left(x^2 + 5\sqrt{\frac{2}{5+\sqrt{5}}}\left(\frac{2}{1+\sqrt{5}}\right)x\right) + \frac{1-\sqrt{5}}{2}\left(y^2 + 5\sqrt{\frac{2}{5-\sqrt{5}}}\left(\frac{2}{1-\sqrt{5}}\right)y\right) = 0$$

$$\Longrightarrow \frac{1+\sqrt{5}}{2}\left(x + 5\sqrt{\frac{2}{5+\sqrt{5}}}\left(\frac{1}{1+\sqrt{5}}\right)\right)^2 + \left(\frac{1-\sqrt{5}}{2}\right)\left(y + 5\sqrt{\frac{2}{5-\sqrt{5}}}\left(\frac{1}{1-\sqrt{5}}\right)\right)^2 =$$

$$= \frac{1+\sqrt{5}}{40+16\sqrt{5}}5^2 + \frac{(1-\sqrt{5})5^2}{40-16\sqrt{5}}$$

$$CY = X \Longrightarrow C\begin{bmatrix} -5\sqrt{\frac{2}{5+\sqrt{5}}}\left(\frac{1}{1+\sqrt{5}}\right) \\ -5\sqrt{\frac{2}{5-\sqrt{5}}}\left(\frac{1}{1-\sqrt{5}}\right)\end{bmatrix} = \begin{bmatrix}\frac{5}{2} \\ \frac{5}{2}\end{bmatrix}$$

Ellipse centered at $(5/2, 5/2)$.
Exercise 10. $y^2 - 2xy + x^2 - 5x = 0$.

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\begin{vmatrix} 1-\lambda & -1 \\ -1 & 1-\lambda \end{vmatrix} = 1 - 2\lambda + \lambda^2 - 1 = \lambda(\lambda - 2) \Longrightarrow \lambda = 0, 2$$

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Longrightarrow \xi_{\lambda=0} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\text{similarly, } \xi_{\lambda=2} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$C = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \Longrightarrow 2x_2^2 - \frac{5x_2}{\sqrt{2}} = \frac{5}{\sqrt{2}}x_1$$

$$\Longrightarrow \boxed{\frac{2\sqrt{2}}{5}\left(x_2 - \frac{5}{4\sqrt{2}}\right)^2 = x_1 + \frac{5}{8\sqrt{2}}}$$

$$C\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} = C\begin{bmatrix} -\frac{5}{8\sqrt{2}} \\ \frac{5}{4\sqrt{2}} \end{bmatrix}$$

$$\Longrightarrow (x, y) = \left(\frac{5}{16}, \frac{-15}{16}\right)$$

The vertex of the parabola in (x, y) coordinates is $\left(\frac{5}{16}, \frac{-15}{16}\right)$.
Exercise 11. $5x^2 - 4xy + 2y^2 - 6 = 0$.

$$\begin{bmatrix} 5 & -2 \\ -2 & 2 \end{bmatrix} = A \qquad \begin{vmatrix} 5-\lambda & -2 \\ -2 & 2-\lambda \end{vmatrix} = 10 - 7\lambda + \lambda^2 - 4 = (\lambda - 6)(\lambda - 1)$$

$$\lambda = 1 \quad \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \qquad \xi_{\lambda=1} = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\lambda = 6 \quad \begin{bmatrix} -1 & -2 \\ -2 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \quad \xi_{\lambda=6} = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$x_1^2 + 6x_2^2 = 6$$

$$\frac{x_1^2}{6} + x_2^2 = 1$$

Ellipse centered at $(0, 0)$ in both sets of coordinates.
Exercise 12. $19x^2 + 4xy + 16y^2 - 212x + 104y = 356$.

$$\begin{bmatrix}19 & 2\\ 2 & 16\end{bmatrix} \quad \begin{vmatrix}19-\lambda & 2\\ 2 & 16-\lambda\end{vmatrix} = (\lambda-15)(\lambda-20)$$

$$\xi_{\lambda=15} = \frac{1}{\sqrt{5}}\begin{bmatrix}-1\\ 2\end{bmatrix}$$

$$\xi_{\lambda=20} = \frac{1}{\sqrt{5}}\begin{bmatrix}2\\ 1\end{bmatrix}$$

$$YC^{-1} = X \text{ so}$$

$$[x \quad y] = [x_1 \quad x_2]\frac{1}{\sqrt{5}}\begin{bmatrix}-1 & 2\\ 2 & 1\end{bmatrix} = \left[\frac{-x_1+2x_2}{\sqrt{5}}, \ \frac{2x_1+x_2}{\sqrt{5}}\right]$$

$$\implies 15x_1^2 + 20x_2^2 + -212\left(\frac{-x_1+2x_2}{\sqrt{5}}\right) + 104\left(\frac{2x_1+x_2}{\sqrt{5}}\right) = 356$$

$$\implies \frac{\left(x_1+\frac{14}{\sqrt{5}}\right)^2}{\left(\frac{403}{5}\right)} + \frac{\left(x_2-\frac{8}{\sqrt{5}}\right)^2}{\left(\frac{1209}{20}\right)} = 1$$

Suppose we want to know what the center is in terms of the original (x, y) coordinates. Use C.

$$C\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix}-1 & 2\\ 2 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix}-x_1+2x_2\\ 2x_1+x_2\end{bmatrix} =$$

$$\xrightarrow{(x_1,x_2)=\left(\frac{-14}{\sqrt{5}},\frac{8}{\sqrt{5}}\right)} = \begin{bmatrix}6\\ -4\end{bmatrix}$$

Thus, we have an ellipse centered at $(6, -4)$.
Exercise 13. $9x^2 + 24xy + 16y^2 - 52x + 14y = 6$

$$\begin{bmatrix}9 & 12\\ 12 & 16\end{bmatrix} \qquad \xi_{\lambda=25} = \tfrac{1}{5}\begin{bmatrix}3\\ 4\end{bmatrix}; \quad \xi_{\lambda=0} = \tfrac{1}{5}\begin{bmatrix}4\\ -3\end{bmatrix}$$

$$X = YC^{-1} \implies [x \quad y] = [x_1 \quad x_2]\frac{1}{5}\begin{bmatrix}4 & -3\\ 3 & 4\end{bmatrix} = \frac{1}{5}[4x_1+3x_2, \quad -3x_1+4x_2]$$

$$\implies 25x_2^2 - 52\left(\frac{4x_1+3x_2}{5}\right) + 14\left(\frac{-3x_1+4x_2}{5}\right) = 6$$

$$\implies \frac{1}{2}(x_2 - \frac{12}{5})^2 = \frac{1}{5} + x_1$$

To get the center in terms of the (x, y) original coordinates,

$$C\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \frac{1}{5}\begin{bmatrix}4 & 3\\ -3 & 4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \frac{1}{5}\begin{bmatrix}4x_1+3x_2\\ -3x_1+4x_2\end{bmatrix}$$

$$\xrightarrow{(x_1,x_2)=\left(\frac{-1}{5},\frac{2}{5}\right)} = \begin{bmatrix}2/25\\ 11/25\end{bmatrix}$$

Thus we have a parabola centered at $\left(\frac{2}{25}, \frac{11}{25}\right)$.
Exercise 14. $5x^2 + 6xy + 5y^2 - 2 = 0$

$$A = \begin{bmatrix}5 & 3\\ 3 & 5\end{bmatrix}$$

$$\lambda = 2, 8 \qquad \xi_{\lambda=2} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ -1\end{bmatrix} \quad \xi_{\lambda=8} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ 1\end{bmatrix}$$

$$\implies 2x_1^2 + 8x_2^2 - 2 = 0$$

$$\implies \boxed{x_1^2 + 4x_2^2 = 1}$$

Thus we have an ellipse centered about the origin in both coordinate axes.
Exercise 15. $x^2 + 2xy + y^2 - 2x + 2y + 3 = 0$

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \qquad \begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = \lambda(\lambda - 2)$$

Directly from the characteristic function,

$$\Longrightarrow \begin{array}{c} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \quad \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \xi_{\lambda=0} \\ \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \quad \xi_{\lambda=2} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{array}$$

$$YC^T = X = [x \quad y] = [x_1 \quad x_2] \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{cases} x &= \frac{1}{\sqrt{2}}(x_1 + x_2) \\ y &= \frac{1}{\sqrt{2}}(-x_1 + x_2) \end{cases}$$

$$\Longrightarrow 2x_2^2 + -\frac{2}{\sqrt{2}}(x_1 + x_2) + \frac{2}{\sqrt{2}}(-x_1 + x_2) + 3 = 0$$

$$\boxed{\frac{\sqrt{2}}{2} x_2^2 + \frac{3\sqrt{2}}{4} = x_1}$$

$$C \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} x_1 + x_2 \\ -x_1 + x_2 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

For vertex at $\left(\frac{3\sqrt{2}}{4}, 0\right)$ in (x_1, x_2) coordinates, vertex has $\left(\frac{3}{4}, \frac{-3}{4}\right)$ as (x, y) coordinates.

Exercise 16. $2x^2 + 4xy + 5y^2 - 2x - y - 4 = 0$.

$$A = \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix}$$

$$\begin{vmatrix} 2-\lambda & 2 \\ 2 & 5-\lambda \end{vmatrix} = (2-\lambda)(5-\lambda) - 4 = 10 - 7\lambda + \lambda^2 - 4 = 6 - 7\lambda + \lambda^2 = (\lambda - 6)(\lambda - 1)$$

$$\xi_{\lambda=1} = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ -1 \end{bmatrix} \quad \xi_{\lambda=6} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$C = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$$

$$YC^T = X = [2x_1 + x_2, \quad -x_1 + 2x_2] \frac{1}{\sqrt{5}}$$

$$x_1^2 + 6x_2^2 - 2\left(\frac{2x_1 + x_2}{\sqrt{5}}\right) - \left(\frac{-x_1 + 2x_2}{\sqrt{5}}\right) - 4 = 0$$

$$\Longrightarrow x_1^2 - \frac{3x_1}{\sqrt{5}} + 6x_2^2 - \frac{4x_2}{\sqrt{5}} = 4 \Longrightarrow \frac{\left(x_1 - \frac{3}{2\sqrt{5}}\right)^2}{\left(\frac{55}{12}\right)} + \frac{\left(x_2 - \frac{1}{3\sqrt{5}}\right)^2}{\left(\frac{55}{72}\right)} = 1$$

To find the center of the ellipse in terms of (x, y),

$$C \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} 2x_1 + x_2 \\ -x_1 + 2x_2 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\xrightarrow{(x_1, x_2) = \left(\frac{3}{2\sqrt{5}}, \frac{1}{3\sqrt{5}}\right)} \left(\frac{2}{3}, \frac{-1}{6}\right)$$

For the center of an ellipse at $\left(\frac{3}{2\sqrt{5}}, \frac{1}{3\sqrt{5}}\right)$ in (x_1, x_2) coordinates, the center of the ellipse in (x, y) coordinates is $\left(\frac{2}{3}, -\frac{1}{6}\right)$.

Exercise 17. $x^2 + 4xy - 2y^2 - 12 = 0$

$$\begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix} \begin{vmatrix} 1-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix} = (1-\lambda)(-2-\lambda) - 4 = (\lambda + 3)(\lambda - 2).$$

$\lambda = 2, -3 \qquad \xi_{\lambda=2} = \frac{1}{\sqrt{5}}\begin{bmatrix}2\\1\end{bmatrix} \quad \xi_{\lambda=-3} = \frac{1}{\sqrt{5}}\begin{bmatrix}-1\\2\end{bmatrix}.$

$$C = \frac{1}{\sqrt{5}}\begin{bmatrix}2 & -1\\1 & 2\end{bmatrix}$$

$$YC^{-1} = X = [x\, y] = [x_1\, x_2]C = \frac{1}{\sqrt{5}}[2x_1 - x_2, x_1 + 2x_2]$$

$$\implies 2x_1^2 - 3x_2^2 - 12 = 0 \implies \boxed{\frac{x_1^2}{6} - \frac{x_2^2}{4} = 1}$$

Exercise 18. $xy + y - 2x - 2 = 0$

$$A = \begin{bmatrix}0 & 1/2\\1/2 & 0\end{bmatrix} \qquad \begin{vmatrix}-\lambda & 1/2\\1/2 & -\lambda\end{vmatrix} = \lambda^2 - \frac{1}{4} = 0$$

$$\lambda = \pm 1/2 \qquad \xi_{\lambda=1/2} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} \quad \xi_{\lambda=-1/2} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$$

$$C = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}$$

$$YC^{-1} = X = [x\, y] = [x_1\, x_2]\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix} = \left(\frac{x_1 + x_2}{\sqrt{2}}, \frac{x_1 - x_2}{\sqrt{2}}\right)$$

$$\implies \frac{1}{2}x_1^2 + \frac{-1}{2}x_2^2 + \frac{x_1 - x_2}{\sqrt{2}} - 2\left(\frac{x_1 + x_2}{\sqrt{2}}\right) = 2$$

$$\implies \left(x_1 - \frac{1}{\sqrt{2}}\right)^2 + -\left(x_2 + \frac{3}{\sqrt{2}}\right)^2 = 4 + \frac{1}{2} - \frac{9}{2} = 0$$

Suppose two lines are the asymptotic limit of a hyperbola. Then these lines are "hyperbolas."

$$\boxed{\pm\left(x_1 - \frac{1}{\sqrt{2}}\right) = \left(x_2 + \frac{3}{\sqrt{2}}\right)}$$

If we want to get what the center of this "hyperbola" is in terms of coordinates in the original x, y axis (we already have them for $(x_1, x_2) = Y$ and that is $\left(\frac{1}{\sqrt{2}}, -\frac{3}{\sqrt{2}}\right)$, then apply C as a transformation.

$$C(x_1, x_2) = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-3\end{bmatrix} = \begin{bmatrix}-1\\2\end{bmatrix}$$

The center is $(-1, 2)$ in (x, y) coordinates.

Exercise 19. $2xy - 4x + 7y + c = 0$.

$$\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$$

$$\begin{vmatrix}\lambda & -1\\-1 & \lambda\end{vmatrix} = \lambda^2 - 1 = 0 \qquad \begin{matrix}\xi_{\lambda=1} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}\\ \xi_{\lambda=-1} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}\end{matrix} \qquad \implies C = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}$$

$$\xrightarrow{CY=X} \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\begin{bmatrix}y_1\\y_2\end{bmatrix} = \begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{pmatrix}\frac{1}{\sqrt{2}}(y_1 + y_2)\\ \frac{1}{\sqrt{2}}(y_1 - y_2)\end{pmatrix}$$

$$x^2 + -y^2 - 4\left(\frac{1}{\sqrt{2}}(x + y)\right) + 7\frac{1}{\sqrt{2}}(x - y) + c = x^2 + \frac{3}{\sqrt{2}}x + -y^2 - \frac{11}{\sqrt{2}}y + c = 0$$

$$\left(x + \frac{3}{2\sqrt{2}}\right)^2 - \left(y + \frac{11}{2\sqrt{2}}\right)^2 + c = 0 \quad \implies \boxed{c = -14}$$

Exercise 20. Note that

$$ax^2 + bxy + cy^2 = 1 = XAX^T \text{ where} A = \begin{bmatrix}a & b/2\\b/2 & c\end{bmatrix}, \quad X = \begin{bmatrix}x & y\end{bmatrix}$$

A symmetric, by theorem, A can be diagonalized into its eigenvalues. By thm., $XAX^T = Y\Lambda Y^T$ where Y is an orthonormal coordinate transformation.

$$|\lambda - A| = (\lambda - a)(\lambda - c) - \frac{b^2}{4} = (\lambda - \lambda_+)(\lambda - \lambda_-)$$

So $\lambda_+\lambda_- = ac - \frac{b^2}{4}$ (this had been the smart way to see this: you can do algebra to get)

$$\lambda_\pm = \frac{(a+c) \pm \sqrt{(a-c)^2 + b^2}}{2}$$

Then using eigenvectors as basis,

$$ax^2 + bxy + cy^2 = \frac{z^2}{1/\lambda_+} + \frac{w^2}{1/\lambda_-} = 1$$

From geometry, the area of an ellipse is πab, where a, (b) is the half of semi-major (minor) axis

$$\text{ellipse area } = \pi \sqrt{\frac{1}{\lambda_+}}\sqrt{\frac{1}{\lambda_-}} = \pi \frac{1}{\sqrt{ac - \frac{b^2}{4}}} = \frac{2\pi}{\sqrt{4ac - b^2}}$$

5.20 Exercises - Eigenvalues of a symmetric transformation obtained as values of its quadratic form; Extremal properties of eigenvalues of a symmetric transformation; The finite-dimensional case; Unitary transformations

Exercise 1.

(1)

$$(T(x), T(x)) = (x, x) = |c|^2 (x, x)$$
$$(x, x) > 0 \text{ if } x \neq 0, \text{ so } \boxed{|c|^2 = 1}$$

(2) V one-dim. $T : V \to V$.
T unitary, T linear. $x \in V$; $x = ae_1$; $T(x) = T(ae_1) = aT(e_1)$. $T(e_1) \in V$; $T(e_1) = \mu e_1$.
$T(x) = a\mu e_1 = \mu x$. Since T unitary, by above $|\mu| = 1$. If V real, μ real, so $\mu = \pm 1$.

Exercise 2.

(1) A real, orthogonal. $AA^* = A\overline{A}^T = AA^T = 1$. Thus, A unitary.
$\implies$ eigenvalue λ of A s.t. $|\lambda| = 1$.
If $\lambda \in \mathbb{R}$, $\lambda = \pm 1$.

(2)

(3) If $n = 2s + 1$ odd, suppose λ_1 is eigenvalue of A. If λ_1 real, done. If λ_1 non-real, then $\overline{\lambda_1}$ is an eigenvalue (by previous part).
Continue, until nth eigenvalue (we've already checked s pairs of eigenvalues to be non-real). If λ_n non-real, $\overline{\lambda}_n$ is an eigenvalue. Then there are $2s + 2$ eigenvalues. But we're given that n odd. Contradiction. Thus λ_n real.

Exercise 3. T or thogonal. Then $m(T) = A$ has at least one real eigenvalue, $|\lambda_n| = 1$.

$$\text{Given } detA = 1,\ detA = 1 = \left(\prod_{i=1}^{s} |\lambda_i|^2\right)\lambda_n = (1)\lambda_n$$

Since suppose there are s complex eigenvalues. Then there are s complex conjugate eigenvalues. Then there are at most $n - 2s = (\text{odd} - \text{even}) = \text{odd}$ number of real eigenvalues. Since $detA = 1$, and $\left(\prod_{i=1}^{s} |\lambda_i|^2\right) = 1$ already (T orthogonal), there can only be an even number of real eigenvalues equal to -1. Then there must be at least one eigenvalue equal to 1.

Exercise 4. A real, orthogonal, then A unitary. Then for eigenvalues λ of A, $|\lambda| = 1$. Consider all complex λ of A; they come in complex conjugate pairs, and so if there are s conjugate pairs, $\prod_{i=1}^{s} |\lambda_s|^2 = 1$.
Consider all real eigenvalues of A. Then $\lambda = \pm 1$. If -1 is an eigenvalue of multiplicity of k, then there are k diagonal entries of -1 for diagonalized A. Thus, all possible eigenvalues are considered, so $detA = \prod_{i=1}^{s} |\lambda_s|^2 (1)(-1)^k = 1(1)(-1)^k = (-1)^k$

Exercise 5. Given that T linear and norm-preserving,

$$(T(x+by),T(x+by)) = \|T(x)\|^2 + b(T(y),T(x)) + \overline{b}(T(x),T(y)) + |b|^2\|T(t)\|^2 = \|T(x+by)\|^2 =$$
$$= \|x+by\|^2 = \|x\|^2 + \overline{b}(x,y) + b(y,x) + |b|^2\|y\|^2$$

$$\begin{aligned}\|T(x)\|^2 &= \|x\|^2 \\ \|T(y)\|^2 &= \|y\|^2\end{aligned} \quad \text{as well} \implies b\left((T(y),T(x)) - (y,x)\right) + \overline{b}\left((T(x),T(y)) - (x,y)\right) = 0$$

$b, \overline{b}$ are independent since $b = s+ti$ and s,t are two arbitrary real numbers. So $(T(x),T(y)) = (x,y)$, so T unitary.

Exercise 6. $T : V \to V$ unitary, Hermitian.

$$(T(x),y) = (x,T(y)) \quad \text{(Hermitian)}$$
$$(T^2(x),y) = (T(x),T(y)) = (x,T^2(y)) = (x,y) \qquad \implies (T^2(x)-x,y) = 0$$

Let $y = x$.

$$((T^2 - I)(x),x) = Q_1(x) = 0 \quad \forall\, x \in V$$

Then $T^2 - I = 0$ (as previously shown for $Q_1(x) = 0 \,\forall\, x \in V$), or $T^2 - I$.

Exercise 7. $(e_1,\dots,e_n)$, $(u_1,\dots,u_n)$ are 2 orthonormal bases for Euclidean space V.

$e_j \in V$ so $e_j \sum_{k=1}^n a_{jk}u_k$.

$$(e_i,e_j) = (e_j,e_i) = \left(\sum_{l=1}^n a_{il}u_l, \sum_{k=1}^n a_{jk}u_k\right) = \sum_{l=1}^n\sum_{k=1}^n a_{il}\overline{a}_{jk}(u_l,u_k) =$$
$$= \sum_{k=1}^n a_{ik}\overline{a}_{jk}$$

$\implies$ A is unitary, T s.t. $m(T) = A$ is unitary (isomorphism).

Exercise 8. $\begin{bmatrix} a & \frac{1}{2}i & \frac{1}{2}a(2i-1) \\ ia & \frac{1}{2}(1+i) & \frac{1}{2}a(1-i) \\ a & \frac{-1}{2} & \frac{1}{2}a(2-i)\end{bmatrix}$

$$\sum_{k=1}^n a_{ki}\overline{a}_{kj} = \sum_{k=1}^3 a_{ki}\overline{a}_{kj} = (e_i,e_j)$$
$$a^2 + ia(-ia) + a^2 = 3a^2 = 1 \qquad \implies a^2 = \frac{1}{3}$$

(a,ia,a), $\left(\frac{1}{2}i,\frac{1}{2}(1+i),\frac{-1}{2}\right)$, $\frac{a}{2}\left(2i-1,1-i,2-i\right)$ are orthogonal to each other through (x,y) inner product on complex Euclidean V. If $a = \pm\sqrt{1/3}$, columsn of A will be normalized.

$\implies A^TA = I$, so A unitary.

Exercise 9. A skew-Hermitian, $\Lambda = C^*AC$.

$$det(1\pm A) = det(C^*C)det(1\pm A) = det(1 \pm C^*AC) = det(1\pm A) = \prod_{j=1}^n (1\pm\lambda_j)$$

If $\lambda_j \in \mathbb{C}$, λ_j purely imaginary and $\overline{\lambda}_j$ is also an eigenvalue.

$$(1\pm\lambda_j)(1\pm\overline{\lambda}_j) = 1 \pm (\lambda_j + \overline{\lambda}_j) + |\lambda_j|^2 = 2$$

If $\lambda_j \in \mathbb{R}$, $\lambda_j = 0$. $1 \pm \lambda_j = 1$

$$\implies det(1\pm A) \neq 0 \text{ so } 1\pm A \text{ nonsingular}$$

Let $B = (1+A)^{-1}$. Use the fact that a left inverse is also a right inverse (theorem) extensively.

$$\begin{aligned}(1+A)B &= B + AB = 1 \\ B(1+A) &= B + BA = 1\end{aligned} \quad \implies AB = BA \qquad \begin{aligned}((1+A)B)* &= B^*(1+A^*) = B^* + B^*A^* = 1 \\ (B(1+A))^* &= (1+A^*)B^* = B^* + A^*B^* = 1\end{aligned} \quad \implies B^*A^* = A^*B^*$$

so

$$B(1-A) = B - BA = B - AB = (1-A)B$$

Thus, using $A = -A^*$, since A skew-Hermitian,

$$(1-A)(1+A)^{-1}((1-A)(1+A)^{-1})* = (1-A)B((B(1-A))^*) = (1-A)B(1-A^*)B^* = (1-A)B(1+A)B^* =$$
$$= (1-A)B^* = (1+A^*)B^* = 1$$

$(1-A)(1+A)^{-1}$ unitary.

Exercise 10. A unitary, $I + A$ nonsingular. Let $(1+A)^{-1} = B$. Using this fact

$$B(1+A) = B(AA^* + A) = BA(1+A^*) = 1 = 1* = (1+A)(A^*B^*)$$
$$\implies A^*B^* = B^*A^* = B$$

Then

$$((1-A)B)^* = B^*(1-A^*) = B^* - A^*B^* = (A-1)A^*B^* = -(1-A)B^*A^* = -(1-A)B$$

Thus $(1-A)B$ is skew-Hermitian.
Exercise 11. A Hermitian, so $A = A^*$. Let $B = (A-i)^{-1}$

$$B(A-i) = 1 = (A-i)B \implies AB = BA$$
$$(B(A-i))^* = 1 = (A^*+i)B^* = A^*B^* + iB^* = B^*A^* + iB^* \quad \implies A^*B^* = B^*A^*$$

Then

$$B(A+i)(B(A+i))^* = (A+i)B(A+i)^*B^* = (A+i)B(A-i)B^* = (A^*+i)1B^* = 1$$

Exercise 12. A unitary, so by theorem, there exists a complete set of orthonormal eigenvectors that form a basis for V, $\{u_1, \dots, u_n\}$.
Suppose A was defined in the $\{e_1, \dots, e_n\}$ basis. Then they are related through some matrix C (most general assumption to make):

$$[u_1, \dots, u_n] = [e_1, \dots, e_n]C \quad \implies u_j = \sum_{i=1}^n \sum_{i=1}^n c_{ij}e_i$$

$$(u_j, u_k) = \left(\sum_{i=1}^n c_{ij}e_i, \sum_{l=1}^n c_{lk}e_l \right) = \sum_{i=1}^n \sum_{l=1}^n c_{ij}\overline{c}_{lk}(e_i, e_l) = \sum_{i=1}^n c_{ij}\overline{c}_{ik} =$$
$$= \sum_{l=1}^n (C^*)_{kl}c_{lj} = (u_k, u_j)$$

Hence $C^*C = 1$, so C is unitary.
Recall what the entries of matrix A are, evaluated from the inner product in a certain chosen basis:

$$Ae_l = \sum_{m=1}^n a_{lm}e_m$$
$$(Ae_k, e_l) = \left(\sum_{m=1}^n a_{km}e_m, e_l \right) = a_{kl}$$

Thus,

$$(CAC^*)_{ij} \sum_{k=1}^n C_{ik}(AC^*)_{kj} = \sum_{k=1}^n C_{ik} \sum_{l=1}^n a_{kl}\overline{C}_{jl} = \sum_{k=1}^n \sum_{l=1}^n c_{ik}a_{kl}\overline{c}_{jl} = \sum_{k=1}^n \sum_{l=1}^n c_{ik}(Ae_k, e_l)\overline{c}_{jl} =$$
$$= \left(A\sum_{k=1}^n c_{ik}e_k, \sum_{l=1}^n c_{jl}e_l \right) = (Au_i)$$

Exercise 13. A square matrix is called *normal* if $AA^* = A^*A$. Determine which of the following types of matrices are normal.

(1) Hermitian matrices. $AA^* = A^*A$ since $A = A*$
(2) Skew-Hermitian matrices. $AA^* = -A^*(-A) = A^*A$
(3) Symmetric matrices.
(4) Skew-symmetric matrices.
(5) Unitary matrices.
(6)

Exercise 14. If A is a normal matrix ($AA^* = A * A$) and if U is a unitary matrix, prove that U^*AU is normal.

$$(U^*AU)(U^*AU)^* = U^*AUU^*A^*U = U^*AA^*U = U^*A^*AU = U^*A^*UU^*AU =$$
$$= (U^*AU)^*(U^*AU)$$

Exercise 1.

a. From (6.35) $(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0$.
If $\alpha = 0$, $(1-x^2)y'' - 2xy' = 0$, $y' = v$, so $\frac{v'}{v} = \frac{2x}{1-x^2} \implies \ln\left(\frac{v}{v_0}\right) = -\ln(1-x^2)$ or $\frac{v}{v_0} = \frac{1}{1-x^2}$

$$y - y_0 = +v_0 \int \frac{1}{1-x^2}dx = +v_0\left(\ln\left(\frac{x}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-x^2}}\right)\right) = \frac{v_0}{2}\ln\left(\frac{1+x}{1-x}\right)$$

We did this integral by considering the following: $\begin{aligned} x &= \sin\theta \\ dx &= \cos\theta d\theta \end{aligned}$. So

$$\int \frac{1}{1-x^2}dx = \int \frac{c(\theta)d\theta}{\cos^2\theta} = \int \sec\theta d\theta = \ln(\tan\theta + \sec\theta)$$

since $(\ln(\tan\theta + \sec\theta))' = \left(\frac{1}{\tan\theta + \sec\theta}\right)(\sec^2\theta + \tan\theta\sec\theta)$

Now by Apostol's notation,
u_1 is the power series solution with $a_0 = 1$, $a_1 = 0$
u_2 is the power series solution with $a_0 = 0$, $a_1 = 1$

My notation:
u_1 is the power series solution with $a_0 = 0$, $a_1 = 1$
u_2 is the power series solution with $a_0 = 1$, $a_1 = 0$

Since $(1-x^2)y'' - 2xy' = 0$, $1-x^2, -2x$ analytic (have power series representation).

$$\begin{aligned} y &= \sum_{n=0}^\infty a_n x^n \\ y' &= \sum_{n=1}^\infty n a_n x^{n-1} \\ y'' &= \sum_{n=2}^\infty n(n-1)a_n x^{n-2} = \sum_{n=0}^\infty (n+2)(n+1)a_{n+2}x^n \end{aligned}$$

$$\implies \begin{aligned} &2a_2 + 2(3)a_3 x + \sum_{n=2}^\infty ((n+2)(n+1)a_{n+2} - n(n-1)a_n)x^n = 2\sum_{n=1}^\infty n a_n x^n \\ &\text{or } 2a_2 + 2(3a_3 - a_1)x = \sum_{n=2}^\infty ((n+2)(n+1)a_{n+2} - n(n+1)a_n)x^n = 0 \end{aligned}$$

So $a_2 = 0$, $a_3 = a_1/3$, $a_{n+2} = \frac{na_n}{n+2}$.

$$a_{2m+1} = \frac{(2m-1)a_{2m-1}}{2m+1} = \frac{(2m-1)}{2m+1}\frac{2m-3}{2m-1}a_{2m-3} = \frac{1}{2m+1}a_1$$

$$\implies \boxed{y = a_1 \sum_{m=0}^\infty \frac{x^{2m+1}}{2m+1}}$$

Indeed, since

$$\begin{aligned} \int \frac{1}{1+x} &= \ln(1+x) = \int \sum (-x)^j = \sum \frac{(-1)^j x^{j+1}}{j+1} \\ \int \frac{1}{1-x} &= -\ln(1-x) = \int \sum (x^j) = \sum \frac{x^{j+1}}{j+1} \end{aligned}$$

So that

$$\frac{1}{2}(\ln(1+x) - \ln(1-x)) = \frac{1}{2}\sum_{m=0}^\infty \frac{2x^{2m+1}}{2m+1}$$

b.

$$u_2' = \frac{1}{2}\left(\frac{1}{1+x} + \frac{1}{1-x}\right) = \frac{1}{2}\left(\frac{1-x+1+x}{1-x^2}\right) = \frac{1}{1-x^2}$$
$$u_2'' = \frac{1}{2}\left(\frac{-1}{(1+x)^2} + \frac{1}{(1-x)^2}\right)$$
$$(1-x^2)u_2'' = \frac{1}{2}\left(\frac{-1(1-x)}{1+x} + \frac{1+x}{1-x}\right) = \frac{1}{2}\left(\frac{-(1-2x+x^2)+1+2x+x^2}{1-x^2}\right) = \frac{2x}{1-x^2}$$

Exercise 2. Let $\alpha = 1$. Then $(1-x^2)y'' - 2xy' + 2y = 0$.

$$f(x) = 1 - \frac{x}{2}\log\frac{1+x}{1-x}$$
$$f'(x) = -\frac{1}{2}\log\frac{1+x}{1-x} - \frac{x}{1-x^2}$$
$$f''(x) = \frac{-1}{1-x^2} - \frac{1}{1-x^2} + \frac{-2x^2}{(1-x^2)^2} = \frac{-2}{1-x^2} + \frac{-2x^2}{(1-x^2)^2}$$

$$\Longrightarrow \frac{-2}{1-x^2} + \frac{-2x^2}{(1-x^2)^2} + \frac{2x^2}{1-x^2} + \frac{2x^4}{(1-x^2)^2} + \frac{2x^2}{1-x^2} + x\log\frac{1+x}{1-x} + \frac{2x^2}{1-x^2} + 2 - x\log\frac{1+x}{1-x} = 0$$

Consider the general theory for Legendre equation: $(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0$. Let $\lambda = \alpha(\alpha+1)$. $1-x^2, -2x, \lambda$ analytic, so $\exists\, y = \sum_{n=0}^{\infty} a_n x^n$.

$$\Longrightarrow 2a_2 + 3(2)a_3x + \sum_{n=2}^{\infty}((n+2)(n+1)a_{n+2} - n(n-1)a_n)x^n + \sum_{n=2}^{\infty}\lambda a_n x^n + \lambda a_1 x + \lambda a_0 =$$
$$= 2\sum_{n=1}^{\infty} n a_n x^n = 2\sum_{n=2}^{\infty} n a_n x^n + 2a_1 x$$

$$\Longrightarrow \begin{aligned} a_2 &= \frac{-\lambda a_0}{2} \\ a_3 &= \frac{(2-\lambda)a_1}{6} \end{aligned} \qquad a_{n+2} = \frac{(n(n+1)-\lambda)a_n}{(n+2)(n+1)} = \frac{(n-\alpha)(n+1+\alpha)a_n}{(n+2)(n+1)}$$

If $\alpha = 1$, $\lambda = 2$,

$$a_{n+2} = \frac{n-1}{n+1}a_n, \quad a_3 = 0, \quad a_2 = -a_0 \Longrightarrow a_{2m} = \frac{1}{2m-1}(-a_0)$$

So that

$$y = -a_0 \sum_{m=0}^{\infty} \frac{x^{2m}}{2m-1}$$

Indeed,

$$\frac{x}{2}\log\left(\frac{1+x}{1-x}\right) = \frac{x}{2}\sum_{m=0}^{\infty}\frac{2x^{2m+1}}{2m+1} = \sum_{m=1}^{\infty}\frac{x^{2m}}{2m-1},$$

so $f(x) = -\sum_{m=0}^{\infty}\frac{x^{2m}}{2m-1}$.

Exercise 3.

a.

$$((x-a)(x-b)y')' - cy = 0 = ((At+B-a)(At+B-b)y')' - cy = 0$$

Let $x = At + B$, $c = \alpha(\alpha+1)$, $\frac{1}{A} = \frac{dt}{dx}$.

$$(At)^2 + 2ABt + B^2 - (At+B)(b+a) + ab = A^2(t^2-1)$$

$$\Longrightarrow 2AB - A(b+a) = 0 = A(2B - (b+a)) \qquad \Longrightarrow \boxed{B = \frac{b+a}{2},\ A = \frac{b-a}{2}}$$

since

$$B^2 - B(b+a) + ab = -A^2$$
$$\Longrightarrow \frac{(b+a)^2}{4} - \frac{(b+a)^2}{2} + ab = \frac{-(b^2+2ab+a^2)}{4} = ab = \frac{-(b^2+-2ab+a^2)}{4} = \frac{-(b-a)^2}{4} = -A^2$$

b.

$$x(x-1)y'' + (2x-1)y' - 2y = 0 = ((x^2-x)y')' - 2y = (x(x-1)y')' - 2y$$

for $x = \frac{t+1}{2}$

$$\implies ((t^2-1)y')' - 2y = 0$$

Exercise 4. $y'' - 2xy' + 2\alpha y = 0$

$$\sum_{n=0}^{\infty}((n+2)(n+1)a_{n+2} - 2na_n + 2\alpha a_n)x^n \implies a_{n+2} = \frac{2(n-\alpha)a_n}{(n+2)(n+1)}$$

For $n = 2m$

$$\begin{aligned} a_{2m} &= \frac{2(2m-2-\alpha)a_{2m-2}}{(2m)(2m-1)} = \frac{-2(\alpha - 2(m-1))}{(2m)(2m-1)}a_{2(m-1)} = \\ &= \frac{(-2)^m(\alpha-2(m-1))(\alpha-2(m-2))\dots\alpha}{(2m)!}a_0 \end{aligned}$$

For $n = 2m+1$

$$\begin{aligned} a_{2m+1} &= \frac{2(2m-1-\alpha)a_{2m-1}}{(2m+1)(2m)} = \frac{(-2)(\alpha-(2m-1))a_{2m-1}}{(2m+1)(2m)} = \\ &= \frac{(-2)^m(\alpha-(2m-1))(\alpha-(2m-3))\dots(\alpha-1)a_1}{(2m+1)!} \end{aligned}$$

$$\begin{aligned} y &= u_1 + u_2 \\ &= \boxed{\sum_{m=1}^{\infty}\frac{(-2)^m(\alpha-(2m-1))(\alpha-(2m-3))\dots(\alpha-1)}{(2m+1)!}x^{2m+1} + \sum_{m=0}^{\infty}\frac{(-2)^m(\alpha-2(m-1))(\alpha-2(m-2))\dots\alpha}{(2m+2)!}x^{2m}} \end{aligned}$$

u_1 has $a_0 = 0$, u_2 has $a_1 = 0$.
Since

$$\begin{aligned} u_1(0) &= 0 \quad u_2(0) = 1 \\ u_1'(0) &= 1 \quad u_2'(0) = 0 \end{aligned}$$

when $\alpha \in \mathbb{Z}^+$, then one of these u_1, u_2 is a polynomial, since $a_{n+2} = \frac{2(n-\alpha)a_n}{(n+2)(n+1)}$

Exercise 5. For $xy'' + (3+x^3)y' + 3x^2y = 0$, assume an analytic expansion.

$$\begin{aligned} y &= \sum_{n=0}^{\infty}a_nx^n = \sum_{n=2}^{\infty}a_{n-2}x^{n-2} \\ y' &= \sum_{n=1}^{\infty}a_nnx^{n-1} = \sum_{n=3}^{\infty}a_{n-2}(n-2)x^{n-3} = \sum_{n=0}^{\infty}a_{n+1}(n+1)x^n \\ y'' &= \sum_{n=2}^{\infty}n(n-1)a_nx^{n-2} = \sum_{n=1}^{\infty}(n+1)na_{n+1}x^{n-1} \end{aligned}$$

$$2a_2x + 3(2)a_3x^2 + 3a_1 + 3a_2(2)x + 3a_3(3)x^2 + 3a_0x^2 + \sum_{n=3}^{\infty}\left((n+1)na_{n+1} + 3a_{n+1}(n+1) + a_{n-2}(n-2) + 3a_{n-2}\right)x^n = 0$$

$$\implies \begin{aligned} a_1 &= 0 \\ 8a_2 &= 0 \\ (15a_3 + 3a_0) &= 0 \text{ or } a_3 = \frac{-a}{5} \end{aligned} \qquad \begin{aligned} (n+1)(n+3)a_{n+1} &= -a_{n-2}(n+1) \\ a_{n+1} &= \frac{-a_{n-2}}{n+3} \end{aligned}$$

$$\implies a_{3j} = \frac{-a_{3(j-1)}}{3j+2} = \frac{(-1)^2a_{3(j-2)}}{(3j+2)(3j-1)} = \frac{(-1)^ja_0}{(3j+2)(3j-1)\dots(8)(5)}$$

$$\boxed{y = a_0\{1 + \sum_{j=1}^{\infty}\frac{(-1)^j}{(3j+2)(3j-1)\dots(8)(5)}x^{3j}\}}$$

To obtain the solution with even-powered terms, consider first possible simple pole at 0 from the form of the differential equation:

$$y'' + \left(\frac{3}{x} + x^2\right)y' + 3xy = 0$$

Then consider the following:

$$y = \sum_{n=0}^{\infty} a_n x^{n-2} = \sum_{n=-1}^{\infty} a_{n+1} x^{n-1}$$
$$y' = \sum n = -1^{\infty}(n-1)a_{n+1}x^{n-2} = \sum_{n=-4}^{\infty}(n+2)a_{n+4}x^{n+1}$$
$$y'' = \sum_{n=-4}^{\infty}(n+2)(n+1)a_{n+4}x^{n+1}$$

Then for the first few terms,

$$(-2)(-3)a_0x^{-4} + (-1)(-2)a_1x^{-3} + 3(-2)a_0x^{-4} + 3(-1)a_1x^{-3}3a^3x^{-1} + (-2a_0)x^{-1} + 3a_0x^{-1} = 0$$
$$\implies a_1 = 0, \quad a_3 = \frac{-a_0}{3}$$

$$\boxed{y = x^{-2}a_0\left(1 + \sum_{j=1}^{\infty} \frac{(-1)^j}{3^j j!} x^{3j}\right)}$$

Exercise 6. $x^2y'' + x^2y' - (\alpha x + 2)y = 0$. $\left(\frac{\alpha x+2}{x^2}\right)$ analytic except at $x = 0$.

$$y = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=1}^{\infty} a_{n-1}x^{n-1}$$
$$y' = \sum_{n=1}^{\infty} na_nx^{n-1} = \sum_{n=2}^{\infty}(n-1)a_{n-1}x^{n-2}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)a_nx^{n-2}$$

$$\sum_{n=2}^{\infty}(n(n-1)a_n + (n-1)a_{n-1})x^n = \sum_{n=1}^{\infty}\alpha a_{n-1}x^n + \sum_{n=0}^{\infty}2a_nx^n =$$
$$= \sum_{n=2}^{\infty}(\alpha a_{n-1} + 2a_n)x^n + \alpha a_1 x + 2a_1x + 2a_0$$

$$(n-2)(n+1)a_n = (\alpha + -n)a_{n-1} \text{ or } a_n = \frac{(\alpha+1-n)a_{n-1}}{(n-2)(n+1)}$$

$\frac{a_n}{a_{n-1}} = \frac{\alpha-(n-1)}{(n-2)(n+1)} \to 0$ as $n \to \infty$, so this power series converges $\forall\, x$.
Also $a_0 = a_1 = 0$.
By recursion,

$$a_n = \frac{(\alpha+1-n)(\alpha+2-n)\dot{(}\alpha-2)}{(n-2)!(n+1)!}(3(2))a_2$$

Then,

$$y = a_2\left(x^2 + \sum_{n=3}^{\infty}\frac{(\alpha+1-n)(\alpha+2-n)\dot{(}\alpha-2)}{(n-2)!(n+1)!}6x^n\right) = a_2\left(x^2 + \sum_{n=1}^{\infty}\frac{(\alpha-n-1)(\alpha-n)\dots(\alpha-2)}{n!(n+3)!}6x^{n+2}\right) =$$
$$= \boxed{a_2x^2\left(1 + \sum_{n=1}^{\infty}\frac{(\alpha-n-1)(\alpha-n)\dots(\alpha-2)}{n!(n+3)!}6x^n\right)}$$

Exercise 7. Leibniz's formula for nth derivative of a product is the following: if $h(x) = f(x)g(x)$, then

$$h^{(n)}(x) = \sum_{k=0}^{n}\binom{n}{k}f^{(k)}(x)g^{(n-k)}(x)$$

a. For

$$A(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$
$$B(x) = \sum_{n=0}^{\infty} b_n (x - x_0)^n$$

and $C(x) = A(X)B(x)$, then

$$C^{(n)}(x) = \sum_{k=0}^{n} \binom{n}{k} A^{(k)}(x) B^{(n-k)}(x)$$

b. Given that

$$A^{(k)}(x_0) = k! a_k$$
$$B^{(n-k)}(x_0) = (n-k)! b_{n-k}$$

Then

$$C^{(n)}(x_0) = \sum_{k=0}^{n} \binom{n}{k} k! a_k (n-k)! b_{n-k} = n! \sum_{k=0}^{n} a_k b_{n-k}$$
$$C^{(n)}(x_0) = n! c_n \text{ so } c_n = \sum_{k=0}^{n} a - k b_{n-k}$$

Exercise 8.

a. By Rodrigues' formula, $P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2-1)^n$. $(x^2-1) = (x-1)(x+1)$. By Leibniz's formula,

$$\frac{d^n}{dx^n}(x-1)^n(x+1)^n = \sum_{k=0}^{n} \binom{n}{k} \frac{d^k}{dx^k}(x-1)^n \frac{d^{n-k}}{dx^{n-k}}(x+1)^n$$

$$P_n(x) = \frac{1}{2^n}(x+1)^n + \frac{1}{2^n n!} \sum_{k=0}^{n-1} \frac{d^k}{dx^k}(x-1)^n \frac{d^{n-k}}{dx^{n-k}}(x+1)^n = \frac{(x+1)^n}{2^n} + (x-1)Q_n(x)$$

$Q_n(x)$ is a polynomial.

b. $P_n(1) = 1$. $P_n(-1) = 0 + \frac{1}{2^n n!}(-2)^n n! = (-1)^n$ where we considered when $k = 0$, for $\frac{d^n}{dx^n}(x+1)^n = n!$.

c.

Exercise 9.

a. Now $(1-x^2)y'' + -2xy' + \alpha(\alpha+1)y = 0$, or $((1-x^2)y')' = -\alpha(\alpha+1)y$.

$$\begin{aligned}
-m(m+1)P_m &= ((1-x^2)P_m')' \\
-m(m+1)P_m P_n &= ((1-x^2)P_m')'P_n = ((1-x^2)P_m' P_n)' - (1-x^2)P_m' P_n' \\
n(n+1)P_n P_m &= -((1-x^2)P_n' P_m)' + (1-x^2)P_n' P_m' \\
\implies ((1-x^2)(P_n P_m' &- P_n' P_m))' = (n(n+1) - m(m+1))P_n P_m
\end{aligned}$$

b. If $n \neq m$, $\int_{-1}^{1} P_n P_m = 0$

Exercise 10.

a. $f(x) = (x^2-1)^n = (x-1)^n(x+1)^n$. Using Leibniz's rule again,

$$f^{(n-1)} = \sum_{k=0}^{n-1} \frac{d^k}{dx^k}(x-1)^n \frac{d^{n-1+k}}{dx^{n-1+k}}(x+1)^n$$

For $f^{(n-1)}(1) = 0$, $f^{(n-1)}(-1) = 0$. Then

$$\int_{-1}^{1} f^{(n)} f^{(n)} = f^{(n-1)} f^{(n)} \Big|_{-1}^{1} - \int_{-1}^{1} f^{(n-1)} f^{(n+1)} = -\int_{-1}^{1} f^{(n-1)} f^{(n+1)}$$

Now

$$f^(2n)(x) = \frac{d^{2n}}{dx^{2n}}(x^2-1)^n = \sum_{k=0}^{2n} \binom{2n}{k} \frac{d^k}{dx^k}(x-1)^n \frac{d^{2n-k}}{dx^{2n-k}}(x+1)^n = (2n)!$$

for the $k = n$ term.

$$\int_{-1}^{1} f^{(n)} f^{(n)} = \int_{-1}^{1} f^{(2n)} f^{(0)} (-1)^n = (2n)! \int_{-1}^{1} (1-x^2)^n dx = 2(2n)! \int_{0}^{1} (1-x^2)^n dx$$

b. Now $P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$.

$$\int_{-1}^{1}(P_n(x))^2dx = \frac{1}{2^{2n}(n!)^2}\int_{-1}^{1} f^{(n)}f^{(n)} = \frac{1}{2^{2n}(n!)^2}2(2n)!\int_0^1(1-x^2)^n dx = \frac{2(2n)!}{2^{2n}(n!)^2}\int_0^{\pi/2}\sin^{2n+1}t dt =$$
$$= \frac{2(2n)!}{2^{2n}(n!)^2}\frac{(2n)!!}{(2n+1)!!} = \frac{2(2n)!2^n n! 2^{n+1}}{2^{2n}(n!)^2(2n+2)!}(n+1)! = \frac{2^2(n+1)}{(2n+2)(2n+1)} =$$
$$= \boxed{\frac{2}{2n+1}}$$

6.24 Exercises - The method of Frobenius, The Bessel equation

Exercise 1.

