THE DIFFERENTIAL GEOMETRY DIFFERENTIAL TOPOLOGY DUMP

ERNEST YEUNG ERNESTYALUMNI@GMAIL.COM

Contents

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Abstract. Everything about Differential Geometry, Differential Topology

Part 1. Combinatorics, Probability Theory

Theorem 1 (4.2. of Feller (1968) [1]). *Let $r_1, \dots r_k \in \mathbb{Z}$, s.t. $r_1 + r_2 + \dots + r_k = n;$, $r_i \geq 0$.*

Let

$$\frac{N!}{r_1! r_2! \dots r_k!} = \tag{1}$$

number of ways in which n elemnts can be divided into k ordered parts (partitioned into k subpopulations). cf. Eq. (4.7) of Feller (1968) [1].

Note that the order of the subpopulations is essential in the sense that $(r_1 = 2, r_2 = 3)$ and $(r_1 = 3, r_2 = 2)$ represent different partitions. However, no attention is paid to the order within the groups.

Proof.

$$\binom{n}{r_1}\binom{n-r_1}{r_2}\binom{n-r_1-r_2}{r_3}\dots\binom{n-r_1-\dots-r_{k-2}}{r_{k-1}} = \frac{n!}{r_1!r_2!\dots r_k!} \tag{2}$$

i.e. in order to effect the desired partition, we have to select r_1 elementsout of n, remaining $n - r_1$ elements select a second group of size r_2, etc. After forming the $(k-1)$st group there remains $n - r_1 - r_2 - \dots - r_{k-1} = r_k$ elements, and these form the last group. □

cf. pp. 37 of Feller (1968) [1] Examples. (g) Bridge. 32 cards are partitioned into 4 equal groups $\to 52!/(13!)^4$.

Probability each player has an ace (?).

The 4 aces can be ordered in $4! = 24$ ways, each order presents 1 possibility of giving 1 ace to each player. Remaining 48 cards distributed $(48!)/(12!)^4$ ways.

$$\to p = 24 \frac{48!}{(12!)^4} / \frac{52!}{(13!)^4}$$

(h) A throw of 12 dice $\to 6^{12}$ different outcomes total. Event each face appears twice can occur in as many ways as 12 dice can be arranged in 6 groups of 2 each.

$$\frac{12!}{(2!)^6} / \frac{52!}{(13!)^4}$$

0.0.1. *Application to Occupancy Problems; binomial coefficients.* cf. Sec. 5 Application to Occupancy Problems of Feller (1968) [1].

Consider randomly placing r balls intos n cells.

Let r_k = occupancy number = number of balls in kth cell.

Every n-tuple of integers satisfying $r_1 + r_2 + \dots + r_n = r$; $r_k \geq 0$. describes a possible configuration of occupancy numbers.

With indistinguishable balls 2 distributions are distinguishable only if the corresponding n-tuples $(r_1, \dots r_n)$ are not identical.

(i) number of distinguishable distributions is

$$A_{r,n} = \binom{n+r-1}{r} = \binom{n+r-1}{n-1} \tag{3}$$

cf. Eq. (5.2) of Feller (1968) [1]

(ii) number of distinguishable distributions in which no cell remains empty is $\binom{r-1}{n-1}$.

Proof. Represent balls by stars, indicate n cells by n spaces between $n+1$ bars. e.g. $\begin{matrix} r = 8 \text{ balls} \\ n = 6 \text{ cells} \end{matrix}$.

$$\begin{matrix} 3 & 1 & 0 & 0 & 0 & 4 \\ |***| & *| & | & | & | & ****| \end{matrix}$$

Such a symbol necessarily starts and ends with a bar, but remaining $n-1$ bars and r starts appear in an arbitrary order. In this way, it becomes apparent that the number of distinguishable distributions equals the number of ways of selecting.

r places out of $n+r-1$, $\frac{(n+r-1)!}{(n-1)!r!} = \binom{n-1+r}{r}$

$$\begin{matrix} ||||\dots|| & n+1 \text{ bars} \\ ****\dots** & r \text{ stars leave } r-1 \text{ spaces} \end{matrix}$$

Condition that no cell be empty imposes the restriction that no 2 bars be adjacent. r stars leave $r-1$ spaces of which $n-1$ are to be occupied by bars. Thus $\binom{r-1}{n-1}$ choices. □

Probability to obtain given occupancy numbers $r_1, \dots r_n = \frac{r!}{r_1!r_2!\dots r_n!}/n^r$, with $\begin{matrix} r \text{ balls} \\ n \text{ cells} \end{matrix}$ given by Thm. 4.2. of Feller (1968) [1], which is the Maxwell-Boltzmann distribution.

(a) Bose-Einstein and Fermi-Dirac statistics. Consider r indistinguishable particles, n cells, each particle assigned to 1 cell.

State of the system - random distribution of r particles in n cells.

If n cells distinguishable, n^r arrangements equiprobable $\to$ Maxwell-Boltzmann statistics.

Bose-Einstein statistics: only distinguishable arrangements are considered, and each assigned probability $\frac{1}{A_{r,n}}$

$$A_{r,n} = \binom{n+r-1}{r} = \binom{n-1+r}{n-1} \tag{4}$$

cf. Eq. 5.2 of Feller (1968) [1]

Fermi-Dirac statistics.

(1) impossible for 2 or more particles to be in the same cell. $\to r \leq n$.
(2) all distinguishable arrangements satisfying the first condition have equal probabilities.
$\to$ an arrangement is completely described by stating which of the n cells contain a particle
r particles $\to \binom{n}{r}$ ways r cells chosen.
Fermi-Dirac statistics, there are $\binom{n}{r}$ possible arrangements, prob. $1/\binom{n}{r}$.

pp. 39. Feller (1968) [1]. Consider cells themselves indistinguishable! Disregard order among occupancy numbers.

cf. Feller (1968) [1]

Part 2. Linear Algebra Review

cf. *Change of Basis*, of Appendix B of John Lee (2012) [3].

Exercise B.22. Suppose V, W, X finite-dim. vector spaces
$S : V \to W, \quad T : W \to X$

(a) $\text{rank}S \leq \text{dim}V$ with $\text{rank}S = \text{dim}V$ iff S injective
(b) $\text{rank}S \leq \text{dim}W$ with $\text{rank}S = \text{dim}W$ iff S surjective
(c) if $\text{dim}V = \text{dim}W$ and S either injective or surjective, then S isomorphism
(d) $\text{rank}TS \leq \text{rank}S$ $\text{rank}TS = \text{rank}S$ iff $\text{im}S \bigcap \text{ker}T = 0$
(e) $\text{rank}TS \leq \text{rank}T$ $\text{rank}TS = \text{rank}T$ iff $\text{im}S + \text{ker}T = W$
(f) if S isomorphism, then $\text{rank}TS = \text{rank}T$
(g) if T isomorphism, then $\text{rank}TS = \text{rank}S$

EY : Exercise B.22(d) is useful for showing the chart and atlas of a Grassmannian manifold, found in the More examples, for smooth manifolds.

Proof. (a) Recall the **rank-nullity theorem**:

Theorem 2 (Rank-Nullity Theorem)**.**

$$dim(im(S)) + dim(ker(S)) = dimV \tag{5}$$

Now

$$\text{rank}(S) + \dim(\ker(S)) \equiv \dim(\text{im}(S)) + \dim(\ker(S)) = \dim V$$

$$\Longrightarrow \boxed{\text{rank}(S) \leq \dim V}$$

If $\text{rank}(S) = \dim V$,
then by rank-nullity theorem, $\dim(\ker(S)) = 0$, implying that $\ker S = \{0\}$.
Suppose $v_1, v_2 \in V$ and that $S(v_1) = S(v_2)$. By linearity of S, $S(v_1) - S(v_2) = S(v_1 - v_2) = 0$, which implies, since $\ker S = \{0\}$, that $v_1 - v_2 = 0$.
$\Longrightarrow v_1 = v_2$. Then by definition of injectivity, S injective.
If S injective, then $S(v) = 0$ implies $v = 0$. Then $\ker S = \{0\}$. Then by rank-nullity theorem, $\text{rank}(S) = \dim V$.

(b) $\forall w \in \text{im}(S)$, $w \in W$. Clearly $\text{rank}S \leq \dim W$.
If S surjective, $\text{im}(S) = W$. Then $\dim(\text{im}(S)) = \text{rank}S = \dim W$.

If $\text{rank}S = \dim W = m$, then $\text{im}(S)$ has basis $\{y_i\}_{i=1}^m$, $y_i \in \text{im}(S)$, so $\exists x_i \in V$, $i = 1 \dots m$ s.t. $S(x_i) = y_i$, with $\{S(x_i)\}_{i=1}^m$ linearly independent.
Since $\{S(x_i)\}_{i=1}^m$ linearly independent and $\dim W = m$, $\{S(x_i)\}_{i=1}^m$ basis for W.
$\forall w \in W$, $w = \sum_{i=1}^m w^i S(x_i) = S(\sum_{i=1}^m w^i x_i)$. $\sum_{i=1}^m w^i x_i \in V$. S surjective.

(c)

(d) Now

$$\dim V = \text{rank}TS + \text{nullity}TS$$
$$\dim V = \text{rank}S + \text{nullity}S$$

$\ker S \subseteq \ker TS$, clearly, so $\text{nullity}S \leq \text{nullity}TS$

$$\Longrightarrow \boxed{\text{rank}TS \leq \text{rank}S}$$

If $\text{rank}TS = \text{rank}S$,
then $\text{nullity}S = \text{nullity}TS$
Suppose $w \in \text{Im}S \bigcap \ker T$, $w \neq 0$
Then $\exists v \in S$, s.t. $w = S(v)$ and $T(w) = 0$
Then $T(w) = TS(v) = 0$. So $v \in \ker TS$
$v \notin \ker S$ since $w = S(v) \neq 0$
This implies $\text{nullity}TS > \text{nullity}S$. Contradiction.
$\Longrightarrow \text{Im}S \bigcap \ker T = 0$

If $\text{Im}S \bigcap \ker T = 0$,
Consider $v \in \ker TS$. Then $TS(v) = 0$.
. Then $S(v) \in \ker T$
$S(v) = 0$; otherwise, $S(v) \in \text{Im}S$, contradicting given $\text{Im}S \bigcap \ker T = 0$
$v \in \ker S$

$\ker TS \subseteq \ker S$
$\Longrightarrow \ker TS = \ker S$
So $\text{nullity}TS = \text{nullity}S$
$\Longrightarrow \text{rank}TS = \text{rank}S$

(e)
(f)
(g)

∎

Part 3. Manifolds

1. Inverse Function Theorem

Shastri (2011) had a thorough and lucid and explicit explanation of the Inverse Function Theorem [5]. I will recap it here. The following is also a blend of Wienhard's Handout 4 `https://web.math.princeton.edu/~wienhard/teaching/M327/handout4.pdf`

Definition 1. *Let (X, a) metric space.*
contraction *$\phi : X \to X$ if $\exists$ constant $0 < c < 1$ s.t. $\forall x, y \in X$*

$$d(\phi(x), \phi(y)) \leq cd(x, y)$$

Theorem 3 (Contraction Mapping Principle)**.** *Let (X, d) complete metric space.*
Then $\forall$ contraction $\phi : X \to X$, $\exists ! y \in X$ s.t. $\phi(y) = y$, y fixed pt.

Proof. Recall def. of complete metric space X, X metric space s.t. $\forall$ Cauchy sequence in X is convergent in X (i.e. has limit in X).

$$\forall x_0 \in X, \text{ Define } \begin{aligned} x_1 &= \phi(x_0) \\ x_2 &= \phi(x_1) \\ &\vdots \\ x_j &= \phi(x_{j-1}) \\ &\vdots \\ x_n &= \phi(x_{n-1}) \end{aligned}$$

$$d(x_{n+1}, x_n) = d(\phi(x_n), \phi(x_{n-1})) \leq cd(x_n, x_{n-1}) \leq \dots \leq c^n d(x_1, x_0)$$

for some $0 < c < 1$.

$$d(x_m, x_n) \leq d(x_n, x_{n-1}) + d(x_{n-1}, x_m) \leq d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + \dots + d(x_{m+1}, x_m) \leq \sum_{k=n-1}^m c^k d(x_1, x_0)$$

Thus, $\forall \epsilon > 0$, $\exists n_0 > 0$, (n_0 large enough) s.t. $\forall m, n \in \mathbb{N}$ s.t. $n_0 < n < m$,

$$d(x_m, x_n) \leq \sum_{k=n-1}^m c^k d(x_1, x_0) < \epsilon d(x_1, x_0)$$

Thus, $\{x_n\}$ Cauchy sequence. Since X complete, $\exists$ limit pt. $y \in X$ of $\{x_n\}$.

$$\phi(y) = \phi(\lim_n x_n) = \lim_n \phi(x_n) = \lim_n x_{n+1} = y$$

Since by def. of y limit pt. of $\{x_n\}$, $\forall \epsilon > 0$, then $\{n | |x_n - y| \leq \epsilon, n \in \mathbb{N}\}$ is infinite.
Consider $\delta > \mathbb{N}$. Consider $\{n | |x_n - y| \leq \delta, n \in \mathbb{N}\}$
$\exists N_\delta \in \mathbb{N}$ s.t. $\forall n > N_\delta$, $|x_n - y| < \delta$; otherwise, $\forall N_\delta$, $\exists n > N_\delta$ s.t. $|x_n - y| \geq \delta$. Then $\{n | |x_n - y| \leq \delta, n \in \mathbb{N}\}$ finite. Contradiction.
ϕ cont. so by def. $\forall \epsilon > 0$, $\exists \delta > 0$ s.t. if $|x_n - y| < \delta$, then $|\phi(x_n) - \phi(y)| < \epsilon$.
Pick N_δ s.t. $\forall n > N_\delta$, $|x_n - y| < \delta$, and so $|\phi(x_n) - \phi(y)| < \epsilon$. There are infinitely many $\phi(x_n)$'s that satisfy this, and so $\phi(y)$ is a limit pt.

If $\exists y_1, y_2 \in X$ s.t. $\phi(y_1) = y_1$, then

$\phi(y_2) = y_2$

$$d(y_1, y_2) = d(\phi(y_1), \phi(y_2)) \leq c d(y_1, y_2) \text{ with } c < 1$$

so $c = 1$ □

Theorem 4 (Inverse Function Theorem)**.** *Suppose open $U \subset \mathbb{R}^n$, let C^1 $f : U \to \mathbb{R}^n$, $x_0 \in U$ s.t. $Df(x_0)$ invertible.*

Then $\exists$ open neighborhoods $V \ni x_0$, $W \ni f(x_0)$ s.t. $V \subseteq U$ and $W \subseteq \mathbb{R}^n$, respectively, and s.t.

(i) *$f : V \to W$ bijection*
(ii) *$g = f^{-1} : V \to U$ differentiable, i.e. $g = f^{-1} : W \to V$ is C^1*
(iii) *$D(f^{-1})$ cont. on W.*
(iv) *$Dg(y) = (Df(g(y)))^{-1} \quad \forall y \in W$*

Also, notice that $f(g(y)) = y \, \forall y \in W$.

Proof. Consider $\widetilde{f}(x) = (Df(x_0))^{-1}(f(x + x_0) - f(x_0))$. Then

$\widetilde{f}(0) = 0$ and

$$D\widetilde{f} = (Df(x_0))^{-1}(Df(x + x_0) - 0)$$
$$D\widetilde{f}(0) = (Df(x_0))^{-1} Df(x_0) = 1$$

So let $\widetilde{f} \to f$ (notation) and so assume, without loss of generality, that $U \ni 0$, $f(0) = 0$, $Df(0) = 1$

Choose $0 < \epsilon \leq \frac{1}{2}$. Let $0 < \delta < 1$ s.t. open ball $V = B_\delta(0) \subseteq U$, and $\|Df(x) - 1\| < \epsilon$. $\forall x \in U$, since Df cont. at 0.

Let $W = f(V)$.

$\forall y \in W$, define $\phi_y : V \to \mathbb{R}^n$

$\phi_y(x) = x + (y - f(x))$

$$D(\phi_y)(x) = 1 + -Df(x) \quad \forall x \in V$$
$$\|D(\phi_y)(x)\| = \|1 - Df(x)\| \leq \epsilon < 1$$

$\forall x_1, x_2 \in V$, by mean value Thm. (not the equality that is only valid in 1-dim., but the inequality, that's valid for $\mathbb{R}^d$,

$$\|\phi_y(x_1) - \phi_y(x_2)\| \leq \|D(\phi_y)(x')\| \|x_1 - x_2\|$$

for some $x' = cx_2 + (1 - c)x_1$, $c \in [0, 1]$. V only needed to be convex set.

$$\Longrightarrow \|\phi_y(x_1) - \phi_y(x_2)\| \leq \epsilon \|x_1 - x_2\|$$

Then ϕ_y contraction mapping.

Suppose $f(x_1) = f(x_2) = y$, $x_1, x_2 \in V$.

$$\phi_y(x_1) = x_1$$
$$\phi_y(x_2) = x_2$$
$$\|\phi_y(x_1) - \phi_y(x_2)\| = \|x_1 - x_2\| \leq \epsilon \|x_1 - x_2\| \quad \forall \epsilon > 0 \Longrightarrow x_1 = x_2$$

$\Longrightarrow f|_U$ injective.

$W = f(V)$, so $f : V \to W$ surjective. f bijective.

Fix $y_0 \in W$, $y_0 = f(x_0)$, $x_0 \in V$.

Let $r > 0$ s.t. $B_r(x_0) \subset V$.

Consider $B_{r\epsilon}(y_0)$. If $y \in B_{r\epsilon}(y_0)$.

$$r\epsilon > \|y - y_0\| = \|y - f(x_0)\| = \|\phi_y(x_0) - x_0\| \text{ with}$$
$$\phi_y(x) = x + (y - f(x))$$

If $x \in B_r(x_0)$,

$$\|\phi_y(x) - x_0\| \leq \|\phi_y(x) - \phi_y(x_0)\| + \|\phi_y(x_0) - x_0\| \leq \epsilon \|x - x_0\| + r\epsilon < 2r\epsilon = r$$

Thus $\phi(B_r(x_0)) = B_r(x_0)$.

By contraction mapping principle, $\exists a \in B_r(x_0)$, s.t. $\phi_y(a) = a$. Then $\phi_y(a) = a + (y - f(a)) = a \Longrightarrow f(a) = y$.

$y \in f(V) = W$.

So $B_{r\epsilon}(y_0) \subset W$. W open.

Let $\text{Mat}(n, n) \equiv$ space of all $n \times n$ matrices; $\text{Mat}(n, n) = \mathbb{R}^{n^2}$. □

There is a proof of the implicit function theorem and its various forms in Shastri (2011) [5], but I found Wienhard's Handout 4 for Math 327 to be clearer.[1]

Theorem 5 (Implicit Function Theorem)**.** *Let open $U \subset \mathbb{R}^{m+n} \equiv \mathbb{R}^m \times \mathbb{R}^n$*

C^1 $f : U \to \mathbb{R}^n$

$(a, b) \in U$ s.t. $f(a, b) = 0$ and $D_y f|_{(a,b)}$ invertible.

Then $\exists$ open $V \ni (a, b)$, $V \subset U$

$\exists$ open neighborhood $W \ni a$, $W \subseteq \mathbb{R}^m$

$\exists!$ C^1 $g : W \to \mathbb{R}^n$ s.t.

$$\{(x, y) \in V | f(x, y) = 0\} = \{(x, g(x)) | x \in W\}$$

Moreover,

$$dg_x = -\left.(d_y f)^{-1}\right|_{(x,g(x))} \left. d_x f\right|_{(x,g(x))}$$

and g smooth if f.

Proof. Define $F : U \to \mathbb{R}^{m+n}$

$F(x, y) = (x, f(x, y))$

Then $F(a, b) = (a, 0)$ (given), and

$$DF = \begin{bmatrix} 1 & \\ \frac{\partial f^i(x,y)}{\partial x^j} & \frac{\partial f^i(x,y)}{\partial y^j} \end{bmatrix} \equiv \begin{bmatrix} 1 & \\ D_x f & D_y f \end{bmatrix}$$

$DF(a, b)$ invertible.

By inverse function theorem, since $DF(a, b)$ invertible at pt. (a, b),

$\exists$ open neighborhoods $V \ni (a, b) \subseteq \mathbb{R}^m \times \mathbb{R}^n$ s.t. F diffeomorphism with $F^{-1} : \widetilde{W} \to V$.
$\widetilde{W} \ni (a, 0) \subseteq \mathbb{R}^m \times \mathbb{R}^n$

Set $W = \{x \in \mathbb{R}^m | (x, 0) \in \widetilde{W}\}$. Then $\pi_1(\widetilde{W}) = W$ open in $\mathbb{R}^m$.

Define $g : W \to \mathbb{R}^n$,

$$g(x) = \pi_2 \circ F^{-1}(x, 0) \text{ or}$$
$$F^{-1}(x, 0) = (h(x), g(x))$$

Now $FF^{-1}(x, 0) = (x, 0) = (h(x), f(h(x), g(x)))$ so $h(x) = x \, \forall x \in W$, $0 = f(x, g(x))$.

Then

$$\{(x, y) \in V | f(x, y) = 0\} = \{(x, y) \in V | F(x, y) = (x, 0)\} = \{(x, g(x)) | x \in W, 0 = f(x, g(x))\}$$

Since π smooth and F^{-1} is C^1, g is C^1.

To reiterate, $f(x, g(x)) = 0$ on W.

[1] https://web.math.princeton.edu/~wienhard/teaching/M327/handout4.pdf

Using chain rule while differentiating $f(x,g(x))=0$,

$$\partial_{x^j} f(x,g(x)) = \frac{\partial f(x,g(x))}{\partial x^k}\frac{\partial x^k}{\partial x^j} + \frac{\partial f(x,g(x))}{\partial y^k}\frac{\partial g^k(x)}{\partial x^j} = D_x f|_{(x,g(x))} + (D_y f)|_{(x,g(x))}\cdot (Dg)_x = 0 \text{ or}$$
$$(Dg)_x = -\left.(D_y f)\right|_{x,g(x)} \left.D_x f\right|_{(x,g(x))}$$

□

2. Immersions

Definition 2 (Immersion)**.** *smooth $f : M \to N$, s.t. $Df(p) : T_pM \to T_{f(p)}N$ injective. Then f* ***immersion*** *at p.*

Absil, Mahony, and Sepulchre [7] pointed out that another definition for a *immersion* can utilize the theorem that rank of $Df \equiv DF = \dim T_pM$. Indeed, recall these facts from linear algebra:
for $T : V \to W$,

It's always true that $\text{rank}T \leq V$, and
$\text{rank}T \leq W$

$\text{rank}T = \dim V$ iff T injective.
$\text{rank}T = \dim W$ iff T surjective.

$$\begin{tikzcd}
T_xM \arrow[r, "DF(x)"] & T_{F(x)}N = T_yN \\
x \in M \arrow[u] \arrow[r, mapsto, "F"] & y = F(x) \in N \arrow[u]
\end{tikzcd}$$

$$M \xrightarrow{F} N$$

Now

$$\dim T_xM = \dim M$$
$$\dim T_{F(x)}N = \dim N$$

And

$$\text{rank}(DF(x)) \equiv \text{rank of } F$$

I know that the notation above is confusing, but this is what all Differential Geometry books apparently mean when they say "rank of F".

Now
$$\text{rank}(DF(x)) = \dim(\text{im}(DF(x))) = \dim T_xM \text{ iff } DF(x) \text{ injective}$$
If $\forall x \in M$, this is the case, then F an **immersion** .
Apply the rank-nullity theorem in this case:

$$\text{rank}(DF(x)) + \dim\ker(DF(x)) = \dim T_xM = \dim M$$
$$\implies \text{rank}(DF(x)) = \dim M \leq \dim T_{F(x)}N = \dim N \text{ or } \dim M \leq \dim N$$

Now

$$\text{rank}(DF(x)) = \dim T_{F(x)}N \text{ iff } DF(x) \text{ surjective}$$

If $\forall x \in M$, this is the case, then F an **submersion** .

$$\text{rank}(DF(x)) = \dim T_{F(x)}N = \dim N \leq \dim M$$

Shastri (2011) has this as the "Injective Form of Implicit Function Theorem", Thm. 1.4.5, pp. 23 and Guillemin and Pollack (2010) has this as the "Local Immersion Theorem" on pp. 15, Section 3 "The Inverse Function Theorem and Immersions" [4].

Theorem 6 (Local immersion Theorem i.e. Injective Form of Implicit Function Theorem)**.** *Suppose $f : M \to N$ immersion at p, $q = f(p)$.*
Then $\exists$ local coordinates around p, q, x, y, respectively s.t. $f(x_1 \dots x_m) = (x_1 \dots x_m, 0 \dots 0)$.

Proof. Choose local parametrizations

$$\begin{tikzcd}
U \subseteq M \arrow[r, "f"] \arrow[d, "\phi"] & N \supseteq V \arrow[d, "\psi"] \\
\phi(U) \arrow[r, "f"] & \psi(V)
\end{tikzcd}
\qquad
\begin{aligned}
\phi(p) &= x \\
\psi(q) &= y
\end{aligned}$$

$D(\psi f \varphi^{-1}) \equiv Df$. $Df(p)$ injective (given f immersion). $Df(p) \in \text{Mat}(n,m)$
By change of basis in $\mathbb{R}^n$, assume $Df(p) = \begin{pmatrix} I_m \\ 0 \end{pmatrix}$.

Now define $G : \phi(U) \times \mathbb{R}^{n-m} \to \mathbb{R}^n$
$G(x,z) = f(x) + (0,z)$
Thus, $DG(x,z) = 1$ and for open $\phi(U) \times U_2$, $G(\phi(U) \times U_2)$ open.
By inverse function theorem, G local diffeomorphism of $\mathbb{R}^n$, at 0.
Now $f = G \circ \mathfrak{i}$, where $\mathfrak{i}$ is canonical immersion.

$$G(x,0) = f(x)$$
$$\implies G^{-1}G(x,0) = (x,0) = G^{-1}f(x)$$

Use $\psi \circ G$ as the local parametrization of N around pt. q. Shrink U, V so that

$$\begin{tikzcd}
U \subseteq M \arrow[r, "f"] \arrow[d, "\phi"] & N \supseteq V \arrow[d, "\psi \circ G"] \\
\phi(U) \arrow[r, "\mathfrak{i}"] & \psi \circ G(V)
\end{tikzcd}$$

□

Theorem 7 (Implicit Function Thm.)**.** *Let open subset $U \subseteq \mathbb{R}^n \times \mathbb{R}^d$, $(x,y) = (x^1 \dots x^n, y^1 \dots y^k)$ on U.*
Suppose smooth $\Phi : U \to \mathbb{R}^k$, $(a,b) \in U$, $c = \Phi(a,b)$

If $k \times k$ matrix $\frac{\partial \Phi^i}{\partial y^j}(a,b)$ nonsingular, then $\exists$ neighborhoods $V_0 \subseteq \mathbb{R}^n$ of a, $W_0 \subseteq \mathbb{R}^k$ of b and smooth $F : V_0 \to W_0$ s.t.

$\Phi^{-1}(c) \bigcap (V_0 \times W_0)$ is graph of F, i.e.
$\Phi(x,y) = c$ for $(x,y) \in V_0 \times W_0$ iff $y = F(x)$.

3. Submersions; Rank Theorem

cf. pp. 20, Sec. 4 "Submersions", Ch. 1 of Guillemin and Pollack (2010) [4].
Consider $X, Y \in \mathbf{Man}$, s.t. $\dim X \geq \dim Y$.

Definition 3 (submersion)**.** *If $f : X \to Y$,*
if $Df_x \equiv df_x$ is surjective*, $f \equiv$* ***submersion*** *at x.*

Recall that,

$$\begin{gathered} Df_x : T_x X \to T_{f(x)} Y \\ \dim T_x X \geq \dim T_{f(x)} Y \\ \text{rank} Df_x \leq \dim T_{f(x)} Y, \text{ in general, while} \\ \text{rank} Df_x = \dim T_{f(x)} Y \text{ iff } Df_x \text{ surjective} \end{gathered}$$

Canonical submersion is standard projection:

If $\dim X = k$, $k \geq l$,
$\dim Y = l$

$$(a_1 \dots a_k) \mapsto (a_1 \dots a_l)$$

Theorem 8 (Local Submersion Theorem)**.** *Suppose $f : X \to Y$ submersion at x, and $y = f(x)$, Then $\exists$ local coordinates around x, y s.t.*

$$f(x_1 \dots x_k) = (x_1 \dots x_l)$$

i.e. f locally equivalent to canonical submersion near x

Proof. I'll have a side-by-side comparison of my notation and the 1 used in Guillemin and Pollack (2010) [4] where I can.
For charts $(U, \phi), (V, \psi)$ for X, Y, respectively, $y = f(x)$ for $x \in X$,

$$\begin{tikzcd} U \subseteq X \arrow[r, "f"] \arrow[d, "\phi"] & Y \supseteq V \arrow[d, "\psi \circ G"] \\ \mathbb{R}^k \arrow[r, "i"] & \mathbb{R}^l \end{tikzcd} \qquad \begin{tikzcd} x \arrow[r, mapsto, "f"] \arrow[d, mapsto, "\phi"] & f(x) = y \arrow[d, mapsto, "\psi"] \\ \phi(x) = (a^1 \dots a^k) \arrow[r, mapsto, "g"] & g(\phi(x)) = g(a^1 \dots a^k) = \psi(y) \end{tikzcd}$$

Dg_x surjective, so assume it's a $l \times k$ matrix $\begin{bmatrix} \mathbf{1}_l & 0 \end{bmatrix}$.
Define

$$\begin{aligned} & G : U \subset \mathbb{R}^k \to \mathbb{R}^k \\ & G(a) \equiv G(a^1 \dots a^k) := (g(a), a_{l+1}, \dots, a_k) \end{aligned} \tag{6}$$

Now

$$DG(a) = \begin{bmatrix} \mathbf{1}_l & 0 \\ & \mathbf{1}_{k-l} \end{bmatrix} = \mathbf{1}_k \tag{7}$$

so G local diffeomorphism (at 0).
So $\exists$ G^{-1} as local diffeomorphism of some U' of a into $U \subset \mathbb{R}^k$.
By construction,

$$g = \mathbb{P}_l \circ G \tag{8}$$

where $\mathbb{P}_l$ is the *canonical submersion*, the projection operator onto $\mathbb{R}^l$.

$$g \circ G^{-1} = \mathbb{P}_l$$

(since G diffeomorphism)

$$\begin{tikzcd} U \subseteq X \arrow[r, "f"] & V \subseteq Y \\ \mathbb{R}^k \arrow[r, "\mathbb{P}_l"] \arrow[u, "\phi^{-1} \circ G^{-1}"] & \mathbb{R}^l \arrow[u, "\psi^{-1}"'] \end{tikzcd}$$

for

$$\begin{tikzcd} \phi^{-1} \circ G^{-1}(a) \equiv \phi^{-1} \circ G^{-1}(a^1 \dots a^k) = x \arrow[r, mapsto, "f"] & f(x) = y = \psi^{-1}(a^1 \dots a^l) \\ (a^1 \dots a^k) \arrow[r, mapsto, "\mathbb{P}_l"] \arrow[u, mapsto, "\phi^{-1} \circ G^{-1}"] & (a^1 \dots a^l) \arrow[u, mapsto, "\psi^{-1}"'] \end{tikzcd}$$

$\Longrightarrow$

□

"An obvious corollary worth noting is that if f is a submersion at x, then it is actually a submersion in a whole neighborhood of x." Guillemin and Pollack (2010) [4]
Suppose f submersion at $x \in f^{-1}(y)$.
By local submersion theorem

$$f(x_1 \dots x_k) = (x_1 \dots x_l)$$

Choose $y = (0, \dots, 0)$.
Then, near x, $f^{-1}(y) = \{(0, \dots 0, x_{l+1} \dots x_k)\}$ i.e. let $V \ni x$ neighborhood of x, define $(x_1 \dots x_k)$ on V.
Then $f^{-1}(y) \bigcap V = \{(0 \dots 0, x_{l+1}, \dots x_k) | x_1 = 0, \dots x_l = 0\}$.
Thus $x_{l+1}, \dots x_k$ form a coordinate system on open set $f^{-1}(y) \bigcap V \subseteq f^{-1}(y)$.
Indeed,

$$\begin{tikzcd} U \subseteq X \arrow[r, "f"] \arrow[d, "\phi"] & V \subseteq Y \arrow[d, "\psi"] \\ \mathbb{R}^k \arrow[r, "\mathbb{P}_l"] & \mathbb{R}^l \end{tikzcd} \qquad \begin{tikzcd} x \arrow[r, mapsto, "f"] \arrow[d, mapsto, "\phi"] & f(x) = y \arrow[d, mapsto, "\psi"] \\ \phi(x) = (x^1 \dots x^k) \arrow[r, mapsto, "\mathbb{P}_l"] & (x^1 \dots x^l) \end{tikzcd}$$

and now

$$\begin{tikzcd} f^{-1}(y) & y \arrow[l, mapsto, "f^{-1}"] \arrow[d, mapsto, "\psi"] \\ \{(0, \dots 0, x^1 \dots x^k)\} \arrow[u, mapsto, "\phi^{-1}"] & (0 \dots 0) \arrow[l, mapsto, "\mathbb{P}_l^{-1}"] \end{tikzcd}$$

3.1. **Rank Theorem.** Lee (2012) [3] in pp. 85, Ch. 4 Submersions, Immersions, and Embeddings, combines Theorems 6, 8 (local immersion and local submersion theorems, respectively) into the "Rank Theorem" (cf. Thm 4.12 "Rank Theorem" of Lee (2012)):

Theorem 9 (Rank Theorem)**.** *Suppose smooth manifolds M, N, $dimM = m$, $dimN = n$, smooth map $F : M \to N$, F has constant rank r.*
$\forall\, p \in M$, $\exists$ smooth charts (U, φ) for M, centered at p, (V, ψ) for N, centered at $F(p)$, s.t.

$$F(U) \subseteq V$$

in which F has coordinate representation of form

$$\widehat{F}(x^1 \dots x^r, x^{r+1} \dots x^m) = (x^1 \dots x^r, 0 \dots 0) \tag{9}$$

Particularly, if F smooth submersion,

$$\widehat{F}(x^1 \dots x^n, x^{n+1} \dots x^m) = (x^1 \dots x^n)$$

and if F smooth immersion

$$\widehat{F}(x^1 \dots x^m) = (x^1 \dots x^m, 0 \dots 0)$$

$$\begin{tikzcd}
\mathbb{R}^m \arrow[r, "\psi F \varphi^{-1}"] & \mathbb{R}^n \\
U \subset M \arrow[u, "\varphi"] \arrow[r, "F"'] & V \subset N \arrow[u, "\psi"]
\end{tikzcd}$$

Also remember that $DF(p) : T_pM \to T_{F(p)}N$

Proof. $DF(p)$ has rank r (given). Then $DF(p)$ is some $r \times r$ submatrix of a $n \times m$ matrix s.t. $\det DF(p)$ nonzero.

By change of basis in $\mathbb{R}^n$, or reordering coordinates, assume $DF(p)$ is upper left submatrix $\left(\frac{\partial F^i}{\partial x^j} \right) \quad \forall\, i, j = 1, \dots r$.

Relabel standard coordinate as

$$(x, y) = (x^1 \dots x^r, y^1 \dots y^{m-r}) \in \mathbb{R}^m$$
$$(v, w) = (v^1 \dots v^r, w^1 \dots w^{n-r}) \in \mathbb{R}^n$$

By initial translations of coordinates, assume without loss of generality $p = (0, 0)$, $F(p) = (0, 0)$

Suppose

$$F(x, y) = (Q(x, y), R(x, y))$$

for some smooth maps $Q : U \to \mathbb{R}^r$, $R : U \to \mathbb{R}^{n-r}$

Define

$$\varphi : U \to \mathbb{R}^m$$
$$\varphi(x, y) = (Q(x, y), y)$$

so

$$D\varphi(0, 0) = \begin{pmatrix} \frac{\partial Q^i}{\partial x^j}(0, 0) & \frac{\partial Q^i}{\partial y^j}(0, 0) \\ 0 & \delta^i_j \end{pmatrix}$$

$D\varphi(0, 0)$ nonsingular, since $\det \frac{\partial Q^i}{\partial x^j} \neq 0$ (by hypothesis).

By inverse function thm., $\exists$ connected neighborhoods U_0 of $(0, 0)$, $\widetilde{U}_0$ of $\varphi(0, 0) = (0, 0)$ s.t.

$$\varphi : U_0 \to \widetilde{U}_0$$

is a diffeomorphism.

By shrinking $U_0, \widetilde{U}_0$, assume $\widetilde{U}_0$ open cube.

Write $\varphi^{-1}(x, y) = (A(x, y), B(x, y))$, for some smooth functions $A : \widetilde{U}_0 \to \mathbb{R}^r$, $B : \widetilde{U}_0 \to \mathbb{R}^{m-r}$

$$(x, y) = \varphi(A(x, y), B(x, y)) = (Q(A(x, y), B(x, y)), B(x, y))$$

$$\Longrightarrow \begin{array}{l} B(x, y) = y \\ \varphi^{-1}(x, y) = (A(x, y), y) \end{array}$$

$$\varphi\varphi^{-1} = 1 \Longrightarrow x = Q(A(x, y), y)$$

Recall that we had hypotehsized that

$$F(x, y) = (Q(x, y), R(x, y))$$

Then

$$F \circ \varphi^{-1}(x, y) = F(A(x, y), y) = (Q(A(x, y), y), R(A(x, y), y)) = (x, R(A(x, y), y))$$

and so

$$F \circ \varphi^{-1}(x, y) = (x, \widetilde{R}(x, y))$$

where $\widetilde{R} : \widetilde{U}_0 \to \mathbb{R}^{n-r}$

$$\widetilde{R}(x, y) = R(A(x, y), y)$$

Compute

$$D(F \circ \varphi^{-1})(x, y) = \begin{pmatrix} \delta^i_j & 0 \\ \frac{\partial \widetilde{R}^i}{\partial x^j}(x, y) & \frac{\partial \widetilde{R}^i}{\partial y^j}(x, y) \end{pmatrix}$$

Since composing with a diffeomorphism doesn't change rank of map, $D(F \circ \varphi^{-1})$ has rank r everywhere in $\widetilde{U}_0$.
$\begin{pmatrix} \delta^i_j \\ \frac{\partial \widetilde{R}^i}{\partial x^j}(x, y) \end{pmatrix}$ $j = 1 \dots r$ are linearly independent, so $\frac{\partial \widetilde{R}^i}{\partial y^j}(x, y) = 0$ on $\widetilde{U}_0$, so $\widetilde{R}^i$ independent of y^j.