(a) Given $g(x) = x^{1/2}f(x)$, $x^2y'' + xy' + (x^2-\alpha^2)y = 0$ that f must satisfy,

$$g' = \frac{1}{2}x^{-1/2}f + x^{1/2}f'$$
$$y'' = \frac{-1}{4}x^{-3/2}f + x^{-1/2}f' + x^{1/2}f''$$

, we want g to satisfy $y'' + \left(1 + \frac{1-4\alpha^2}{4x^2}\right)y = 0$.

Now

$$f'' + \frac{f'}{x} + \left(1 - \frac{\alpha^2}{x^2}\right)f = 0$$
$$g'' = \frac{-1}{4x^{-3/2}}f + x^{1/2}\left(\frac{\alpha^2}{x^2} - 1\right)f$$
$$g'' + g = x^{1/2}\left(\frac{\alpha^2}{x^2} - \frac{1}{4x^2}\right)f = g\left(\frac{4\alpha^2-1}{4x^2}\right)$$

(b)

(c) $J_p(x) = \left(\frac{x}{2}\right)^p\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(n+1+\alpha)}\left(\frac{x}{2}\right)^{2n}$. Also,
$\Gamma\left(n+1+\frac{1}{2}\right) = \left(n+\frac{1}{2}\right)\left(n-1+\frac{1}{2}\right)\dots\left(1+\frac{1}{2}\right)\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)$

$$J_{1/2}(x) = \left(\frac{2}{x}\right)^{1/2}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(n+\frac{1}{2})(n-1+\frac{1}{2})\dots(1+\frac{1}{2})(\frac{1}{2})\Gamma\left(\frac{1}{2}\right)}\left(\frac{x}{2}\right)^{2n+1} =$$
$$= \left(\frac{2}{\pi x}\right)^{1/2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1} = \left(\frac{2}{\pi x}\right)^{1/2}\sin x$$

Now for $\alpha = \frac{-1}{2}$, consider
$\Gamma(n+1-1/2) = \Gamma\left(n+\frac{1}{2}\right) = \left(n+\frac{1}{2}-1\right)\left(n-2+\frac{1}{2}\right)\dots\left(2+\frac{1}{2}\right)\left(1+\frac{1}{2}\right)\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)$

$$J_{-1/2}(x) = \left(\frac{2}{x}\right)^{1/2}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma\left(n+\frac{1}{2}\right)}\left(\frac{x}{2}\right)^{2n} = \left(\frac{2}{x}\right)^{1/2}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!\Gamma(1/2)} = \left(\frac{2}{\pi x}\right)^{1/2}\cos x$$

8.3 Exercises - Functions from $\mathbb{R}^n$ to $\mathbb{R}^m$. Scalar and vector fields, Open balls and Open sets

Exercise 1. Let f be a scalar field defined on a set S and let c be a given real number. The set of all points x in S such that $f(x) = c$ is called a level set of f.

See sketch.

Exercise 2. Let S be the set of all points (x,y) in the plane satisfying the given inequalities.

See sketch.

Exercise 3. *Proofs are hard!* I *read* the *examples at the end of Section 8.2*, particularly the example on the 2-dim. *Cartesian product*: it helps. In fact, we'll review it right now.

$A_1, A_2 \subseteq \mathbb{R}^1$
$A_1 \times A_2 = \{(a_1,a_2)|a_1 \in A_1,\ A_2 \in A_2\}$
If A_1, A_2 are open subsets of $\mathbb{R}^1$,
Choose any $a \in A_1 \times A_2$

Want: a is an int. pt. of $A_1 \times A_2$

Since

A_1, A_2 open in $\mathbb{R}^1$, $\exists\, B(a_1; r_1)$, $\exists\, B(a_2; r_2)$

Let $r = min\{r_1, r_2\}$

Want: $B(a; r) \subseteq A_1 \times A_2$

If $(x_1, x_2) = x \in B(a; r)$,

$\|x - a\| < r$, so $|x_1 - a_1| < r$, $|x_2 - a_2| < r_2$,

then $\begin{matrix} x_1 \in B(a_1; r_1) \\ x_2 \in B(a_2; r_2) \end{matrix} \implies \begin{matrix} x_1 \in A_1 \\ x_2 \in A_2 \end{matrix}$

We then get what we want: $(x_1, x_2) \in A_1 \times A_2$ so that any $x \in B(a; r)$ belongs in S, which means, by def., that $B(a; r) \subseteq S$

Onward with the problem:

Let S be the set of all points (x, y, z) in 3-space.

(1) $z^2 - x^2 - y^2 - 1 > 0$

(2) $|x| < 1,\ |y| < 1,$ and $|z| < 1$ Consider $a \in S$ We must use the fact that an open rectangular box is a basic open set.

Let $a \in S$, $a = (a_1, a_2, a_3)$

Let $\rho_i = \begin{cases} 1 - a_i & \text{if } a_i \geq 0 \\ |-1 - a_i| & \text{if } a_i < 0 \end{cases}$ and $R_a = \prod_{i=1}^{3} (a_i - \rho_i, a_i + \rho_i)$

Consider $b \in R_a$.

If $a_i \geq 0$,

if $b_i \geq a_i$, $b_i - a_i < 1 - a_i$ or $b_i < 1$

if $b_i < a_i$, $\begin{matrix} \text{if } b_i > 0, & a_i - b_i > 0 \text{ or } 1 > a_i > b_i \\ \text{if } b_i < 0, & -b_i < a_i - b_i < 1 - a_i < 1 \end{matrix}$

If $a_i < 0$,

if $b_i > a_i$, $\begin{matrix} \text{if } b_i > 0, & b_i - a_i < 1 + a_i \text{ or } b_i < 1 + 2a_i < 1 \\ \text{if } b_i < 0, & -b_i < -a < 1\ (\text{since} |a_i| < 1) \end{matrix}$ if $b_i < a_i$, $a_i - b_i < 1 + a_i$ or $-b_i < a_i - b_i < 1 + a_i < 1$.

Then, $\boxed{|b_i| < 1}$ for *each and every possible case*. Then $R_a \subseteq S$, so $\forall\, a \in S$ is an int. pt. (since $\forall\, a$, $\exists$ open rectangle R_a, that is completely contained in S). Then S is open.

(3) $x + y + z < 1$

(4) $|x| \leq 1,\ |y| < 1,$ and $|z| < 1$

Consider $a_0 = (1, a_2, a_3)$, $1 > a_2, a_3 > 0$.

$a_0 \in S$, but for $B(a_0, 1/2)$, $(5/4, a_2, a_3) \in B(a_0, 1/2)$ and $(5/4, a_2, a_3) \notin S$. $\boxed{\text{So } S \text{ is not open.}}$

Or...

An open set has every element to be interior to it (definition).
An interior pt. is a pt. s.t. $\exists$ some open basic set containing the pt. and is a subset of the set.

We must show $\nexists$ any open basic set for a pt. in this set.
Since every open rectangle contains an open ball and every open ball contains an open rectangle, we only need to consider open rectangles.

Consider $(1, y_0, z_0) \in S$.
Consider open rectangle containing 1. I claim that at best, $\left(1 - \frac{1}{n}, 1 + \frac{1}{n}\right)$, $n \in \mathbb{Z}^+$, since by Archimedes property of real numbers,

Theorem 1 (Apostol's Archimedes property of real numbers, pp. 26, Thm. 1.30, Vol. 1)**.** *If $x > 0$ and if y is an arbitrary real number, $\exists\, n \in \mathbb{Z}^+$ s.t. $nx > y$. We want $i \in (a_i, b_i)$ i.e. $a_i < 1 < b_i$*

Consider open interval containing 1; $1 \in (a_i, b_i)$. Then $a_i < 1 < b_i$. But (a_i, b_i) always contains pts. not belonging to S_i: $nx < y$ (existence of n *guaranteed by Archimedes prop. of reals thm.*).

$$nb_i > (n+1) \Longrightarrow b_i > 1 + \frac{1}{n} > 1$$

Then $\nexists$ open interval containing 1 completely contained in S. $(1, y_0, z_0)$ is not an interior pt. S is not open.

(5) $x + y + z < 1$ and $x > 0,\ y > 0,\ z > 0$

Consider $x \in S$. Then $x + y + z < 1$.
Consider $\prod_{j=1}^{3}(x_j - \delta_j, x_j + \delta_j)$.

$$\sum_{j=1}^{3} x_j + \delta_j = \sum_{j=1}^{3} x_j + \sum_{j=1}^{3} \delta_j$$

$$0 < x + y + z < 1 \text{ so let } 1 - \sum_{j=1}^{3}(x_j) = \epsilon(x) > 0$$

$$\text{We can choose } \sum_{j=1}^{3} \delta_j = \delta \text{ s.t. } \epsilon(x) > \delta > 0$$

Furthermore, by Archimedes axiom, we can choose $\delta_j > 0$ s.t. $x_j - \delta_j > 0$
$\Longrightarrow \forall\, x \in S$, we can construct an open rectangle $\prod_{j=1}^{3}(x_j - \delta_j, x_j + \delta_j) = R(x)$ s.t. $R(x) \subseteq S$.

(6) $x^2 + 4y^2 + 4z^2 - 2x + 16y + 40z + 113 < 0$

$$x^2 + 4y^2 + 4z^2 - 2x + 16y + 40z + 113 = (x-1)^2 - 1 + 4(y+2)^2 - 16 + 4(z+5)^2 - 100 + 113 =$$

$$= (x-1)^2 + 4(y+2)^2 + 4(z+5)^2 - 4 < 0 \quad \Longrightarrow \frac{(x-1)^2}{2^2} + (y+2)^2 + (z+5)^2 < 1$$

Thus, S is, by definition, a **basic open set**, a basic open sphere. By theorem, **a basic open sphere is an open set**.

Exercise 4.

(1) A is an open set in n-space and $x \in A$. Given A is open, $A - \{x\} \subset A$.

Consider $a \in A - \{x\}$. Then $a \in A$ so, $\exists B_a(a, r_a) \subseteq A$.
If $x \notin B_a(a, r_a)$, we're done.
If $x \in B_a(a, r_a)$, consider $r_{ax} = \|a - x\|$
$\forall\, x_a \in B_a(a, r_{ax})$, $\|x_a - a\| < r_{ax}$ also $\|x_a - a\| < r_{ax} < r_a$, so $x_a \in A$
$\Longrightarrow B_a(a, r_{ax}) \subseteq A$ and $B_a(a, r_{ax}) \subseteq A - \{x\}$ since we constructed B_a s.t. $x \notin B_a$

(2) A open. Let B have endpoints $b_1,\ b_2$

$A - \{b_1\}$ open. $A - \{b_1\} - \{b_2\} = A'$ open.
$A' - intB = A - B$ open since open set minus an open set is open, since union of 2 open sets is an open set.

We could also directly say, since we're dealing with intervals in one-dimension, $A = (a_A, b_A)$, $B = [a_B, b_B]$.
$b_B < b_A$
$a_A < a_B$.
B is a closed subinterval of A.
$A - B = (a_A, a_B) \bigcup (b_B, b_A)$

(3) $\forall\, x \in A \bigcup B$, $x \in A$ or $x \in B$. Since A, B are open, x is int. to A, or int. to B.
Explicitly, if $\exists\, B(x, r) \subseteq A,\ B$ then $B(x, r) \subseteq A,\ B \subseteq A \bigcup B$.
Then x is interior to $A \bigcup B$. $\Longrightarrow A \bigcup B$ open.

$\forall\, x \in A \bigcap B$, $x \in A$ and $x \in B$.
Since $x \in A$ and $x \in B$, $\exists\, B(x, r_A) \subseteq A$; $\ B(x, r_B) \subseteq B$ or
$(x - r_A, x + r_A) \subseteq A\,; \qquad (x - r_B, x + r_B) \subseteq B$
Let $r_m = \min(r_A, r_B)$
So then $(x - r_m, x + r_m) \subseteq A$ and $(x - r_m, x + r_m) \subseteq B \Longrightarrow (x - r_m, x + r_m) \subseteq A \bigcap B$

(4) $\mathbb{R}^1$ is open (since $\forall\, B(x; r) \subseteq \mathbb{R}^1$)
$A = [a_A, b_A]$ is a closed interval.
Let $\mathbb{R}^1 - A = \mathbb{R}^-$
$\forall\, x \in \mathbb{R}^-$, $x \in \mathbb{R}^1$ so $B(x; r) \subseteq \mathbb{R}^1$.

Suppose $x \in \mathbb{R}^-$, so $x > b_A$ or $x < a_A$ (otherwise $x \in A$).

If $x > b_A$, then let $r_1 = x - b_A$,
$(x - r_1, x + r_1) \subseteq \mathbb{R}^-$ since $(x - r_1, x + r_1) \subseteq \mathbb{R}$ and $\forall\, x_1 \in (x - r_1, x + r_1)$, $x_1 > b_A$
If $x < a_A$, then let $r_1 = a_A - x$
$(x - r_1, x + r_1) \subseteq \mathbb{R}^-$ since $(x - r_1, x + r_1) \subseteq \mathbb{R}$ and $\forall\, x_1 \in (x - r_1, x + r_1)$, $x_1 < a_A$

Exercise 5. Prove the following properties of open sets in $\mathbb{R}^n$

(1) The empty set $\emptyset$ is open.

Let $a \in \emptyset$

Then $B(a; r) \subseteq \emptyset$ since there are no $a \in \emptyset$, $\implies$ a is an interior pt. of $\emptyset$.

Or ...

Consider $a \in \emptyset$

Consider $B(a; r) = \emptyset$. Then $B(a; r) \subseteq \emptyset$. So $\emptyset$ is open.

(2) $\mathbb{R}^n$ is open.

Consider $a \in \mathbb{R}^n$. Consider $B(a; r)$. So for $x \in B(a; r)$, then $\|x - a\| < r$.
$\implies |x_j - a_j| < r_j$
$x_j \in \mathbb{R} \quad \forall\, x_j$ s.t. $|x_j - a_j| < r_j$ defines an open interval on $\mathbb{R}^1$, and so by induction, the Cartesian product of n open intervals is an open n-ball. So $\mathbb{R}^n$ is open.

Or ...

Consider $a \in \mathbb{R}^n$.

Consider $B(a; r)$. Since $\forall\, y \in B(a; r)$, $y \in \mathbb{R}^n$. $B(a; r) \subseteq \mathbb{R}^n$. $\mathbb{R}^n$ open.

(3) Consider $\{W_j\}$, collection of open sets.
Consider $y \in \bigcup_j W_j$. Then $y \in W_j$ for some j. Since W_j open, $\exists\, B(y; \rho) \subseteq W_j$.
$W_j \subseteq \bigcup_j W_j$, so $B(y, \rho) \subseteq \bigcup_j W_j$. $\bigcup_j W_j$ is open.

(4) Consider $\{W_j | j = 1, \ldots, n\}$, finite collection of open sets.
Consider $y \in \bigcap_{j=1}^n W_j$.
$y \in W_i, \ \forall\, i = 1, \ldots, n$. Then since $\forall\, i$, W_i open, $\exists\, B_i(y, \rho_i) \subseteq W_i$.
By Thm., $\exists$ open set $B(y, \rho) \subseteq \bigcap_{i=1}^n B_i(y, \rho_i)$. Then $B(y, \rho) \subseteq \bigcap_{i=1}^n B_i(y, \rho_i) = \bigcap_{i=1}^n W_i$

(5) Let $W_k = \left(\frac{-1}{k}, \frac{1}{k}\right)$; $k \geq 1$
Then $\bigcap_k W_k = \{0\}$, which is not open.

Exercise 7.

(1) $(A \bigcup \{x\})^c = A^c \bigcup (\mathbb{R} - x)$
A^c open. $\mathbb{R}^n - x$ open. (since $\{x\}$ is closed). Then, by thm., the intersection of these two open sets, $A^c \bigcap (\mathbb{R} - x)$ is open. Then, by definition, $A \bigcup \{x\}$ is closed.

(2) $\mathbb{R} - [a, b] = [a, b]^c$.
Consider $y \in \mathbb{R} - [a, b]$

If $y > b$, then
$y - b > 0$, so $\exists\, N \in \mathbb{Z}^+$ s.t. $y - b > \frac{1}{N}$. $y > \frac{1}{N} + b$ (Archimedes prop. of real numbers).
$y \in \left(\frac{1}{N} + b, y + 1\right)$ is open and $\left(\frac{1}{N} + b, y + 1\right) \subseteq \mathbb{R} - [a, b]$

If $y < a$, then
$a - y > 0$, so $\exists\, N \in \mathbb{Z}^+$ s.t. $a - y > \frac{1}{N}$ or $a - \frac{1}{N} > y$
$y \in \left(y - 1, a - \frac{1}{N}\right) \subseteq \mathbb{R} - [a, b]$
then $\mathbb{R} - [a, b]$ open. $[a, b]$ closed (by definition).

(3) $(A \bigcup B)^c = A^c \bigcap B^c$. A, B closed, so A^c, B^c open. $A^c \bigcap B^c$, intersection of 2 open sets, is open.
then $A \bigcup B$ closed.

$(A \bigcap B)^c = A^c \bigcap B^c$. A^c, B^c open. $A^c \bigcup B^c$ open. Then $(A \bigcap B)$ closed.

Exercise 8.

(1) $\emptyset^c = \mathbb{R}^n$ and $\mathbb{R}^n$ open. $\emptyset$ closed.
(2) $(\mathbb{R}^n)^c = \emptyset$ and $\emptyset$ open. $\mathbb{R}^n$ closed.
(3) $(\bigcap_i A_i)^c = \bigcup_i A_i^c$. A_i^c open, so $\bigcup_i A_i^c$ open. $\bigcap_i A_i$ closed.
(4) $(\bigcup_{i=1}^n A_i)^c = \bigcap_{i=1}^n A_i^c$. A_i^c open, so $\bigcap_{i=1}^n A_i^c$ open. $\bigcup_{i=1}^n A_i$ closed.
(5) $\bigcup_{i=1}^\infty \{i\} = \mathbb{Z}^+$ is closed since $\left(\bigcup_{i=1}^\infty \{i\}\right)^c = \bigcup_{i=1}^\infty (i, i+1)$ is open.

Exercise 9. Let S be a subset of $\mathbb{R}^n$

(1) Prove that both $intS$ and $extS$ are open sets.

Want: $intS$ is open, i.e. $\forall\, a \in intS, \exists\, B(a;r) \subseteq intS$ i.e.
$\forall\, x_1 \in B(a;r),\ x_1 \in intS$
$x_1 \in intS$ if $\exists\, B(x_1;r_1) \subseteq S$

Consider $a \in intS$, then $\exists\, B(a;r) \subseteq S$
Consider $x_1 \in B(a;r)$. If $\|x_1 - a\| < r$ consider $\forall\, x_2$ s.t. $\|x_2 - x_1\| < \|x_1 - a\| = r_1 < r$.
Then $x_2 \in B(a;r)$, so $B(x_1, r_1) \subseteq B(a;r) \subseteq S$
$\implies \forall\, x_1 \in intS$ for $x_1 \in B(a;r)$, so $B(a;r) \subseteq intS$.

Want: $extS$ is open, i.e. $\forall\, a \in extS, \exists\, B(a;r) \subseteq extS$ i.e.
$\forall\, x_1 \in B(a;r),\ x_1 \in extS$
$x_1 \in extS$ if $\exists\, B(x_1;r_1)$ s.t. $\forall\, x_2 \in B(x_1, r_1),\ x_2 \notin S$.

Consider $a \in extS$, then $\exists\, B(a;r)$ s.t. $\forall\, x_1 \in B(a,r),\ x_1 \notin S$
Consider $x_1 \in B(a;r)$. If $\|x_1 - a\| < r$ consider $\forall\, x_2$ s.t. $\|x_2 - x_1\| < \|x_1 - a\| = r_1 < r$.
Then $x_2 \in B(a;r)$, so $x_2 \notin S$. so then $\exists\, B(x_1, r_1)$ s.t. $\forall\, x_2 \in B(x_1, r_1),\ x_2 \notin S$
$\implies \forall\, x_1 \in extS$, so $B(a;r) \subseteq extS$. $extS$ open.

(2) Prove that $\mathbb{R}^n = (intS) \bigcup (extS) \bigcup \partial S$, a union of disjoint sets, and use this to deduce that boundary ∂S is always a closed set.

Suppose $a_e \in extS$. Then $\exists\, B(a_e, r)$ s.t. $\forall\, x_e \in B(a_e, r),\ x_e \notin S$.
Then $\forall\, R > 0$, $B(a_e, R)$ will contain $x_{eR} \in B(a_e, R)$ s.t. $x_{eR} \notin S$. (all open n-balls will either contain $B(a_e, r)$ or be a part of $B(a_e, r)$). So $\nexists\, B(a_e, R)$ s.t. $B(a_e, r) \subseteq S$. $a_e \notin intS$

If $a_{in} \in intS$, suppose $a_{in} \in extS$. Then $a_{in} \notin intS$. Contradiction. $a_{in} \notin extS$.

$intS$, $extS$ are open and disjoint.

Suppose $a_{bd} \in \partial S$. $\begin{array}{l} a_{bd} \text{ is not interior to } S, \text{ so } a_{bd} \notin intS \\ a_{bd} \text{ is not exterior to } S, \text{ so } a_{bd} \notin extS \end{array}$

Let $x \in \mathbb{R}^n$. Consider $B(x, r_0)$. If $B(x, r_0) \subseteq S$, then $x \in intS$. If $\forall\, x_1 \in B(x, r_0),\ x_1 \notin S$. Then $x \in extS$. Otherwise, $B(x, r_0)$ may contain $x_{1a} \in S$ and $x_{1b} \in S^c$. Then x is neither interior or exterior to S. So $x \in \partial S$

$\implies x \in intS \bigcup extS \bigcup \partial S$, $\mathbb{R}^n \subseteq intS \bigcup extS \bigcup \partial S$.

Since $\forall\, x \in intS \bigcup extS \bigcup \partial S$, $x \in \mathbb{R}^n$, then $intS \bigcup extS \bigcup \partial S \subseteq \mathbb{R}^n$.
$\implies \mathbb{R}^n = intS \bigcup extS \bigcup \partial S$ a union of disjoint sets.

$(intS \bigcup extS)^c = \partial S$, $intS \bigcup extS$ is open so ∂S is closed, since its complement is open.

Exercise 10. **Want:** $x =$ boundary pt. of $S =$ b.p. of S, neither interior nor exterior to S.

$\forall\, B(x), \exists\, a_i \in B(x)$, s.t. $a_i \in intS \subseteq S$. Then x cannot be an exterior pt.
$\forall\, B(x), \exists\, a_e \in B(x)$, s.t. $a_e \in extS$. Then $a_e \notin S$ and so x is not interior, by definition. x is a boundary pt.

Exercise 11. $\mathbb{R}^n - S = S^c$.

Let $x \in intS^c$. Then $\exists$ open V s.t. $x \in V$ and $V \subseteq S^c$.

Then $\forall\, x_1 \in V$, $x_1 \notin S$, so x is an exterior pt. to S. $x \in extS$, so $intS^c \subseteq extS$.
Let $x \in extS$. Then $\exists\, B(x)$ s.t. $B(x) \subseteq S^c$. By def., $x \in intS^c$.
$extS \subseteq intS^c$

$$\implies extS = intS^c$$

Exercise 12. Suppose S closed. Let y be a boundary pt. of S.

Suppose $y \notin S$. Then $y \in S^c$, S^c open.
So by def. of open set, $\exists\, U$ s.t. $y \in U$ and $U \subseteq S^c$. But y is then an exterior pt., contradicting the definition of a boundary pt. for y.

Then $y \in S$, so that $S = intS \bigcup \partial S$

Suppose $intS \bigcup \partial S = S$
Consider any $z \in S^c$.
Then z has to be either a boundary pt. of S^c or interior pt. of S^c.
z cannot be a boundary pt. of S^c (we already showed that $\begin{matrix} extS = intS^c \\ extS^c = intS \end{matrix}$), because then $z \in \partial S$ and hence belong to S.
Then z is interior to S^c. $S^c = intS^c$, so S^c open. S closed.

8.5 Exercises - Limits and continuity

Exercise 1.

(1) $f(x,y)$ is continuous $\forall\,(x,y) \in \mathbb{R}^2$
(2) $(x,y) \neq (0,0)$
(3) $y \neq 0$
(4) $y \neq 0$, $\frac{x^2}{y} \neq \frac{\pi}{2} + \pi k$, $k \in \mathbb{Z}$
(5) $x \neq 0$
(6) $(x,y) \neq (0,0)$
(7) $\frac{x+y}{1-xy} \neq \frac{\pi}{2} + \pi k, \;\; k \in \mathbb{Z}, xy \neq 1$
(8) $(x,y) \neq (0,0)$
(9) $f = \exp{(y^2 \ln x)}$, $x \neq 0$
(10) $y \neq 0$, $\frac{x}{y} \geq 0$

Exercise 2. $\lim_{x\to a} f(x,y)$, $\lim_{y \to b} f(x,y)$ exist, so $\lim_{x\to a} f(x,y) = f(a,y)$ and $\lim_{y\to b} f(x,y) = f(x,b)$.

Since $\lim_{x \to x_0} f(x) = f(x_0)$, where $x_0 = (a,b)$, which is equivalent to saying

$$\forall\,\epsilon > 0,\ \exists\,\delta > 0 \text{ s.t. } |f(x) - L| < \epsilon \text{ if } \|x - x_0\| < \delta$$

then
Consider $\frac{\epsilon}{2} > 0$. $\exists\,\delta_y > 0$ s.t. $|f(x,y) - f(x,b)| < \frac{\epsilon}{2}$ if $|y - b| < \delta_y$ (since $\lim_{y \to b} f(x,y)$ exists).
Consider $\frac{\epsilon}{2} > 0$. $\exists\,\delta_{xy} > 0$ s.t. $|f(x,y) - L| < \frac{\epsilon}{2}$ if $\|(x,y) - (a,b)\| < \delta_{xy}$ (since $\lim_{x\to x_0} f(x) = L$).

$$|f(x,b) - L| = |f(x,b) - f(x,y) + f(x,y) - L| < |f(x,y) - f(x,b)| + |f(x,y) - L| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \text{ whenever}$$
$$|y - b| < \delta_y \text{ and } \sqrt{(x-a)^2 + (y-b)^2} < \delta_{xy}$$
$$\text{then } |x - a| < \delta_{xy}$$

So $\forall\,\epsilon > 0, \exists\,\delta_{xy} = \delta_x(\epsilon)$ s.t. $|f(x,b) - L| < \epsilon$ whenever $|x - a| < \delta_x(\epsilon)$.
We just proved $\lim_{x\to a} \lim_{y \to b} f(x,y) = \lim_{x \to a} f(x,b) = L$
Similarly, we get the same result for $lim_{y\ tob} f(a,y)$. Thus, $\lim_{x\to a} \lim_{y\to b} f(x,y) = \lim_{y\to b} \lim_{x \to a} f(x,y) = L$ whenever $\lim_{x\to x_0} f(x) = L$

Exercise 3. $f(x,y) = \frac{(x-y)}{x+y}$

$$\lim_{y\to 0} f = 1$$
$$\lim_{x\to 0} f = -1$$

Exercise 4. $f(x,y) = \frac{x^2y^2}{x^2y^2 + (x-y)^2}$

$$\lim_{x \to 0} f = 0 \qquad \text{but if } y = x, f = \frac{x^4}{x^4 + 0} = 1$$
$$\lim_{y\to 0} f = 0$$

Exercise 5. $0 < x \sin{\frac{1}{y}} < x \quad x \to 0$, so by squeeze principle, $x \sin{\frac{1}{y}} \to 0$.
$\to \lim_{x\to 0} f = 0$
$\lim_{y \to 0} f$ undefined, since
Consider $|y| < \frac{1}{n}$ or $n < \frac{1}{|y|}$
For $y > 0$, $\sin{\frac{1}{y}} > \sin{n}$
For $y < 0$, $|\sin{\frac{1}{y}}| = \sin{\frac{-1}{y}} > \sin{n}$
$\forall\,\delta = \frac{1}{n}, \exists\,\epsilon = \epsilon(\delta) = \sin{(1/\delta)}$ s.t. $|\sin{1/y}| > \epsilon$ if $|y| < \frac{1}{n}$
Then $\lim_{y\to 0} \lim_{x \to 0} f = 0 \neq \lim_{x\to 0}\lim_{y\to 0} f$

Exercise 6. $(x, y) \neq (0, 0)$, let $f(x, y) = \frac{x^2 - y^2}{x^2 + y^2}$

If $y = mx$, $f \to \frac{x^2(1-m^2)}{x^2(1+m^2)} = \frac{1-m^2}{1+m^2}$. If $y = 0$, $f = 1$. If $x = 0$, $f = -1$, so there's no way to define $f(0, 0)$ to be single valued.

Exercise 7. Consider $y = kx$; $k \in \mathbb{R}$.

For $k = 0$, $y = 0$ and $f(x, y) = 0$, if $y = 0$
For $k \gtrless 0$, $x \lessgtr 0$, $y < 0$, so $f(x, y) = 0$.

Consider the limit as $x \to 0$. ϵ can be as small as you want.
Then we must have $|x| < \epsilon < |k|$.
then $x^2 < kx$
Thus $y = kx > x^2$, so $f(x, y) = 0$ for any straight line through the origin.
Consider $|x| < \epsilon = 1$

$$x^4 < x^2$$

So $y = x^4 < x^2 \to f(x, y) = 1$
So, f is discontinuous at $(0, 0)$. $f(0, 0)$ depends upon path taken.

Exercise 8. Change to polar coordinates. Then

$$f(x, y) = \frac{\sin(x^2 + y^2)}{(x^2 + y^2)} = f(r, \theta) = \frac{\sin r^2}{r^2}$$

Then regardless of what value of θ, $\lim_{r \to 0} \frac{\sin r^2}{r^2} = \boxed{1}$.

Exercise 9. Let f be a scalar field continuous at an interior pt. a of a set S in $\mathbb{R}^n$. $f(a) \neq 0$ (given).

Continuity of f at a means that

$$\lim_{x \to a} f(x) = f(a) \Longrightarrow \forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } |f(x) - f(a)| < \epsilon \text{ whenever } \|x - a\| < \delta$$

Let $\epsilon = \frac{f(a)}{2}$, $\exists \delta = \delta(\epsilon; a)$ s.t. $\begin{cases} f(x) - f(a) < \frac{f(a)}{2} & \text{if } f(x) > f(a) \\ -f(x) + f(a) < \frac{f(a)}{2} & \text{if } f(x) < f(a) \end{cases}$,

so $\frac{f(a)}{2} < f(x) < \frac{3f(a)}{2}$ for $\forall x$ s.t. $\|x - a\| < \delta(\epsilon; a)$.

$\delta(\epsilon; a)$ defines a $B(a) \subseteq \mathbb{R}^n$ s.t. $f(x)$ has the same sign as $f(a)$.

8.9 Exercises - The derivative of a scalar field with respect to a vector, Directional derivatives and partial derivatives, Partial derivatives of higher order

Exercise 1. $f(x) = a \cdot x$

$$f'(x; y) = \lim_{h \to 0} \frac{f(x + hy) - f(x)}{h} = \lim_{h \to 0} \frac{a \cdot (x + hy) - a \cdot x}{h} = \boxed{a \cdot y}$$

Exercise 2. $f(x) = \|x\|^4$

(1)

$$f'(x, y) = \lim_{h \to 0} \frac{f(x + hy) - f(x)}{h} = \lim_{h \to 0} \frac{\|x + hy\|^4 - \|x\|^4}{h} = \boxed{4x^2(x \cdot y)}$$

(2) $n = 2$

$$f'(2i + 3j; xi + yj) = 6 = 4(13)(2x + 3y) \Longrightarrow \frac{3}{26} = 2x + 3y \Longrightarrow \boxed{y = \frac{-2x}{3} + \frac{1}{26}}$$

(3) $n = 3$

$$f'(i + 2j + 3k; xi + yj + zk) = 0 = 4(1^2 + 2^2 + 9)(x + 2y + 3z) \Longrightarrow \boxed{x + 2y + 3z = 0}$$

Exercise 3.

$$f'(x, y) = \lim_{h \to 0} \frac{f(x + hy) - f(x)}{h} = \lim_{h \to 0} \frac{(x + hy) \cdot T(x + hy) - x \cdot T(x)}{h} = \lim_{h \to 0} \frac{(x + hy) \cdot (T(x) + hT(y)) - x \cdot T(x)}{h} =$$

$$= \lim_{h \to 0} \frac{hy \cdot T(x) + hx \cdot T(y) + h^2 yT(y)}{h} = \boxed{y \cdot T(x) + x \cdot T(y)}$$

Exercise 4. $f(x, y) = x^2 + y^2 \sin(xy)$

$$\partial_x f = 2x + y^3 \cos(xy)$$

$$\partial_y f = 2y \sin(xy) + xy^2 \cos(xy)$$

Exercise 5. $f(x,y) = \sqrt{x^2+y^2}$

$$\partial_x f = \frac{x}{f}; \quad \partial_y f = \frac{y}{f}$$

Exercise 6. $f(x,y) = \frac{x}{\sqrt{x^2+y^2}}$

$$\partial_x f = \frac{1}{f} + \frac{-x^2}{(x^2+y^2)^{3/2}}$$
$$\partial_y f = \frac{-xy}{(x^2+y^2)^{3/2}}$$

Exercise 7. $f(x,y) = \frac{x+y}{x-y}, \quad x \neq y$

$$\partial_x f = \frac{1}{x-y} + \frac{-(x+y)}{(x-y)^2} = \frac{-2y}{(x-y)^2}$$
$$\partial_y f = \frac{1}{x-y} + \frac{-(x+y)}{(x-y)^2}(-1) = \boxed{\frac{2x}{(x-y)^2}}$$

Exercise 8. $f(x) = a \cdot x$; a fixed.

$$\boxed{\partial_{x_j} f = a_j}$$

Exercise 9. $f(x) = \sum_{i=1}^n \sum_{j=1}^n a_{ij} x_i x_j$, $a_{ij} = a_{ji}$

$$\partial_{x_k} f(x) = \sum_{i=1}^n \sum_{j=1}^n a_{ij}\delta_{ik}x_j + \sum_{i=1}^n \sum_{j=1}^n a_{ij}x_i\delta_{jk} = \sum_{j=1}^n a_{kj}x_j + \sum_{i=1}^n a_{ik}x_i = \boxed{2\sum_{j=1}^n a_{kj}x_j}$$

Exercise 10. $f(x,y) = x^4 + y^4 - 4x^2y^2$

$$\partial_x f = 4x^3 - 8xy^2 \qquad \partial^2_{yx} f = -16xy$$
$$\partial_y f = 4y^3 - 8x^2y \qquad \partial^2_{xy} f = -16xy$$

Exercise 11. $f(x,y) = \log{(x^2+y^2)}$

$$\partial_x f = \frac{2x}{x^2+y^2} \qquad \partial_{yx} f = \frac{-4xy}{(x^2+y^2)^2}$$
$$\partial_y f = \frac{2y}{x^2+y^2} \qquad \partial_{xy} f = \frac{-4xy}{(x^2+y^2)^2}$$

Exercise 12. $f(x,y) = \frac{1}{y}\cos x^2 \ ; \ \ y \neq 0$

$$\partial_x f = \frac{-2x\sin x^2}{y} \qquad \partial_{yx} f = \frac{2x\sin x^2}{y^2}$$
$$\partial_y f = \frac{-1}{y^2}\cos x^2 \qquad \partial_{xy} f = \frac{2x\sin x^2}{y^2}$$

Exercise 13. $f(x,y) = \tan{(x^2/y)}; \ \ y \neq 0$

$$\partial_x f = \frac{2x}{y}\sec{(x^2/y)}$$
$$\partial_y f = \frac{-x^2}{y^2}\sec{(x^2/y)}$$

Exercise 14. $f(x,y) = \arctan{(y/x)}$

$$\partial_x f = \frac{1}{1+(y/x)^2}\left(\frac{-y}{x^2}\right)$$
$$\partial_y f = \frac{1}{1+(y/x)^2}\left(\frac{1}{x}\right)$$

Exercise 15. $f(x,y) = \arctan\left(\frac{x+y}{1-xy}\right)$

$$\partial_x f = \frac{1}{1+\left(\frac{x+y}{1-xy}\right)^2} = \left(\frac{1}{1-xy} + \frac{-(x+y)}{(1-xy)^2}((-y)\right) = \frac{1-xy+xy+y^2}{1+x^2y^2+x^2+y^2} = \frac{1+y^2}{1+x^2y^2+x^2+y^2}$$

$$\partial_y f = \frac{1+x^2}{1+x^2y^2+x^2+y^2} \quad \text{(by label symmetry!)}$$

Exercise 16. $f(x,y) = e^{y^2}\ln x \quad x > 0$

$$\partial_x f = y^2 x^{y^2-1}$$
$$\partial_y f = x^{y^2} 2y \ln x$$

Exercise 17. $f(x,y) = \arccos\sqrt{x/y}; \quad y \neq 0$

$$\partial_x f = \frac{-1}{\sqrt{1-x/y}}\frac{1/2}{\sqrt{xy}} = \frac{-1/2}{\sqrt{xy-x^2}}$$
$$\partial_y f = \frac{-1}{\sqrt{1-x/y}}\frac{\sqrt{x}(-1/2)}{y^{3/2}} = \frac{(1/2)\sqrt{x}/y}{\sqrt{y-x}}$$

Exercise 18. $v(r,t) = t^n e^{-r^2/4t}$

$$\begin{aligned}
\partial_r v &= \frac{-1r}{2t}t^n e^{-r^2/4t} = \frac{-r}{2t}t^n e^{-r^2/4t} \\
r^2\partial_r v &= \frac{-r^3}{2t}t^n e^{-r^2/4t} \\
\partial_r(r^2\partial_r v) &= \frac{-3r^2}{2t}t^n e^{-r^2/4t} + \frac{-r^3}{2t}t^n\left(\frac{-r}{2t}\right)e^{-r^2/4t} \\
\frac{1}{r^2}\partial_r(r^2\partial_r v) &= \frac{-3}{2t}t^n e^{-r^2/4t} + \frac{r^2t^n}{4t^2}e^{-r^2/4t}
\end{aligned}$$

$$\partial_t v = nt^{n-1}e^{-r^2/4t} + t^n\left(\frac{r^2}{4t^2}\right)e^{-r^2/4t}$$

$$\boxed{n = -3/2}$$

Exercise 19. $z = u(x,y)e^{ax+by}; \quad \frac{\partial^2 u}{\partial x \partial y} = 0$

$$\begin{aligned}
&\partial_x z = \partial_x u e^{ax+by} + az \\
&\partial_y z = (\partial_y u)e^{ax+by} + bz \\
&\partial^2_{xy} z = a(\partial_y u)e^{ax+by} + b(\partial_x u)e^{ax+by} + baue^{ax+by} \\
&\partial^2_{xy} z - \partial_x z - \partial_y z + z = a(\partial_y u)e^{ax+by} + b(\partial_x u) + abz - (\partial_x u)e^{ax+by} - az - (\partial_y u)e^{ax+by} - bz + z = 0 \\
&\boxed{a = 1;\ b = 1}
\end{aligned}$$

Exercise 20.

(1) $f'(x,y) = 0 \quad \forall\, x \in B(a) \quad \forall\, y$
Recall Thm. 8.4, Mean-value Thm. for derivatives of scalar fields. Assume $\exists\, f'(a+ty;y) \quad \forall\, t \in [0,1]$. Then $\exists$ some $\theta \in (0,1)$ s.t.

$$f(a+y) - f(a) = f'(z;y), \text{ where } z = a + \theta y$$

Proof. Let $g(t) = f(a+ty)$
Use one-dim. mean-value thm. to g on $[0,1]$.

$$g(1) - g(0) = g'(\theta),\ \theta \in (0,1)$$

□

$$\begin{gathered}
x = a + y \\
y = x'; \quad 0 \leq |x'| < r \\
\exists\, f'(x,y) = f'(a+ty;y)\,\forall\, t \in [0,1] \\
(\text{ since } |ty| = t|y| < tr < r) \\
\Longrightarrow f'(a+ty;y) = 0 = f(a+y) - f(a) \\
f(a) = f(x), \forall\, x \in B(a)
\end{gathered}$$

(2) Suppose we consider $x = a + x'$ where $|x'| < r$ and $x' \parallel y$.

$$f'(x,y) = f'(a+x',y) = f'(a+|x'|\frac{y}{|y|},y) =$$
$$= \lim_{h\to 0} \frac{f(a+|x'|e_y, +h|y|e_y) - f(a+|x'|e_y)}{h} = \lim_{|y|h\to 0} \frac{f(a+|x'|e_y+(h|y|)e_y) - f(a+|x'|e_y)}{h|y|/|y|} =$$
$$= |y| f'(a+|x'|e_y, e_y)$$

$$\exists\, |y|f'(a+t|x'|e_y, e_y)\, \forall\, t \in [0,1], \text{ since } f'(x,y) = 0 \quad \forall\, x \in B(a)$$
$$0 = |y|f'(a+t|x'|e_y, e_y) = f(a+|x'|e_y) - f(a) \Longrightarrow f(a+|x'|e_y) = f(a)$$

$f = f(a) =$ constant $\forall\, x \in B(a)$ s.t. $x = a + ke_y \quad 0 \leq |k| < r$

Exercise 21.

(1) A set S in $\mathbb{R}^n$ is convex if $\forall\, a, b \in S$,

$$ta + (1-t)b \in S \quad \forall\, t \in [0,1]$$

Consider $x_1, x_2 \in b(a)$; $\quad \begin{matrix} x_1 = a + x_1' \\ x_2 = a + x_2' \end{matrix}$ and $\begin{matrix} \|x_1 - a\| < r \\ \|x_2 - a\| < r \end{matrix}$

$$tx_2 + (1-t)x_1 - a = t(a+x_2') + (1-t)(a+x_1') - a = at + x_2't + a - at + x_1' - tx_1' - a =$$
$$= x_1'(1-t) + x_2't$$
$$\|tx_2 + (1-t)x_1 - a\| = \|x_1'(1-t) + x_2't\| \leq (1-t)\|x_1'\| + t\|x_2'\| < (1-t)r + tr = r$$

So $tx_2 + (1-t)x_1 \in S \quad \forall\, t \in [0,1]$. So an n-ball is a convex set.

(2) Consider $x \in S$. Then for some $a, b \in S$, $k \in [0,1]$, $x = a + k(b-a)$.

$$f'(x;y) = f'(a+k(b-a),y) \xrightarrow{\text{choose } y=b-a} f'(a+k(b-a), b-a) \text{ exists}$$
$$\Longrightarrow f'(a+\theta(b-a), b-a) = 0 = f(b) - f(a) \Longrightarrow f(b) = f(a)$$

This must be true for all pairs of $a, b \in S$ since x was arbitrarily chosen from S. f is constant on S.

Exercise 22.

(1)

$$f'(a;y) = \lim_{h\to 0} \frac{f(a+hy) - f(a)}{h}$$
$$f'(a,-y) = \lim_{h\to 0} \frac{f(a-hy) - f(a)}{h} = -\lim_{-h\to 0} \frac{f(a+(-h)y) - f(a)}{-h} = -f'(a,y)$$
$$\text{so if } f'(a,y) > 0, \quad f'(a,-y) < 0$$

(2) $f(x) = x \cdot y$ because

$$f'(x;y) = \lim_{h\to 0} \frac{f(x+hy) - f(x)}{h} = \lim_{h\to 0} \frac{x\cdot y + hy^2 - x\cdot y}{h} = y^2 > 0$$

8.14 Exercises - Directional derivatives and continuity, The total derivative, The gradient of a scalar field, A sufficient condition for differentiability

Let's review a number of important concepts with R^n fields. Differentiability must be redefined through a $n - dim$ Taylor expansion.

Definition 2 (Definition of a Differentiable Scalar Field)**.**
Let $f: S \to \mathbb{R}$
Let a be an int. pt. of S.
Let $B(a;r)$ s.t. $B(a;r) \subseteq S$
Let v s.t. $\|v\| < r$, so $a + v \in B(a;r)$ Then

f diff. at a
if $\exists\, T_a,\ E$ s.t.
linear $T_a : \mathbb{R}^n \to \mathbb{R}$
scalar $E(a,v),\ E(a,v) \to 0$ as $\|v\| \to 0$ and

$$f(a+v) = f(a) + T_a(v) + \|v\|E(a,v) \tag{1}$$

The next theorem shows that if the total derivative exists, it is unique. It also tells us how to compute $T_a(y)$, $\forall\, y \in \mathbb{R}^n$.

Theorem 2 (Uniqueness of total derivative)**.** *Assume f diff. at a with total derivative T_a*
Then $\exists\, f'(a;y)\, \forall y \in \mathbb{R}^n$ and

$$T_a(y) = f'(a;y)$$

Also,

$$f'(a;y) = \sum_{j=1}^{n} D_j f(a) y_j \text{ for}$$
$$y = (y_1, \dots, y_j, \dots, y_n)$$

Proof.
If $y = 0$, $\quad T_a(0) = 0$ and $f'(a;0) = 0$. Done.

Suppose $y \neq 0$

$$f(a+v) = f(a) + T_a(v) + \|v\| E(a,v) \qquad \text{(since we assume } f \text{ diff.)}$$
$$v = hy$$
$$\implies \frac{f(a+hy) - f(a)}{h} = \frac{1}{h} T_a(hy) + \frac{\|hy\|}{h} E(a,hy) \xrightarrow{h \to 0} f'(a,y) = T_a(y) + 0$$

Now use linearity of T_a:

$$T_a(y) = \sum T_a(y_j e_j) = \sum y_j T_a(e_j) = \sum y_j f'(a;e_j) = \sum y_j D_j f(a)$$

□

Then the gradient was introduced, $\nabla f(a) = (\partial_1 f(a), \dots, \partial_n f(a))$ so that
$f'(a;y) = \sum_{j=1}^{n} \partial_j f(a) y_j = \nabla f(a) \cdot y$ so then also

$$\implies f(a+v) = f(a) + \nabla f(a) \cdot v + \|v\| E(a;v)$$

Theorem 3 (Differentiability implies Continuity)**.**
If a scalar field f is differentiable at a, then f is cont. at a

Proof. Since f is diff.
$|f(a+v) - f(a)| = |\nabla f(a) \cdot v + \|v\| E(a,v)|$
By Cauchy-Schwarz inequality,

$0 \leq |f(a+v) - f(a)| \leq \|\nabla f(a)\| \|v\| + \|v\| |E(a;v)|$

As $v \to 0$, $|f(a+v) - f(a)| \to 0$ so f cont. at a. □

If f is diff. at a, then all its partials exist (but the converse isn't true).
existence of partials doesn't necessarily imply f is diff.
e.g. $f(x,y) = \frac{xy^2}{x^2+y^4}$

Theorem 4 (Sufficient Condition for Differentiability)**.**
Assume $\exists\, \partial_1 f, \dots, \partial_n f$ in some n-ball $B(a)$ and are cont. at a. Then f diff. at a.

Proof.
Let $\lambda = \|v\|$; then $v = \lambda u$, $\quad \|u\| = 1$
Express $f(a+v) - f(a)$ as a telescoping sum.

$$f(a+v) - f(a) = f(a+\lambda u) - f(a) = \sum_{k=1}^{n} \left(f(a + \lambda v_k) - f(a + \lambda v_{k-1}) \right)$$

where $\{v_k\}$ s.t. $\begin{matrix} v_0 = 0 \\ v_n = u \end{matrix}$. Then choose the v_k's s.t.

$$v_k = v_{k-1} + u_k e_k$$
$$v_1 = u_1 e_1; \quad v_2 = u_1 e_1 + u_2 e_2, \dots, \; v_n = u_1 e_1 + \dots + u_n e_n$$
$$f(a + \lambda v_k) - f(a + \lambda v_{k-1}) = f(a + \lambda v_{k-1} + \lambda u_k e_k) - f(a + \lambda v_{k-1}) =$$
$$= f(b_k + \lambda u_k e_k) - f(b_k)$$

$b_k,\ b_k+\lambda u_k e_k$ differ only by their kth component so apply the mean value theorem

$$\implies f(b_k+\lambda u_k e_k)-f(b_k)=(\lambda u_k)\partial_k f(c_k)$$

$$\text{as } b_k\to a\text{, as } \lambda\to 0\text{, so } c_k\to a$$

$$\implies f(a+v)-f(a)=\lambda\sum_{k=1}^{n}u_k\partial_k f(c_k)$$

Now $\nabla f(a)\cdot v=\lambda\sum u_k\partial_k f(a)$.
$\implies f(a+v)-f(a)-\nabla f(a)\cdot v=\lambda\sum u_k(\partial_k f(c_k)-\partial_k f(a))=E(a,v)$

$c_k\to a$ as $\|v\|\to 0$, and given $\partial_k f$ are cont., $E(a,v)\to 0$ as $\|v\|\to 0$.
By def. of diff., f is diff. □

Exercise 1.