Let $S(x) = \widetilde{R}(x, 0)$, then

$$F \circ \varphi^{-1}(x, y) = (x, S(x))$$

Let open $V_0 \subseteq V$, $(0, 0) \in V$ be an open subset $V_0 = \{(v, w) \in V : (v, 0) \in \widetilde{U}_0\}$.
Then V_0 is a neighborhood of $(0, 0)$.

Because $\widetilde{U}_0$ is a cube, $F \circ \varphi^{-1}(x, y) = (x, S(x))$,

$$F \circ \varphi^{-1}(\widetilde{U}_0) \subseteq V_0$$

so $F(U_0) \subseteq V_0$.

Define $\psi : V_0 \to \mathbb{R}^n$

$$\psi(v, w) = (v, w - S(v))$$

Because $\psi^{-1}(s, t) = (s, t + S(s))$,
it is a diffeomorphism.

Thus (V_0, ψ) is a smooth chart.

$$\psi \circ F(\varphi^{-1}(x, y)) = \psi(x, S(x)) = (x, S(x) - S(x)) = (x, 0)$$

□

Definition 4 (regular value)**.** *For smooth $f : X \to Y$, $X, Y \in$ **Man**,*
*$y \in Y$ is a **regular value** for f if $Df_x : T_xX \to T_yY$ surjective $\forall\, x$ s.t. $f(x) = y$.*
*$y \in Y$ **critical value** if y not a regular value of f.*

Absil, Mahony, and Sepulchre [7] pointed out that another definition for a *regular value* can utilize the theorem that rank of $Df \equiv DF = \dim T_pN = \dim N$, iff $DF(p)$ surjective, for $p \in M$, $F : M \to N$. Then

regular value $y \in N$, of F, if rank of $F \equiv \text{rank}(DF(x)) = \dim N$, $\forall x \in F^{-1}(y)$, for $F : M \to N$.

Theorem 10 (Preimage theorem)**.** *If y regular value of $f : X \to Y$,*
$f^{-1}(y)$ is a submanifold of X, with $dim f^{-1}(y) = dimX - dimY$

Proof. Given y is a regular value of $f : X \to Y$,
$\forall x \in f^{-1}(y)$, $Df_x : T_xX \to T_yY$ is surjective. By local submersion theorem,

$$f(x^1 \dots x^k) = (x^1 \dots x^l) = y$$

Since $x \in f^{-1}(y)$, $(x^1 \dots x^k) = (y^1 \dots y^l, x^{l+1} \dots x^k)$.

For this chart for (U, φ), $U \ni x$, consider $(U \cap f^{-1}(y), \psi)$ with $\psi(x) = (x^{l+1} \dots x^k) \quad \forall x \in U \cap f^{-1}(y)$.

$\forall f^{-1}(y)$ submanifold with $\dim f^{-1}(y) = k - l = \dim X - \dim Y$. □

Examples for emphasis

If $\dim X > \dim Y$,

if $y \in Y$, regular value of $f : X \to Y$,

f submersion, $\forall x \in f^{-1}(y)$

If $\dim X = \dim Y$,

f local diffeomorphism $\forall x \in f^{-1}(y)$

If $\dim X < \dim Y$, $\forall y \in f(X)$ is a critical value.

Example: $O(n)$ as a submanifold of Mat(n,n)

Given $\text{Mat}(n,n) \equiv M(n) = \{ n \times n \text{ matrices } \}$ is a manifold; in fact $\text{Mat}(n,n) \cong \mathbb{R}^{n^2}$,
Consider $O(n) = \{ A \in \text{Mat}(n,n) | AA^T = 1 \}$.

$$AA^T \in \text{Sym}(n) \equiv S(n) = \{ S \in \text{Mat}(n,n) | S^T = S \} = \{ \text{ symmetric } n \times n \text{ matrices } \} \tag{10}$$

$\text{Sym}(n)$ submanifold of $\text{Mat}(n,n)$, $\text{Sym}(n)$ diffeomorphic to $\mathbb{R}^k$ (i.e. $\text{Sym}(n) \cong \mathbb{R}^k$), $k := \frac{n(n+1)}{2}$.

$$\begin{aligned} & f : \text{Mat}(n,n) \to \text{Sym}(n) \\ & f(A) = AA^T \end{aligned}$$

Notice f is smooth,

$$f^{-1}(1) = O(n)$$

$$Df_A(B) = \lim_{s \to 0} \frac{f(A + sB) - f(A)}{s} = \lim_{s \to 0} \frac{(A + sB)(A^T + sB^T) - AA^T}{s} = AB^T + BA^T$$

If $Df_A : T_A\text{Mat}(n,n) \to T_{f(A)}\text{Sym}(n)$ surjective when $A \in f^{-1}(1) = O(n)$ (???).

Proposition 1. *If smooth $g_1 \dots g_l \in C^\infty(X)$ on X are independent $\forall x \in X$, s.t. $g_i(x) = 0$, $\forall i = 1 \dots l$, then $Z = \{ x \in X | g_1(x) = \dots = g_l(x) = 0 \}$ = set of "common zeros" is a* submanifold *of X s.t. $dim Z = dim X - l$.*

Take note *that $g_1 \dots g_l$ are independent at x means, really, that $D(g_1)_x \dots D(g_l)_x$ are linearly independent on T_xX.*

Proof. Suppose smooth $g_1 \dots g_l \in C^\infty(X)$ on manifold X s.t. $\dim X = k \geq l$.

Consider $g = (g_1 \dots g_l) : X \to \mathbb{R}^l$, $Z \equiv g^{-1}(0)$.

Since $\forall g_i$ smooth, $D(g_i)_x : T_xX \to \mathbb{R}$ linear.

Now for

$$Dg_x = (D(g_1)_x \dots D(g_l)_x) : T_xX \to \mathbb{R}^l$$

By rank-nullity theorem (linear algebra), Dg_x surjective iff $\text{rank} Dg_x = l$ i.e. l functionals $D(g_1)_x \dots D(g_l)_x$ are linearly independent on T_xX.

"We express this condition by saying the l functions $g_1 \dots g_l$ are independent at x." (Guillemin and Pollack (2010) [4]) □

4. Submanifolds; immersed submanifold, embedded submanifolds, regular submanifolds

Definition 5 (Embedded Submanifold)**.**

Recall immersion:
$F : M \to N$ immersion iff DF injective, i.e. iff $\text{rank}DF = \dim M$.

Consider manifolds $M \subseteq N$.

Consider inclusion map $i : M \to N$.

$$i : x \mapsto x$$

If i immersion, $Di(x) = \frac{\partial y^i}{\partial x^j} = \delta_j^{\ i}$ if $y^i = x^i$, $\forall i = 1, \dots \dim M$.

Definition 6 (immersed submanifold)**.** ***immersed submanifold*** *$M \subseteq N$ if inclusion $i : M \to N$ is an immersion.*

cf. 3.3 Embedded Submanifolds of Absil, Mahony, and Sepulchre [7], also Ch. 5 Submanifolds, pp. 108, **Immersed Submanifolds** of John Lee (2012) [3].

Immersed submanifolds often arise as images of immersions.

Proposition 2 (Images of Immersions as submanifolds)**.** *Suppose smooth manifold M,*
smooth manifold with or without boundaries N,
injective, smooth immersion $F : M \to N$ (F injective itself, not just immersion)

Let $S = F(M)$.

Then S has unique topology and smooth structure of smooth submanifolds of N s.t. $F : M \to S$ diffeomorphism.

cf. Prop. 5.18 of John Lee (2012) [3].

Proof. Define topology of S: set $U \subseteq S$ open iff $F^{-1}(U) \subseteq M$ open $(F^{-1}(U \cap V) = F^{-1}(U) \cap F^{-1}(V), F^{-1}(U \cup V) = F^{-1}(U) \cup F^{-1}(V))$.

Define smooth structure of S: $\{ F(U), \varphi \circ F^{-1} | (U, \varphi) \in \text{atlas for } M, \text{ i.e. } (U, \varphi) \text{ any smooth chart of } M \}$.

"smooth compatibility condition":

$$(\varphi_2 \circ F^{-1})(\varphi_i F^{-1})^{-1} = \varphi_2 \circ F^{-1} F \varphi_1^{-1} = \varphi_2 \varphi_1^{-1}$$

since $\varphi_2 \varphi_1^{-1}$ diffeomorphism ($\varphi_2 \varphi_1^{-1}$ bijection and it and inverse is differentiable)

F diffeomorphism onto $F(M)$.

and these are the only topology and smooth structure on S with this property:

$$S \xrightarrow{F^{-1}} M \xrightarrow{F} N \qquad = \qquad S \hookrightarrow M$$

and F^{-1} diffeomorphism, F smooth immersion, so $i : S \to M$ smooth immersion. □

5. Curves, Integral Curves, and Flows

cf. John Lee (2012) [3], Ch. 9, deals with time-dependent vector fields and I don't see other texts or references handling such an important, but overlooked, case.

5.1. **Curves in Euclidean space.** cf. 4.1 Jeffrey Lee (2009) [2]

If C is 1-dim. submanifold of $\mathbb{R}^n$, $p \in C$, $\exists$ chart (V, y) of C, $p \in V$ s.t. $y(V)$ is a connected open interval $I \subset \mathbb{R}$, inverse map $y^{-1} : I \to V \subset M$ is a local parametrization.
idea is to extract information that's appropriately independent of parametrization.

If $\gamma : I \to \mathbb{R}^n$, $c : J \to \mathbb{R}^n$ curves with same image,
c is a **positive reparametrization** of γ if $\exists$ smooth $h : J \to I$ with $h' > 0$ s.t. $c = \gamma \circ h$,
in this case, γ, c have same **sense** and same **orientation**.

Assume $\gamma : I \to \mathbb{R}^n$ has $\|\gamma'\| > 0$,
Such a curve is **regular**, i.e. curve is an immersion (Recall, an immersion would be smooth $\gamma : I \to \mathbb{R}^n$, s.t. $D\gamma(p) : T_p I \to T_{\gamma(p)} \mathbb{R}^n$ injective).

Definition 7 (unit tangent field in Euclidean space, 4.3 Lee (2009) [2]). *If $\gamma : I \to \mathbb{R}^n$ regular curve, then $\mathbf{T}(t) := \frac{\gamma'(t)}{\|\gamma'(t)\|}$ defines* ***unit tangent field*** *along γ ($\|T\| = 1$)*

length of a curve defined on closed interval $\gamma : [t_1, t_2] \to \mathbb{R}^n$,

$$L = \int_{t_1}^{t_2} \|\gamma'(t)\| dt$$

Define **arc length function** for curve $\gamma : I \to \mathbb{R}^n$ by choosing $t_0 \in I$,

$$s = h(t) := \int_{t_0}^{t} \|\gamma'(t)\| d\tau$$

If curve is smooth and regular, then $h' = \|\gamma'(\tau)\| > 0$, so by inverse function theorem, $\exists$ smooth h^{-1} (since h' invertible (with $1/h'$)).
If $c(s) = \gamma h^{-1}(s)$, then $\|c'\|(s) := \|c'(s)\| = 1$, $\forall s$

$$c'(s) = (\gamma(h^{-1}(s)))' = \gamma'(h^{-1}(s)) \cdot \frac{dh^{-1}}{ds}(s)$$
$$\|c'(s)\| = \|\gamma'(t)\| \| \frac{dh^{-1}}{ds}(s)\| = h' \cdot \frac{1}{h'} = 1$$

Curves parametrized by arc length are **unit speed curves**.
For a unit speed curve, $\frac{dc}{ds}(s) = \mathbf{T}(s)$.

Definition 8 (curvature vector, curvature function, principal normal in Euclidean space, 4.4 Lee (2009) [2]). *Let $c : I \to \mathbb{R}^n$ be a unit speed curve. vector valued function*

$$\kappa(s) := \frac{d\mathbf{T}}{ds}(s)$$

is called the ***curvature vector****.*

$$\kappa(s) := \|\kappa(s)\| = \|\frac{d\mathbf{T}}{ds}(s)\|$$

is called the ***curvature function****.*
If $\kappa(s) > 0$, then define principal normal

$$\mathbf{N}(s) = \|\frac{d\mathbf{T}}{ds}(s)\|^{-1} \frac{d\mathbf{T}}{ds}(s)$$

s.t. $\frac{d\mathbf{T}}{ds} = \kappa \mathbf{N}$

6. Tensors

I'll go through Ch.7 *Tensors* of Jeffrey Lee (2009) [2].

Definition 9 (7.1[2]). *Let V, W be modules over commutative ring R, with unity.*
Then, algebraic W-valued tensor on V is multilinear map.

$$\tau : V_1 \times V_2 \times \dots \times V_m \to W \tag{11}$$

where $V_i = \{V, V^*\}$ $\quad \forall i = 1, 2, \dots m$.
If for r, s s.t. $r + s = m$, there are r $\; V_i = V^*$, $s\, V_i = V$, tensor is r-contravariant, s-covariant; also say tensor of total type $\binom{r}{s}$.

EY : 20170404 Note that

$$(\tau^{i\alpha}_{\beta} \frac{\partial}{\partial x^i} \text{ or } \tau^{i\alpha}_{\beta} e_i)(\omega_j dx^j \text{ or } \omega_j e^j \in V^*)$$
$$(\tau^{\beta}_{i\alpha} dx^i \text{ or } \tau^{\beta}_{i\alpha} e^i)(X^j \frac{\partial}{\partial x^j} \text{ or } X^j e_j \in V)$$

$\exists$ natural map $\begin{matrix} V \to V^{**}, \\ v \mapsto \widetilde{v} \end{matrix}$ $\widetilde{v} : \alpha \mapsto \alpha(v)$. If this map is an isomorphism, V is **reflexive** module, and identify V with V^{**}.

Exercise 7.5. Given vector bundle $\pi : E \to M$, open $U \subset M$, consider sections of π on U, i.e. cont. $s : U \to E$, where $(\pi \circ s)(u) = u$, $\forall u \in U$.
Consider $E^* \ni \omega = \omega_i e^i$.
$\forall s \in \Gamma(E)$, $\omega(s) = \omega_i(s(x))^i$, $\forall x \in U \subset M$. So define $\widetilde{s} : \omega, x \mapsto \omega(s(x))$, $\forall x \in U$.
If $\widetilde{s} = 0$, $\widetilde{s}(\omega, x) = \omega(s(x)) = 0$ $\quad \forall \omega \in E^*, \forall x \in U$, and so $s = 0$. (Let $\omega_i = \delta_{iJ}$ for some J, and so $s^J(x) = 0$ $\quad \forall J$).
$s = 0$. So $\ker(s \mapsto \widetilde{s}) = \{0\}$ (so condition for injectivity is fulfilled).
Since $\widetilde{s} : \omega, x \mapsto \omega(s(x))$, $\forall \omega \in E^*$, $\forall x \in U$, $s \mapsto \widetilde{s}$ is surjective.
$s \mapsto \widetilde{s}$ is an isomorphism so $\Gamma(E)$ is a *reflexive* module.

Proposition 3. *For R a ring (special case), $\exists$ module homomorphism:*

tensor product space $\to$ tensor, as a multilinear map, i.e. $\exists$

$$\begin{gathered} (\otimes_{i=1}^r V) \otimes (\otimes_{j=1}^s V^*) \to T^r{}_s(V; R) \\ u_1 \otimes \dots \otimes u_r \otimes \beta^1 \otimes \dots \otimes \beta^s \in (\otimes^r V) \otimes (\otimes^s V^*) \mapsto (\alpha^1 \dots \alpha^r, v_1 \dots v_s) \mapsto \alpha^1(u_1) \dots \alpha^r(u_r) \beta^1(v_1) \dots \beta^s(v_s) \end{gathered} \tag{12}$$

Indeed, consider

$$(\alpha^1 \dots \alpha^r, v_1 \dots v_s) \in \underbrace{V^* \times \dots \times V^*}_{r} \times \underbrace{V \times \dots \times V}_{s} \mapsto \alpha^1(u_1) \dots \alpha^r(u_r) \beta^1(v_1) \dots \beta^s(v_s)$$

and so for

$$\begin{aligned} \alpha^i &= \alpha^i_\mu e^\mu, \quad i = 1, 2, \dots r, \, \mu = 1, 2, \dots \dim V^* \qquad \alpha^i(u_i) = \alpha^i_\mu u^\mu_i \\ v_i &= v^\mu_i e_\mu, \quad i = 1, 2, \dots s, \, \mu = 1, 2, \dots \dim V \qquad \beta^i(v_i) = \beta^i_\mu v^\mu_i \end{aligned}$$

So that

$$\begin{aligned} \alpha^1(u_1) \dots \alpha^r(u_r) \beta^1(v_1) \dots \beta^s(v_s) &= \alpha^1_{\alpha_1} u^{\alpha_1}_1 \dots \alpha^r_{\alpha_r} u^{\alpha_r}_r \beta^1_{\mu_1} v^{\mu_1}_1 \dots \beta^s_{\mu_s} v^{\mu_s}_s = \\ &= (u^{\alpha_1}_1 \dots u^{\alpha_r}_r \beta^1_{\mu_1} \dots \beta^s_{\mu_s})(\alpha^1_{\alpha_1} \dots \alpha^r_{\alpha_r} v^{\mu_1}_1 \dots v^{\mu_s}_s) \end{aligned}$$

Identify $u_1 \otimes \dots \otimes u_r \otimes \beta^1 \otimes \dots \otimes \beta^s$ with this multiplinear map.

Proposition 4. *If V is finite-dim. vector space, or if $V = \Gamma(E)$, for vector bundle $E \to M$, map*

$$(\otimes_{i=1}^r V) \otimes (\otimes_{j=1}^s V^*) \to T^r{}_s(V; R) \tag{13}$$

is an isomorphism.

Definition 10. *tensor that can be written as*

$$u_1 \otimes \dots \otimes u_r \otimes \beta^1 \otimes \dots \otimes \beta^s \equiv u_1 \otimes \dots \otimes \beta^s \quad (14)$$

*is **simple** or **decomposable**.*

Now well that not *all* tensors are simple.

Definition 11 (7.7[2], tensor product)**.** $\forall\, S \in T^{r_1}_{s_1}(V)$, $\forall\, T \in T^{r_2}_{s_2}(V)$,
define tensor product

$$\begin{gathered} S \otimes T \in T^{r_1+r_2}_{s_1+s_2}(V) \\ S \otimes T(\theta^1 \dots \theta^{r_1+r_2}, v_1 \dots v_{s_1+s_2}) := S(\theta^1 \dots \theta^{r_1}, v_1 \dots v_{s_1}) T(\theta^{r_1+1} \dots \theta^{r_1+r_2}, v_{s_1+1} \dots v_{s_1+s_2}) \end{gathered} \quad (15)$$

Proposition 5 (7.8[2])**.**

$$\tau^{i_1 \dots i_r}{}_{j_1 \dots j_s} e_{i_1} \otimes \dots \otimes e_{i_r} \otimes e^{j_1} \otimes \dots \otimes e^{j_s} = \tau(e^{i_1} \dots e^{i_r}, e_{j_1} \dots e_{j_s}) e_{i_1} \otimes \dots \otimes e_{i_r} \otimes e^{j_1} \otimes \dots \otimes e^{j_s} = \tau$$

So $\{ e_{i_1} \otimes \dots \otimes e_{i_r} \otimes e^{j_1} \otimes \dots \otimes e^{j_s} | i_1 \dots i_r, j_1 \dots j_s \in 1 \dots n \}$ spans $T^r_s(V;R)$

Exercise 7.11. Let basis for V $\;e_1 \dots e_n$, corresponding dual basis for V^* $\;e^1 \dots e^n$
Let basis for V $\;\overline{e}_1 \dots \overline{e}_n$, corresponding dual basis for V^* $\;\overline{e}^1 \dots \overline{e}^n$
s.t.

$$\begin{aligned} \overline{e}_i &= C^k_{\ i} e_k \\ \overline{e}^i &= (C^{-1})^i_{\ k} e^k \end{aligned}$$

EY:20170404, keep in mind that

$$\begin{gathered} Ax = e_i A^i_{\ k} e^k (x^j e_j) = e_i A^i_{\ j} x^j = A^i_{\ j} x^j e_i \\ A e_j = e_k A^k_{\ i} e^i(e_j) = A^k_{\ j} e_k = \overline{e}_j \\ \overline{\tau}^i_{\ jk} \overline{e}_i \otimes \overline{e}^j \otimes \overline{e}^k = \overline{\tau}^i_{\ jk} C^l_{\ i} e_l (C^{-1})^j_{\ m} e^m (C^{-1})^k_{\ n} e^n = \overline{\tau}^i_{\ jk} C^l_{\ i} (C^{-1})^j_{\ m} (C^{-1})^k_{\ n} = \tau^l_{\ mn} \\ \overline{\tau}^i_{\ jk} = C^c_{\ k} C^b_{\ j} (C^{-1})^i_{\ a} \tau^a_{\ bc} \end{gathered}$$

On Remark 7.13 of Jeffrey Lee (2009) [2]: first, egregious typo for $L(V,V)$; it shoudl be $L(V,W)$. Onward, for $L(V,W)$,
consider $W \otimes V^* \ni w \otimes \alpha$ s.t.

$$(w \otimes \alpha)(v) = \alpha(v) w \in W, \forall\, v \in V, \text{ so } w \otimes \alpha \in L(V,W)$$

Now consider (category of) left R-module,

$$_R\mathbf{Mod} \ni {}_{\mathrm{Mat}_{\mathbb{K}}(N,M)} \mathbb{K}^N \quad (16)$$

where

$$\begin{aligned} V &= \mathbb{K}^N \\ W &= \mathbb{K}^M \end{aligned}$$

For $A \in \mathrm{Mat}_{\mathbb{K}}(N,M)$, $x \in \mathbb{K}^N$,

$$\begin{gathered} e_i A^i_{\ ,\mu} e^\mu (x^\nu e_\nu) = Ax = e_i A^i_{\ \mu} x^\mu, \quad i = 1,2, \dots M,\ \mu = 1,2,\dots N \\ A \in \mathrm{Mat}_{\mathbb{K}}(N,M) \cong W \otimes V^* \cong L(V,W) \end{gathered}$$

Consider

$$\begin{gathered} \alpha \in (\mathbb{K}^N)^* = V^* \qquad \alpha = \alpha_\mu e^\mu \\ w \in \mathbb{K}^M = W \qquad w = w^i e_i \\ \alpha \otimes w = w \otimes \alpha = w^i \alpha_\mu e_i \otimes e^\mu \end{gathered}$$

(remember, isomoprhism between $\mathrm{Mat}_{\mathbb{K}}(N,M)$ and $W \otimes V^*$ guaranteed, if V,W are free R-modules, $R = \mathbb{K}$).

Let V,W be left R-modules, i.e. $V,W \in {}_R\mathbf{Mod}$.

$$V^* \in \mathbf{Mod}_R$$

For $V^* \otimes W \in \mathbf{Mod}_R \otimes {}_R\mathbf{Mod}$

$$\alpha \in V^*, w \in W$$

$$(\alpha \otimes w)(v) = \alpha(v) w, \text{ for } v \in V \in {}_R\mathbf{Mod}$$

But $(w \otimes \alpha)(v) = w\alpha(v)$.
Note $\alpha(v) \in R$.
Let V,W be right R-modules, i.e. $V,W \in \mathbf{Mod}_R$.

$$V^* \in {}_R\mathbf{Mod}$$

For $W \otimes V^* \in \mathbf{Mod}_R \otimes {}_R\mathbf{Mod}$.

$$\alpha \in V^*,\ w \in W$$

$$(v)(w \otimes \alpha) = w\alpha(v), \text{ with } \alpha(v) \in R,\ v \in V$$

So $W \otimes V^* \cong L(V,W)$, for $V,W \in \mathbf{Mod}_R$

Definition 12 (7.20[2], **contraction**)**.** *Let $(e_1, \dots e_n)$ basis for V, $(e^1 \dots e^n)$ dual basis.*
If $\tau \in T^r_s(V)$, then for $k \leq r$, $l \leq s$, define

$$\begin{gathered} C^k_l \tau \in T^{r-1}_{s-1}(V) \\ C^k_l \tau(\theta^1 \dots \theta^{r-1}, w_1 \dots w_{s-1}) := \\ \sum_{a=1}^n \tau(\theta^1 \dots \underbrace{e^a}_{kth\ position} \dots \theta^{r-1}, w_1 \dots \underbrace{e_a}_{ith\ position} \dots w_{s-1}) \end{gathered} \quad (17)$$

*C^k_l is called **contraction**, for some single $1 \leq k \leq r$, some single $1 \leq l \leq s$,*

$$C^k_l : T^r_s(V) \to T^{r-1}_{s-1}(V)$$

s.t.

$$(C^k_l \tau)^{i_1 \dots \widehat{i}_k \dots i_r}{}_{j_1 \dots \widehat{j}_l \dots j_s} := \tau^{i_1 \dots a \dots i_r}{}_{j_1 \dots a \dots j_s}$$

Universal mapping properties can be invoked to give a basis free definition of contraction (EY : 20170405???).
IN general,

$$\forall\, v_1 \dots v_s \in V, \forall\, \alpha^1 \dots \alpha^r \in V^*$$

so that

$$\begin{aligned} v_j &= v^\mu_j e_\mu \qquad j = 1 \dots s, \quad \mu = 1, \dots \mathrm{dim} V \\ \alpha^i &= \alpha^i_\mu e^\mu \qquad i = 1 \dots r, \quad \mu = 1 \dots \mathrm{dim} V^* \end{aligned}$$

then $\forall\, \tau \in T^r_s(V)$,

$$\begin{aligned} \tau(\alpha^1 \dots \alpha^r, v_1 \dots v_s) &= \tau(\alpha^1_{\mu_1} e^{\mu_1} \dots \alpha^r_{\mu_r} e^{\mu_r}, v^{\nu_1}_1 e_{\nu_1} \dots v^{\nu_s}_s e_{\nu_s}) = \\ &= \alpha^1_{\mu_1} \dots \alpha^r_{\mu_r} v^{\nu_1}_1 \dots v^{\nu_s}_s \tau(e^{\mu_1} \dots e^{\mu_r}, e_{\nu_1} \dots e_{\nu_s}) = \alpha^1_{\mu_1} \dots \alpha^r_{\mu_r} v^{\nu_1}_1 \dots v^{\nu_s}_s \tau^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s} \end{aligned}$$

which is equivalent to

$$\begin{tikzcd}
\tau \in T^r_s(V) \arrow[r, mapsto, "\alpha^1 \otimes \dots \otimes \alpha^r \otimes v_1 \otimes \dots \otimes v_s \otimes"] & \alpha^1 \otimes \dots \otimes \alpha^r \otimes v_1 \otimes \dots \otimes v_s \otimes \tau \arrow[d, mapsto, "C^1_{s+1} C^2_{s+2} \dots C^r_{r+s} C^r_1 C^{r+1}_2 \dots C^{r+s}_s"'] \\
& \tau(\alpha^1 \dots \alpha^r, v_1 \dots v_s) \in R
\end{tikzcd}$$

where I've tried to express the right-R-module, "right action" on $\alpha^1 \otimes \dots \otimes \alpha^r \otimes v_1 \otimes \dots \otimes v_s \in V^* \otimes \dots \otimes V$.

Conlon (2008) [16]

Part 4. Lie Groups, Lie Algebra

7. Lie Groups

- **:** Lie Groups
- **:** Groups
- **:** Ring
- **:** group algebra
- **:** Group Ring
- **:** Representation Theory
- **:** Modules
- **:** kG-modules

From Sec. 8.1 "Noncommutative Rings" of Rotman (2010) [9]:

Definition 13. *ring R - additive abelian group equipped with multiplication* $\begin{aligned} R \times R &\to R \\ (a,b) &\mapsto ab \end{aligned}$ *s.t.* $\forall\, a, b \in R$

(i) $a(bc) = (ab)c$
(ii) $a(b+c) = ab + ac$, $(b+c)a = ba + ca$
(iii) $\exists\, 1 \in R$ *s.t.* $\forall\, a \in R$, $1a = a = a1$

Example 8.1[9]

(ii) group algebra kG, k commutative ring, G group, "its additive abelian group is free k-module having basis labeled by elements of G,
i.e. $\forall\, a \in kG$, $a = \sum_{g \in G} a_g g$, $a_g \in k$, $\forall\, g \in G$, $a_g \neq 0$ for only finitely many $g \in G$.

define (ring) multiplication $\begin{aligned} kG \times kG &\to kG \\ ab &= ab \end{aligned}$ $\forall\, a, b \in kG$, $\begin{aligned} a &= \sum_{g \in G} a_g g \\ b &= \sum_{h \in G} b_h g \end{aligned}$ to be

$$\left(\sum_{g \in G} a_g g \right) \left(\sum_{h \in G} b_h h \right) = \sum_{z \in G} \left(\sum_{gh = z} a_g b_h \right) z$$

Definition 14. *Given R ring, left R-module is (additive) abelian group M equipped with scalar multiplication* $\begin{aligned} R \times M &\to M \\ (r,m) &\mapsto rm \end{aligned}$ *s.t.* $\forall\, m, m' \in M$, $\forall\, r, r', 1 \in R$

(i) $r(m + m') = rm + rm'$
(ii) $(r + r')m = rm + r'm$
(iii) $(rr')m = r(r'm)$
(iv) $1m = m$

EY : 20150922 Example : for kG-module V^σ, for $r \in kG$, so $r = \sum_{g \in G} a_g g$

$$\begin{aligned} R \times M &\to M \\ (r,m) &\mapsto rm \end{aligned} \Longrightarrow \begin{aligned} kG \times V &\to V \\ (r,v) &\mapsto tv \end{aligned}$$

For some representation $\sigma : G \to GL(V)$,

$$rv = \sum_{g \in G} a_g g \cdot v = \sum_{g \in G} a_g \sigma_g(v)$$

So a kG-module needs to be associated with some chosen representation.

Note for V as an additive abelian group, $\forall\, u, v, w \in V$,

$$\begin{aligned} & v + w = w + v, \ (u+v) + w = u + (v + w) \\ & v + 0 = v \quad \forall\, v \in V \text{ for } 0 \in V \\ & v + (-v) = 0 \quad \forall\, v \in V \end{aligned}$$

So a vector space can be an additive abelian group.

Note that

$$r(v+w) = \left(\sum_{g \in G} a_g g \right) (v + w) = \left(\sum_{g \in G} a_g \sigma_g \right) (v+w) = \sum_{g \in G} a_g \sigma_g(v) + \sum_{g \in G} a_g \sigma_g(w) = rv + rw$$

$$(r + r')v = \left(\sum_{g \in G} a_g g + b_g g \right) v = \sum_{g \in G} (a_g \sigma_g + b_g \sigma_g) v = \sum_{g \in G} a_g \sigma_g(v) + \sum_{g \in G} b_g \sigma_g(v) = rv + r'v$$

$$(rr')v = \left(\sum_{g \in G} a_g g \sum_{h \in G} b_h h \right) v = \left(\sum_{z \in G} \sum_{gh = z} a_g b_h z \right) v = \sum_{z \in G} \sum_{gh \in z} a_g b_h \sigma_z(v) = \sum_{g \in G} \sum_{h \in G} a_g b_h \sigma_g \sigma_h(v)$$

since $\sigma(gh) = \sigma(g)\sigma(h) = \sigma_g \sigma_h = \sigma_{gh}$ (σ homomorphism)

$1v = \sigma(1)v = 1v = v$

From Sec. 8.3 "Semisimple Ring" of Rotman (2010) [9]:

Definition 15. *k-representation of group G is homomorphism*

$$\sigma : G \to GL(V)$$

where V is vector field over field k

Proposition 6 (8.37 Rotman (2010)[9])**.** *$\forall\, k$-representation $\sigma : G \to GL(V)$ equips V with structure of left kG-module, denote module by V^σ.*
Conversely, $\forall$ left kG-module V determines k-representation $\sigma : G \to GL(V)$

Proof. Given $\sigma : G \to GL(V)$,

$$\sigma_g =: \sigma(g) : V \to V$$

define

$$kG \times V \to V$$
$$\left(\sum_{g\in G} a_g g \right) v = \sum_{g \in G} a_g \sigma_g(v)$$

$v, w \in V$

Let $r, r', 1 \in kG$

$$r = \sum_{g \in G} a_g g$$

$$r(v+w) = \left(\sum_{g\in G} a_g g\right)(v+w) = \left(\sum_{g\in G} a_g \sigma_g \right)(v+w) = \sum_{g\in G} a_g\sigma_g(v) + \sum_{g\in G} a_g \sigma_g(w) = rv + rw$$
$$(r+r')v = \left(\sum_{g\in G} a_g g + b_g g \right) v = \sum_{g\in G} (a_g \sigma_g + b_g\sigma_g)v = \sum_{g\in G} a_g\sigma_g(v) + \sum_{g\in G} b_g\sigma_g(v) = rv + r'v$$
$$(rr')v = \left(\sum_{g\in G} a_g g \sum_{h\in G} b_h h \right) v = \left(\sum_{z\in G} \sum_{gh=z} a_g b_h z \right) v = \sum_{z\in G}\sum_{gh\in z} a_g b_h \sigma_z(v) = \sum_{g\in G}\sum_{h\in G} a_g b_h \sigma_g\sigma_h(v)$$

since $\sigma(gh) = \sigma(g)\sigma(h) = \sigma_g\sigma_h = \sigma_{gh}$ (σ homomorphism)

$1v = \sigma(1)v = 1v = v$

Conversely, assume V left kG-module.

If $g \in G$, then $v \mapsto gv$ defines $T_g : V \to V$. T_g nonsingular since $\exists T_g^{-1} = T_{g^{-1}}$

Define $\sigma : G \to GL(V)$

$\sigma : g \mapsto T_g$

σ k-representation

$$\begin{aligned} \sigma(gh) &= T_{gh} = T_g T_h = \sigma(g)\sigma(h) \\ \sigma(gh)(v) &= T_{gh}v = ghv = T_gT_hv = \sigma(g)\sigma(h)v \quad \forall v \in V \end{aligned}$$

□

Proposition 7. *Let group G, let $\sigma, \tau : G \to GL(V)$ be k-representations, field k. If V^σ, V^τ corresponding kG-modules in Prop. 6 (Prop. 8.37 in Rotman (2010) [9]), then $V^\sigma \simeq V^\tau$ as kG-modules iff $\exists$ nonsingular $\varphi : V \to V$ s.t.*

$$\varphi\tau(g) = \sigma(g)\varphi \quad \forall g \in G$$

Proof. If $\varphi : V^\tau \to V^\sigma$ kG-isomorphism, then $\varphi : V \to V$ isomorphism s.t.

$$\varphi(\sum a_g g v) = (\sum a_g g)\varphi(v) \quad \forall v \in V, \forall g \in G$$

in V^τ, $kG \times V \to V$ $\quad$ in V^σ, $kG \times V \to V$ scalar multiplication

$gv = \tau(g)(v)$ $\quad$ $gv = \sigma(g)(v)$

$$\Longrightarrow \forall g \in G, v \in V, \quad \varphi(\tau(g)(v)) = \sigma(g)(\varphi(v))$$

I think

$$\varphi(gv) = \varphi(\tau(g)(v)) = g\varphi(v) = \sigma(g)\varphi(v)$$
$$\Longrightarrow \varphi\tau(g) = \sigma(g)\varphi \quad \forall g \in G$$

Conversely, if $\exists$ nonsingular $\varphi : V \to V$ s.t. $\varphi\tau(g) = \sigma(g)\varphi \quad \forall g \in G$

$$\varphi\tau(g)v = \varphi(\tau(g)v) = \sigma(g)\varphi(v) \quad \forall g \in G, \forall v \in V$$

Consider scalar multiplication

$$kG \times V \to V$$
$$\sum_{g\in G} a_g g(v) = \sum_{g\in G} a_g \tau_g(v)$$

$$\varphi\left(\sum_{g\in G} a_g\tau_g(v) \right) = \varphi\left(\sum_{g\in G} a_g \tau(g) v\right) = \sum_{g\in G} a_g\sigma(g)\sigma(g)\varphi(v) = \left(\sum_{g\in G} a_g g\right)\varphi(v)$$

□

Admittedly, after this exposition from Rotman (2010) [9], I still didn't understand how kG-modules relate to representation theory and group rings. I turned to Baker (2011) [10], which we'll do right now. Note that I found a lot of links to online resources on representation theory from Khovanov's webpage `http://www.math.columbia.edu/~khovanov/resource/`.

Note,

Definition 16. *vector subspace $W \subseteq V$ is called a G-submodule, G-subspace, EY : 20150922 "invariant" subspace? if $\forall g \in G$, for representation $\rho : G \to GL_k(V)$, $\rho_g(w) \in W$, $\forall w \in W, \forall g \in G$ i.e. closed under "action of elements of G" with $\rho_g =: \rho(g) : V \to V$*

Given basis $\mathbf{v} = \{v_1 \dots v_n\}$ for V, $\dim_k V = n$, $\forall g \in G$,

$$\rho_g v_j = \rho(g) v_j = r_{kj}(g)v_k$$

for, indeed,

$$\rho_g x^j v_j = \rho(g) x^j v_j = x^j \rho(g) v_j = x^j r_{kj}(g) v_k = r_{kj} x^j v_k$$

so that

$$\begin{aligned} \rho &: G \to GL_k(V) \\ \rho(g) &= [r_{ij}(g)] \end{aligned}$$

Example 2.1 (Baker (2011) [10]): Let $\rho : G \to GL_k(V)$ where $\dim_k V = 1$

$$\forall v \in V,\ v \neq 0, \forall g \in G,\ \lambda_g \in k \text{ s.t. } g\cdot v = \rho_g(v) = \lambda_g v$$
$$\rho(hg)v = \rho_h\rho_g v = \lambda_{hg} v = \lambda_h\lambda_g v \Longrightarrow \lambda_{hg} = \lambda_h\lambda_g$$

$\Longrightarrow \exists$ homomorphism $\Lambda : G \to k^\times$

$\Lambda(g) = \lambda_g$

From Sec. 2.2 "G-homomorphisms and irreducible representations" of Baker (2011) [10], suppose $\begin{aligned}\rho &: G \to GL_k(V) \\ \sigma &: G \to GL_k(W)\end{aligned}$ are 2 representations

Many names for the same thing: G-equivalent, G-linear, G-homomorphism, EY : 20150922 kG-isomorphic?