(1) $f(x,y)=x^2+y^2\sin(xy)$

$$\nabla f=(2x+y^2\cos(xy),2y\sin(xy)+y^2x\cos(xy))$$

(2) $f(x,y)=e^x\cos y$

$$\nabla f=(e^x\cos y,-e^x\sin y)$$

(3) $f(x,y,z)=x^2y^3z^4$

$$\nabla f=(2xy^3z^4,3x^2y^2z^4,4x^2y^3z^3)$$

(4) $f(x,y,z)=x^2-y^2+2z^2$

$$\nabla f=(2x,-2y,4z)$$

(5) $f(x,y,z)=\log(x^2+2y^2-3z^2)$

$$\nabla f=\frac{1}{f}(2x,4y,-6z)$$

(6) $f(x,y,z)=e^{(lnx)e^{z\ln y}}$

$$\nabla f=f\left(\frac{e^z\ln y}{x},(\ln x)e^{z\ln y}\left(\frac{z}{y}\right),(\ln x)(\ln y)e^{z\ln y}\right)$$

Exercise 2.

(1) $f(x,y,z)=x^2+2y^2+3z^2$ at $(1,1,0)$ in the direction of $i-j+2k$.

$$f'(a,y)=\nabla f(a)\cdot y$$

(2) $\nabla f=(2x,4y,6z)\ \nabla f(1,1,0)=(2,4,0)$. $\nabla f(a)\cdot y=\boxed{-2}$

Exercise 3. $f(x,y)=3x^2+y^2$; $x^2+y^2=1$

$$(\nabla f)\cdot y=|\nabla f||y|\cos\theta$$

$$|\nabla f|=\sqrt{36x^2+4(1-x^2)}=\sqrt{32x^2+4}=2\sqrt{8x^2+1};\quad |\nabla f|\text{ maximized when } x=\pm 1$$

$$(\pm1,0),\quad y\parallel(\pm1,0)$$

Exercise 4. $(1,2)$, $+2$ towards $(2,2)$; -2 towards $(1,1)$.

$$\nabla f(a)\cdot y=\nabla f(a)\cdot(1,0)=2;\quad \nabla f(a)\cdot(0,-1)=-2\implies\nabla f(a)=2(1,1)$$

$$\nabla f(a)\cdot\frac{(4,6)-(1,2)}{5}=\nabla f(a)\cdot(3,4)/5=\boxed{\frac{14}{5}}$$

Exercise 5. a,b,c s.t. $f(x,y,z)=axy^2+byz+cz^2x^3,\quad(1,2,-1)$

$$\nabla f=(ay^2+3cz^2x^2,2axy+bz,by+2czx^3)$$

$$\nabla f(1,2,-1)=(4a+3c,4a+-b,2b-2c)$$

$$\nabla f(1,2-1)\cdot e_z=2b-2c=64\implies b-c=32$$

Maximum value means ∇f only has components in the z-direction.

$$\partial_x f=4a+3c=0$$
$$\partial_y f=4a-b=0$$

$$\boxed{c=-8;\quad b=24;\quad a=6}$$

Exercise 6. $f'(a,y)=1$; $f'(a,z)=2$ where $y=2i+3j$, $z=i+j$

$$(\partial_x f)(2) + (\partial_y f)(3) = 1 \qquad \partial_y f = -3$$
$$(\partial_x f)(1) + (\partial_y f)(1) = 2 \qquad \partial_x f = 5$$

Exercise 7. Let f and g denote scalar fields that are differentiable on an open set S.

(1)
$$\begin{gathered} \nabla f(a) = \sum (\partial_j f)(a) e_j \\ (\partial_j f)(a) = f'(a_j e_j) \\ \text{if } f \text{ const., } f'(a; e_j) = 0 \\ \Longrightarrow \nabla f(a) = 0 \end{gathered}$$

We can also do the following: if $\nabla f = 0$, $f'(a;y) = \nabla f(a) \cdot y = 0, \ \ \forall y$. Then, from Exercise 20 of Sec. 8.9, f is constant on this open set S.

If f is constant on S, $f(a+v) = f(a)$ for $f(a+v) = f(a) + T_a(v) + \|v\| E(a,v)$
$$\begin{gathered} T_a(y) = -\|y\| E(a,y) \\ E(a,y) \to 0 \text{ as } y \to 0 \end{gathered}$$

By uniqueness of the total derivative, $\nabla f(a) = 0, \ \ \forall\, a \in S$.

(2) ∇ is a linear transformation. $\Longrightarrow \nabla(f+g) = \nabla f + \nabla g$

(3) ∇ is a linear transformation. $\Longrightarrow \nabla(cf) = c\nabla f$

(4)
$$\begin{aligned} (fg)(a+v) - (fg)(a) = \nabla(fg)(a) \cdot v + E_{fg}(a;v) &= f(a+v)g(a+v) - f(a)g(a) = \\ &= f(a+v)g(a+v) - f(a)g(a+v) + f(a)g(a+v) - f(a)g(a) = \\ &= g(a+v)(f(a+v) - f(a)) + f(a)(g(a+v) - g(a)) = \\ &= g(a+v)((\nabla f)(a) + E_f(a;v)) + f(a)((\nabla g)(a) + E_g(a;v)) \\ &\text{Let } \|v\| \to 0, \text{ so that } (\nabla(fg))(a) = g(a)(\nabla f)(a) + f(a)(\nabla g)(a) \end{aligned}$$

(5)
$$\begin{aligned} \left(\frac{f}{g}\right)(a+v) - \left(\frac{f}{g}\right)(a) &= \frac{f(a+v)}{g(a+v)} - \frac{f(a)}{g(a)} = \frac{g(a)f(a+v) - g(a+v)f(a)}{g(a)g(a+v)} = \\ &= \frac{g(a)f(a+v) - g(a)f(a) + g(a)f(a) - g(a+v)f(a)}{g(a)g(a+v)} = \\ &= \frac{g(a)((\nabla f)(a) \cdot v + E_f(a;v)) - f(a)((\nabla g)(a) \cdot v + E_g(a;v))}{g(a)g(a+v)} = \\ &= \nabla\left(\frac{f}{g}\right) \cdot v + E_{f/g}(a;v) \\ \text{Let } \|v\| \to 0 \ \Longrightarrow &\ \frac{g(a)\nabla f(a) - f(a)\nabla g(a)}{g^2(a)} = \nabla\left(\frac{f}{g}\right)(a) \end{aligned}$$

Exercise 8. In $\mathbb{R}^3$, let $r(x,y,z) = xi + yj + zk$, and let $r(x,y,z) = \|r(x,y,z)\|$

(1) $\nabla r = \nabla \sqrt{x^2+y^2+z^2} = \frac{1}{\sqrt{x^2+y^2+z^2}}(x,y,z) = \frac{\vec{r}}{r}$

(2) Use induction.
$$\begin{aligned} \nabla(r^2) &= r\frac{\vec{r}}{r} + r\frac{\vec{r}}{r} = 2\vec{r} \\ \nabla(r^3) &= 2\vec{r}r + r^2\frac{\vec{r}}{r} = 3r\vec{r} \\ \nabla(r^{n+1}) &= nr^{n-2}\vec{r}r + r^{n-1}\frac{\vec{r}}{r} = (n+1)r^{n-1}\vec{r} \end{aligned}$$

(3) $n = 0$. $\nabla(1) = 0$
$$\begin{aligned} \nabla(r^{-1}) &= \nabla\frac{1}{\sqrt{x^2+y^2+z^2}} = \frac{-\vec{r}}{r^2} \\ \nabla(r^{-2}) &= \nabla\frac{1}{x^2+y^2+z^2} = (-2)\vec{r}r^{-4} \end{aligned}$$

Then $\nabla(r^{n+1}) = (n+1)r^{n-1}\vec{r}$, where we reuse the induction step above, because no reference was made to whether n was positive or negative.

So the formula is still valid when n is a negative integer (by induction).

(4) $\nabla f = \vec{r}$

$$\partial_x f = x \quad \partial_y f = y \quad \partial_z f = z$$
$$\boxed{\frac{1}{2}x^2 + \frac{1}{2}y^2 + \frac{1}{2}z^2 = f}$$

Exercise 9. Given n independent vectors, $y_1, \dots, y_n$, then by Thm., $y_1, \dots, y_n$ for a basis for $\mathbb{R}^n$.

$f'(x,y) = \nabla f(x) \cdot y$, so $f'(x,y)$ is linear. Then $\forall\, y \in \mathbb{R}^n$, $y = \sum a_j y_j$ and

$$f'(x,y) = \nabla f(x) \cdot \sum a_j y_j = \sum a_j \nabla f(x) \cdot y_j = 0$$

Then from Exercise 20 of Sec. 8.9, f is constant on $B(a)$.

Exercise 10.

(1) Consider $x \in B(a)$, $\ x = a + x'$.

$$f'(x;y) = f'(a+x',y) = \nabla f(x) \cdot y$$
$$\text{Let } y = x' \Longrightarrow f'(a+x';x')$$
$$\text{By mean value thm., } f'(a+\theta x';x') = f(a+x') - f(a)$$
$$f'(a+x';x') = \nabla f(x) \cdot x' = 0 \Longrightarrow f(a+x') = f(a)$$

This must be true $\forall\, x'$ s.t. $|x'| < r$ for $B(a;r)$. Then f is constant on $B(a)$.

(2)

$$\lim_{h\to 0} \frac{f(a+hy) - f(a)}{h} = f'(a;y) \leq 0$$
$$f'(a;y) = \nabla f(a) \cdot y = |\nabla f(a)||y| \cos\theta \leq 0$$

Consider when $\frac{-\pi/2}{<}\theta < \pi/2$, $|y| \neq 0$. Then $|\nabla f(a)| = 0$.

Exercise 11. Consider the following six statements about a scalar field $f : S \to \mathbb{R}$, where $S \subseteq \mathbb{R}^n$ and $a \in intS$.

(1) (a) f continuous at a
(b) f is differentiable at a
(c) $\exists\, f'(a,y) \quad \forall\, y \in \mathbb{R}^n$.
(d) All the first-order partial derivatives of f exist in a neighborhood of a and are continuous at a.
(e) $\nabla f(a) = 0$
(f) $f(x) = \|x - a\|$ for all x in $\mathbb{R}^n$.

(b) imples (a),(c), because differentiability implies continuity and differentiability through the total derivative gave what the directional derivative would be $\forall\, y \in \mathbb{R}^n$. (d) imples (a),(b),(c) because (d), by theorem, is a sufficient condition for differentiability, and thus differentiability implies (a),(c). (e) doesn't tell us anything because we need a scalar function $E(a;v)$ as well for differentiability. (f) is a continuous function, so (f) imples (a).

8.17 Exercises - A chain rule for derivatives of scalar fields, Applications to geometry. Level sets. Tangent planes.

Exercise 1.

(1)

$$\begin{aligned} u &= f(x,y) \qquad & x &= X(t) \\ u &= F(t) \qquad & y &= Y(t) \end{aligned}$$

$$\nabla f(r) \cdot r'(t) = (\partial_x f)x' + (\partial_y f)y' = F'(t) = u'$$

(2)

$$F(t) = u = f(x,y) = f(r(t))$$
$$F'(t) = \nabla f(r) \cdot r'(t) = (\partial_x f)x' + (\partial_y f)y'$$
$$\nabla f(r) = \nabla f(r(t))$$
$$\frac{d}{dt}\nabla f(r(t)) = \frac{d}{dt}(\partial_x f, \partial_y f) = ((\partial^2_{xx} f)x' + (\partial^2_{yx} f)y', (\partial^2_{xy} f)x' + (\partial^2_{yy} f)y')$$
$$\begin{aligned} F''(t) &= \left(\frac{d}{dt}\nabla f(r(t))\right) \cdot r'(t) + \nabla f(r) \cdot r''(t) = \\ &= (\partial^2_{xx} f)x'^2 + \left((\partial^2_{yx} f) + \partial^2_{xy} f\right) x'y' + (\partial^2_{yy} f)y'^2 + (\partial_x f)x'' + \partial_y f y'' \end{aligned}$$

Exercise 2.

(1) $f(x,y) = x^2 + y^2, \quad X(t) = t, Y(t) = t^2$

$$\partial_x f = 2x \qquad X' = 1$$
$$\partial_y f = 2y \qquad Y' = 2t$$
$$F'(t) = 2x(1) + 2y2t = 2t + 4t^3$$
$$F''(t) = 2 + 12t^2$$

(2) $f(x,y) = e^{xy}\cos(xy^2); X(t) = \cos t, Y(t) = \sin t$

$$\partial_x f = yf + -e^{xy}\sin(xy^2)y^2 \qquad X' = -s = -y \qquad X'' = -c = -x$$
$$\partial_y f = xf + -e^{xy}\sin(xy^2)2yx \qquad Y' = c = x \qquad Y'' = -s = -y$$
$$F'(t) = -y^2 f + e^{xy}\sin(xy^2)y^3 + x^2 f + -e^{xy}\sin(xy^2)2yx^2$$

This is the answer I got. Note that I tried it 2 ways: using the formula $(\partial^2_{xx}f)x'^2 + (\partial^2_{yx}f + \partial^2_{xy}f)x'y' + (\partial^2_{yy}f)y'^2 + (\partial_x f)x'' + (\partial_y f)y''$, and second, taking our answer $F'(t) = G(t)$ and then applying $\partial g_x x' + \partial g_y y'$ on it (which seemed clever).

$$\partial_x g = 4xf + (2x^2-1)(yf - y^2e^{xy}\sin(xy^2)) + (3y^3-2y)ye^{xy}\sin(xy^2) + (3y^3-2y)e^{xy}\cos(xy^2)y^2 =$$
$$= 4xf + 2x^2yf - yf - 2x^2y^2e^{xy}\sin(xy^2) + y^2e^{xy}\sin(xy^2) + 3y^4e^{xy}\sin(xy^2) - 2y^4e^{xy}\sin(xy^2) + 3y^5f - 2y^3f$$
$$= f(4x + 2x^2y - y + 3y^5 - 2y^3) + e^{xy}\sin(xy^2)(-2x^2y^2 + y^2 + y^4)$$
$$\partial_y g = (2x^2-1)(xf - 2yxe^{xy}\sin(xy^2)) + (9y^2-2)e^{xy}\sin(xy^2) + (3y^3-2y)e^{xy}x\sin(xy^2) + (3y^3-2y)e^{xy}\cos(xy^2)2yx =$$
$$= (2x^3 - x + 6y^4x - 4y^2x)f + e^{xy}\sin(xy^2)(-2yx(2x^2-1) + 9y^2 - 2 + 3y^3x - 2yx)$$

$$\partial_x g x' + \partial_y g y' = (-12y^5 + 14y^3 - 4y + 7x - 9x^3)e^{xy}\sin(xy^2) + f(9x^6 - 11x^4 + 3x^2 - 4xy)$$

(3) $f(x,y) = \log\left(\frac{(1+e^{x^2})}{1+e^{y^2}}\right) = \log(1+e^{x^2}) - \log(1+e^{y^2})$

$$X(t) = e^t \qquad X' = X \qquad X'' = X$$
$$Y(t) = e^{-t} \qquad Y' = -Y \qquad Y'' = Y$$
$$\partial_x f = \frac{1}{1+e^{x^2}}(2xe^{x^2}) \qquad F'(t) = \boxed{\frac{2x^2e^{x^2}}{1+e^{x^2}} + \frac{2y^2e^{y^2}}{1+e^{y^2}}}$$
$$\partial_y f = \frac{-2ye^{y^2}}{1+e^{y^2}}$$
$$\partial_{xx} f = (2)\left(\frac{(e^{x^2} + 2x^2e^{x^2})(1+e^{x^2}) - (2xe^{x^2})(xe^{x^2})}{(1+e^{x^2})^2}\right) = (2)\left(\frac{e^{x^2} + 2x^2e^{x^2} + e^{2x^2}}{(1+e^{x^2})^2}\right)$$
$$\partial_{yy} f = (-2)\left(\frac{e^{y^2} + 2y^2e^{y^2} + e^{2y^2}}{(1+e^{y^2})^2}\right)$$
$$\partial_{xy} f = 0$$

$$(\partial^2_{xx}f)x'^2 + ((\partial^2_{yx}f + \partial^2_{xy}f)x'y' + (\partial^2_{yy}f)y'^2 + (\partial x f)x'' + \partial y f y'' =$$
$$= (2)\frac{e^{x^2} + 2x^2e^{x^2} + e^{2x^2}}{(1+e^{x^2})^2}(x^2) + (-2)\frac{e^{y^2} + 2y^2e^{y^2} + e^{2y^2}}{(1+e^{y^2})^2}y^2 + \frac{2xe^{x^2}(1+e^{x^2})}{(1+e^{x^2})^2}x + \frac{-2ye^{y^2}(1+e^{y^2})}{(1+e^{y^2})^2}y =$$
$$= \boxed{\frac{4x^2e^{x^2}(1+x^2+e^{x^2})}{(1+e^{x^2})^2} + \frac{-4y^2e^{y^2}(1+y^2+e^{y^2})}{(1+e^{y^2})^2}}$$

Exercise 3.

(1)

$$\nabla f = (3,-5,2)$$
$$N = (2x, 2y, 2z) = 2r$$
$$\nabla f \cdot \frac{(2,2,1)}{3} = (3,-5,2)\cdot(2,2,1)/3 = \boxed{-2/3}$$

It should be noted that the normal to a sphere is the position vector.

(2)

$$\nabla f = (2x, -2y, 0)$$
$$x^2 + y^2 + z^2 = 4 \text{ is a sphere}$$
$$\nabla f \cdot \frac{(x, y, z)}{r} = \frac{2x^2 - 2y^2}{r}$$

(3) $x^2 + y^2 = 25$ and $2x^2 + 2(z^2 - x^2) - z^2 = 25$

$$\nabla f = (2x, 2y, -2z)$$
$$T = \frac{(1, \frac{\mp x}{\sqrt{25-x^2}}, 0)}{\sqrt{1 + \frac{x^2}{25-x^2}}} = \frac{\left(1, \frac{\mp x}{\sqrt{25-x^2}}, 0\right)}{\sqrt{25/(25-x^2)}} = \left(\frac{\sqrt{25-x^2}}{5}, \frac{\mp x}{5}, 0\right)$$
$$\implies \nabla f \cdot T = \left(\frac{2x\sqrt{25-x^2}}{5}, \frac{-2\sqrt{25-x^2}x}{5}, 0\right) = \left(\frac{6}{5}4, \frac{-2(4)3}{5}, 0\right) = \left(\frac{24}{5} + \frac{-24}{5} + 0\right) = 0$$

Exercise 4.

(1) Find a vector $V(x, y, z)$ normal to the surface

$$z = \sqrt{x^2 + y^2} + (x^2 + y^2)^{3/2}$$

at a general point (x, y, z) of the surface, $(x, y, z) \neq (0, 0, 0)$

$$0 = \sqrt{x^2 + y^2} + (x^2 + y^2)^{3/2} - z = f(r)$$
$$\implies \nabla f = \left(\frac{x}{\sqrt{x^2 + y^2}} + 3x(x^2 + y^2)^{1/2}, \frac{y}{\sqrt{x^2 + y^2}} + 3y(x^2 + y^2)^{1/2}, -1\right)$$

(2)

$$\nabla f \cdot e_z = |\nabla f| \cos\theta_z \implies \cos\theta_z = \frac{-1}{\sqrt{\frac{(x+3x^3+3xy^2)^2}{x^2+y^2} + \frac{(y+3yx^2+3y^3)^2}{x^2+y^2} + 1}}$$
$$\cos\theta_z = \frac{-1}{\sqrt{(1 + 3(x^2 + y^2))^2 + 1}}$$

$$\lim_{y\to 0} \cos\theta_z = \frac{-1}{\sqrt{(1 + 3(x^2))^2 + 1}} \xrightarrow{x\to 0} = \frac{-1}{\sqrt{2}}$$
$$\lim_{x\to 0}\lim_{y\to 0} \cos\theta_z = \boxed{\sqrt{-1}\sqrt{2}}$$

$\cos\theta_z$ is differentiable at $(x, y) = (0, 0)$ (we can observe that the partial derivatives exist and are continuous at $(0, 0)$), so $\cos\theta_z$ is continuous at $(x, y) = (0, 0)$.

Exercise 5. $\begin{array}{ll} e^u \cos v = x & u = u(x, y) \\ e^u \sin v = y & v = v(x, y) \end{array}$

$$\begin{array}{l} x^2 + y^2 = e^{2u} \\ \ln(x^2 + y^2) = 2u \end{array} \implies \boxed{\frac{1}{2}\ln(x^2 + y^2) = u} \qquad \begin{array}{l} \sin v = \frac{y}{\sqrt{x^2 + y^2}} \\ \cos v = \frac{x}{sqrtx^2 + y^2} \end{array} \implies \tan v = \frac{y}{x}$$

$$\begin{array}{l} \nabla U = \left(\frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2}\right) \\ \nabla V = \left(\frac{-y/x^2}{1 + (y/x)^2}, \frac{1/x}{1 + (y/x)^2}\right) \end{array} \implies \nabla U \cdot \nabla V = 0$$

Exercise 6. $f(x, y) = \sqrt{|xy|}$

(1)

$$\begin{array}{llll}
\text{if } x \geqslant 0,\ y \geqslant 0, & f=(xy)^{1/2} & \partial_x f = \dfrac{1}{2}\sqrt{\dfrac{y}{x}} & \partial_y f = \dfrac{1}{2}\sqrt{\dfrac{x}{y}} \\
\text{if } x > 0,\ y < 0, & f=(x(-y))^{1/2} & \partial_x f = \dfrac{1}{2}\left(\dfrac{|y|}{x}\right)^{1/2} & \partial_y f = \dfrac{-1}{2}\left(\dfrac{x}{|y|}\right)^{1/2} \\
\text{if } x < 0,\ y > 0, & f=(-xy)^{1/2} & \partial_x f = \dfrac{-1}{2}\left(\dfrac{y}{|x|}\right)^{1/2} & \partial_y f = \dfrac{1}{2}\left(\dfrac{|x|}{y}\right)^{1/2}
\end{array}$$

(2) Does the surface $z = f(x,y)$ have a tangent plane at the origin?

$$z = f(x,y)$$
$$g = f(x,y) - z \Longrightarrow \nabla g = (\partial_x f, \partial_y f, -1)$$

For $x = y$,

$$\nabla g(0,0,0) = \left(\frac{1}{2}, \frac{1}{2}, -1\right)$$

But when approaching from the x or y axis, $\nabla g = (0,0,-1)$. A tangent plane cannot be defined at the origin.

Exercise 7. Given surface $z = xy$, $z = y_0 x,\ y = y_0$ and $z = x_0 y,\ x = x_0$ intersect at (x_0, y_0, z_0) and lie on the surface. We want to show that the tangent plane to this surface at (x_0, y_0, z_0) contains these 2 lines.
Note that the 2 lines could be reexpressed in vector form:

$$x(1,0,y_0) + (0,y_0,0)$$
$$y(0,1,x_0) + (x_0,0,0)$$

Rewrite the surface equation so to get the gradient

$$0 = xy - z$$
$$\nabla f = (y, x, -1)$$

$$\nabla f(r_0) = (y_0, x_0, -1)$$
$$\nabla f(r_0) \cdot (x,y,z) = y_0 x + x_0 y - z = x_0 y_0$$

$$\nabla f(r_0) \cdot (1,0,y_0) = \nabla f(r_0) \cdot (0,1,x_0) = 0 \text{ and note that } \begin{array}{l} (0,y_0,0) \in S \\ (x_0,0,0) \in S \end{array}$$

So indeed, the tangent plane contains these lines.

Exercise 8. $xyz = a^3$ (x_0, y_0, z_0), $\nabla f = (yz, xz, xy)$

$$y_0 z_0 x + x_0 z_0 y + x_0 y_0 z = 3a^3$$

Volume of the tetrahedron:

$$V = \frac{1}{3}Bh = \frac{1}{3}\left(\frac{1}{2}xy\right)h = \frac{1}{6}xyz = \frac{1}{6}(3x_0)(3y_0)(3z_0) = \boxed{\frac{9a^3}{2}}$$

Exercise 9. We want a pair of linear Cartesian equations for the line tangent to $x^2+y^2+2z^2 = 4$, $z = e^{x-y}$ at pt. $(1,1,1) = P_1$
We calculate the gradients for the 2 surfaces, so to get the normal to these surfaces.

$$\nabla f = (2x, 2y, 4z) \qquad \nabla g = (e^{x-y}, -e^{x-y}, -1)$$
$$\xrightarrow{(1,1,1)} \nabla f(1,1,1) = 2(1,1,2) \qquad \nabla g(1,1,1) = (1,-1,-1)$$

We know the general form of the equation for the line with these normals, N, will be $X \cdot N = X \cdot P_1$. Then

$$\boxed{\begin{array}{c} x - y - z = -1 \\ x + y + 2z = 4 \end{array}}$$

Exercise 10. Find a constant c s.t. at any pt. of intersection, the corresponding tangent planes will be $\perp$ to each other.

$$(x-c)^2 + y^2 + z^2 = 3 \qquad \nabla f = 2(x-c, y, z)$$
$$x^2 + (y-1)^2 + z = 1 \qquad \nabla g = 2(x, y-1, z)$$

tangent planes are $\perp$ to each other $\Longrightarrow \nabla f \cdot \nabla g = x(x-c) + y(y-1) + z^2 = x^2 - xc + y^2 - y + z^2 = 0$

$$\Longrightarrow y = xc$$

We have the intersection condition, and so solving for the system of 2 linear equations, with $y = xc$,

$$\boxed{c = \pm\sqrt{3}}$$

Exercise 11. Without loss of generality, choose the origin to make the ellipse symmetrical and the major axis to lie on the x axis.

$$R = (x, y)$$

$$\begin{aligned} F_1 &= (ae, 0) & |R - F_1| &= r_1 = \sqrt{(x-ae)^2 + y^2} & R - F_1 &= (x - ae, y) \\ F_2 &= (-ae, 0) & |R - F_2| &= r_2 = \sqrt{(x+ae)^2 + y^2} & R - F_2 &= (x + ae, y) \end{aligned}$$

$$\nabla |R - F_1| = \frac{(x-ae, y)}{\sqrt{(x-ae)^2 + y^2}} = \frac{R - F_1}{r_1}$$

$$\nabla |R - F_2| = \frac{(x+ae, y)}{\sqrt{(x+ae)^2 + y^2}} = \frac{R - F_2}{r_2}$$

$$r_1 + r_2 = K$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \begin{aligned} y^2 &= b^2\left(1 - \frac{x^2}{a^2}\right) \\ y &= \pm b\sqrt{1 - \frac{x^2}{a^2}} \end{aligned} \qquad \frac{dy}{dx} = \frac{\mp bx/a^2}{\sqrt{1-(x/a)^2}} = \frac{-b^2}{a^2}\frac{x}{y}$$

$$R' = (1, \frac{-b^2}{a^2}\frac{x}{y})$$

$$R' \cdot \nabla(|R - F_1| + |R - F_2|) = R' \cdot \left(\frac{R-F_1}{r_1} + \frac{R - F_2}{r_2} \right) = \frac{R' \cdot kR + R' \cdot (-F_1 r_2 - F_2 r_1)}{r_1 r_2}$$

Consider $R' \cdot kR + r_2(-F_1 \cdot R') - r_1(R' \cdot F_2) = K(x - \frac{b^2}{a^2}x) + r_2(-ae) - r_1(-ae)$

$$r_{1,2} = \sqrt{(x \mp ae)^2 + b^2 - \frac{b^2x^2}{a^2}} = \sqrt{x^2 \mp 2xae + a^2e^2 + b^2 - \frac{b^2x^2}{a^2}} = \sqrt{e^2x^2 \mp 2xae + a^2} = a \mp xe$$

$$r_2 - r_1 = 2xe$$

$$r_2 + r_1 = 2a$$

$$\Longrightarrow R' \cdot kR + r_2(-F_1 \cdot R') - r_1(R' \cdot F_2) = 2axe^2 + (-ae)(2xe) = 0$$

Thus $T \cdot (\nabla r_1 + r_2) = 0$. This means that $T \cdot \nabla r_1 = -T \cdot \nabla r_2$. As we had shown above, $\nabla r_{1,2}$ is in the direction from the respective foci to the arbitrary point (x, y) and both ∇r_1 and ∇r_2 are of length 1. Thus $T \cdot \nabla r_1 = -T \cdot \nabla r_2$ geometrically says that the angle between ∇r_1 and the tangent line is equal to the angle between ∇r_2 and the tangent line.

Exercise 12. $f = f(x, y, z)$

$$\partial_z f = k_0 z \Longrightarrow f = k_1 z^2 g(x, y) + h(x, y)$$

$$f(0, 0, a) = k_1 a^2 g(0, 0) + h(0, 0) = f(0, 0, -a)$$

8.22 Exercises - Derivatives of vector fields, Differentiability implies continuity, The chain rule for derivatives of vector fields, Matrix form of the chain rule

Exercise 1. Recall

$$(Dh(a))_{jk} = \sum_{l=1}^n (Df(b))_{jl}(Dg(a))_{lk} = (\partial_k h_j(a)) = \sum_{l=1}^n (\partial_l f_j(b))(\partial_k g_l(a))$$

(1)

$$\partial_x f = \partial_t F \partial_x g \Longrightarrow \frac{\partial f}{\partial x} = F'(g(x, y))\frac{\partial g}{\partial x}$$

$$\frac{\partial f}{\partial y} = F'(g(x, y))\frac{\partial g}{\partial y}$$

(2)

$$\begin{aligned} F(t) &= e^{\sin t} & \frac{\partial f}{\partial x} &= -\cos t e^{\sin t} \sin{(x^2 + y^2)} 2x \\ g(x, y) &= \cos{(x^2 + y^2)} & \frac{\partial f}{\partial y} &= -\cos t e^{\sin t} \sin{(x^2 + y^2)} 2y \end{aligned}$$

Exercise 2. Given

$$u = \frac{x-y}{2} \qquad f(u,v) \to F(x,y), \quad F(x,y) = f(u(x,y), v(x,y))$$
$$v = \frac{x+y}{2}$$

$$\begin{aligned}\partial_x F = \partial_u f \partial_x u + \partial_v f \partial_x v &= \frac{1}{2}\partial_u f + \frac{1}{2}\partial_v f \\ \partial_y F = \partial_u f \partial_y u + \partial_v f \partial_y v &= -\frac{1}{2}\partial_u f + \frac{1}{2}\partial_v f\end{aligned}$$

Exercise 3. Given

$$u = f(x,y) \qquad \begin{aligned} x &= X(s,t) \\ y &= Y(s,t) \end{aligned} \qquad u = F(s,t) = f(x(s,t), y(s,t))$$

(1)
$$\begin{aligned}\partial_s F &= \partial_x f \partial_s x + \partial_y f \partial_s y \\ \partial_t F &= \partial_x f \partial_t x + \partial_y f \partial_t y\end{aligned}$$

(2) To get to the second order partial derivatives, it seems that a direct application of the partial derivatives is needed: there's not a way to reformulate a matrix chain rule for second order partial derivatives, until maybe the Hessian matrix.

$$\partial^2_{ss} f = \partial^2_{xs} F \partial_s x + \partial_x f \partial^2_{ss} x + \partial^2_{ys} F \partial_s y + \partial_y f \partial^2_{ss} y$$

$$\begin{aligned} \partial_s f &= \partial_x f \partial_s x + \partial_y f \partial_s y \\ \implies \partial_{xs} f &= \partial_{xx} f \partial_s x + \partial_x f \partial_s \partial_x x + \partial_{xy} f \partial_s y + \partial_y f \partial_s \partial_x y = \\ &= \partial_{xx} f \partial_s x + \partial_{xy} f \partial_s y \end{aligned}$$

$$\begin{aligned}\implies \partial^2_{ss} f &= (\partial^2_{xx} f \partial_s x + \partial^2_{xy} f \partial_s y)\partial_s x + \partial_x f \partial^2_{ss} x + (\partial^2_{yx} f \partial_s x + \partial^2_{yy} f \partial_s y)\partial_s y + (\partial_y f)(\partial^2_{ss} y) = \\ &= (\partial^2_{xx} f)(\partial_s x)^2 + 2\partial^2_{xy} f \partial_s y \partial_s x + \partial^2_{yy} f (\partial_s y)^2 + (\partial_x f)\partial^2_{ss} x + (\partial_y f)(\partial^2_{ss} y)\end{aligned}$$

(3) By label symmetry:

$$\partial^2_{tt} f = (\partial^2_{xx} f)(\partial_t x)^2 + 2\partial^2_{xy} f \partial_t y \partial_t x + \partial^2_{yy} f (\partial_t y)^2 + (\partial_x f)\partial^2_{tt} x + (\partial_y f)(\partial^2_{tt} y)$$

Let's calculate $\partial^2_{st} F$

$$\begin{aligned}\partial^2_{st} F &= \partial^2_{xs} f \partial_t x + \partial_x f \partial^2_{st} x + \partial^2_{ys} f \partial_t y + \partial_y f \partial^2_{st} y = \\ &= (\partial^2_{xx} f \partial_s x + \partial^2_{xy} f \partial_s y)\partial_t x + \partial_x f \partial^2_{st} x + (\partial^2_{yx} f \partial_s x + \partial^2_{yy} f \partial_s y)\partial_t y + \partial_y f \partial^2_{st} y = \\ &= \partial^2_{xx} f \partial_s x \partial_t x + \partial^2_{xy} f (\partial_s y \partial_t x + \partial_s x \partial_t y) + \partial^2_{yy} f \partial_s y \partial_t y + \partial_x f \partial^2_{st} x + \partial_y f \partial^2_{st} y\end{aligned}$$

Exercise 4.

(1)
$$\begin{array}{lll} X(s,t) = s+t & \partial_s X = \partial_t X = 1 & \partial_s F = \partial_x f + t\partial_y f \\ Y(s,t) = st & \partial_s Y = t \quad \partial_t Y = s & \partial_t F = \partial_x f + s \partial_y f \end{array}$$

$$\begin{aligned}\partial^2_{ss} f &= (\partial^2_{xx} f) + 2\partial^2_{xy} f t + \partial^2_{yy} f t^2 \\ \partial^2_{tt} f &= (\partial^2_{xx} f) + 2\partial^2_{xy} f s + \partial^2_{yy} f s^2 \\ \partial^2_{st} f &= \partial^2_{xx} f + \partial^2_{xy} f (t+s) + \partial^2_{yy} f ts\end{aligned}$$

(2)
$$\begin{array}{llll} X(s,t) = st & X_s = t & Y_s = 1/t & \partial_s F = \partial_x f t + \partial_y f (1/t) \\ Y(x,t) = s/t & X_t = s & Y_t = -s/t & \partial_t F = \partial_x f s + \partial_y f (-s/t^2) \end{array}$$

$$\partial^2_{ss} F = (\partial^2_{xx} f)(t^2) + 2\partial^2_{xy} f \left(\frac{1}{t}\right) t + \partial^2_{yy} f (1/t)^2$$

$$\partial^2_{tt} F = (\partial^2_{xx} f) s^2 + 2\partial^2_{xy} f \left(\frac{-s}{t}\right) s + \partial^2_{yy} f \left(\frac{-s}{t^2}\right)^2$$

$$\partial^2_{st} F = \partial^2_{xx} f ts + \partial^2_{xy} f \left(\frac{1}{t} s + t\left(\frac{-s}{t^2}\right)\right) + \partial^2_{yy} f \frac{1}{t}\left(\frac{-s}{t^2}\right) + \partial_x f + \partial_y f \left(\frac{-1}{t^2}\right)$$

(3)

$$\begin{aligned} X(s,t) &= \frac{s-t}{2} & x_s &= 1/2 & Y_s &= 1/2 & \partial_s F &= \partial_x f 1/2 + \partial_y f 1/2 \\ Y(s,t) &= \frac{s+t}{2} & x_t &= -1/2 & Y_t &= 1/2 & \partial_t F &= \partial_x f(-1/2) + \partial_y f 1/2 \end{aligned}$$

$$\begin{aligned} \partial^2_{ss} F &= \left(\frac{1}{2}\right)((\partial^2_{xx} f + \partial_{xy} f)1/2 + (\partial^2_{yx} f + \partial^2_{yy})1/2) = \frac{1}{4}(\partial^2_{xx} f + \partial^2_{xy} f + \partial^2_{yx} f + \partial^2_{yy} f) \\ \partial^2_{tt} F &= \frac{1}{4}(\partial^2_{xx} f - \partial^2_{xy} f - \partial^2_{yx} f + \partial^2_{yy} f) \\ \partial^2_{st} F &= \frac{1}{4}(-\partial^2_{xx} f + \partial^2_{xy} - \partial^2_{yx} f + \partial^2_{yy} f) \end{aligned}$$

Exercise 5. You cannot interchange ∂_x and ∂_r, ∂_x and ∂_θ, etc.

$$\begin{aligned} \partial_r \phi &= \partial_x f \partial_r x + \partial_y f \partial_\theta y = f_x c + f_y s \\ \partial_\theta \phi &= \partial_x f \partial_\theta x + \partial_y f \partial_\theta y = -f_x rs + f_y rc \end{aligned}$$

Notice that the above formulas give a prescription or algorithm for computing the ∂_r or ∂_θ of functions of x, y. Notice also that f_x, f_y are each composite functions.

$$\begin{aligned} \partial^2_{r\theta}\phi &= \partial_r(-f_x rs + f_y rc) = -\partial_r f_x rs - f_x s + \partial_r f_y f c + f_y c = \\ &= -(f_{xx} c + f_{yx} s) rs - f_x s + (f_{xy} c + f_{yy} s) rc + f_y c = \\ &= -f_{xx} rcs - f_{yx} rs^2 + f_{xy} rc^2 + f_{yy} rsc - f_x s + f_y c \end{aligned}$$

$$\begin{aligned} \partial^2_{\theta r}\phi &= \partial_\theta(f_x c + f_y s) = \partial_\theta f_x c - f_x s + \partial_\theta f_y s + f_y c = \\ &= (-f_{xx} rs + f_{yx} rc)c - f_x s + (-f_{xy} rs + f_{yy} rc)s + f_y c = \\ &= -f_{xx} rsc + f_{yx} rc^2 - f_x s - f_{xy} rs^2 + f_{yy} rcs + f_y c \end{aligned}$$

$$\begin{aligned} \partial^2_{rr}\phi &= \partial_r(f_x c + f_y s) = \\ &= \partial_r f_x c + \partial_r f_y s = (f_{xx} c + f_{yx} s)c + (f_{xy} c + f_{yy} s)s = f_{xx} c^2 + f_{yx} sc + f_{xy} cs + f_{yy} s^2 \end{aligned}$$

Exercise 6. Given $u = f(x, y, z)$, $\begin{aligned} x &= X(r,s,t) \\ y &= Y(r,s,t) \\ z &= Z(r,s,t) \end{aligned}$ $\quad u = F(r,s,t)$

$$\begin{aligned} \partial_r F &= \partial_x F \partial_r x + \partial_y F \partial_r y + \partial_z F \partial_r z \\ \partial_s F &= \partial_x F \partial_s x + \partial_y F \partial_s y + \partial_z F \partial_s z \\ \partial_t F &= \partial_x F \partial_t x + \partial_y F \partial_t y + \partial_z F \partial_t z \end{aligned}$$

Exercise 7.

(1) Given $\begin{aligned} X(r,s,t) &= r + s + t \\ Y(r,s,t) &= r + -2s + 3t \\ Z(r,s,t) &= 2r + s + -t \end{aligned}$

$$\begin{aligned} \partial_r F &= \partial_x F + \partial_y F + 2\partial_z F \\ \partial_s F &= \partial_x F + -2\partial_y F + \partial_z F \\ \partial_t F &= \partial_x F + 3\partial_y F - \partial_z F \end{aligned}$$

(2) Given $\begin{aligned} X(r,s,t) &= r^2 + s^2 + t^2 \\ Y(r,s,t) &= r^2 - s^2 - t^2 \\ Z(r,s,t) &= r^2 - s^2 + t^2 \end{aligned}$

$$\begin{aligned} \partial_r F &= 2r(\partial_x F + \partial_y F + \partial_z F) \\ \partial_s F &= 2s(\partial_x F + -\partial_y F - \partial_z F) \\ \partial_t F &= 2t(\partial_x F - \partial_y F + \partial_z F) \end{aligned}$$

Exercise 8. $u = f(x, y, z) \quad \begin{aligned} x &= X(s,t) \\ y &= Y(s,t) \\ z &= Z(s,t) \end{aligned} \quad u = F(s,t)$

$$\begin{aligned} \partial_s F &= \partial_x F \partial_s X + \partial_y F \partial_s Y + \partial_z F \partial_s Z \\ \partial_t F &= \partial_x F \partial_t X + \partial_y F \partial_t y + \partial_z F \partial_t Z \end{aligned}$$

Exercise 9.

(1) $\begin{aligned} X(s,t) &= s^2 + t^2 \\ Y(s,t) &= s^2 - t^2 \\ Z(s,t) &= 2st \end{aligned} \qquad \begin{aligned} \partial_s F &= 2s(\partial_x F + \partial_y F) + 2t\partial_z F \\ \partial_t F &= 2t(\partial_x F - \partial_y F) + 2s\partial_z F \end{aligned}$

(2) $\begin{aligned} X(s,t) &= s + t \\ Y(s,t) &= s - t \\ Z(s,t) &= st \end{aligned} \qquad \begin{aligned} \partial_s F &= (\partial_x F + \partial_y F) + t\partial_z F \\ \partial_t F &= (\partial_x F - \partial_y F) + s\partial_z F \end{aligned}$

Exercise 10. Given $u = f(x, y)$; $\begin{aligned} x &= X(r,s,t) \\ y &= Y(r,s,t) \end{aligned} \quad u = F(r,s,t) \implies \begin{aligned} \partial_r F &= \partial_x F \partial_r x + \partial_y F \partial_r y \\ \partial_s F &= \partial_x F \partial_s x + \partial_y F \partial_s y \\ \partial_t F &= \partial_x F \partial_t x + \partial_y F \partial_t y \end{aligned}$

Exercise 11.

(1) Given $X(r, s, t) = r + s, Y(r, s, t) = t \implies \begin{aligned} \partial_r F &= \partial_x F \\ \partial_s F &= \partial_x F \\ \partial_t F &= \partial_y F \end{aligned}$

(2) Given $X(r, s, t) = r + s + t, Y(r, s, t) = r^2 + s^2 + t^2 \implies \begin{aligned} \partial_r F &= \partial_x F + \partial_y F(2r) \\ \partial_s F &= \partial_x F + 2s\partial_y F \\ \partial_t F &= \partial_x F + 2t\partial_y F \end{aligned}$

(3) Given $X(r, s, t) = r/s, Y(r, s, t) = s/t \implies \begin{aligned} \partial_r F &= \frac{1}{s}\partial_x F \\ \partial_s F &= \partial_x F(-r/s^2) + \partial_y F/t \\ \partial_t F &= \partial_y F(-s/t^2) \end{aligned}$

Exercise 12. $h(x) = f(g(x)) \qquad g = (g_1, \ldots, g_n)$

$$\nabla h(a) \implies \partial_k h(a) = \sum_{l=1}^{n} \partial_l f \partial_k g_l \text{ or } \nabla h(a) = \sum_{l=1}^{n} \partial_l f \nabla g_l$$

Exercise 13.

(1) $f(x, y, z) = xi + yj + zk$

$$Df(x) = \begin{bmatrix} \nabla f_x(x) \\ \nabla f_y(x) \\ \nabla f_z(x) \end{bmatrix} = \begin{bmatrix} \partial_x f_x & \partial_y f_x & \partial_z f_x \\ \partial_x f_y & \partial_y f_y & \partial_z f_y \\ \partial_x f_z & \partial_y f_z & \partial_z f_z \end{bmatrix} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$$

(2) $f = (x + c_x, y + c_y, z + c_z)$ where c_z, c_y, c_z are constants.

(3)

$$Df(x) = \begin{bmatrix} \nabla f_x(x) \\ \nabla f_y(x) \\ \nabla f_z(x) \end{bmatrix} = \begin{bmatrix} \partial_x f_x & \partial_y f_x & \partial_z f_x \\ \partial_x f_y & \partial_y f_y & \partial_z f_y \\ \partial_x f_z & \partial_y f_z & \partial_z f_z \end{bmatrix} = \begin{bmatrix} p(x) & & \\ & q(y) & \\ & & r(z) \end{bmatrix}$$

$$\implies f(x) = ((\int p(x)dx + x_0), (\int q(y)dy + y_0), (\int r(z)dz + z_0))$$

where x_0, y_0, z_0 are constants.