If $\forall g \in G$,

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "\tau_g"'] & V \arrow[d, "\sigma_g"] \\ V \arrow[r, "\varphi"] & V \end{tikzcd} \iff V^\tau \overset{\varphi}{\simeq} V^\sigma$$

Indeed, define

$$\begin{aligned} &\varphi : V^\tau \to V^\sigma \\ &\varphi(v+w) = \varphi(v) + \varphi(w) \\ &\varphi(rv) = \varphi(\sum_{g\in G} a_g g\cdot v) = \varphi(\sum_{g\in G} a_g \tau_g(v)) = \sum_{g\in G} a_g \varphi(\tau_g(v)) = \sum_{g\in G} a_g\sigma_g \cdot \varphi(v) = r\varphi(v) \end{aligned}$$

EY : 20150922 So φ is a kG-isomorphism between left kG modules V^τ and V^σ if it's bijective and is "linear" in "scalars" $r \in kG$, i.e. $\varphi(rv) = r\varphi(v)$.

Define action of G on $\text{Hom}_k(V,W)$ ($\text{Hom}_k(V,W)$ is the vector space of k-linear transformations $V \to W$)

$$\begin{aligned} V &\xrightarrow{f} W \\ v &\overset{f}{\longmapsto} f(v) \end{aligned}$$

$\forall f \in \text{Hom}_k(V,W)$, $f : V \to W$

$f(v) \in W$

Consider

$$\begin{aligned} &G \times \text{Hom}_k(V,W) \to \text{Hom}_k(V,W) \\ &(g\cdot f) \mapsto (\sigma_g f)\circ \rho_{g^{-1}} \text{ i.e. } (g\cdot f)(v) = \sigma_g f(\rho_{g^{-1}}v) \quad (f \in \text{Hom}_k(V,W)) \end{aligned}$$

Let $g,h \in G$,

$$(gh\cdot f)(v) = g\cdot \sigma_h f(\rho_{h^{-1}}v) = \sigma_g\sigma_h f\rho_{h^{-1}}\rho_{g^{-1}}(v) = (\sigma_{gh}f\rho_{(gh)^{-1}})(v)$$

Thus, $G\times \text{Hom}_k(V,W) \to \text{Hom}_k(V,W)$ is thus another G-representation of G.

$(g\cdot f) \mapsto (\sigma_g f)\circ \rho_{g^{-1}}$

For k-representation ρ, if the only G-subspaces of V are $\{0\}$, V, ρ **irreducible** or **simple**.

$$\begin{aligned} &\rho_g(\{0\}) = \{0\} \\ &\rho_g(V) = V \end{aligned}$$

given subrepresentation $W \subseteq V$, V/W admits linear action of G, $\overline{\rho}_W : G \to GL_k(V/W)$ quotient representation

$$\overline{\rho}_W(g)(v+W) = \rho(g)(v) + W$$

if $v' - v \in W$

$$\rho(g)(v') + W = \rho(g)(v + (v'-v)) + W = (\rho(g)(v) + \rho(g)(v'-v)) + W = \rho(g)(v) + W$$

Proposition 8 (2.7 Baker (2011)[10]). *if $f : V \to W$ G-homomorphism, then*

(a) *$\ker f$ is G-subspace of V*
(b) *$\text{im} f$ is G-subspace of W*

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\rho_g"'] & W \arrow[d, "\sigma_g"] \\ V \arrow[r, "f"] & W \end{tikzcd}$$

Proof. Recall

(a) Let $v \in \ker f$. Then $\forall g \in G$,

$$f(\rho_g v) = \sigma_g f(v) = 0$$

so $\rho_g v \in \ker f$, $\forall g \in G$. So $\ker f$ is G-subspace of V

(b) Let $w \in \text{im} f$. So $w = f(u)$ for some $u \in V$

$$\sigma_g w = \sigma_g f(u) = f(\rho_g u) \in \text{im} f$$

So $\text{im} f$ is G-subspace of W

□

Theorem 11 (Schur's Lemma). *Let $\rho : G \to GL_{\mathbb{C}}(V)$, $\sigma : G \to GL_{\mathbb{C}}(W)$ be irreducible representations of G over field $k = \mathbb{C}$; let $f : V \to W$ be G-linear map.*

(a) *if $f \neq 0$, f isomorphism. True $\forall k$ field, not just $\mathbb{C}$*
(b) *if $V = W$, $\rho = \sigma$, then for some $\lambda \in \mathbb{C}$, f given by $f(v) = \lambda v$ ($v \in V$) (true for algebraically closed fields)*

Proof. (a) By Prop. 8, $\ker f \subseteq V$, $\text{im} f \subseteq W$ are G-subspaces.
For ρ, only G-subspaces are 0 or V, so if $\ker f = V$, $f = 0$. If $\ker f = 0$, f injective.
For σ, only G-subspaces are 0 or V, so $\text{im} f = 0$, $f = 0$. If $\text{im} f = V$, f surjective.
$\implies f$ isomorphism.

(b) Let $\lambda \in \mathbb{C}$ be an eigenvalue of f, $f(v_0) = \lambda v_0$ eigenvector, $v_0 \neq 0$.
Let linear $f_\lambda : V \to V$ s.t.

$$f_\lambda(v) = f(v) - \lambda v \quad (v \in V)$$

$\forall g \in G$

$$\rho_g f_\lambda(v) = \rho_g f(v) - \rho_g\lambda v = f(\rho_g v) - \lambda\rho_g v = f_\lambda(\rho_g v)$$

So f_λ is G-linear, for

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\rho_g"'] & V \arrow[d, "\rho_g"] \\ V \arrow[r, "f"] & V \end{tikzcd}$$

Since $f_\lambda(v_0) = 0$, by Prop. 8, $\ker f_\lambda = V$, (for $\ker f_\lambda \neq 0$ and so $\ker f_\lambda = V$)
By rank-nullity theorem, $\dim V = \dim\ker f_\lambda + \dim\text{im} f_\lambda$.
So $\text{im} f_\lambda = 0$, and so $f_\lambda(v) = 0$ ($\forall v \in V$) $\implies f(v) = \lambda v$

□

Schur's lemma, at least the first part, implies that the left kG-modules associated with representations ρ, σ are kG-isomorphic, i.e.

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\rho_g"'] & W \arrow[d, "\sigma_g"] \\ V \arrow[r, "f"] & W \end{tikzcd} \iff V^\rho \overset{f}{\simeq} V^\sigma$$

with f being an isomorphism between V^ρ and V^σ s.t.

$$\begin{aligned} f(v+w) &= f(v) + f(w) \quad \forall\, v, w \in (V^\sigma, +) \\ f(rv) &= rf(v) \quad \forall\, r = \sum_{g\in G} a_g g \in kG \end{aligned}$$

Kosmann-Schwarzbach's **Groups and Symmetries**[11] is a very lucid text that's mathematically rigorous enough and practical for physicists. It's really good and very clear. Let's follow its development for $SU(2), SO(3), SL(2,\mathbb{C})$ and corresponding Lie algebras $\mathfrak{su}(2)$, $\mathfrak{so}(3)$, $\mathfrak{sl}(2,\mathbb{C})$.

From Chapter 2 "Representations of Finite Groups" of Kosmann-Schwarzbach (2010) [11]

Definition 17 (2.1 Kosmann-Schwarzbach (2010)[11])**.** *On $L^2(G)$, scalar product defined by*

$$\langle f_1 | f_2 \rangle = \frac{1}{|G|} \sum_{g\in G} \overline{f_1(g)} f_2(g)$$

$f_1, f_2 \in \mathcal{F}(G) \equiv \mathbb{C}[G]$ vector space of functions on G taking values on $\mathbb{C}$

Definition 18 (2.3 Kosmann-Schwarzbach (2010)[11])**.** *Let (E,ρ) be representation of G*

character of $\begin{aligned} &\rho \equiv \chi_\rho : G \to \mathbb{C} \\ &\chi_\rho(g) = tr(\rho(g)) = \sum_{i=1}^n (\rho(g))_{ii} \end{aligned}$

Note: equivalent representations have same character
each conjugacy class of G, function χ_p is constant

Looking at Def. 17

$$\langle \chi_{\rho_1} | \chi_{\rho_2} \rangle = \frac{1}{|G|} \sum_{g\in G} \chi_{\rho_1}(g^{-1}) \chi_{\rho_2(g)}$$

since $\overline{\chi_{\rho_1(g)}} = \chi_{\rho_1}(g^{-1})$ by unitarity of representation with respect to scalar product $\langle\, , \,\rangle$

Proposition 9 (2.7 Kosmann-Schwarzbach (2010)[11])**.** *Let $\begin{matrix}(E_1,\rho_1)\\(E_2,\rho_2)\end{matrix}$ be representations of G, let linear $u : E_1 \to E_2$.*

Then $\exists$ linear T_u s.t.

$$\begin{aligned} &T_u : E_1 \to E_2 \\ &T_u = \frac{1}{|G|} \sum_{g\in G} \rho_2(g) u \rho_1(g)^{-1} \end{aligned} \tag{18}$$

so that $\rho_2(g) T_u = T_u \rho_1(g) \quad \forall\, g \in G$

Proof.

$$\rho_2(g) T_u = \frac{1}{|G|} \sum_{h\in G} \rho_2(gh) u \rho_1(h^{-1}) = \frac{1}{|G|} \sum_{k \in G} \rho_2(k) u \rho_1(k^{-1}g) = T_u \rho_1(g)$$

□

Thus, diagrammatically, we have that

$$E_1 \xrightarrow{u} E_2 \implies \begin{tikzcd} E_1 \arrow[r, "T_u"] \arrow[d, "\rho_1(g)"] & E_2 \arrow[d, "\rho_2(g)"] \\ E_1 \arrow[r, "T_u"] & E_2 \end{tikzcd}$$

From Definition 1.12 of Kosmann-Schwarzbach [11], "representations ρ_1 and ρ_2 are called **equivalent** if there is a bijective intertwining operator for ρ_1 and ρ_2." So I will interpret this as if an intertwining operator is not bijective, then the representations ρ_1, ρ_2 are not equivalent.

Proposition 10 (2.8 Kosmann-Schwarzbach (2010)[11])**.** *Let $\begin{matrix}(E_1,\rho_1)\\(E_2,\rho_2)\end{matrix}$ be irreducible representations of G, let linear $u : E_1 \to E_2$, define T_u by $T_u = \frac{1}{|G|}\sum_{g\in G} \rho_2(g) u \rho_1(g)^{-1}$ by Eq. 18.*

(i) *If ρ_1, ρ_2 inequivalent, then $T_u = 0$*
(ii) *If $E_1 = E_2 = E$ and $\rho_1 = \rho_2 = \rho$, then*

$$T_u = \frac{tr(u)}{dim E} 1_E$$

Proof. (i) if ρ_1, ρ_2 are inequivalent, by definition, T_u is not isomorphic. Then by Schur's lemma (first part), $T_u = 0$
(ii) By Schur's lemma, $T_u(v) = \lambda v \quad \forall\, v \in E = E_1 = E_2$. So $T_u = \lambda 1_E$. $\mathrm{tr} T_u = \lambda \mathrm{dim} E$ or $\lambda = \frac{\mathrm{tr} T_u}{\mathrm{dim} E}$. Thus, $T_u = \frac{\mathrm{tr} T_u}{\mathrm{dim} E} 1_E$ □

Let $(e_1 \dots e_n)$ basis of E
$(f_1 \dots f_p)$ basis of F

$$\forall\, u \in \mathcal{L}(E,F), \begin{aligned} &u : E \to F \\ &u(x) = u(x^j e_j) = x^j u(e_j) = x^j u^i_{\ j} f_i \\ &u = u^i_{\ j} e^j \otimes f_i \end{aligned} \text{ for } \begin{aligned} x &= x^j e_j \in E \\ y &= y^i f_i \in F \end{aligned}$$

For

$$\begin{aligned} &T : E^* \otimes F \to \mathcal{L}(E,F) \\ &T(\xi \otimes y) = u^i_{\ j} e^j \otimes f_i \text{ i.e. set } T(\xi \otimes y) \text{ to this } u \\ &T(\xi \otimes y) = T(\xi_l e^l \otimes y^k f_k) = \xi_l y^k T(e^l \otimes f_k) = (\xi_l y^k T^{li}_{kj}) e^j \otimes f_i \implies \xi_l y^k T^{li}_{kj} = u^i_{\ j} \end{aligned}$$

Exercises. Exercises of Ch. 2 Representations of Finite Groups [11]

Exercise 2.6. [11] *The dual representation.*

Let (E, π) representation of group G.
$\forall\, g \in G$, $\xi \in E^*$, $x \in E$, set $\langle \pi^*(g)(\xi), x\rangle = \langle \xi, \pi(g^{-1})(x)\rangle$

(a) *dual* (or *contragredient*) of π, $\pi^* : G \to \text{End}(E^*)$, π^* is a representation, since

$$\begin{aligned} \langle \pi^*(gh)(\xi), x\rangle = \langle \xi, \pi((gh)^{-1})(x)\rangle = \langle \xi, \pi(h^{-1}g^{-1})(x)\rangle = \langle \xi, \pi(h^{-1})\pi(g^{-1})(x)\rangle = \langle \xi, \pi(h^{-1})(\pi(g^{-1})(x))\rangle = \\ = \langle \pi^*(h)(\xi), \pi(g^{-1})(x)\rangle = \langle \pi^*(g)\pi^*(h)(\xi), x\rangle \end{aligned}$$

since this is true, $\forall\, x \in E$, $\forall\, \xi \in E^*$, $\pi^*(gh) = \pi^*(g)\pi^*(h)$.
dual π^* of π is a representation.

(b) Consider $\begin{aligned} & G \times \mathcal{L}(E,F) \to \mathcal{L}(E,F). \\ & g \cdot u = \rho(g) \circ u \circ \pi(g^{-1}) \end{aligned}$

Define

$$\begin{aligned} & \sigma : G \to \text{End}(\mathcal{L}(E,F)) \\ & \sigma(g) : \mathcal{L}(E,F) \to \mathcal{L}(E,F) \\ & \sigma(g)(u) = \rho(g) \circ u \circ \pi(g^{-1}) \end{aligned}$$

Let $(e_1 \dots e_n)$ be a basis of E. Let $\xi = \xi_i e^i \in E^*$, $x = x^j e_j \in E$.
Consider the isomorphism $T : E^* \otimes F \to \mathcal{L}(E,F)$ defined as[2]

$$\begin{aligned} & T : E^* \otimes F \to \mathcal{L}(E,F) = \text{Hom}(E,F) \\ & \xi \otimes y \mapsto (x \mapsto \xi(x)y) \end{aligned}$$

Choose bases $\begin{aligned} & (e_1 \dots e_n) \text{ of } E \\ & (e^1 \dots e^n) \text{ of } E^* \\ & (f_1 \dots f_p) \text{ of } F \end{aligned}$. Then

$$\begin{aligned} & T(e^j \otimes f_i)(x) = T(e^j \otimes f_i)(x^k e_k) = \delta^j_{\ k} x^k f_i = x^j f_i \\ & T(e^j \otimes f_i)(e_k) = \delta^j_{\ k} f_i \end{aligned}$$

Consider

$$\begin{aligned} & u \in \mathcal{L}(E,F) \\ & u : E \to F \\ & u(x) = u(x^j e_j) = x^j u(e_j) = x^j u^i_{\ j} f_i \\ & u(e_j) = u^i_{\ j} f_i \text{ i.e. } u : e_j \to u^i_{\ j} f_i \end{aligned}$$

Then $\forall u \in \mathcal{L}(E,F)$,

$$T(u^i_{\ j} e^j \otimes f_i)(e_k) = u^i_{\ j}\delta^j_{\ k} f_i = u^i_{\ k} f_i = u(e_k) \Longrightarrow u = T(u^i_{\ j} e^j \otimes f_i)$$

so T is surjective.
With $T(\xi \otimes y) = T(\xi' \otimes y')$,

$$\begin{aligned} & T(\xi \otimes y)(x) = T(\xi' \otimes y')(x) \\ & \xi(x)y = \xi'(x)y' \Longrightarrow \xi(x)y - \xi'(x)y' = 0 \end{aligned}$$

which implies that $\xi \otimes y = \xi' \otimes y'$. So T is injective. Or, one could consider that $T^{-1} : \mathcal{L}(E,F) \to E^* \otimes F$, $T^{-1} : u \mapsto u^i_{\ j} e^j \otimes f_i$, which is the inverse of T.

Remark 1.

$$\begin{aligned} & E^* \otimes F \overset{T}{\simeq} \mathcal{L}(E,F) = Hom(E,F) \\ & (\xi, y) \mapsto (x \mapsto \xi(x)y) \end{aligned}$$

and so $(e^j \otimes f_i) \mapsto (x \mapsto e^j(x) f_i = x^j f_i)$
So $E^* \otimes F$ *is isomorphic to* $\mathcal{L}(E,F) = Hom(E,F)$

For representation π,

$$\begin{aligned} & \pi : G \to \text{End}(E) \\ & \pi(g) : E \to E \\ & \pi(g)(x) = \pi(g)(x^j e_j) = x^j \pi(g)(e_j) = x^j \pi(g)^i_{\ j} e_i = (\pi(g)^i_{\ j} x^j e_i \end{aligned}$$

Consider this matrix formulation:

$$\begin{aligned} & \pi^*(g)(\xi) = \pi^*(g)(\xi_i e^i) = \xi_i \pi^*(g)(e^i) = \xi_i (\pi^*(g))^i_{\ j} e^j \\ & \Longrightarrow \langle \pi^*(g)(\xi), x\rangle = \xi_i (\pi^*(g))^i_{\ j} x^j \end{aligned}$$

and

$$\langle \xi, \pi(g^{-1})(x)\rangle = \xi_i \pi(g^{-1})^i_{\ j} x^j$$

so that

$$\langle \pi^*(g)(\xi), x\rangle = \langle \xi, \pi(g^{-1})(x)\rangle \Longrightarrow \pi(g^{-1})^i_{\ j} = (\pi^*(g))^i_{\ j}$$

Thus, given a choice of basis for E, the *dual* of π, $\pi^*(g)^i_{\ j}$, and $\pi(g^{-1})^i_{\ j}$ are formally equal.
So for a choice of basis of E and of F,

$$(\pi^* \otimes \rho)(g)(\xi, y) = (\pi^*(g) \otimes \rho(g))(\xi, y) = \pi^*(g)\xi \otimes \rho(g)y = \xi_l \pi(g^{-1})^l_{\ j} e^j \otimes \rho(g)^i_{\ k} y^k f_i = \rho(g)^i_{\ k} y^k \xi_l \pi(g^{-1})^l_{\ j} e^j \otimes f_i$$

Applying T,

$$T(\pi^* \otimes \rho)(g)(\xi, \rho) = \rho(g)^i_{\ k} y^k \xi_l \pi(g^{-1})^l_{\ j} = \rho(g) T(\xi, y) \pi(g^{-1})$$

$$\begin{tikzcd} E^* \otimes F \arrow[r, "T"] \arrow[d, "(\pi^* \otimes \rho)(g)"'] & \mathcal{L}(E,F) \arrow[d, "\sigma(g)"] \\ E^* \otimes F \arrow[r, "T"'] & \mathcal{L}(E,F) \end{tikzcd}$$

$$\begin{tikzcd} (\xi, y) \arrow[r, mapsto, "T"] \arrow[d, mapsto, "(\pi^* \otimes \rho)(g)"'] & (x \mapsto \xi(x)y) = y^i \xi_j \arrow[d, mapsto, "\sigma(g)"] \\ \pi^*(g)(\xi) \otimes \rho(g)y \arrow[r, mapsto, "T"'] & \rho(g) y^i \xi_j \pi(g^{-1}) = \rho(g) T(\xi, y) \pi(g^{-1}) \end{tikzcd}$$

Thus
Thus, representation $\sigma(g)$ is equivalent to representation $(\pi^* \otimes \rho)$, a tensor product of representations.

Exercise 2.15. *Representation of* $GL(2, \mathbb{C})$ *on the polynomials of degree 2*

Let group G, let representation ρ of G on $V = \mathbb{C}^n$, i.e. $\rho : G \to \text{End}(V)$
Let $P^{(k)}(V)$ vector space of complex polynomials on V that are homogeneous of degree k.
For $f \in P^{(k)}(V)$, the general form is

$$f = \sum_{\substack{i_1 + i_2 + \dots + i_n = k \\ 0 \leq i_j \leq k}} a_{i_1 i_2 \dots i_d} x_1^{i_1} x_2^{i_2} \dots x_n^{i_n}$$

---

[2] *Mathematics stackexchage* Isomorphism between Hom and tensor product [duplicate] http://math.stackexchange.com/questions/428185/isomorphism-between-hom-and-tensor-product http://math.stackexchange.com/questions/57189/understanding-isomorphic-equivalences-of-tensor-product

Given

$$\binom{n+k}{k} = \binom{k-1}{k-1} + \binom{k}{k-1} + \dots + \binom{n+k-1}{k-1} = \sum_{i=0}^{n} \binom{k-1+i}{k-1}$$

$\binom{k+n-1}{n-1}$ is number of monomials of degree k.

So $\dim P^{(k)}(V) = \binom{k+n-1}{n-1}$. This is a very lucid and elementary exposition on the basics of polynomials which I found was useful for the basic facts I forgot[3].

So we have the graded algebra

$$P(V) = \bigoplus_{k=0}^{\infty} P^{(k)}(V)$$
$$\rho^{(k)} : G \to \text{End}(P^{(k)}(V))$$
$$\rho^{(k)}(g) : P^{(k)}(V) \to P^{(k)}(V)$$
$$\rho^{(k)}(g)(f) = f \circ \rho(g^{-1})$$

This is a representation of G since

(a)
$$\begin{gathered} \rho^{(k)}(gh)(f) = f \circ \rho((gh)^{-1}) = f \circ \rho(h^{-1}g^{-1}) = f \circ \rho(h^{-1}\rho(g^{-1}) \\ \rho^{(k)}(g)\rho^{(k)}(h)(f) = \rho^{(k)}(g)(f \circ \rho(h^{-1})) = f \circ \rho(h^{-1}) \circ \rho(g^{-1}) \end{gathered} \Longrightarrow \rho^{(k)}(gh) = \rho^{(k)}(g)\rho^{(k)}(h)$$

(b) Choose basis $(e_1 \dots e_n)$ of V, $x = x^j e_j \in V$, $\rho : G \to \text{End}(V)$, and so $\rho(g)(x) = \rho(g)(x^j e_j) = x^j \rho(g)(e_j) = x^j (\rho(g))^i{}_j e_i$.
With $\xi(e_i) = \xi_i \Longrightarrow \langle \xi, \rho(g^{-1})x \rangle = \xi_i x^j (\rho(g^{-1}))^i{}_j$
$\forall \xi \in V^*$, $\xi = \xi_i e^i$,

$$\rho^*(g)(\xi) = \rho^*(g)(\xi_i e^i) = \xi_i \rho^*(g)^i{}_j e^j$$
$$\Longrightarrow \langle \rho^*(g)(\xi), x \rangle = \xi_i x^j (\rho^*(g))^i{}_j \Longrightarrow (\rho^*(g))^i{}_j = (\rho(g^{-1}))^i{}_j$$

So $\forall f \in P^{(1)}(V)$, $x \in V$, $\rho(g^{-1})x = x^j (\rho(g^{-1}))^i{}_j e_i$. So $f \circ \rho(g^{-1})(x) = \sum_{i=1}^n a_i (\rho(g^{-1}))^i{}_j x^j = \sum_{i=1}^n a_i (\rho^*(g))^i{}_j x^j$
$\Longrightarrow \rho^{(1)}(g)(f) = f \circ \rho^*(g)$

(c) Suppose $G = GL(2, \mathbb{C})$, $V = \mathbb{C}^2$, ρ fundamental representation $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $g^{-1} = \frac{1}{\det g} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ for $\det g = ad - bc$.
Let $k = 2$, $\dim P^{(2)}(\mathbb{C}^2) = \binom{2+2-1}{2-1} = \binom{3}{1} = 3$
$\forall f \in P^{(2)}(\mathbb{C}^2)$, $f(x, y) = Ax^2 + 2Bxy + Cy^2$
Let

$$P^{(2)}(\mathbb{C}^2) \to \mathbb{C}^3$$
$$f(x, y) = Ax^2 + 2Bxy + Cy^2 \mapsto \begin{pmatrix} A \\ B \\ C \end{pmatrix} \in \mathbb{C}^3$$

Call this transformation T, $T : P^{(2)}(\mathbb{C}^2) \to \mathbb{C}^3$.
$\forall \begin{pmatrix} A \\ B \\ C \end{pmatrix} \in \mathbb{C}^3$, $f(x, y) = Ax^2 + 2Bxy + Cy^2$ and $Tf(x, y) = \begin{pmatrix} A \\ B \\ C \end{pmatrix}$. T surjective.
Suppose $Tf(x, y) = Tf'(x, y)$,

$$\Longrightarrow Ax^2 + 2Bxy + Cy^2 = A'x^2 + 2B'xy + C'y^2$$
$$\Longrightarrow (A - A')x^2 + 2(B - B')xy + (C - C')y^2 = 0$$

Then since the monomials form a basis, and its basis elements are independent (by definition), then $A = A'$, $B = B'$, $C = C'$. T injective. So T is bijective, an isomorphism.

[3]Polynomials. Math 4800/6080 Project Course http://www.math.utah.edu/~bertram/4800/PolyIntroduction.pdf

(This is all in `groups.sage`)

```
sage: P2CC.<x,y> = PolynomialRing(CC,2) # this declares a PolynomialRing of field of complex numbers,
# of order 2 (i.e. only 2 variables for a polynomial, such as x, y)
sage: A = var('A')
sage: assume(A,''complex'')
sage: B = var('B')
sage: assume(B,''complex'')
sage: C = var('C')
sage: assume(C,''complex'')
sage: f(x,y) = A*x**2 +2*B*x*y + C*y**2

sage: a = var('a')
sage: assume(a,''complex'')
sage: b = var('b')
sage: assume(b,''complex'')
sage: c = var('c')
sage: assume(c,''complex'')
sage: d = var('d')
sage: assume(d,''complex'')
sage: g = Matrix([[a,b],[c,d]] )
sage: X = Matrix([[x],[y]])
sage: f( (g.inverse()*X)[0,0], (g.inverse()*X)[1,0] ).expand()
sage: f( (g.inverse()*X)[0,0], (g.inverse()*X)[1,0] ).expand().coefficient(x^2).full_simplify()
(C*c^2 - 2*B*c*d + A*d^2)/(b^2*c^2 - 2*a*b*c*d + a^2*d^2)
sage: f( (g.inverse()*X)[0,0], (g.inverse()*X)[1,0] ).expand().coefficient(x*y).full_simplify()
-2*(C*a*c + A*b*d - (b*c + a*d)*B)/(b^2*c^2 - 2*a*b*c*d + a^2*d^2)
sage: f( (g.inverse()*X)[0,0], (g.inverse()*X)[1,0] ).expand().coefficient(y^2).full_simplify()
(C*a^2 - 2*B*a*b + A*b^2)/(b^2*c^2 - 2*a*b*c*d + a^2*d^2)
```

So

$$\rho^{(2)}(g)(f)(x, y) = f \circ \rho(g^{-1})(x, y) =$$
$$= \frac{Cc^2 - 2Bcd + Ad^2}{(ad - bc)^2} x^2 + -2 \frac{(Cac + Abd - (bc + ad)B)}{(ad - bc)^2} xy + \frac{Ca^2 - 2Bab + Ab^2}{(ad - bc)^2} y^2$$

So define $\widetilde{\rho} : G \to \text{End}(\mathbb{C}^3)$. $\widetilde{\rho}$ is a representation, for

$$\forall v = \begin{pmatrix} A \\ B \\ C \end{pmatrix} \in \mathbb{C}^3, \quad \widetilde{\rho}(gh)(v) = T \circ f \circ \rho((gh)^{-1}) = T \circ f \circ \rho(h^{-1}g^{-1}) = T \circ f \circ \rho(h^{-1})\rho(g^{-1})$$
$$\text{Now } \widetilde{\rho}(h)(v) = T \circ f \circ \rho(h^{-1})$$
$$\Longrightarrow \widetilde{\rho}(g)\widetilde{\rho}(h)(v) = T \circ (f \circ \rho(h^{-1})) \circ \rho(g^{-1}) = T \circ f \circ \rho(h^{-1})\rho(g^{-1}) \text{ and so}$$
$$\widetilde{\rho}(gh) = \widetilde{\rho}(g)\widetilde{\rho}(h)$$

And so

$$\widetilde{\rho}^*(g)(v) = Tf\rho(g^{-1})$$

and consider this commutation diagram, that (helped me at least and) clarifies the relationships:

$$\begin{tikzcd} P^{(2)}(\mathbb{C}^2) \arrow[r, "T"] \arrow[d, "\rho^{(2)}(g)"'] & \mathbb{C}^3 \arrow[d, "\widetilde{\rho}(g)"] \\ P^{(2)}(\mathbb{C}^2) \arrow[r, "T"'] & \mathbb{C}^3 \end{tikzcd} \qquad \begin{tikzcd} f \arrow[r, mapsto, "T"] \arrow[d, mapsto, "\rho^{(2)}(g)"'] & \begin{pmatrix} A \\ B \\ C \end{pmatrix} \arrow[d, mapsto, "\widetilde{\rho}(g)"] \\ f \circ \rho(g^{-1}) \arrow[r, mapsto, "T"'] & \begin{pmatrix} D \\ E \\ F \end{pmatrix} \end{tikzcd}$$

with

$$\begin{pmatrix} D \\ E \\ F \end{pmatrix} = \begin{pmatrix} \frac{Cc^2 - 2Bcd + Ad^2}{(ad-bc)^2} \\ -2\frac{(Cac + Abd - (bc+ad)B)}{(ad-bc)^2} \\ \frac{Ca^2 - 2Bab + Ab^2}{(ad-bc)^2} \end{pmatrix}$$

Now define the dual $\widetilde{\rho}^*$ as such:

$$\begin{gathered} \widetilde{\rho}^*(g) : (\mathbb{C}^3)^* \to (\mathbb{C}^3)^* \\ \widetilde{\rho}^*(g) = \widetilde{\rho}(g^{-1}) \\ \forall\, \xi \in (\mathbb{C}^3)^* \\ \widetilde{\rho}^*(g)\xi = \xi_i(\widetilde{\rho}^*(g))^i{}_j e^j = \xi_i(\widetilde{\rho}(g^{-1}))^i{}_j e^j \end{gathered}$$

So for $v = \begin{pmatrix} A \\ B \\ C \end{pmatrix} \in \mathbb{C}^3$, $f = T^{-1}v = Ax^2 + 2Bxy + Cy^2 \in P^2(\mathbb{C}^2)$,

$$\widetilde{\rho}(g^{-1})(v) = T \circ (f\rho(g)) = \begin{bmatrix} Aa^2 + 2Bac + Cc^2 \\ Aab + Bbc + Bad + Ccd \\ Ab^2 + 2Bbd + Cd^2 \end{bmatrix}$$

which was found using Sage Math:

```
sage: f((g*X)[0,0],(g*X)[1,0])
(a*x + b*y)^2*A + 2*(a*x + b*y)*(c*x + d*y)*B + (c*x + d*y)^2*C
sage: f((g*X)[0,0],(g*X)[1,0]).expand()
A*a^2*x^2 + 2*B*a*c*x^2 + C*c^2*x^2 + 2*A*a*b*x*y + 2*B*b*c*x*y + 2*B*a*d*x*y + 2*C*c*d*x*y + A*b^2*y^2 + 2*B*b*d*y^2 + C*d^2*y^2
sage: f((g*X)[0,0],(g*X)[1,0]).expand().coefficient(x^2)
A*a^2 + 2*B*a*c + C*c^2
sage: f((g*X)[0,0],(g*X)[1,0]).expand().coefficient(x*y)
2*A*a*b + 2*B*b*c + 2*B*a*d + 2*C*c*d
sage: f((g*X)[0,0],(g*X)[1,0]).expand().coefficient(y^2)
A*b^2 + 2*B*b*d + C*d^2
```

or

```
sage: T( f((g*X)[0,0],(g*X)[1,0]).expand() )
[A*a^2 + 2*B*a*c + C*c^2,
2*A*a*b + 2*B*b*c + 2*B*a*d + 2*C*c*d,
A*b^2 + 2*B*b*d + C*d^2]
```

So then

$$\widetilde{\rho}(g^{-1}) = \begin{bmatrix} a^2 & 2ac & c^2 \\ 2ab & 2(ad+bc) & 2cd \\ b^2 & 2bd & d^2 \end{bmatrix}$$

So then

$$\widetilde{\rho}^*(g) = \begin{bmatrix} a^2 & 2ac & c^2 \\ 2ab & 2(ad+bc) & 2cd \\ b^2 & 2bd & d^2 \end{bmatrix}$$

and operate on row vectors $\xi \in (\mathbb{C}^3)^*$ with $\widetilde{\rho}^*(g)$ from the row vector's right.

More: Let $G = SU(2)$. Then $U = e^{i\phi}\begin{bmatrix} a & b \\ -\overline{b} & \overline{a} \end{bmatrix}$

$$\begin{gathered} \widetilde{\rho} : SU(2) \to \mathrm{End}(\mathbb{C}^3) \\ \widetilde{\rho}(U) : \mathbb{C}^3 \to \mathbb{C}^3 \\ \widetilde{\rho}(U)(v) = e^{-2i\varphi}\begin{bmatrix} A\overline{a}^2 + 2B\overline{a}\overline{b} + C\overline{b}^2 \\ -A\overline{a}b + B + Ca\overline{b} \\ Ab^2 - 2Bab + Ca^2 \end{bmatrix} \\ \Longrightarrow \widetilde{\rho}(U) = e^{-2i\varphi}\begin{bmatrix} -\overline{a}^2 & 2\overline{a}\overline{b} & \overline{b}^2 \\ -\overline{a}b & 1 & a\overline{b} \\ b^2 & -2ab & a^2 \end{bmatrix} \end{gathered}$$

cf. Ch. 5 Lie Groups of Jeffrey Lee (2009) [2]

Definition 19 (Lie Group)**.** ***Lie Group*** $G :=$ *smooth manifold G is a* ***Lie Group*** *if G is a group (abstract group), s.t.*

multiplication map $\begin{aligned} \mu &: G \times G \to G \\ \mu(g,h) &= gh \end{aligned}$

inverse map $\begin{aligned} inv &: G \to G \\ inv(g) &= g^{-1} \end{aligned}$

are C^∞ maps.

If group is abelian, use additive notation $g + h$ for group operation.

Definition 20 ($GL(n,\mathbb{R})$)**.** *$GL(n,\mathbb{R}) :=$ group of all invertible real $n \times n$ matrices.*

global chart on $GL(n,\mathbb{R}) = \{x^i_j\}$, n^2 functions x^i_j, where if $A \in GL(n,\mathbb{R})$, then $x^i_j(A)$ is ijth entry of A.

Claim: $GL(n,\mathbb{R})$ is a Lie group.

Proof. multiplication is clearly smooth: $(AB)_{ij} = A_{ik}B_{kj}$,

$$\frac{\partial}{\partial x^l_m}(x^i_k(A)x^k_j(B)) = \delta^i_l\delta^m_k x^k_j(B) + x^i_k(A)\delta^k_l\delta^m_j$$

inversion map; appeal to formula for A^{-1}, $A^{-1} = \mathrm{adj}(A)/\det(A)$, $\mathrm{adj}(A) \equiv$ adjoint matrix (whose entries are cofactors).
$\Longrightarrow A^{-1}$ depends smoothly on entries of A.

Similarly, $GL(n,\mathbb{C})$, group of invertible $n \times n$ complex matrices, is a Lie group. □

Exercise 5.5. Let subgroup H of G, consider cosets gH, $g \in G$.

Recall G is disjoint union of cosets of H.

Claim: if H open, so are all its cosets. And H closed.

Proof. cf. stackexchange: Open subgroups of a topological group are closed

$gH = \{gh | h \in H\}$ is an open neighborhood of g (since $1 \in H$, and mapping $h \mapsto gh$ sends open sets to open sets, since its inverse, $gh \mapsto h$, is C^∞ (so continuous)).

$$gH \to H \qquad H \to gH$$

$$gh \xrightarrow{g^{-1}} h = \mu(g^{-1}, gh) \qquad h \xrightarrow{g} gh = \mu(g,h)$$

Then $\forall$ coset gH, gH is open.
Suppose $g' \in H^c \equiv G - H \equiv G \backslash H$.
Consider $h \in H$, if $g'h \in H$, then $g' = (g'h)h^{-1} \in H$ (recall $h^{-1} \in H$, and H is a subgroup).
Contradiction.
$\Longrightarrow \forall g' \in H^c$, $\exists$ open neighborhood $g'H \subset H^c$, so H^c open (by definition). Then H closed. □

cf. Thm. 5.6 in Jeffrey Lee (2009) [2].

Theorem 12. *If G connected Lie group, U neighborhood of identity element e, then U generates the group, i.e. $\forall g \in G$, g is a product of elements of U.*

Proof. Note $V = \text{inv}(U) \cap U$ is an open neighborhood of e. Note $\text{inv}(V) = V$. $\text{inv}(V) \equiv V^{-1} = \{V^{-1} | v \in V\}$. We say that V is *symmetric*.
Claim: V generates G.
$\forall$ open W_1, open $W_2 \subset G$,
$W_1 W_2 = \{w_1 w_2 | w_1 \in W_1, w_2 \in W_2\}$ is an open set being a union of open sets $\bigcup_{g \in W_1} gW_2$.
Thus, inductively defined sets

$$V^n = VV^{n-1}, \quad n = 1,2,3,\dots$$

are open.

$$e \in V \subset V^2 \subset \dots V^n \subset \dots$$

It's easy to check that each V^n is symmetric.

$$\begin{aligned} &\text{inv}(V) = V \\ &\text{inv}(V^2) = \text{inv}(\bigcup_{v \in V} vV) = V\text{inv}(V) = V = V^2 \\ &\text{inv}(V^{n+1}) = \text{inv}\left(\bigcup_{v \in V} vV^n \right) = V\text{inv}(V^n) = VV^n = V^{n+1} \end{aligned}$$

so $V^\infty := \bigcup_{n=1}^\infty V^n$ is symmetric.
V^∞ closed under inversion, also multiplication. Thus V^∞ is an open subgroup.
From Exercise 5.5, Jeffrey Lee (2009) [2], i.e. Exercise 7, V^∞ also closed, since G is connected, $V^\infty = G$. (a topological space X is **connected** iff the only open and closed (clopen) sets are $\emptyset$ and X). □

Definition 21. *Identity component of G, G_0.*
$G_0 :=$ connected component of Lie group G that contains identity;
G_0 is a Lie group, and is generated by any open neighborhood of the identity.

Definition 22. *For Lie group G, fixed element $g \in G$,*
left translation (by g) $L_g : G \to G$, $L_g x = gx$, $\quad \forall x \in G$
right translation (by g) $R_g : G \to G$, $R_g x = xg$, $\quad \forall x \in G$

L_g, R_g are diffeomorphisms with $L_g^{-1} = L_{g^{-1}}$, $R_g^{-1} = R_{g^{-1}}$.

Definition 23 (Product Lie group)**.** *If G, H are Lie groups, then product manifold $G \times H$ is a Lie group, where multiplication*

$$(g_1, h_1) \cdot (g_2, h_2) = (g_1 g_2, h_1 h_2)$$

Lie group $G \times H$ is called ***product Lie group***

e.g. product group $S^1 \times S^1 \equiv$ 2-torus group.
Generally, higher torus groups $T^n = S^1 \times \dots \times S^1$ (n factors).

Definition 24 (Lie subgroup of G, H)**.** *Let H be an abstract subgroup of Lie group G.*
If H is a Lie group s.t. inclusion map $i : H \to G \equiv H \hookleftarrow G$ is an immersion, then H is a ***Lie subgroup*** *of G.*

Recall $i : H \to G$ immersion iff Di injective, i.e. iff $\text{rank}Di = \text{dim}H$
cf. Prop. 5.9 in Jeffrey Lee (2009) [2].

Proposition 11. *If H abstract subgroup of Lie group G, that's also a regular submanifold $\equiv$ embedded submanifold, then H closed Lie subgroup.*

Recall that

embedded submanifold $\equiv$ regular submanifold

Each name is used frequently and we shouldn't be biased against one or the other; we'll have to refer to both, to emphasize they're *exactly the same*.
embedded submanifold $\equiv$ regular submanifold is an immersed submanifold s.t. inclusion map i is a topological embedding,
i.e. embedded submanifold $\equiv$ regular submanifold $S \subset M$,
immersed submanifold S if $i : S \to M \equiv S \hookrightarrow M$ is an immersion, i.e. Di injective, i.e. $\text{rank}Di \equiv \text{dim}S$.
topological embedding := homeomorphism onto its image, i.e.
injective cont. map $f : X \to Y$, X, Y topological spaces, is a **topological embedding**
if f is a homeomorphism between X and $f(X)$.
f homeomorphism is a bijection, continuous, and f^{-1} continuous.

e.g. $\forall$ embedding $f : M \to N$, $f(M) \subset N$ naturally has the structure of an embedding submanifold $\equiv$ regular submanifold.
Useful, intrinsic definition of **embedded submanifold** $\equiv$ regular submanifold.
Let manifold M, $\text{dim}M = n$, let $k \in \mathbb{Z}^+$, s.t. $0 \leq k \leq n$.
A k-dim. embedded submanifold $\equiv$ regular submanifold S is subset $S \subset M$ s.t. $\forall p \in S$, $\exists$ chart $(U \subset M, \varphi : U \to \mathbb{R}^n \ni 0)$, s.t. $\varphi(S \cap U)$ is the intersection of a k-dim. plane with $\varphi(U)$.
(pairs $(S \cap U, \varphi|_{S \cap U})$ form an atlas for differential structure on S).
Proof 1:

Proof. H subgroup of G, so
multiplication map $H \times H \to H$
inversion map $H \to H$
are restrictions of multiplication and inversion maps on G.

Since H regular submanifold, maps are smooth.
Recall H regular submanifold iff H immersive submanifold (i.e. $H \hookleftarrow G$ is an immersion) and H topological subspace of G, i.e. submanifold topology on H is same as subspace topology.

Claim: H closed.
Let $x_0 \in \overline{H}$
Let (U, x) be a chart adapted to H, whose domain contains e.
Let

$$\begin{aligned} &\delta : G \times G \to G \\ &\delta(g_1, g_2) = g_1^{-1} g_2 \end{aligned}$$

Choose open set V s.t. $e \in V \subset \overline{V} \subset U$.
By continuity map δ, find open neighborhood O of identity e s.t. $O \times O \subset \delta^{-1}(V)$
If $\{h_i\}$ sequence in H converging to $x_0 \in \overline{H}$, then $x_0^{-1} h_i \to e$ and $x_0^{-1} h_i \in O$ for all sufficiently large i.

Since $h_j^{-1}h_i = (x_0^{-1}h_j)^{-1}x_0^{-1}h_i$, $h_j^{-1}h_i \in V$ for sufficiently large i, j.

For any sufficiently large fixed j,

$$\lim_{i\to 0} h_j^{-1}h_i = h_j^{-1}x_0 \in \overline{V} \subset U$$

Since U is domain of a single-slice chart, $U \subset H$ closed in U.

Thus, since $\forall\, h_j^{-1}h_i \in U \cap H$, $h_j^{-1}x_0 \in U \cap H \subset H$, $\quad\forall$ sufficiently large j.
$\Longrightarrow x_0 \in H$, and since x_0 arbitrary, done.

$\square$

Proof 2:
cf. 9.2 The Closed Subgroup Theorem I of 427 Notes[4]

Proof. **Claim**: Since H is an embedded submanifold $\equiv$ regular submanifold, $\exists$ neighborhood U of 1, $1 \in G$, s.t. $U \cap H$ closed in U.

Let $x_0 \in \overline{H}$, $\overline{H} \equiv$ closure of x_0.
Then $x_0U^{-1} \subseteq G$ is a neighborhood of x_0 in G (since $1 \in U^{-1}$, $x_0 1 = x_0 \in x_0U^{-1}$)

$$\Longrightarrow x_0U^{-1} \cap H \neq \emptyset$$

$\forall\, x \in x_0U^{-1} \cap H$, $x = x_0U^{-1}$ for some $u \in U$. Thus, $x^{-1}x_0 = u \in U$.

Now
$L_{x^{-1}} : G \to G$ is a homeomorphism, so $L_{x^{-1}}(H) = H$. By continuity, $L_{x^{-1}}(\overline{H}) = \overline{H}$. Thus $x^{-1}x_0 \in \overline{H}$.
Claim: $x^{-1}x_0 \in H \cap U$.
Since $x^{-1}x_0 \in \overline{H} \cap U$, $\exists$ sequence $\{h_i\} \subset H \cap U$ s.t. $h_0 \to x^{-1}x_0$.
But recall $H \cap U$ closed in U, so $x^{-1}x_0 \in H \cap U$.

$$\Longrightarrow x_0 \in xH = H, \quad \overline{H} \subseteq H$$

Thus H closed.

Claim: If H abstract subgroup of Lie group G, that's also an embedded submanifold $\equiv$ regular submanifold, then H is a Lie subgroup.

Recall that by definition, Lie group has group multiplication and inverse map to be C^∞. Then, just show group multiplication is C^∞, first.

Since G is a Lie group, then

$$\begin{gathered} \mu : G \times G \to G \\ \mu(x, y) = xy \end{gathered}$$

is C^∞ (by definition).

Then $\mu : G \times G \to G$ cont.
Consider subgroup $H \subseteq G$ and $\mu : H \times H \to H$.
Since $H \times H \subseteq G \times G$, $\forall\,(x, y) \in H \times H$ (fix $(x, y) \in H \times H$), $\forall$ neighborhood V of $\mu(x, y) = xy$, $V \subset G$, $\exists$ neighborhood U of (x, y) s.t. $\mu(U) \subseteq V$ (by $\mu : G \times G \to G$ cont.).

[4] https://faculty.math.illinois.edu/~lerman/519/s12/427notes.pdf

Since H embedded submanifold $\equiv$ regular submanifold of G,
$\exists$ neighborhood $V' \subseteq V$ of $xy \in G$, coordinate map $\varphi : V' \to \mathbb{R}^n$ ($n = \dim G$) s.t.

$$\varphi(H \cap V') = \varphi(V') \cap (\mathbb{R}^k \times \{0\})$$

where $k = \dim H$
(since H is a k-dim. embedded submanifold $\equiv$ regular submanifold, $H \subseteq G$, s.t. $\forall\, p \in H$, $\exists$ chart $(V \subset G, \varphi : U \to \mathbb{R}^n \ni 0)$, s.t. $\varphi(U \cap V) = \varphi(V) \cap (\mathbb{R}^k \times \{0\})$).
Now

$$\varphi \circ \mu : \mu^{-1}(V') \cap U \to \mathbb{R}^n \text{ is } C^\infty, \text{ and } \varphi \circ \mu(\mu^{-1}(V') \cap U) \subseteq \mathbb{R}^k \times \{0\}$$

Let projection $\pi : \mathbb{R}^n \to \mathbb{R}^k$ be the standard projection,

$$\pi \circ \varphi \circ \mu : \mu^{-1}(V') \cap U \to \mathbb{R}^k \text{ is } C^\infty$$

$\Longrightarrow \mu$ is C^∞ $\square$

From Chapter 4 "Lie Groups and Lie Algebras" of Kosmann-Schwarzbach (2010) [11]
While Proposition 2.6 of Kosmann-Schwarzbach (2010) [11] states that

$$\det(\exp(X)) = \exp(\mathrm{tr}X)$$

here are some other resources online that gave further discussion on the characteristic polynomial, $\det(A - \lambda 1)$ and the different terms of it, called Newton identities:

- http://scipp.ucsc.edu/~haber/ph116A/charpoly_11.pdf
- http://math.stackexchange.com/questions/1126114/how-to-find-this-lie-algebra-proof-that-mathfraksl-is-tra
- http://mathoverflow.net/questions/131746/derivative-of-a-determinant-of-a-matrix-field

Theorem 13 (5.1 [11])**.** *Consider* $\mathfrak{g} = \{X = \gamma'(0) | \gamma : 1 \to G \text{ of class } C^1, \gamma(0) = 1\}$
Let Lie group G

(i) $\mathfrak{g}$ *vector subspace of* $\mathfrak{gl}(n, \mathbb{R})$
(ii) $X \in \mathfrak{g}$ *iff* $\forall\, t \in \mathbb{R}$, $\exp(tX) \in G$
(iii) *if* $X \in \mathfrak{g}$, *if* $g \in G$, *then* $gXg^{-1} \in \mathfrak{g}$
(iv) $\mathfrak{g}$ *closed under matrix commutator, i.e. if* $X, Y \in \mathfrak{g}$, $[X, Y] \in \mathfrak{g}$

Proof. (i)
(ii) If $\exp(tX) \in G$, then $X \left.\frac{d}{dt}\exp(tX)\right|_{t=0} \in \mathfrak{g}$ (by def.)
If $X \in \mathfrak{g}$, then by def., $X = \left.\frac{d}{dt}\gamma(t)\right|_{t=0}$ with $\gamma(t) \in G$.
Now Taylor expand; $\forall\, k \in \mathbb{Z}^+$

$$\gamma\left(\frac{t}{k}\right) = 1 + \frac{t}{k}X + O\left(\frac{1}{k^2}\right) = \exp\left(\frac{t}{k}X + O\left(\frac{1}{k^2}\right)\right)$$
$$\Longrightarrow \left(\gamma\left(\frac{t}{k}\right)\right)^k = \exp(tX)$$
$$\gamma\left(\frac{t}{k}\right) \in G \quad \forall\, k \in \mathbb{Z}^+$$

G closed subgroup, so $\lim_{k\to\infty}(\gamma\left(\frac{t}{k}\right))^k = \exp(tX) \in G$
(iii)
(iv)

$\square$

Definition 25 (Lie algebra). *Lie algebra $\mathfrak{g}$, tangent space to G at 1, i.e. $\mathfrak{g} := T_1 G$ is called* Lie algebra *of Lie group G.*

$$\mathfrak{g} := \{ X = \gamma'(0) | \gamma : 1 \to G \text{ of class } C^1 \ , \gamma(0) = 1 \} = T_1 G$$

This is based on Proposition 5.3 of Kosmann-Schwarzbach (2010) [11].

For Lie group

$$U(n) = \{ U \in GL(n, \mathbb{C}) | UU^\dagger = 1 \}$$

If $X \in \mathfrak{u}(n)$, then $\exp{(tX)} \in U(n)$. Then

$$\exp{(tX)}\exp{(tX)}^\dagger = (1 + tX + O(t^2))(1 + tX^\dagger + O(t^2)) = 1 + t(X + X^\dagger) + O(t^2) = 1 \forall t \in \mathbb{R} \Longrightarrow X + X^\dagger = 0$$

i.e. $X \in \mathfrak{u}(n)$ is an anti-Hermitian complex $n \times n$ matrix.

$\mathfrak{u}(n) = \{ X \in \mathfrak{gl}(n, \mathbb{C}) | X + X^\dagger = 0 \}$

Physicists: $X = iA$ and so $A - A^\dagger$. $A \in \mathfrak{u}(n)$ is a Hermitian complex $n \times n$ matrix.

$\mathfrak{u}(n) = \{ A \in \mathfrak{gl}(n, \mathbb{C}) | A - A^\dagger = 0 \}$

Regardless, $\dim_{\mathbb{R}} \mathfrak{u}(n) = n^2 = 2n^2 - n^2$

For Lie group

$$SU(n) = \{ U \in GL(n, \mathbb{C}) | UU^\dagger = 1, \det U = 1 \}$$

Then

$$\mathfrak{su}(n) = \{ X \in \mathfrak{gl}(n, \mathbb{C}) | X + X^\dagger = 1, \, \text{tr} X = 0 \}$$

is the Lie algebra of traceless anti-Hermitian complex $n \times n$ matrices, and that

$$\dim_{\mathbb{R}} \mathfrak{su}(n) = n^2 - 1$$

In summary,

$$\begin{tikzcd} \mathfrak{u}(n) = \{ X \in \mathfrak{gl}(n, \mathbb{C}) | X + X^\dagger = 0 \} \arrow[d, "\exp{(tX)}"'] & \mathfrak{su}(n) = \{ X \in \mathfrak{gl}(n, \mathbb{C}) | X + X^\dagger = 0, \text{tr} X = 0 \} \arrow[d, "\exp{(tX)}"'] \\ U(n) = \{ U \in GL(n, \mathbb{C}) | UU^\dagger = 1 \} & SU(n) = \{ U \in GL(n, \mathbb{C}) | UU^\dagger = 1, \det U = 1 \} \end{tikzcd}$$

$$\dim_{\mathbb{R}} \mathfrak{u}(n) = n^2 \qquad \dim_{\mathbb{R}} \mathfrak{su}(n) = n^2 - 1$$

From Chapter 5 "Lie Groups $SU(2)$ and $SO(3)$" of Kosmann-Schwarzbach (2010) [11],

7.0.1. *Bases of $\mathfrak{su}(2)$, Subsection 1.1 of Chapter 5of Kosmann-Schwarzbach (2010)* [11]. Recall that

$$\begin{tikzcd} \mathfrak{su}(n) = \{ X \in \mathfrak{gl}(n, \mathbb{C}) | X + X^\dagger = 0, \text{tr} X = 0 \} \arrow[d, "\exp{(tX)}"'] \\ SU(n) = \{ U \in GL(n, \mathbb{C}) | UU^\dagger = 1, \det U = 1 \} \end{tikzcd}$$

$$\dim_{\mathbb{R}} \mathfrak{su}(n) = n^2 - 1$$

and so

$$\begin{tikzcd} \mathfrak{su}(2) = \{ X \in \mathfrak{gl}(2, \mathbb{C}) | X + X^\dagger = 0, \text{tr} X = 0 \} \arrow[d, "\exp{(tX)}"'] \\ SU(2) = \{ U \in GL(n, \mathbb{C}) | UU^\dagger = 1, \det U = 1 \} \end{tikzcd}$$

$$\dim_{\mathbb{R}} \mathfrak{su}(2) = 3$$

Also, recall that $\mathfrak{g} \subseteq \mathfrak{gl}(n, \mathbb{C})$ is a vector subspace (13) and that
$X \in \mathfrak{g}$ iff $\forall t \in \mathbb{R}$, $\exp{(tX)} \in G$.
if $X \in \mathfrak{g}$, if $g \in G$, then $gXg^{-1} \in \mathfrak{g}$

$\mathfrak{g}$ closed under $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$
$(X, Y) \mapsto [X, Y]$

and so with $\mathfrak{g}$ as a vector space, we can have a choice of bases.

(a) $\xi_1 = \frac{i}{2} \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$
$\xi_2 = \frac{1}{2} \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$
$\xi_3 = \frac{i}{2} \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$
satisfying

$$[\xi_k, \xi_l] = \epsilon_{klm} \xi_m$$

(b) *Physics*
$\sigma_1 = -2i\xi_1 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$
$\sigma_2 = 2i\xi_2 = \begin{pmatrix} & -i \\ i & \end{pmatrix}$
$\sigma_3 = -2i\xi_3 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$

satisfying

$$[\sigma_k, \sigma_l] = 2i \epsilon_{klm} \sigma_m$$

EY : 20151001 Sage Math 6.8 doesn't run on Mac OSX El Capitan: I suspect that it's because in Mac OSX El Capitan, `/usr` cannot be modified anymore, even in an Administrator account. The TUG group for MacTeX had a clear, thorough, and useful (i.e. copy UNIX commands, paste, and run examples) explanation of what was going on:

http://tug.org/mactex/elcapitan.html

So keep in mind that my code for Sage Math is for Sage Math 6.8 that doesn't run on Mac OSX El Capitan. I'll also use `sympy` in Python as an alternative and in parallel.

One can check in `sympy` the traceless anti-Hermitian (or Hermitian) property of the bases and Pauli matrices, and the commutation relations (see `groups.py`):

```
import itertools
from itertools import product, permutations
```

```
import sympy
from sympy import I, LeviCivita
from sympy import Rational as Rat

from sympy.physics.matrices import msigma # <class 'sympy.matrices.dense.MutableDenseMatrix'>

def commute(A,B):
"""
commute = commute(A,B)
commute takes the commutator of A and B
"""
return (A*B - B*A)

def xi(i):
"""
xi = xi(i)
xi is a function that returns the independent basis for
Lie algebra su(2)\equiv su(2,\mathbb{C}) of Lie group SU(2) of
traceless anti-Hermitian matrices, based on msigma of sympy
cf. http://docs.sympy.org/dev/_modules/sympy/physics/matrices.html#msigma
"""
if i not in [1,2,3]:
raise IndexError("Invalid Pauli index")
elif i==1:
return I/Rat(2)*msigma(1)
elif i==2:
return -I/Rat(2)*msigma(2)
elif i==3:
return I/Rat(2)*msigma(3)
```

```
## check anti-Hermitian property and commutation relations with xi
# xi is indeed anti-Hermitian
xi(1) == -xi(1).adjoint() # True
xi(2) == -xi(2).adjoint() # True
xi(3) == -xi(3).adjoint() # True

# xi obeys the commutation relations

for i,j in product([1,2,3],repeat=2): print i,j

for i,j in product([1,2,3],repeat=2): print i,j, "\t Commutator: ", commute(xi(i),xi(j))

## check traceless Hermitian property and commutation relations with Pauli matrices
# Pauli matrices i.e. msigam is indeed traceless Hermitian

msigma(1) == msigma(1).adjoint() # True
msigma(2) == msigma(2).adjoint() # True
msigma(3) == msigma(3).adjoint() # True

msigma(1).trace() == 0 # True
msigma(2).trace() == 0 # True
msigma(3).trace() == 0 # True

# Pauli matrices obey commutation relation
print "For Pauli matrices, the commutation relations are:\n"
for i,j in product([1,2,3],repeat=2): print i,j, "\t Commutator: ", commute(msigma(i),msigma(j))

for i,j,k in permutations([1,2,3],3): print "Commute: ", i,j,k, msigma(i), msigma(j), \
": and is 2*i of ", msigma(k), commute(msigma(i),msigma(j)) == 2*I*msigma(k)*LeviCivita(i,j,k)
```

And finally the traceless property of the Pauli matrices:

```
>>> msigma(1).trace()
0
>>> msigma(2).trace()
0
>>> msigma(3).trace()
0
```

7.1. **Spin.** Let's follow the development by Baez and Muniain (1994) on pp. 175 of the Section II.1 "Lie Groups", the second (II) chapter on "Symmetry" [8].

Let $V = \mathbb{C}^2$, $G = SU(2)$. Then consider the graded algebra of polynomials on $V = \mathbb{C}^2 \ni (x,y)$

$$P(V) = \bigoplus_{k=0}^{\infty} P^{(k)}(V) = \bigoplus_{\substack{j=0 \\ 2j\in\mathbb{Z}}}^{\infty} P^{(2j)}(V) = \bigoplus_{\substack{j=0 \\ j\in\mathbb{Z}}}^{\infty} P^{(2j)}(V) \oplus \bigoplus_{\substack{j=1/2 \\ 2j \text{ odd}}}^{\infty} P^{(2j)}(V)$$

$$P^{(2j)}(V) \equiv \text{ vector space of complex polynomials of degree } 2j$$

and recall this representation on $P^{(2j)}(V)$

$$\begin{aligned}
&\rho^{(2j)} : G \to \mathrm{End}(P^{(2j)}(V)) \\
&\rho^{(2j)} : P^{(2j)}(V) \to P^{(2j)}(V) \\
&\rho^{(2j)}(g)(f) = f \circ \rho(g^{-1}) \text{ where } \rho \text{ is the fundamental representation of } G = SU(2) \\
&\rho^{(2j)}(g)(f)(v) = f \circ \rho(g^{-1})(v) \quad \forall\, f \in P^{(2j)}(V), \forall\, v \in V = \mathbb{C}^2
\end{aligned}$$

Note,
$\dim P^{(2j)} = \binom{2j+2-1}{2-1} = 2j+1$

Exercise 21. [8] *spin-0* Consider the trivial representation τ:

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "T"] \arrow[d, "\tau(g)"'] & P^{(0)}(V) \arrow[d, "\rho^{(0)}(g)"] \\
\mathbb{C} \arrow[r, "T"] & P^{(0)}(V)
\end{tikzcd}$$

$$\begin{aligned}
&\tau : G \to \mathrm{End}(\mathbb{C}) \\
&\tau(g) : \mathbb{C} \to \mathbb{C} \\
&\tau(g) = 1_{\mathbb{C}}
\end{aligned}$$

Clearly, $P^{(0)}(V) = \mathbb{C}$, since $P^{(0)}(V)$ consists of polynomials of constants in $\mathbb{C}$.

Consider $c_0 \in \mathbb{C}$, $f = k_0 \in P^{(0)}(V)$
$\rho^{(0)}(g)(f) = f \circ \rho(g^{-1}) = k_0$
$\Longrightarrow \rho^0(g)T(c_0) = T \circ \tau(g)c_0 = T(c_0)$. Let $T = 1_{\mathbb{C}} = 1_{P^0(V)}$

So $\rho^{(0)}(g) = \tau(g) = 1$. $T = 1$. So representations $\rho^{(0)}$ and trivial representation τ on G are equivalent.

Exercise 22. [8] *spin-$\frac{1}{2}$* For spin-$\frac{1}{2}$, $j = \frac{1}{2}$, $2j = 1$.

$\forall\, f \in P^{(1)}(V)$, $V = \mathbb{C}^2$. So in general form, $f(x,y) = ax + by \in P^{(1)}(V)$, $\begin{pmatrix} x \\ y \end{pmatrix} \in V = \mathbb{C}^2$

Recall the fundamental representation $\begin{aligned} &\rho : G \to GL(2,\mathbb{C}) \equiv GL(\mathbb{C}^2) \\ &\rho(g) : \mathbb{C}^2 \to \mathbb{C}^2 \\ &\rho(g) = g \end{aligned}$

So consider T such that

$$\begin{tikzcd} \mathbb{C}^2 \arrow[r, "T"] \arrow[d, "\rho(g)"'] & P^{(1)}(V) \arrow[d, "\rho^{(1)}(g)"] \\ \mathbb{C}^2 \arrow[r, "T"] & P^{(1)}(V) \end{tikzcd}$$

Consider $\forall\, v \in \mathbb{C}^2$, $v = \begin{pmatrix} x \\ y \end{pmatrix}$, then

$$\rho(g)v = gv = \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix}$$

```
sage: g*X
[a*x + b*y]
[c*x + d*y]
```

For notation, let $U \in G = SU(2)$ s.t. $UU^\dagger = 1$.
Consider $(\rho^{(2j)}(U)(f))(x) = f(U^{-1}x)$, $\forall\, x \in \mathbb{C}^2$.

Choose $f(x,y) = x$. So for $f(x,y) = Ax+By$, $A = 1, B = 0$. Choose $U = \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix}$ so $U^{-1} = \begin{pmatrix} \overline{a} & -b \\ \overline{b} & a \end{pmatrix}$. Then $U^{-1}x = \begin{pmatrix} \overline{a}x - by \\ \overline{b}x + ay \end{pmatrix}$

So

$$(\rho^{(1)}(U)(f))(x) = f(U^{-1}x) = \overline{a}x - by$$
$$(\rho^{(1)}(U)(f))(x) = f(U^{-1}x) = \overline{b}x + ay \text{ for } f(x,y) = y$$

Let $f(x,y) = Ax+By$

$$(\rho^{(1)}(U)(f))(x) = f(U^{-1}x) = (A\overline{a} + B\overline{b})x + (Ba - Ab)y = (\overline{a}x - by)A + (\overline{b}x + ay)B = (A\overline{a} + B\overline{b})x + (Ba - Ab)y$$

which was calculated with the assistance of Sage Math:

```
sage: U_try1 = Matrix( [[a.conjugate(),-b],[b.conjugate(),a ] ] )
sage: f1( U_try1*X).coefficient(x)
A*conjugate(a) + B*conjugate(b)
sage: f1( U_try1*X).coefficient(y)
B*a - A*b
```

Treating $P^{(1)}(\mathbb{C}^2)$ as a vector space, in its matrix formulation, then $f(x,y) = Ax+By \in P^{(1)}(\mathbb{C}^2)$ is treated as $\begin{bmatrix} A \\ B \end{bmatrix}$, then $(\rho^{(1)}(U)f)$ is

$$\Longrightarrow \begin{bmatrix} \overline{a} & \overline{b} \\ -b & a \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} A\overline{a} + B\overline{b} \\ -Ab + Ba \end{bmatrix}$$

so conclude in general that $\rho^{(1)}(U) = (U^\dagger)^T$.

Now, as Kosmann-Schwarzbach (2010) [11] says, on pp. 13, Chapter 2 Representations of Finite Groups, "Two representations (E_1,ρ_1) and (E_2,ρ_2) are equivalent if and only if there is a basis B_1 of E_1 and a basis B_2 of E_2 such that for every $g \in G$, the matrix of $\rho_1(g)$ in the basis B_1 is equal to the matrix of $\rho_2(g)$ in the basis B_2. In particular, if the representations (E_1,ρ_1) and (E_2,ρ_2) are equivalent, then E_1 is isomorphic to E_2." So we need a change of basis between $\rho(U) = U$ and $\rho^{(1)}(U)$. What's the linear transformation T s.t.

$$T^{-1}\rho^{(1)}(U)T = U?$$

By intuition,

$$T = \sigma_x\sigma_z \equiv \sigma_1\sigma_3$$

where σ_i's are Pauli matrices.

Indeed,

```
sage: Paulimat[3] * Paulimat[1]*U_try*Paulimat[1] * Paulimat[3]
[conjugate(a) conjugate(b)]
[          -b            a]
```

Then $\rho^{(1)}(U) \circ T = TU$, so this $T = \sigma_1\sigma_3$ is an "intertwining operator" between $\rho^{(1)}(U)$ and fundamental representation $\rho(U) = U$, with $T = \begin{bmatrix} & -1 \\ 1 & \end{bmatrix}$, and $T^{-1} = \begin{bmatrix} & 1 \\ -1 & \end{bmatrix}$.

T is an isomorphism between $\mathbb{C}^2$ and $P^{(1)}(\mathbb{C}^2)$. So fundamental representation ρ of $G = SU(2)$ is equivalent to $\rho^{(1)}(U)$ on $P^{(1)}(\mathbb{C}^2)$.

Exercise 23. [8] (Also from Exercise 2.6 of Kosmann-Schwarzbach (201) [11])
Let (E,π) representation of group G.
$\forall\, g \in G$, $\xi \in E^*$, $x \in E$, set $\langle \pi^*(g)(\xi), x\rangle = \langle \xi, \pi(g^{-1})(x)\rangle$
dual (or *contragredient*) of π, $\pi^* : G \to \text{End}(E^*)$, π^* is a representation, since

$$\begin{aligned} \langle \pi^*(gh)(\xi), x\rangle = \langle \xi, \pi((gh)^{-1})(x)\rangle = \langle \xi, \pi(h^{-1}g^{-1})(x)\rangle = \langle \xi, \pi(h^{-1})\pi(g^{-1})(x)\rangle = \langle \xi, \pi(h^{-1})(\pi(g^{-1})(x))\rangle = \\ = \langle \pi^*(h)(\xi), \pi(g^{-1})(x)\rangle = \langle \pi^*(g)\pi^*(h)(\xi), x\rangle \end{aligned}$$

since this is true, $\forall\, x \in E$, $\forall\, \xi \in E^*$, $\pi^*(gh) = \pi^*(g)\pi^*(h)$.
dual π^* of π is a representation.

7.2. Adjoint Representation.

I will first follow Sec. 7.3 The Adjoint Representation of Ch. 4 Lie Groups and Lie Algebras of Kosmann-Schwarzbach (201) [11]).

The *conjugation action* $\mathcal{C}_g : G \to G$ is defined as

$$\begin{aligned} \mathcal{C}_g &: G \to G \\ \mathcal{C}_g &: h \mapsto ghg^{-1} \end{aligned}$$

So

$$\begin{aligned} \mathcal{C} &: G \to \text{Aut}(G) \\ \mathcal{C}g &= \mathcal{C}_g \end{aligned}$$

Now define the *adjoint action* of g as the differential or push forward of $\mathcal{C}_g$:

$$\text{Ad}_g := D_1\mathcal{C}_g \equiv (\mathcal{C}_g)_{*1} \equiv (\mathcal{C}_g)_*|_{g=1} \qquad \text{(adjoint action of } g\text{)}$$

Now $\text{Ad}_g : \mathfrak{g} \to \mathfrak{g}$, so $\text{Ad} : G \to \text{End}(\mathfrak{g})$

$$\text{Ad}(g) \equiv \text{Ad}_g$$

Note $\mathcal{C}_{gg'} = \mathcal{C}_g\mathcal{C}_{g'} \equiv \mathcal{C}(gg') = \mathcal{C}(g) \circ \mathcal{C}(g')$ and so

$$\xrightarrow{D_1} \text{Ad}_{gg'} = \text{Ad}_g \circ \text{Ad}_{g'}$$

Kosmann-Schwarzbach (201) [11]) claims, because $\text{Ad}_g = 1_{\mathfrak{g}}$ when $g = 1$,

$\text{Ad} : G \to GL(\mathfrak{g})$ is a representation of G on $\mathfrak{g}$. (EY : 20160505 ???)
$\text{Ad} : g \mapsto \text{Ad}_g$

Definition 26. *representation Ad of G on $V = \mathfrak{g}$ is called adjoint representation of Lie group G.*

Denote adjoint representation of Lie algebra $\mathfrak{g}$, ad.
By definition, $\text{Ad}_{\exp(tX)} = \exp{(t\text{ad}_X)}$
cf. Prop. 7.8 of Kosmann-Schwarzbach (201) [11])

Proposition 12. (1) *Let A invertible matrix, $A \in$ Lie group G.*
Let X matrix s.t. $X \in \mathfrak{g}$. Then

$$Ad_A(X) = AXA^{-1}$$

(2) *Let* $X, Y \in \mathfrak{g}$. *Then*

$$ad_X(Y) = [X, Y]$$

(3) *Let* $X, Y \in \mathfrak{g}$. *Then*

$$ad_{[X,Y]} = [ad_X, ad_Y]$$

Proof. (1) By def., $\forall B \in G$, $\mathcal{C}_A(B) = ABA^{-1}$, and thus

$$\mathrm{Ad}_A(X) = \frac{d}{dt} A \exp{(tX)} A^{-1} \Big|_{t=0} = AXA^{-1}$$

(2)

$$\mathrm{ad}_X(Y) = \frac{d}{dt} \mathrm{Ad}_{\exp{(tX)}}(Y) \Big|_{t=0} = \frac{d}{dt} \exp{(tX)} Y \exp{(tX)} \Big|_{t=0} =$$
$$= XY - YX = [X, Y]$$

(3) Use Jacobi identity:

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0 \text{ or}$$
$$[[A, B], C] = [A, [B, C]] - [B, [A, C]]$$

$$\mathrm{ad}_{[X,Y]} C = [[X, Y], C] = [X, [Y, C]] - [Y, [X, C]] = [X, \mathrm{ad}_Y C] - [Y, \mathrm{ad}_X C] \text{ and that}$$
$$\mathrm{ad}_X \mathrm{ad}_Y C = [X, [Y, C]] \Longrightarrow \mathrm{ad}_{[X,Y]} C = [\mathrm{ad}_X, \mathrm{ad}_Y] C$$

□

Part 5. Cohomology; Stoke's Theorem

8. Stoke's Theorem

Theorem 14 (Stoke's Theorem). *Let M be oriented, smooth n-manifold with boundary, let ω be a compactly supported smooth $(n-1)$-form on M, or if $\omega \in A_c^{n-1}(M)$,*
Then

(19)
$$\int_M d\omega = \int_{\partial M} \omega$$

If $\partial M = \emptyset$, then $\int_{\partial M} \omega = 0$
$\int_{\partial M} \omega$ interpreted as $\int_{\partial M} i^*_{\partial M} \omega = \int_{\partial M} i^* \omega$ so

(20)
$$\int_M d\omega = \int_{\partial \mathbf{M}} i^*(\omega)$$

where inclusion $i : \partial M \hookrightarrow M$

Proof. Begin with very special case:
Suppose $M = \mathbb{H}^n$ (upper half space), $\partial M = \mathbb{R}^{n-1}$
ω has compact support, so $\exists\, R > 0$ s.t. $\mathrm{supp}\omega \subseteq$ rectangle $A = [-R, R] \times \cdots \times [-R, R] \times [0, R]$.
$\forall\, \omega \in A_c^{n-1}(\mathbb{H}^n)$

(21)
$$\omega = \sum_{j=1}^n (-1)^{j-1} f_j dx^1 \wedge \cdots \wedge \widehat{dx}^j \wedge \cdots \wedge dx^n \equiv \sum_{i=1}^n \omega_i dx^1 \wedge \cdots \wedge \widehat{dx}^i \wedge \cdots \wedge dx^n$$

with Conlon (2008) [16] and John Lee (2012) [3]'s notation, respectively, and where f_j has compact support.

$$i^*\omega = (f_1 \circ i) dx^2 \wedge \cdots \wedge dx^n \in A_c^{n-1}(\partial \mathbb{H}^n)$$

$$d\omega = \sum_{i=1}^n d\omega_i \wedge dx^1 \wedge \cdots \wedge \widehat{dx}^i \wedge \cdots \wedge dx^n = \sum_{i,j=1}^n \frac{\partial \omega_i}{\partial x^j} dx^j \wedge dx^1 \wedge \cdots \wedge \widehat{dx}^i \wedge \cdots \wedge dx^n =$$
$$= \sum_{i=1}^n (-1)^{i-1} \frac{\partial \omega_i}{\partial x^i} dx^1 \wedge \cdots \wedge dx^n$$

i.e. (for another notation)

$$d\omega = \left(\sum_{j=1}^n \frac{\partial f_j}{\partial x^j} \right) dx^1 \wedge \cdots \wedge dx^n \in A_c^n(\mathbb{H}^n)$$
$$d\omega = \left(\sum_{j=1}^n \frac{\partial f_j}{\partial x^j} \right) dx^1 \wedge \cdots \wedge dx^n \in A_c^n(\mathbb{H}^n)$$

$$\int_{\mathbb{H}^n} d\omega = \sum_{i=1}^n (-1)^{i-1} \int_A \frac{\partial \omega_i}{\partial x^i} dx^1 \wedge \cdots \wedge dx^n = \sum_{i=1}^n (-1)^{i-1} \int_0^R \int_{-R}^R \cdots \int_{-R}^R dx^1 \dots dx^n \frac{\partial \omega_i}{\partial x^i}(x)$$

We can change order of integration in each term so to do x^i integration first.
By fundamental thm. of calculus, terms for which $i \neq n$ reduce to

$$\sum_{i=1}^{n-1} (-1)^{i-1} \int_0^R \int_{-R}^R \cdots \int_{-R}^R \frac{\partial \omega_i}{\partial x^i}(x) dx^1 \dots dx^n = \sum_{i=1}^{n-1} (-1)^{i-1} \int_0^R \int_{-R}^R \cdots \int_{-R}^R \frac{\partial \omega_i}{\partial x^i}(x) dx^i dx^1 \dots \widehat{dx}^i \dots dx^n =$$
$$= \sum_{i=1}^{n-1} (-1)^{i-1} \int_0^R \int_{-R}^R \cdots \int_{-R}^R [\omega_i(x)]_{x^i=-R}^{x^i=R} dx^1 \dots \widehat{dx}^i \dots dx^n = 0$$

because we've chosen R large enough that $\omega = 0$ when $x^i = \pm R$.