Exercise 14. Given $f : \mathbb{R}^2 \to \mathbb{R}^2, g : \mathbb{R}^3 \to \mathbb{R}^2 \quad \begin{aligned} f(x,y) &= (e^{x+2y}, \sin(y+2x)) \\ g(u,v,m) &= ((u + 2v^2 + 3w^3), (2v - u^2)) \end{aligned}$

(1) $Df(x, y), Dg(u, v, w) \implies Df = \begin{bmatrix} e^{x+2y} & 2e^{x+2y} \\ 2\cos(y+2x) & \cos(y+2x) \end{bmatrix} \qquad Dg = \begin{bmatrix} 1 & 4v & 9w^2 \\ -2u & 2 & 0 \end{bmatrix}$

(2) $h(u,v,w) = f(g(u,v,w))$

$$f(g(u,v,w)) = (e^{u+2v^2+3w^3+2(2v-u^2)}, \sin(2v-u^2+2(u+2v^2+3w^3))) =$$
$$= (e^{u+2v^2+3w^3+4v-2u^2}, \sin(2v-u^2+2u+4v^2+6w^3))$$

(3)

$$Dh = DfDg = \begin{bmatrix} e^{x+2y} & 2e^{x+2y} \\ 2\cos(y+2x) & \cos(y+2x) \end{bmatrix} \begin{bmatrix} 1 & 4v & 9w^2 \\ -2u & 2 & 0 \end{bmatrix} =$$
$$= \begin{bmatrix} e^{x+2y}(1-4u) & e^{x+2y}(4v+4) & 9w^2 e^{x+2y} \\ \cos(y+2x)(2-2u) & \cos(y+2x)(8v+2) & 18w^2\cos(y+2x) \end{bmatrix}$$
$$Dh(1,-1,1) = \begin{bmatrix} -3 & 0 & 9 \\ 0 & -6\cos 9 & 18\cos 9 \end{bmatrix}$$

Exercise 15. Given
$f = ((x^2+y+z), (2x+y+z^2))$
$g = (uv^2w^2, w^2\sin v, u^2e^v)$

(1) $Df = \begin{bmatrix} 2x & 1 & 1 \\ 2 & 1 & 2z \end{bmatrix}$ $\qquad Dg = \begin{bmatrix} v^2w^2 & 2uvw^2 & 2uv^2w \\ 0 & w^2\cos v & 2w\sin v \\ 2ue^v & u^2e^v & 0 \end{bmatrix}$

(2)

$$h(u,v,w) = f[g(u,v,w)] = ((uv^2w^2)^2 + w^2\sin v + u^2e^v, 2uv^2w^2 + w^2\sin v + u^4e^{2v}) =$$
$$= (u^2v^4w^4 + w^2\sin v + u^2e^v, 2uv^2w^2 + w^2\sin v + u^4e^{2v})$$

(3)

$$Dh(u,0,w) = \begin{bmatrix} 2x & 1 & 1 \\ 2 & 1 & 2z \end{bmatrix} \begin{bmatrix} v^2w^2 & 2uvw^2 & 2uv^2w \\ 0 & w^2\cos v & 2w\sin v \\ 2ue^v & u^2e^v & 0 \end{bmatrix} =$$
$$= \begin{bmatrix} 2xv^2w^2 + 2ue^v & 4xuvw^2 + w^2\cos v + u^2e^v & 4xuv^2w + 2w\sin v \\ 2v^2w^2 + 4zue^v & 4uvw^2 + w^2\cos v + 2zu^2e^v & 4uv^2w + 2w\sin v \end{bmatrix} = \begin{bmatrix} 2u & w^2+u^2 & 0 \\ 4(u^3) & w^2+2u^4 & 0 \end{bmatrix}$$

8.24 Miscellaneous exercises - Sufficient conditions for the equality of mixed partial derivatives

Exercise 2. $f = \frac{y(x^2-y^2)}{x^2+y^2}$.

$$f_1 = (y)\frac{2x(x^2+y^2)-(2x)(x^2-y^2)}{(x^2+y^2)^2} = \frac{4x^3y}{(x^2+y^2)^2}$$
$$f_2 = \frac{((x^2-y^2)-2y^2)(x^2+y^2)-2y^2(x^2-y^2)}{(x^2+y^2)^2} = \frac{x^4-4x^2y^2-y^4}{(x^2+y^2)^2}$$
$$D_{2,1}f = 4x^3\frac{(x^2+y^2)^2-2(x^2+y^2)(2y)y}{(x^2+y^2)^4} = \frac{x^4-4x^2y^2-y^4}{(x^2+y^2)^2}$$
$$D_{1,2}f = \frac{(4x^3-8xy^2)(x^2+y^2)^2-2(x^2+y^2)(2x)(x^4-4x^2y^2-y^4)}{(x^2+y^2)^4} =$$
$$= \frac{4x(x^2-2y^2)(x^2+y^2)-4x(x^4-4x^2y^2-y^4)}{(x^2+y^2)^3} = \frac{4xy^2(3x^2-y^2)}{(x^2+y^2)^3}$$

From the above results, clearly,

$$\lim_{x\to 0} f_1 = 0, \quad \lim_{y\to 0} f_1 = 0$$

So that $\boxed{f_1(0,0)=0}$, while

$$\lim_{x\to 0} f_2 = -1 \quad \lim_{y\to 0} f_2 = 1$$

so $f_2(0,0)$ undefined.

$$\lim_{x\to 0} f_{12} = 0 \quad \lim_{y\to 0} f_{12} = 0$$

so that $\boxed{f_{12}(0,0)=0}$, but

$$\lim_{x\to 0} f_{21} = 0 \quad \lim_{y\to 0} f_{21} = \frac{4x^3}{x^4} = \frac{4}{x} \xrightarrow{x\to 0} \infty$$

Exercise 3. Given $f(x,y) = \frac{xy^3}{x^3+y^6}$ if $(x,y) \neq (0,0)$, $f(0,0) = 0$

a.

$$f'(0;a) = \lim_{h\to 0} \frac{f(x+ha) - f(x)}{h} = \lim_{h\to 0} \frac{f(ha) - f(0)}{h} = \lim_{h\to 0} \left(\frac{ha_x h^3 a_y^3}{h^3 a_x^3 + h^6 a_y^6} \right) /h = \lim_{h\to 0} \frac{h^3 a_x a_y^3}{h^3 a_x^3 + h^6 a_y^6} =$$

$$= \lim_{h\to 0} \frac{a_x a_y^3}{a_x^3 + h^3 a_y^6} = \frac{a_y^3}{a_x^2}$$

So $f'(0;a) = \frac{a_y^3}{a_x^2}$ if $a_x \neq 0$, $f'(0;a) = 0$ if $a_x = 0$

b. If $x = y^2$, then

$$f(x,y) = \frac{xy^3}{x^3+y^6} = \frac{y^5}{2y^6} = \frac{1}{2y} \xrightarrow{y\to 0} \infty \text{ not } 0$$

So $f(x,y)$ is not continuous at $(0,0)$.

Exercise 4. $f(x,y) = \int_0^{\sqrt{xy}} e^{-t^2} dt$ for $x > 0$; $y > 0$.

Let $u = u(x,y) = \sqrt{xy}$ and then we can use chain rule.

$$\partial_x f = \partial_u f \partial_x u = \boxed{e^{-xy} \frac{1}{2} \sqrt{y/x}} \qquad \partial_y f = e^{-xy} \frac{1}{2} \sqrt{x/y} \text{ by label symmetry}$$

Exercise 5. Given $u = f(x,y)$; $\begin{aligned} x &= x(t) \\ y &= y(t) \end{aligned}$ and $u = F(t)$.

$$F'(t) = (\partial_x u)x' + (\partial_y u)y' = x'u_x + y'u_y$$
$$F''(t) = x''u_x + x'(x'u_{xx} + y'u_{yx}) + y''u_y + y'(x'u_{xy} + y'u_{yy}) = x''u_x + y''u_y + x'^2 u_{xx} + y'^2 u_{yy} + (x'y')(u_{yx} + u_{xy})$$

$$F'''(t) = \begin{aligned} & x''u_x + x''(x'u_{xx} + y'u_{yx}) + y'''u_y + y''(x'u_{xy} + y'u_{yy}) + \\ & + 2x'x''u_{xx} + 2y'y''u_{yy} + x'^2(x'u_{xxx} + y'u_{yxx}) + y'^2(x'u_{xyy} + y'u_{yyy}) + \\ & + (x''y' + x'y'')(u_{yx} + u_{xy}) + (x'y')(x'u_{xyx} + y'u_{yyx} + x'u_{xxy} + y'u_{yxy}) = \end{aligned}$$

$$= \begin{aligned} & x''u_x + y'''u_y + x'^3 u_{xxx} + y'^3 u_{yyy} + 3x''x'u_{xx} + 3y''y'u_{yy} + \\ & + 2x''y'u_{yx} + 2y''x'u_{xy} + x'y''u_{yx} + x''y'u_{xy} + \\ & + x'^2 y'u_{yxx} + y'^2 x'u_{xyy} + x'^2 y'u_{xyx} + x'y'^2 u_{yxy} + x'y'^2 u_{yyx} + x'^2 y'u_{xxy} \end{aligned}$$

Exercise 6. Given $\begin{aligned} x &= u+v \\ y &= uv^2 \end{aligned}$ $\quad f(x,y)$ into $g(u,v)$, and

$$\frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial^2 x} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = 1$$

So

$$\begin{aligned} \partial_u y &= v^2 & \qquad \partial_u g &= \partial_x f \partial_u x + \partial_y f \partial_u y = \partial_x f + v^2 \partial_y f \\ \partial_u x &= 1 & \qquad \partial_v g &= \partial_x f \partial_v x + \partial_y f \partial_v y = \partial_x f(1) + 2vu \partial_y f = \partial_x f + 2vu \partial_y f \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 g}{\partial v \partial u} &= \partial_v(\partial_x f) + 2v\partial_y f + v^2 \partial_v(\partial_y f) = \\ &= (\partial_{xx}^2 f + 2vu\partial_{yx}^2 f) + 2v(\partial_y f) + v^2(\partial_{xy}^2 f + 2vu\partial_{yy}^2 f) = \\ &= (\partial_{xx}^2 f + 2vu\partial_{yx}^2 f + v^2\partial_{xy}^2 f + 2vu\partial_{yy}^2 f + 2v(\partial_y f) \end{aligned}$$

So for $u = 1$, $v = 1$

$$\frac{\partial^2 g}{\partial v \partial u} = 1 + 2 + 1(1) + 2(1)(1)(1) + 2(1)(1) = \boxed{8}$$

Exercise 7. Given $\begin{aligned} x &= uv \\ y &= \frac{1}{2}(u^2 - v^2) \end{aligned}$

(1) Assume equality of mixed partials.

$$\begin{array}{lll} \partial_u x = v & \partial_u y = u & \dfrac{\partial g}{\partial u} = v\partial_x f + u\partial_y f \\ \partial_v x = u & \partial_v y = -v & \dfrac{\partial g}{\partial v} = u\partial_x f - v\partial_y f \end{array}$$

$$\begin{aligned} \partial_u \partial_v g &= \partial_x f + u\partial_u(\partial_x f) + -v\partial_u(\partial_y f) = \\ &= \partial_x f + u(v\partial^2_{xx} f + u\partial^2_{yx} f) + -v(v\partial^2_{xy} f + u\partial^2_{yy} f) = \\ &= \boxed{\partial_x f + uv\partial^2_{xx} f + 2y\partial^2_{xy} f - x\partial^2_{yy} f} \end{aligned}$$

(2) Given $\|\nabla f(x,y)\|^2 = (\partial_x f)^2 + (\partial_y f)^2 = 2$

$$a\left(\frac{\partial g}{\partial u}\right)^2 + -b\left(\frac{\partial g}{\partial v}\right)^2 = u^2 + v^2 = a(v^2(\partial_x f)^2 + u^2(\partial_y f)^2 + 2vu\partial_x f\partial_y f) + -b(u^2(\partial_x f)^2 + v^2(\partial_y f)^2 - 2uv\partial_x f\partial_y f)$$

$$\Longrightarrow a = -b \text{ since } u, v \text{ are independent}$$

$$(\partial_x f)^2(av^2 + au^2) + (\partial_y f)^2(au^2 + av^2) = a((\partial_x f)^2 + (\partial_y f)^2)(u^2 + v^2) = u^2 + v^2$$

$$\Longrightarrow \boxed{a = 1/2}$$

Exercise 8. Given that

$$(F(x) + G(y))^2 e^{z(x,y)} = 2F'(x)G'(y); \quad F(x) + G(y) \neq 0 \text{ or } e^{z(x,y)} = \frac{2F'(x)G'(y)}{(F+G)^2}$$

so

$$\begin{aligned} z(x,y) &= \ln(2F'G'/(F+G)^2) = \ln F' + \ln G' - 2\ln(F+G) \\ \partial_x z &= \frac{1}{F'}F'' - \frac{2}{F+G}F' \\ \Longrightarrow \partial^2_{yx} z &= \frac{-2F'G'}{F+G} = -e^{z(x,y)} \neq 0 \end{aligned}$$

Exercise 9.

Exercise 11.

$$\begin{aligned} (\nabla f)_i &= \partial_i((r\times A)_j (r\times B)_j) = \partial_i(\epsilon_{jkl}x_k A_l)(\epsilon_{jmn}x_m B_n) = \\ &= \epsilon_{jik}A_k\epsilon_{jmn}x_m B_n + \epsilon_{jkl}x_k A_l \epsilon_{jim}B_m = \\ &= \epsilon_{ijk}A_j\epsilon_{kmn}x_m B_n + \epsilon_{ijk}B_j\epsilon_{klm}x_l A_m \end{aligned}$$

So for $f(x,y,z) = (r\times A)\cdot(r\times B)$,

$$\boxed{\nabla f(x, yz) = B\times(r\times A) + A\times(r\times B)}$$

Exercise 12.

(1)

$$\partial_i\left(\frac{1}{r}\right) = \frac{-1}{r^2}\frac{1}{2}\left(\frac{2x_i}{r}\right) = \frac{-x_i}{r^3}$$

$$\boxed{A\cdot\nabla\left(\frac{1}{r}\right) = \frac{-A\cdot r}{r^3}}$$

(2)

$$\partial_i\left(\frac{-a_j x_j}{r^3}\right) = \frac{-a_i}{r^3} + -a_j x_j\left(\frac{-3}{r^4}\right)\left(\frac{x_i}{r}\right) = \frac{-a_i}{r^3} + \frac{3a_j x_j x_i}{r^5}$$

$$\boxed{B\cdot\nabla\left(A\cdot\nabla\left(\frac{1}{r}\right)\right) = \frac{-A\cdot B}{r^3} + \frac{3(A\cdot x)(x\cdot B)}{r^5}}$$

Exercise 13.

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = 1 \text{ or } x^2 + y^2 + z^2 - 2xa - 2by - 2zc + a^2 + b^2 + c^2 = 1$$

$$\text{consider pts. of intersection, with } x^2 + y^2 + z^2 = 1 \Longrightarrow 2xa + 2by + 2zc = a^2 + b^2 + c^2$$

$$\text{Let } f(x,y,z) = x^2+y^2+z^2-1 \quad \Longrightarrow \nabla f = 2(x,y,z)$$
$$\text{Let } g(x,y,z) = (x-a)^2+(y-b)^2+(z-c)^2-1 \quad \Longrightarrow \nabla g = 2(x-a,y-b,z-c)$$
$$\text{orthogonality condition: } \nabla g \cdot \nabla f = 0 = x(x-a)+y(y-b)+(z-c)z = 1-ax-by-cz =$$
$$= 1 - \left(\frac{a^2+b^2+c^2}{2} \right) = 0$$
$$\Longrightarrow \boxed{2 = a^2+b^2+c^2}$$

$2 = a^2+b^2+c^2$ describes a sphere of radius $\sqrt{2}$ and center at the origin.

Exercise 14.

$$z^2+2xz+y = 0 \quad \Longrightarrow \quad \begin{aligned} z &= \frac{-2x \pm \sqrt{4x^2-4(1)(y)}}{2(1)} = \\ &= -x \pm \sqrt{x^2-y} \end{aligned}$$

Consider parametrizing the position vector for the surface by the x coordinate:

$$r = (x, f(x), -x \pm \sqrt{x^2-y})$$
$$r' = (1, f', -1 \pm \frac{1}{2\sqrt{x^2-y}}(2x-y'))$$

Now consider the position vector for points contained in the "cylinder." Note that the z coordinate does not depend upon x for this cylinder because it looks the same in each $x-y$ plane for each z coordinate.

$$r = (x,y,z)$$
$$r' = (1,f',0)$$

These two tangent vectors must coincide (since the x coordinate and y coordinate are the same, 1 and f', respectively).

$$0 = -1 \pm \frac{1}{2\sqrt{x^2-y}}(2x-y') \text{ or } 4 = \frac{4x^2-4xy'+y'^2}{x^2-y}$$
$$\Longrightarrow 4x^2-4y = 4x^2-4xy'+y'^2 \text{ or } y'^2-4xy'+4y = 0$$
$$\Longrightarrow y' = \frac{4x \pm \sqrt{16x^2-4(1)(4y)}}{2} = 2x \pm 2\sqrt{x^2-y}$$

A solution to this ordinary differential equation is $\boxed{y = x^2}$

9.3 Exercises - Partial differential equations, A first-order partial differential equation with constant coefficients

Exercise 1. $4\partial_x f + 3\partial_y f = 0$

$$g(3x-4y) = f(x,y)$$
$$f(x,0) = \sin x = g(3x) \Longrightarrow \boxed{g(3x-4y) = \sin{(x - \frac{4}{3}y)}}$$

Exercise 2. $5\partial_x f - 2\partial_y f = 0$

$$g(2x+5y) = f(x,y)$$
$$\partial_x f(x,0) = 2\, g'(2x+5y)|_{y=0} = 2g'(2x) = e^x \text{ or } g(u) = e^{u/2}+C$$
$$\boxed{g(2x+5y) = e^{(x+\frac{5}{2}y)} + -1}$$

Exercise 3.

(1) If $u(x,y) = f(xy)$, then consider that $xy = $ const. represent level curves for f (because if $f(xy) = f(const.)$, then, "obviously," $f(const.) = $ another constant.

Parametrize r by x

$$r = (x,y) = (x, \frac{+k}{m}); \quad r' = (1, \frac{-k}{x^2}) = (1, \frac{-y}{x}) \text{ where } y = \frac{k}{x}$$
$$(\nabla u) \cdot r' = \partial_x u + \frac{-y}{x}\partial_y u = 0 \text{ or } x\partial_x u - y\partial_y u = 0$$

$$u(x,x) = x^4 e^{x^2} \qquad \partial\, x$$

$$u(x,x) = f(xx) = f(x^2) = (x^2)^2 e^{x^2} \implies \boxed{f(xy) = (xy)^2 e^{xy}}$$

(2) $v(x,y) = f\left(\frac{x}{y}\right)$ for $y \neq 0$
$\frac{x}{y} =$ const., then f const. $\nabla f \cdot r' = 0$ on these level curves with $\frac{x}{y} =$ const.; $r = (x,y) = (x, \frac{x}{k})$; $r' = (1, 1/k)$ or $(1, \frac{y}{x})$

$$\implies \partial_x v + \frac{y}{x}\partial_y v = 0 \text{ or } x\partial_x v + y\partial_y v = 0$$

$$\partial_x v(x, 1/x) = 1/x; \quad \partial_x v = \frac{1}{y} f'\left(\frac{x}{y}\right) \xrightarrow{y=1/x} xf'(x^2) = \frac{1}{x} \text{ or } f'(x^2) = \frac{1}{x^2} \text{ or } f(x) = \ln x + C$$

$$f\left(\frac{x}{y}\right) = \ln\frac{x}{y} + C = v(x,y)$$

$$\text{Since } v(1,1) = 2 \implies \boxed{v(x,y) = \ln\frac{x}{y} + 2}$$

Exercise 4. $\frac{\partial^2 g(x,y)}{\partial x \partial y} = 0$
$\partial_y g(x,y) = \psi_2(y)$ for $\partial^2_{xy} g = 0$; $\quad g(x,y) = \phi_2(y) + \phi_1(x)$ for $\phi_2'(y) = \psi_2(y)$
Exercise 5. $a = 1, \;\; b = -2, \;\; c = -3$

Consider the general problem: $a\partial^2_{xx} f + b\partial^2_{xy} f + c\partial^2_{yy} f = 0 \qquad \begin{matrix} x = Au + Bv \\ y = Cu + Dv \end{matrix}, \qquad g(u,v) = f(Au + Bv, Cu + Dv)$

$\frac{\partial^2 g}{\partial u \partial v} = 0$ (assume equality of mixed partials)

$$\begin{aligned} \partial_v g &= B\partial_x f + D\partial_y f & \partial^2_{uv} g &= B(A\partial_{xx} f + C\partial_{yx} f) + D(A\partial_{xy} f + C\partial_{yy} f) = 0 = \\ \partial_u g &= A\partial_x f + C\partial_y f & &= AB\partial^2_{xx} f + (BC + DA)\partial^2_{xy} f + DC\partial^2_{yy} f = 0 \end{aligned}$$

$$\implies \begin{aligned} AB &= a \\ BC + DA &= b \\ DC &= c \end{aligned}$$

$$\partial^2_{uv} g = 0 \implies \partial_v g = h(v) \text{ or } g = H(v) + l(u)$$

$$\begin{aligned} g(u,v) &= H_1(v) + l_1(u) = H_1\left(\frac{Cx - Ay}{BC - AD}\right) + l_1\left(\frac{Dx - By}{AD - BC}\right) \\ &= H(Cx - Ay) + l(Dx - By) \end{aligned}$$

Exercise 6. $u(x,y) = xyf\left(\frac{x+y}{xy}\right)$

$$x^2\partial_x u + -y^2\partial_y u = G(x,y)u \implies \begin{aligned} \partial_x u(x,y) &= yf\left(\frac{x+y}{xy}\right) + xyf'\left(\frac{1}{y} + \frac{1}{x}\right)\left(\frac{-1}{x^2}\right) = yf\left(\frac{x+y}{xy}\right) + \frac{-1}{x}yf'\left(\frac{1}{y} + \frac{1}{x}\right) \\ \partial_y u(x,y) &= xf\left(\frac{x+y}{xy}\right) + xyf'\left(\frac{x+y}{xy}\right)\left(\frac{-1}{y^2}\right) \end{aligned}$$

$$\implies x^2\partial_x - y^2\partial_y u = x^2yf - xyf' - y^2xf + xyf' = (x-y)u \quad \boxed{G = x - y}$$

Exercise 7. $\begin{matrix} x = e^s \\ y = e^t \end{matrix} \qquad f(x,y) \implies g(s,t) \qquad g(x,t) = f(e^s, e^t)$

$$x^2\partial^2_{xx} f + y^2\partial^2_{yy} f + x\partial_x f + y\partial_y f = 0$$

$$\begin{aligned} \partial_s g &= x\partial_x f & \partial_t g &= y\partial_y f \\ \partial^2_{ss} g &= x\partial_x f + x(x\partial^2_{xx} f) & \partial^2_{tt} g &= y\partial_y f + y^2\partial^2_{yy} f \end{aligned}$$

$$\boxed{\partial^2_{ss} g + \partial^2_{tt} g = x\partial_x f + x^2\partial^2_{xx} f + y\partial_y f + y^2\partial^2_{yy} f = 0}$$

Exercise 8. $f(tx) = t^p f(x) \quad \forall\, t > 0, \forall\, x \in S$ s.t. $tx \in S$ For fixed x, define $g(t) = f(tx)$;

$$\begin{aligned} g(t) &= f(tx) = t^p f(x) \\ g'(t) &= pt^{p-1} f(x) \end{aligned} \quad \implies g'(1) = pf(x) = f'(x) = (\nabla f) \cdot x \quad \text{(by definition of total derivative)}$$

Exercise 9. Given $g(t) = f(tx) - t^p f(x)$, note that we want $g(t) = 0$.

$$g'(t) = \frac{d}{dt} f(tx) - pt^{p-1} f(x)$$

It is very **useful** to recall the *total derivative* definition.

$$\frac{d}{dt} f(tx) = \lim_{\Delta t \to 0} \frac{f((t+\Delta t)x) - f(tx)}{\Delta t} = \lim_{\Delta t \to 0} \frac{f(tx + \Delta tx) - f(tx)}{\Delta t} = ((\nabla f)(tx)) \cdot x$$

Use the fact that we're given: $x \cdot (\nabla f)(x) = pf(x)$, so that $tx \cdot \nabla f(tx) = pf(tx)$

$$\begin{aligned} g'(t) &= x \cdot (\nabla f)(tx) - pt^{p-1} f(x) = \frac{p}{t}(f(tx) - t^p f(x)) = \\ &= \frac{p}{t} g(t) \end{aligned} \qquad \Longrightarrow \qquad \begin{aligned} \frac{g'}{g} &= \frac{p}{t} \\ \ln g &= p \ln t + C \\ g &= Kt^p \end{aligned}$$

Now $g(1) = 0$ (by plugging into the given $g(t) = f(tx) - t^p f(x)$). But $g(1) = 0$ if $K = 0$
$\Longrightarrow g = 0 \quad \forall t$

Exercise 10.

$$\begin{aligned} g'(1) &= pf = x\partial_x f + y \partial_y f \\ g''(1) &= p(x\partial_x f + y\partial_y f) = x\partial_x(x\partial_x f + y\partial_y f) + y\partial_y(x\partial_x f + y \partial_y f) = \\ &= x(\partial_x f + x\partial^2_{xx} f + y\partial^2_{xy} f) + y(x\partial^2_{yx} f + \partial_y f + y\partial^2_{yy} f) = x\partial_x f + x^2 \partial^2_{xx} f + 2xy\partial^2_{xy} f + y\partial_y f + y^2 \partial^2_{yy} f \\ &\Longrightarrow x^2\partial^2_{xx} f + 2xy\partial^2_{xy} f + y^2\partial^2_{yy} f + (pf) = p^2 f \text{ or } \boxed{x^2\partial^2_{xx} f + 2xy\partial^2_{xy} f + y^2\partial^2_{yy} f = p(p-1)f} \end{aligned}$$

9.5 EXERCISES - THE ONE-DIMENSIONAL WAVE-EQUATION

Exercise 4.

$$\begin{aligned} \partial^2_{xx} f &= \frac{1}{r}\partial_r g + \frac{-x}{r^2}\left(\frac{x}{r}\right)\partial_r g + \frac{x}{r}\left(\frac{x}{r}\partial^2_{rr} g + \frac{-y}{r^2}\partial^2_{\theta r} g\right) + \frac{2y}{r^3}\left(\frac{x}{r}\right)\partial_\theta g + \frac{-y}{r^2}\left(\frac{x}{r}\partial^2_{r\theta} g + \frac{-y}{r^2}\partial^2_{\theta\theta} g\right) \\ \partial^2_{yy} f &= \frac{1}{r}\partial_r g + \frac{-y}{r^2}\left(\frac{y}{r}\right)\partial_r g + \frac{y}{r}\left(\frac{y}{r}\partial^2_{rr} g + \frac{x}{r^2}\partial^2_{\theta r} g\right) + \frac{-2x}{r^3}\left(\frac{y}{r}\right)\partial_\theta g + \frac{x}{r^2}\left(\frac{y}{r}\partial^2_{r\theta} g + \frac{x}{r^2}\partial^2_{\theta\theta} g\right) \end{aligned}$$

$$\begin{aligned} \partial^2_{xx} f + \partial^2_{yy} f &= \frac{2}{r}\partial_r g - \frac{\partial_r g}{r} + \partial^2_{rr} g + \frac{1}{r^2}\partial^2_{\theta\theta} g = \frac{1}{r}\partial_r g + \partial^2_{rr} g + \frac{1}{r^2}\partial^2_{\theta\theta} g = \\ &= \frac{1}{r}\partial_r(r\partial_r g) + \frac{1}{r^2}\partial^2_{\theta\theta} g \end{aligned}$$

Exercise 5. We want for

$$\begin{aligned} x &= \rho\cos\theta\sin\phi \\ y &= \rho\sin\theta\sin\phi \qquad f(x,y,z) \to F(\rho,\theta,\phi) \\ z &= \rho\cos\phi \end{aligned}$$

But first, consider $\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}$ so that $f(x,y,z) \to g(r,\theta,z)$

(1)

$$\nabla^2 f = \frac{1}{r}\partial_r(r\partial_r g) + \frac{1}{r^2}\partial^2_{\theta\theta} g + \partial^2_{zz} g = \frac{\partial_r g}{r} + \partial^2_{rr} g + \frac{1}{r^2}\partial^2_{\theta\theta} g + \partial^2_{zz} g$$

(2) $\begin{aligned} z &= \rho\cos\phi \\ r &= \rho\sin\phi \end{aligned}$, so

$$\frac{1}{r^2}\partial^2_{\theta\theta} g = \frac{1}{\rho^2\sin^2\phi}\partial^2_{\theta\theta} g$$

Note that, except for a change in notation, this transformation is the same as that used in (a).

$$\partial^2_{zz} g + \partial^2_{rr} g = \frac{1}{\rho}\partial_\rho(\partial\partial_\rho g) + \frac{1}{\rho^2}\partial^2_{\phi\phi} g$$

$$\frac{1}{r}\partial_r g = \frac{1}{\rho\sin\phi}\left(\rho\sin\phi\partial_\rho g + \frac{\rho\cos\phi}{\rho^2}\partial_\phi g\right) = \frac{1}{\rho}\partial_\rho g + \frac{\cos\phi}{\rho^2\sin\phi}\partial_\phi g$$

$$\nabla^2 f = \partial^2_{\rho\rho} F + \frac{2}{\rho}\partial_\rho F + \frac{1}{\rho^2}\partial^2_{\phi\phi} F + \frac{\cos\phi}{\rho^2\sin\phi}\partial_\rho F + \frac{1}{\rho^2\sin\phi}\partial^2_{\theta\theta} g$$

Exercise 1.

$$\begin{aligned} x+y &= uv & x &= X(u,v) \\ xy &= u-v & y &= Y(u,v) \end{aligned} \qquad \partial_u : \begin{aligned} x_u + y_u &= v \\ x_u y + x y_u &= 1 \end{aligned} \qquad \partial_v : \begin{aligned} x_v + y_v &= u \\ x_v y + xy + v &= -1 \end{aligned}$$

$$\begin{bmatrix} 1 & 1 \\ y & x \end{bmatrix} \begin{bmatrix} x_u \\ y_u \end{bmatrix} = \begin{bmatrix} v \\ 1 \end{bmatrix} \qquad \frac{1}{x-y} \begin{bmatrix} x & -1 \\ -y & 1 \end{bmatrix} \begin{bmatrix} v \\ 1 \end{bmatrix} = \begin{pmatrix} xv - 1 \\ -vy + 1 \end{pmatrix} \left(\frac{1}{x-y} \right)$$

$$\begin{bmatrix} 1 & 1 \\ y & x \end{bmatrix} \begin{bmatrix} x_v \\ y_v \end{bmatrix} = \begin{bmatrix} u \\ -1 \end{bmatrix} \qquad \frac{1}{x-y} \begin{bmatrix} x & -1 \\ -y & 1 \end{bmatrix} \begin{bmatrix} u \\ -1 \end{bmatrix} = \begin{bmatrix} ux + 1 \\ -uy - 1 \end{bmatrix} \frac{1}{x-y}$$

$$\boxed{\begin{aligned} x_u &= \frac{1}{x-y}(xv-1) & x_v &= \frac{1}{x-y}(ux+1) \\ y_u &= \frac{-vy+1}{x-y} & y_v &= \frac{1}{x-y}(-uy-1) \end{aligned}}$$

Exercise 5. Given

$$\begin{aligned} & F(u,v) = 0 & & \partial_x u = y & & \partial_x v = \frac{1}{2\sqrt{x^2+z^2}}(2x) = \frac{x}{\sqrt{x^2+z^2}} \\ & u = u(x,y,z) = xy & & \partial_y u = x & & \partial_y v = 0 \\ & v = v(x,y,z) = \sqrt{x^2+z^2} & & \partial_z u = 0 & & \partial_z v = \frac{z}{\sqrt{x^2+z^2}} \end{aligned}$$

$$\begin{aligned} \partial_x f &= \partial_x u \partial_u F + \partial_x v \partial_v F = y \partial_u F + \frac{x}{v} \partial_v F \\ \partial_y f &= \partial_y u \partial_u F + \partial_y v \partial_v F = x \partial_u F \\ \partial_z f &= \partial_z u \partial_u F + \partial_z v \partial_v F = \frac{z}{v} \partial_v F \end{aligned}$$

Since $F = f = 0$, $\nabla f \cdot R' = 0$, so ∇f is a normal vector to this surface.
We're given

$$x = 1,\ y = 1,\ z = \sqrt{3} \qquad \begin{aligned} D_1 F(1,2) &= 1 \\ D_2 F(1,2) &= 2 \end{aligned}$$

so then

$$\begin{aligned} \nabla f &= (y\partial_u F + \frac{x}{v}\partial_v F, x\partial_u F, \frac{z}{v}\partial_v F) = \\ &= (1 + \frac{1}{2}2, 1(1), \frac{\sqrt{3}}{2}2) = \boxed{(2,1,\sqrt{3})} \\ \Longrightarrow & \frac{(2,1,\sqrt{3})}{2\sqrt{2}} = \boxed{\left(\frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}, \frac{1}{2}\sqrt{\frac{3}{2}} \right)} \end{aligned}$$

Exercise 6.

$$\begin{aligned} & x^2 - y\cos(uv) + z^2 = 0 & & x = x(u,v) \\ & x^2 + y^2 - \sin(uv) + 2z^2 = 2 & & y = y(u,v) \\ & xy - \sin u \cos v + z = 0 & & z = z(u,v) \end{aligned} \qquad \text{we want } \begin{aligned} \partial_u x \\ \partial_v x \end{aligned} \text{ at } \begin{aligned} x = y &= 1 \\ u &= \pi/2 \\ v &= 0 \\ z &= 0 \end{aligned}$$

$$\partial_u : \begin{aligned} & 2xx_u - y_u \cos(uv) + y\sin(uv)v + 2zz_u = 0 \\ & 2xx_u + 2yy_u - \cos(uv)(v) + 4zz_u = 0 \\ & x_u y + xy_u - \cos u \cos v + z_u = 0 \end{aligned} \qquad \text{or}$$

$$(2x, 2x, y)x_u + (-\cos(uv), 2y, x)y_u + (2z, 4z, 1)z_u = (-y\sin(uv)v, v\cos(uv), \cos u \cos v)$$

$$x_u = \frac{\begin{vmatrix} -y\sin(uv)v & v\cos(uv) & \cos u \cos v \\ -\cos(uv) & 2y & x \\ 2z & 4z & 1 \end{vmatrix}}{\begin{vmatrix} 2x & 2x & y \\ -\cos(uv) & 2y & x \\ 2z & 4z & 1 \end{vmatrix}}$$

Note that $-y\sin(uv)v = v\cos(uv) = \cos u \cos v = 0$

$$\Longrightarrow x_u = 0$$

$$\begin{aligned} & 2xx_v - y_v\cos(uv) + y\sin(uv)u + 2zz_v = 0 \\ \partial_v : \ & 2xx_v + 2yy_v - \cos(uv)u + 4zz_v = 0 \qquad \text{or} \\ & x_v y + xy_v + \sin u \sin v + z_v = 0 \end{aligned}$$

$$(2x,2x,y)x_v + (-\cos(uv), 2y, x)y_v + (2z,4z,1)z_v = (-y\sin(uv)u, \cos(uv)u, -\sin u \sin v)$$

$$x_v = \frac{\begin{vmatrix} -yu\sin(uv) & u\cos(uv) & -\sin u\sin v \\ -\cos(uv) & 2y & x \\ 2z & 4z & 1 \end{vmatrix}}{\begin{vmatrix} 2x & 2x & y \\ -\cos(uv) & 2y & x \\ 2z & 4z & 1 \end{vmatrix}} = \frac{\begin{vmatrix} 0 & \pi/2 & 0 \\ -1 & 2 & 1 \\ 0 & 0 & 1 \end{vmatrix}}{\begin{vmatrix} 2 & 2 & 1 \\ -1 & 2 & 1 \\ 0 & 0 & 1 \end{vmatrix}}$$
$$= \boxed{\pi/12}$$

9.13 Exercises - Maxima, minima, and saddle points. Second-order Taylor formula for scalar fields. The nature of a stationary point determined by the eigenvalues of the Hessian matrix. Second-derivative test for extrema of functions of two variables.

Exercise 1. $z = x^2 + (y-1)^2$

$$\begin{aligned} & f(x,y) = x^2 + (y-1)^2 \\ & \nabla f = (2x, 2(y-1)) = 0 \text{ where } (x,y) = (0,1) \end{aligned} \quad f \geq 0 \text{ and } f = 0 \text{ when } (x,y) = (0,1)$$

$(0,1)$ is an abs. min.

Exercise 2. $z = x^2 - (y-1)^2$

$$\nabla f = (2x, -2(y-1)) = 0 \text{ when } (x,y) = (0,1) \quad \begin{aligned} & \text{For } (x,y) = (t,1),\ f(t,1) = t^2 \geq 0 \\ & \text{For } (x,y) = (0,1+\delta),\ f(0,1+\delta) = -\delta^2 < 0 \end{aligned}$$

So $(0,1)$ is a saddle pt.

Exercise 3. $z = 1 + x^2 - y^2$

$$\nabla f = (2x, -2y) = 0 \text{ when } (x,y) = (0,0) \quad \begin{aligned} & \text{For } (x,y) = (0,u),\ z = 1 + -u^2 \leq 1 \\ & \text{For } (x,y) = (t,0),\ z = 1 + t^2 \geq 1 \end{aligned}$$

So $(0,0)$ is a saddel pt.

Exercise 4. $z = (x-y+1)^2$

$$\begin{gathered} \nabla f = (2(x-y+1), 2(x-y+1)(-1)) = 0 \text{ when } y = x+1 \\ f \geq 0 \, \forall (x,y), \text{ so } (x,x+1) \text{ is an abs. min. since } f(x,x+1) = 0 \end{gathered}$$

Exercise 5. $z = 2x^2 - xy - 3y^2 - 3x + 7y$

$$\nabla f = (4x - y - 2, -x + 2y + 1) = 0 \text{ where } \begin{aligned} & y = 4x - 2 \\ & y = \frac{x-1}{2} \end{aligned} \text{ so } \begin{aligned} & x = 3/7 \\ & y = -2/7 \end{aligned}$$

$$H = \begin{bmatrix} 4 & -1 \\ -1 & 2 \end{bmatrix} \Longrightarrow (\lambda - 4)(\lambda - 2) - 1 = (\lambda - 7)(\lambda + 1)$$

So we have a saddle point at $(3/7, -2/7)$.

Exercise 6. For $z = x^2 - xy + y^2 - 2x + y$,

$$\nabla f = (2x - y - 2, -x + 2y + 1) = 0$$

so the critical point is at $(x,y) = (1,0)$.
The Hessian matrix is

$$H = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

So $\lambda = 1, +3$ are the eigenvalues. $(1,0)$ is a relative minimum.

Exercise 7. For $z = x^3 - 3xy^2 + y^3$,

$$\nabla f = (3x^2 - 3y^2, -6xy + 3y^2) = 3(x^2 - y^2, -2xy + y^2) = 0$$

Then because $y^2 - 2xy = 0$, $(x,y) = (0,0)$.

The Hessian matrix is

$$H = \begin{bmatrix} 6x & -6y \\ -6y & -6x+6y \end{bmatrix}$$

For $(x, y) = (0, 0)$, by theorem, the Hessian matrix doesn't give a definite conclusion. Then resort to the definitions of saddle points, relative minima, relative maxima.
Now $z = x^3 - 3xy^2 + y^3 = y^2(y - 3x) + x^3$. $z(0, 0) = 0$.
Consider $(x, y) = (\delta, \epsilon)$. Then

$z = \epsilon^2(\epsilon - 3\delta) + \delta^3$
So for $y = 3x$,

$\forall\, E > 0$, $\exists \delta > 0$ s.t.
if $0 < x < \delta$, then $0 < z < 2\delta = E(\delta)$,
if $-\delta < x < 0$, then $-E(\delta) = -2\delta^3 < z < 0$
So in the neighborhood of $(0, 0)$, there exist pts. above and below $z = 0$. By definition, $(0, 0)$ is a saddle point.
Exercise 8. For $z = x^2y^3(6 - x - y)$,

$$\nabla f = (12xy^3 - 3x^2y^3 - 2xy^4, 18x^2y^2 - 3x^3y^2 - 4x^2y^3) = 0$$

With

$$\begin{aligned} &12xy^3 - 3x^2y^3 - 2xy^4 = xy^3(12 - 3x - 2y) = 0 \\ &x^2y^2(18 - 3x - 4y) = 0 \end{aligned}$$

So $(x, y) = (2, 3)$ or $(x, 0)$, $(0, y)$ are the critical points.
The Hessian matrix is

$$H = \begin{bmatrix} 12y^3 - 6xy^3 - 2y^4 & 36xy^2 - 9x^2y^2 - 8xy^3 \\ 36xy^2 - 9x^2y^2 - 8xy^3 & 36x^2y - 6x^3y - 12x^2y^2 \end{bmatrix}$$

Now

$$\begin{aligned} A &= D_{1,1}f = 2y^3(6 - 3x - y) \\ B &= D_{1,2}f = xy^2(36 - 9x - 8y) \\ C &= D_{2,2}f = 6x^2y(6 - x - 2y) \end{aligned}$$

For $(2, 3)$, $\Delta < 0$. $(2, 3)$ is a saddle point.
Looking at the Hessian matrix, the definitions must be used to determine if the critical points are saddle points, relative maxima, or relative minima.
Consider $(x, 0)$, $z(x, 0) = 0$.
Consider $|y| < \delta_2$.
$z = y^3x^2(6 - x - y)$
Choose δ_2, s.t. $\delta_2 < |6 - x|$ (since δ_2 is arbitrarily small, we can make this choice).
Then for fixed x, either $6 - x < 0$, or $6 - x > 0$.
But for $|y| < \delta_2$, $y > 0$, or $y < 0$, either $z < 0$, $z > 0$, since y or $-y$ allowed, for $|y| < \delta_2$.

$$|z| = |y^3||x^2||6 - x - y| < \delta_2^3 2|6 - x|x^2 = E(x, \delta_2)$$

$\forall\, E > 0$, $\exists \delta_2 > 0$ s.t. for $|y| < \delta_2$, $|z| < E(x, \delta_2)$ and in this neighborhood, $\exists\,(x, y)$ s.t. $z < 0$ and (x, y) s.t. $z > 0$.
$(x, 0)$ are saddle points.
$(0, y)$, $z(0, y) = 0$.
For $|x| < \delta_1$,
$z = x^2y^3(6 - y - x)$.
Consider $\delta_1 < |6 - y|$.
For $y < 0$, $y > 6$, $z < 0$ for infinitesimal neighborhood about z (with $\delta_1 < |6 - y|$).
For $(0, y)$, $y < 0$, $y > 6$, $z(0, y)$ a relative minimum.
Likewise, for $0 < y < 6$, $z > 0$ for infinitesimal neighborhood about z, (with $\delta_1 < |6 - y|$), so $z(0, y)$ a relative maximum.
For $(0, 6)$, $z(x, 6) = 216x^2(-x) = -216x^3$. $\forall\, E > 0$, $\exists\, \delta_1 > 0$ s.t. $|z| < E$ when $|x| < \delta_1$,

For $|x| < \delta_1$, both $x, -x$ fulfill the condition, so that
$\exists$ pts. $(x, 6)$, $(-x, 6)$ in this neighborhood such that $z < 0$, $z > 0$, respectively.
$(0, 6)$ a saddle point.
Exercise 9. $z = x^3 + y^3 - 3xy$

$$\nabla f = (3x^2 - 3y, 3y^2 - 3x) = 0 \Longrightarrow \begin{matrix} x^2 = y \\ y^2 = x \end{matrix} \Longrightarrow (0,0), (1,1)$$

$$|Df| = \begin{vmatrix} 6x & -3 \\ -376y & \end{vmatrix} = 36xy - 9$$

$(1,1)$ minimum since $Df(1,1) = 27$ and $D_{1,1}f(1,1) = 6 > 0$

$(0,0)$ saddle pt. since $Df(0,0) = -9 < 0$

Exercise 10. $z = \sin x \cosh y$

$$\nabla f = (\cos x \cosh y, \sin x \sinh y) \qquad |Df| = \begin{vmatrix} -\sin x \cosh y & \cos x \sinh y \\ \cos x \sinh y & \sin x \cosh y \end{vmatrix} = -\sin^2 x \cosh^2 y + -\cos^2 x \sinh^2 y$$

$$\nabla f = 0 \Longrightarrow (x,y) = \left(\left(\frac{2j-1}{2} \right) \pi, 0 \right)$$

$$Df\left(\left(\frac{2j-1}{2} \right) \pi, 0 \right) = -1 \quad \Longrightarrow \left(\left(\frac{2j-1}{2} \right) \pi, 0 \right) \text{ is a saddle pt.}$$

Exercise 11. $z = e^{(2x+3y)}(8x^2 - 6xy + 3y^2) = fe^g$. It helps alot to make these notation substitutions.

$$\begin{aligned} f_x &= 16x - 6y \\ f_y &= -6x + 6y \end{aligned} \qquad \nabla z = (f_x e^g + 2fe^g, f_y e^g + 3fe^g)$$

$$\xrightarrow{\nabla z} \begin{aligned} (f_x + 2f)e^g &= 0 \\ (f_y + 3f)e^g &= 0 \end{aligned} \Longrightarrow (0,0),\ (\frac{-1}{4}, \frac{-1}{2})$$

$$D_{ij}z = \begin{vmatrix} 16e^g + 4f_x e^g + 4fe^g & -6e^g + 2f_y e^g + 3f_x e^g + 6fe^g \\ -6e^g + 3f_x e^g + 2f_y e^g + 6fe^g & 6e^g + 6f_y e^g + 9fe^g \end{vmatrix}$$

$$D_{ij}z(0,0) = \begin{vmatrix} 16 & -6 \\ -6 & 6 \end{vmatrix} = 96 - 36 = 60 \qquad (0,0) \text{ is a minimum}$$

$$D_{ij}z(\frac{-1}{4}, \frac{-1}{2}) = e^{-4} \begin{vmatrix} 16 & -9 \\ -6 & \frac{-3}{2} \end{vmatrix} = e^{-4}(-24 - 54) < 0 \qquad (\frac{-1}{4}, \frac{-1}{2}) \text{ is a saddle pt.}$$

Exercise 12. $z = (5x+7y-25)e^{-(x^2+xy+y^2)} = fe^{-g}$. Using these shorthand, substitution notation helps with the calculation.

$$\nabla z = (5e^{-g} + fe^{-g}(-2x - y), 7e^{-g} + fe^{-g}(-x - y))$$

$$\nabla z = 0 \Longrightarrow \begin{aligned} 5 + (-2x - y)f &= 0 \\ 7 + (-x - 2y)f &= 0 \end{aligned} \Longrightarrow \boxed{(x,y) = (1,3), \left(\frac{-1}{26}, \frac{-3}{26} \right)}$$

$$D_{ij}z =$$

$$e^{-g} \begin{bmatrix} 5(-2-y)2 + f(-2x-y)^2 + f(-2) & 7(-2x-y) + 5(-x-2y) + f(-x-2y)(-2x-y) - f \\ 7(-2x-y) + 5(-x-2y) + f(-x-2y)(-2x-y) - f & 7(-x-2y)2 + (-x-2y)^2 f + -2f \end{bmatrix}$$

$$D_{ij}z(1,3) = e^{-2g} \begin{vmatrix} -27 & -36 \\ -36 & -51 \end{vmatrix} > 0 \Longrightarrow (1,3) \text{ is a maximum}$$

$$D_{ij}z(\frac{-1}{26}, \frac{-3}{26}) = \begin{vmatrix} \frac{25}{26} + 52 & \frac{35}{26} + 26 \\ \frac{35}{26} + 26 & \frac{49}{26} + 52 \end{vmatrix} > 0 \Longrightarrow (\frac{-1}{26}, \frac{-3}{26}) \text{ is a minimum.}$$

Exercise 13. $z = \sin x \sin y \sin (x+y), \quad 0 \leq x \leq \pi, \ 0 \leq y \leq \pi$

Exercise 21. *Method of least squares.* Given n distinct numbers $x_1, \dots, x_n$ and n further numbers $y_1, \dots, y_n$, and $f(x) = ax + b$ fitting form,

$$E(a,b) = \sum_{i=1}^n (f(x_i) - y_i)^2 = \sum_{i=1}^n (ax_i + b - y_i)^2$$

$$\nabla E = 0 \Longrightarrow \begin{aligned} \partial_a E &= \sum_{i=1}^n 2(ax_i + b - y_i)x_i = 2\left(a \sum x_i^2 + b \sum x_i - \sum y_i x_i \right) = 0 \\ \partial_b E &= \sum_{i=1}^n 2(ax_i + b - y_i) = 2\left(a \sum x_i + nb + - \sum y_i \right) = 0 \end{aligned}$$

$$X^2 = \sum x_i^2$$
$$\overline{X} = \frac{1}{n}\sum x_i \qquad \Longrightarrow \begin{bmatrix} X^2 & n\overline{X} \\ n\overline{X} & n \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \sum y_i x_i \\ n\overline{Y} \end{bmatrix}$$
$$\overline{Y} = \frac{1}{n}\sum y_i$$

$$\Longrightarrow \begin{bmatrix} a \\ b \end{bmatrix} = \left(\frac{1}{nX^2 - n^2\overline{X}^2}\right) \begin{bmatrix} n & -n\overline{X} \\ -n\overline{X} & X^2 \end{bmatrix} \begin{bmatrix} \sum y_i x_i \\ n\overline{Y} \end{bmatrix} = \begin{bmatrix} n\sum y_i x_i - n^2\overline{XY} \\ -n\overline{X}\sum y_i x_i + nX^2\overline{Y} \end{bmatrix} \left(\frac{1}{nX^2 - n^2\overline{X}^2}\right)$$

Suppose
$$u_i = x_i - \overline{X}$$
$$u_i^2 = x_i^2 - 2x_i\overline{X} + \overline{X}^2 \qquad \Longrightarrow \sum u_i^2 = X^2 - 2n\overline{X}^2 + n\overline{X}^2 = X^2 - n\overline{X}^2$$
$$\sum y_i u_i = \sum y_i(x_i - \overline{X}) = \sum y_i x_i - \overline{XY}n$$

$$\Longrightarrow a = \sum y_i u_i / \sum u_i^2$$

Then use $an\overline{X} + nb - n\overline{Y} = 0$ or $b = \overline{Y} - a\overline{X}$ to get b.