□

Part 6. Prástaro

Prástaro (1996) [12]

8.0.1. *Affine Spaces.* cf. Sec. 1.2 - *Affine Spaces* of Prástaro (1996) [12]

Definition 27 (affine space).

(22)
$$\begin{gathered} \textit{affine space} \qquad (M, \mathbf{M}, \alpha) \\ \textit{with} \\ M \equiv \textit{ set (set of pts.)} \\ \mathbf{M} \equiv \textit{ vector space (space of free vectors)} \\ \alpha \equiv \mathbf{M} \times M \to M \equiv \textit{ translation operator} \\ \alpha : (v, p) \mapsto p' \equiv p + v \end{gathered}$$

*Note: α is a **transitive** action and without fixed pts. (free).*
i.e. $\forall\, p \in M$,

$\forall$ pt. $O \in M$, $\alpha : (v, O) \mapsto O' \equiv O + v$, $\alpha(\cdot, O) \equiv \alpha_O \equiv \alpha(O)$. $\alpha_O(v) = O' = O + \mathbf{v}$ $\quad \forall\, O' \in M$, $\exists\, \mathbf{v} \in \mathbf{M}$ s.t. $O' = O + \mathbf{v}$ $\Longrightarrow M \equiv \mathbf{M}$.
$\forall\, (O, \{e_i\})_{1 \leq i \leq n}$, where $\{e_i\}$ basis of $\mathbf{M}$, $M \equiv \mathbf{M} = \mathbb{R}^n$ so isomorphism $M \simeq \mathbb{R}^n$

i.e. α is **without fixed pts.**, meaning,
Given pointed space (M, O), where base pt. $O \in M$, we can associate $\forall\, p \in M$, vector $\mathbf{x} \in \mathbf{M}$, by 1-to-1 mapping $M \to \mathbf{M}$.

So for

$$\alpha : \mathbf{M} \times M \to M$$
$$\alpha(\mathbf{x}, p) = p' = p + \mathbf{x}$$

Consider

$$\alpha(\mathbf{x}, O) = p = \alpha_O(\mathbf{x}) = p \Longrightarrow \exists \alpha_O^{-1}(p) = \mathbf{x} \in \mathbf{M}$$

(1) tangent space of M in $p \in M$ is vector space $T_pM \equiv (\mathbf{M}, p) \cong M$
(2) If $\mathbf{M}$ Euclidean space, affine space $(M, \mathbf{M}, \alpha)$ is Euclidean
(3) Call dim. of affine space $(M, \mathbf{M}, \alpha)$, dim. of $\mathbf{M} \equiv \mathrm{dim}\mathbf{M}$

$\{\mathbf{e}_i\}$ basis of $\mathbf{M}$

Definition 28. $(O, \{e_i\}) \equiv$ *affine frame.*
$\forall$ *affine frame* $(O, \{e_i\})$, $\exists$ *coordinate system* $x^\alpha : M \to \mathbb{R}$,
where $x^\alpha(p)$ *is* α*th component, in basis* $\{e_i\}$, *of vector* $p - O$

Proposition 13 (1.6, Prástaro (1996) [12])**.** $\forall O \in M$, *we have canonical identification* $M \equiv \mathbf{M}$, *since*

$$\alpha_O^{-1} : M \to \mathbf{M} \qquad \alpha_O : \mathbf{M} \to M$$
$$\alpha_O^{-1}(p) = \mathbf{x} \qquad \alpha_O : \mathbf{x} = \alpha(\mathbf{x}, O) = p$$

Furthermore,
$\forall$ ***affine frame*** $(O, \{\mathbf{e}_i\})_{1 \leq i \leq d}$, *where* $\{\mathbf{e}_i\}$ *basis of* $\mathbf{M}$,
$\exists$ *isomorphism* $M \cong \mathbb{R}^d$,
Then, $\forall (O, \{\mathbf{e}_i\})_{1 \leq i \leq d}$,
$\exists$ *coordinate system* $x^\alpha : M \to \mathbb{R}$,
where $x^\alpha(p) = \alpha$*th component, in basis* $\{\mathbf{e}_i\}$, *of vector* $p - O$.

Theorem 15 (1.4 Prástaro (1996) [12])**.** *Let* $(x^\alpha), (\overline{a}^\alpha)$ *2 coordinate systems correspond to affine frames* $(O, \{e_i\})$, $(\overline{O}, \{\overline{e}_i\})$, *respectively.*

$$\overline{x}^\alpha = A^\alpha_{\ \beta} x^\beta + y^\alpha \tag{23}$$

where

$$y^\alpha \in \mathbb{R}^n, \qquad A^\alpha_{\ \beta} \in GL(n; \mathbb{R})$$

Definition 29 (1.10 Prástaro (1996) [12])**.**

$$A(n) \equiv Gl(n, \mathbb{R}) \times \mathbb{R}^n \tag{24}$$

affine group of dim. n

Theorem 16 (1.5)**.** *symmetry group of* n*-dim. affine space, called affine group* $A(M)$ *of* M. $\exists$ *isomorphism,*

$$A(M) \simeq A(n), \qquad f \mapsto (f^\alpha_{\ \beta}, y^\alpha)\,; \qquad f^\alpha \equiv x^\alpha \circ f = f^\alpha_{\ \beta} x^\beta + y^\alpha \tag{25}$$

cf. Eq. 1.4 Prástaro (1996) [12]

Definition 30 (metric)**.** *Let smooth manifold* M, $\mathrm{dim}M = n$, $\forall p \in M$, $\exists$ *vector space* T_pM, *and so for*

$$\begin{aligned} & g_p (T_pM)^2 \to \mathbb{R} \\ & g_p : (X_p, Y_p) \mapsto g_p(X_p, Y_p) \in \mathbb{R} \end{aligned} \tag{26}$$

with g_p *being bilinear, symmetric (in* X_p, Y_p*), nondegenerate (i.e. if* $g_p(X_p, Y_p) = 0$*, then* X_p *or* $Y_p = 0$*)*
Note that

$$g \in \Gamma((TM \otimes TM)^*)$$

and that for $X = X^i \dfrac{\partial}{\partial x^i}$ *so*

$$Y = Y^i \frac{\partial}{\partial x^i}$$

$$g(X, Y) = g_{ij} X^i Y^j$$

Now for

$$F : M \to N \qquad DF \equiv F_* : T_pM \to T_{F(p)}N$$
$$F : x \mapsto y = y(x) \qquad DF : X_p \mapsto (DF)(X^j \frac{\partial}{\partial x^j}) = X^j \frac{\partial y^i}{\partial x^j} \frac{\partial}{\partial y^i}$$

$$(F^* g')(X, Y) = (F^* g')(X^i \frac{\partial}{\partial x^i}, Y^j \frac{\partial}{\partial x^j}) = (F^* g')_{ij} X^i Y^j = g'(F_* X, F_* Y) = $$
$$=$$

Part 7. Connections

9. Connections of Vector Bundles

[23]

Definition 31 (Connection in a vector bundle)**.** ***connection in a vector bundle*** $\pi : E \to M$ *over* C^∞ *manifold* M, *is a bilinear map*

$$\nabla : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$$

satisfying

(i) $\nabla_{fX} s = f \nabla_X s$
(ii) $\nabla_X(fs) = f \nabla_X s + (Xf)s$ *where* $f \in C^\infty(M)$, $X \in \mathfrak{X}(M)$, $s \in \Gamma(E)$

$\nabla_X s$ *is covariant derivative of* s *relative to* X *(for Morita (2001)*[23]*)*

Claim: *Any vector bundle admits a connection.*
e.g. product bundle $M \times \mathbb{R}^n$. Let $x_1, \dots, x_n$ be canonical coordinates in $\mathbb{R}^n$. Take frame field $(s_1, \dots, s_n)$, where $s_i(p) = \frac{\partial}{\partial x^i}$.
Set $\nabla_X s_i = 0$ $(i = 1, \dots, m)$ $\quad \forall$ vector space X,
$\forall s = \sum_i a_i s_i$, $\forall X \in \mathfrak{X}(M)$, set

$$\nabla_X s = \sum_{i=1}^n (X a_i) s_i$$

For this connection $\nabla_X s$ is the partial derivative in direction of X if s is considered $\mathbb{R}^n$-valued function on M. Call it **trivial connection** in product bundle.

Indeed,

$$\nabla_X s = \nabla_{X^i \frac{\partial}{\partial x^i}} \left(s^m \frac{\partial}{\partial x^m} \right) = X^i \nabla_{\frac{\partial}{\partial x^i}} \left(s^m \frac{\partial}{\partial x^m} \right) = X^i \left(\nabla_{\frac{\partial}{\partial x^i}} s^m \right) \frac{\partial}{\partial x^m} + X^i s^m \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^m} =$$
$$= X^i \left(\frac{\partial}{\partial x^i} s^m \right) \frac{\partial}{\partial x^m} + X^i s^m \Gamma^q_{\ mi} \frac{\partial}{\partial x^q} = X^i \frac{\partial s^m}{\partial x^i} \frac{\partial}{\partial x^m} + 0$$

if $\Gamma^q_{\ mi} = 0$ at a chosen point p.

For arbitrary vector bundle $\pi : E \to M$, take locally finite open covering $\{U_\alpha\}_{\alpha \in A}$ s.t. $\pi^{-1}(U_\alpha)$ trivial. Denote ∇^α trivial connection $\forall \pi^{-1}(U_\alpha)$. Let $\{f_\alpha\}$ be a partition of unity for covering U_α, define

$$\nabla_X s := \sum_\alpha f_\alpha \nabla^\alpha_X s$$

Verify this defines connection in E:

$$\nabla_X(gs) = \sum_\alpha f_\alpha \nabla_X^\alpha(gs) = \sum_\alpha f_\alpha \left[X^i \left(\frac{\partial g}{\partial x^i} \right) s + X^i g \frac{\partial s^m}{\partial x^i} \frac{\partial}{\partial x^m} \right] =$$
$$= g \sum_\alpha f_\alpha \nabla_X^\alpha s + \sum_\alpha f_\alpha (Xg) s$$

Proposition 14 (5.18 Morita (2001)[23]). *Let ∇_i $(1 \leq i \leq k)$ be k connections in a given vector bundle. Then $\forall$ linear combination $\sum_{i=1}^k t_i \nabla_i$, where $t_1 + \dots + t_k = 1$ is a connection.*

Proof. TODO: Ex. 5.5 □

Part 8. Holonomy

Definition 32 (Conlon, 10.1.2). *If $X, Y \in \mathfrak{X}(M)$, $M \subset \mathbb{R}^m$, **Levi-Civita connection** on $M \subset \mathbb{R}^m$*

$$\nabla : \mathfrak{X}(M) : \mathfrak{X}(M) \to \mathfrak{X}(M)$$
$$\nabla_X Y := p(D_X Y) \tag{27}$$

with

$$D_X Y := \sum_{j=1}^m X(Y^j) \frac{\partial}{\partial x^j} = \sum_{i,j=1}^m X^i \frac{\partial Y^j}{\partial x^i} \frac{\partial}{\partial x^j} \qquad \begin{aligned} \forall\, X &= \sum_{i=1}^m X^i \frac{\partial}{\partial x^i}, \\ \forall\, Y &= \sum_{i=1}^m Y^i \frac{\partial}{\partial x^i} \end{aligned}$$

$$\nabla_{fX} Y = f(D_{fX} Y) = p(f D_X Y) = f p D_X Y = f \nabla_X Y$$
$$\nabla_X fY = p(D_X fY) = p \left(\sum_{i,j=1}^m \left(X^i f \frac{\partial Y^j}{dx^i} + X^i Y^j \frac{\partial f}{\partial x^i} \right) \frac{\partial}{\partial x^j} \right) = f \nabla_X Y + p \sum_{j=1}^m X(f) Y^j \frac{\partial}{\partial x^j} = f \nabla_X Y + X(f) p(Y)$$

Definition 33 (Conlon, 10.1.4; Christoffel symbols).

$$\begin{aligned} &\nabla_{\frac{\partial}{\partial x^i}} \tfrac{\partial}{\partial x^j} = \Gamma^k_{ij} \tfrac{\partial}{\partial x^k} && \textit{(Conlon's notation)} \\ &\nabla_{\frac{\partial}{\partial x^j}} \tfrac{\partial}{\partial x^i} = \Gamma^k_{ij} \tfrac{\partial}{\partial x^k} && \textit{(F. Schuller's notation)} \end{aligned} \tag{28}$$

Definition 34 (torsion).

$$\begin{aligned} &T : \mathfrak{X}(M) \in \mathfrak{X}(M) \to \mathfrak{X}(M) \\ &T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] \end{aligned} \tag{29}$$

If $T = 0$, ∇ torsion-free or symmetric.

$$T(fX, Y) = f \nabla_X Y - (f \nabla_Y X + Y(f) X) - \{ (fXY - (Y(f)X + fYX) \} = fT(X,Y)$$
$$T(X, fY) = f \nabla_X Y + X(f) Y - f \nabla_Y X - \{ ((X(f)Y + fXY) - fYX \} = fT(X,Y)$$

Thus, $T(X,Y)$ $C^\infty(M)$-bilinear.
$T \in \tau_1^2(M)$.
$T(v,w) \in T_x M$ defined, $\forall\, v, w \in T_x M$, $\forall\, x \in M$.
Thus, torsion is a **tensor**.

Exercise 10.1.7 Conlon (2008)[16] . .
If $T(X,Y) = 0$,

$$T(e_i, e_j) = \Gamma^k_{ji} e_k - \Gamma^k_{ij} e_k - 0 = 0 \Longrightarrow \Gamma^k_{ji} = \Gamma^k_{ij}$$

If $\Gamma^k_{ij} = \Gamma^k_{ji}$, $T(e_i, e_j) = 0$.

Exercise 10.1.8, Conlon (2008)[16].
If $M \subset \mathbb{R}^m$ smoothly embedded submanifold, $\forall\, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^i} \in T_x M$, spanning $T_x M$, consider $\frac{\partial}{\partial x^j} = X^k_j \frac{\partial}{\partial \widetilde{x}^k}$, $\frac{\partial}{\partial x^i} = X^k_i(\widetilde{x}) \frac{\partial}{\partial \widetilde{x}^k}$

$$\nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i} = p D_{X^k_j \frac{\partial}{\partial \widetilde{x}^k}} X^l_i \frac{\partial}{\partial \widetilde{x}^l} = p \left(X^k_j \frac{\partial X^l_i}{\partial \widetilde{x}^k} \frac{\partial}{\partial \widetilde{x}^l} \right) = X^k_j p \left(\frac{\partial X^l_i}{\partial \widetilde{x}^k} \frac{\partial}{\partial \widetilde{x}^l} \right)$$
$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = X^k_i p \left(\frac{\partial X^l_j}{\partial \widetilde{x}^k} \frac{\partial}{\partial \widetilde{x}^l} \right)$$

If $X \in \mathfrak{X}(M)$, smooth $s : [a,b] \to M$,
then $\forall\, s(t)$,

$$X'_{s(t)} = \nabla_{\dot{s}(t)} X \in T_{s(t)} M$$

In fact, it's often natural to consider fields $X_{s(t)}$ along s, parametrized by parameter t, allowing

$$X_{s(t_1)} \neq X_{s(t_2)}$$

each of $s(t_1) = s(t_2)$.

Definition 35 (10.1.9). *Let smooth $s : [a,b] \to M$.*
Vector field along s is smooth $v : [a,b] \to TM$ s.t.

$$\begin{tikzcd} & TM \arrow[d, "\pi"] \\ {[a,b]} \arrow[ur, "v"] \arrow[r, "s"] & M \end{tikzcd}$$

commutes.
Note that $v \in \mathfrak{X}(s) \subset \mathfrak{X}(M)$

e.g. $(Y|s)(t) = Y_{s(t)}$, restriction of $Y \in \mathfrak{X}(M)$ to s.
e.g. $\dot{s}(t) \in \mathfrak{X}(M)$.
$\forall\, v, w \in \mathfrak{X}(s)$, $v + w \in \mathfrak{X}(s)$,

$$(fv + gv)(t) := (f(s(t)) + g(s(t)))v(t) = f(s(t))v(t) + g(s(t))v(t) = (f+g)v(t)$$

Likewise,

$$f(v+w) = fv + fw$$

$\mathfrak{X}(s)$ is a real vector space and $C^\infty[a,b]$-module.

Definition 36 (10.1.10). *Let conection ∇ on M.*
***Associated covariant derivative** is operator*

$$\frac{\nabla}{dt} \mathfrak{X}(s) \to \mathfrak{X}(s)$$

$\forall$ smooth s on M, s.t.
(1) $\frac{\nabla}{dt}$ *$\mathbb{R}$-linear*
(2) $\left(\frac{\nabla}{dt} \right) (fv) = \frac{df}{dt} v + f \frac{\nabla}{dt} v$, $\forall\, f \in C^\infty[a,b]$, $\forall\, v \in \mathfrak{X}(s)$
(3) *If $Y \in \mathfrak{X}(M)$, then*

$$\frac{\nabla}{dt}(Y|s)(t) = \nabla_{\dot{s}(t)} Y \in T_{s(t)} M, \quad a \leq t \leq b$$

Theorem 17 (Conlon Thm. 10.1.11[16]). *$\forall$ connection ∇ on M, $\exists !$ associated covariant derivative $\frac{\nabla}{dt}$*

Proof. Consider arbitrary coordinate chart $(U, x^1 \dots x^n)$.
Consider smooth curve $s : [a, b] \to U$.
Let $v \in \mathfrak{X}(s)$, $v(t) = v^i(t)\frac{\partial}{\partial x^i}$; $\dot{s}(t) = s^j \frac{\partial}{\partial x^j}$.

$$\frac{\nabla v}{dt} = \frac{dv^i(t)}{dt}\frac{\partial}{\partial x^i} + v^i(t)\frac{\nabla}{dt}\frac{\partial}{\partial x^i} = \frac{dv^i}{dt}\frac{\partial}{\partial x^i} + v^i \nabla_{\dot{s}(t)}\frac{\partial}{\partial x^i} = \dot{v}^i\frac{\partial}{\partial x^i} + v^i\dot{s}^j\Gamma^k_{ij}\frac{\partial}{\partial x^k} = \left(\dot{v}^k + v^i\dot{s}^j\Gamma^k_{ij}\right)\frac{\partial}{\partial x^k}$$

This is an explicit, local formula in terms of connection, proving uniqueness.
Existence: $\forall$ coordinate chart $(U, x^1 \dots x^n)$, $\left(\dot{v}^k + v^i\dot{s}^j\Gamma^k_{ij}\right)\frac{\partial}{\partial x^k} =: \frac{\nabla v}{dt}$.

$$\frac{\nabla}{dt}(fv) = \dot{f}v^k + f\dot{v}^k + fv^i\dot{s}^j = \dot{f}v + f\frac{\nabla v}{dt}$$

If f constant, then $\frac{\nabla}{dt}$ is $\mathbb{R}$-linear. □

Definition 37 (10.1.12 Conlon (2008)[16]). *Let (M, ∇). Let $v \in \mathfrak{X}(s)$ for smooth $s : [a, b] \to M$.*
*If $\frac{\nabla v}{dt} \equiv 0$ on s, then v is **parallel** along s.*

Theorem 18 (10.1.13). *Let (M, ∇), smooth $s : [a, b] \to M$, $c \in [a, b]$, $v_0 \in T_{s(c)}M$.*
Then $\exists !$ parallel field $v \in \mathfrak{X}(s)$ s.t. $v(c) = v_0$.
v parallel transport along s.

Proof.

$$\begin{aligned} \dot{s}(t) &= \dot{s}^j(t)e_j \\ v(t) &= v^i(t)e_i \\ v_0 &= a^i e_i \end{aligned}$$

$$0 = \left(\frac{dv^k}{dt}(t) + v^i(t)\dot{s}^j(t)\Gamma^k_{ij}(s(t))\right)e_k$$

or equivalently

$$\frac{dv^k}{dt} = -v^i\dot{s}^j\Gamma^k_{ij}, \qquad 1 \leq k \leq n \qquad (10.1) \tag{30}$$

with initial conditions $v^k(c) = a^k$, $1 \leq k \leq n$.
By existence and uniqueness of solutions of O.D.E.
$\exists\, \epsilon > 0$ s.t. $\exists !$ solutions $v^k(t)$. For $c - \epsilon < t < c + \epsilon$.
In fact, these ODEs being linear in v^k, by ODE theory (Appendix C, Thm. C.4.1).
$\nexists$ restriction on ϵ, so $\exists !\, v^k(t) \quad \forall\, t \in [a, b]$, $1 \leq k \leq n$ □

9.1. **Principal bundle, vector bundle case for parallel transport.** Recall the 2 different forms or viewpoints for Lie-algebra valued 1-forms, or vector-valued 1-forms, or sections of 1-form-valued endomorphisms:

$$\omega^k_{\ i\mu}dx^\mu \equiv \omega^k_{\ i} \in \Omega^1(M, \mathfrak{gl}(n, \mathbb{F})) = \Gamma(\mathfrak{gl}(n, \mathbb{R} \otimes T^*M|_U))$$

for $i, k = 1 \dots n = \dim E$.
$\mu = 1 \dots d = \dim E$
Now

$$D_X\mu = X^\mu D_{\frac{\partial}{\partial x^\mu}}\mu = X^\mu\left[\left(\frac{\partial}{\partial x^\mu}\mu^k\right)e_k + \mu^i\omega^k_{\ i\mu}e_k\right] = \left(X(\mu^k) + \mu^i\omega^k_{\ i}(X)\right)e_k = \left(d\mu^k(X) + \mu^i\omega^k_{\ i}(X)\right)e_k$$

So then define

$$\begin{aligned} &D : \Gamma(E) \to \Gamma(E) \otimes \Gamma(T^*M) \\ &D\mu = D(\mu^i e_i) = e_k(d\mu^k + \mu^i\omega^k_{\ i}) \equiv (d + A)\mu \end{aligned} \tag{31}$$

Also, D can be defined for this case:

$$D : \Gamma(\mathrm{End}(E)) \to \Gamma(\mathrm{End}E) \otimes \Gamma(T^*M)$$

Let $\sigma = \sigma^i_{\ j}e_i \otimes e^j \in \Gamma(\mathrm{End}(E))$

$$\begin{aligned} D\sigma = D(\sigma^i_{\ j}e_i) \otimes e^j + \sigma^i_{\ j}e_i \otimes D^*e^j = \left(d\sigma^k_{\ j} + \sigma^i A^k_{\ i}\right)e_k \otimes e^j + \sigma^i_{\ j}e_i \otimes (A^*)_k^{\ j}e^k = \\ = (d\sigma^k_{\ j} + \sigma^i_{\ j}A^k_{\ i})e_k \otimes e^j + \sigma^k_{\ i}e_j \otimes (-A^i_{\ j})e^j = (d\sigma^k_{\ j} + [A, \sigma]^k_{\ j})e_k \otimes e^j \end{aligned} \tag{32}$$

cf. Def. 4.1.4 of Jost (2011), pp. 138.
For $\mu \in \Gamma(E)$, smooth $s : [a, b] \to M$, $X(t) = \dot{s}(t)$,

$$D_{\dot{s}(t)}\mu = \dot{s}^\mu D_{\frac{\partial}{\partial x^\mu}}\mu = \dot{s}^\mu\left[\frac{\partial\mu^k}{\partial x^\mu}e_k + \mu^i\omega^k_{\ i\mu}e_k\right] = \left[\dot{s}^\mu\frac{\partial\mu^k}{\partial x^\mu} + \dot{s}^\mu\mu^i\omega^k_{\ i\mu}\right]e_k = \frac{d}{dt}\mu(s(t)) + \mu^i\dot{s}^\mu\omega^k_{\ i\mu}e_k \tag{33}$$

Let $D_{\dot{s}(t)}\mu = 0$. Then,

$$\frac{d}{dt}\mu(s(t)) = -\mu^i\dot{s}^\mu\omega^k_{\ i\mu}e_k \tag{34}$$

Recall, given vector bundle $E \xrightarrow{\pi} N$, given $\varphi : M \to N$, then pullback

$$\varphi^*E \to M \tag{35}$$

i.e.

$$\begin{tikzcd} \varphi^*E \arrow[d, "\psi"] & E \arrow[l, "\varphi^*"'] \arrow[d, "\pi"] & (\varphi^*E)_x = E_{\varphi(x)} \\ M \arrow[r, "\varphi"] & N & x \in M \arrow[u] \end{tikzcd}$$

i.e. if $s \in \Gamma(E)$,

$$\varphi^*s = s \circ \varphi \in \Gamma(\varphi^*E)$$

Thus,

$$\begin{tikzcd} \gamma^*E \arrow[d] & E \arrow[l, "\gamma^*"'] \arrow[d, "\pi"] & (\varphi^*E)_c = E_{\gamma(c)} \\ {[a,b]} \arrow[r, "\gamma"] & M & c \in [a,b] \arrow[u] \end{tikzcd}$$

For

$$\begin{aligned} \dot{v}^k &= -v^i\dot{s}^j\Gamma^k_{ij} \\ v^k(c) &= v^k_0 \qquad 1 \leq k \leq m \\ \dot{v} &= -v^i\dot{s}^j\Gamma_{ij} \\ (v \dot{+} w) &= -(v^i + w^i)\dot{s}^j\Gamma_{ij}(v + w)(c) = v(c) + w(c) = v_0 + w_0 \end{aligned}$$

so $v + w \in \mathfrak{X}(s)$ is parallel transport of $v_0 + w_0$.
Likewise, $\forall\, a \in \mathbb{F}$, $av \in \mathfrak{X}(s)$ is the parallel transport of av_0.

$$\dot{\mu}^k = -\mu^i \dot{s}^\mu \omega^k_{\ i\mu} = -\mu^i \omega^k_{\ i}(\dot{s}^\mu)$$

Suppose $\gamma^* E$ trivialized over $[a,b]$.
Closed interval is contractible, so this is always possible.
For chart (U,φ),

$$\begin{tikzcd} \gamma^* E \arrow[d] & E \arrow[l, "\gamma^*"'] \arrow[d, "\pi"] \\ {[a,b]} \arrow[r, "\gamma"] & M \end{tikzcd} \qquad \begin{tikzcd} E|_U \arrow[r, "\psi"] & U \times V \\ U \subset M \arrow[u, "\pi^{-1}"] \arrow[ur] & \end{tikzcd}$$

Consider

$$\varphi : [a,b] \times V \to \gamma^* E$$
$$\varphi(t,\cdot) = \gamma^* \circ \psi^{-1}(\gamma(t),\cdot)$$

$\forall\, \mu \in \Gamma(E|_{x\in M})$,
$\mu = \mu^i e_i$.
$\varphi(t,e_i) = \epsilon_i$ is a basis for $\gamma^* E$.
$\forall\, \sigma \in \Gamma(\gamma^* E)$,

$$\sigma = \sigma^i \epsilon_i, \quad \sigma^i : [a,b] \to \mathbb{F}$$

$$\nabla_{\frac{\partial}{\partial x^\mu}} \sigma = \frac{\partial \sigma^k}{\partial x^\mu}\epsilon_k + \omega^k_{\ j\mu}\sigma^j \epsilon_k = \left(\frac{\partial \sigma^k}{\partial x^\mu} + \omega^k_{\ ,j\mu}\sigma^j \right)\epsilon_k$$
$$\nabla \sigma = \epsilon_k \otimes (d\sigma^k + \omega^k_{\ j\mu} dx^\mu \sigma^j) = \epsilon_k \otimes (d\sigma^k + \omega^k_{\ j}\sigma^j)$$
$$\nabla_{\frac{d}{dt}} \sigma = \epsilon_k \otimes \left(\frac{d\sigma^k}{dt} + \omega^k_{\ j\mu}\dot{x}^\mu \sigma^j \right)$$

Now

$$\frac{d}{dt} = \dot{x}^\nu \frac{\partial}{\partial x^\nu}$$

Then σ parallel along γ if

$$\frac{d\sigma^k}{dt} + \omega^k_{\ j\mu}\dot{x}^\mu \sigma^j = 0$$

Definition 38 (3.1.4 [17])**.** *Parallel transport along γ is*

(36)
$$P_\gamma : E_{\gamma(a)} \to E_{\gamma(b)}$$
$$P_\gamma(v) \mapsto \sigma(b)$$

where $\sigma \in \Gamma(\gamma^* E)$, σ unique and s.t. $\sigma(a) = v$.

Lemma 1 (10.1.16[16])**.** *holonomy*

$$h_s : T_x M \to T_{x_0} M$$

if ∇ around piecewise smooth loop s is a linear transformation.

Lemma 2 (10.1.18 Conlon (2008)[16])**.** *Let piecewise smooth loop $s : [a,b] \to M$ at x_0.*
Let weak reparametrization $\widetilde{s} = s \circ r : [c,d] \to M$.
If reparametrization is orientation-preserving, then $h_{\widetilde{s}} = h_s$,
If reparametrization is orientation-reversing, then $h_{\widetilde{s}} = h_s^{-1}$,

Proof. Without loss of generality, assume smooth s, r

$$\widetilde{s}(\tau) = s(r(\tau))$$
$$\widetilde{v}(\tau) = v(r(\tau))$$

$$\widetilde{w}^j(\tau) = \frac{dt}{d\tau}(\tau) u^j(r(\tau))$$
$$\frac{d\widetilde{v}^k}{d\tau}(\tau) = \frac{dr}{d\tau}(\tau)\frac{dv^k}{dt}(r(\tau))$$
$$\frac{d\widetilde{v}^k}{d\tau} = -\widetilde{v}^i \widetilde{w}^j \Gamma^k_{ij}$$

since

$$\frac{dv^k}{dt} = -v^i u^j \Gamma^k_{ij}; \qquad 1 \leq k \leq n$$
$$v^k(c) = a^k; \qquad 1 \leq k \leq a$$
$$\frac{dr}{d\tau}\frac{dv^k}{dt} = -v^i \frac{dr}{d\tau} u^j \Gamma^k_{\ ij} = \frac{d\widetilde{v}^k}{d\tau} = -\widetilde{v}^i \widetilde{w}^j \Gamma^k_{\ ij}$$

Thus, if $r(c) = a$, $r(d) = b$

$$h_{\widetilde{s}}(v_0) = \widetilde{v}(d) = v(b) = h_s(v_0)$$

If $r(c) = a$, $r(d) = b$, then

$$\widetilde{v}(c) = v(b) = h_s(v_0)$$

and

$$h_{\widetilde{s}}(h_s(v_0)) = h_{\widetilde{s}}(v(b)) = \widetilde{v}(d) = v(a) = v_0$$

At this point, I will switch to my notation because it clarified to me, at least, what was going on, in that a holonomy h_s is *invariant* under orientation-preserving reparametrization, and its inverse is well-defined.
For $\widetilde{s} = s \circ t : [c,d] \to M$,
piecewise smooth t is reparametrized, i.e.

(37)
$$t : [c,d] \to [a,b]$$

Now,

$$\frac{d}{d\tau}\widetilde{s}(\tau) = \frac{d}{d\tau}\widetilde{s}(t(\tau)) = \dot{s}(t)\frac{dt}{d\tau}(\tau) \equiv \dot{s}\frac{dt}{d\tau}$$
$$v^k(t) = v^k(t(\tau)) = v^k(\tau)$$
$$\frac{dv^k}{d\tau}(t(\tau)) = \frac{dv^k}{dt}\frac{dt}{d\tau} = \frac{dt}{d\tau}(-v^i(\tau)\dot{s}^j(t)\Gamma^k_{\ ij}) = -v^i(\tau)\frac{d\widetilde{s}^j}{d\tau}\Gamma^k_{\ ij}$$

Consider

$$h_s(v_0) = v(b)$$

If $t(c) = a$,
$t(d) = b$

$$h_{\widetilde{s}}(v_0) = \widetilde{v}(d) = v(t(d)) = v(b) = h_s(v_0)$$

If $t(c) = b$,
$t(d) = a$

$$h_{\widetilde{s}}(h_s(v_0)) = h_{\widetilde{s}}(v(b)) = h_{\widetilde{s}}(v(t(c))) = h_{\widetilde{s}}(\widetilde{v}(c)) =$$
$$= \widetilde{v}(d) = v(t(d)) = v(a) = v_0$$

Thus,

$$\boxed{h_{\widetilde{s}} = h_s^{-1}}$$

□

I am working through Conlon (2008) [16] , Clarke and Santoro (2012) [17], and Schreiber and Waldorf (2007)[18], concurrently, for holonomy.

10. Path Groupoid of a smooth manifold; generalization of paths

cf. Schreiber and Waldorf (2007)[18].

Definition 39 (path)**.** ***path*** *is a smooth map $\gamma : [0,1] \to M$, between 2 pts. $x, y \in M$, which has a sitting instant; i.e. number $0 < \epsilon < \frac{1}{2}$ s.t.*

$$\gamma(t) = \begin{cases} x & \text{for } 0 \leq t < \epsilon \\ y & \text{for } 1-\epsilon < t \leq 1 \end{cases} \tag{38}$$

Denote the set of such paths by PM,

$$PM \equiv \{ \gamma \in \Gamma(M) | \text{ smooth } \gamma : [0,1] \to M \text{ s.t. } \exists 0 < \epsilon < \frac{1}{2} \text{ s.t. } \begin{cases} x & \text{for } 0 \leq t < \epsilon \\ y & \text{for } 1-\epsilon < t \leq 1 \end{cases} \} \tag{39}$$

cf. Def. 2.1. of Schreiber and Waldorf (2007)[18]
Define *composition*:

Given paths γ_1, γ_2; $\gamma_1(0) = x, \quad \gamma_2(0) = y,$
$\gamma_1(1) = y \quad \gamma_2(1) = z$
define composition to be path

$$\gamma_2 \circ \gamma_1$$
$$(\gamma_2 \circ \gamma_1)(t) := \begin{cases} \gamma_1(2t) & \text{for } 0 \leq t \leq \frac{1}{2} \\ \gamma_2(2t-1) & \text{for } \frac{1}{2} \leq t \leq 1 \end{cases} \tag{40}$$

$\gamma_2 \circ \gamma_1$ smooth since γ_1, γ_2 both constant near gluing pt., due to sitting instants ϵ_1, ϵ_2, respectively.
Define *inverse*:

$$\begin{gathered} \gamma^{-1} : [0,1] \to M \\ \gamma^{-1}(t) := \gamma(1-t) \end{gathered} \tag{41}$$

(so that $\gamma^1(t) = \begin{cases} y & \text{for } 1-\epsilon < 1-t \leq 1 \text{ or } 0 \leq t < \epsilon \\ x & \text{for } 0 \leq 1-t < \epsilon \text{ or } 1-\epsilon < t \leq 1 \end{cases}$)

Definition 40 (thin homotopy equivalent)**.** *2 paths γ_1, γ_2 s.t. $\gamma_1(0) = \gamma_2(0) = x$, $\gamma_1(1) = \gamma_2(1) = y$, γ_1, γ_2 are thin homotopy equivalent, if $\exists$ smooth $h : [0,1] \times [0,1] \to M$ s.t.*

(1) *$\exists 0 < \epsilon < \frac{1}{2}$ with*
 (a) *$h(s,t) = x$ for $0 \leq t < \epsilon$*
 $h(s,t) = y$ for $1-\epsilon < t \leq 1$
 (b)
 (c) *$h(s,t) = \gamma_1(t)$ for $0 \leq s < \epsilon$*
 $h(s,t) = \gamma_2(t)$ for $1-\epsilon < s \leq 1$

(2) *differential of h has at most rank 1 everywhere, i.e.*

$$\text{rank}(dh|_{(s,t)}) \leq 1 \qquad \forall (s,t) \in [0,1] \times [0,1] \tag{42}$$

cf. Def. 2.2. of Schreiber and Waldorf (2007)[18]

$h(s,t) = \gamma_1(t)$ for $0 \leq s < \epsilon$ is the homotopy from γ_1 to γ_2, i.e. $h(0,t) = \gamma_1(t)$
$h(s,t) = \gamma_2(t)$ for $1-\epsilon < s \leq 1$ $\quad h(1,t) = \gamma_2(t)$
and define an equivalence relation on PM.
Note that for $h : [0,1] \times [0,1] \to M$,

$$(Dh)|_{(s,t)} = \left[\frac{\partial h^i}{\partial s}, \frac{\partial h^i}{\partial t} \right]$$

$$\begin{gathered} P^1 M \equiv \text{ set of thin homotopy classes of paths, i.e.} \\ P^1M = \{ [\gamma] | \gamma_1 \in PM, \text{ if } \exists \text{ smooth } h : [0,1] \times [0,1] \to M \text{ s.t. } h \text{ thin homotopy of } \gamma_1 \text{ and } \gamma_2, \gamma_1 \sim \gamma_2 \} \end{gathered} \tag{43}$$

$\text{pr} : PM \to P^1M$ is projection to classes.

Denote thin homotopy class of path γ, $\gamma(0) = x$, $\gamma(1) = y$ by $\overline{\gamma}$, or $[\gamma]$.

10.1. Reparametrization of thin homotopies.

Let $\beta : [0,1] \to [0,1]$, $\beta(0) = 0$, $\beta(1) = 1$.

Then $\forall$ path γ, $\gamma(0) = x$, $\gamma(1) = y$, $\gamma \circ \beta$ is also a path $\gamma \circ \beta(0) = x$ and $\gamma \circ \beta(1) = y$

$$h(s,t) := \gamma(t\beta(1-s) + \beta(t)\beta(s) \tag{44}$$

defines a homotopy from γ to $\gamma \circ \beta$.

$$\gamma_1 \circ \gamma_2 \in PM \xmapsto{\text{pr}} [\gamma_1 \circ \gamma_2] = [\gamma_1][\gamma_2] \in P^1 M$$

Composition of thin homotopy classes of paths obeys following rules:

Lemma 3. *$\forall$ path γ, $\gamma(0) = x$, $\gamma(1) = y$*

(1) $\overline{\gamma} \circ \overline{id_x} = \overline{\gamma} = \overline{id_y} \circ \overline{\gamma} \equiv [\gamma]1_x = [\gamma] = 1_y[\gamma]$

(2) *for paths γ'; $\gamma'(0) = y$, $\quad \gamma''(0) = z$*
 $\gamma'(1) = z \quad \gamma''(1) = w$

$$(\overline{\gamma}'' \circ \overline{\gamma}') \circ \overline{\gamma} = \overline{\gamma}'' \circ (\overline{\gamma}' \circ \overline{\gamma}) \equiv ([\gamma''][\gamma'])[\gamma] = [\gamma'']([\gamma'][\gamma]) \tag{45}$$

(3) $\overline{\gamma} \circ \overline{\gamma^{-1}} = \overline{id_y}$ *and* $\overline{\gamma^{-1}} \circ \overline{\gamma} = \overline{id_x} \equiv [\gamma][\gamma^{-1}] = 1_y$ *and* $[\gamma^{-1}][\gamma] = 1_x$

cf. Lemma 2.3. of Schreiber and Waldorf (2007)[18]

Definition 41 (path groupoid)**.** *$\forall$ smooth manifold M,*
consider category whose set of objects is M,

whose set of morphisms is P^1M, where class $[\gamma]$, $[\gamma](0)=x$ is a morphism from x to y and

$$[\gamma](1)=y$$

composition $[\gamma_1][\gamma_2]=[\gamma_1\circ\gamma_2]\in P^1M$ Lemma 3 are axioms of a category, 3rd. property says $\forall$ morphism is invertible.
*Hence, we've defined a groupoid, called **path groupoid** of M, $\mathcal{P}_1(M)$.*

So

$$\mathrm{Obj}(\mathcal{P}_1(M))=M$$
$$\mathrm{Mor}(\mathcal{P}_1(M))=P^1M$$

$\forall$ smooth $f:M\to N$, denote functor f_*

(46) $$f_*:\mathcal{P}_1(M)\to\mathcal{P}_1(N)$$

with

$$f_*(x)=f(x)$$
$$(f_*)([\gamma]):=[f\circ\gamma]$$

If $\gamma\sim\gamma'$, for $f\circ\gamma$, $f\circ\gamma'$,

$f\circ h(s,t)$ with $f\circ h(0,t)=f\circ\gamma(t)$,
$f\circ h(1,t)=f\circ\gamma'(t)$
so $f\circ h$ is a thin homotopy between $f\circ\gamma$, $f\circ\gamma'$ and so $[f\circ\gamma]$ *well-defined.*

Part 9. Complex Manifolds

EY : 20170123 I don't see many good books on Complex Manifolds for physicists other than Nakahara's. I will supplement this section on Complex Manifolds with external links to the notes of other courses that I found useful to myself.
Complex Manifolds - Lecture Notes Koppensteiner (2010) [13]
Lectures on Riemannian Geometry, Part II: Complex Manifolds by Stefan Vandoren
Vandoren (2008) [14]

Part 10. Jets, Jet bundles, h-principle, h-Prinzipien

cf. Eliashberg and Misahchev (2002) [19]
cf. Ch. 1 Jets and Holonomy, Sec. 1.1 Maps and sections of Eliashberg and Misahchev (2002) [19].
Visualize $f:\mathbb{R}^n\to\mathbb{R}^q$ as graph $\Gamma_f\subset\mathbb{R}^n\times\mathbb{R}^q$.

Consider this graph as image of $\mathbb{R}^n\to\mathbb{R}^n\times\mathbb{R}^q$, i.e.

$$x\mapsto(x,f(x))$$

$\mathbb{R}^n\to\mathbb{R}^n\times\mathbb{R}^q$ is called section (by mathematicians),
$x\mapsto(x,f(x))$
is called *field* or $\mathbb{R}^q$-valued field (by physicists).
cf. Ch. 1 Jets and Holonomy, Sec. 1.2 Coordinate definition of jets of Eliashberg and Misahchev (2002) [19].

Definition 42 (r-jet)**.** *Given (smooth) $f:\mathbb{R}^n\to\mathbb{R}^q$, given $x\in\mathbb{R}^n$.*
r-jet of f at x - sequence of derivatives of f, up to order r, $\equiv$

(47) $$J^r_f(x)=(f(x),f'(x)\dots f^{(r)}(x))$$

$f^{(q)}$ consists of all partial derivatives $D^\alpha f$, $\alpha=(\alpha_1\dots\alpha_n)$, $|\alpha|=\alpha_1+\dots+\alpha_n=s$, ordered lexicographically.
e.g. $q=1$,
$f:\mathbb{R}^n\to\mathbb{R}$.
1-jet of f at $x=J^1_f(x)=(f(x),f^{(1)}(x))$.

$$f^{(1)}(x)=\{D^\alpha f|\alpha=(\alpha_1\dots\alpha_n),|\alpha|=\alpha_1+\dots+\alpha_n=1\}=\left(\frac{\partial f}{\partial x^1},\frac{\partial f}{\partial x^2},\dots\frac{\partial f}{\partial x^n}\right)$$

Let $d_r=d(n,r)=$ number of all partial derivatives D^α of order r of function $\mathbb{R}^n\to\mathbb{R}$.
Consider r-jet $J^r_f(x)$ of map $f:\mathbb{R}^n\to\mathbb{R}^q$ as pt. of space $\mathbb{R}^q\times\mathbb{R}^{qd_1}\times\mathbb{R}^{qd_2}\times\dots\times\mathbb{R}^{qd_r}=\mathbb{R}^{qN_r}$, where $N_r=N(n,r)=1+d_1+d_2+\dots+d_r$, i.e.

$$J^r_f(x)=(f(x),f^{(1)}(x),\dots f^{(r)}(x))\in\mathbb{R}^q\times\mathbb{R}^{qd_1}\times\dots\times\mathbb{R}^{qd_r}=\mathbb{R}^{qN_r}$$

Exercise 1.
Given order r, consider n-tuple of (positive) integers $(r_1,r_2\dots r_n)$ s.t. $r_1+r_2+\dots+r_n=r$, and $r_k\geq 0$.
Imagine $r_k=$ occupancy number, num ber of balls in kth cell. $(r_1\dots r_n)$ describes a positive ocnfiguration of occupancy numbers, with indistinguishable balls; 2 distributions are distinguishable only if corresponding n-tuples $(r_1\dots r_n)$ not identical.
Represent balls by stars, and indicate n cells by n spaces between $n+1$ bars.
With $n+1$ bars, r stars, 2 bars are fixed. $n-1$ bars and r stars to arrange linearly, so a total of $n-1+r$ objects to arrange. r stars indistinguishable amongst themselves, so choose r out of $n-1+r$ to be stars.

(48) $$\Longrightarrow d_r=d(n,r)=\binom{n-1+r}{r}$$

Use *induction* (cf. Ch. 4 Binomial Coefficients).

$$N_0=N(n,0)=\binom{n-1+0}{0}=1$$
$$N_1=N(n,1)=1+\binom{n-1+1}{1}=1+n=\frac{(n+1)!}{n!1!}$$

Induction step:

$$N_{r-1}=N(n,r-1)=\sum_{k=1}^{r-1}d_k+1=\binom{n+r-1}{r-1}$$

and so

$$N_r=N(n,r)=\sum_{k=1}^{r}d_k+1=\sum_{k=1}^{r}\binom{n-1+k}{k}+1=\sum_{k=1}^{r-1}\binom{n-1+k}{k}+\binom{n-1+r}{r}+1=$$
$$=\binom{n+r-1}{r-1}+\binom{n-1+r}{r}=\frac{(n+r-1)!}{(r-1)!n!}+\frac{(n-1+r)!}{r!(n-1)!}=\frac{(n+r)!}{n!r!}=\binom{n+r}{r}$$

$$\begin{tikzcd} \mathbb{R}^{qN_r} & & & J^r_f(x) & \\ \mathbb{R}^n \arrow[u, "J^r_f"] \arrow[r, "f"] & \mathbb{R}^q & & x \arrow[u, "J^r_f"] \arrow[r, mapsto, "f"] & f(x) \end{tikzcd}$$

Definition 43 (space of r-jets)**.** *space of r-jects of maps $\mathbb{R}^n\to\mathbb{R}^q$ or space of r-jets of sections $\mathbb{R}^n\to\mathbb{R}^n\times\mathbb{R}^q\equiv$*

(49) $$J^r(\mathbb{R}^n,\mathbb{R}^q)=\mathbb{R}^n\times\mathbb{R}^{qN_r}\equiv\mathbb{R}^n\times\mathbb{R}^q\times\mathbb{R}^{qd_1}\times\mathbb{R}^{qd_2}\times\dots\times\mathbb{R}^{qd_r}$$

e.g. $J^1(\mathbb{R}^n,\mathbb{R}^q)=\mathbb{R}^n\times\mathbb{R}^q\times M_{q\times n}$, where $M_{q\times n}=\mathbb{R}^{qn}$ is the space of $(q\times n)$-matrices.

Part 11. Morse Theory

11. Morse Theory introduction from a physicist

I needed some physical motivation to understand Morse theory, and so I looked at Hori, et. al. [15].

cf. pp. 43, Sec. 3.4 Morse Theory, from Ch. 3. Differential and Algebraic Topology of Hori, et. al. [15].

Consider smooth $f : M \to \mathbb{R}$, with non-degenerate critical points.

If no critical values of f between a and b ($a < b$), then subspace on which f takes values less than a is deformation retract of subspace where f less than b, i.e.

$$\{x \in M | f(x) < b\} \times [0,1] \xrightarrow{F} \{x \in M | f(x) < b\}$$

$\forall\, x \in M$ s.t. $f(x) < b$,

$$\begin{aligned} & F(x,0) = x \\ & F(x,1) \in \{x \in M | f(x) < a\} \end{aligned} \qquad \text{and } F(a',1) = a' \qquad \forall\, a' \in M \text{ s.t. } f(a') < a$$

To show this, consider $-\nabla f / |\nabla f|^2$

Morse lemma: $\forall$ critical pt. p s.t. $\exists$ choice of coordinates s.t.

$$f = -(x_1^2 + x_2^2 + \dots + x_\mu^2) + x_{\mu+1}^2 + \dots + x_n^2 \tag{50}$$

where $f(p) = 0$ and p is at origin of these coordinates.

- difference between
$$f^{-1}(\{x \leq -\epsilon\}),\ f^{-1}(\{x \leq +\epsilon\})$$
can be determined by local analysis and only depends on μ, $\mu \equiv$ "Morse index" $=$ number of negative eigenvalues of Hessian of f at critical pt.

  Answer:

  $f^{-1}(\{x \leq +\epsilon\})$ can be obtained from $f^{-1}(\{x \leq -\epsilon\})$ by "attaching μ-cell" along boundary $f^{-1}(0)$
- "attaching μ-cell to X mean, take μ-ball $B_\mu = \{|x| \leq 1\}$ in μ-dim. space,
identity pts. on boundary $S^{\mu-1}$ with pts. in the space X, through
cont. $f : S^{\mu-1} \to X$, i.e. take
$$X \coprod B_\mu$$
with $x \sim f(x) \quad \forall\, x \in \partial B_\mu = S^{\mu-1}$.
- find homology of M,

  f defines chain complex C_f^*, kth graded piece C^{α_k}, α_k is number of critical pts. with index k.
$$\begin{aligned} & \partial : C_p^k \to C_p^{k-1} \\ & \partial x_a = \sum_b \Delta_{a,b} x_b \end{aligned} \tag{51}$$
where $\Delta_{a,b} :=$ signed number of lines of gradient flow from x_a to x_b, b labels pts. of index $k-1$.

  Gradient flow line is path $x(t)$ s.t. $\dot{x} = \nabla(f)$, with $\begin{aligned} x(-\infty) &= x_a \\ x(+\infty) &= x_b \end{aligned}$
- To define this number ($\Delta_{a,b}$?), construct moduli space of such lines of flow (???)
by intersecting outward and inward flowing path spaces from each critical point, and then show this moduli space is oriented, 0-dim. manifold (pts. with signs)
- $\partial^2 = 0$ proof

  ∂, boundary of space of paths connecting critical points, whose index differs by 2 = union over compositions of paths between critical pts. whose index differs by 1.

  $\implies$ coefficients of ∂^2 are sums of signs of pts. in 0-dim. space, which is boundary of 1-dim. space.

  These signs must therefore add to 0, so $\partial^2 = 0$.

Hori, et. al. [15] is good for physics, but there isn't much thorough, step-by-step explanations of the math. I will look at Hirsch (1997) [6] and Shastri (2011) [5] at the same time.

11.1. Introduction, definitions of Morse Functions, for Morse Theory.

cf. Ch. 6, Morse Theory of Hirsch (1997) [6], Section 1. Morse Functions, pp. 143-

Recall for TM, $T_xM \xrightarrow{\varphi} \mathbb{R}^n$.

Cotangent bundle T^*M defined likewise:

$$T_x^*M \xrightarrow{\varphi} \text{ dual vector space } (\mathbb{R}^n)^* = L(\mathbb{R}^n, \mathbb{R})$$

i.e.

$$T^*M = \bigcup_{x \in M} (M_x^*) \qquad M_x^* = L(M_x, \mathbb{R})$$

If chart (φ, U) on M, natural chart on T^*M is

$$\begin{aligned} & T^*U \to \varphi(U) \times (\mathbb{R}^n)^* \\ & \lambda \in M_x^* \mapsto (\varphi(x), \lambda \varphi_x^{-1}) \end{aligned}$$

Projection map

$$\begin{aligned} & p : T^* \to M \\ & M_x^* \mapsto x \end{aligned}$$

Let C^{r+1} map, $1 \leq r \leq \omega$, $f : M \to \mathbb{R}$, $\forall\, x \in M$, linear map $T_x f : M_x \to \mathbb{R}$ belongs to M_x^*

$$T_x f = Df_x \in M_x^*$$

Then

$$\begin{aligned} & Df : M \to T^*M \\ & x \mapsto Df_x = Df(x) \end{aligned}$$

is C^r section of T^*M.

Definition 44. ***critical point*** *x of f is zero of Df, i.e.*

$$Df(x) = 0 \tag{52}$$

of vector space M_x^*.

Thus, set of critical pts. of f is counter-image of submanifold $Z^* \subset T^*M$ of zeros.

Note $Z^* \approx M$, codim. of Z^* is $n = \dim M$.

Definition 45. ***Morse function*** *f if $\forall$ critical pts. of f are nondegenerate.*

Note set of critical pts. closed discrete subset of M.

Let open $U \subset \mathbb{R}^n$, let C^2 map $g : U \to \mathbb{R}$,

critical pt. $p \in U$ nondegenerate iff

- linear $D(Dg)(p) : \mathbb{R}^n \to (\mathbb{R}^n)^*$ bijective
- identify $L(\mathbb{R}^n, (\mathbb{R}^n)^*)$ with space of bilinear maps $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $\implies$ equivalent to condition that symmetric bilinear $D^2 g(p) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ non-degenerate
- $n \times n$ *Hessian matrix*
$$\left[\frac{\partial^2 g}{\partial x^i \partial x^j}(p) \right]$$
has rank n

Hessian of g at critical pt. p is quadratic form H_pf associated to bilinear form $D^2g(p)$

$$\Longrightarrow H_pf(y) = D^2g(p)(y,y) = \sum_{i,j} \frac{\partial^2 g}{\partial x^i \partial x^j}(p) y^i y^j$$

Let open $V \subset \mathbb{R}^n$, suppose C^2 diffeomorphism $h : V \to U$.
Let $q = h^{-1}(p)$, so q is critical pt. of $gh : V \to \mathbb{R}$.

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "H_q(gh)"] \arrow[d, "Dh(q)"'] & \mathbb{R} \\ \mathbb{R}^n \arrow[ur, "H_pg"'] & \end{tikzcd}$$

(quadratic) form (H_pf) invariant under diffeomorphisms.
Let C^2 $f : M \to \mathbb{R}$.
$\forall$ critical pt. x of f, define
Hessian quadratic form

$$H_xf : M_x \to \mathbb{R}$$
$$H_xf : M_x \xrightarrow{D\varphi_x} \mathbb{R}^n \xrightarrow{H_{\varphi(x)}(f\varphi^{-1})} \mathbb{R}$$

where φ is any chart at x.
Thus, critical pt. of a C^2 real-valued function nondegenerate iff associated Hessian quadratic form is nondegenerate.
Let Q nondegenerate quadratic form on vector space E.
Q negative definite on subspace $F \subset E$ if $Q(x) < 0$ whenever $x \in F$ nonzero.
Index of $Q \equiv \text{Ind}Q$, is largest possible dim. of subspace on which Q is negative definite.
cf. 1.1. Morse's Lemma of Ch. 6, pp. 145, Morse Theory of Hirsch (1997) [6]

Lemma 4 (Morse's Lemma)**.** *Let $p \in M$ be nondegenerate critical pt. of index k of C^{r+2} map $f : M \to \mathbb{R}$, $1 \leq r \leq \omega$.*
Then $\exists\, C^r$ chart (φ, U) at p s.t.

$$f\varphi^{-1}(u_1 \dots u_n) = f(p) - \sum_{i=1}^k u_i^2 + \sum_{i=k+1}^n u_i^2 \qquad (53)$$

Let ${}^TQ \equiv Q^T$ denote tranpose of matrix Q.

Lemma 5. *Let $A = diag\{a_1, \dots, a_n\}$ diagonal $n \times n$ matrix, with diagonal entries ± 1.*
Then $\exists$ neighborhood N of A in vector space of symmetric $n \times n$ matrices, C^∞ map

$$P : N \to GL(n, \mathbb{R}) \qquad (54)$$

s.t. $P(A) = I$, and if $P(B) = Q$, then $Q^TBQ = A$

Proof. Let $B = [b_{ij}]$ be symmetri matrix near A s.t. $b11 \neq 0$ and b_{11} has same sign as a_1.
Consider $x = Ty$ where

$$x_1 = \left[y_1 - \frac{b_{12}}{b_{11}} y_2 - \dots - \frac{b_{1n}}{b_{11}} y_n \right] / \sqrt{|b_n|}$$
$$x_k = y_k \text{ for } k = 2, \dots n$$

$\square$

12. Lagrange multipliers

From *wikipedia:Lagrange multiplier*, https://en.wikipedia.org/wiki/Lagrange_multiplier, find local minima (maxima), pt. $a \in N$, s.t. $\exists$ neighborhood U s.t. $f(x) \geq f(a)$ $(f(x) \leq f(a))$ $\forall x \in U$.
For $f : U \to \mathbb{R}$, open $U \subset \mathbb{R}^n$, find $x \in U$ s.t. $D_xf \equiv Df(x) = 0$, check if Hessian $H_xf < 0$.
Maxima may not exit since U open.
References:
Relative Extrema and Lagrange Multipliers
Other interesting links:
The Lagrange Multiplier Rule on Manifolds and Optimal Control of nonlinear systems

Part 12. Classical Mechanics applications

cf. Arnold, Kozlov, Neishtadt (2006) [20].
If known forces $\mathbf{F}_1 \dots \mathbf{F}_n$ acts on points, then

$$\sum_{i=1}^n \langle m_i \ddot{\mathbf{r}}_i - \mathbf{F}_i, \xi_i \rangle = 0 \qquad (55)$$

cf. Eq. (1.26) of Arnold, Kozlov, Neishtadt (2006) [20], where $\xi_1, \dots \xi_n$ are arbitrary tangent vectors to M, $\xi_i, \dots \xi_n \in TM$.
$\sum_{i=1}^n \langle m_i \ddot{\mathbf{r}}_i - \mathbf{F}_i, \xi_i \rangle$ called "general equation of dynamics" or d'Alembert-Lagrange principle.

Part 13. Classical Mechanics

13. Classical Mechanics

13.1. **Structure of Galilean Space-Time.** cf. Sec. 3.1 - *Structure of Galilean Space-Time* of Prástaro (1996) [12].
Mechanics assumes a particular simple formulation if formulated with respect to some spacetime manifold.
In Galilean spacetime, it's possible to naturally recognize absolute objects, and others that depend on frames.
cf. Def. 3.1 of Prástaro (1996) [12]

Definition 46 (Galilean spacetime structure)**.** (1) *Galilean spacetime structure* $:= (\mathcal{G}, g)$ *where*
$\mathcal{G}$ is (fiber bundle space-time)

$$\mathcal{G} \equiv \{\tau : M \to T\} \qquad (56)$$

where
*M = 4-dim. affine manifold (**space-time**); corresponding structure is $(M, \mathbf{M}, \alpha)$,*

(2) *T = 1-dim. affine space (**time**), corresponding affine structure is $(T, \mathbf{T}, \beta)$*
(3) *τ = surjective affine mapping, of constant rank 1, s.t. $\forall\, p \in M$ associates its time $\tau(p) \in T$*

Put $\mathbf{S} = ker(\underline{\tau}) \equiv ker(D\tau) \in M$,
where $\underline{\tau} \equiv D\tau$, D is symbol of derivative.
Define

$$g : M \to vS_2^0(M) \equiv M \times S_2^0(\mathbf{S})$$
$$g(p) = (p, \underline{g}) \equiv (p, Dg), \forall\, p \in M$$

where $\underline{g} \equiv Dg$ is a Euclidean structure on $\mathbf{S}$. g is called vertical metric field.

Thus, given $(M, \mathbf{M}, \alpha)$, $\forall\, (O, \{\mathbf{e}_i\}_{1 \leq i \leq d}, \{\mathbf{e}_i\}_{i=1 \dots d}$, is basis of $\mathbf{M}$,

$$M \cong \mathbb{R}^4, \text{and } \exists \{x^\alpha : M = \mathbb{R}^4 \to \mathbb{R}\}_{\alpha = 1 \dots 4}$$

13.2. **Fundamental Theorems of (Classical) Dynamics.** cf. Sec. 3.4 - *Fundamental Theorems of Dynamics* of Prástaro (1996) [12].

cf. Thm. 3.20 of Prástaro (1996) [12]

Theorem 19 (Momentum Theorem)**.** *Variation of the free part of momentum of the observed motion of 1 body, in time interval* $\Delta t \equiv [0,t]$ *is equal to the corresponding impulse:*

$$I[0,t] \equiv \int_0^t F dt \equiv \left(\int_0^t F^j dt \right) \mathbf{e}_j \tag{57}$$

where $\{\mathbf{e}_j\}_{1\leq j \leq 3}$ *is a fixed basis of* $\mathbf{S}$

$$\overline{p}_\psi(t) - \overline{p}_\psi(0) = I[0,t] \tag{58}$$

Proof.

$$\overline{p}_\psi = \mu \ddot{m}_\psi = \overline{f}_\psi \Longrightarrow \dot{\overline{p}}_\psi = \dot{p}^j \mathbf{e}_j = F^j \mathbf{e}_j \Longrightarrow \int_{[0,t]} \dot{p}^j dt = \int_{[0,t]} F^j dt$$

□

14. Fluid Mechanics, Fluid Flow

14.1. **Mass Conservation for Fluid Flow, Continuum media.** The mass of fluid in some volume $V_0 \subset N$ is $\int_{V^0} \rho \text{vol}^n$, where ρ is fluid density, $\rho \in C^\infty(N)$.

The total mass of fluid flowing out of volume V_0 is

$$\frac{d}{dt}\int_{V_0} \rho \text{vol}^n = \int_{V_0} \mathcal{L}_{\frac{\partial}{\partial t} + \mathbf{u}}(\rho \text{vol}^n) = \int_{V_0} \frac{\partial}{\partial t} \rho \text{vol}^n + \int_{V_0} \mathcal{L}_u \rho \text{vol}^n$$

$$\int_{V_0} \mathcal{L}_u \rho \text{vol}^n = \int_{V_0} d i_{\mathbf{u}} \rho \text{vol}^n + i_{\mathbf{u}} d\rho \text{vol}^n = \int_{V_0} d i_{\mathbf{u}} \rho \text{vol}^n + 0 = \int_{V_0} d i_{\mathbf{u}} \rho \text{vol}^n = \int_{\partial V_0} i_{\mathbf{u}} \rho \text{vol}^n$$

Now

$$i_u \text{vol}^n = i_u \frac{\sqrt{g}}{n!} \epsilon_{i_1 \dots i_n} dx^{i_1} \wedge \dots \wedge dx^{i_n}$$

$$i_u dx^{i_1} \wedge \dots \wedge dx^{i_n} = u^{i_1} dx^{i_2} \wedge \dots \wedge dx^{i_n} - dx^{i_1} \wedge u^{i_2} dx^{i_3} \wedge \dots \wedge dx^{i_n} + \dots + (-1)^{n+1} dx^{i_1} \wedge \dots \wedge dx^{i_{n-1}} u^{i_n} = \epsilon^{i_1 \dots i_n}_{j_1 \dots j_n} u^{j_1} dx^{j_2} \wedge \dots \wedge dx^{j_n}$$

$$\Longrightarrow i_u \text{vol}^n = \frac{\sqrt{g}}{(n-1)!} \epsilon_{j_1 \dots j_n} u^{j_1} dx^{j_2} \wedge \dots \wedge dx^{j_n} \tag{59}$$

We can also rewrite Eq. 59 to be a "surface differential":

$$\begin{gathered} \int_{V_0} \mathbf{d} i_{\mathbf{u}} \rho \text{vol}^n = \int_{\partial V_0} i_{\mathbf{u}} \rho \text{vol}^n = \\ \int_{\partial V_0} \rho \frac{\sqrt{g}}{(n-1)!} u^{j_1} \epsilon_{j_1 j_2 \dots j_n} dx^{j_2} \wedge \dots \wedge dx^{j_n} \equiv \int_{\partial V_0} \rho \mathbf{u} \cdot d\mathbf{S} \equiv \int_{\partial V_0} \rho \langle \mathbf{u}, d\mathbf{S} \rangle \end{gathered} \tag{60}$$

If $\sqrt{g} = 1$, $n=2$,

$$i_u \text{vol}^2 = (u^1 dx^2 - u^2 dx^1) = u \cdot n_1 dx^2 + u \cdot n_2 dx^1 = u \cdot n dS$$

with $n_1 = e_1$ and $n_2 = -e_2$.

Now

$$d i_u \rho \text{vol}^n =$$

$$= \frac{\partial(\sqrt{g} \rho u^{j_1})}{\partial x^k} \frac{\epsilon_{j_1 \dots j_n}}{(n-1)!} dx^k \wedge dx^{j_2} \wedge \dots \wedge dx^{j_n} = \frac{\partial(\sqrt{g}\rho u^k)}{\partial x^k} \frac{\epsilon_{j_1 \dots j_n}}{n!} dx^{j_1} \wedge \dots \wedge dx^{j_n} = \frac{1}{\sqrt{g}} \frac{\partial (\sqrt{g} \rho u^k)}{\partial x^k} \text{vol}^n =$$

$$= \frac{\partial(\rho u^k)}{\partial x^k} \text{vol}^n + \rho u^k \frac{\partial \ln \sqrt{g}}{\partial x^k} \text{vol}^n = \text{div}(\rho u) \text{vol}^n + \rho u^k \frac{\partial \ln \sqrt{g}}{\partial x^k} \text{vol}^n$$

Now if $\sqrt{g} = 1$, then

$$\frac{d}{dt} \int_{V_0} \rho \text{vol}^n = \int_{V_0} \frac{\partial \rho}{\partial t} \text{vol}^n + \int_{V_0} d i_u \rho \text{vol}^n = \int_{V_0} \frac{\partial \rho}{\partial t} \text{vol}^n + \int_{V_0} \text{div}(\rho u) \text{vol}^n \Longrightarrow \frac{\partial \rho}{\partial t} + \text{div}(\rho u) = 0$$

which is the so-called mass continuity equation. $j = \rho u$ is the mass flux density.

Thus,

$$\begin{gathered} \text{(mass conservation)} \\ m = m(t) := \int_{V_0} \rho \text{vol}^n, \, V_0 \subset N \\ \dot{m} \equiv \frac{d}{dt} m(t) = \int_{V_0} \left(\frac{\partial \rho}{\partial t} \text{vol}^n + \mathbf{d} i_{\mathbf{u}} \rho \text{vol}^n \right) = \boxed{\int_{V_0} \frac{\partial \rho}{\partial t} \text{vol}^n + \int_{\partial V_0} \rho \mathbf{u} \cdot d\mathbf{S}} \\ \text{if } \sqrt{g} = 1, \text{ and } \dot{m} = 0, \text{ then} \\ \frac{\partial \rho}{\partial t} + \text{div}(\rho \mathbf{u}) = 0 \end{gathered} \tag{61}$$

TODO: 20190804 Frankel (2012) [22] in pp. 138 and onwards, for Sec. 4.3. Differentiation of Integrals posed the rightful question, "How does one compute the rate of change of an integral when the domain of integration is also changing?" Revisit the derivation from a Lie derivative and 1-parameter flow point of view.

Force should not be represented by a vector but rather by a 1-form. Then,

$$f \in \Omega^1(N), \, f = f(t, \mathbf{x}) = f_j dx^j \,\, j = 1, \dots n, \, \mathbf{x} = (x^1, \dots x^n), \, \text{dim} N = n$$

Indeed, the reason for f to be a 1-form is that we integrate differential forms, we don't integrate vectors.

$$W = \int_C f \qquad \text{(line integral)} \tag{62}$$

If f conservative, $\exists$ scalar $U = U(x)$; $x \in N$ s.t. $f = -dU$.

Let $\mathbf{u} = u^j(t, \mathbf{x}) \frac{\partial}{\partial x^j} \in \mathfrak{X}(N)$ generate flow ϕ_t.

The Lie derivative along vector field $\mathbf{u}$, $\mathcal{L}_{\mathbf{u}}$, can be calculated in at least 2 ways: from the definition:

$$\mathcal{L}_{\mathbf{u}} \omega = \frac{d}{dt} |_{t=0} \phi_t^* \omega$$

or Cartan's magic formula:

$$\begin{aligned} \mathcal{L}_{\mathbf{u}} \text{vol}^n &= (i_{\mathbf{u}} \mathbf{d} + \mathbf{d} i_{\mathbf{u}}) \text{vol}^n = 0 + \mathbf{d} i_{\mathbf{u}} \text{vol}^n = \mathbf{d} i_{\mathbf{u}} \left(\frac{\sqrt{g}}{n!} \epsilon_{i_1 \dots i_n} dx^{i_1} \wedge \dots \wedge dx^{i_n} \right) = \\ &= \mathbf{d} \left(\frac{\sqrt{g}}{(n-1)!} \epsilon_{j_1 \dots j_n} u^{j_1} dx^{j_2} \wedge \dots \wedge dx^{j_n} \right) = \frac{\partial(\sqrt{g} u^{j_1})}{\partial x^k} \frac{\epsilon_{j_1 \dots j_n}}{(n-1)!} dx^k \wedge dx^{j_2} \wedge \dots \wedge dx^{j_n} = \\ &= \frac{\partial(\sqrt{g} u^{j_1})}{\partial x^k} \frac{\epsilon_{j_1 \dots j_n}}{(n-1)!} \epsilon^{kj_2 \dots j_n}_{k_1, \dots, k_n} dx^{k_1} \wedge \dots \wedge dx^{k_n} \underbrace{\left(\frac{\sqrt{g}}{\sqrt{g}} \right)}_{=1} = \frac{1}{\sqrt{g}} \frac{\partial (\sqrt{g} u^k)}{\partial x^k} \text{vol}^n = \\ &= (\text{div} \mathbf{u}) \text{vol}^n \end{aligned} \tag{63}$$

cf. https://math.stackexchange.com/questions/2566381/lie-derivative-of-volume-form?rq=1

Note that

$$\text{div} : \mathfrak{X}(N) \to C^\infty(N)$$

$$\mathcal{L}_{\mathbf{u}} : \Omega^n(N) \to \Omega^n(N)$$

Frankel (2012) [22] on pp. lvii of the "Elasticity and Stresses" section offered this analogy: "While work in particle mechanics pairs a force covector (f_i) with a contravariant tangent vector (dx^i/dt) to a curve, work done by traction in elasticity pairs the contravariant stress force 2-form $\mathcal{S}$ with the covector valued deformation 1-form $\mathcal{E}$, to yield a scalar valued 3-form.

Instead of pushing along a curve (line) particle to do work, what does it mean to do work on a moving fluid element? One could possibly try to consider the deformation of a volume element under fluid flow. Frankel (2012) [22] in Sec. 4.2. The Lie Derivative of a Form, on pp. 132, asks "If a flow deforms some attribute, say volume, how does one measure the deformation? $\mathcal{L}_{\mathbf{u}}\text{vol}^n$ "is the n-form that reads off the rate of change of volume of a parallelopiped spanned by n vectors that are pushed forward by the flow ϕ_t." So, "in other words, $\mathcal{L}_{\mathbf{u}}\text{vol}^n$ *measures how volumes are changing under the flow* ϕ_t *generated by* $\mathbf{X}$" on pp. 133 on Frankel (2012) [22].

Then rate of work done on a fluid element could be the following:

$$W = \int_V \mathcal{L}_{\mathbf{u}}(p\text{vol}^n) = \int_V \mathbf{d}i_{\mathbf{u}}p\text{vol}^n = \int_{\partial V} i_{\mathbf{u}}p\text{vol}^n = \int_{\partial V} p\mathbf{u}\cdot d\mathbf{S}$$

TODO: work on a fluid element?

15. Thermodynamics

Let Σ be a (topological) manifold.
Suppose U is a global coordinate on Σ:

(First Law of Thermodynamics (energy conservation))

$$\boxed{dU = Q - W \text{ or } Q = dU + W} \tag{64}$$

where $dU, Q, W \in \Omega^1(\Sigma)$ (i.e. dU, Q, W are 1-forms over manifold Σ).

Consider a path in Σ, γ, $\gamma : \mathbb{R} \to \Sigma$,
and using a chart $(U, S^1, \dots, S^n)$ (e.g. $n=1$, $S^1 = v$ for volume)

$$\gamma(t) = (U(t), S^1(t), \dots, S^n(t))$$

$$\dot{\gamma} \in \mathfrak{X}(\Sigma),\ \dot{\gamma} = \dot{U}\frac{\partial}{\partial U} + \dot{S}^i \frac{\partial}{\partial S^i}$$

Now

$$dU(\dot{\gamma}) = \dot{\gamma}(U) = \dot{U}\frac{\partial}{\partial U}U + 0 = \dot{U}$$

$$Q(\dot{\gamma}) = Q(t)dt\left(\dot{\gamma}\frac{\partial}{\partial t}\right) = Q(t)\dot{\gamma}$$

$$\implies \dot{U} = Q(t)\dot{\gamma} - W(t)\dot{\gamma}$$

Recall that for enthalpy H, $H := U + pV$, $H = H(\sigma, p)$

TODO 20190804 Derive and check convection form of enthalpy against both Kittle and Kroemer plus thermodynamics and Sonntag, et. al.

Incomplete:

$$dH = Q + Vdp$$

$$W = pdV = pdV + Vdp - Vdp = d(pV) - Vdp =$$

$$dE = Q - W + \mu dH$$

$$dE = Q - W + \widehat{h}dN$$

$U \to E$ notation is to promote the internal energy U to include kinetic and potential energies, so that possibly, $E = U + \text{K.E.} + \text{P.E.}$, or, i.e., E includes internal energy and mechanical energy.

$G = \mu N$

$G = U - \tau\sigma + pV \xrightarrow{U\to E} G = E - \tau\sigma + pV \implies \tau\check{h}$

$$dE = Q + W + \mu dN = Q + W + (\check{h} - \tau\check{\sigma})dN$$

$$Q = \tau d\sigma = \tau d(N\check{\sigma}) = \tau N d\check{\sigma} + \tau\check{\sigma}dN$$

$\tau N d\check{\sigma}$ is the entropy change due to change in entropy per particle; i.e. **conduction term**
$\tau\check{\sigma}dN$ is entropy change due to change in number of particles, i.e. **convection term**

$$dE = Q + W + \check{h}dN - \tau\check{\sigma}dN = \tau N d\check{\sigma} + \tau\check{\sigma}dN + W + \check{h}dN - \tau\check{\sigma}dN = \tau N d\check{\sigma} + W + \check{h}dN$$

$m(t) = MN$

Assume only 1 chemical species:

$$\check{Q} := \tau N d\check{\sigma} \qquad \widehat{h} := \frac{H}{m} = \frac{H}{MN}$$

$$\implies dE = \check{Q} + W + \check{h}dN\frac{M}{M} = \check{Q} + W + \widehat{h}dm$$

The Gibbs free energy equilibrium is given by,

$$dG = \mu dN - \sigma d\tau + Vdp$$

For a throttling valve (don't all valves throttle?) the pressure drop is accounted for by the Gibbs free energy, *not* by an isenthalpy condition.

Part 14. General Relativity

Part 15. WE Heraeus International Winter School on Gravity and Light

Introduction (from EY)

The International Winter School on Gravity and Light held *central lectures* given by Dr. Frederic P. Schuller. These lectures on General Relativity and Gravity are unequivocally and undeniably, the best and most lucid and well-constructed lecture series on General Relativity and Gravity. The mathematical foundation from topology and differential geometry from which General Relativity arises from is solid, well-selected in rigor. The lectures themselves are well-thought out and clearly explained.

Even more so, the International Winter School provided accompanying Tutorial Sessions for each of the lectures. I had given up hopes in seeing this component of the learning process ever be put online so that anyone and everyone in the world could learn through the Tutorial process as well. I was afraid that nobody would understand how the Tutorial or "Office Hours" session was important for students to digest and comprehend and work out-doing exercises-the material presented in the lectures. This International Winter School gets it and shows how online education has to be done, to do it in an excellent manner, moving forward.

For anyone who is serious about learning General Relativity and Gravity, I would simply point to these video lectures and tutorials.