Exercise 22. $f(x,y) = ax + by + c$. $E(a,b,c) = \sum_{i=1}^n (f(x_i,y_i) - z_i)^2 = \sum_{i=1}^n (ax_i + by_i + c - z_i)^2$. (x_i, y_i) are n given distinct pts. $z_1, \ldots, z_n$ are n given real numbers.

$$\partial_a E = 0 = 2\sum(ax_i + by_i + c - z_i)x_i \Longrightarrow \qquad a\sum x_i^2 + b\sum x_i y_i + c\sum x_i - \sum z_i x_i = 0$$
$$\partial_b E = 0 = 2\sum(ax_i + by_i + c - z_i)y_i \Longrightarrow \qquad a\sum x_i y_i + b\sum y_i^2 + c\sum y_i - \sum z_i y_i = 0$$
$$\partial_c E = 0 = 2\sum(ax_i + by_i + c - z_i) \Longrightarrow \qquad a\sum x_i + b\sum y_i + nc - \sum z_i = 0 \text{ or } c = \overline{Z} - a\overline{X} - b\overline{Y}$$

Then rewrite the above equations substituting the expression for c.

$$aX^2 + b\sum x_i y_i + cn\overline{X} = \sum z_i x_i = aX^2 + b\sum x_i y_i + n\overline{X}(\overline{Z} - a\overline{X} - b\overline{Y}) =$$
$$= a(X^2 - n\overline{X}^2) + b(\sum x_i y_i - n\overline{XY}) + n\overline{XZ}$$
$$a\sum x_i y_i + bY^2 + cn\overline{Y} = \sum z_i y_i = a(\sum x_i y_i - n\overline{XY}) + b(Y^2 - n\overline{Y}^2) + n\overline{YZ}$$

Then

$$\begin{bmatrix} X^2 - n\overline{X}^2 & \sum x_i y_i - n\overline{XY} \\ \sum x_i y_i - n\overline{XY} & Y^2 - n\overline{Y}^2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \sum z_i x_i - n\overline{XZ} \\ \sum z_i y_i - n\overline{YZ} \end{bmatrix} = \begin{bmatrix} \sum u_i z_i \\ \sum v_i z_i \end{bmatrix}$$

Note that for $u_i = x_i - \overline{X}$, $v_i = y_i - \overline{Y}$, we already showed in the previous exercise, Exercise 21, that
$$X^2 - n\overline{X}^2 = \sum u_i^2$$
$$Y^2 - n\overline{Y}^2 = \sum v_i^2$$

$$\sum u_i v_i = \sum (x_i - \overline{X})(y_i - \overline{Y}) = \sum x_i y_i - n\overline{XY} - n\overline{XY} + \overline{XY}n = \sum x_i y_i - n\overline{XY}$$

So let $\Delta = \begin{vmatrix} \sum u_i^2 & \sum u_i v_i \\ \sum u_i v_i & \sum v_i^2 \end{vmatrix}$ and use Cramer's rule to obtain

$$\boxed{a = \frac{1}{\Delta}\begin{vmatrix} \sum u_i z_i & \sum u_i v_i \\ \sum v_i z_i & \sum v_i^2 \end{vmatrix}}$$
$$\boxed{b = \frac{1}{\Delta}\begin{vmatrix} \sum v_i z_i & \sum u_i v_i \\ \sum u_i z_i & \sum u_i^2 \end{vmatrix}}$$
$$\boxed{c = \overline{Z} - a\overline{X} - b\overline{Y}}$$

Exercise 23. $z_1, \ldots, z_n$ are n distinct pts. in m-space.

Let $x \in \mathbb{R}^m$, let $f(x) = \sum_{k=1}^n \|x - z_k\|^2 = \sum_{k=1}^n \sum_{j=1}^m (x_j - (z_k)_j)^2$

$$\text{Now } \partial_i f = \sum_{k=1}^n 2(x_i - (z_k)_i)$$

$$\text{Conditions we want: } \nabla f = 0 \Longrightarrow \sum_{k=1}^n (x_i - (z_k)_i) = 0 \Longrightarrow nx_i - \sum_{k=1}^n (z_k)_i = 0$$

$$\Longrightarrow x_i = \frac{1}{n}\sum_{k=1}^n (z_k)_i \text{ or } x = \frac{1}{n}\sum_{k=1}^n z_k$$

$$\partial_{ij} f = \partial_i \sum_{k=1}^n 2(x_j - (z_k)_j) = \sum_{k=1}^n 2(1)\delta_{ij} = 2n\delta_{ij}$$

H is diagonalized and $\lambda_j > 0, \forall j = 1, \dot{;} m$. By Thm., $a = \frac{1}{n}\sum_{k=1}^n z_k$, (the centroid) is a minimum.

Exercise 25. $f(x,y,z) = x^4 + y^4 + z^4 - 4xyz$.

$$\nabla f = (4x^3 - 4yz, 4y^3 + -4xz, 4z^3 - 4xy) = 0 = (x^3 - yz, y^3 - xz, z^3 - xy) \qquad \Longrightarrow \begin{aligned} x^3 &= yz \\ y^3 &= xz \\ z^3 &= xy \end{aligned} \qquad \text{thus } \nabla f(1,1,1) = 0$$

$$H = \begin{bmatrix} 12x^2 & -4z & -4y \\ -4z & 12y^2 & -4x \\ -4y & -4x & 12z^2 \end{bmatrix} = 4\begin{bmatrix} 3x^2 & -z & -y \\ -z & 3y^2 & -x \\ -y & -x & 3z^2 \end{bmatrix} \qquad H(a) = 4\begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}$$

$$|\lambda I - H| = \begin{vmatrix} \lambda - 12 & 4 & 4 \\ 4 & \lambda - 12 & 4 \\ 4 & 4 & \lambda - 12 \end{vmatrix} = (\lambda - 16)^2(\lambda - 4) = 0$$

$$\Longrightarrow \lambda = 16, 4$$

$a = (1,1,1)$ is a minimum, by theorem.

9.15 Exercises - Extrema with constraints. Lagrange's multipliers

Exercise 1. Given $f(x,y) = z = xy$ and $g(x,y) = 0 = x + y - 1$

$$\begin{aligned} \nabla f &= (y,x) \\ \nabla g &= (1,1) \end{aligned} \quad \Longrightarrow \nabla f = \lambda \nabla g = (y,x) = \lambda(1,1) = (1-x, x)$$

$$\Longrightarrow 1 - x = x \text{ so that } \begin{aligned} x &= \frac{1}{2} \\ y &= \frac{1}{2} \end{aligned} \qquad \boxed{z = \frac{1}{4}}$$

Exercise 2.

$$\begin{aligned} f(x,y) &= r = \sqrt{x^2 + y^2} \\ g(x,y) &= 5x^2 + 6xy + 5y^2 - 8 \end{aligned} \qquad \begin{aligned} \nabla f &= \frac{(x,y)}{r} = \lambda(10x + 6y, 10y + 6x) \\ \nabla g &= (10x + 6y, 10y + 6x) \end{aligned}$$

$$\nabla f = \frac{(x,y)}{r} = \lambda(10x + 6y, 10y + 6x) \qquad \Longrightarrow \frac{x}{r}\left(\frac{1}{10x + 6y}\right) = \frac{y}{r}\left(\frac{1}{10y + 6x}\right) \text{ or } x^2 = y^2$$

$$g(x, \pm x) = 5x^2 \pm 6x^2 + 5x^2 - 8 = 0 \Longrightarrow 10x^2 \pm 6x^2 = 8 \text{ or } x = \pm\frac{1}{\sqrt{2}}, \pm\sqrt{2}$$

$$r \text{ maximum }, 2, \text{ when } (\sqrt{2}, \pm\sqrt{2}), (-\sqrt{2}, \pm\sqrt{2})$$

$$r \text{ minimum }, 1, \text{ when } (\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}), (\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}})$$

Exercise 3. $a, b > 0$

(1)

$$\begin{aligned} f = z &= \frac{x}{a} + \frac{y}{b} \\ \nabla f &= \left(\frac{1}{a}, \frac{1}{b}\right) \end{aligned} \qquad \begin{aligned} g(x,y) &= x^2 + y^2 = 1 \\ \nabla g &= (2x, 2y) \end{aligned}$$

$$\nabla f = \lambda \nabla g \Longrightarrow \begin{array}{l} \frac{1}{a} = \lambda 2x \\ \frac{1}{b} = \lambda 2y \end{array} \text{ or } \frac{y}{x} = \frac{a}{b} \qquad \text{so then } x^2\left(1 + \frac{a^2}{b^2}\right) = 1 \text{ or } \begin{array}{l} x = \frac{\pm b}{\sqrt{a^2+b^2}} \\ y = \frac{\pm a}{\sqrt{a^2+b^2}} \end{array}$$

$$\boxed{z = \frac{\sqrt{b^2+a^2}}{ab}, -\frac{\sqrt{b^2+a^2}}{ab}}$$

Geometrically, consider lines of $bz - \frac{b}{a}x = y$ inside a circular region of $x^2 + y^2 = 1$.

(2)

$$\begin{array}{l} f = z = x^2 + y^2 \\ \nabla f = 2(x,y) \end{array} \qquad \begin{array}{l} g(x,y) = \frac{x}{a} + \frac{y}{b} - 1 \\ \nabla g = \left(\frac{1}{a}, \frac{1}{b}\right) \end{array} \qquad \nabla f = \lambda \nabla g \Longrightarrow 2(x,y) = \lambda\left(\frac{1}{a}, \frac{1}{b}\right) \text{ or } \frac{a}{b} = \frac{y}{x}$$

$$\text{Plug back into } g(x,y)\text{:} \Longrightarrow \begin{array}{l} x = \frac{ab^2}{a^2+b^2} \\ y = \frac{ba^2}{a^2+b^2} \end{array} \qquad \text{minimum at } \left(\frac{ab^2}{b^2+a^2}, \frac{ba^2}{a^2+b^2}\right) \qquad z = \frac{a^2b^2}{a^2+b^2}$$

Geometrically, consider points on a line defined by $g(x,y)$, $y = b - \frac{bx}{a}$. Then f defines circles of increasing radius. Obviously, we can make the radius for f, z, as large as we want.

Exercise 4.

Exercise 5.

$$\begin{array}{l} f(x,y,z) = x - 2y + 2z \\ \nabla f = (1,-2,2) \end{array} \qquad \begin{array}{l} g = x^2 + y^2 + z^2 - 1 = 0 \\ \nabla g = (2x, 2y, 2z) \end{array} \xrightarrow{\nabla f = \lambda \nabla g} \frac{1}{2x} = \frac{-1}{y} = \frac{1}{z}$$

$$\begin{array}{l} 2x = z \\ z = -y \end{array} \Longrightarrow z = \frac{\pm 2}{3}$$

$$f\left(\frac{1}{3}, \frac{-2}{3}, \frac{2}{3}\right) = 3$$

$$f\left(\frac{-1}{3}, \frac{2}{3}, \frac{-2}{3}\right) = -3$$

Exercise 6.

$$\begin{array}{l} f = \sqrt{x^2+y^2+z^2} \\ \nabla f = \frac{(x,y,z)}{4} \end{array} \qquad \begin{array}{l} g = z^2 - xy - 1 \\ \nabla g = (-y, -x, 2z) \end{array}$$

$$\xrightarrow{\nabla f = \lambda \nabla g} \frac{x}{-y} = \frac{y}{-x} = \frac{1}{2} \Longrightarrow \begin{array}{l} x^2 = y^2 \\ y = \frac{-x}{2} \end{array} \text{ so } x = y = 0$$

$\Longrightarrow z = 1, -1$ or $(0,0,\pm 1)$ are the points where the distance is minimized.

Exercise 7.

$$\begin{array}{l} f(x,y) = \sqrt{(x-1)^2+y^2} \\ \nabla f = \frac{(x-1,y)}{\sqrt{(x-1)^2+y^2}} \end{array} \qquad \begin{array}{l} g(x,y) = 4x - y^2 \\ \nabla g = (4, -y) \end{array} \xrightarrow{\nabla f = \lambda \nabla g} \begin{array}{l} \frac{(x-1)}{\sqrt{(x-1)^2+y^2}} = 4\lambda \\ \frac{y}{\sqrt{(x-1)^2+y^2}} = -2y\lambda \end{array}$$

$x > 0$, but if $y \neq 0$, the equations imply $x = -1$.

$\Longrightarrow (x,y) = (1,0)$ is the point of shortest distance on the parabola to $(1,0)$

Exercise 8. Given the constraining surfaces

$$\begin{array}{r} x^2 - xy + y^2 - z^2 = 1 \\ x^2 + y^2 = 1 \end{array} \text{ or } xy + z^2 = g = 0$$

Then we want to minimize $f = \sqrt{x^2+y^2+z^2}$. $\nabla f = \frac{(x,y,z)}{f}$

$$\nabla g_1 = (y,x,2z) \xrightarrow{\nabla f = \lambda \nabla g} \begin{aligned} \frac{x}{f} &= \lambda y \\ \frac{y}{f} &= \lambda x \\ \frac{z}{f} &= \lambda 2z \end{aligned}$$

$$\text{Suppose } z \neq 0, \text{ then } \frac{1}{2f} = \lambda \Longrightarrow \begin{aligned} \frac{y}{f} &= \frac{x}{2f} \\ \frac{x}{f} &= \frac{1}{2f}y \end{aligned} \text{ or } \begin{aligned} y &= \frac{x}{2} \\ x &= \frac{y}{2} \end{aligned} \text{ Contradiction.}$$

Then $z = 0$.

$$\text{Suppose } x, y \neq 0 \Longrightarrow \frac{x}{y} = \frac{y}{x} \text{ or } x^2 = y^2$$

$$\Longrightarrow 2x^2 = 1 \text{ or } x = \frac{\pm 1}{\sqrt{2}} \text{ but } z^2 = -xy = 0 \quad \text{Contradiction.}$$

Then $x = 0$ or $y = 0$, so that $\lambda = 0$

$$\boxed{(0,\pm 1, 0), (\pm 1, 0, 0)}$$

Exercise 9.

$$\begin{aligned} f(x,y,z) &= x^a y^b z^c \\ \nabla f &= f\left(\frac{a}{x}, \frac{b}{y}, \frac{c}{z}\right) \end{aligned} \qquad \begin{aligned} g &= x+y+z-1 \\ \nabla g &= (1,1,1) \end{aligned} \xrightarrow{\nabla f = \lambda \nabla g} \frac{af}{x} = \frac{bf}{y} = \frac{cf}{z}$$

If $x, y, z \neq 0$ then

$$\begin{aligned} y &= \frac{bx}{a} \\ z &= \frac{cx}{a} \end{aligned} \Longrightarrow x + \frac{bx}{a} + \frac{cx}{a} = 1 \text{ so that the maximum occurs at } (x,y,z) = \frac{1}{a+b+c}(a,b,c) \text{ and } f = \frac{a^a b^b c^c}{(a+b+c)^{a+b+c}}$$

Exercise 10. Consider the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = g_1$. Then ellipsoid will have the same normal at a point on the ellipsoid as the tangent plane through the same point. Thus, we want

$$\nabla g_1 = 2\left(\frac{x}{a^2}, \frac{y}{b^2}, \frac{z}{c^2}\right)$$

to be a normal that defines a plane through (x,y,z), that's on the ellipsoid, in the uvw plane.

$$\Longrightarrow \frac{xu}{a^2} + \frac{yv}{b^2} + \frac{zw}{c^2} = 1$$

The volume of the tetrahedron is

$$V = \int_0^{a^2/x} du \int_0^{\frac{b^2}{y}\left(1-\frac{xu}{a^2}\right)} dv \int_0^{\frac{c^2}{z}\left(1-\frac{xu}{a^2}-\frac{yv}{b^2}\right)} dv = \frac{(abc)^2}{6xyz} \qquad \nabla V = \frac{(abc)^2}{6}\left(\frac{-1}{x^2yz}, \frac{-1}{xy^2z}, \frac{-1}{xyz^2}\right)$$

$$\xrightarrow{\nabla V = \lambda \nabla g_1} \begin{aligned} \frac{(abc)^2}{6}\left(\frac{-1}{x^2yz}\right) &= \lambda 2\frac{x}{a^2} \\ \frac{(abc)^2}{6}\left(\frac{-1}{xy^2z}\right) &= \lambda 2\frac{y}{b^2} \\ \frac{(abc)^2}{6}\left(\frac{-1}{xyz^2}\right) &= \lambda 2\frac{z}{c^2} \end{aligned} \text{ so } \begin{aligned} y^2 &= \left(\frac{bx}{a}\right)^2 \\ z^2 &= \left(\frac{c}{a}x\right)^2 \end{aligned} \xrightarrow{\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1} \begin{aligned} x &= \frac{a}{\sqrt{3}} \\ y &= \frac{b}{\sqrt{3}} \\ z &= \frac{c}{\sqrt{3}} \end{aligned}$$

$$\boxed{V = abc\frac{\sqrt{3}}{2}}$$

Exercise 12. Consider the conic section as a quadratic form.

$$Ax^2 + 2Bxy + Cy^2 = 1 \text{ where } A > 0 \text{ and } B^2 < AC \quad \Longrightarrow T = \begin{bmatrix} A & B \\ B & C \end{bmatrix}$$

Note that $detT = AC - B^2$, so normalize T by $detT$ so to obtain only pure rotation, no "amplification."

$$T \to T = \frac{1}{AC - B^2}\begin{bmatrix} \lambda - A & -B \\ -B & \lambda - C \end{bmatrix} = 0 \Longrightarrow \lambda_\pm = \frac{(A+C) \pm \sqrt{(A-C)^2 + 4B^2}}{2(AC - B^2)}$$

We don't need to find the eigenvectors. By theorem, we can find a C s.t. $Y = XC$ and Y are the coordinates in which T is diagonalized.

$$\lambda_+ u^2 + \lambda_- v^2 = 1 \Longrightarrow \frac{u^2}{1/\lambda_+} + \frac{v^2}{1/\lambda_-} = 1$$

Immediately we recognize this to be the equation of an ellipse. The T rotation does not amplify distances since $detT = 1$, and so distances from the origin to the conic section are preserved. Then we can immediately name the minimum and maximum distances:

$$M^2, m^2 = 1/\frac{(A+C) + \sqrt{(A-C)^2 + 4B^2}}{2(AC - B^2)}, 1/\frac{(A+C) - \sqrt{(A-C)^2 + 4B^2}}{2(AC - B^2)}$$

Exercise 13. Let $X = (x, y)$ be a point on the ellipse.
The line is given by the set $\{(0,4) + s(1,-1) | s \in \mathbb{R}\}$.
The normal to the line is $(1,1)/\sqrt{2}$, so connect a point on the ellipse to the line by a perpendicular distance t by the following:

$$X = t\frac{(1,1)}{\sqrt{2}} = (0,4) + s(1,-1)$$

so that

$$\frac{t(1,1)}{\sqrt{2}} = (0,4) + (s,-s) - (x,y) = (s-x, 4-s-y)$$

$$\begin{aligned} \frac{t}{\sqrt{2}} &= s - x \\ \frac{t}{\sqrt{2}} &= 4 - s - y \end{aligned} \Longrightarrow \sqrt{2}t = 4 - x - y \text{ so let } f(x,y) = t = \frac{4-x-y}{\sqrt{2}}$$

$$\nabla f = \left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}} \right) \qquad \begin{aligned} g(x,y) &= \frac{x^2}{4} + y^2 - 1 = 0 \\ \nabla g &= \left(\frac{x}{2}, 2y \right) \end{aligned} \qquad \Longrightarrow \frac{x}{2} = 2y \qquad \Longrightarrow 16y^2/4 + y^2 = 5y^2 = 1$$

Thus, the points that extremize f are $(x,y) = \left(\frac{4}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right), \left(\frac{-4}{\sqrt{5}}, \frac{-1}{\sqrt{5}} \right)$.

$$\boxed{t_{min} = \frac{4-\sqrt{5}}{\sqrt{2}}, \quad t_{max} = \frac{4+\sqrt{5}}{\sqrt{2}}}$$

10.5 Exercises - Introduction, Paths and line integrals, Other notations for line integrals, Basic properties of line integrals

Exercise 1. $f = ((x^2 - 2xy), (y^2 - 2xy))$ from $(-1,1)$ to $(1,1)$; $\quad y = x^2$
$\alpha(x) = (x, x^2); \quad \alpha' = (1, 2x)$

$$\int ((x^2 - 2xy), (y^2 - 2xy)) \cdot (1, 2x) dx = \int (x^2 - 2xy + 2xy^2 - 4x^2 y) dx =$$

$$\int_{-1}^{1} (x^2 - 2x^3 + 2x^5 - 4x^4) dx = \left(\frac{1}{3}x^3 - \frac{2}{4}x^4 + \frac{2}{6}x^6 - \frac{4}{5}x^5 \right) \bigg|_{-1}^{1} = \frac{1}{3}(1 - (-1)) - \frac{4}{5}(1 - (-1)) = \frac{2}{3} - \frac{8}{5} = \boxed{\frac{-14}{15}}$$

Exercise 2. $f = (2a - y, x)$ along the path described by $\alpha = (a(t - \sin t), a(1 - \cos t)) \quad 0 \leq t \leq 2\pi$
$f(t) = (2a - a(1 - \cos t), a(t - \sin t)) = (a + a\cos t, a(t - \sin t))$
$\alpha'(t) = (a(1 - \cos t), a \sin t)$

$$\int f(t) \cdot \alpha'(t) dt = \int_0^{2\pi} (a^2(1 - \cos^2 t) + a^2(t \sin t - \sin^2 t)) dt = a^2 \int_0^{2\pi} (1 + t\sin t - 1) dt =$$

$$= a^2 \int_0^{2\pi} t \sin t dt = a^2 \left((-t\cos t)|_0^{2\pi} - \int_0^{2\pi} -\cos t dt \right) = a^2(-2\pi) + 0 = \boxed{-2\pi a^2}$$

Exercise 3. $f(x,y,z) = ((y^2 - z^2), 2yz, -x^2); \quad \alpha(t) = (t, t^2, t^3), \quad 0 \leq t \leq 1$
$f[\alpha(t)] = ((t^4 - t^6), 2t^5, -t^2) \qquad \alpha'(t) = (1, 2t, 3t^2)$

$$\int_0^1 f(\alpha(t)) \cdot \alpha'(t) dt = \int_0^1 (t^4 - t^6 + 4t^6 - 3t^4) dt = \left((-2)\frac{1}{5}t^5 + \frac{1}{7}3t^7 \right) \bigg|_0^1 = \frac{-2}{5} + \frac{3}{7} = \boxed{\frac{1}{35}}$$

Exercise 4. $f = (x^2 + y^2, x^2 - y^2)$ from $(0,0)$ to $(2,0)$ along the curve $y = 1 - |1 - x|$.

$$|1-x| = \begin{cases} 1-x & \text{if } 1-x>0 \text{ or } 1>x \\ -(1-x) & \text{if } 1-x<0 \text{ or } 1<x \end{cases} \qquad y = \begin{cases} x & 1>x \\ 2-x & 1<x \end{cases}$$

For$x<1$, For $x>1$,

$$\alpha(x) = (x,x) \qquad \alpha(x) = (x,(2-x))$$
$$\alpha'(x) = (1,1) \qquad \alpha'(x) = (1,-1)$$

$$\int f\cdot\alpha'(x)dx = \int_0^1 ((x^2+y^2)+(x^2-y^2))dx + \int_1^2 ((x^2+y^2)-(x^2-y^2))dx = \int_0^1 2x^2dx + \int_1^2 2(2-x)^2dx =$$
$$= \frac{2}{3}x^3\Big|_0^1 + 2\left(\frac{1}{3}(2-x)^3(-1)\right)\Big|_1^2 = \frac{2}{3} + \left(\frac{-2}{3}\right)(0-1^3) = \frac{2}{3}+\frac{2}{3} = \boxed{\frac{4}{3}}$$

Exercise 5. $f=(x+y,x-y) \quad b^2x^2+a^2y^2=a^2b^2 \quad \implies \left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2=1$. From the ellipse equation, parametrize by $\theta. \quad \implies \begin{matrix} x=a\cos\theta \\ y=b\sin\theta \end{matrix}$

$$\alpha(\theta) = (a\cos\theta, b\sin\theta)$$
$$\alpha'(\theta) = (-a\sin\theta, b\cos\theta)d\theta$$

$$\int_0^{2\pi} f(\theta)\cdot\alpha'(\theta)d\theta = \int_0^{2\pi} ((a\cos\theta+b\sin\theta)(-a\sin\theta)d\theta + (a\cos\theta-b\sin\theta)b\cos\theta d\theta) =$$
$$= (-a^2-b^2)\int_0^{2\pi}\cos\theta\sin\theta d\theta + ab\int_0^{2\pi}(\cos^2\theta-\sin^2\theta)d\theta =$$
$$= (-a^2-b^2)\left(\frac{\sin^2\theta}{2}\right)\Big|_0^{2\pi} + ab\left(\frac{\sin 2\theta}{2}\right)\Big|_0^{2\pi} = \boxed{0}$$

Exercise 6. Given $f=(2xy,x^2+z,y)$ from $x_1=(1,0,2)$ to $x_2=(3,4,1)$, use vector calculus and analytic geometry to form a line with direction vector $P=x_2-x_1=(2,4,-1)$. Thus, the line is described by $x=tP+x_1$, where t is the parameter.

$$x = tP+x_1 \quad \implies \begin{matrix} x=2t+1 \\ y=4t \\ z=-t+2 \end{matrix}$$

$$f(t) = (2(2t+1)(4t),((2t+1)^2+(-t+2)),4t) = (16t^2+8t,4t^2+3t+3,4t)$$
$$\alpha'(t) = (2,4,-1)$$

$$\int_0^1 f(t)\cdot\alpha'(t)dt = \int_0^1 (32t^2+16t+16t^2+12t+12-4t)dt = \int_0^1 dt(48t^2+24t+12) =$$
$$= \left(\frac{48}{3}t^3+\frac{24}{2}t^2+12t\right)\Big|_0^1 = 16+12+12 = \boxed{40}$$

Exercise 7. $f(x,y,z)=(x,y,(xz-y))$ from $(0,0,0)$ to $(1,2,4)$ along a line segment.

$x_1=(0,0,0)$
$x_2=(1,2,4)$ $\quad P=x_2-x_1=x_2$. The line is described by $Pt+x_1=x=Pt$, so $\begin{bmatrix} x\\y\\z \end{bmatrix} = \begin{bmatrix} t\\2t\\4t \end{bmatrix}$

$$\int_0^1 f(t)\cdot\alpha'(t)dt = \int(t+4t+(4t^2-2t)4)dt = \int(t+4t+16t^2-8t)dt = \int_0^1(-3t+16t^2)dt = \left(\frac{-3t^2}{2}+\frac{16t^3}{2}\right)\Big|_0^1 = \boxed{\frac{23}{6}}$$

Exercise 8. Given $f(x,y,z)=(x,y,(xz-y))$ along the path described by $\alpha(t)=(t^2+2t,4t^3)$; $\quad 0\le t\le 1$

$$\alpha'(t) = (2t,2,12t^2)$$
$$f(t) = (t^2,2t,(4t^2t^3-2t)) = (t^2,2t,(4t^5-2t))$$

$$\int f(t)\cdot\alpha'(t)dt = \int_0^1(2t^3+4t+12t^2(4t^5-2t))dt = \left(\frac{2}{4}t^4+\frac{4}{2}t^2+\frac{48}{8}t^8-\frac{24t^4}{4}\right)\Big|_0^1 = \boxed{\frac{5}{2}}$$

Exercise 9. Given $\int_C(x^2-2xy)dx+(y^2-2xy)dy$; where C is a path from $(-2,4)$ to $(1,1)$ along the parabola $y=x^2$

parametrize to x.

$$\int_C ((x^2-2x(x^2))dx+(x^4-2x(x^2))2xdx) = \int_{-2}^1 (x^2-2x^3+2x^5-4x^4)dx = \left(\frac{1}{3}x^3-\frac{2}{4}x^4+\frac{2}{6}x^6-\frac{4x^5}{5}\right)\Big|_{-2}^1 =$$
$$= \frac{1}{3}(1-(-8))-\frac{1}{2}(1-16)+\frac{1}{3}(1-64)-\frac{4}{5}(1-(-32)) = 3+\frac{15}{2}-\frac{63}{3}-\frac{4}{5}(33) = \boxed{\frac{-369}{10}}$$

Exercise 10. $\int_C \frac{(x+y)dx-(x-y)dy}{x^2+y^2}$ where C is the circle $x^2+y^2=a^2$ or $\left(\frac{x}{a}\right)^2+\left(\frac{y}{a}\right)^2=1$. Let $\begin{aligned} \frac{x}{a} &= \cos\theta & \frac{dx}{d\theta} &= -a\sin\theta \\ \frac{y}{a} &= \sin\theta & \frac{dy}{d\theta} &= a\cos\theta \end{aligned}$

$$\implies \int_0^{2\pi} \left((a\cos\theta+a\sin\theta)(-a\sin\theta)d\theta-(a\cos\theta-a\sin\theta)a\cos\theta d\theta\right)/a^2 = (-1)(2\pi) = \boxed{-2\pi}$$

Exercise 11. $\int_F \frac{dx+dy}{|x|+|y|}$, where $F=A+B+C+D$, where

$$\begin{aligned} A &: \, y=-x+1 \\ B &: \, y=x+1 \\ C &: \, y=-x-1 \\ D &: \, y=x-1 \end{aligned}$$

So

$$\begin{aligned} &\int_A \left(\frac{dx}{x+(-x+1)}+\frac{-dx}{x+(-x+1)}\right)=0 \\ &\int_B \frac{2dx}{-x+x+1} = 2\int_0^{-1}\frac{dx}{1} = 2(-1)=-2 \\ &\int_C \frac{dx-dx}{-x+x+1}=0 \\ &\int_D \frac{2dx}{x+(-x+1)} = 2\int_0^1 \frac{dx}{1} = 2(1)=2 \end{aligned} \qquad \implies \boxed{\int_F \frac{dx+dy}{|x|+|y|}=0}$$

Exercise 12.

(1) Given that we want to compute $\int_C ydx+zdy+xdz$, on the intersection of $x+y=2$ or $y=2-x$ and $x^2+y^2+z^2=2(x+y)=2(2)=4$, then

$$\begin{aligned} dx &= dx \\ dy &= -dx \\ \frac{dz}{dx}2z &= -4x+4 \text{ or } dz = \frac{-4x+4}{2\sqrt{4x-2x^2}}dx \end{aligned}$$

and

$$z^2 = 4-x^2-y^2 = 4-x^2-(2-x)^2 = -2x^2+4x$$

$$\int_0^2 (2-x)dx+\sqrt{4x-2x^2}(-dx)+x\frac{2(1-x)}{\sqrt{4x-2x^2}}dx + \int_2^0\left((2-x)dx+-\sqrt{4x-2x^2}(-dx)+\frac{x2(1-x)}{-\sqrt{4x-2x^2}}dx\right) =$$
$$= 2\int_0^2 \frac{-(4x-2x^2)+2x-2x^2}{\sqrt{4x-2x^2}}dx = (-4)\int_0^2 (x/\sqrt{4x-2x^2})dx$$

Let $u=\frac{x}{2}$. Then $2du=dx$.

$$\implies (-4)\int_0^1 (2u)(2du)/\sqrt{2}\sqrt{4u-(2u)^2} = -4\sqrt{2}\int_0^1 \frac{\sqrt{u}}{\sqrt{1-u}}du = -4\sqrt{2}\int_0^{\pi/2}\frac{\sin\theta}{\cos\theta}2\sin\theta\cos\theta d\theta =$$
$$= -8\sqrt{2}\left(\frac{1}{2}\left(\frac{\pi}{2}-9\right)\frac{-\sin 2\theta}{4}\right)\Big|_0^{\pi/2} = \boxed{-2\sqrt{2}\pi}$$

since $\begin{aligned} u &= \sin^2\theta \\ du &= 2\sin\theta\cos\theta d\theta \end{aligned}$ and

$$\sqrt{1-u} = \sqrt{1-\sin^2\theta} = \cos\theta$$

(2) With $x^2+y^2=1$, $z=xy$, let $x=\cos\theta$, $y=\sin\theta$, $z=\cos\theta\sin\theta$, so that

$$\int_C ydx+zdy+xdz=\int_0^{2\pi}(\sin\theta(-\sin\theta)d\theta+\cos\theta\sin\theta\cos\theta d\theta+\cos\theta(-\sin^2\theta+\cos^2\theta)d\theta=$$

$$=\int_0^{2\pi}(-\sin^2\theta+\cos^2\theta\sin\theta-\sin^2\theta\cos\theta+\cos^3\theta)d\theta=$$

$$=\left((-1)\left(\frac{1-\cos 2\theta}{2}\right)+\frac{-\cos^3\theta}{3}-\frac{1}{3}\sin^3\theta+\sin\theta-\frac{1}{3}\sin^3\theta\right)\Big|_0^{2\pi}=\boxed{-\pi}$$

10.9 Exercises - The concept of work as a line integral, Line integrals with respect to arc length, Further applications of line integrals

Exercise 1. $f(x,y,z)=(x,y,(xz-y))$

$$\begin{aligned} x_1&=(0,0,0) &\quad P&=x_2-x_1=x_2\\ x_2&=(1,2,4) &\quad x&=tx_2+x_1;\quad 0\le t\le 1\end{aligned}$$

$$\int f\cdot ds=\int f\cdot\frac{ds}{dt}dt=\int_0^1(t,2t,t(4t)-2t)\cdot(1,2,4)dt=\int_0^1(t+4t+16t^2-8t)dt=\frac{-3}{2}+\frac{16}{3}=\boxed{\frac{23}{6}}$$

Exercise 2. $f(x,y)=((x^2-y^2),2xy)$

$$\begin{aligned} A&:\ r=(a,ta)\\ B&:\ r=(a(1-t),a)\\ C&:\ r=(0,a(1-t))\\ D&:\ r=(at,0)\end{aligned}$$

$$A:\ \int_0^1(a^2-a^2t^2,2ta^2)\cdot(0,a)dt=\int_0^1 2ta^3dt=a^3$$

$$B:\ \int_0^1((a^2(1-2t+t^2)-a^2),2a(1-t)a)\cdot(-a,0)dt=-a\int_0^1 a^2(-2t+t^2)dt=-a^3\left(-t^2+\frac{1}{3}t^3\right)\Big|_0^1=\frac{2}{3}a^3$$

$$C:\ \int_0^1(-a^2(1-2t+t^2),0)\cdot(0,-a)=0$$

$$D:\ \int_0^1(a^2t^2,0)\cdot(a,0)dt=a^3\frac{1}{3}$$

$$\implies\int_C f\cdot ds=\boxed{2a^3}$$

Exercise 3. $f(x,y)=(cxy,x^6y^2)$ $\quad c>0$. $(0,0)$ to line $x=1$ via $y=ax^b$; $\ a>0,\ b>0$.

$$W=\int f\cdot ds=\int(cx(ax^b),x^6a^2x^{2b})\cdot(1,bax^{b-1})=\int(acx^{b+1}+a^3bx^{3b+5})dx=$$

$$=\frac{ac}{b+2}+\frac{a^3b}{3b+6}=\frac{3ac}{3(b+2)}+\frac{a^3b}{3(b+2)}=\frac{3ac+a^3b}{3(b+2)}$$

$$\implies\frac{3c}{a^2}=2\implies\boxed{a=\sqrt{\frac{3c}{2}}}$$

Exercise 4. $f=(yz,xz,x(y+1))$. $\quad a=(0,0,0)$, $b=(1,1,1)$, $c=(-1,1,-1)$

$$
\begin{aligned}
&A: (b-a) = b; && r = tb\\
&B: (c-b) = (-2,0,-2); && r = t(-2,0,-2)+b\\
&C: (a-c) = -c; && r = t(-c)+c = c(1-t)
\end{aligned}
$$

$$A: \int (t^2, t^2, t(t+1))\cdot(1,1,1)dt = \int_0^1 t^2+t^2+t^2+t = \left(t^3+\frac{1}{2}t^2\right)\Big|_0^1 = \frac{3}{2}$$

$$
B: \begin{aligned}
&\int (1(-2t+1), (-2t+1), (-2t+1)(2))\cdot(-2,0,-2)dt = \int (-2t+1)(-2)+(-2)(-2t+1) =\\
&= \int (-2)(-2t+1)(3) = -6\left(-t^2+t\right)\Big|_0^1 = 0
\end{aligned}
$$

$$
C: \begin{aligned}
&\int ((1-t)(-1)(1-t), -1(1-t)(-1)(1-t), (-1)(1-t)((1-t)+1))\cdot(1,-1,1) =\\
&= \int -(1-t)^2+(1-t)^2(-1)+(-1)(1-t)^2-(1-t) = \int -3(1-2t+t^2)-1+t = \int -3t^2+7t-4 =\\
&= \left(-t^3+\frac{7t^2}{2}-4t\right)\Big|_0^1 = \frac{-3}{2}
\end{aligned}
$$

$$\int f\cdot ds = \boxed{0}$$

Exercise 5. $f = (y-z, z-x, x-y)$

$$
\begin{aligned}
x^2+y^2+z^2 &= 4\\
x^2+y^2\sec^2\theta &= 4
\end{aligned}
\qquad z = y\tan\theta \qquad
\begin{aligned}
x &= 2\cos\phi\\
y\sec\theta &= 2\sin\phi\\
z &= 2\sin\phi\sin\theta
\end{aligned}
$$

$$f = \left(\frac{2\sin\phi}{\sec\theta} - 2\sin\phi\sin\theta, 2\sin\phi\sin\theta - 2\cos\phi, 2\cos\phi - \frac{2\sin\phi}{\sec\theta}\right)$$

$$\frac{ds}{d\phi} = (-2\sin\phi, \frac{2\cos\phi}{\sec\theta}, 2\cos\phi\sin\theta)$$

$$
\begin{aligned}
\int f\cdot ds &= \int 4(\sin\phi(\cos\theta-\sin\theta)(-\sin\phi) + (\sin\phi\sin\theta-\cos\phi)(\cos\phi\cos\theta) + (\cos\phi-\sin\phi\cos\theta)\cos\phi\sin\theta)d\phi =\\
&= 4\int (-\sin^2\phi(\cos\theta-\sin\theta) + -\cos^2\phi(\cos\theta-\sin\theta) = 4\int(-\cos\theta+\sin\theta)d\phi = \boxed{8\pi(\sin\theta-\cos\theta)}
\end{aligned}
$$

Exercise 6. $f = (y^2, z^2, x^2)$. $\begin{aligned} x^2+y^2+z^2 &= a^2\\ x^2+y^2 &= ax\end{aligned} \implies \left(x-\frac{a}{2}\right)^2+y^2 = \left(\frac{a}{2}\right)^2.$ $\begin{aligned} x-\frac{a}{2} &= \frac{a}{2}\cos\phi\\ y &= \frac{a}{2}\sin\phi\\ z^2 &= a^2 - a\left(\frac{a}{2}(\cos\phi+1)\right)\end{aligned}$

$$
\begin{aligned}
f &= \left(\left(\frac{a}{2}\right)^2\sin^2\phi, \frac{a^2}{2}(1-\cos\phi), \left(\frac{a}{2}\right)^2(\cos\phi+1)^2\right)\\
r' &= \left(\frac{-a}{2}\sin\phi, \frac{a}{2}\cos\phi, \frac{a^2}{4}\sin\phi/z\right)\\
f\cdot r' &= \left(\frac{a}{2}\right)^3\sin^3\phi + \frac{a^3}{4}(\cos\phi-\cos^2\phi) + \left(\frac{a}{2}\right)^4\frac{(\cos\phi+1)^2\sin\phi}{z}
\end{aligned}
$$

$$\sin^3\phi = \sin\phi(1-\cos^2\phi) \xrightarrow{\int} -\cos\phi + \frac{1}{3}\cos^3\phi \xrightarrow{\int_0^{2\pi}} 0$$

$$\int_0^{2\pi}\cos\phi = 0$$

$$\int_0^{2\pi}\cos^2\phi = \boxed{\pi}$$

$$z\geq 0, \quad \implies z = \frac{a}{2}\sqrt{1+\cos\phi} = \frac{a}{2}\sqrt{2\cos^2\phi/2} = \frac{a}{\sqrt{2}}|\cos\phi/2| \text{ and so}$$

$$\left(\frac{a}{2}\right)^4(1+\cos\phi)^2 = \left(\frac{a^2}{4}+\frac{a^2}{4}\cos\phi\right)^2 = \left(\frac{a^2}{4}+\frac{a^2}{4}-\frac{z^2}{2}\right)^2 = \frac{(a^2-z^2)^2}{2^2} \quad \text{so that}$$

$$\int\left(\frac{a}{2}\right)^4\frac{(1+\cos\phi)^2\sin\phi}{z} = \int\frac{(a^2-z^2)^2\sin\phi}{z} = \int\frac{(a^4-2a^2z^2+z^4)\sin\phi}{z}$$

$$\int\sin\phi/2\cos\phi/2/|\cos\phi/2| = \int_0^\pi\sin\phi/2 - \int_\pi^{2\pi}\sin\phi/2 = -2\cos\phi/2|_0^\pi + 2\cos\phi/2|_\pi^{2\pi} = (-2)(-1)+2(-1) = 0$$

$$\int z\sin\phi = \int_0^{2\pi}\frac{a}{\sqrt{2}}\sqrt{1-\cos\phi}\sin\phi = \frac{a}{\sqrt{2}}\left.(1-\cos\phi)^{3/2}\frac{2}{3}\right|_0^{2\pi} = 0$$

$$\int z^3\sin\phi = \int_0^{2\pi}\left(\frac{a}{\sqrt{2}}\sqrt{1-\cos\phi}\right)^3\sin\phi = \left.\frac{a^3}{2^{3/2}}(1-\cos\phi)^{5/2}\frac{2}{5}\right|_0^{2\pi} = 0$$

So we finally get

$$\int f\cdot r'd\phi = -a^3(\pi)/4$$

Since we had gone counterclockwise, the exercise asked for the clockwise direction, so reverse the sign to get $\boxed{\frac{a^3\pi}{4}}$

Exercise 7. $\int_C(x+y)ds$.

$$\begin{array}{lllll} a=(0,0) & A:\, b-a=b & r=bt+a=bt & |r'|=|b|=1 & (x+y)=t \\ b=(1,0) & B:\, c-b=(-1,1) & r=(-1,1)t+(1,0) & |r'|=\sqrt{2} & (x+y)=-t+1+t=1 \\ c=(0,1) & C:\, a-c=-c & r=-ct+c=c(1-t) & |r'|=1 & (x+y)=1-t \end{array}$$

$$\int_C(x+y)ds = \frac{1}{2}t^2+t\sqrt{2}+t-\frac{1}{2}t^2 = \sqrt{2}+1$$

Exercise 8. $\int_C y^2dx \quad \alpha(t)=(a(t-\sin t),a(1-\cos t)), \quad 0\leq t\leq 2\pi$

$$\begin{aligned} \alpha' &= (a(1-c(t)),a(s(t))) \\ \alpha'^2 &= a^2(1-2c+c^2+s^2) = a^22(1-c) \end{aligned}$$

$$\int a^2(1-2c+c^2)(a\sqrt{2}\sqrt{1-c})dt = a^3\sqrt{2}\int(1-c)^{5/2}dt =$$

$$= a^3\sqrt{2}\int_0^{2\pi}\left(2\sin^2\left(\frac{t}{2}\right)\right)^{5/2}dt = a^32^{1/2}2^{5/2}\int_0^{2\pi}\sin^5\left(\frac{t}{2}\right)dt$$

$$\text{since} \quad \begin{aligned}\cos(2t) &= \cos^2t-\sin^2t \\ &= 1-2\sin^2t\end{aligned} \quad \text{or } 2\sin^2t = 1-\cos(2t)$$

Now

$$\sin^5\left(\frac{t}{2}\right) = (1-\cos^2\left(\frac{t}{2}\right))^2\sin\left(\frac{t}{2}\right) = \left(1-2\cos^2\left(\frac{t}{2}\right)+\cos^4\left(\frac{t}{2}\right)\right)\sin\left(\frac{t}{2}\right)$$

So

$$\int_0^{2\pi}\sin^5\left(\frac{t}{2}\right)dt = (-2\cos(t/2))|_0^{2\pi} + \left.\left((4)(1/3)\cos^3(t/2)+\frac{-2}{5}\cos^5(t/2)\right)\right|_0^{2\pi} = (-2)\left((-2)+\frac{4}{3}+\frac{-2}{5}\right) = \frac{32}{15}$$

$$\int_C y^2ds = 8a^3\frac{32}{15} = \boxed{\frac{256}{15}a^3}$$

Exercise 9. $\int_C(x^2+y^2)ds$ where C has the vector equation $\alpha(t)=(a(\cos t+t\sin t),a(\sin t-t\cos t)), \;\; 0\leq t\leq 2\pi$

$$\alpha'(t) = (a((-s+s+tc),c-c+ts)) = a(tc,ts) = at(c,s) \qquad \|a'\|at$$

$$x^2+y^2 = a^2(c^2+2tcs+t^2s^2+s^2-2tsc+t^2c^2) = a^2(1+t^2)$$

$$\int_C a^2(1+t^2)atdt = a^3\int(t+t^3)dt = a^3\left(\frac{1}{2}(2\pi)^2+\frac{1}{4}(2\pi)^4\right) = \boxed{a^3\frac{(2\pi)^2}{2}\left(1+\frac{(2\pi)^3}{2}\right)}$$

Exercise 10. $\int_C zdx \qquad \alpha(t)=(t\cos t,t\sin t,t)$

$$\alpha' = (\cos t + -t\sin t, \sin t + t\cos t, 1) \qquad |\alpha'|^2 = c^2 - 2tsc + t^2s^2 + s^2 + 2sct + t^2c^2 + 1 = 2 + t^2$$

$$\int t\sqrt{2+t^2}dt = \left(\frac{1}{3}(2+t^2)^{3/2}\right)\Big|_0^{t_0} = \boxed{\frac{(2+t_0^2)^{3/2} - 2^{3/2}}{3}}$$

Exercise 11.

(1)

$$\begin{aligned} r &= a(\cos t, \sin t) \\ r' &= a(-\sin t, \cos t) \qquad t \in [0,\pi] \\ \|\alpha'(t)\| &= \|r'(t)\| = a \end{aligned} \qquad \begin{aligned} \overline{z} &= 0 \\ \overline{y} &= \frac{1}{\pi a}\int_0^\pi a\sin t a dt = \frac{-a}{\pi}\cos t\Big|_0^\pi = \frac{2a}{\pi} \\ \overline{x} &= \frac{1}{\pi a}\int_0^\pi a\cos t a dt = 0 \end{aligned}$$

(2)

$$\left(\frac{M}{\pi a}\right)\int_0^\pi a^2\sin^2 t(a)dt = \frac{a^2M}{\pi}\int_0^\pi \frac{1-\cos 2t}{2}dt = \boxed{\frac{1}{2}Ma^2}$$

Exercise 12. $\rho = |x| + |y|$

$$\begin{aligned} r &= a(\cos t, \sin t) \\ r' &= a(-s(t), c(t)) \qquad t \in [0, 2\pi] \\ \|\alpha'(t)\| &= \|r'(t)\| = a \end{aligned}$$

$$\begin{aligned} &\int_0^{\pi/2}(a\cos t + a\sin t)adt = a^2(\sin t - \cos t)\big|_0^{\pi/2} = a^2(1-(-1)) = 2a^2 \\ &\int_{\pi/2}^{\pi}(-a\cos t + a\sin t)adt = a^2(-\sin t - \cos t)\big|_{\pi/2}^{\pi} = -a^2(-1+(-1)) = 2a^2 \\ &\int_{\pi}^{3\pi/2}(-a\cos t - a\sin t)adt = -a^2(\sin t - \cos t)\big|_{\pi}^{3\pi/2} = -a^2(-1-(-(-1))) = 2a^2 \\ &\int_{3\pi/2}^{2\pi}(a\cos t - a\sin t)adt = a^2(\sin t + \cos t)\big|_{3\pi/2}^{2\pi} = a^2(1+1) = 2a^2 \end{aligned} \implies \boxed{M = 8a^2}$$

$$\begin{aligned} &\int_0^{\pi/2}(a^2\sin^2 t)(a\cos t + a\sin t)adt = a^4\int_0^{\pi/2}\sin^2 t\cos t + \sin t(1-\cos^2 t)dt = a^4\left(\frac{\sin^3 t}{3} + -\cos t + \frac{1}{3}\cos^3 t\right)\Big|_0^{\pi/2} = a^4 \\ &\int_{\pi/2}^{\pi}(a^2\sin^2 t)(-a\cos t + a\sin t)adt = a^4\int_{\pi/2}^{\pi}(-\sin^2 t\cos t + \sin t(1-\cos^2 t))dt = \\ &= a^4\left(\frac{-\sin^3 t}{3} + -\cos t + \frac{1}{3}\cos^3 t\right)\Big|_{\pi/2}^{\pi} = a^4 \\ &\int_{\pi}^{3\pi/2}(-a^2\sin^2 t)(a\cos t + a\sin t)adt = -a^4\left(\frac{-\sin^3 t}{3} + -\cos t + \frac{1}{3}\cos^3 t\right)\Big|_{\pi}^{3\pi/2} = -a^4\left(\frac{-1}{3} + -1 + \frac{1}{3}\right) = a^4 \\ &\int_{3\pi/2}^{2\pi}(a^2\sin^2 t)(a\cos t - a\sin t)adt = a^4\left(\frac{\sin^3 t}{3} + \cos t - \frac{1}{3}\cos^3 t\right)\Big|_{3\pi/2}^{2\pi} = a^4 \\ &\implies \boxed{I = 4a^4} \end{aligned}$$

Exercise 13. Notice that we have the plane cut through the center of the sphere: like a conic section, we obtain a circle with perpendicular $\frac{1}{\sqrt{3}}(1,1,1)$.

$$\begin{aligned} x^2+y^2+z^2 &= 1 \\ x+y+z &= 0 \end{aligned} \quad \implies x^2+y^2+xy = \frac{1}{2} \implies \quad \begin{aligned} 2x + 2yy_x + y + xy_x &= 0 \\ y_x &= \frac{-y-2x}{x+2y} \end{aligned}$$

$$
\begin{aligned}
r &= (x, y, z) \\
r_x &= \left(1, \frac{-y-2x}{x+2y}, -1 - \left(\frac{-y-2x}{x+2y}\right)\right) = \left(1, -\frac{(y+2x)}{x+2y}, \frac{x-y}{x+2y}\right) \\
r_x^2 &= 1 + \frac{(y+2x)^2}{(x+2y)^2} + \frac{(x-y)^2}{(x+2y)^2} = \left(\frac{3}{2-3x^2}\right) \\
|r_x| &= \frac{\sqrt{3}}{\sqrt{2-3x^2}}
\end{aligned}
$$

I guess that x ranges between $\pm\sqrt{\frac{2}{3}}$.