What I want to do is to build upon the material presented in this International Winter School. Why it's important to me, and to the students and practicing researchers out there, is that the material presented takes the student from an introduction to the research frontier. That is the stated goal of the International Winter School. I want to dig into and help contribute to the cutting edge in research and this entire program with lectures and tutorials appears to be the most direct and sensible route directly to being able to do research in General Relativity and Gravity. -EY 20150323

16. Lecture 1: Topology

16.1. Lecture 1: Topological Spaces.

Definition 47. *Let M be a set.*
*A **topology** $\mathcal{O}$ is a subset $\mathcal{O} \subseteq \mathcal{P}(M)$, $\mathcal{P}(M)$ power set of M: set of all subsets of M. satisfying*

(i) $\emptyset \in \mathcal{O}$, $M \in \mathcal{O}$
(ii) $U \in \mathcal{O}, \quad V \in \mathcal{O} \Longrightarrow U \bigcap V \in \mathcal{O}$
(iii) $U_\alpha \in \mathcal{O}, \quad \alpha \in \mathcal{A} \quad \Longrightarrow \left(\bigcup_{\alpha \in \mathcal{A}} U_\alpha \right) \in \mathcal{O}$

$\mathcal{O}\}$ utterly useless

Definition 48. $\mathcal{O}_{standard} \subseteq \mathcal{P}(\mathbb{R}^d)$

EY : 20150524
I'll fill in the proof that $\mathcal{O}_{\text{standard}}$ is a topology.

Proof. $\emptyset \in \mathcal{O}_{\text{standard}}$
since $\forall\, p \in \emptyset$, $\exists\, r \in \mathbb{R}^+$: $\mathcal{B}_r(p) \subseteq \emptyset$ (i.e. satisfied "vacuously")

Suppose $U, V \in \mathcal{O}_{\text{standard}}$.

Let $p \in U \bigcap V$. Then $\exists\, r_1, r_2 \in \mathbb{R}^+$ s.t. $\mathcal{B}_{r_1}(p) \subseteq U$
$\mathcal{B}_{r_2}(p) \subseteq V$

Let $r = \min\{r_1, r_2\}$.
Clearly $\mathcal{B}_r(p) \subseteq U$ and $\mathcal{B}_r(p) \subseteq V$. Then $\mathcal{B}_r(p) \subseteq U \bigcap V$. So $U \bigcap V \in \mathcal{O}_{\text{standard}}$.

Suppose, $U_\alpha \in \mathcal{O}_{\text{standard}}$, $\forall\, \alpha \in \mathcal{A}$.
Let $p \in \bigcup_{\alpha \in \mathcal{A}} U_\alpha$. Then $p \in U_\alpha$ for at least 1 $\alpha \in \mathcal{A}$.
$\exists\, r_\alpha \in \mathbb{R}^+$ s.t. $\mathcal{B}_{r_\alpha}(p) \subseteq U_\alpha \subseteq \bigcup_{\alpha \in \mathcal{A}} U_\alpha$. So $\bigcup_{\alpha \in \mathcal{A}} U_\alpha \in \mathcal{O}_{\text{standard}}$ □

16.2. 2. Continuous maps.

16.3. 3. Composition of continuous maps.

16.4. 4. Inheriting a topology.

EY : 20150524
I'll fill in the proof that given f continuous (cont.), then the restriction of f onto a subspace S is cont. If you want a reference, check out Klaus Jänich [?, pp. 13, Ch. 1 Fundamental Concepts, Sec. Continuous Maps]
If cont. $f : M \to N$, $S \subseteq M$, then $f|_S$ cont.

Proof. Let open $V \subseteq N$, i.e. $V \in \mathcal{O}_N$ i.e. V in the topology $\mathcal{O}_N$ of N.

$$f|_S^{-1}(V) = \{ m \in M | \, f|_S(m) \in V \}$$

Now $f^{-1}(V) = \{ m \in M | f(m) \in V \}$.
So $f^{-1}(V) \bigcap S = f|_S^{-1}(V)$
Now f cont. So $f^{-1}(V) \in \mathcal{O}_N$.
and recall $\mathcal{O}_S| := \{ U \bigcap S | U \in \mathcal{O}_M \}$.
so $f^{-1}(V) \bigcap S = f|_S^{-1}(V) \in \mathcal{O}_S$ i.e. $f|_S^{-1}(V)$ open.
$\Longrightarrow f|_S$ cont. □

Topology Tutorial Sheet

filename : `main.pdf`
The WE-Heraeus International Winter School on Gravity and Light: Topology

EY : 20150524
What I won't do here is retype up the solutions presented in the Tutorial (cf. `https://youtu.be/_XkhZQ-hNLs`): the presenter did a very good job. If someone wants to type up the solutions and copy and paste it onto this LaTeX file, in the spirit of open-source collaboration, I would encourage this effort.
Instead, what I want to encourage is the use of as much CAS (Computer Algebra System) and symbolic and numerical computation because, first, we're in the 21st century, second, to set the stage for further applications in research. I use Python and Sage Math alot, mostly because they are open-source software (OSS) and fun to use. Also note that the structure of Sage Math modules matches closely to Category Theory.
In checking whether a set is a topology, I found it strange that there wasn't already a function in Sage Math to check each of the axioms. So I wrote my own; see my code snippet, which you can copy, paste, edit freely in the spirit of OSS here, titled `topology.sage`:
gist github ernestyalumni topology.sage
Download topology.sage
Loading `topology.sage`, after changing into (with the usual Linux terminal commands, cd, ls) by

```
sage: load(''topology.sage'')
```

Exercise 2: Topologies on a simple set.

Question Does $\mathcal{O}_1 := \dots$ constitute a topology ...?.

Solution: Yes, since we check by typing in the following commands in Sage Math:

```
emptyset in O_1
Axiom2check(O_1) # True
Axiom3check(O_1) # True
```

Question What about $\mathcal{O}_2$ ...?.

Solution: No since the 3rd. axiom fails, as can be checked by typing in the following commands in Sage Math:

```
emptyset in O_2
Axiom2check(O_2) # True
Axiom3check(O_2) # False
```

17. Lecture 2: Topological Manifolds

Lecture 2: Manifolds. Topological spaces: $\exists$ so man that mathematicians cannot even classify them.
For spacetime physics, we may focus on topological spaces $(M, \mathcal{O})$ that can be <u>charted</u>, analogously to how the surface of the earth is charted in an <u>atlas</u>.

17.1. Topological manifolds.

Definition 49. *A topological space $(M, \mathcal{O})$ is called a d-**dimensional topological method** if*
$\forall\, p \in M : \exists\, U \in \mathcal{O}, U \ni p : \exists\, x : U \subseteq M \to x(U) \subseteq \mathbb{R}^d \qquad (M, \mathcal{O}), (\mathbb{R}^d, \mathcal{O}_{std})$

(i) x ***invertible** :*

$$x^{-1} : x(U) \to U$$

(ii) x ***continuous***
(iii) x^{-1} ***continuous***

17.2. **Terminology.**

17.3. **3. Chart transition maps.** Imagine 2 charts (U, x) and (V, y) with overlapping regions.

17.4. **4. Manifold philosophy.** Often it is desirable (or indeed the way) to define properties ("continuity") of real-world object ("$\mathbb{R} \xrightarrow{\gamma} M$") by judging suitable coordinates not on the "real-world" object itself, but on a chart-representation of that real world object.

EY's add-ons. This lecture gives me a good excuse to review Topology and Topological Manifolds from a mathematician's point of view. I find John M. Lee's **Introduction to Topological Manifolds** book good because it's elementary and thorough and it's fairly recent (2010) so it's up to date [?]. See my notes and solutions for the book; it's a file titled `LeeJM_IntroTopManifolds_sol.pdf` of which I'll try to keep the pdf and LaTeX file available for download on my ernestyalumni Google Drive (so try to search for it on Google).

TUTORIAL TOPOLOGICAL MANIFOLDS

filename: `Sheet_1.2.pdf`

Exercise 4: Before the invention of the wheel.
Another one-dimensional topological manifold. Another one?
Consider set $F^1 := \{(m,n) \in \mathbb{R}^2 | m^4 + n^4 = 1\}$, equipped with subset topology $\mathcal{O}_{\text{std}}|_{F^1}$

Question $x : F^1 \to \mathbb{R}$ **is what?.**

Solution . EY : 20150525 The tutorial video https://youtu.be/ghfEQ3u_B6g is really good and this solution is how I'd write it, but it's really the same (I needed the practice).

$$\boxed{\begin{aligned} x : F^1 &\to \mathbb{R} \\ (m,n) &\mapsto m \end{aligned}}$$

If $m = 0$, $n^4 = 1$ so $n = \pm 1$ so it's not injective.
Let the closed n-dim. upper half-space $\mathbb{H}^n \subseteq \mathbb{R}^1$. Then

$$\begin{aligned} \mathbb{H}^n &= \{(x_1 \dots x_n) \in \mathbb{R}^n | x_n \geq 0\} \\ \text{int}\mathbb{H}^n &= \{(x_1 \dots x_n) \in \mathbb{R}^n | x_n > 0\} \\ -\mathbb{H}^n &= \{(x_1 \dots x_n) \in \mathbb{R}^n | x_n \leq 0\} \\ -\text{int}\mathbb{H}^n &= \{(x_1 \dots x_n) \in \mathbb{R}^n | x_n < 0\} \end{aligned}$$

Question This map x may be made injective by restricting its domain to either of 2 maximal open subsets of F^1. Which ones?.

Solution .
Let

$$\begin{aligned} U_+ &= F^1 \cap \text{int}\mathbb{H}^2 \\ U_- &= F^1 \cap -\text{int}\mathbb{H}^2 \end{aligned}$$

Look at

$$\begin{aligned} x^4 &= 1 - n^4 \\ \Longrightarrow x &= \pm (1 - n^4)^{1/4} \end{aligned}$$

Then for

$$\begin{aligned} x_+^{-1} : (-1,1) \subseteq \mathbb{R} &\to U_+ \\ m &\mapsto (m, (1-m^4)^{1/4}) \\ x_-^{-1} : (-1,1) \subseteq \mathbb{R} &\to U_- \\ m &\mapsto (m, -(1-m^4)^{1/4}) \end{aligned}$$

x_+,x_- injective (since left inverse exists).

Question Construct injective y.

Solution .
Let

$$\begin{aligned} V_+ &= F^1 \cap \text{int}\mathbb{H}^1 \\ V_- &= F^1 \cap -\text{int}\mathbb{H}^1 \end{aligned}$$

Then

$$\begin{aligned} y_+ : V_+ &\to (-1,1) \subseteq \mathbb{R} \\ (m,n) &\mapsto n \\ y_- : V_- &\to (-1,1) \subseteq \mathbb{R} \\ (m,n) &\mapsto n \end{aligned}$$

Question Construct inverse y^{-1}. Solution .

For

$$\begin{aligned} y_+^{-1} : (-1,1) \subseteq \mathbb{R} &\to V_+ \\ n &\mapsto ((1-n^4)^{1/4}, n) \\ y_-^{-1} : (-1,1) \subseteq \mathbb{R} &\to V_- \\ n &\mapsto (-(1-n^4)^{1/4}, n) \end{aligned}$$

y_+,y_- injective (since left inverse exists).

Note $(-1,0) \notin U_+, U_-$
$(1,0) \notin U_+, U_-$
and

$(0,1) \notin V_+, V_-$
$(0,-1) \notin V_+, V_-$

Question construct *transition map* $x \circ y^{-1}$.

Solution .

$$\begin{aligned} x_+y_+^{-1} : (0,1) \subseteq \mathbb{R} &\to (0,1) \subseteq \mathbb{R} \\ n &\mapsto (1-n^4)^{1/4} \\ x_-y_+^{-1} : (-1,0) \subseteq \mathbb{R} &\to (0,1) \subseteq \mathbb{R} \\ n &\xrightarrow{y_+^{-1}} ((1-n^4)^{1/4}, n) \xrightarrow{x_-} (1-n^4)^{1/4} \\ x_+y_-^{-1} : (0,1) \subseteq \mathbb{R} &\to (-1,0) \subseteq \mathbb{R} \\ n &\mapsto -(1-n^4)^{1/4} \\ x_-y_-^{-1} : (-1,0) \subseteq \mathbb{R} &\to (-1,0) \subseteq \mathbb{R} \\ n &\mapsto -(1-n^4)^{1/4} \end{aligned}$$

Question . . . Does the collection of these domains and maps form an atlas of F^1?.

Yes, with atlas

$$\mathcal{A} = \{ \begin{matrix} (U_+, x_+) \\ (U_-, x_-) \end{matrix} , \begin{matrix} (V_+, y_+) \\ (V_-, y_-) \end{matrix} \}$$

Clearly

$$\begin{gathered} U_+ \cup U_- \cup V_+ \cup V_- = (F^1 \cap \text{int}\mathbb{H}^2) \cup (F^1 \cap -\text{int}\mathbb{H}^2) \cup (F^1 \cap \text{int}\mathbb{H}^1) \cup (F^1 \cap -\text{int}\mathbb{H}^1) = \\ = F^1 \cap \mathbb{R}^2 \backslash \{(0,0)\} = F^1 \end{gathered}$$

and (the point is that) $x_\pm, y_\pm$ are homeomorphisms of open sets of F^1 onto open sets of 1 dim. $\mathbb{R}^1$ (namely $(-1,1) \subseteq \mathbb{R}^1$), and so $\mathcal{A}$ is an atlas of F^1.

18. Lecture 3: Multilinear Algebra

Lecture 3: Multilinear Algebra (International Winter School on Gravity and Light 2015)

We will **not** equip space(time) with a vector space structure. Do you know where

$$\begin{matrix} 5 \cdot \text{ Paris } =? \\ \text{lie ?} \end{matrix} \qquad \text{Paris } + \text{ Vienna } =?$$

Moreover, the tangent spaces T_pM (lecture 5) smooth manifolds (Lecture 4)

Beneficial to first study vector spaces abstractly for two reason

(i) for construction of T_pM one needs an intermediate vector space $C^\infty(M)$
(ii) tensor technique are most easily understood in an abstract setting.

18.1. Vector spaces.

Definition 50. *A vector space $(V,+,-)$ is*

(i) *a set V*
(ii) $+ : V \times V \to V$ *"addition"*
(iii) $\cdot : \mathbb{R} \times V \to V$ *"s-multiplication" EY : 20160317 s for "scalar"*

satisfying:

CANIADDU

$$\begin{aligned} C^+ : &\quad v + w = w + v \\ A^+ : &\quad (u+v)+w = u+(v+w) \\ N^+ : &\quad \exists\, 0 \in V : \forall\, v \in V : v + 0 = v \\ I^+ : &\quad \forall\, v \in V : \exists\, (-v) \in V : v + (-v) = 0 \\ A : &\quad \lambda \cdot (\mu + v) = (\lambda \cdot \mu) \cdot v \qquad\qquad (\forall\, \lambda, \mu \in \mathbb{R}) \\ D : &\quad (\lambda + \mu) \cdot v = \lambda \cdot v + \mu \cdot v \\ D : &\quad \lambda \cdot v + \lambda \cdot w = \lambda \cdot (v + w) \\ U : &\quad 1 \cdot v = v \end{aligned}$$

Terminology. An element of a vector space is often referred to, informally as a vector.

Example. def. **set** of polynomials (fixed) degree $\mathcal{P} := \{ p : (-1,+1) \to \mathbb{R} | p(x) = \sum_{n=0}^N p_n \cdot x^n \}$

Thought bubble: is $\square$ a vector?

$$\square(x) = x^2$$

No $\square \in \mathcal{P}$.

$$\begin{aligned} + : &\mathcal{P} \times \mathcal{P} \to \mathcal{P} \\ &(p,q) \mapsto p + q \end{aligned}$$

where $(p+q)(x) = p(x) +_{\mathbb{R}} q(x)$

$$\begin{aligned} \cdot : &\mathbb{R} \times \mathcal{P} \to \mathcal{P} \\ &(\lambda, p) \mapsto \lambda \cdot p \end{aligned}$$

where $(\lambda \cdot p)(x) := \lambda \cdot_{\mathbb{R}} p(x)$

Thought bubble: $\square$ a vector? Yes, but who cares?

$(\mathcal{P}, +, \cdot)$ is a vector space.

$\square \in \mathcal{P}$

18.2. Linear maps.

These are the structure-respecting maps between vector spaces.

EY : 20160316 out of tradition, they're called "linear" maps

Definition 51. *$(V, +_V, \cdot_V)$ and $(W, +_W, \cdot_W)$ vector spaces*
Then a map

$$\varphi : V \to W$$

*is called **linear** if*

(i) $\varphi(v +_V \widetilde{v}) = \varphi(v) +_W \varphi(\widetilde{v})$
(ii) $\varphi(\lambda \cdot_V v) = \lambda \cdot_W \varphi(v)$

Example. :
$$\begin{aligned} \delta : &\mathcal{P} \to \mathcal{P} \\ &p \mapsto \delta(p) := p' \end{aligned}$$

<u>linear</u>:

(i) $\delta(p+q) = (p +_{\mathcal{P}} q)' \underbrace{=}_{\text{sum rule}} p' +_{\mathcal{P}} q' = \delta(p) +_{\mathcal{P}} \delta(q)$
(ii) $\delta(\lambda p) = (\lambda p)' = \lambda \cdot p' = \lambda \cdot \delta(p)$

<u>Notation</u>: $\varphi : V \to W$ linear $\Longleftrightarrow: \varphi : V \xrightarrow{\sim} W$

$$\begin{tikzcd} V \arrow[r, "\psi", "\sim"'] \arrow[rr, bend right, "\sim", "\varphi \circ \psi"'] & W \arrow[r, "\varphi", "\sim"'] & U \end{tikzcd}$$

18.2.1. *Example*.* $\delta \circ \delta : \mathcal{P} \xrightarrow{\sim} \mathcal{P}$

18.3. **Vector space of Homomorphisms.** fun fact: $(V,+,\cdot)$ $(W,+,\cdot)$ vector spaces
def. $\mathrm{Hom}(V,W) := \{\varphi : V \xrightarrow{\sim} W\}$ set.
We can make this into a vector spaces.

$$\oplus : \mathrm{Hom}(V,W) \times \mathrm{Hom}(V,W) \to \mathrm{Hom}(V,W)$$
$$(\varphi,\psi) \mapsto \varphi \oplus \psi$$

where $(\varphi \otimes \psi)(v) := \varphi(v) +_W \psi(v)$
$\otimes$: ... similarly.
$(\mathrm{Hom}(V,W), \oplus, \otimes)$ **is** a vector space.

18.3.1. *Example*.* $\mathrm{Hom}(\mathcal{P},\mathcal{P})$ is a vector space.
$\delta \in \mathrm{Hom}(\mathcal{P},\mathcal{P})$
$\delta \circ \delta \in \mathrm{Hom}(\mathcal{P},\mathcal{P})$
$\vdots$
$\underbrace{\delta \circ \cdots \circ \delta}_{M} \in \mathrm{Hom}(\mathcal{P},\mathcal{P})$

$$\implies 5 \circ \delta \oplus_{\mathrm{Hom}(\mathcal{P},\mathcal{P})} \delta \circ \delta \in \mathrm{Hom}(\mathcal{P},\mathcal{P})$$

18.4. **Dual vector space.** heavily used special case:
$(V,+,\cdot)$ vector space:

Definition 52.

$$V^* := \{\varphi : V \xrightarrow{\sim} \mathbb{R}\} = Hom(V,\mathbb{R})$$

$\underbrace{(V^*, \oplus, \otimes)}_{\text{dual vector space (to } V)}$ *is a vector space*

<u>Terminology</u>: $\varphi \in V^*$ is called, informally, a covector.

Example. $I : \mathcal{P} \xrightarrow{\sim} \mathbb{R}$
i.e. $I \in \mathcal{P}^*$
<u>def</u>. $I(p) := \int_0^1 dx p(x)$

linear: $I(p+q) = \int_0^1 dx \underbrace{(p+q)(x)}_{p(x)+q(x)}$
$= \cdots = I(q) + I(p)$

$I(\lambda p) = \lambda \cdot I(p)$
i.e. $I = \int_0^1 dx$

18.5. **Tensors.**

Definition 53. *Let $(V,+,\cdot)$ be a vector space.*
An <u>(r,s)-tensor T over V</u> $r, s \in \mathbb{N}_0$
<u>is a multi-linera map</u>

$$T : \underbrace{V^* \times \cdots \times V^*}_{r} \times \underbrace{V \times \cdots \times V}_{s} \xrightarrow{\sim \vdots\}r+s \, \sim} \mathbb{R}$$

18.5.1. *Example.* : T $(1,1)$-tensor

$$T(\varphi+\psi, v) = T(\varphi,v) + T(\psi,v) \qquad T(\varphi, v+w) = T(\varphi,v) + T(\varphi,w)$$
$$T(\lambda\varphi, v) = \lambda \cdot T(\varphi,v) \qquad T(\varphi, \lambda \cdot v) = \lambda T(\varphi,v)$$
$$T(\varphi+\psi, v+w) =$$
$$= t(\varphi,v) + T(\varphi,w) + T(\psi,v) + t(\psi,w)$$

<u>Excursion</u>: Given $T : V^* \times V \xrightarrow{\sim} \mathbb{R}$
$\phi_T :$ $V \xrightarrow{\sim} (V*)^* \underbrace{=}_{\dim V<\infty} V$
Define $v \mapsto \underbrace{T(\cdot,v)}_{V^* \xrightarrow{\sim} \mathbb{R}}$
Given $\phi : V \xrightarrow{\sim} V$

Construct $T_\phi : V^* \times V \xrightarrow{\sim} \mathbb{R}$
$(\varphi, v) \mapsto \varphi(\phi(v))$
$\implies$ given $T : T = T_{\varphi_T}$
given $\phi : \phi = \phi_{T_\phi}$

Example. $g : P \times P \xrightarrow{\sim} \mathbb{R}$
$(p,q) \mapsto \int_{-1}^1 dx p(x) q(x)$
is a $(0,2)$-tensor over P.
<u>Info</u>: If $T \in \mathrm{Hom}(V,W)$

18.6. **Vectors and covectors as tensors.**

Theorem 20. *(including proof)*
"covector" $\varphi \in V^* \iff \varphi : V \xrightarrow{\sim} \mathbb{R} \iff \varphi (0,1)$-tensor.

Theorem 21. *$v \in V \underbrace{=}_{\dim V<\infty} (V^*)^* \iff v : V^* \xrightarrow{\sim} \mathbb{R} \iff v$ is $(1,0)$-tensor.*

18.7. **Bases.**

Definition 54. *$(V,+,\cdot)$ vector space.*
A subset $B \subset V$ is called
a basis if
Thought bubble: Hamel (L.A.) EY : 20160316 Hamel basis, Linear Algebra

$$\forall v \in V \quad \exists \ \underline{\text{finite}} \ \underbrace{F}_{\{f_1,\ldots,f_n\}} \subset B : \exists! \underbrace{v^1, v^2, \ldots, v^n}_{\in\mathbb{R}}, \qquad v = v^1 f_1 + \cdots + v^n f_n$$

Definition 55. *If $\exists$ basis $\mathcal{B}$ with finitely many elements, say d many, then we call $d =: dimV$*
This is <u>well-defined</u>.

<u>Remark</u>: $(V,+,\cdot)$ be a finite-dim. vector space.
Having chosen a basis $e_1, \ldots, e_n$ of $(V,+,\cdot)$ we may uniquely associate
(Thought bubble: this requires a chosen basis)

$$v \mapsto (v^1, \ldots, v^n) \text{ called the components of } v \text{ w.r.t. chosen basis}$$

where: $v^1 e_1 + \cdots + v^n e + n = v$

18.8. **Basis for the dual space.** choose Basis $e_1, \dots, e_n$ for V
can choose Basis $\epsilon^1, \dots, \epsilon^n$ for V^*

However, more economical to require
once $e_1, \dots, e_n$ on V has been chosen, that

$$\epsilon^a(e_b) = \delta^a_b$$

This uniquely determines choice of $\epsilon^1, \dots, \epsilon^n$ from choice of $e_1, \dots, e_n$

Definition 56. *If a basis $\epsilon^1, \dots, \epsilon^n$ of V^* satisfies this, it is called the* ***dual basis*** *(of the dual space)*

Example: P $(N = 3)$

$$e_0, e_1, e_2, e_3 \text{ basis if } \begin{aligned} e_0(x) &= 1 \\ e_1(x) &= x \\ e_2(x) &= x^2 \\ e_3(x) &= x^3 \end{aligned} \{ e_a(x) := x^a$$

$\epsilon^0, \epsilon^1, \epsilon^2, \epsilon^3$ dual basis $\epsilon^a := \frac{1}{a!} \partial^a|_{x=0}$

Proof. $\epsilon^a(e_b) = \delta^a_b$ □

18.9. **Components of tensors.** Let T be an (r, s)-tensor on a finite-dim. vs. V. Then define the $(r + s)^{\dim V}$ many real numbers.

$$\underbrace{T^{i_1 \dots i_r}{}_{j_1 \dots j_s}}_{\in \mathbb{R}} := T(\epsilon^{i_1}, \dots, \epsilon^{i_r}, e_{j_1}, e_{j_2}, \dots, e_{j_s})$$

$i_1 \dots i_r, j_1 \dots j_s \in \{1, \dots, \dim V\}$

Thought bubble: $\underbrace{T^{i_1 \dots i_r}{}_{j_1 \dots j_s}}_{\in \mathbb{R}}$ are the components of the tensor w.r.t. chosen basis

Useful: Knowing components (and basis) one can reconstruct the entire tensor.

Example. T $(1, 1)$- tensor

$$T^i{}_j := T(\epsilon^i, e_j)$$

reconstruct

$$\begin{aligned} T(\varphi, v) &= T(\sum_{i=1}^{\dim V} \varphi_i \epsilon^i, \sum_{j=1}^{\dim V} v^j e_j) \qquad \varphi_i \in \mathbb{R}, v^j \in \mathbb{R} \\ &= \sum_{i=1}^{\dim V} \sum_{j=1}^{\dim V} \varphi_i v^j \underbrace{T(\epsilon^i, e_j)}_{T^i{}_j} \\ &=: \varphi_i v^j T^i{}_j \end{aligned}$$

19. Lecture 4: Differentiable Manifolds

so far: top. mfd. $(M, \mathcal{O})$
$\dim M = d$

we wish to define a notion of differentiable
curves $\mathbb{R} \to M$
function $M \to \mathbb{R}$
maps $M \to N$

19.1. **1. Strategy.** choose a chart (U, x)

$\gamma : \mathbb{R} \to M$ portion of curve in chart domain

$$\begin{tikzcd} \gamma : \mathbb{R} \arrow[r] \arrow[dr, "x \circ \gamma"'] & U \arrow[d, "x"] \\ & x(U) \subseteq \mathbb{R}^d \end{tikzcd}$$

idea. try to "lift" the undergraduate notion of differentiability of a curve on $\mathbb{R}^d$ to a notion of differentiability of a curve on M

Problem Can this be well-defined under change of chart?

$$\begin{tikzcd} & y(U \cap V) \subseteq \mathbb{R}^d \\ \gamma : \mathbb{R} \arrow[ur, "y \circ \gamma"] \arrow[r] \arrow[dr, "x \circ \gamma"'] & U \cap V \neq \emptyset \arrow[u, "y"'] \arrow[d, "x"] \\ & x(U \cap V) \subseteq \mathbb{R}^d \arrow[uu, bend right, "y \circ x^{-1}"'] \end{tikzcd}$$

$x \circ \gamma$ undergraduate differentiable ("as a map $\mathbb{R} \to \mathbb{R}^d$")

$$\underbrace{y \circ \gamma}_{\text{maybe only continuous, but not undergraduate differentiable}} = \underbrace{\overbrace{(y \circ x^{-1})}^{\mathbb{R}^d \to \mathbb{R}^d}}_{\text{continuous}} \circ \underbrace{\overbrace{(x \circ \gamma)}^{\mathbb{R} \to \mathbb{R}^d}}_{\text{undergrad differentiable}} = y \circ (x^{-1} \circ x) \circ \gamma$$

At first sight, strategy does not work out.

19.2. **2. Compatible charts.** In section 1, we used any imaginable charts on the top. mfd. $(M, \mathcal{O})$.

To emphasize this, we may say that we took U and V from the *maximal atlas* $\mathcal{A}$ of $(M, \mathcal{O})$.

Definition 57. *Two charts (U, x) and (V, y) of a top. mfd. are called ❀-compatible if either*

(a) $U \cap V = \emptyset$ *or*
(b) $U \cap V \neq \emptyset$

chart transition maps have undergraduate ❀ property.

EY : 20151109 e.g. since $\mathbb{R}^d \to \mathbb{R}^d$, can use undergradate ❀ property such as continuity or differentiability.

$$\begin{aligned} y \circ x^{-1} &: x(U \cap V) \subseteq \mathbb{R}^d \to y(U \cap V) \subseteq \mathbb{R}^d \\ x \circ y^{-1} &: y(U \cap V) \subseteq \mathbb{R}^d \to x(U \cap V) \subseteq \mathbb{R}^d \end{aligned}$$

Philosophy:

Definition 58. *An atlas $\mathcal{A}_{❀}$ is a ❀-compatible atlas if any two charts in $\mathcal{A}_{❀}$ are ❀-compatible.*

Definition 59. *A* ***❀-manifold*** *is a triple* $(\underbrace{M, \mathcal{O}}_{\text{top. mfd.}}, \mathcal{A}_{❀}) \qquad \mathcal{A}_{❀} \subseteq \mathcal{A}_{maximal}$

❀	undergraduate ❀	
C^0	$C^0(\mathbb{R}^d \to \mathbb{R}^d) =$	continuous maps w.r.t. $\mathcal{O}$
C^1	$C^1(\mathbb{R}^d \to \mathbb{R}^d) =$	differentiable (once) and is continuous
C^k		k-times continuously differentiable
D^k		k-times differentiable
$\vdots$		
C^∞	$C^\infty(\mathbb{R}^d \to \mathbb{R}^d)$	
$\cup\!\vert$		
C^ω	$\exists$ multi-dim. Taylor exp.	
$\mathbb{C}^\infty$	satisfy Cauchy-Riemann equations, pair-wise	

EY : 20151109 Schuller says: C^k is easy to work with because you can judge k-times cont. differentiability from existence of all partial derivatives **and** their continuity. There are examples of maps that partial derivatives exist but are not D^k, k-times differentiable.

Theorem 22 (Whitney). *Any $C^{k\geq 1}$-atlas, $\mathcal{A}_{C^{k\geq 1}}$ of a topological manifold* contains *a C^∞-atlas.*

Thus we may w.l.o.g. always consider C^∞-manifolds, "smooth manifolds", unless we wish to define Taylor expandibility/-complex differentiability ...

EY : 20151109 Hassler Whitney [5]

Definition 60. *A smooth manifold* $(\underbrace{M,\mathcal{O}}_{top.\ mfd.}, \underbrace{\mathcal{A}}_{C^\infty-atlas})$

$$\begin{tikzcd} \mathbb{R} \arrow[r, "\gamma"] \arrow[dr, "x\circ\gamma"'] & M \arrow[d, "x"] \\ & \mathbb{R}^d \end{tikzcd}$$

EY: 20151109 Schuller was explaining that the trajectory is real in M; the coordinate maps to obtain coordinates is $x\circ\gamma$

19.3. **4. Diffeomorphisms.** $M \xrightarrow{\phi} N$

If M, N are naked sets, the structure preserving maps are the bijections (invertible maps).

e.g. $\{1,2,3\} \to \{a,b\}$

Definition 61. $M \cong_{set} N$ *(set-theoretically) isomorphic if $\exists$ bijection $\phi : M \to N$*

Examples. $\mathbb{N} \cong_{\text{set}} \mathbb{Z}$

$\mathbb{N} \cong_{\text{set}} \mathbb{Q}$ (EY: 20151109 Schuller says from diagonal counting)

$\mathbb{N} \not\cong_{\text{set}} \mathbb{R}$

Now $(M,\mathcal{O}_M) \cong_{\text{top}} (N,\mathcal{O}_N)$ (topl.) isomorphic = "homeomorphic" $\exists$ bijection $\phi : M \to N$

ϕ, ϕ^{-1} are continuous.

$(V,+,\cdot) \cong_{\text{vec}} (W,+_w,\cdot_w)$ (EY: 20151109 vector space isomorphism) if $\exists$ bijection $\phi : V \to W$ linearly

finally

Definition 62. *Two C^∞-manifolds $(M,\mathcal{O}_M,\mathcal{A}_M)$ and $(N,\mathcal{O}_N,\mathcal{A}_N)$ are said to be* ***diffeomorphic*** *if $\exists$ bijection $\phi : M \to N$ s.t.*

$$\phi : M \to N$$
$$\phi^{-1} : N \to M$$

are both C^∞-maps

$$\begin{tikzcd} \mathbb{R}^d \arrow[rr, "\widetilde{y}\circ\phi\circ\widetilde{x}^{-1}"] & & \mathbb{R}^e \\ M \supseteq U \arrow[u, "\widetilde{x}"] \arrow[rr, "\phi"] \arrow[d, "x"'] & & V \subseteq N \arrow[u, "\widetilde{y}"'] \arrow[d, "y"] \\ \mathbb{R}^d \arrow[rr, "y\circ\phi\circ x^{-1}", "undergraduate\ C^\infty"'] \arrow[uu, bend left=60, "C^\infty"] & & \mathbb{R}^e \arrow[uu, bend right=60] \end{tikzcd}$$

Theorem 23. *$\#$ = number of C^∞-manifolds one can make out of a given C^0-manifolds (if any) - up to diffeomorphisms.*

dimM	*#*	
1	*1*	*Morse-Radon theorems*
2	*1*	*Morse-Radon theorems*
3	*1*	*Morse-Radon theorems*
4	*uncountably infinitely many*	
5	*finite*	*surgery theory*
6	*finite*	*surgery theory*
$\vdots$	*finite*	*surgery theory*

EY : 20151109 cf. http://math.stackexchange.com/questions/833766/closed-4-manifolds-with-uncountably-many-differe The wild world of 4-manifolds

Tutorial 4 Differentiable Manifolds

EY : 20151109 The gravity-and-light.org website, where you can download the tutorial sheets *and* the full length videos for the tutorials and lectures, are no longer there. = (

Hopefully, the YouTube video will remain: https://youtu.be/FXPdKxOq1KA?list=PLFeEvEPtX_0RQ1ys-7VIsKlBWz7RX-FaL

Exercise 1: True or false?. *These basic questions are designed to spark discussion and as a self-test.*

Tick the correct statements, but not the incorrect ones!

(a) The function $f : \mathbb{R} \to \mathbb{R}$, ...

-
-
- ..., defined by $f(x) = |x^3|$, lies in $C^3(\mathbb{R} \to \mathbb{R})$.

EY : 20151109 **Solution 1a3.** For $f : \mathbb{R} \to \mathbb{R}$, $f(x) = |x^3| = \begin{cases} x^3 & \text{if } x \geq 0 \\ -x^3 & \text{if } x < 0 \end{cases}$

$$f'(x) = \begin{cases} 3x^2 & \text{if } x \geq 0 \\ -3x^2 & \text{if } x < 0 \end{cases}$$

$$f''(x) = \begin{cases} 6x & \text{if } x \geq 0 \\ -6x & \text{if } x < 0 \end{cases}$$

Thus,

$$\boxed{f(x) = |x^3| \in C^1(\mathbb{R}) \text{ but } f(x) \notin C^2(\mathbb{R}) \subseteq C^3(\mathbb{R})}$$

[5] http://mathoverflow.net/questions/8789/can-every-manifold-be-given-an-analytic-structure

-
-

(b)
(c)

Short Exercise 4: Undergraduate multi-dimensional analysis .

A good notation and basic results for partial differentiation.
For a map $f : \mathbb{R}^d \to \mathbb{R}$ we denote by the map $\partial_i f : \mathbb{R}^d \to \mathbb{R}$ the partial derivative with respect to the i-th entry.

Question :. Given a function

$$f : \mathbb{R}^3 \to \mathbb{R}; (\alpha, \beta, \delta) \mapsto f(\alpha, \beta, \delta) := \alpha^3\beta^2 + \beta^2\delta + \delta$$

calculate the values of the following derivatives:
Solution :.

- $(\partial_2 f)(x, y, z) =$
- $(\partial_1 f)(\square, \circ, *) =$
- $(\partial_1\partial_2 f)(a, b, c) =$
- $(\partial_3^2 f)(299, 1222, 0) =$

EY: 20151110
For $f(\alpha, \beta, \delta) := \alpha^3\beta^2 + \beta^2\delta + \delta$, or $f(x, y, z) = x^3y^2 + y^2z + z$,

$$\begin{aligned} (\partial_2 f) &= 2(x^3y + yz) \\ (\partial_1 f) &= 3x^2y^2 \\ (\partial_1\partial_2 f) &= 6x^2y \\ (\partial_3^2 f) &= 0 \end{aligned}$$

and so

- $(\partial_2 f)(x, y, z) = 2(x^3y + yz)$
- $(\partial_1 f)(\square, \circ, *) = 3\square^2\circ^2$
- $(\partial_1\partial_2 f)(a, b, c) = 6a^2b$
- $(\partial_3^2 f)(299, 1222, 0) = 0$

Exercise 5: Differentiability on a manifold.
How to deal with functions and curves in a chart
Let $(M, \mathcal{O}, \mathcal{A})$ be a smooth d-dimensional manifold. Consider a chart (U, x) of the atlas $\mathcal{A}$ together with a smooth curve $\gamma : \mathbb{R} \to U$ and a smooth function $f : U \to \mathbb{R}$ on the domain U of the chart.