We're given that the mass density is x^2. If we go around the circle on one branch, one semicircle, from $-\sqrt{2/3}$ to $\sqrt{2/3}$ in x, and then around in the same direction on the other branch, other semicircle, from $\sqrt{2/3}$ to $-\sqrt{2/3}$ in x, then we calculate for this branch the same number. So do the calculation for one semicircle.

$$
\begin{aligned}
\int \frac{x^2}{\sqrt{2-3x^2}} dx = \int \frac{\sqrt{\frac{3}{2}} x^2}{\sqrt{1 - \left(\sqrt{\frac{3}{2}} x\right)^2}} = \int_{-\pi/2}^{\pi/2} \frac{\sqrt{\frac{3}{2}} \frac{2}{3} \sin^2\theta \sqrt{\frac{2}{3}} \cos\theta d\theta}{\sqrt{1-\sin^2\theta}} = \\
= \int_{-\pi/2}^{\pi/2} \frac{2}{3} \sin^2\theta d\theta = \frac{2}{3}\left(\frac{\theta - \sin 2\theta/2}{2}\right)\Bigg|_{-\pi/2}^{\pi/2} = \boxed{\pi/3}
\end{aligned}
\qquad \text{where} \quad
\begin{aligned}
\sqrt{\frac{3}{2}} x &= \sin\theta \\
\sqrt{\frac{3}{2}} dx &= \cos\theta d\theta
\end{aligned}
$$

$\Longrightarrow M = \boxed{\frac{2\pi}{3}}$

Exercise 14. ??? (work on it) Given $x^2 + y^2 = z^2$, $y^2 = x$, $(0,0,0)$ to $(1,1,\sqrt{2})$, curve is parametrized s.t.

$$
\alpha = \alpha(y) = (y^2, y, y\sqrt{1+y^2})
$$

We want z-coordinate of the centroid for a uniform wire.

Consider mass on infinitesimal segment λds.

Weight each λds with corresponding z-coordinate: $z\lambda dx$

Now $\alpha'(y) = (2y, 1, \sqrt{1+y^2} + \frac{y^2}{\sqrt{1+y^2}})$.

$$
\begin{aligned}
\|\alpha'(y)\|^2 &= 7y^2 + 2 + \frac{y^4}{1+y^2} \\
ds &= \|\alpha'(y)\| dy \\
z &= y\sqrt{1+y^2}
\end{aligned}
$$

Then

$$
\begin{aligned}
\int z ds = \int y\sqrt{8y^4 + 9y^2 + 2} dy = \int y 2\sqrt{2} \sqrt{(y^2 + \frac{9}{16})^2 - \frac{17}{16^2}} dy \\
u = y^2 + 9/16 \qquad y = 0 \Longrightarrow u = 9/16 \\
du = 2y dy \qquad y = 1 \Longrightarrow u = 25/16
\end{aligned}
$$

Now

$$
\int \sqrt{u^2 + \beta} = \frac{1}{2}(u\sqrt{u^2+\beta} + \beta \ln(u + \sqrt{u^2+\beta}))
$$

So then

$$
\int z ds = \sqrt{2} \int_{9/16}^{25/16} \sqrt{u^2 - \frac{17}{16^2}} du = \sqrt{2}/2 \left(\frac{25}{16^2} 4\sqrt{38} - \frac{9}{16}\frac{8}{16} + \frac{-17}{16} \ln\left(\frac{25 + 4\sqrt{38}}{17}\right)\right)
$$

since $\begin{aligned} 25^2 - 17 = 608 = 38 * 16 \\ 81 - 17 = 64 \end{aligned}$

However, $M = \lambda \int ds = \lambda \int \frac{\sqrt{8y^4+9y^2+2}}{\sqrt{y^2+1}} dy$ and we need to divide by M.

Exercise 15. Given $r = (a\cos t, a\sin t, bt)$, recall that

$$
M = \sqrt{a^2+b^2} \int_0^{2\pi} (a^2 + b^2 t^2) dt = \sqrt{a^2+b^2}(2\pi a^2 + \frac{8}{3}\pi^3 b^2)
$$

$$\overline{x}M = \int_C x(x^2+y^2+z^2)ds = \sqrt{a^2+b^2}\int_0^{2\pi} a\cos t(a^2+b^2t^2)dt = \sqrt{a^2+b^2}\int_0^{2\pi} ab^2t^2\cos t dt =$$
$$= \left(\sqrt{a^2+b^2}ab^2\right)\left(t^2\sin t + 2t\cos t - 2\sin t\right)\Big|_0^{2\pi} = \sqrt{a^2+b^2}ab^2(2(2\pi)) = \boxed{4\pi\sqrt{a^2+b^2}ab^2}$$
$$\overline{y}M = \int_C y(x^2+y^2+z^2)ds = \sqrt{a^2+b^2}\int_0^{2\pi} a\sin t(a^2+b^2t^2)dt =$$
$$= \sqrt{a^2+b^2}ab^2\left(-t^2\cos t + 2t\sin t + 2\cos t\right)\Big|_0^{2\pi} = \boxed{\sqrt{a^2+b^2}ab^2(-(2\pi)^2)}$$

Exercise 16.

$$I_x = \int_C (y^2+z^2)(x^2+y^2+z^2)ds = \sqrt{a^2+b^2}\int_C (a^2\sin^2 t + b^2t^2)(a^2+b^2t^2)dt =$$
$$= \rho_0^2\int_C (a^4\sin^2 t + a^2b^2t^2 + a^2b^2t^2\sin^2 t + b^4t^4)dt = \rho_0^2\int_C a^4\left(\frac{1-\cos 2t}{2}\right) + a^2b^2t^2 + a^2b^2t^2\left(\frac{1-\cos(2t)}{2}\right) + b^4t^4 dt =$$
$$= \rho_0^2\left(\frac{a^4(2\pi)}{2} + \frac{a^2b^2(2\pi)^3}{3} + \frac{a^2b^2(2\pi)^3}{2(3)} + \frac{b^4(2\pi)^5}{5} - \frac{\pi a^2b^2}{2}\right) = \boxed{\sqrt{a^2+b^2}\left(\pi a^4 + \frac{(2\pi)^3a^2b^2}{2} + \frac{(2\pi)^5b^4}{5} - \frac{\pi(a^2b^2)}{2}\right)}$$
$$I_y = \int_C (x^2+z^2)(x^2+y^2+z^2)ds = \rho_0^2\int_C (a^2\cos^2 t + b^2t^2)(a^2+b^2t^2)dt =$$
$$= \rho_0^2\int_C (a^4\cos^2 t + a^2b^2t^2 + a^2b^2t^2\cos^2 t + b^4t^4)dt = \rho_0^2\int_C a^4\left(\frac{1+\cos 2t}{2}\right) + a^2b^2t^2 + a^2b^2t^2\left(\frac{1+\cos(2t)}{2}\right) + b^4t^4 dt =$$
$$= \rho_0^2\left(\frac{a^4(2\pi)}{2} + \frac{a^2b^2(2\pi)^3}{3} + \frac{a^2b^2(2\pi)^3}{2(3)} + \frac{a^2b^2\pi}{2} + \frac{b^4(2\pi)^5}{5}\right) = \boxed{\sqrt{a^2+b^2}\left(\pi a^4 + \frac{(2\pi)^3a^2b^2}{2} + \frac{(2\pi)^5b^4}{5} + \frac{\pi a^2b^2}{2}\right)}$$

10.13 Exercises - Open connected sets. Independence of the path. The second fundamental theorem of calculus for line integrals. Applications to mechanics.

Exercise 1. Recall the lesson of the preceding sections.

Let S be an open set in $\mathbb{R}^n$. The set S is called connected if every pair of points in S can be joined by a piecewise smooth path whose graph lies in S. That is, for every pair of points a and b in S there is a piecewise smooth path α defined on an interval $[a,b]$ such that $\alpha(t) \in S$ for each $t \in [a,b]$ with $\alpha(a) = a$, $\alpha(b) = b$.

(1) $S = \{(x,y) | x^2+y^2 \geq 0\}$ connected.
Consider $(x_1,y_1) = (r_1,\theta_1)$ $(x_2,y_2) = (r_2,\theta_2)$
Suppose $r_2 > r_1$. Let $\alpha_1 = (r\cos\theta_2, r\sin\theta_2)$ s.t. $r_1 \leq r \leq r_2$
Then consider $\alpha_2 = (r_1\cos\theta, r_1\sin\theta)$ s.t. $\theta : \theta_2 \to \theta_1$
(2) $S = \{(x,y) | x^2+y^2 > 0\}$ connected.
See part(a), with α_1, α_2
(3) $S = \{(x,y) | x^2+y^2 < 1\}$ connected.
See part(a), with α_1, α_2 but with $r_1, r_2 < 1$
(4) $S = \{(x,y) | 1 < x^2+y^2 < 2\}$ connected.
See part (a), with α_1, α_2, but with $1 < r_1, r_2 < 2$

Exercise 2. $f = (P,Q) = \left(\frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}\right)$

Since $\frac{\partial P}{\partial y}$, $\frac{\partial Q}{\partial x}$ continuous, then by Apostol Vol. 2, Thm. 8.12, $\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial\varphi}{\partial x}\right) = \frac{\partial^2\varphi}{\partial x\partial y} = \frac{\partial Q}{\partial x}$

Exercise 3.

(1) $\frac{\partial P}{\partial y} = 1$
$\frac{\partial Q}{\partial x} = -1$ $f(x,y) = y\mathbf{i} - x\mathbf{j}$
Consider a square contour lying on x and y axes.

$$\int_{(0,0)}^{(1,0)} f dx = 0 \qquad \int_{(1,0)}^{(1,1)} (-1)dy = -1$$
$$\int_{(1,1)}^{(0,1)} 1 dy = -1 \qquad \int_{(0,1)}^{(0,0)} 0 = 0$$

So then $\int_C f \cdot ds = -2$

(2) $f(x,y) = y\mathbf{i} + (xy-x)\mathbf{j}$ $\frac{\partial P}{\partial y} = 1$ $\frac{\partial Q}{\partial x} = y-1$

$$\int_{(0,0)}^{(1,0)} f dx = 0 \qquad \int_{(1,0)}^{(1,1)} (y-1)dy = 1/2 - 1 = -1/2$$
$$\int_{(1,1)}^{(0,1)} 1dx = -1 = -1 \qquad \int_{(0,1)}^{(0,0)} 0 = 0$$

$\int_C f \cdot ds = -3/2$

Exercise 4. $f(x,y,z) = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k}$

We're given that

$$\partial_y P, \partial_z P, \partial_x Q, \partial_z Q, \partial_x P, \partial_y R$$

are cont.

Now

$$f = \nabla\varphi = (P,Q,R) = (\partial_x\varphi, \partial_y\varphi, \partial_z\varphi)$$
$$\partial_y P = \partial_y\partial_x\varphi = \partial_x\partial_y\varphi = \partial_x Q$$
$$\partial_z P = \partial_z\partial_x\varphi = \partial_x\partial_z\varphi = \partial_x R$$
$$\partial_z Q = \partial_z\partial_y\varphi = \partial_y\partial_z\varphi = \partial_x R$$

Exercise 6.

(1) $P = y, \quad Q = z.$
$\partial_y P = 1 \quad \partial_x Q = 0$ Not conservative.

(2) Since $f \cdot d\alpha = f \cdot \alpha'(t)dt$

$$\alpha(t) = (\cos t, \sin t, e^t) \implies \alpha'(t) = (-\sin t, \cos t, e^t)$$

$f = (y, z, yz).$

$$f \cdot \alpha'(t) = (-\sin^2 t + e^t \cos t + \sin t e^{2t})$$

$$\int_0^\pi (-\sin^2 t + e^t\cos t + \sin t e^{2t})dt = -\pi + \left(\frac{e^t \sin t + e^t \cos t}{2} + \left(e^{2t}\sin t + \frac{-e^{2t}\cos t}{2} \right) /(5/2) \right)\Big|_0^\pi =$$
$$= \boxed{-\pi + \left(\frac{e^\pi(-1) - 1}{2} \right) + \frac{2}{5}\left(\frac{-e^{2\pi}+1}{2} \right)}$$

11.9 Exercises - Introduction. Partitions of rectangles. Step functions. The double integral of a step function. The definition of the double integral of a function defined and bounded on a rectangle. Upper and lower double integrals. Evaluation of a double integral by repeated one-dimensional integration. Geometric interpretation of the double integral as a volume. Worked examples.

Exercise 1.

$$A(y) = \int_0^1 yx(x+y)dx = \frac{y}{3} + \frac{y^2}{2}$$
$$\int_0^1 A(y)dy = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$$

Exercise 2.

$$A(y) = \int_0^1 dx(x^3 + 3x^2y + y^3) = \frac{1}{4} + y + y^3$$
$$\int_0^1 A(y) = \frac{1}{4} + \frac{1}{2} + \frac{1}{4} = \boxed{1}$$

Exercise 3.

$$A(y) = \int_0^1 (\sqrt{y} + x - 3xy^2)dx = \sqrt{y} + \frac{1}{2} - \frac{3}{2}y^2$$
$$\int_1^3 \sqrt{y} + \frac{1}{2} - \frac{3}{2}y^2 = \left(\frac{2}{3}y^{3/2} + \frac{y}{2} - \frac{y^3}{2} \right)\Big|_1^3 = 2\sqrt{3} + \frac{-2}{3} + \frac{3}{2} - \frac{1}{2} - \frac{27}{2} + \frac{1}{2} = \boxed{2\sqrt{3} - \frac{38}{3}}$$

Exercise 4.

$$A(y) = \int_0^\pi \sin^2 x \sin^2 y dx = \sin^2 y \frac{\pi}{2}$$
$$\int A(y)dy = \frac{\pi}{2}\left(\frac{\pi}{2}\right) = \frac{\pi^2}{4}$$

Exercise 5.

$$A(y) = \int_0^{\pi/2} \sin{(x+y)}dx = -\cos{(x+y)}|_0^{\pi/2} = \cos y - \cos\left(\frac{\pi}{2}+y\right) = \cos y + \sin y$$
$$\int_0^{\pi/2} A(y)dy = +\sin y|_0^{\pi/2} - \cos y|_0^{\pi/2} = \boxed{2}$$

Exercise 6. Split the integral up into 4 parts. Given $\iint_Q |\cos{(x+y)}|dxdy$ where $Q = [0,\pi]\times[0,\pi]$

$$A(x) = \int_0^{-x+\pi/2} \cos{(x+y)}dy = 1 - \sin x$$
$$A(x) = \int_{-x+\pi/2}^{\pi} -\cos{(x+y)}dy = \sin{(x+y)}|_\pi^{\pi/2-x} = 1 - \sin{(x+\pi)} = 1 + \sin x \qquad \Longrightarrow 2$$
$$A(x) = \int_0^{-x+3\pi/2} -\cos{(x+y)}dy = \sin{(x+y)}|_{3\pi/2-x}^{0} = \sin x - (-1) = 1 + \sin x$$
$$A(x) = \int_{-x+3\pi/2}^{\pi} \cos{(x+y)}dy = \sin{(x+y)}|_{-x+3\pi/2}^{\pi} = \sin{(x+\pi)} - (-1) = 1 - \sin x \qquad \Longrightarrow 2$$
$$\int_0^\pi A(x) = \boxed{2\pi}$$

Exercise 7. $\iint_Q f(x+y)dxdy$ and $Q = [0,2]\times[0,2]$, $f(t)$ greatest integer $\leq t$

$y < 1$

$$A(y) = \int_{1-y}^{2-y} 1dx + \int_{2-y}^{2} 2dx = 2 - y - (1-y) + 2(2-(2-y)) = 2y + 1$$

$y > 1$

$$A(y) = \int_0^{2-y} dx + \int_{2-y}^{3-y} 2dx = \int_{3-y}^{2} 3dy = 2 - y + 2(3-y-(2-y)) + 3(2-(3-y)) = 2y+1$$
$$\int_0^2 (2y+1) = \boxed{6}$$

Exercise 8. $\iint_Q y^{-3}e^{tx/y}dxdy$, and $Q = [0,t]\times[1,t]$, $t > 0$

$$\int_0^t y^{-3}e^{tx/y}dx = \left(\frac{y^{-3}e^{tx/y}}{t/y}\right)\Bigg|_0^t = \frac{y^{-3}e^{t^2/y}}{t/y} - \frac{y^{-2}}{t}$$
$$\int_1^t y^{-2}e^{t^2/y}dy = \frac{-e^{t^2/y}}{t^2}\Bigg|_1^t = \frac{-e^t}{t^2} + \frac{e^{t^2}}{t^2}$$
$$\int_1^t y^{-2}/tdy = \frac{-1}{yt}\Bigg|_1^t = \frac{-1}{t^2} + \frac{1}{t}$$
$$\Longrightarrow \boxed{-\frac{e^t}{t^3} + \frac{e^{t^2}}{t^3} + \frac{-1}{t} + \frac{1}{t^2}}$$

Exercise 9. Q rectangle, $Q = [a,b]\times[c,d]$.

$$\iint_Q f(x)g(y)dxdy = \int\left(\int_a^b f(x)dx\right)g(y)dy = \left(\int_a^b f(x)dx\right)\int_c^d g(y)dy$$

Assume $\int_a^b f(x)dx = A$ exists.

Exercise 10. $f(x,y) = \begin{cases} 1 - x - y & \text{if } x + y \leq 1 \\ 0 & \text{otherwise} \end{cases}$

$$A(y) = \int_0^{1-y} (1-x-y)dx = 1-y-\frac{1}{2}(1-2y+y^2) - y(1-y) = \frac{1}{2} - y + \frac{1}{2}y^2$$

$$\int_0^1 A(y)dy = \frac{1}{2} - \frac{1}{2} + \frac{1}{6} = \boxed{1/6}$$

Indeed, vol. of tetrahedron $= \frac{1}{3}Bh = \frac{1}{3}\left(\frac{1}{2}(1)(1)\right) = 1/6$

Exercise 11. If $x < \frac{1}{\sqrt{2}}$

$$A(x) = \int_{x^2}^{2x^2} (x+y)dy = xx^2 + \frac{1}{2}(4x^4 - x^4) = x^3\left(1 + \frac{3}{2}x\right)$$

$$\implies \int_0^{1/\sqrt{2}} A(x)dx \left(\frac{1}{4}x^4 + \frac{3}{10}x^5\right)\Big|_0^{1/\sqrt{2}} = \frac{1}{4}\left(\frac{1}{4}\right) + \frac{3}{10}\left(\frac{1}{4}\right)\left(\frac{\sqrt{2}}{2}\right) = \frac{1}{16} + \frac{3\sqrt{2}}{80}$$

When $x > 1/\sqrt{2}$,

$$A(x) = \int_{x^2}^{1}(x+y)dy = (x)(1-x^2) + \frac{1}{2}(1-x^4) = x - x^3 + \frac{1}{2} - \frac{x^4}{2}$$

$$\int_{1/\sqrt{2}}^{1}(x - x^3 + \frac{1}{2} - \frac{x^4}{2})dx = \frac{1}{2}(1-\frac{1}{2}) - \frac{1}{4}(1-\frac{1}{4}) + \frac{1}{2}(1-\frac{1}{\sqrt{2}}) - \frac{1}{10}(1-\frac{1}{4\sqrt{2}}) = \frac{84}{160} + \frac{-16\sqrt{2}}{80}$$

So we get $\boxed{\frac{21}{40} - \frac{\sqrt{2}}{5}}$

Exercise 12. $f(x,y) = \begin{cases} x^2+y^2 & \text{if } x^2+y^2 \le 1 \\ 0 & \text{otherwise} \end{cases}$

$$A(x) = \int_{-\sqrt{1=x^2}}^{\sqrt{1-x^2}}(x^2+y^2)dy = 2x^2\sqrt{1-x^2} + \frac{1}{3}\left((\sqrt{1-x^2})^3 - (-\sqrt{1-x^2})^3\right) = \frac{4}{3}x^2\sqrt{1-x^2} + \frac{2}{3}\sqrt{1-x^2}$$

Now

$$\int_{-1}^{1} x^2\sqrt{1-x^2} = \int_{-\pi/2}^{\pi/2} \sin^2\theta\cos^2\theta d\theta = \int_{-\pi/2}^{\pi/2}\left(\frac{\sin 2\theta}{2}\right)^2 d\theta = \frac{1}{4}\int_{-\pi/2}^{\pi/2}\left(\frac{1-\cos 4\theta}{2}\right) = \pi/8$$

$$\int \sqrt{1-x^2}dx = \int_{-\pi/2}^{\pi/2} \cos^2\theta d\theta = \pi/2$$

$$\implies \frac{4}{3}\left(\frac{\pi}{8}\right) + \frac{2}{3}\left(\frac{\pi}{2}\right) = \boxed{\pi/2}$$

Exercise 13. Split the integral up into 2 parts.

$$\int_1^y (x+y)^{-2}dx = -\left.(x+y)^{-1}\right|_1^y = -(2y)^{-1} + (y+1)^{-1}$$

$$\int_1^2\int_1^y (x+y)^{-2}dxdy = \frac{-1}{2}\ln 2 + \ln\left(\frac{3}{2}\right)$$

$$\int_{y/2}^{2}(x+y)^{-2}dx = -\left.(x+y)^{-1}\right|_{y/2}^{2} = -(2+y)^{-1} + \left(\frac{3y}{2}\right)^{-1}$$

$$\int_2^4\int_{y/2}^{2}(x+y)^{-2}dxdy = -\left.\ln(2+y)\right|_2^4 + \frac{2}{3}\ln 2 = -\ln 6 + 2\ln 2 + \frac{2}{3}\ln 2$$

Add the two up to get $\boxed{\frac{\ln 2}{6}}$ (it may help to remember that $\ln 3 + -\ln 6 = \ln 3 + \ln 1/6 = \ln 1/2 = -\ln 2$)

Exercise 14. $Q = [0,1]\times[0,1]$ and $f(x,y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}$

Let $D = \{(x,y)|x=y\}$, $D =$ “diagonal” of Q.
D has content zero as it's the graph of a continuous function $y = x,\ \ 0 \le x \le 1$.

f discontinuous only on D.
so f continuous on D, with D of content zero, so f integrable on R.

$$\iint_{Q-D} f = 0 \text{ since } f = 0 \quad \forall (x,y) \in Q - D$$

11.15 Exercises - Integrability of continuous functions. Integrability of bounded functions with discontinuities. Double integrals extended over more general regions. Applications to area and volume. Worked examples.

Exercise 1.

$$\int_0^x x\cos{(x+y)}dy = x\sin{(x+y)}|_0^x = x(\sin{2x} - \sin{x})$$

$$\int_0^\pi dx x(\sin{2x} - \sin{x}) = \left(\frac{x\cos{2x}}{-2} + \frac{\sin{2x}}{4} + x\cos{x} - \sin{x} \right)\Big|_0^\pi = \frac{\pi}{-2} - \pi = \boxed{\frac{-3\pi}{2}}$$

Exercise 2.

$$\int_0^{x+1} (1+x)\sin{y}dy = (1+x)(1-\cos{(x+1)})$$

$$\int_0^1 (1+x) - (1+x)\cos{(x+1)} = 1 + \frac{1}{2} - (1+x)\sin{(x+1)} - \cos{(x+1)}|_0^1 = \frac{3}{2} - 2\sin{2} - \cos{2} + \sin{1} + \cos{1}$$

Exercise 3. $\iint_S e^{x+y}dxdy$ where $S = \{(x,y)||x|+|y| \leq 1\}$

$$\int_{-1+x}^{1-x} e^{x+y}dy = e^x(e^{1-x} - e^{-1+x}) = e^1 - e^{-1+2x}$$

$$\int_0^1 (e^{+1} - e^{-1+2x})dx = e - \frac{e^{-1}}{2}(e^2-1) = \frac{e}{2} + \frac{e^{-1}}{2}$$

$$\int_{-x-1}^{1+x} e^{x+y}dy = e^x(e^{1+x} - e^{-x-1}) = e^{1+2x} - e^{-1}$$

$$\int_{-1}^0 (e^{1+2x} - e^{-1})dx = \frac{e^1}{2}(1-e^{-2}) - e^{-1}(1) = \frac{-3e^{-1}}{2} + \frac{e^1}{2}$$

$$\Longrightarrow \boxed{e - e^{-1}}$$

Exercise 4. $\iint_S x^2y^2dxdy$ and
$$\begin{aligned} xy &= 1 \\ xy &= 2 \\ y &= x \\ y &= 4x \end{aligned} \qquad y = \frac{1}{x} \; y \quad = \frac{2}{x}$$

$I: \; (1/2, 2) \to (1/\sqrt{2}, 2\sqrt{2})$

$$\int_{1/x}^{4x} x^2y^2dy = x^2\frac{1}{3}\left(64x^3 - \frac{1}{x^3}\right)$$

$$\frac{1}{3}\int_{1/2}^{1/\sqrt{2}} \left(64x^5 - \frac{1}{x}\right)dx = \frac{1}{3}\left(\frac{64x^6}{6} - \ln{x}\right)\Big|_{1/2}^{1/\sqrt{2}} = \frac{1}{3}\left(\frac{4}{3} - \frac{1}{6}\right) - \frac{1}{3}\left(\ln{\frac{1}{\sqrt{2}}} - \ln{\frac{1}{2}}\right) =$$

$$= \frac{7}{18} - \frac{1}{3}\left(\frac{1}{2}\ln{2}\right)$$

$II: \qquad (1/\sqrt{2}, 2\sqrt{2}) \to (1,1)$

$$\int_{1/x}^{2/x} x^2y^2dy = x^2\frac{1}{3}\left(\frac{8}{x^3} - \frac{1}{x^3}\right) = \frac{7}{3}\frac{1}{x}$$

$$\int_{1/\sqrt{2}}^1 \frac{7}{3}\frac{1}{x} = \frac{7}{3} - \ln{\frac{1}{\sqrt{2}}} = \frac{7}{6}\ln{2}$$

$III: \qquad (1,1) \to (\sqrt{2}, \sqrt{2})$

$$\int_x^{2/x} x^2y^2dy = x^2\frac{1}{3}\left(\frac{8}{x^3} - x^3\right) = \frac{8}{3}\frac{1}{x} - \frac{1}{3}x^5$$

$$\int_1^{\sqrt{2}} \frac{8}{3}\frac{1}{x} - \frac{1}{3}x^5 = \frac{8}{3}\ln{\sqrt{2}} - \frac{1}{18}x^6\Big|_1^{\sqrt{2}} = \frac{4}{3}\ln{2} - \frac{7}{18}$$

$$\Longrightarrow \frac{4}{3}\ln{2} - \frac{7}{18} + \frac{7}{6}\ln{2} + \frac{7}{18} - \frac{1}{6}\ln{2} = \boxed{\frac{7}{3}\ln{2}}$$

Exercise 5. $\iint_S (x^2-y^2)dxdy$.

$$\int_0^{\sin x}(x^2-y^2)dy = x^2\sin x - \frac{1}{3}\sin^3 x = x^2\sin x - \frac{1}{3}(1-\cos^2 x)\sin x$$

$$\int x^2 \sin x = -x^2\cos x + 2x\sin x + 2\cos x$$

$$\iint_S (x^2-y^2)dxdy = -\pi^2(-1) + 2((-1)-1) + \frac{1}{3}(-1-1) + \left(\frac{\cos^3 x}{-9}\right)\Bigg|_0^{\pi} = \pi^2 + -4 - \frac{2}{3} + \frac{2}{9} = \boxed{\frac{-40}{9} + \pi^2}$$

Exercise 6. $x+2y+3z=6 \Longrightarrow z = \frac{6-x-2y}{3}$

$$\int_0^{3-\frac{x}{2}} \left(\frac{6-x-2y}{3}\right)dy = \frac{1}{3}\left(6\left(3-\frac{x}{2}\right) - x\left(3-\frac{x}{2}\right) - \left(3-\frac{x}{2}\right)^2\right) = \frac{1}{3}\left(18-3x-3x+\frac{x^2}{2} - \left(9-3x+\frac{x^2}{4}\right)\right) =$$

$$= \frac{1}{3}\left(9-3x+\frac{x^2}{4}\right)$$

$$\int_0^6 \frac{1}{3}\left(9-3x+\frac{x^2}{4}\right) = \frac{1}{3}\left(9(6) - \frac{3(6)^2}{2} + \frac{1}{12}6^3\right) = \frac{1}{3}(6)(9-9+3) = \boxed{6}$$

Indeed, $\frac{1}{3}BH = \frac{1}{3}(9)(2) = 6$

Exercise 7.

$$\int_{-x}^{x}(x^2-y^2)dy = x^2(x-(-x)) - \frac{1}{3}(x^3-(-x)^3) = 2x^3 - \frac{1}{3}(x^3+x^3) = \frac{4}{3}x^3$$

$$\int_1^3 \frac{4}{3}x^3 dx = \frac{1}{3}x^4\Bigg|_1^3 = \frac{1}{3}(81-1) = \boxed{\frac{80}{3}}$$

Exercise 8.

(1)

$$\int_{-1}^{1}(x^2+y^2)dy = x^2(2) + \frac{1}{3}(1^3-(-1)^3) = 2x^2 + \frac{2}{3}$$

$$\int_{-1}^{1} 2x^2 + \frac{2}{3} = \frac{2}{3}(2) + \frac{2}{3}(2) = \boxed{8/3}$$

(2) $f(x,y) = 3x+y$. $S = \{(x,y)|4x^2+9y^2 \leq 36,\ x>0,\ y>0\}$. Thus, $y^2 \leq 4 - \frac{4}{9}x^2$

$$\int_0^{2\sqrt{1-(x/3)^2}}(3x+y)dy = 3x2\sqrt{1-\left(\frac{x}{3}\right)^2} + \frac{1}{2}4(1-\left(\frac{x}{3}\right)^2) = 6x\sqrt{1-\left(\frac{x}{3}\right)^2} + 2(1-\left(\frac{x}{3}\right)^2)$$

$$\int_0^3 6x\sqrt{1-\left(\frac{x}{3}\right)^2} + 2(1-\frac{x^2}{9}) = -18(1-\frac{x^2}{9})^{3/2}\Bigg|_0^3 + 2(3) - \frac{2}{27}(27) = \boxed{22}$$

$$\text{since } ((1-\frac{x^2}{9})^{3/2})' = \frac{3}{2}(1-\frac{x^2}{9})^{1/2}(\frac{-2x}{9}) = \frac{-x}{3}(1-\frac{x^2}{9})^{1/2}$$

(3)

$$\int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}}(y+2x+20)dy = (2x+20)(2\sqrt{16-x^2}) = 4x\sqrt{16-x^2} + 40\sqrt{16-x^2}$$

$$\int_{-4}^{4} 4x\sqrt{16-x^2} + 40\sqrt{16-x^2} = \int_0^{\pi}(16\cos\theta 4\sin\theta + 40(4)\sin\theta)4\sin\theta d\theta = \frac{640}{2}\pi = \boxed{320\pi}$$

$$\text{since} \quad \begin{aligned} x &= 4\cos\theta \\ dx &= -4\sin\theta d\theta \end{aligned}$$

Exercise 9. $\int_0^1\left(\int_0^y f(x,y)dx\right)dy = \int_0^1\left(\int_x^1 f(x,y)dy\right)dx$ **Exercise 10.** $\int_0^2\left(\int_{y^2}^{2y} f(x,y)dx\right)dy = \int_0^4\left(\int_{x/2}^{\sqrt{x}} f(x,y)dy\right)dx$

Exercise 11. $\int_1^4\left(\int_{\sqrt{x}}^2 f(x,y)dy\right)dx = \int_0^2\left(\int_0^{y^2} f(x,y)dx\right)dy$

Exercise 12. $\int_1^2\left(\int_{2-x}^{\sqrt{2x-x^2}} f(x,y)dy\right)dx = \int_0^1\left(\int_{2-y}^{\sqrt{1-y^2}+1} f(x,y)dx\right)dy$ **Exercise 13.** $\int_{-6}^{2}\left(\int_{(x^2-4)/4}^{2-x} f(x,y)dy\right)dx =$ $\int_{-1}^{0}\left(\int_{-\sqrt{4y+4}}^{\sqrt{4y+4}} f(x,y)dx\right)dy + \int_0^8\left(\int_{-\sqrt{4y+4}}^{2-y} f(x,y)dx\right)dy$ **Exercise 14.** $\int_1^e\left(\int_0^{\log x} f(x,y)dy\right)dx = \int_0^1\left(\int_{e^y}^{e} f(x,y)dx\right)dy$

Exercise 15. $\int_{-1}^{1}\left(\int_{-\sqrt{1-x^2}}^{1-x^2} f(x,y)dy\right)dx = \int_0^1\left(\int_{-\sqrt{1-y}}^{\sqrt{1-y}} f(x,y)dx\right)dy + \int_{-1}^{0}\left(\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x,y)dx\right)dy$ **Exercise 16.**

$\int_0^1 \left(\int_{x^3}^{x^2} f(x,y)dy\right) dx = \int_0^1 \left(\int_{\sqrt{y}}^{y^{1/3}} f(x,y)dx\right) dy$

Exercise 17. Consider that $\sin(\pi - x) = -1\sin(-x) = \sin x = y$. This way, we get the "branch" of values for $\pi/2 < x < \pi$ and $0 < y < 1$.

$$\int_0^\pi \left(\int_{-\sin(x/2)}^{\sin x} f(x,y)dy\right) dx = \int_0^1 \left(\int_{\arcsin y}^{\pi - \arcsin y} f(x,y)dx\right) dy + \int_{-1}^0 \int_{2\arcsin(-y)}^{\pi} f(x,y)dxdy$$

Exercise 18. $\int_0^4 \left(\int_{-\sqrt{4-y}}^{(y-4)/2} f(x,y)dx\right) dy = \int_{-2}^0 \left(\int_{2x+4}^{4-x^2} f(x,y)dy\right) dx$

Exercise 19.

$$V = \int_0^1 \left(\int_0^y (x^2+y^2)dx\right) dy + \int_1^2 \left(\int_0^{2-y} (x^2+y^2)dx\right) dy = \int_0^1 \left(\int_x^{-2x} (x^2+y^2)dy\right) dx$$

$$\int_x^{2-x} (x^2+y^2)dy = x^2(2-x-x) + \frac{1}{3}((2-x)^3 - x^3) = x^2(2-2x) + \frac{1}{3}(8 - 4x(3) + 3(2)x^2 - x^3 - x^3) =$$

$$= 2x^2 - 2x^3 + \frac{8}{3} - 4x + 2x^2 - \frac{2x^3}{3} = \frac{-8}{3}x^3 + 4x^2 - 4x + \frac{8}{3}$$

$$\xrightarrow{\int_0^1} \frac{-2}{3} + \frac{4}{3} - 2 + \frac{8}{3} = \boxed{\frac{4}{3}}$$

Exercise 21.

(1) $V = \int_1^2 \left(\int_x^{x^3} f(x,y)dy\right) dx + \int_2^8 \left(\int_x^8 f(x,y)dy\right) dx = \int_1^8 \left(\int_{y^{1/3}}^y f(x,y)dx\right) dy$

(2)

Exercise 22. $I = \int_{-1/2}^1 \left(\int_0^x e^{-y^2}dy\right) dx$

$$\int_{-1/2}^1 \left(\int_0^x e^{-y^2}dy\right) dx = \int_0^1 \left(\int_0^x e^{-y^2}dy\right) dx + \int_{-1/2}^0 \left(\int_0^x e^{-y^2}dy\right) dx$$

$$\int_0^1 \left(\int_0^x e^{-y^2}dy\right) dx = \int_0^1 \left(\int_y^1 e^{-y^2}dx\right) dy = \int_0^1 e^{-y^2}(1-y)dy = A + \int_0^1 -ye^{-y^2}dy =$$

$$= A + \left.\left(\frac{e^{-y^2}}{2}\right)\right|_0^1 = A + \frac{e^{-1}}{2} - \frac{1}{2}$$

$$\text{since} \begin{array}{c} z = -x \\ dz = -dx \end{array}$$

$$\int_{-1/2}^0 \left(\int_0^x e^{-y^2}dy\right) dx = -\int_{1/2}^0 \left(\int_0^{-x} e^{-y^2}dy\right) dx = \int_0^{1/2} \left(\int_0^{-x} e^{-y^2}dy\right) dx = -\int_0^{1/2} \left(\int_{-x}^0 e^{-y^2}dy\right) dx =$$

$$= -\int_{-1/2}^0 \left(\int_{-y}^{1/2} e^{-y^2}dx\right) dy = -\int_{-1/2}^0 e^{-y^2}\left(\frac{1}{2} - (-y)\right) dy =$$

$$= \frac{-1}{2}\int_{-1/2}^0 e^{-y^2}dy + -\int_{-1/2}^0 ye^{-y^2}dy = \frac{1}{2}\int_{1/2}^0 e^{-y^2}(+dy) + -\left.\left(\frac{-e^{-y^2}}{2}\right)\right|_{-1/2}^0 =$$

$$= \frac{-1}{2}\int_0^{1/2} e^{-y^2}dy + \frac{1}{2} + \frac{-e^{-1/4}}{2}$$

$$\Longrightarrow I = 2A + e^{-1} - 1 + -B + 1 - e^{-1/4} = \boxed{2A - B + e^{-1} - e^{-1/4}}$$

Exercise 23.

(1) Suppose S is a type I region, without loss of generality.
Use the geometry of similar triangles. Thus, observe that the cross-sectional area is a projection of plane region S (from the geometry of similar triangles). Now express this mathematically.

$$A = \int_a^b dx \int_{\phi_1(x)}^{\phi_2(x)} dy = \int_a^b \int_{\phi_1(x)}^{\phi_2 x} dydx$$

$$\int_{at/h}^{bt/h} dx \int_{\phi_1(x)\left(\frac{t}{h}\right)}^{\phi_2(x)\left(\frac{t}{h}\right)} dy = \int_{at/h}^{bt/h} dx \int_{\phi_1}^{\phi_2} dY \frac{t}{h} = \left(\frac{t}{h}\right)^2 \int_a^b dX \int_{\phi_1}^{\phi_2} dY = \left(\frac{t}{h}\right)^2 A$$

(2) $\int_0^h \frac{t^2}{h^2} A = \boxed{\frac{1}{3} hA}$

Exercise 24.

$$\int_0^1 \left(\int_x^a e^{m(a-x)} f(x) dy\right) dx = \int_0^1 e^{m(a-x)} f(x)(a-x) dx$$

11.18 Exercises - Further applications of double integrals, Two theorems of Pappus

Exercise 1.

$$\int_{-2}^1 \int_{x^2}^{2-x} x dy dx = \int_{-2}^1 x((2-x)-x^2) = \left(x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4\right)\Big|_{-2}^1 = 1 - 4 - \frac{1}{3}(1+8) - \frac{1}{4}(1-16) = -9/4$$

$$\int_{-2}^1 \int_{x^2}^{2-x} y dy dx = \int_{-2}^1 \frac{1}{2}((2-x)^2 - x^4) dx = \int_{-2}^1 \frac{1}{2}(4 - 4x + x^2 - x^4) dx = 2(1+2) - x^2\big|_{-2}^1 + \frac{1}{8}x^3\Big|_{-2}^1 - \frac{1}{10}x^5\Big|_{-2}^1 =$$

$$= 6 + \frac{1}{6}(1+8) - \frac{1}{10}(1+32) = \frac{72}{10}$$

$$\int_2^1 \int_{x^2}^{2-x} dy dx = \int_{-2}^1 (2 - x - x^2) = 2(1+2) - \frac{1}{2}(1-4) - \frac{1}{3}(1-(-8)) = 9/2$$

$$\overline{x} = -1/2, \ \overline{y} = 8/5$$

Exercise 2.
$$\begin{aligned} y^2 &= x+3 \qquad (-3,0),\ (1,\pm 2) \\ y^2 &= 5-x \qquad (1,\pm 2),\ (5,0) \end{aligned}$$

$$\int_{-2}^2 \left(\int_{y^2-3}^{5-y^2} dx\right) dy = \int_{-2}^2 5 - y^2 - (y^2-3) dy = \int_{-2}^2 8 - 2y^2 dy = 8(4) - \frac{2}{3}y^3\Big|_{-2}^2 = 32 - \frac{2}{3}(8-(-8)) =$$

$$= \frac{96-32}{3} = \boxed{\frac{64}{3}}$$

$$\int_{-2}^2 \left(\int_{y^2-3}^{5-y^2} x dx\right) dy = \int_{-2}^2 \frac{1}{2}(25 - 10y^2 + y^4 - (y^4 - 6y^2 + 9)) dy = \frac{1}{2}\int_{-2}^2 (16 - 4y^2) dy = \frac{1}{2}\left(16(4) - \frac{4}{3}y^3\Big|_{-2}^2\right) =$$

$$= \frac{1}{2}(64 - 4/3(16)) = \frac{1}{2}\left(\frac{192-64}{3}\right) = \frac{1}{6}(128) = \frac{64}{3}$$

$$\int_{-2}^2 \int_{y^2-3}^{5-y^2} y dx dy = \int_{-2}^2 8y - 2y^3 = 0$$

$$\overline{x} = 1, \quad \overline{y} = 0$$

Exercise 3.
$$\begin{aligned} x - 2y + 8 &= 0 \qquad x = -2 \\ x + 3y + 5 &= 0 \qquad x = 4 \end{aligned}$$

$$\int_{-2}^4 \int_{\frac{5+x}{-3}}^{\frac{x+8}{2}} dy dx = \int_{-2}^4 \left(\frac{3x + 24 + 10 + 2x}{6}\right) dx = \int_{-2}^4 \left(\frac{5x+34}{6}\right) dx = \frac{5x^2}{12}\Big|_{-2}^4 + \frac{17}{3}(6) = \frac{5}{12}(12) + 34 = -39$$

$$\overline{x}A = \int_{-2}^4 dx \left(\frac{5x^2 + 34x}{6}\right) = \frac{1}{6}\left(\frac{5}{3}x^3 + 17x^2\right)\Big|_{-2}^4 = \frac{1}{6}\left(\frac{5}{3}(64+8) + 17(16-4)\right) = \frac{1}{6}(120 + 17(12)) = 54$$

$$\overline{y}A = \int_{-2}^4 dx \frac{1}{2}\left(\frac{(x^2 + 16x + 64)}{4} - \frac{25 + 10x + x^2}{9}\right) = \int_{-2}^4 \frac{dx}{2}\left(\frac{9x^2 + 9(16x) + 9(64) - 100 - 40x - 4x^2}{36}\right) =$$

$$= \frac{1}{2(36)} \int_{-2}^4 dx (5x^2 + 18(8x) - 8(5x) + 9(4)(16) - 4(25)) = \frac{1}{36}\left(\frac{5}{3}x^3 + 52x^2 + 119(4x)\right)\Big|_{-2}^4 =$$

$$= \frac{1}{2(36)}\left(\frac{5}{3}(64+8) + 52(16-4) + 119(4)(6)\right) = \boxed{50}$$

$$\boxed{\overline{y} = \frac{50}{39} \quad \overline{x} = \frac{18}{13}}$$

Exercise 4. $y = \sin^2 x; \quad y = 0; \;\; 0 \leq x \leq \pi$

$$\int_0^\pi \int_0^{\sin^2 x} dydx = \int_0^\pi \sin^2 x dx = \int_0^\pi \left(\frac{1-\cos 2x}{2} \right) dx = \frac{\pi}{2}$$

$$\int_0^\pi \int_0^{\sin^2 x} xdydx = \int_0^\pi x\sin^2 x dx = \int_0^\pi x \left(\frac{1-\cos 2x}{2} \right) dx = \frac{\pi^2}{4} - \frac{1}{2} \left(\frac{x\sin 2x}{2} + \frac{\cos 2x}{4} \right) \Big|_0^\pi = \frac{\pi^2}{4}$$

$$\int_0^\pi \int_0^{\sin^2 x} ydydx = \int_0^\pi \frac{1}{2} \sin^4 x dx = \frac{1}{2} \int_0^\pi \left(\frac{1-\cos 2x}{2} \right)^2 dx = \frac{1}{8} \int_0^\pi 1 - 2\cos 2x + \cos^2 2x = \frac{1}{8}(\pi + \frac{\pi}{2}) = \frac{3\pi}{16}$$

$$\overline{x} = \pi^2/4/\pi/2 = \boxed{\pi/2} \quad \overline{y} = \frac{\frac{3\pi}{16}}{\frac{\pi}{2}} = \boxed{\frac{3}{8}}$$

Exercise 5.

$$\int_0^{\pi/4} \int_{\sin x}^{\cos x} dydx = \int_0^{\pi/4} \cos x - \sin x = (\sin x + \cos x)|_0^{\pi/4} = \sqrt{2} - 1$$

$$\int_0^{\pi/4} \int_{\sin x}^{\cos x} xdydx = \int_0^{\pi/4} x(\cos x - \sin x)dx = (x\sin x + \cos x + x\cos x - \sin x)|_0^{\pi/4} = \frac{\pi}{2}\frac{\sqrt{2}}{2} - 1 \qquad \boxed{\overline{x} = \frac{\frac{\pi\sqrt{2}}{4} - 1}{\sqrt{2}-1}}$$

$$\int_0^{\pi/4} \frac{1}{2}(\cos^2 x - \sin^2 x)dx = \frac{1}{2}\int_0^{\pi/4} \cos 2x dx = \frac{1}{4}\sin 2x \Big|_0^{\pi/4} = \frac{1}{4} \qquad \boxed{\overline{y} = \frac{1/4}{\sqrt{2}-1}}$$

Exercise 6. $\begin{matrix} y = \log x \\ y = 0 \end{matrix}, \quad 1 \leq x \leq a.$

$$\int_1^a \int_0^{\log x} dydx = \int_1^a \log x dx = (x\log x - x)|_1^a = a\log a - a + 1$$

$$\int_1^a \int_0^{\log x} xdydx = \int_1^a xdx \log x = \left(\frac{x^2 \log x}{2} - \frac{x^2}{4} \right)\Big|_1^a = \frac{a^2\log a}{2} - \frac{(a^2-1)}{4} \qquad \Longrightarrow \boxed{\overline{x} = \frac{\left(\frac{2a^2\log a - (a^2-1)}{4} \right)}{a\log a - a + 1}}$$

$$\int_1^a \int_0^{\log x} ydydx = \int_1^a \frac{1}{2}(\log x)^2 dx = \frac{1}{2}\left((\log x)^2 x - 2x\log x + 2x \right)\Big|_1^a = \frac{1}{2}(a(\log a)^2 - 2a\log a + 2a - 2) =$$

$$= \frac{1}{2}a(\log a)^2 - a\log a + a - 1$$

$$\begin{aligned} ((\log x)^2 x)' &= (\log x)^2 + 2(\log x) \\ (x\log x - x)' &= \log x \end{aligned} \qquad \Longrightarrow \boxed{\overline{y} = \frac{\frac{1}{2}a(\log a)62 - a\log a + a - 1}{a\log a - a + 1}}$$

Exercise 7. $\sqrt{x} + \sqrt{y} = 1$ or $\sqrt{y} = 1 - \sqrt{x}$. $x = 0, y = 0$. So $y = 1 - 2\sqrt{x} + x$.

$$\int_0^1 \int_0^{1-2\sqrt{x}+x} dydx = \int_0^1 (1 - 2\sqrt{x} + x) = (x - \frac{4}{3}x^{3/2} + \frac{1}{2}x^2)\Big|_0^1 = 1 - \frac{4}{3} + \frac{1}{2} = \boxed{\frac{1}{6}}$$

$$\int_0^1 \int_0^{1-2\sqrt{x}+x} ydydx = \int_0^1 dx\frac{1}{2}(1 - 4\sqrt{x} + 2x + 4x - 4x^{3/2} + x^2) = \int_0^1 dx\frac{1}{2}(1 - 4x^{1/2} + 6x - 4x^{3/2} + x^2) =$$

$$= \frac{1}{2}\left(x - \frac{8}{3}x^{3/2} + 3x^2 - \frac{8x^{5/2}}{5} + \frac{1}{3}x^3)\right|_0^1 = \frac{1}{30}$$

$$\Longrightarrow \boxed{\overline{y} = 1/5}$$

$$\int_0^1 \int_0^{1-2\sqrt{x}+x} xdydx = \int_0^1 x(1 - 2\sqrt{x} + x)dx = \left(\frac{1}{2}x^2 - 2\frac{2}{5}x^{5/2} + \frac{1}{3}x^3 \right)\Big|_0^1 = \frac{1}{3}$$

$$\Longrightarrow \boxed{\overline{x} = 1/5}$$

Exercise 8. $x^{2/3} + y^{2/3} = 1$ or $\begin{matrix} y^{2/3} = 1 - x^{2/3} \\ y = (1 - x^{2/3})^{3/2} \end{matrix}$ and $x = 0, \;\; y = 0$.

$$u = 1 - x^{2/3} \qquad \text{or } x^{2/3} = 1 - u$$

$$\text{Since } du = \frac{-2}{3}x^{-1/3}dx$$

$$\frac{-3}{2}du(\sqrt{1-u}) = dx$$

$$\int_0^1 \int_0^{(1-x^{2/3})^{3/2}} dydx = \int_0^1 (1-x^{2/3})^{3/2}dx = \int_1^0 u^{3/2}\left(\frac{-3}{2}\right)du(1-u)^{1/2}$$

$$\int u^{3/2}(1-u)^{1/2} = \int u^{3/2}\left(\frac{-2}{3}(1-u)^{3/2}\right)' = u^{3/2}\frac{-2}{3}(1-u)^{3/2} - \int \frac{3}{2}u^{1/2}\frac{-2}{3}(1-u)(1-u)^{1/2} =$$

$$= \frac{-2}{3}u^{3/2}(1-u)^{3/2} + \int (u^{1/2}(1-u)^{1/2} - u^{3/2}(1-u)^{1/2})$$

$$\implies 2\int u^{3/2}(1-u)^{1/2} = \frac{-2}{3}u^{3/2}(1-u)^{3/2} + \int u^{1/2}(1-u)^{1/2}$$

$$\int u^{1/2}(1-u)^{1/2} = \int (u-u^2)^{1/2} = \int \left(\frac{1}{4} - \left(\frac{1}{2}-u\right)^2\right)^{1/2} = \frac{1}{2}\int \sqrt{1-(1-2u)^2}$$

$$\text{Since } \begin{aligned} x &= (1-2u) \\ dx &= -2du \end{aligned} \quad \text{and} \quad \begin{aligned} x &= \sin\theta \\ dx &= \cos\theta d\theta \end{aligned} \text{ then}$$

$$\int \sqrt{1-(1-2u)^2} = \int \frac{dx}{-2}\sqrt{1-x^2} = \int \frac{\cos\theta d\theta}{-2}\cos\theta = \int \frac{1+\cos 2\theta}{-4} = \frac{-1}{4}(\theta + \frac{\sin 2\theta}{2})$$

$$\int_0^1 u^{3/2}(1-u)^{1/2}du = \frac{1}{2}\int_0^1 u^{1/2}(1-u)^{1/2} = \frac{1}{4}\int_0^1 \sqrt{1-(1-2u)^2} = \frac{-1}{8}\int_1^{-1} dx\sqrt{1-x^2} =$$

$$= \frac{-1}{8}\int_{\pi/2}^{-\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)d\theta =$$

$$= \frac{-1}{16}\left(\theta + \frac{\sin 2\theta}{2}\right)\Bigg|_{\pi/2}^{-\pi/2} = \frac{-1}{16}\left(\frac{-\pi}{2} - \frac{\pi}{2}\right) = \boxed{\pi/16}$$

$$\implies A = \frac{3}{2}\frac{\pi}{16} = \boxed{\frac{3\pi}{32}}$$

$$\int_0^1 \int_0^{(1-x^{2/3})^{3/2}} xdydx = \int_0^1 x(1-x^{2/3})^{3/2}dx = \int_1^0 \frac{-3}{2}du\sqrt{1-u}u^{3/2}(1-u)^{3/2} = \frac{3}{2}\int_0^1 du(1-2u+u^2)u^{3/2} =$$

$$= \frac{3}{2}\int_0^1 du(u^{3/2} - 2u^{5/2} + u^{7/2}) =$$

$$= \frac{3}{2}\left(\frac{2}{5} - 2\left(\frac{2}{7}\right) + \frac{2}{9}\right) = \frac{3}{2}\left(\frac{2}{5} - \frac{4}{7} + \frac{2}{9}\right) = \frac{8}{105}$$

$$\implies \boxed{\overline{x} = 256\pi/315}$$

$$\int_0^1 \int_0^{(1-x^{2/3})^{3/2}} ydydx = \int_0^1 \frac{1}{2}(1-x^{2/3})^3dx = \frac{1}{2}\int_0^1 (1+3(-x^{2/3}) + 3x^{4/3} - x^2)dx =$$

$$= \frac{1}{2}\left(1 + -3\left(\frac{3(1)^{5/3}}{5}\right) + 3\frac{3}{7} - \frac{1}{3}\right) = \frac{1}{2}\left(1 + \frac{-9}{5} + \frac{9}{7} + \frac{-1}{3}\right) = \frac{1}{2}\left(\frac{-4}{5} + \frac{9}{7} - \frac{1}{3}\right) = \frac{8}{105}$$

$$\implies \boxed{\overline{y} = 256\pi/315}$$

Exercise 9. $\int_0^2 \int_0^{x(2-x)} \left(\frac{1-y}{1+x}\right) dydx = \int_0^2 \frac{1}{1+x}\int_0^{x(2-x)}(1-y)dydx$

$$\frac{1}{2}(x(2-x))^2 = \frac{1}{2}x^2(4-4x+x^2) = 2x^2 - 2x^3 + \frac{1}{2}x^4$$

$$\int_0^{x(2-x)}(1-y)dy = 2x - x^2 - 2x^2 + 2x^3 - \frac{1}{2}x^4 = 2x - 3x^2 + 2x^3 - \frac{1}{2}x^4 =$$

$$= \frac{-1}{2}x^4 - \frac{1}{2}x^3 + \frac{5}{2}x^3 + \frac{5}{2}x^2 - \frac{11}{2}x^2 - \frac{11}{2}x + \frac{15}{2}x = \left(\frac{-1}{2}x^3 + \frac{5}{2}x^2 - \frac{11}{2}x\right)(x+1) + \frac{15}{2}x$$

$$\int_0^2 \frac{1}{1+x}\int_0^{x(2-x)}(1-y)dydx = \int_0^2 \frac{1}{1+x}\left(\left(\frac{-1}{2}x^3+\frac{5}{2}x^2-\frac{11x}{2}\right)(x+1)+\frac{15x}{2}\right) =$$
$$= \int_0^2 dx\left(\left(\frac{-1}{2}x^3+\frac{5}{2}x^2-\frac{11x}{2}\right)+\frac{15x}{12(1+x)}\right) =$$
$$= \frac{-1}{8}(16)+\frac{5}{6}(8)-\frac{11}{4}(4)+\frac{15}{2}\left(2+\ln(1+x)|_0^2\right) = -2+\frac{20}{3}-11+\frac{15}{2}(2-\ln(3)) = \boxed{\frac{26}{3}+\frac{-15}{2}\ln 3}$$

Exercise 10.

$$\int_0^a\int_0^b (xy)dydx = \int_0^a x\frac{1}{2}b^2dx = \frac{1}{4}a^2b^2$$
$$\int_0^a\int_0^b (xy^2)dydx = \int_0^a x\frac{1}{3}b^3dx = \frac{1}{6}a^2b^3$$
$$\int_0^a\int_0^b x^2ydydx = \int_0^a x^2\frac{1}{2}b^2 = \frac{1}{6}b^2a^3$$

$$\boxed{\overline{x} = \frac{\frac{1}{6}b^2a^3}{\frac{1}{4}a^2b^2} = \frac{2}{3}a \quad \overline{y} = \frac{\frac{1}{6}a^2b^3}{\frac{1}{4}a^2b^2} = \frac{2}{3}b}$$

Exercise 11.

$$\begin{aligned} y &= \sin^2 x \\ y &= -\sin^2 x \end{aligned} \qquad -\pi \le x \le \pi; \;\; f(x,y)=1$$

$$I_x = \int_{-\pi}^{\pi}\left(\int_{-\sin^2 x}^{\sin^2 x} y^2dy\right)dx = \int_{-\pi}^{\pi}\frac{1}{3}2\sin^6 xdx = \frac{2}{3}\int_{-\pi}^{\pi}\left(\frac{1-\cos 2x}{2}\right)^3 =$$
$$= \frac{1}{12}\int_{-\pi}^{\pi} 1+-3\cos 2x+3\cos^2 2x+\cos^3 2x = \boxed{\frac{5\pi}{12}}$$

$$I_y = \int_{-\pi}^{\pi}\int_{-\sin^2 x}^{\sin^2 x} x^2dydx = \int_{-\pi}^{\pi} x^2 2\sin^2 xdx = 2\int_{-\pi}^{\pi} 2x^2\left(\frac{1-\cos 2x}{2}\right)dx =$$
$$= \int_{-\pi}^{\pi} x^2 - x^2\cos 2x = \boxed{\frac{2\pi^3}{3}-\pi}$$

Exercise 12. $\frac{x}{a}+\frac{y}{b}=1$, $\frac{x}{c}+\frac{y}{b}=1$, $y=0$, $0<c<a$, $b>0$; $f(x,y)=1$

$$I_y = \int_0^b\left(\int_{c(1-y/b)}^{a(1-y/b)} x^2dx\right)dy = \int_0^b\left(\frac{a^3(1-y/b)^3-c^3(1-y/b)^3}{3}\right)dy = \left(\frac{a^3-c^3}{3}\right)(-b)\left(1-\frac{y}{b}\right)^4/b\Big|_a^b = \boxed{(a^3-c^3)\frac{b}{12}}$$

$$I_x = \int_0^b y^2(a-c)(1-y/b)dy = (a-c)\left(\frac{1}{3}b^3-\frac{1}{4}\frac{b^4}{b}\right) = \boxed{(a-c)(b^3)/12}$$

Exercise 13. $(x-r)^2+(y-r)^2=r^2$, $y=0$, $x=0$. $0\le x\le r$, $0\le y\le r$. So we want the piece of the graph that is the "lower left-hand corner" of the "complement" of the circle centered at (r,r), with radius r. Be careful about this point.

$$\text{since} \quad \begin{aligned} x-r &= r\sin\theta \\ dx &= r\cos\theta d\theta \end{aligned}$$

$$I_x = \int_0^r\int_0^{r-\sqrt{r^2-(x-r)^2}} y^2dydx = \int_0^r\frac{1}{3}r^3\left(1-\sqrt{1-\left(\frac{x-r}{r}\right)^2}\right)^3dx = \frac{r^4}{3}\int_{-\pi/2}^{0}(1-\cos\theta)^3\cos\theta d\theta =$$
$$= \frac{r^4}{3}\int_{-\pi/2}^{0} c-3c^2+3c^3-c^4 = \boxed{r^4\left(1-\frac{5\pi}{16}\right)}$$

Since

$$\int_{-\pi/2}^{0} c = s|_{-\pi/2}^{0} = -(-1) = 1$$

$$\int_{-\pi/2}^{0} c^2 = \int_{-\pi/2}^{0} \frac{1+\cos 2\theta}{2} d\theta = \frac{\pi}{4}$$

$$\int_{-\pi/2}^{0} c^3 = \int_{-\pi/2}^{0} c(1-s^2) = 1 - \frac{1}{3}(0-(-1)) = 2/3$$

$$\int_{-\pi/2}^{0} c^4 = \int_{-\pi/2}^{0} \left(\frac{1+\cos 2\theta}{2} \right)^2 d\theta = \int_{-\pi/2}^{0} \frac{1}{4}(1 + 2\cos 2\theta + \cos^2 2\theta) d\theta = \frac{1}{4}\left(\frac{\pi}{2} + \frac{\pi}{4} \right) = \frac{3\pi}{16}$$

$$I_y = \int_0^r \int_0^{r-\sqrt{r^2-(x-r)^2}} x^2 dy dx = \int_0^r x^2 r \left(1 - \sqrt{1 - \left(\frac{x-r}{r} \right)^2} \right) dx = \int_{-\pi/2}^{0} (r^2)(\sin\theta + 1)^2 r (1-\cos\theta) r\cos\theta d\theta =$$

$$= r^4 \int_{-\pi/2}^{0} (s^2+2s+1)(1-c)c d\theta = r^4 \int_{-\pi/2}^{0} (s^2c + 2sc + c - c^2s^2 - 2sc^2 - c^2) d\theta = \boxed{r^4\left(1 - \frac{5\pi}{16}\right)}$$

Since

$$\int_{-\pi/2}^{0} s^2 c = \frac{1}{3}s^3\Big|_{-\pi/2}^{0} = \frac{1}{3} \qquad \int c^2 s^2 = \int \left(\frac{\sin 2\theta}{2} \right)^2 = \frac{1}{4}\int_{-\pi/2}^{0} \frac{1-\cos 4\theta}{2} = \frac{\pi}{16}$$

$$\int c = \frac{1}{2}s^2\Big|_{-\pi/2}^{0} = \frac{-1}{2} \qquad \int sc^2 = \left(\frac{-1}{3}c^3 \right)\Big|_{-\pi/2}^{0} = -1/3$$

$$\int c = 1 \qquad \int c^2 = \pi/4$$

Exercise 16. $y = \sqrt{2x}, \;\; y=0, \;\; 0 \leq x \leq 2, \;\; f(x-y) = |x-y|$

$$\begin{aligned}
I_y &= \int_0^2 \int_0^x x^2 dy dx (x-y) + \int_0^2 \int_x^{\sqrt{2x}} x^2 dy dx (y-x) = \\
&= \int_0^2 \left(x^3(x) - x^2 \frac{1}{2} x^2 \right) + \int_0^2 x^2 \left(\frac{1}{2}((2x) - x^2) \right) - x^3(\sqrt{2x} - x) = \\
&= \frac{1}{10}\, x^5|_0^2 + \frac{1}{4}\, x^4|_0^2 - \frac{1}{10}\, x^5|_0^2 - \sqrt{2}\, \frac{2x^{9/2}}{9}\Big|_0^2 + \frac{1}{5}2^5 = 4 - \frac{2^6}{9} + \frac{2^5}{5} = \boxed{\frac{148}{45}}
\end{aligned}$$

$$\begin{aligned}
I_x &= \int_0^2 \int_0^x y^2 dy dx (x-y) + \int_0^2 \int_x^{\sqrt{2x}} y^2 dy dx (y-x) = \\
&= \int_0^2 \left(\frac{1}{3}y^3\Big|_0^x x - \frac{1}{4}\, y^4|_0^4 \right) dx + \int_0^2 \left(\frac{1}{4}y^4\Big|_x^{\sqrt{2x}} - x\, \frac{1}{3}y^3\Big|_x^{\sqrt{2x}} \right) dx = \\
&= \int_0^2 \left(\frac{1}{3}x^4 - \frac{1}{4}x^4 \right) dx + \int_0^2 \frac{1}{4}(4x^2 - x^4) - \frac{x}{3}(2^{3/2}x^{3/2} - x^3) = \\
&= \int_0^2 \left(\frac{1}{12}x^4 + x^2 - \frac{x^4}{4} - \frac{2^{3/2}}{3}x^{5/2} + \frac{x^4}{3} \right) = \int_0^2 \frac{1}{6}x^4 + x^2 - \frac{2^{3/2}}{3}x^{5/2} = \\
&= \frac{1}{30}(2^5) + \frac{1}{3}2^3 - \frac{2^{3/2}}{3}\frac{2}{7}2^{7/2} = \boxed{\frac{24}{35}}
\end{aligned}$$

Exercise 17. Let S be thin plate of mass m.
Let the center of mass of thin plate S be located at the coordinate axis origin.
Let L_0, L be parallel to the x axis.
Let h be the perpendicular distance of L from L_0, with the sign of h included.
Since m is the mass, $\int_a^b \int_{\phi_1(x)}^{\phi_2(x)} dy dx = m$
Since CM is at $(0,0)$, $\overline{y} = 0 = \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} y dy dx = 0$

$$\text{moment of inertia about } L = \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} (y-h)^2 dy dx = \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} y^2 - 2yh + h^2 dy = \boxed{I_{L_0} + mh^2}$$

Exercise 18. The perpendicular direction is given by the following:

$$(\cos(\alpha+\pi/2), \sin(\alpha+\pi/2)) = (-\sin\alpha, \cos\alpha)$$

Then

$$(x,y)\cdot(-\sin\alpha,\cos\alpha) = -x\sin\alpha + y\cos\alpha = \delta$$

We want to find the square of the above quantity, δ^2.

$$\int_{-b\sqrt{1-\left(\frac{x}{a}\right)^2}}^{b\sqrt{1-\left(\frac{x}{a}\right)^2}} (x^2\sin^2\alpha + -2xy\sin\alpha\cos\alpha + y^2\cos^2\alpha)dy = 2x^2\sin^2\alpha b\sqrt{1-(x/a)^2} + \frac{\cos^2\alpha}{3}\left(2b^3\left(1-\left(\frac{x}{a}\right)^2\right)^{3/2}\right)$$

$$\text{Let } \left(\frac{x}{a}\right) = \sin t$$

$$\int a^2\sin^2 t\cos^2 tadt = a^2\int\left(\frac{\sin 2t}{2}\right)^2 dt = \frac{a^2}{4}\int\frac{1-\cos 4t}{2}dt = \frac{a^2\pi}{8}$$

$$\int_{-\pi/2}^{\pi/2}\cos^3 ta\cos tdt = a\int_{-\pi/2}^{\pi/2}\frac{1+2\cos 2t+\cos^2 2t}{4} = a\left(\frac{\pi}{4}+\frac{\pi}{8}\right) = \frac{a3\pi}{8}$$

$$\Longrightarrow 2b\sin^2\alpha\frac{a^3\pi}{8} + \frac{2}{3}\cos^2\alpha b^3 a\frac{3\pi}{8} = \frac{1}{4}\pi ab(a^2\sin^2\alpha + b^2\cos^2\alpha)$$

With $m=\pi ab$, the area of the ellipse, we get the desired answer.

Exercise 19. We want $\int_0^h\int_0^h\sqrt{x^2+y^2}dxdy$.

$$\int_0^h\sqrt{x^2+y^2}dx = \left(\frac{x}{2}\sqrt{x^2+y^2}+\frac{y^2}{2}\ln(x+\sqrt{x^2+y^2})\right)\Bigg|_0^h = \frac{1}{2}\left(h\sqrt{h^2+y^2}+y^2\ln(\frac{h}{y}+\sqrt{1+\left(\frac{h}{y}\right)^2})\right)$$

since, recall

$$(\ln(x+\sqrt{x^2+y^2}))' = \frac{1+\frac{x}{\sqrt{x^2+y^2}}}{x+\sqrt{x^2+y^2}} = \frac{1}{\sqrt{x^2+y^2}} \quad \text{and} \quad (x\sqrt{x^2+y^2})' = \sqrt{x^2+y^2}+\frac{x^2}{\sqrt{x^2+y^2}}$$

$$\int_0^h y^2\ln\left(\frac{h}{y}+\sqrt{1+\left(\frac{h}{y}\right)^2}\right)dy \xrightarrow{u=\frac{h}{y}} \int_\infty^1\left(\frac{h}{u}\right)^2\ln(u+\sqrt{1+u^2})\frac{-h}{u^2}du = h^3\int_1^\infty\frac{\ln(u+\sqrt{1+u^2})}{u^4}du$$

$$\int\frac{\ln(u+\sqrt{1+u^2})}{u^4} = \int\left(\frac{u^{-3}}{-3}\right)'\ln(u+\sqrt{1+u^2}) = \frac{u^{-3}}{-3}\ln(u+\sqrt{1+u^2}) - \int\frac{u^{-3}}{-3}\frac{1}{\sqrt{1+u^2}}$$

If we make the following substitution, $\begin{aligned} y &= \frac{1}{u} \\ du &= \frac{-1}{y^2}dy \end{aligned}$

$$\int\frac{u^{-3}}{\sqrt{1+u^2}} = \int\frac{u^{-4}}{\sqrt{1+\left(\frac{1}{u}\right)^2}} = \int\frac{y^4\left(\frac{-1}{y^2}\right)dy}{\sqrt{1+y^2}} = \int\frac{-y^2}{\sqrt{1+y^2}}dy = -\left(y\sqrt{1+y^2}-\int\sqrt{1+y^2}\right) =$$

$$= -y\sqrt{1+y^2}+\frac{1}{2}(y\sqrt{1+y^2}+\ln(y+\sqrt{1+y^2})) = \frac{-1}{2}y\sqrt{1+y^2}+\frac{1}{2}\ln(y+\sqrt{1+y^2})$$

$$h^3\left(\frac{u^{-3}}{-3}\ln(u+\sqrt{1+u^2})\Bigg|_1^\infty + \frac{1}{3}\left(\frac{-\sqrt{1+\left(\frac{1}{u}\right)^2}}{2u}+\frac{1}{2}\ln\left(\frac{1}{u}+\sqrt{1+\left(\frac{1}{u}\right)^2}\right)\right)\right)\Bigg|_1^\infty = \frac{h^3}{6}(\sqrt{2}+\ln(1+\sqrt{2}))$$

$$\Longrightarrow \frac{1}{4}(h^3\sqrt{2}+h^3\ln(1+\sqrt{2})) + \frac{h^3}{12}(\sqrt{2}+\ln(1+\sqrt{2})) = \boxed{\frac{h^3}{3}(\sqrt{2}+\ln(1+\sqrt{2}))}$$

Exercise 20. Let $P_0=(0,h)$. We want

$$\int_{-R}^R\int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}}(x^2+(y-h)^2)dydx$$

$$\int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}}(x^2+(y-h)^2)dy = 2x^2\sqrt{R^2-x^2}+\frac{1}{3}((\sqrt{R^2-x^2}-h)^3-(-\sqrt{R^2-x^2}-h)^3) =$$

$$= 2x^2\sqrt{R^2-x^2}+\frac{2}{3}((R^2-x^2)^{3/2}+3\sqrt{R^2-x^2}h^2) = \frac{4}{3}x^2\sqrt{R^2-x^2}+\left(\frac{2}{3}R^2+2h^2\right)\sqrt{R^2-x^2}$$

$$\text{Since} \begin{aligned} x &= R\sin\theta \\ dx &= R\cos\theta d\theta \end{aligned}$$

$$\int_{-R}^{R} x^2\sqrt{R^2-x^2} = \int_{-\pi/2}^{\pi/2} R^2\sin^2\theta R^2\cos^2\theta d\theta = R^4\int_{-\pi/2}^{\pi/2}\left(\frac{\sin 2\theta}{2}\right)^2 d\theta = \frac{R^4}{4}\int_{-\pi/2}^{\pi/2}\left(\frac{1-\cos 4\theta}{2}\right)d\theta = \frac{R^4}{8}\pi$$

$$\int_{-\pi/2}^{\pi/2}\sqrt{R^2-x^2} = \int_{-\pi/2}^{\pi/2}\cos\theta R\cos\theta d\theta(R) = R^2\int_{-\pi/2}^{\pi/2}\frac{1+\cos 2\theta}{2}d\theta = R^2\frac{\pi}{2}$$

$$\Longrightarrow \frac{4}{3}\frac{R^4\pi}{8} + \left(\frac{2}{3}R^2+2h^2\right)R^2\frac{\pi}{2} = \frac{R^4\pi}{6} + \frac{\pi R^4}{3} + h^2R^2\pi = \frac{\pi R^4}{2} + \pi R^2h^2$$

Now

$$\iint dydx = \pi r^2$$

So then the average of δ^2 is $\boxed{\frac{R^2}{2}+h^2}$

Exercise 21.

$$\begin{aligned} A &= [0,4]\times[0,1] \\ B &= [2,3]\times[1,3] \\ C &= [2,4]\times[3,4] \end{aligned}$$

(1) $A\cup B$

$$\frac{4(2,\frac{1}{2})+(2)(\frac{5}{2},\frac{4}{2})}{4+2} = \frac{\frac{1}{2}(16+10,10)}{6} = \left(\frac{13}{6},1\right)$$

(2) $A\cup C$

$$\frac{4\left(\frac{4}{2},\frac{1}{2}\right)+2\left(\frac{6}{2},\frac{7}{2}\right)}{4+2} = \frac{(8,2)+(6,7)}{6} = \left(\frac{7}{3},\frac{3}{2}\right)$$

(3) $B\cup C$,

$$\frac{2\left(\frac{5}{2},\frac{4}{2}\right)+2\left(\frac{6}{2},\frac{7}{2}\right)}{2+2} = \left(\frac{11}{4},\frac{11}{4}\right)$$

(4) $A\cup B\cup C$

$$\frac{4(2,\frac{1}{2})+4(\frac{11}{4},\frac{11}{4})}{8} = \boxed{\left(\frac{19}{8},\frac{13}{8}\right)}$$

Exercise 22.

rectangle $R:$ area $A_R \quad = 1(2) \quad (\overline{x},\overline{y})_R = (0,-1)$

triangle $T:$ area$A_T \quad = \frac{1}{2}(1)h = \frac{h}{2} \quad \overline{x}_T = 0$

$$\overline{y}_TA_T = \int_0^h\int_0\left(\frac{y}{h}-1\right)/2^{\left(\frac{y}{h}-1\right)/(-2)} ydxdy = \int_0^h\frac{y}{-2}\left(\frac{y}{h}-1+\left(\frac{y}{h}-1\right)\right) = \int_0^h y-\frac{y^2}{h} = \frac{1}{6}h^2$$

$$\overline{y}_T = \frac{\frac{1}{6}h^2}{h/2} = \frac{1}{3}h$$

Condition for centroid to lie on the common edge, with the common edge located at the origin:

$$0 = \frac{\left(\frac{h}{2}\right)\left(\frac{h}{3}\right)+2(-1)}{h/2+2} \Longrightarrow \boxed{h=2\sqrt{3}}$$

Exercise 23.

isosceles triangle $T \ :A_T = \frac{1}{2}2rh = rh$

$$\overline{y}_TA_T = \int_0^h\int_{\left(\frac{y}{h}-1\right)(r)}^{\left(\frac{y}{h}-1\right)(-r)} ydxdy = \int_0^h yr\left(\left(\frac{y}{h}-1\right)(-1)-\left(\frac{y}{h}-1\right)\right)dy =$$

$$= 2(-r)\int_0^h\frac{y^2}{h}-ydy = (-r)\left(\frac{1}{h}\frac{1}{3}h^3-\frac{1}{2}h^2\right) = \frac{1}{3}h^2r$$

$$\Longrightarrow \overline{y}_T = h^2r/3rh = h/3$$

semicircular disk $D \ :A_D = \frac{1}{2}\pi r^2; \quad (2\pi\overline{y})A = 2\pi\overline{y}\frac{\pi}{2}r^2 = \frac{4}{3}\pi r^3$

Condition for centroid to lie in triangle:

$$\frac{(rh)\left(\frac{h}{3}\right)+\left(\frac{1}{2}\pi r^2\right)\left(\frac{-4r}{3\pi}\right)}{(rh)+\frac{1}{2}\pi r^2}\geq 0 \quad \text{or } \frac{r}{3}\left(h^2+-2r^2\right)\geq 0 \implies \boxed{h>\sqrt{2}r}$$

11.22 Green's theorem in the plane. Some applications of Green's theorem. A necessary and sufficient condition for a two-dimensional vector field to be a gradient.

Exercise 1. Green's theorem: $\int_C Pdx+Qdy=\iint_R\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy$. Thus

$$\oint_C y^2dx+xdy=\iint_R(1-2y)dxdy$$

(1) $\int_0^2\int_0^2(1-2y)dydx=4-4(2)=-4$
(2) $\int_{-1}^1\int_{-1}^1(1-2y)dydx=4$
(3) $\int_{-2}^0\int_{-2-x}^{2+x}(1-2y)dydx+\int_0^2\int_{-2+x}^{2-x}(1-2y)dydx=\int_{-2}^0(2(2+x))dx+\int_0^2(4-2x)dx=\int_{-2}^0(4+2x)dx+4(2)-4=\boxed{8}$
(4) $\int_{-2}^2\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}(1-2y)dydx=\int_{-2}^2 2\sqrt{4-x^2}dx=\int_{-\pi/2}^{\pi/2}8\cos^2\theta d\theta=\boxed{4\pi}$ where we used
$x=2\sin\theta$
$dx=2\cos\theta d\theta$
(5) $\alpha(t)=(2\cos^3 t,2\sin^3 t)=2(\cos^3 t,\sin^3 t),\ \ 0\leq t\leq 2\pi$

Exercise 2.

$$P(x,y)=xe^{-y^2}$$
$$Q(x,y)=-x^2ye^{-y^2}+\frac{1}{x^2+y^2}$$

$$\oint Pdx+Qdy=\iint\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy$$

$$Q_x=-2xye^{-y^2}+\frac{-2x}{(x^2+y^2)^2}$$
$$P_y=xe^{-y^2}(-2y)$$

$$\int_{-a}^a\int_{-a}^a\frac{-2x}{(x^2+y^2)^2}dxdy=\int_{-a}^a\left(\frac{1}{x^2+y^2}\right)\Big|_{-a}^a dy=\int_{-a}^a\frac{1}{a^2+y^2}-\frac{1}{a^2+y^2}=0$$

Exercise 3. $nI_z=\oint_C x^3dy-y^3dx=\iint_R(3x^2+3y^2)dydx=3I_z\quad n=3$

Exercise 4. $f=(v,u),\ \ g=((u_x-u_y),v_x-v_y)$

$$(f\cdot g)=vu_x-vu_y+uv_x-uv_y=(uv)_x-(uv)_y=Q_x-P_y$$

$$\iint_R(f\cdot g)dxdy=\int(uv,uv)\cdot ds=\int(uv,uv)\cdot(-\sin t,\cos t)dt=\int_0^{2\pi}(-\sin^2 t+\sin t\cos tdt=\boxed{-\pi}$$

Exercise 5. $f,g\in\mathcal{C}^1$, f,g on open connected set S in the plane.

$$\oint_C f\nabla g\cdot d\alpha=\oint_C fg_xdx+fg_ydy=\iint_R(f_xg_y+fg_{xy})-(f_yg_x+fg_{yx})=$$
$$=\iint_R-(g_xf_y+gf_{xy})+(g_yf_x+gf_{yx})=\iint_R(-gf_y)_x-(-gf_x)_y=\oint-gf_xdx-gf_ydy=-\oint g(\nabla f)\cdot d\alpha$$

Since $f_{xy}=f_{yx};\ \ g_{xy}=g_{yx}$.

Exercise 6.

(1) $\oint_C uvdx+uvdy=\iint(\partial_x(uv)-\partial_y(uv))dxdy=\iint v(\partial_xu-\partial_yu)+u(\partial_xv-\partial_yv)dxdy$
(2)

$$\frac{1}{2}\oint_C(v\partial_xu-u\partial_xv)dx+(u\partial_yv-v\partial_yu)dy=\frac{1}{2}\iint(u_xv_y+uv_{xy}-v_xu_y-vu_{xy})-(v_yu_x+vu_{yx}-u_yv_x-uv_{yx})=$$
$$=\frac{1}{2}\iint u(v_{xy}+v_{yx})-v(u_{xy}+u_{yx})=\iint u\partial_{yx}v-v\partial_{yx}u$$

Note the formulation of normal derivatives. Note that ds refers to the arc length.

$$\int_C(Pdx+Qdy)=\int_C f\cdot Tds \qquad \alpha(t)=(X(t),Y(t))$$

$T\equiv$ unit tangent vector to C $\qquad n(t)=\frac{1}{\|\alpha'(t)\|}(Y'(t),X'(t))$ whenever $\|\alpha'(t)\|\neq 0$

So the normal derivative is defined as

$$\frac{\partial \psi}{\partial n} = \nabla \psi \cdot n$$

Exercise 7.

$$\int_C Pdx + Qdy = \int_C (P,Q)\cdot\left(\frac{ds}{dt}\right)dt = \int_C \left(P\frac{dx}{dt} + Q\frac{dy}{dt}\right)dt = \int_C (QY' + (-P)(-X'))dt =$$
$$= \int_C (Q,-P)\cdot \frac{(Y',-X')}{\|\alpha'(t)\|}\|\alpha'(t)\|dt = \int_C f\cdot nds$$

Exercise 8.

(1)

$$\oint_C \frac{\partial g}{\partial n}ds = \oint_C \nabla\cdot nds = \oint_C \nabla g\cdot\frac{(Y',-X')}{\|\alpha'(t)\|}\|\alpha'(t)\|dt =$$
$$= \oint_C (g_xY' + -g_yX')dt = \oint_C g_xdy + -g_ydx = \iint (g_{xx} - (-g_{yy}))dxdy = \iint \nabla^2 gdxdy$$

(2)

$$\oint_C f\nabla g\cdot nds = \oint_C f\nabla g\cdot \frac{(Y',-X')}{\|\alpha'(t)\|}\|\alpha'(t)\|dt = \oint_C f(g_xdy - g_ydx) =$$
$$= \iint (fg_x)_x - (-fg_y)_y = \iint f_xg_x + fg_{xx} + f_yg_y + fg_{yy} = \iint (\nabla f\cdot\nabla g + f(\nabla^2 g))dxdy$$
$$\Longrightarrow \oint_C f\frac{\partial g}{\partial n}ds = \iint_R (f\nabla^2 g + \nabla f\cdot\nabla g)dxdy$$

(3) Use previous part, (b), of this exercise, Exercise 8.

$$\begin{aligned} \oint_C \left(f\frac{\partial g}{\partial n}\right)ds &= \iint_R (f\nabla^2 g + \nabla f\cdot\nabla g)dxdy \\ \oint_C \left(g\frac{\partial f}{\partial n}\right)ds &= \iint_R (g\nabla^2 f + \nabla g\cdot\nabla f)dxdy \end{aligned} \quad \Longrightarrow \oint_C \left(f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n}\right)ds = \iint_R (f\nabla^2 g - g\nabla^2 f)dxdy$$

Exercise 9. $P(x,y)dx + Q(x,y)dy = 0$.

$\mu(x,y)$ is an integration factor, so $\mu Pdx + \mu Qdy = 0$ leads to $\phi(xy) = C$ s.t. $\begin{aligned}\phi_x &= \mu P \\ \phi_y &= \mu Q\end{aligned}$

Slope of $\phi(x,y) = c$ at (x,y) is $\tan\theta$, so $\tan\theta = \frac{dY/dt}{dX/dt}$

$n = (\sin\theta, -\cos\theta)$

$$\frac{\partial\phi}{\partial n} = \nabla\phi\cdot n = (\nabla\phi)\cdot(\sin\theta,-\cos\theta) = \phi_x\sin\theta - \phi_y\cos\theta = \mu P\sin\theta - \mu Q\cos\theta =$$
$$= \mu(P\sin\theta - Q\cos\theta) = \mu(x,y)g(x,y)$$
$$\Longrightarrow g = P\sin\theta - Q\cos\theta$$

$$\begin{aligned} \sin\theta &= \frac{-P}{\sqrt{P^2+Q^2}} \text{ or } \frac{P}{\sqrt{P^2+Q^2}} \\ \cos\theta &= \frac{Q}{\sqrt{P^2+Q^2}} \text{ or } \frac{Q}{\sqrt{P^2+Q^2}} \end{aligned} \Longrightarrow g = -\sqrt{P^2+Q^2} \text{ or } \sqrt{P^2+Q^2}$$

11.25 Exercises - Green's theorem for multiply connected regions. The winding number.

Exercise 1.

(1) Note that

$$\partial_x Q = \frac{-x}{x^2+y^2}\left(\frac{1}{x} + \frac{(-1)(2x)}{x^2+y^2}\right) = \frac{-x}{x^2+y^2}\left(\frac{x^2+y^2-2x^2}{x(x^2+y^2)}\right) = \frac{x^2-y^2}{(x^2+y^2)^2}$$
$$\partial_y P = \partial_y\left(\frac{y}{x^2+y^2}\right) = -\left(\frac{y^2-x^2}{(x^2+y^2)^2}\right)$$

Note that $\partial_x Q,\ \partial_y P$ is not continuous at $(0,0)$.

$$\text{So then for } \begin{array}{l} x=\cos t \\ y=\sin t \end{array} \qquad \begin{array}{l} P(x,y)=\dfrac{y}{x^2+y^2} \\ Q(x,y)=\dfrac{-x}{x^2+y^2} \end{array}$$

$$\int_C Pdx = \int_0^{2\pi} \sin t(-\sin t)dt = -\pi$$

$$\int_C Qdx = \int_0^{2\pi} -\cos t(\cos t)dt = -\pi$$

$$\int_C Pdx + Qdy = -2\pi$$

$+$ sign occurs when C is in the clockwise direction, since if $x=\cos t$, $y=-\sin t$, then $\int_C Pdx+Qdy = 2\pi$ and all clockwise direction, piecewise smooth Jordan curves whose interior contains $(0,0)$ can be deformed into a circle (by Thm.).

(2) By Thm., we can pick any piecewise smooth Jordan curve whose interior doesn't contain $(0,0)$.
Recall Green's theorem.

$$\int_C Pdx+Qdy = \iint_R \left(\frac{\partial Q}{\partial y} - \frac{\partial P}{\partial x} \right) = 0$$

Since $\frac{\partial Q}{\partial y}$, $\frac{\partial P}{\partial x}$, is continuous everywhere in $C \cup intC$, where $(0,0) \notin C$

Exercise 2.

$$f = \left(\frac{\partial(\ln r)}{\partial y}, -\frac{\partial(\ln r)}{\partial x} \right) \qquad \begin{array}{l} x=a\cos t \\ y=a\sin t \end{array} \qquad \alpha=(x,y) \qquad \sqrt{x'^2+y'^2} = \|\alpha'\| = a$$

$$\ln r = \ln \sqrt{x^2+y^2} = \frac{1}{2}\ln(x^2+y^2)$$

$$(\ln r)_x = \frac{1}{2}\frac{1}{x^2+y^2}(2x) = \frac{x}{x^2+y^2}$$

$$\int f\cdot ds = \int f\cdot \frac{dr}{dt}dt = \int \left(\frac{\partial(\ln r)}{\partial y}a(-\sin t) + \frac{-\partial(\ln r)}{\partial x}a\cos t \right)dt = \int \frac{\partial(\ln r)}{\partial y}\frac{dx}{dt} + \frac{-\partial(\ln r)}{\partial x}\frac{dy}{dt}dt =$$

$$= \int \frac{y}{x^2+y^2}dx + \frac{-x}{x^2+y^2}dy = \boxed{-2\pi} \text{ as shown in the previous exercise, Exercise 1.}$$

Exercise 5.

(1) $I_1 - I_3 = 12 - 15 = -3$
(2) One possible solution is this: Draw a large curve around all 3 points, for $I_1+I_2+I_3 = 37$. Then circle around, inside, point 1, three times, in a clockwise fashion, to obtain $-3I_1 = -36$.

Exercise 6. $\alpha(t) = (X(t), Y(t))$ if $a \leq t \leq b$ $\qquad n = \frac{(Y'(t), X'(t))}{\sqrt{X'^2+Y'^2}}$

$$W(\alpha_0; P_0) = \frac{1}{2\pi}\int_a^b \frac{(X(t)-x_0)Y'(t) - (Y(t)-y_0)X'(t)}{(X(t)-x_0)^2+(Y(t)-y_0)^2}dt =$$

$$= \frac{1}{2\pi}\int_a^b \frac{(r(t)-P_0)\cdot n}{\|r(t)-P_0\|^2}\|\alpha'(t)\|dt = \frac{1}{2\pi}\int_a^b \left(\frac{r(t)-P_0}{\|r(t)-P_0\|} \right) \frac{n}{\|r(t)-P_0\|}ds$$

$$I_k = \oint_{C_k} Pdx + Qdy$$

$$P(x,y) = -y\left(\frac{1}{(x-1)^2+y^2} + \frac{1}{x^2+y^2} + \frac{1}{(x+1)^2+y^2} \right) =$$

$$= -y\left(\frac{1}{\|(x,y)-(1,0)\|^2} + \frac{1}{\|(x,y)-(0,0)\|^2} + \frac{1}{\|(x,y)-(-1,0)\|^2} \right)$$

$$Q(x,y) = \frac{(x-1)}{\|(x,y)-(1,0)\|^2} + \frac{x}{\|(x,y)-(0,0)\|^2} + \frac{x+1}{\|(x,y)-(-1,0)\|^2}$$

$$C_1 \text{ is the smallest circle}, x^2+y^2 = \left(\frac{1}{2\sqrt{2}}\right)^2$$
$$C_2: x^2+y^2 = 2^2$$

$$\begin{aligned} & (x-1)^2+y^2 = \left(\frac{1}{2}\right)^2 \\ C_3: \quad & x^2+y^2 = \left(\frac{1}{2}\right)^2 \\ & (x+1)^2+y^2 = \left(\frac{1}{2}\right)^2 \end{aligned} \qquad \begin{aligned} I_2 &= 6\pi \\ I_3 &= 2\pi \end{aligned}$$

$I_k = \oint_{C_k} Pdx + Qdy = \int \left(\frac{(x,y)-(1,0)}{\|(x,y)-(1,0)\|^2} + \frac{(x,y)-(0,0)}{\|(x,y)-(0,0)\|^2} + \frac{(x,y)-(-1,0)}{\|(x,y)-(-1,0)\|^2} \right) \cdot nds$
I_2 wraps around 3 holes, $(1,0),(0,0),(-1,0)$.
I_3 wraps around $(0,0)$ clockwise and around $(1,0),(-1,0)$ counterclockwise. The wrap around $(0,0)$ and $(1,0)$ cancel each other and so the result is we wrap around $(-1,0)$.

$$\implies \boxed{I_1 = 2\pi}$$

11.28 Exercises - Change of variables in a double integral, Special cases of the transformation formula

Exercise 1. $S = \{(x,y)|x^2+y^2 \leq a^2\}$ where $a > 0$.
Recall,

$$\int \cdots \int_{\mathcal{D}} f(\phi(u))|detD\phi(u)|du_1,\dots,du_n = \int \cdots \int_{\mathcal{D}} f(x)dx_1,\dots,dx_n \qquad \phi(\mathcal{D}) = \mathcal{D}^*$$

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned} \qquad \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r > 0 \qquad \iint_S f(x,y)dxdy = \int_0^{2\pi}\int_0^a f(r\cos\theta, r\sin\theta)rdrd\theta$$

Exercise 2. $S = \{(x,y)|x^2+y^2 \leq 2x\}$

$$\begin{aligned} (x-1)^2+y^2 &= 1 \\ (1,0)& \end{aligned} \qquad \begin{aligned} x-1 &= r\cos\theta \\ x &= r\cos\theta + 1 \\ y &= r\sin\theta \end{aligned} \qquad \iint_S f(x,y)dxdy = \int_0^{2\pi}\int_0^1 f(r\cos\theta+1, r\sin\theta)rdrd\theta$$

Exercise 3. $S = \{(x,y)|a^2 \leq x^2+y^2 \leq b^2\}$ where $0 < a < b$

$$\iint_S f(x,y)dxdy = \int_0^{2\pi}\int_a^b f(r\cos\theta, r\sin\theta)rdrd\theta$$

Exercise 4. $S = \{(x,y)|0 \leq y \leq 1-x,\ 0 \leq x \leq 1\}$

$$r\sin\theta \leq 1 - r\cos\theta \implies r \leq \frac{1}{\sin\theta+\cos\theta}, \text{ since } \sin\theta, \cos\theta \geq 0$$

$$\int_0^{\pi/4}\int_0^{\frac{1}{\sin\theta+\cos\theta}} f(r\cos\theta, r\sin\theta)rdrd\theta$$

Exercise 5. $S = \{(x,y)|x^2 \leq y \leq 1,\ -1 \leq x \leq 1\}$ Consider imaginary angular wedges dividing up the parabolic region. Then we identify 3 regions since each region have different boundaries.

$$y = 1 = r\sin\theta \text{ or } r = \csc\theta$$
$$x^2 = r^2\cos^2\theta = r\sin\theta \text{ or } r = \tan\theta\csc\theta$$

Observe that $y = x$ or $\theta = \frac{\pi}{4}$, and $y = -x$ or $\theta = \frac{-\pi}{4}$, divide up the regions.

$$\int_0^{\pi/4}\int_0^{\tan\theta\csc\theta} f(r\cos\theta, r\sin\theta)rdrd\theta + \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\int_0^{\csc\theta} f(r\cos\theta, r\sin\theta)rdrd\theta + \int_{\frac{3\pi}{4}}^{\pi}\int_0^{\tan\theta\csc\theta} f(r\cos\theta, r\sin\theta)rdrd\theta$$

Exercise 6. Note that $\sqrt{2ax-x^2} = \sqrt{a^2-(a-x)^2}$. Then

$$\begin{aligned} y^2 &= a^2-(a-x)^2 \\ (x-a)^2+y^2 &= a^2 \end{aligned} \qquad \begin{aligned} x-a &= r\cos\theta \\ y &= r\sin\theta \\ x^2 = (a+r\cos\theta)^2 &= a^2+2ar\cos\theta+r^2\cos^2\theta \end{aligned}$$

$$\int_0^{2a}\left(\int_0^{\sqrt{2ax-x^2}}(x^2+y^2)dy\right)dx = \int_0^{\pi}d\theta\int_0^a rdr(a^2+2ar\cos\theta+r^2) = a^2\frac{a^2\pi}{2}+\int_0^{\pi}\frac{2a\cos\theta a^3}{3}d\theta+\frac{a^4\pi}{4} = \boxed{\frac{3a^4\pi}{4}}$$

Exercise 7. $x = a \Longrightarrow r\cos\theta = a$ or $r = a\sec\theta$

$$\int_0^a\left(\int_0^x\sqrt{x^2+y^2}dy\right)dx = \int_0^{\pi/4}d\theta\int_0^{a\sec\theta}r^2dr = \int_0^{\pi/4}\frac{a^3\sec^3\theta}{3}$$

$$\text{Now }\int\sec^3\theta = \int\sec\theta(1+\tan^2\theta)\text{ and}$$

$$\int\sec\theta = \ln|\sec\theta+\tan\theta|$$

$$\int(\sec\theta\tan^2\theta) = \int(\sec\theta)'\tan\theta = \sec\theta\tan\theta - \int\sec\theta\sec^2\theta \quad \Longrightarrow \int\sec^3\theta = \frac{\sec\theta\tan\theta+\ln|\sec\theta+\tan\theta}{2}$$

$$\Longrightarrow \frac{a^3}{3}\int_0^{\pi/4}\sec^3\theta d\theta = \frac{a^3(\sqrt{2}+\ln|\sqrt{2}+1|)}{6}$$

Exercise 8.

$$\text{Since } y = r\sin\theta = x^2 = r^2\cos^2\theta \text{ or } r = \tan\theta\sec\theta$$

$$\int_0^{\pi/4}d\theta\int_0^{\tan\theta\sec\theta}\frac{1}{r}rdr = \int_0^{\pi/4}d\theta\tan\theta\sec\theta = \boxed{\sqrt{2}-1}$$

Exercise 9.

$$\int_0^a\left(\int_0^{\sqrt{a^2-y^2}}(x^2+y^2)dx\right)dy = \int_0^{\pi/2}\int_0^a r^2rdrd\theta = \frac{a^4\pi}{8}$$

Exercise 10. After sketching a box with vertices at $(0,0),(1,0),(0,1),(1,1)$, it's very clear that in polar coordinates, we must divide the region into 2 parts by the $y = x$ line since each region have different boundaries for r.

$$\int_0^1\left(\int_0^x f(x,y)dy\right)dx = \int_0^{\pi/4}d\theta\int_0^{\sec\theta}f(r\cos\theta,r\sin\theta)rdr + \int_0^{\pi/4}d\theta\int_0^{\csc\theta}f(r\cos\theta,r\sin\theta)rdr \text{ since}$$

$$x = 1 = r\cos\theta$$
$$y = 1 = r\sin\theta$$

Exercise 11. $\int_0^2\left(\int_x^{x\sqrt{3}}f(\sqrt{x^2+y^2})dy\right)dx = \int_{\pi/4}^{\pi/3}\int_0^{2\sec\theta}f(r)rdrd\theta$

$$x = 2 = r\cos\theta$$
$$r = 2\sec\theta$$

Exercise 12. $\int_0^1\left(\int_{1-x}^{\sqrt{1-x^2}}f(x,y)dy\right) = \int_0^{\pi/2}d\theta\int_{\frac{1}{\sin\theta+\cos\theta}}^1 f(r\cos\theta,r\sin\theta)rdr$ since

$$y = 1-x = r\sin\theta = 1-r\cos\theta$$
$$r = \frac{1}{\sin\theta+\cos\theta}$$

Exercise 13. $\int_0^1\left(\int_0^{x^2}f(x,y)dy\right)dx = \int_0^{\pi/4}\int_{\sec\theta}^{\tan\theta\sec\theta}f(r\cos\theta,r\sin\theta)rdrd\theta$ since

$$y = x^2 = r\sin\theta = r^2\cos^2\theta \qquad 1 = x = r\cos\theta$$
$$\Longrightarrow r = \tan\theta\sec\theta \qquad r = \sec\theta$$

Exercise 14. Let $\begin{matrix} x+y=u \\ x-y=v \end{matrix}$

$$\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}u\\v\end{bmatrix} \quad \Longrightarrow \begin{bmatrix}x\\y\end{bmatrix} = \frac{1}{-2}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{bmatrix}u\\v\end{bmatrix}$$

$$J = \begin{vmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{-1}{2}\end{vmatrix} = \frac{-1}{2}$$

Sketch the transformation of S to obtain a rectangle with vertices $(\pi,\pi),(3\pi,\pi),(\pi,-\pi),(3\pi,-\pi)$.

$$\int_{\pi}^{3\pi}du\int_{-\pi}^{\pi}dvv^2\sin^2u\left(\frac{-1}{2}\right) = \frac{-1}{2}\int_{\pi}^{3\pi}\left(\frac{1-\cos 2u}{2}\right)\frac{2}{3}\pi^3 = \boxed{-\frac{\pi^4}{3}}$$

Exercise 15.

$$(0,0),(2,10),(3,17),(1,7) \Longrightarrow (0,0),(4,2)$$

$$\begin{aligned} u &= ax+by \\ v &= cx+dy \end{aligned} \qquad \begin{pmatrix}4\\0\end{pmatrix} = \begin{pmatrix}a2+b10\\c2+d10\end{pmatrix} \quad \begin{pmatrix}0\\2\end{pmatrix} = \begin{pmatrix}a+7b\\c+7d\end{pmatrix} \Longrightarrow \begin{aligned} b=-1 \quad a &= 7 \\ d=1 \quad c &= -5 \end{aligned}$$

$$\begin{bmatrix}7 & -1\\ -5 & 1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}u\\v\end{bmatrix} \qquad \frac{1}{2}\begin{bmatrix}1 & 1\\ 5 & 7\end{bmatrix}\begin{bmatrix}u\\v\end{bmatrix} = \begin{pmatrix}\frac{u+v}{2}\\ \frac{5u+7v}{2}\end{pmatrix} = \begin{pmatrix}x\\y\end{pmatrix}$$

$$\int_0^2\int_0^4 \frac{5u^2+12uv+7v^2}{8}dudv = \int_0^2 \frac{5}{24}\, u^3\Big|_0^4 + \frac{3}{4}\, u^2v\Big|_0^4 + \frac{7}{8}v^2(4)dv = \frac{5}{3}16+24+\frac{28}{3} = \boxed{60}$$

Exercise 16. If $r>0$, let $I(r) = \int_{-r}^{r} e^{-u^2}du$.

(1)

$$I^2(r) = \left(\int_{-r}^{r} e^{-u^2}du\right)^2 = \left(\int_{-r}^{r} e^{-x^2}dx\right)\left(\int_{-r}^{r} e^{-y^2}dy\right) = \int_{-r}^{r}dy\int_{-r}^{r}dy\int_{-r}^{r}e^{-x^2-y^2}dx = \int_{-r}^{r}\int_{-r}^{r}e^{-x^2-y^2}dxdy$$

(2) Let C_1 have radius a, C_2 have radius b, $C_1 \subset R \subset C_2$, and since $e^{-(x^2+y^2)} > 0, \ \forall\, x,y \in \mathbb{R}$, then

$$\iint_{C_1} e^{-(x^2+y^2)}dxdy < I^2(r) < \iint_{C_2} e^{-(x^2+y^2)}dxdy$$

(3)

$$\int_0^{2\pi}\int_0^a e^{-r^2}drrd\theta = \int_0^{2\pi}\left(\frac{e^{-r^2}}{-2}\right)\Bigg|_0^a d\theta = \left(\frac{e^{-a^2}-1}{-2}\right)\Bigg|_0^{2\pi} = 2\pi\left(\frac{1-e^{-a^2}}{2}\right) \xrightarrow{a\to\infty} \pi$$

$$I(r) \to \sqrt{\pi} \text{ as } r\to\infty$$

$$I(r) = \int_{-r}^{r} e^{-u^2}du = 2\int_0^r e^{-u^2}du = \sqrt{\pi} \qquad \int_0^r e^{-u^2}du = \frac{\sqrt{\pi}}{2}$$

(4)

$$\Gamma(s) = \int_{0+}^{\infty} t^{s-1}e^{-t}dt$$

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty t^{-1/2}e^{-t}dt = \int_0^\infty u^{-1}e^{-u^2}2udu = 2\int_0^\infty e^{-u^2}du = 2\frac{\sqrt{\pi}}{2} = \sqrt{\pi}$$

$$\text{since} \quad \begin{aligned} t &= u^2 \\ dt &= 2udu \end{aligned}$$

Exercise 17. $\begin{aligned} x &= u+v \\ y &= v-u^2 \end{aligned}$

(1) $detD\phi = \begin{vmatrix}1 & 1\\ -2u & 1\end{vmatrix} = 1+2u$

(2)

$$\begin{aligned} (u,0) &\Longrightarrow (u,-u^2) \quad u && \in [0,2] \\ (0,v) &\Longrightarrow (v,v) \quad v && \in [0,2] \\ (u,2-u) &\Longrightarrow (u+2-u, 2-u-u^2) = (2, -(u+\frac{1}{2})^2+\frac{9}{4}) \quad u && \in [0,2] \end{aligned}$$

(3)

$$\iint_T dudv(1+2u) = \int_0^2\int_0^{2-u}(1+2u)dvdu = \left.(2u-\frac{1}{2}u^2)\right|_0^2 + \int_0^2 2u(2-u)du = \frac{14}{3}$$

$$\int_0^2 dx\int_{-x^2}^{x}dy = \int_0^2 dx(x+x^2) = \left(\frac{1}{2}x^2+\frac{1}{3}x^3\right)\Bigg|_0^2 = \frac{14}{3}$$

(4)

$$\iint_S (x-y+1)^{-2}dxdy = \int_0^2\int_0^{2-u}(u+v-v+u^2+1)^{-2}(1+2u)dudv = \int_0^2 dv\int_0^{2-v}\frac{1+2u}{(u^2+u+1)^2}du =$$

$$= \int_0^2 dv\left(\frac{-1}{u^2+u+1}\right)\Bigg|_0^{2-v} = \int_0^2 dv\left(1-\frac{1}{4-4v+v^2+2-v+1}\right)$$

Now

$$\int_0^2 \frac{1}{7-5v+v^2} = \int_0^2\frac{1}{(v-\frac{5}{2})^2+\frac{3}{4}} = \int\frac{4/3}{\left(\frac{2}{\sqrt{3}}(v-\frac{5}{2})\right)^2+1} = \frac{4}{3}\left(\frac{\arctan\frac{2}{\sqrt{3}}(v-\frac{5}{2})}{2/\sqrt{3}}\right)\Bigg|_0^2 =$$

$$= \frac{2\sqrt{3}}{3}\left(\arctan\left(\frac{-1}{\sqrt{3}}\right)-\arctan\left(\frac{5}{\sqrt{3}}\right)\right)$$

$$\Longrightarrow \iint_S (x-y+1)^{-2}dxdy = 2-\frac{2\sqrt{3}}{3}\left(\arctan\left(\frac{-1}{\sqrt{3}}\right)-\arctan\left(\frac{5}{\sqrt{3}}\right)\right)$$

Exercise 18. $\begin{matrix} x = u^2-v^2 \\ y = 2uv \end{matrix}$

(1) $J(u,v) = detD\phi = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} = 4u^2+4v^2 = 4(u^2+v^2)$

(2) Note the transformation of the boundaries.

$$\begin{array}{llll}
(u,1),\ u \ \in [1,2] & \begin{matrix} x = u^2-1 \\ y = 2u \end{matrix} & x = \frac{y^2}{4}-1 & \begin{matrix} y\in[2,4] \\ x\in[0,3] \end{matrix} \\
(2,v),\ v \ \in [1,3] & \begin{matrix} x = 4-v^2 \\ y = 4v \end{matrix} & x = 4-\frac{y^2}{16} & \begin{matrix} y\in[4,12] \\ x\in[-5,3] \end{matrix} \\
(u,3),\ u \ \in [1,2] & \begin{matrix} x = u^2-9 \\ y = 6u \end{matrix} & x = \frac{y^2}{36}-9 & \begin{matrix} y\in[6,12] \\ x\in[-8,-5] \end{matrix} \\
(1,v),\ v \ \in [1,3] & \begin{matrix} x = 1-v^2 \\ y = 2v \end{matrix} & x = 1-\frac{y^2}{4} & \begin{matrix} y\in[2,6] \\ x\in[-8,0] \end{matrix}
\end{array}$$

(3)

$$x^2+y^2 = u^4-2u^2v^2+v^4+4u^2v^2 = (u^2+v^2)^2 = 1 \Longrightarrow u^2+v^2 = 1$$

Circle is invariant under "hyperbolic" transformation.

$$\int_{-1}^1\int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}}(u^2-v^2)(2uv)4(u^2+v^2)dudv = 8\int_{-1}^1 du\int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}}(u^2-v^2)(uv)dv =$$

$$= 8\int_{-1}^1 du\left(\frac{u^3}{2}((1-u^2)-(1-u^2))-u\frac{1}{4}(0)\right) = \boxed{0}$$

Exercise 19.

$$I(p,r) = \iint_R \frac{dxdy}{(\rho^2+x^2+y^2)^p} = \int_0^{2\pi}d\theta\int_0^R\frac{rdr}{(p^2+r^2)^p} = \begin{cases} \int_0^{2\pi}d\theta\,\frac{(p^2+r^2)^{-p+1}}{2(1-p)}\Big|_0^R = \frac{\pi}{1-p}((p^2+R^2)^{-p+1}-(p^2)^{-p+1}) \\ \int_0^{2\pi}d\theta\frac{\ln 1^2+R^2}{2} = \pi\ln(1+R^2) \end{cases}$$

$$R\to\infty,\ \text{if } p>1,\ \lim_{R\to\infty} I(p,r) = \frac{-\pi}{1-p}p^{2-2p}$$

Exercise 20. Let $u = x+y$. Note that the region in xy is a rectangle starting from $P = (0,-1)$ and spanned by $a = (1,1)$, $b = (-1,1)$.

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} + s\begin{pmatrix} 1 \\ 1 \end{pmatrix} + t\begin{pmatrix} -1 \\ 1 \end{pmatrix} \qquad detD\phi = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 1+1 = 2$$

$$\begin{matrix} x = s+-t \\ y = -1+s+t \end{matrix} \qquad x+y = 2s-1$$

$$\Longrightarrow \iint_S f(x+y)dydx = \int_0^1\int_0^1 f(2s-1)2dsdt = 2\int_0^1 f(2s-1)ds = \int_{-1}^1 f(u)du$$

Exercise 21. Look at what we want. We eventually want $ax + by = u\sqrt{a^2+b^2}$. Then try that substitution.
Also note that we want $J(x,y) = detD\phi \neq 0$ for all points considered. We are given that $a^2 + b^2 \neq 0$. Then try to get that as a nonzero factor for J, the Jacobian.

$$\begin{aligned} ax + by &= Au \\ cx + dy &= Av \end{aligned} \quad \implies \quad \begin{aligned} &A = \sqrt{a^2+b^2} \\ &\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} Au \\ Av \end{bmatrix} \end{aligned}$$

$$\implies \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} Au \\ Av \end{bmatrix}$$

We want $ad - bc \neq 0$ so simply use our hypothesis: $a^2 + b^2 \neq 0$. Then $d = a$, $c = -b$

$$\implies \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{A}\begin{pmatrix} ua - bv \\ ub + av \end{pmatrix} \qquad detD\phi = \begin{vmatrix} \frac{a}{A} & \frac{-b}{A} \\ \frac{b}{A} & \frac{a}{A} \end{vmatrix} = 1$$

Let's observe how the circular region in xy changes with uv.

$$x^2 + y^2 = 1 = \frac{a^2u^2 - 2abuv + b^2v^2}{A^2} + \frac{b^2u^2 + 2abuv + a^2v^2}{A^2} = u^2 + v^2 = 1$$

Amazing! The circle is invariant under a normalized linear transformation of nonzero determinant.

$$\implies \int_{-1}^{1}\int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} f(u+c)dvdu = 2\int_{-1}^{1} \sqrt{1-u^2} f(u\sqrt{a^2+b^2})du$$

Exercise 22. From the given problem, we obviously want to make the substitution $u = yx$. Consider the transformed boundaries and the Jacobian for this transformation.

$$\begin{aligned} xy &= 1 & u &= 1 \\ xy &= 2 & u &= 2 \\ y &= x & u &= x^2 \\ y &= 4x & u &= 4x^2 \end{aligned} \qquad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ u/x \end{pmatrix} detD\phi = \begin{vmatrix} 1 & 0 \\ -u/x^2 & 1/x \end{vmatrix} = \frac{1}{x}$$

Sketch the transformed region in the xu plane, with boundaries as described above. Then clearly,

$$\int_1^2 du \int_{\frac{\sqrt{u}}{2}}^{\sqrt{u}} f(u)\frac{1}{x}dx = \int_1^2 du\left(\ln(\sqrt{u}) - \ln\left(\frac{\sqrt{u}}{2}\right)\right) f(u) = \int_1^2 du\left(\frac{1}{2}\ln u - \frac{1}{2}\ln u + \ln 2\right) f(u) = \ln 2 \int_1^2 f(u)du$$

11.34 Exercises - Proof of the transformation formula in a special case, Proof of the transformation formula in the general case, Extensions to higher dimensions, Change of variables in an n-fold integral, Worked examples

Exercise 1. $z = xy$. Note that $z = 0$ implied $x = 0$ or $y = 0$.

$$\iiint_S xy^2z^3 dxdydz = \int_0^1\int_0^x\int_0^{xy} xy^2z^3 dzdydx = \int_0^1\int_0^x \frac{xy^2}{4}(xy)^4 dydx = \int_0^1 \frac{x^5}{4}\frac{x^7}{7}dx = \frac{1}{28}\frac{1}{13} = \frac{1}{364}$$

Exercise 2. $z = 1 - x - y$. $z = 0$ defines a boundary, so $y = 1 - x$.

$$\iiint_S (1+x+y+z)^{-3}dxdydz = \int_0^1 dx\int_0^{1-x} dy \int_0^{1-x-y}(1+x+y+z)^{-3}dz =$$

$$= \int_0^1 dx\int_0^{1-x} dy\left(\frac{1/-2}{(1+x+y+(1-x-y))^2} - \frac{1/-2}{(1+x+y)^{-2}}\right) =$$

$$= \frac{-1}{2}\int_0^1 dx\left(\frac{1}{4}(1-x) + (1+x+y)^{-1}\Big|_0^{1-x}\right) = \frac{-1}{2}\left(\frac{1}{4}(1-1/2) + 1/2(1) - \ln 2\right) = \boxed{\frac{-1}{2}\left(\frac{5}{8} - \ln 2\right)}$$

Exercise 3. $J = r^2\sin\theta$ (polar coordinates). Note $x \leq 0$, $y \leq 0$, $z \leq 0$

$$\int_0^{\pi/2} d\theta\int_0^{\pi/2} d\phi\int_0^1 r\cos\phi\sin\theta r\sin\phi\sin\theta r\cos\theta r^2\sin\theta dr = \int_0^{\pi/2}\int_0^{\pi/2} d\phi\frac{1}{6}\sin^3\theta\cos\theta\cos\phi\sin\phi =$$

$$= \frac{1}{6}\int_0^{\pi/2} d\theta\sin^3\theta\cos\theta\frac{1}{2} = \boxed{\frac{1}{48}}$$

Exercise 4. $\begin{aligned} x &= au \\ y &= bv \qquad J = abc \\ z &= cw \end{aligned}$

$$\Longrightarrow \int_0^\pi d\theta \int_0^{2\pi} d\phi \int_0^1 (u^2+v^2+w^2) abcr^2 \sin\theta dr = \int_0^\pi d\theta \int_0^{2\pi} d\phi \sin\theta \frac{abc}{5} = \frac{2\pi}{5} abc(2) = \boxed{\frac{4\pi abc}{5}}$$

Exercise 5. $z^2 = x^2+y^2 = r^2$. $z = r$

$$\iiint_S \sqrt{x^2+y^2} dxdydz = \int_0^1 dz \int_0^{2\pi} d\phi \int_0^z r^2 dr = \frac{1}{3}\frac{1}{4} 2\pi = \boxed{\frac{\pi}{6}}$$

For exercises 6,7,8, I think you have to sketch the region and surmise the new boundaries intuitively from the sketch. I don't see a formula you could simply plug in to determine the new regions and boundaries.

Exercise 6.

$$\iiint_S (x^2+y^2) dxdydz = \int_0^2 \int_0^{2\pi} \int_0^{\sqrt{2z}} r^2 (rdr) d\phi dz = \int_0^2 2\pi \frac{1}{4}(2z)^2 = \frac{16\pi}{3}$$

Exercise 7.

Exercise 10.

$$\iiint_S (x^2+y^2) dxdydz = \int_0^2 \int_0^{2\pi} \int_0^{\sqrt{2z}} r^2 (rdr) d\phi dz = \int_0^2 2\pi \frac{1}{4}(2z)^2 = \boxed{\frac{16\pi}{3}}$$

Exercise 13.

$$\iiint_S dxdydz = \int_0^\pi d\theta \int_0^{2\pi} d\phi \int_0^a r^2 \sin\theta = \frac{a^3}{3}(2\pi)(2) = \frac{4\pi a^3}{3}$$

Exercise 14.

$$\iiint_S dxdydz = \int_0^\pi d\theta \int_0^{2\pi} d\phi \int_a^b r^2 \sin\theta = \frac{4\pi}{3}(b^3 - a^3)$$

Exercise 15. Let $\sqrt{a^2+b^2+c^2} = \delta$ s.t. $\delta > R$.

Since the sphere S of integration is rotationally symmetric, **do a rotation so that** so that $(a,b,c) = (0,0,\delta)$ it's a lot easier!

Tip: Take advantage of symmetries, particularly spherical symmetries, and make problems easier by choosing a convenient rotation of the coordinate axes.

$$\int_0^\pi d\theta \int_0^{2\pi} d\phi \int_0^R r^2 \sin\theta dr (x^2+y^2+(z-\delta)^2)^{-1/2} = \int_0^\pi d\theta \int_0^{2\pi} d\phi \int_0^R r^2 \sin\theta dr (r^2+\delta^2 - 2r\delta\cos\theta)^{-1/2} =$$

$$= \int_0^{2\pi} d\phi \int_0^R r \left. \frac{(r^2+\delta^2-2r\delta\cos\theta)^{1/2}}{\delta} \right|_0^\pi = \int_0^{2\pi} d\phi \int_0^R \frac{r}{\delta}\left((r^2+\delta^2+2r\delta)^{1/2} - (r^2+\delta^2-2r\delta)^{1/2} \right)$$

since $\delta > R$, then $((r-\delta)^2)^{1/2} = |r-\delta| = \delta - r$, so then

$$= 2\pi \int_0^R \frac{r}{\delta}((r+\delta) - (\delta - r)) = \boxed{\frac{4\pi}{3}\frac{R^2}{\delta}}$$

Exercise 16. $\begin{aligned} x &= a\rho \cos^m\theta \sin^n\phi \\ y &= b\rho \sin^m\theta \sin^n\phi \\ z &= c\rho\cos^n\phi \end{aligned}$

$$detJ = \begin{vmatrix} ac^m\theta s^n\phi & na\rho c^m\theta s^{n-1}\phi c\phi & -ma\rho c^{m-1}\theta s\theta s^n\phi \\ bs^m\theta s^n\phi & nb\rho s^m\theta s^{n-1}\phi c\phi & mb\rho s^{m-1}\theta s^n\phi c\theta \\ cc^n\phi & -nc\rho c^{n-1}\phi s\phi & 0 \end{vmatrix} =$$

$$= ac^m\theta s^n\phi(mnbc\rho^2 c^{n-1}\phi s\phi s^{m-1}\theta s^n\phi c\theta + -na\rho(c^m\theta s^{n-1}\phi c\phi)(-mbc\rho s^{m-1}\theta s^n\phi c^n\phi c\theta) +$$
$$+ - ma\rho c^{m-1}\theta s\theta s^m\phi((-bnc\rho s^m\theta c^{n-1}\phi s^n\phi s^\phi) - nb\rho c c^n\phi s^m\theta s^{n-1}\phi c\phi) =$$
$$= abcmn\rho^2(c^{m+1}(\theta)s^{m-1}\theta s^{2n+1}\phi c^{n-1}\phi + c^{m+1}\theta s^{m-1}\theta s^{2n-1}\phi c^{n+1}\phi +$$
$$+ c^{m-1}\theta s\theta s^n\phi(s^m\theta s^{n+1}\phi c^{n-1}\phi + s^m\theta s^{n-1}\phi c^{n+1}\phi)) =$$
$$= abcmn\rho^2(c^{m+1}\theta s^{m-1}\theta s^{2n-1}\phi c^{n-1}\phi + c^{m-1}\theta s^{m+1}\theta s^{2n-1}\phi c^{n-1}\phi) =$$
$$= \boxed{abcmn\rho^2(c^{m-1}\theta s^{m-1}\theta s^{2n-1}\phi c^{n-1}\phi)}$$

Exercise 17.

$$\begin{aligned}
I_x &= \iiint_S (y^2+z^2)f(x,y,z)dxdydz = \iiint_S y^2 f(x,y,z)dxdydz + \iiint_S z^2 f(x,y,z)dxdydz = I_{xy}+I_{xz} \\
I_y &= \iiint_S (x^2+z^2)f(x,y,z)dxdydz = \iiint_S x^2 f(x,y,z)dxdydz + \iiint_S z^2 f(x,y,z)dxdydz = I_{yz}+I_{yx} \\
I_z &= \iiint_S (x^2+y^2)f(x,y,z)dxdydz = \iiint_S x^2 f(x,y,z)dxdydz + \iiint_S y^2 f(x,y,z)dxdydz = I_{zy}+I_{zx}
\end{aligned}$$

Exercise 18. The condition for the paraboloid and sphere to meet is the following:

$$x^2+y^2 = 4z = 5-z^2 \Longrightarrow z^2+4z-5=0 \text{ or } (z+5)(z-1)=0$$

$$\begin{aligned}
V &= \int_0^2 \int_0^{2\pi} \int_{\frac{r^2}{4}}^{\sqrt{5-r^2}} rdzd\phi dr = \int_0^2 2\pi r\left(\sqrt{5-r^2}-\frac{r^2}{4}\right) = \\
&= 2\pi\left(\frac{-1}{3}(5-r^2)^{3/2} - r^4/16\right)\Big|_0^2 = 2\pi(\frac{-1}{3}(1-5^{3/2}) - 16/16) = \boxed{\frac{2\pi}{3}(5^{3/2}-4)}
\end{aligned}$$

Exercise 20.

$$\int_0^{2\pi}\int_0^{\pi}\int_a^b r^2(r^2\sin\theta)drd\theta d\phi = \frac{1}{5}(b^5-a^5)(2)(2\pi) = \boxed{\frac{4\pi(b^5-a^5)}{5}}$$

Exercise 21.

$$\begin{aligned}
\int_0^h\int_0^{2\pi}\int_0^z rdrd\phi dz &= \int_0^h \frac{2\pi}{2}z^2dz = \frac{\pi}{3}h^3 = M \\
\overline{z}M = \int_0^h\int_0^{2\pi}\int_0^z rzdrd\phi dz &= \int_0^h dz 2\pi\frac{1}{2}z^3 = \frac{\pi}{4}h^4
\end{aligned}$$

$\overline{z} = \frac{3h}{4}$ so centroid is $\frac{h}{4}$ away from base.

Exercise 22. Note the symmetry in ϕ and r.

$$\begin{aligned}
M = \int_0^h\int_0^{2\pi}\int_0^z (h-z)rdrd\phi dz &= \int_0^h dz(2\pi)(h-z)\frac{1}{2}z^2 = (2\pi)\left(\frac{h}{6}h^3 - \frac{h^4}{8}\right) = (2\pi h^4)\left(\frac{1}{24}\right) = \boxed{\frac{\pi h^4}{12}} \\
\overline{z}M = \int_0^h\int_0^{2\pi}\int_0^z (h-z)zrdrd\phi dz &= 2\pi\int_0^h dz(h-z)z\frac{1}{2}z^2 = \pi\left(\frac{z^4}{4}h - \frac{1}{5}z^5\right)\Big|_0^h = \\
&= \pi(h^5)(1/4-1/5)
\end{aligned}$$

$$\Longrightarrow \overline{z} = \frac{3}{5}h$$

Center of mass is $\frac{2}{5}h$ from the base.

Exercise 23. Note symmetry in ϕ and r.

$$\begin{aligned}
M &= \int_0^h\int_0^{2\pi}\int_0^z rrdrd\phi dz = \int_0^h 2\pi\left.\frac{r^3}{3}\right|_0^z dz = \frac{2\pi}{3}\frac{h^4}{4} = \frac{\pi h^4}{6} \\
\overline{z}M &= \int_0^h\int_0^{2\pi}\int_0^z rzrdrd\phi dz = 2\pi\int_0^h z\frac{1}{3}z^3 = \left.\frac{2\pi}{3}\frac{1}{5}z^5\right|_0^h = \frac{2\pi}{15}h^5
\end{aligned} \qquad \Longrightarrow \overline{z} = \frac{4h}{5}$$

$\frac{h}{5}$ from base.

Exercise 24. Consider concentric hemispheres of radii a and b, where $0<a<b$.

$$\begin{aligned}
M &= \int_0^{\pi/2}d\theta\int_0^{2\pi}d\phi\int_a^b r^2\sin\theta = \frac{b^3-a^3}{3}(2\pi) \\
\overline{z}M &= \int_0^{\pi/2}d\theta\int_0^{2\pi}d\phi\int_a^b r^2\sin\theta(r\cos\theta) = \int_0^{\pi/2}d\theta 2\pi\frac{b^4-a^4}{4}\sin\theta\cos\theta = \frac{\pi}{4}(b^4-a^4)
\end{aligned} \qquad \Longrightarrow \overline{z} = \frac{3}{8}\frac{b^4-a^4}{b^3-a^3}$$

Exercise 25. I tried cylindrical coordinates first. Didn't help.

Tip: quickly switch and try another way, another set of coordinates, if one way doesn't work.

$$M = \int_0^1 dz \int_0^1 dy \int_0^1 dx(x^2+y^2+z^2) = 1$$

$$\overline{x}M = \int_0^1 dz \int_0^1 dy \int_0^1 dxx(x^2+y^2+z^2) = \frac{1}{4} + \frac{1}{2}\left(\frac{1}{3}\right) + \frac{1}{2}\left(\frac{1}{3}\right) = \frac{7}{12}$$

By label symmetry of x, y, z, $\overline{x} = \overline{y} = \overline{z} = \frac{7}{12}$

Exercise 26. Note that $\frac{r}{z} = \frac{a}{h}$

$$I_{cone,z} = \int_0^h \int_0^{2\pi} \int_0^{\frac{a}{h}z} r^2(rdr)dz\frac{M}{V} = 2\pi\int_0^h \frac{1}{4}\left(\frac{a}{h}\right)^4 z^4 dz\frac{M}{V} = \frac{\pi}{10}\left(\frac{a}{h}\right)^4 h^5\frac{M}{V} = \frac{3a^2}{10}M$$

$$V = \int_0^h \int_0^{2\pi} d\phi \int_0^{az/h} rdrdz = 2\pi\frac{a^2}{2h^2}\frac{1}{3}h^3 = \frac{\pi a^2 h}{3}$$

$$I_x + I_y = \iiint \frac{M}{V}(y^2+z^2+x^2+z^2)dxdydz = \frac{M}{V}\iiint(x^2+y^2+2z^2)dxdydz =$$

$$= \frac{M}{V}\left(\frac{\pi a^4 h}{10} + 2\int_0^h \int_0^{2\pi} d\phi \int_0^{\frac{az}{h}} z^2 rdrdz\right) = \frac{M}{V}\left(\frac{\pi a^4 h}{10} + 2\int_0^h 2\pi\frac{1}{2}\frac{a^2}{h^2}z^4 dz\right) = \frac{M}{V}\left(\frac{\pi a^4 h}{10} + \frac{2\pi a^2}{5h^2}h^5\right) =$$

$$= 2I_x$$

$$\Longrightarrow I_x = \frac{M}{2\left(\frac{\pi a^2 h}{3}\right)}(\pi a^2 h)\left(\frac{a^2}{10} + \frac{2h^2}{5}\right) = \boxed{\frac{3M}{2}\left(\frac{a^2}{10} + \frac{2h^2}{5}\right)}$$

Exercise 27. $f = M/\frac{4}{3}\pi R^3 = M/V$

$$I = \iiint (x^2+y^2)\frac{M}{V}dxdydz = \int_0^\pi d\theta \int_0^{2\pi} d\phi \int_0^R r^2(\sin\theta)\frac{M}{V}r^2\sin^2\theta = \frac{2\pi M}{V}\int_0^\pi \sin\theta(1-\cos^2\theta)\frac{R^5}{5} =$$

$$= \frac{2\pi M}{5V}R^5\left(2 + \frac{1}{3}(-2)\right) = \frac{2\pi M R^5}{5\frac{4\pi}{3}R^3}\left(\frac{4}{3}\right) = \boxed{\frac{2}{5}MR^2}$$

Another way, which is quite clever, is the following. Consider that

$$2(x^2+y^2+z^2) = x^2+y^2+z^2+x^2+x^2+y^2$$

Then

$$2\iiint (x^2+y^2+z^2)\frac{M}{V}dxdydz = 2\int_0^\pi d\theta \int_0^{2\pi} d\phi \int_0^R r^2\frac{M}{V}r^2\sin\theta = \frac{8\pi M}{5V}R^5 = I_z + I_y + I_x$$

By spherical symmetry, $I_z = I_y$. So then $I = \frac{8\pi M R^5}{15\left(\frac{4\pi R^3}{3}\right)} = \frac{2MR^2}{5}$

Exercise 28.

$$M = \int_{-h}^h dz \int_0^{2\pi} d\phi \int_0^a rrdr = 2h(2\pi)\frac{a^3}{3}$$

$$I_z = \int_{-h}^h dz \int_0^{2\pi} d\phi \int_0^a r^2 rdrr = (2h)(2\pi)\frac{1}{5}a^5 = \boxed{\frac{3Ma^2}{5}}$$

Exercise 29.

$$V_{cap} = \left(\frac{4\pi R^3}{3}\right)\frac{1}{2} = \frac{2\pi R^3}{3} \qquad c = \frac{M}{V} = \text{ mass density}$$

$$V_{cylinder} = \pi\left(\frac{1}{2}\right)^2 2 = \frac{\pi}{2}$$

$$\overline{z}M = \int_0^{\pi/2} d\theta \int_0^{2\pi} d\phi \int_0^R r\cos\theta r^2 \sin\theta dr\frac{M}{V} = \frac{2\pi R^4}{4}\frac{1}{2}\frac{M}{V} = \frac{\pi R^4}{4}\frac{M}{V}$$

$$\overline{z} = \frac{\pi R^4}{4\left(\frac{2\pi R^3}{3}\right)} = \frac{3R}{8}$$

Condition wanted is for center of mass of the mushroom to be at $z = 0$, for this particular choice of coordinates.

$$\frac{c\frac{\pi R^4}{4} + \left(c\frac{\pi}{2}\right)(-1)}{cV + c\frac{\pi}{2}} = \frac{\frac{\pi R^4}{4} + \frac{-\pi}{2}}{\frac{2\pi R^3}{3} + \frac{\pi}{2}} = 0 \Longrightarrow R^4 = 2 \text{ or } \boxed{R = 2^{1/4}}$$

Exercise 1. *Plane*:

$$\mathbf{r}(u,v) = (x_0 + a_1 u + b_1 v)\mathbf{i} + (y_0 + a_2 u + b_2 v)\mathbf{j} + (z_0 + a_3 u + b_3 v)\mathbf{k} \implies \begin{aligned} x &= x_0 + a_1 u + b_1 v \\ y &= y_0 + a_2 u + b_2 v \\ z&z_0 + a_3 u + b_3 v \end{aligned}$$

Then get u, v in terms of x, y.

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix}$$

$$\implies \begin{bmatrix} u \\ v \end{bmatrix} = \frac{1}{a_1 b_@ - b_1 a_2} \begin{bmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{bmatrix} \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix} = \frac{1}{a_1 b_2 - b_1 a_2} \begin{bmatrix} b_2(x - x_0) - b_1(y_1 - y_0) \\ -a_2(x - x_0) + a_1(y - y_0) \end{bmatrix}$$

So then

$$\begin{gathered} (a_1 b_2 - b_1 a_2)(z - z_0) = (a_3 b_2 - b_3 a_2)(x - x_0) + (b_3 a_1 - a_3 b_1)(y - y_0) \\ \partial_u r_u = (a_1, a_2, a_3) \\ \partial_v r_v = (b_1, b_2, b_3) \\ \partial_u r \times \partial_v r = \begin{vmatrix} e_1 & e_2 & e_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1) \end{gathered}$$

Exercise 2. *Elliptic paraboloid*: $\mathbf{r}(u,v) = au\cos v\mathbf{i} + bu\sin v\mathbf{j} + u^2\mathbf{k}$. Then

$$\begin{aligned} x &= au\cos v \\ y &= bu\sin v \implies \boxed{\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = z} \\ z &= u^2 \end{aligned}$$

$$\begin{aligned} \partial_u r &= (a\cos v, b\sin v, 2u) \\ \partial_v r &= (-au\sin v, bu\cos v, 0) \end{aligned} \implies \begin{vmatrix} e_1 & e_2 & e_3 \\ a\cos v & b\sin v & 2u \\ -au\sin v & bu\cos v & 0 \end{vmatrix} = \boxed{(-2bu^2\cos v, -2u^2 a\sin v, abu)}$$

Exercise 3. *Ellipsoid*: $\mathbf{r}(u,v) = (a\sin u\cos v, b\sin u\sin v, c\cos u)$.

$$\implies \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1$$

Now

$$\begin{aligned} \partial_u r &= (ac(u)c(v), bc(u)s(v), -cs(u)) \\ \partial_v r &= (-as(u)s(v), bs(u)c(v), 0) \end{aligned}$$

$$\implies \begin{vmatrix} e_1 & e_2 & e_3 \\ ac(u)c(v) & bc(u)s(v) & -cs(u) \\ -as(u)s(v) & bs(u)c(v) & 0 \end{vmatrix} = \boxed{(bcs^2(u)c(v), acs^2(u)s(v), ab(c(u))s(u))}$$

Exercise 4. *Surface of revolution:* $\mathbf{r}(u,v) = (u\cos v, u\sin v, f(u))$. $x^2 + y^2 = u^2$ so then

$$\boxed{f(\sqrt{x^2 + y^2}) = z}$$

$$\begin{gathered} \partial_u r = (\cos v, \sin v, f'(u)) \\ \partial_V r = (-u\sin v, u\cos v, 0) \\ \partial_u r \times \partial_v r = \begin{vmatrix} e_1 & e_2 & e_3 \\ \cos v & \sin v & f'(u) \\ -u\sin v & u\cos v & 0 \end{vmatrix} = \boxed{(-f'u\cos v, -f'u\sin v, u)} \end{gathered}$$

Exercise 5. *Cylinder*: $\boxed{y^2 + z^2 = a^2}$

$$\mathbf{r}(u,v) = (u, a\sin v, a\cos v) \begin{aligned} \partial_u r &= (1, 0, 0) \\ \partial_v r &= (0, a\cos v, -a\sin v) \end{aligned}$$

$$\implies \begin{vmatrix} e_1 & e_2 & e_3 \\ 1 & 0 & 0 \\ 0 & a\cos v & -a\sin v \end{vmatrix} = (0, a\sin v, a\cos v)$$

Exercise 6. *Torus*: $\mathbf{r}(u,v) = ((a+b\cos u)\sin v, (a+b\cos u)\cos v, b\sin u),\ \ 0<b<a.$

$$x^2+y^2 = (a+b\cos u)^2 = (a+\sqrt{b^2-z^2})^2$$

$$\begin{aligned}\partial_u r &= (-bs(u)s(v), -bs(u)c(v), bc(u))\\ \partial_v r&((a+bc(u))c(v), (a+bc(u))(-s(v)), 0)\end{aligned}$$

$$\begin{vmatrix} e_1 & e_2 & e_3 \\ -bs(u)s(v) & -bs(u)c(v) & bc(u) \\ (a+bc(u))c(v) & (a+bc(u))(-s(v)) & 0 \end{vmatrix} =$$
$$= (b(a+bc(u))c(u)s(v), b(a+bc(u))c(u)c(v), (a+bc(u))(bs(u)s^2(v)+bs(u)c^2(v)))$$

$$\implies \boxed{(a+b\cos(u))b(\cos(u)\sin(v), \cos(u)\cos(v), \sin(u))}$$

Exercise 7. $\mathbf{r}(u,v) = (a\sin u\cosh v, b\cos u\cosh v, c\sinh v)$

$$\begin{aligned}\partial_u r &= (ac(u)\cosh(v), -bs(u)\cosh v, 0)\\ \partial_v r &= (as(u)\sinh(v), bc(u)\sinh v, c\cosh v)\end{aligned}$$

$$\partial_u r\times\partial_v r = \begin{vmatrix} e_1 & e_2 & e_3 \\ ac(u)\cosh(v) & -bs(u)\cos(v) & 0 \\ as(u)\sinh(v) & bc(u)\sinh v & c\cosh v \end{vmatrix} =$$
$$= (-bc\sin u\cosh^2 v, ac\cos u\cosh^2 v, ab\cos^2 u\cosh v\sinh v + ab\sin^2 u\cosh v\sinh v)$$

$$\|\partial_u r\times\partial_v r\| = \boxed{abc\cosh v\left(\left(\frac{\sin^2 u}{a^2}+\frac{\cos^2 u}{b^2}\right)\cosh^2 v + \frac{\sinh^2 v}{c^2}\right)^{1/2}}$$

Exercise 8. $\mathbf{r}(u,v) = (u+v, u-v, 4v^2)$

$$\begin{aligned}\partial_u r &= (1,1,0)\\ \partial_v r &= (1,-1,8v)\end{aligned} \implies \begin{vmatrix} e_1 & e_2 & e_3 \\ 1 & 1 & 0 \\ 1 & -1 & 8v \end{vmatrix} = (8v, -8v, -2)$$

$$\|\partial_u r\times\partial_v r\| = \sqrt{64v^2+64v^2+4} = 2\sqrt{1+32v^2}$$

Exercise 9. $\mathbf{r}(u,v) = ((u+v), u^2+v^2, u^3+v^3)$

$$\begin{aligned}\partial_u r &= (1,2u,3u^2)\\ \partial_v r&(1,2v,3v^2)\end{aligned} \implies \begin{vmatrix} e_1 & e_2 & e_3 \\ 1 & 2u & 3u^2 \\ 1 & 2v & 3v^2 \end{vmatrix} = (6uv^2-6vu^2, 3u^2-3v^2, 2v-2u) =$$
$$= (v-u)(6uv, -3(u+v), 2)$$

$$\implies |v-u|\sqrt{36u^2v^2+9(u^2+2uv+v^2)+4}$$

Exercise 10. $\mathbf{r}(u,v) = (u\cos v, u\sin v, \frac{1}{2}u^2\sin 2v).$

$$\begin{aligned}\partial_u r &= (c(v), s(v), us(2v))\\ \partial_v r &= (-us(v), uc(v), u^2c(2v))\end{aligned} \implies \partial_u r\times\partial_v r = \begin{vmatrix} e_1 & e_2 & e_3 \\ c(v) & s(v) & us(2v) \\ -us(v) & uc(v) & u^2c(2v) \end{vmatrix} =$$
$$= (u^2s(v)c(2v) - u^2c(v)s(2v), -u^2c(2v)c(v) - u^2s(2v)s(v), u) = (u^2s(-v), -u^2c(v), u)$$

$$\|\partial_u r\times\partial_v r\| = \sqrt{u^4s^2(v)+u^4c^2(v)+u^2} = \boxed{u\sqrt{u^2+1}}$$

Exercise 2. $S = r(T)$
$x^2 + y^2 = a^2$ represents T.
$x + y + z = a$ is S.

Using $z = a - x - y$, $r = (x, y, a - x - y)$. Then $\begin{matrix} \partial_x r = (1, 0, -1) \\ \partial_y r = (0, 1, -1) \end{matrix} \Longrightarrow \begin{vmatrix} e_x & e_y & e_z \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = (1, 1, 1)$

$$\|\partial_x r \times \partial_y r\| = \sqrt{1+1+1} = \sqrt{3}$$

$$\iint_T \sqrt{3}dxdy = \boxed{\sqrt{3}\pi a^2}$$

Exercise 4. $z^2 = 2xy$ $x = 2$, $y = 1$. Then $2z\partial_x z = 2y$ or $\partial_x z = \frac{y}{z}$. Similarly $\partial_y z = \frac{x}{z}$.

$$\begin{vmatrix} e_1 & e_2 & e_3 \\ 1 & 0 & \frac{y}{z} \\ 0 & 1 & \frac{x}{z} \end{vmatrix} = (\frac{-y}{z}, \frac{-x}{z}, 1) \Longrightarrow \|\partial_x r \times \partial_y r\|^2 = \frac{y^2}{z^2} + \frac{x^2}{z^2} + \frac{z^2}{z^2} = \frac{(x+y)^2}{z^2}$$

$$\|\partial_x r \times \partial_y\| = \frac{x+y}{z} = \frac{x+y}{\sqrt{2}\sqrt{xy}} = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{x}{y}} + \sqrt{\frac{y}{x}}\right)$$

$\int a(S) = \iint \frac{1}{\sqrt{2}}\left(\sqrt{\frac{x}{y}} + \sqrt{\frac{y}{x}}\right) dxdy$ so with

$$\int \sqrt{\frac{x}{y}}dx = \left.\frac{\frac{2}{3}x^{3/2}}{\sqrt{y}}\right|_0^2 = \frac{\frac{2}{3}2\sqrt{2}}{\sqrt{y}} = \frac{4\sqrt{2}}{3}y^{-1/2} \xrightarrow{\int dy} \left.\frac{4\sqrt{2}}{3}2y^{1/2}\right|_0^1 = \frac{8\sqrt{2}}{3}$$

$$\int \sqrt{\frac{y}{x}} = \left.\frac{\frac{2}{3}y^{3/2}}{\sqrt{x}}\right|_0^1 = \frac{2}{3\sqrt{x}} \xrightarrow{\int dx} \frac{2}{3}\left. 2x^{1/2}\right|_0^2 = \frac{4\sqrt{2}}{3}$$

$$\Longrightarrow a(S) = \frac{1}{\sqrt{2}}\left(\frac{8\sqrt{2}}{3} + \frac{4\sqrt{2}}{3}\right) = \boxed{4}$$

Exercise 5. $\mathbf{r} = (u\cos v, u\sin v, u^2)$

a. $x^2 + y^2 = z$. u is radius, v is angle in $x - y$ plane.
b.

$$\begin{matrix} \partial_u r = (c, s, 2u) \\ \partial_v r = (-us, uc, 0) \end{matrix} \Longrightarrow \partial_u r \times \partial_v r = \begin{vmatrix} e_1 & e_2 & e_3 \\ c & s & 2u \\ -us & uc & 0 \end{vmatrix} = (-2u^2c, -2u^2s, u)$$

c. $\|\partial_u r \times \partial_v r\|^2 = 4u^4 + u^2$

$$a(S) = \iint u\sqrt{(1+4u^2)}dudv = 2\pi\frac{2}{3}(1+4u^2)^{3/2}\left.\left(\frac{1}{8}\right)\right|_0^4 = \frac{\pi}{6}((1+64)^{3/2} - 1) = \frac{\pi}{6}(65\sqrt{65} - 1)$$

$\boxed{n = 6}$

Exercise 6. $x^2 + y^2 = z^2$.
$x^2 + y^2 + z^2 = 2ax \Longrightarrow x^2 - 2ax + a^2 + y^2 + z^2 = a^2 = (x-a)^2 + y^2 + z^2 = a^2$
Determine where sphere and cone intersect: $z^2 = ax$, so then $y^2 = ax - x^2 = x(a - x)$. Since $x > 0$, $a - x > 0$, $a > x$

$$\Longrightarrow y = \pm\sqrt{x(a-x)} = \pm\sqrt{\frac{-a^2}{4} + ax - x^2 + \frac{a^2}{4}} = \sqrt{\frac{a^2}{4} - \left(\frac{a}{2} - x\right)^2}$$

Now $2z\partial_x z = 2x$. Then

$$\begin{vmatrix} e_1 & e_2 & e_3 \\ 1 & 0 & \frac{x}{z} \\ 0 & 1 & \frac{y}{z} \end{vmatrix} = (\frac{-x}{z}, \frac{-y}{z}, 1) \quad \Longrightarrow \|\partial_x r \times \partial_y r\|^2 = \frac{x^2}{z^2} + \frac{y^2}{z^2} + \frac{z^2}{z^2} = 2$$

$$a(S) = \iint \sqrt{2}dxdy = \sqrt{2}\int_0^1 \int_{-\sqrt{x(a-x)}}^{\sqrt{x(a-x)}} dydx = 2\sqrt{2}\int_0^a dx\sqrt{\frac{a^2}{4} - \left(\frac{a}{2} - x\right)^2} = 2\sqrt{2}\int_{-a/2}^{a/2} \sqrt{\frac{a^2}{4} - x^2} =$$

$$u = \frac{2x}{a}$$

$$= a\sqrt{2}\int_{-a/2}^{a/2} \sqrt{1 - \left(\frac{2x}{a}\right)^2} \xrightarrow{\frac{a}{2}du = dx} \frac{a^2\sqrt{2}}{2}\int_{-1}^{1} \sqrt{1-u^2}du$$

$$\xrightarrow[\;]{\substack{u = \sin\theta \\ du = \cos\theta d\theta}} = \frac{a^2\sqrt{2}}{2}\int_{-\pi/2}^{\pi/2} \cos^2\theta d\theta = \frac{a^2\sqrt{2}}{2}\int_{-\pi/2}^{\pi/2} \frac{1+\cos 2\theta}{2} = \boxed{\frac{\sqrt{2}\pi a^2}{4}}$$