Question :. Draw a commutative diagram containing the chart domain, chart map, function, curveand the respective represen-

tatives of the function and the curve in the chart.
Solution :.

$$\begin{tikzcd}
\mathbb{R} \arrow[r, "\gamma"] \arrow[rr, bend left, "x \circ \gamma"] & U \arrow[r, "x"] \arrow[d, "f"] & \mathbb{R}^d \arrow[l, bend left, "x^{-1}"] \arrow[dl, "(f \circ x^{-1})"] \\
& \mathbb{R} &
\end{tikzcd}
\qquad
\begin{tikzcd}
\tau \in \mathbb{R} \arrow[r, mapsto, "\gamma"] \arrow[rr, bend left, "x \circ \gamma"] & p \in U \arrow[r, mapsto, "x"] \arrow[d, mapsto, "f"] & x(p) = (x \circ \gamma)(\tau) \in \mathbb{R}^d \arrow[l, mapsto, bend left, "x^{-1}"] \arrow[dl, mapsto, "(f \circ x^{-1})"] \\
& f(p) \in \mathbb{R} &
\end{tikzcd}$$

Question :. Consider, for $d = 2$,

$$(x \circ \gamma)(\lambda) := (\cos(\lambda), \sin(\lambda)) \text{ and } (f \circ x^{-1})((x, y)) := x^2 + y^2$$

Using the chain rule, calculate

$$(f \circ \gamma)'(\lambda)$$

explicitly.
Solution :.
EY : 20151109 Indeed, the domains and codomains of this $f\gamma$ mapping makes sense, from $\mathbb{R} \to \mathbb{R}$ for

$$\begin{tikzcd}
\mathbb{R} \arrow[r, "\gamma"] \arrow[rr, bend left, "x \circ \gamma"] \arrow[dr, "f \circ \gamma"'] & U \arrow[r, "x"] \arrow[d, "f"] & \mathbb{R}^d \arrow[l, bend left, "x^{-1}"] \arrow[dl, "(f \circ x^{-1})"] \\
& \mathbb{R} &
\end{tikzcd}
\qquad
\begin{tikzcd}
\tau \in \mathbb{R} \arrow[r, mapsto, "\gamma"] \arrow[rr, bend left, "x \circ \gamma"] \arrow[dr, mapsto, "f \circ \gamma"'] & p \in U \arrow[r, mapsto, "x"] \arrow[d, mapsto, "f"] & x(p) = (x \circ \gamma)(\tau) \in \mathbb{R}^d \arrow[l, mapsto, bend left, "x^{-1}"] \arrow[dl, mapsto, "(f \circ x^{-1})"] \\
& f(p) \in \mathbb{R} &
\end{tikzcd}$$

$$(f \circ \gamma)'(\lambda) = (Df) \cdot \dot{\gamma}(\lambda) = \frac{\partial f}{\partial x^j}\dot{\gamma}^j(\lambda) = 2x(-\sin\lambda) + 2y\cos\lambda = 2(-\cos\lambda\sin\lambda + \sin\lambda\cos\lambda) = 0$$

20. Lecture 5: Tangent Spaces

lead question: "what is the velocity of a curve γ point p?

20.1. Velocities.

Definition 63. *$(M, \mathcal{O}, \mathcal{A})$ smooth mfd.*
curve $\gamma : \mathbb{R} \to M$ at least C^1.
Suppose $\gamma(\lambda_0) = p$
*The **velocity** of γ p is the linear map*

$$v_{\gamma,p} : C^\infty(M) \xrightarrow{\sim} \mathbb{R}$$

$$C^\infty(M) := \{f : M \to \mathbb{R} | f \text{ smooth function }\} \text{ equipped with } \begin{aligned} (f \oplus g)(p) &:= f(p) + g(p) \\ (\lambda \otimes g)(p) &:= \lambda \cdot g(p) \end{aligned}$$

$\sim$ denotes linear map on top of $\to$.

$$f \mapsto v_{\gamma,p}(f) := (f\circ\gamma)'(\lambda_0)$$

$$\begin{tikzcd} \mathbb{R} \arrow[r, "\gamma"] \arrow[rr, bend right, "f\circ\gamma"'] & M \arrow[r, "f"] & \mathbb{R} \end{tikzcd}$$

intuition

Schuller says: children run around the world. Temperature function as temperature contour lines. You feel the temperature. You observe the rate of change of temperature as you run around. f is temperature.

past: " $\underbrace{v^i}(\partial_i f) = (\underbrace{v^i\partial_i}_{\text{vector}})f$

20.2. Tangent vector space.

Definition 64. *For each point $p\in M$ def the* ***set*** *"tangent space $\neq_0$ M p "*

$$T_pM := \{v_{\gamma,p} | \gamma \textit{ smooth curves } \}$$

picture:
rather M than (embedded) p T_pM EY : 20151109 see https://youtu.be/pepU_7NJSGM?t=12m38s for the picture

Observation: T_pM can be made into a vector space.

$$\begin{aligned} \oplus : & T_pM\times T_pM \to \\ & (v_{\gamma,p}\oplus v_{\delta,p})(\underbrace{f}_{\in C^\infty(M)}) := v_{\gamma,p}(f) +_{\mathbb{R}} v_{\delta,p}(f) \\ \odot : & \mathbb{R}\times T_pM \to \mathrm{Hom}(C^\infty(\mathbb{R}),\mathbb{R}) \\ & (\alpha\odot v_{\gamma,p})(f) := \alpha \cdot_{\mathbb{R}} v_{\gamma,p}(f) \end{aligned}$$

Remains to be shown that

(i) $\exists\,\sigma$ curve : $v_{\gamma,p}\oplus v_{\delta,p} = v_{\sigma,p}$
(ii) $\exists\,\tau$ curve : $\alpha\odot v_{\gamma,p} = v_{\tau,p}$

Claim: $\begin{aligned} \tau : & \mathbb{R}\to M \\ \mapsto & \tau(\lambda) := \gamma(\alpha\lambda+\lambda_0) = (\gamma\circ\mu_\alpha)(\lambda)\end{aligned}$ where $\begin{aligned}\mu_\alpha : & \mathbb{R}\to\mathbb{R} \\ & r\mapsto \alpha\cdot r+\lambda_0\end{aligned}$, does the trick.

$\tau(0) = \gamma(\lambda_0) = p$

$$\begin{aligned} v_{\tau,p} & := (f\circ\tau)'(0) = (f\circ\gamma\circ\mu_\alpha)'(0) \\ & = (f\circ\gamma)'(\lambda_0)\cdot\alpha = \\ & = \alpha\cdot v_{\gamma,p} \end{aligned}$$

Now for the sum:
$v_{\gamma,p}\oplus v_{\delta,p} \overset{?}{=} v_{\sigma,p}$
make a choice of chart $(\underbrace{U}_{\ni p}, x)$ In cloud: ill definition alarm bells.

and define:
Claim:

$$\begin{aligned} & \sigma : \mathbb{R}\to M \\ & \sigma(\lambda) := x^{-1}(\underbrace{(x\circ\gamma)(\lambda_0+\lambda)}_{\mathbb{R}\to\mathbb{R}^d} + (x\circ\delta)(\lambda_1+\lambda) - (x\circ\gamma)(\lambda_0)) \end{aligned}$$

does the trick.

Proof. Since:

$$\begin{aligned} \sigma_x(0) & = x^{-1}((x\circ\gamma)(\lambda_0) + (x\circ\delta)(\lambda_1) - (x\circ\gamma)(\lambda_0)) \\ & = \delta(\lambda_1) = p \end{aligned}$$

Now:

$$\begin{aligned} v_{\sigma_x,p}(f) & := (f\circ\sigma_x)'(0) = \\ & = (\underbrace{(f\circ x^{-1})}_{\mathbb{R}^d\to\mathbb{R}}\circ\underbrace{(x\circ\sigma_x)}_{\mathbb{R}\to\mathbb{R}^d})'(\gamma) = \underbrace{(x\circ\sigma_x)'(0)}_{(x\circ\gamma)'(\lambda_0)+(x\circ\delta)'(\lambda_1)} \cdot (\partial_i(f\circ x^{-1}))\,(x(\underbrace{\sigma(0)}_{p})) = \\ & = (x\circ\gamma)'(\lambda_0)(\partial_i(f\circ x^{-1}))(x(p)) + (x\circ\delta)(\lambda_1)(\partial_i(f\circ x^{-1}))(x(p)) \\ & = (f\circ\gamma)'(\lambda_0) + (f\circ\delta)'(\lambda_1) = \\ & = v_{\gamma,p}(f) + v_{\delta,p}(f) \quad \forall\, f\in C^\infty(M) \end{aligned}$$

$$\boxed{v_{\gamma,p}\oplus v_{\delta,p} = v_{\sigma,p}}$$

□

$$\begin{tikzcd} \mathbb{R} \arrow[r, "\sigma"] \arrow[dr, "x\circ\sigma"'] & M \arrow[r, "f"] \arrow[d, "x"] & \mathbb{R} \\ & \mathbb{R}^d \arrow[ur, "f\circ x^{-1}"'] & \end{tikzcd}$$

picture: (cf. https://youtu.be/pepU_7NJSGM?t=39m5s)

$$\begin{aligned} \gamma & : \mathbb{R}\to M \\ \delta & : \mathbb{R}\to M \end{aligned}$$

$(\gamma\oplus)(\lambda) := \gamma(\lambda)+\delta(\lambda)$
EY : 20151109 Schuller says adding trajectories is chart dependent, bad. Adding velocities is good.

20.3. Components of a vector wrt a chart.

Definition 65. *Let $(U,x)\in\mathcal{A}_{smooth}$.*
Let $\begin{aligned}\gamma & :\mathbb{R}\to U \\ \gamma(0) & = p\end{aligned}$.
Calculate

$$\begin{aligned} v_{\gamma,p}(f) := (f\circ\gamma)'(0) & = (\underbrace{(f\circ x^{-1})}_{\mathbb{R}^d\to\mathbb{R}}\circ\underbrace{(x\circ\gamma)}_{\mathbb{R}\to\mathbb{R}^d})'(0) \\ & = \underbrace{(x\circ\gamma)^{i'}(0)}_{\dot{\gamma}^i_x(0)}\cdot\underbrace{(\partial_i(f\circ x^{-1}))(x(p))}_{=:\left(\frac{\partial f}{\partial x^i}\right)_p} \end{aligned}$$

think cloud $f: M\to\mathbb{R}$

$$= \boxed{\dot{\gamma}^i_x(0)\cdot\left(\frac{\partial}{\partial x^i}\right)_p} f \quad \forall\, f\in C^\infty(M)$$

$\therefore$ as a map.

$$v_{\gamma,p} \underbrace{=}_{\textit{use of chart}} \underbrace{\gamma^i_x(0)}_{\textit{"components of the velocity } v_{\gamma,p}\textit{"}} \underbrace{\left(\frac{\partial}{\partial x^i}\right)}_{\substack{\textit{basis elements of the } T_pM \textit{ wrt which the components need to be understood.} \\ \textit{"chart induced basis of } T_pM\textit{"}}}$$

Picture: https://youtu.be/pepU_7NJSGM?t=1h16s

20.4. 4. Chart-induced basis.

Definition 66. $(U,x) \in \mathcal{A}_{smooth}$
the $\left(\frac{\partial}{\partial x^1}\right)_p, \dots, \left(\frac{\partial}{\partial x^d}\right)_p \in T_pU \subseteq T_pM$
constitute a ***basis*** *of* T_pU

Proof. remains: linearly independent

$$\lambda^i \left(\frac{\partial}{\partial x^i}\right)_p \overset{!}{=} 0$$

$$\Longrightarrow \lambda^i \left(\frac{\partial}{\partial x^i}\right)_p (x^j) = \lambda^i \partial_i (\underbrace{x^j \circ x^{-1}})(x(p)) = \qquad \begin{aligned} x^j \circ x^{-1} &: \mathbb{R}^d \to \mathbb{R} \\ (\alpha^1, \dots, \alpha^d) &\mapsto \alpha^j \end{aligned}$$

$$= \lambda^i \delta_i^{\ j} = \lambda^j \qquad j = 1, \dots, d$$

in cloud: $x^j : U \to \mathbb{R}$ differentiable

Corollary 1. $dim T_pM = d = dim M$

Terminology: $X \in T_pM \to \exists \gamma : \mathbb{R} \to M : X = v_{\gamma,p}$ and
$\exists \underbrace{X_1^1, \dots, X^d}_{\in \mathbb{R}} : X = X^i \left(\frac{\partial}{\partial x^i}\right)_p$

20.5. 5. Change of vector ***components*** under a change of chart. ✘ vector does **not** change under change of chart.

Let (U,x) and (V,y) be overlapping charts and $p \in U \cap V$.
Let $X \in T_pM$

$$X^i_{(y)} \cdot \left(\frac{\partial}{\partial y^i}\right)_p \underbrace{=}_{(V,y)} X \underbrace{=}_{(U,x)} X^i_x \left(\frac{\partial}{\partial x^i}\right)_p$$

to study change of components formula:

$$\begin{aligned} \left(\frac{\partial}{\partial x^i}\right)_p f &= \partial_i (f \circ x^{-1})(x(p)) = \\ &= \partial_i (\underbrace{(f \circ y^{-1})}_{\mathbb{R}^d \to \mathbb{R}} \circ \underbrace{(y \circ x^{-1})}_{\mathbb{R}^d \to \mathbb{R}^d})(x(p)) \\ &= (\partial_i (y^i \circ x^{-1}))(x(p)) \cdot (\partial_j (f \circ y^{-1}))(y(p)) = \\ &= \boxed{\left(\frac{\partial y^p}{\partial x^i}\right)_p \cdot \left(\frac{\partial f}{\partial y^j}\right)_p} f \end{aligned}$$

$$\Longrightarrow X^i_{(x)} \left(\frac{\partial y^j}{\partial x^i}\right)_p \left(\frac{\partial}{\partial y^j}\right)_p = X^j_{(y)} \left(\frac{\partial}{\partial y^j}\right)_p$$

$$\Longrightarrow \boxed{X^j_{(y)} = \left(\frac{\partial y^j}{\partial x^i}\right)_p X^i_{(x)}}$$

20.6. 6. Cotangent spaces. $T_pM = V$

trivial $(T_pM)^* := \{\varphi : T_pM \xrightarrow{\sim} \mathbb{R}\}$
Example: $f \in C^\infty(M)$

$$\begin{aligned} (df)_p : & T_pM \xrightarrow{\sim} \mathbb{R} \\ & X \mapsto (df)_p(X) \end{aligned}$$

i.e. $\boxed{(df)_p \in T_pM^*}$
$(df)_p$ called the gradient of $f \quad p \in M$.
Calculate components of gradient w.r.t. chart-induced basis (U,x)

$$\begin{aligned} ((df)_p)_j &:= (df)_p \left(\left(\frac{\partial}{\partial x^j}\right)_p \right) \\ &= \left(\frac{\partial f}{\partial x^j}\right)_p = \partial_j (f \circ x^{-1})(x(p)) \end{aligned}$$

□

Theorem 24. *Consider chart* $(U,x) \Longrightarrow x^i : U \to \mathbb{R}$
Claim: $(dx^1)_p, (dx^2)_p, \dots, (dx^d)_p$ *basis of* T_p^*M
$\Longrightarrow$ *In fact: dual basis:*

$$(dx^a)_p \left(\left(\frac{\partial}{\partial x^b}\right)_p \right) = \left(\frac{\partial x^a}{\partial x^b}\right)_p = \dots = \delta^a_b$$

20.7. 7. Change of ***components*** of a covector under a change of chart:

$$\underbrace{T_p^*M}_{\ni \omega} \text{ with } \omega_{(y)}(dy^j)_p = \omega = \omega_{(x)i}(dx^i)_p$$

$$\Longrightarrow \boxed{\omega_{(y)i} = \frac{\partial x^j}{\partial y^i} \omega_{(x)j}}$$

Lecture 6: Fields

cf. Lecture 6: Fields (International Winter School on Gravity and Light 2015)
So far:

$$\begin{array}{c} T_pM \\ \vdots\downarrow \\ T_p^*M \\ \vdots\downarrow \\ \vdots \end{array},$$

now
in Thought Cloud: theory of bundles

20.8. Bundles.

Definition 67. *A* ***bundle*** *is a triple*

$$E \xrightarrow{\pi} M$$

E smooth manifold ***"total space"***
π smooth map (surjective) "projection map"
M smooth manifold "base space"

Example $E =$ cylinder $M =$ circle

Definition 68.

$$E \xrightarrow{T} M$$

bundle.
$p \in M$
define ***fibre over*** p
$:= preim_\pi(\{p\})$

Definition 69. *A* ***section*** σ *of a bundle*

$$\begin{tikzcd} E \arrow[d, "\pi"'] \\ M \arrow[u, bend right, "\sigma"'] \end{tikzcd}$$

require $\pi \circ \sigma = id_M$

Schuller says: in quantum mechanics, Aside: $\psi : M \to \mathbb{C}$

20.9. **Tangent bundle of smooth manifold.** $(M, \mathcal{O}, \mathcal{A})$ smooth manifold

(a) as a **set** $TM := \dot{\bigcup}_{p \in M} T_pM$

(b) surjective $\pi : TM \to M$ the *unique* point $p \in M$, $X \in T_pM$
$X \mapsto p$
situation: $\underbrace{TM}_{\text{set}} \quad \underbrace{\xrightarrow{\pi}}_{\text{surjective map}} \quad \underbrace{M}_{\text{smooth manifold}}$

(c) Construct topology on TM that is the coarsest topology such that π (just) continuous. ("initial topology with respect to π").

$$\mathcal{O}_{TM} := \{\text{preim}_\pi(U) | \mathcal{U} \in \mathcal{O}\}$$

Show: Tutorial $\mathcal{O}_{TM}$ Schuller says this is shown in the tutorial
$(TM, \mathcal{O}_{TM})$

Construction of a C^∞-atlas on TM from the C^∞-atlas $\mathcal{A}$ on M.

$$\mathcal{A}_{TM} := \{(T\mathcal{U}, \xi_x) | (U, x) \in \mathcal{A}\}$$

where

$$\begin{aligned} \xi_x : & T\mathcal{U} \to \mathbb{R}^{2 \cdot \dim M} \\ & X \mapsto (\underbrace{(x^1 \circ \pi)(X), \dots, (x^d \circ \pi)(X)}_{(U,x)-\text{ coords of } \pi(X) \ (d \text{ many })}, (dx^1)_{\pi(X)}(X), \dots, (dx^d)_{\pi(X)}(X)) \end{aligned}$$

where $X \in T_{\pi(X)}M$
$X = X^i_{(x)} \left(\frac{\partial}{\partial x^i}\right)_{\pi(X)}$

$$(dx^j)_{\pi(X)}(X) = (dx^j)_{\pi(X)} \left(X^i_{(x)} \left(\frac{\partial}{\partial x^i} \right)_{\pi(X)} \right) =$$
$$= X^i_{(x)} \delta^j_i = X^j_{(x)}$$

Write $\xi_x^{-1} : \xi_x(TU) \subseteq \mathbb{R}^{2\dim M} \to TU$

$$(\alpha^1, \dots, \alpha^d, \beta^1, \dots, \beta^d) := \beta^i \left(\frac{\partial}{\partial x^i} \right)_{\underbrace{x^{-1}(\alpha^1, \dots, \alpha^d)}_{\pi(X)}}$$

Check:

$$(\xi_y \circ \xi_x^{-1})(\alpha^1, \dots, \alpha^d, \beta^1, \dots, \beta^d) =$$
$$= \xi_y \left(\beta^i \left(\frac{\partial}{\partial x^i} \right)_{x^{-1}(\alpha^1, \dots, \alpha^d)} \right)$$
$$= \left(\dots, (y^i \circ \pi)(\beta^m \cdot \left(\frac{\partial}{\partial x^m} \right)_{x^{-1}(\alpha^1 \dots \alpha^d)}), \dots, \dots (dy^i)_{x^{-1}(\alpha^1, \dots \alpha^d)} \left(\beta^m \left(\frac{\partial}{\partial x^m} \right)_{x^{-1}(\alpha^1 \dots \alpha^d)} \right), \dots \right) =$$
$$= (\dots, (y^i \circ x^{-1})(\alpha^1, \dots, \alpha^d), \dots, \dots, \underbrace{\beta^m (dy^i)_{x^{-1}(\alpha^1, \dots, \alpha^d)} \left(\left(\frac{\partial}{\partial x^m} \right)_{x^{-1}(\alpha^1 \dots \alpha^d)} \right)}_{\beta^m \left(\frac{\partial y}{\partial x^m} \right)_{x^{-1}(\alpha^1 \dots \alpha^d)}})$$

Check transition map: $(U, x), (V, y)$, $U \cap V \neq 0$
$\left(\frac{\partial y}{\partial x^m} \right)_{x^{-1}(\alpha^1 \dots \alpha^d)} = \partial_m (y^i \circ x^{-1})(x \circ (x^{-1}(\alpha^1 \dots \alpha^d))) = \partial_m (y^i \circ x^{-1})(\alpha^1 \dots \alpha^d)$ smooth.
upshot

$$\underbrace{TM}_{\text{smooth manifold}} \quad \underbrace{\xrightarrow{\pi}}_{\text{smooth map}} \quad \underbrace{M}_{\text{smooth manifold}}$$

bundle, called the tangent bundle.

3. Vector fields.

Definition 70. *A* ***smooth vector field*** χ *is a* smooth *map, (where)*

$$\begin{tikzcd} TM \arrow[d, "\pi"'] \\ M \arrow[u, bend right, "\chi"'] \end{tikzcd}$$

Example:

$$\begin{tikzcd} TS^1 \arrow[d, "\pi"] \\ S^1 \end{tikzcd}$$

4. The $C^\infty(M)$-module $\Gamma(TM)$.
$C^\infty(M)$-module $\leftarrow (C^\infty(M), +, \cdot)$ (satisfies) $C^+, A^+, N^+, I+, C^+, A^+, N^+, D^+$. Not a ~~field~~. A *ring*.

set $\Gamma(TM) = \{\chi \quad M \to TM |$ smooth section $\}$
$(\chi \oplus \widetilde{\chi})(f) := (\chi f) \underbrace{+}_{C^\infty(M)} \widetilde{\chi}(f)$

$(\underbrace{g}_{C^\infty(M)} \odot \xi)(f) := \underbrace{g}_{C^\infty(M)} \cdot \chi(f)$

$$\begin{aligned} \chi :& M \to TM \\ & p \mapsto \chi(p) \\ \chi f :& M \to \mathbb{R} \\ & p \mapsto \chi(p) f \end{aligned}$$

$(\Gamma(TM), \oplus, \odot)$ $C^\infty(M)$ **- module**

upshot: set of all smooth vector fields can be made into a $C^\infty(M)$-module.
Fact:

(1) ZFC $\Longrightarrow$ every vector space has a basis. (You have to have C - axiom of choice in set theory)
(2) no such result exists for modules.

This is a shame, because otherwise, we could have chosen (for any manifolds) vector fields,

$$\chi_{(1)}, \dots, \chi_{(d)} \in \Gamma(TM)$$

and would be able to write every vector field Ξ

$$\chi = \underbrace{f^i}_{\text{component functions}} \cdot \chi_{(i)}$$

Simple counterexample
Schuller says: Take a sphere, Morse Theorem, every smooth vector field must vanish at 2 pts. "mustn't choose a global basis"

However: $\dfrac{\partial}{\partial x^i} : U \xrightarrow{\text{smooth}} TU$
$p \mapsto \left(\dfrac{\partial}{\partial x^i} \right)_p$

20.10. **Tensor fields.** so far
$\Gamma(M) =$"set of vector fields" $C^\infty(M)$-module
$\Gamma(T^*M) =$ "covector fields" $C^\infty(M)$-module

Definition 71. *An (r,s)-tensor field T is a multi-linear map*

$$T : \underbrace{\Gamma(T^*M) \times \dots \times \Gamma(T^*M)}_{r} \times \Gamma(TM) \times \dots \times \Gamma(TM) \xrightarrow{\sim} C^\infty(M)$$

Example: $f \in C^\infty(M)$

$$\begin{aligned} df :& \Gamma(TM) \xrightarrow{\sim} C^\infty(M) \\ & \chi \mapsto df(\chi) := \chi[f] \end{aligned}$$

df $(0,1)$-T.F. (tensor field)
where $(\chi f)(\underbrace{p}_{\in M}) := \underbrace{\chi(p)}_{\in T_pM} f$
can check: df is $C^\infty-$linear

Lecture 7: Connections

cf. Lecture 7: Connections (International Winter School on Gravity and Light 2015)
So far: saw that a vector field X can be used to provide a directional derivative

$$\nabla_X f := Xf$$

of a function $f \in C^\infty(M)$.
Remark: from now on: consider mostly vector fields.
Notational overkill?

$$\nabla_X f = Xf = (df)(X)$$

In Thought Bubble: $\nabla_X(f \cdot g) = X(fg) = (Xf) \cdot g + fX(g)$ Product rule, because it's a derivative.
not quite:

$$\begin{aligned} X &: C^\infty(M) \to C^\infty(M) \\ df &: \Gamma(TM) \to C^\infty(M) \\ \nabla_X &: C^\infty(M) \to C^\infty(M) \end{aligned}$$

$$\begin{array}{ccc} \nabla_X : C^\infty(M) & \longrightarrow & C^\infty(M) \\ \vdots\downarrow & & \vdots\downarrow \\ \nabla_X : \begin{matrix} TM^p \otimes T^*M^q \text{ i.e.} \\ \binom{p}{q} \text{ tensor field} \end{matrix} & \longrightarrow & \begin{matrix} TM^p \otimes T^*M^q \text{ i.e.} \\ \binom{p}{q} \text{ tensor field} \end{matrix} \end{array}$$

1. Directional derivatives of tensor fields. We formulate a wish list of properties which the ∇_X acting on a tensor field should have.
In Thought Bubble: Any remaining freedom in choosing ∇, will need to be provided as additional structure beyond $(M, \mathcal{O}, \mathcal{A})$

Definition 72 (connection)**.** *In Thought Bubble: linear connection, covariant derivatve, affine connection*
A ***connection*** ∇ *on a smooth manifold $(M, \mathcal{O}, \mathcal{A})$ is a map that takes a pair consisting of a vector (field) X and a (p,q)-tensor field T and sends them to a (p,q)-tensor (field) $\nabla_X T$ satisfying*

(i) $\nabla_X f = Xf \quad \forall f \in C^\infty(M)$
(ii) $\nabla_X(T+S) = \nabla_X T + \nabla_X S$
(iii) $\nabla_X(\underbrace{T(\omega, Y)}_{\in C^\infty(M)}) = (\nabla_X T)(\omega, Y) + T(\nabla_X \omega, Y) + T(\omega, \nabla_X Y)$
In Thought Bubble:for $(1,1)$-TF T, but analogously for any (p,q) - TF

"Leibnitz" rule.
(iv) $\nabla_{fX+Z} T = f \nabla_X T + \nabla_Z T$
$f \in C^\infty(M)$

A manifold with connection is quadruple $(M, \mathcal{O}, \mathcal{A}, \nabla)$
Remark: ∇_X is the extension of X.
∇ —— " ———- of d

2. New structure on $(M,\mathcal{O},\mathcal{A})$ required to fix ∇. Q: How much freedom do we have in choosing such a structure.

Consider X,Y vector fields

$$\nabla_X Y \underbrace{=}_{\text{In Thought Bubble:}(U,x)} \nabla_{X^i\frac{\partial}{\partial x^i}}\left(Y^m \frac{\partial}{\partial x^m}\right)$$

$$\underbrace{=}_{\text{(iii)}} X^i\left(\nabla_{\frac{\partial}{\partial x^i}} Y^m\right)\frac{\partial}{\partial x^m} + X^i Y^m \underbrace{\nabla_{\frac{\partial}{\partial x^i}}\left(\frac{\partial}{\partial x^m}\right)}$$

$$\underbrace{=}_{\text{(i)}} X^i\left(\frac{\partial}{\partial x^i}Y^m\right)\frac{\partial}{\partial x^m} + X^i Y^m \underbrace{\Gamma^q{}_{mi}(x)}_{\text{connection coefficient functions (on } M\text{) of } \nabla \text{ wrt } (U,x)} \frac{\partial}{\partial x^q}$$

$$(\underbrace{T}_{(p,q)} \otimes \underbrace{S}_{(r,s)})(\omega,\dots,X,\dots) := T(\omega,\dots,X,\dots) \underbrace{\cdot}_{C^\infty(M)} S(\dots,\dots)$$

$$\nabla_X(T\otimes S) = (\nabla_X T)\otimes S + T\otimes(\nabla_X S)$$

Definition 73 (Connection coefficient functions)**.** $(M,\mathcal{O},\mathcal{A},\nabla)$, $(\mathcal{U},x)\in\mathcal{A}$.
*Then the **connection coefficient functions** ("Γ"s) with respect to (wrt) (U,x) on the $(dim(M))^3$ many functions*

$$\Gamma^i{}_{jk} : \mathcal{U}\to\mathbb{R}$$
$$p \mapsto \left(dx^i\left(\nabla_{\frac{\partial}{\partial x^k}}\frac{\partial}{\partial x^j}\right)\right)(p)$$

Thus:

$$(\nabla_X Y)^i = X^m\left(\frac{\partial}{\partial x^m}Y^i\right) + \Gamma^i{}_{nm} \underbrace{\cdot}_{C^\infty(M)} Y^n X^m$$

Remark: On a chart domain U, choice of the $(\dim M)^3$ functions $\Gamma^i{}_{jk}$ suffices to fix the action of ∇ on a *vector* field.
Fortunately, the same $(\dim M)^3$ functions fix the action of ∇ on any tensor field.
key observation:

$$\nabla_{\frac{\partial}{\partial x^m}}(dx^i) = \sum_{jm}^{i} dx^j$$

but now:

$$\underbrace{\nabla_{\frac{\partial}{\partial x^m}}(\underbrace{dx^i\left(\frac{\partial}{\partial x^j}\right)}_{\delta^i{}_j})}_{\| \quad \text{(iii)}} = \frac{\partial}{\partial x^m}(\delta^i{}_j) = 0$$

$$= \left(\nabla_{\frac{\partial}{\partial x^m}} dx^i\right)\left(\frac{\partial}{\partial x^j}\right) + dx^i(\underbrace{\nabla_{\frac{\partial}{\partial x^m}}\frac{\partial}{\partial x^j}}_{\Gamma^q{}_{jm}\frac{\partial}{\partial x^q}}) = 0$$

$$\implies \left(\nabla_{\frac{\partial}{\partial x^m}} dx^i\right)\left(\frac{\partial}{\partial x^j}\right) = -\Gamma^i_{jm}$$

Summary so far:

$$(\nabla_X Y)^i = X(Y^i) \underbrace{+}_{\text{act on \textbf{vector} field}} \Gamma^i_{jm} Y^j X^m$$

$$(\nabla_X \omega)_i = X(\omega_i) + -\Gamma^j_{im}\omega_j X^m$$

Note that for the immediately above expression for $(\nabla_X Y)^i$, in the second term on the right hand side, Γ^i_{jm} has the last entry at the bottom, m going in the direction of X, so that it matches up with X^m. This is a good mnemonic to memorize the index positions of Γ.

similarly, by further application of Leibnitz

T a $(1,2)$-TF (tensor field)

$$(\nabla_X T)^i{}_{jk} = X(T^i{}_{jk}) + \Gamma^i{}_{sm}T^s{}_{jk}X^m - \Gamma^s{}_{jm}T^i{}_{sk}X^m - \Gamma^s{}_{km}T^i{}_{js}X^m$$

<u>Question</u>: If in a Euclidean space, the Γs all vanish in a (then existing) global chart.

<u>Answer</u>: Yes, but: What is a Euclidean space:
$(M=\mathbb{R}^n,\mathcal{O}_{\text{st}},\mathcal{A})$ smooth manifold.
Assume $(\mathbb{R}^n,\mathrm{id}_{\mathbb{R}^n})\in\mathcal{A}$ and

$$(\Gamma^i_{(x)})_{jk} = dx^i\left((\nabla_{\underline{\mathrm{E}}})_{\frac{\partial}{\partial x^k}}\frac{\partial}{\partial x^j}\right) \overset{!}{=} 0$$

Intuition:

$\mathbb{R}^2$: $\nabla_{\text{Euclidean}}$

$\mathbb{R}^2$: $\nabla_{\text{Hyperbolic}}$

Definition 74. *X vector field on $(M,\mathcal{O},\mathcal{A},\nabla)$.*
Then divergence of X is the function:

$$div(X) := \left(\nabla_{\frac{\partial}{\partial x^i}} X\right)^i$$

Claim: chart-independent.

3. Change of Γ's under change of chart.

(U,x), $(V,y)\in\mathcal{A}$ and $U\cap V\neq\emptyset$

$$\Gamma^i_{jk}(y) := dy^i\left(\nabla_{\frac{\partial}{\partial y^k}}\frac{\partial}{\partial y^j}\right) = \frac{\partial y^i}{\partial x^q}dx^q\left(\nabla_{\frac{\partial x^p}{\partial y^k}\frac{\partial}{\partial x^p}}\frac{\partial x^s}{\partial y^j}\frac{\partial}{\partial x^s}\right)$$

Note ∇_{fX} is C^∞-linear for fX
covector dy^i is C^∞-linear in its argument

$$\implies \Gamma^i_{jk}(y) = \frac{\partial y^i}{\partial x^q}dx^q\left(\frac{\partial x^p}{\partial y^k}\left[\left(\nabla_{\frac{\partial}{\partial x^p}}\frac{\partial x^s}{\partial y^j}\right)\frac{\partial}{\partial x^s} + \frac{\partial x^s}{\partial y^j}\left(\nabla_{\frac{\partial}{\partial x^p}}\frac{\partial}{\partial x^s}\right)\right]\right) =$$
$$= \frac{\partial y^i}{\partial x^q}\underbrace{\frac{\partial x^p}{\partial y^k}\frac{\partial}{\partial x^p}}_{\frac{\partial}{\partial y^k}}\frac{\partial x^s}{\partial y^j}\delta^q{}_s + \frac{\partial y^i}{\partial x^q}\frac{\partial x^p}{\partial y^k}\frac{\partial x^s}{\partial y^j}\Gamma^q_{sp}(x)$$

in summary:

$$\Gamma^i_{jk}(y) = \frac{\partial y^i}{\partial x^q}\frac{\partial^2 x^q}{\partial y^j\partial y^k} + \frac{\partial y^i}{\partial x^q}\frac{\partial x^s}{\partial y^j}\frac{\partial x^p}{\partial y^k}\Gamma^q_{sp}(x) \tag{65}$$

Eq. (65) is the change of connection coefficient function under the change of chart $(U\cap V, x) \to (U\cap V, y)$

4. Normal Coordinates. Let $p\in M$ of $(M,\mathcal{O},\mathcal{A},\nabla)$

Then one can construct a chart (U,x) with $p\in U$ such that

$$\Gamma(x)^i_{\ (jk)}(p) = 0$$

at the point p. **Not** necessarily in any neighborhood.

Proof. Let (V,y) be any chart (with) $p\in V$.

Thus, in general: $\Gamma(y)^i_{\ jk} \neq 0$

Then consider a new chart (U,x) to which one transits by virtue of

$$(x\circ y^{-1})^i(\alpha^1,\dots,\alpha^d) := \alpha^i - \frac{1}{2}\Gamma(y)^i_{\ (jk)}(p)\alpha^j\alpha^k$$

$p = x^{-1}(\alpha^1,\dots,\alpha^d)$

$$\left(\frac{\partial x^i}{\partial y^j}\right)_p = \partial_j(x^i\circ y^{-1}) = \delta^i_j - \Gamma(y)^i_{\ mj}(p)\ \alpha^m|_{\alpha=0} = \delta^i_j$$

$$\frac{\partial x^i}{\partial y^k\partial y^j}(p) = -\Gamma(y)^i_{\ kj}(p)$$

$$\Longrightarrow \Gamma(x)^i_{jk}(p) = \Gamma(y)^i_{\ jk}(p) - \Gamma(y)^i_{\ kj}(p) = 0$$

$$= \Gamma(y)^i_{[jk]}(p) = T(y)^i_{\ jk}$$

Terminology: (U,x) is called a **normal coordinate chart** of ∇ at $\mathbf{p}\in\mathbf{M}$. □

Tutorial 7 Connections. Exercise 1. : True or false?

(a)
- $\nabla_{fX}Y = f\nabla_X Y$ by definition so $\nabla_{fX} = f\nabla_X$ i.e. ∇_X is $C^\infty(M)$-linear in X
- $f\in C^\infty(M)$ is a $(0,0)$-tensor field. $\nabla_X f = Xf \equiv X(f)$ by definition.
- If the manifold is flat, I'm assuming that means that the manifold is globally a Euclidean space, and by definition, $\Gamma = 0$.

$$\nabla_X Y = X^j\frac{\partial}{\partial x^j}(Y^i)\frac{\partial}{\partial x^i} + \Gamma^i_{jk}Y^kX^k\frac{\partial}{\partial x^i} = X^j\frac{\partial Y^i}{\partial x^j}\frac{\partial}{\partial x^i} + 0$$

and similarly for any (p,q)-tensor field, i.e.

$$\nabla_X T = X^j\frac{\partial T^{i_1\dots i_p}_{j_1\dots j_q}}{\partial x^j}$$

- $$\nabla_X f = X^j\frac{\partial f}{\partial x^j} = X\cdot \text{grad}(f)$$
- $\forall (U,x)\in\mathcal{A}$, locally (after working out the first few cases, and doing induction, one can look up the expression for the local form; I found it in Nakahara's **Geometry, Topology and Physics**, Eq. 7.26, and it needs to be modified for the convention of order of bottom indices for Γ:

$$\nabla_\nu t^{\lambda_1\dots\lambda_p}_{\mu_1\dots\mu_q} = \partial_\nu t^{\lambda_1\dots\lambda_p}_{\mu_1\dots\mu_q} + \Gamma^{\lambda_1}_{\kappa\nu}t^{\kappa\lambda_2\dots\lambda_p}_{\mu_1\dots\mu_q} + \dots + \Gamma^{\lambda_p}_{\kappa\nu}t^{\lambda_1\dots\lambda_{p-1}\kappa}_{\mu_1\dots\mu_q} - \Gamma^\kappa_{\mu_1\nu}t^{\lambda_1\dots\lambda_p}_{\kappa\mu_2\dots\mu_q} - \dots - \Gamma^\kappa_{\mu_q\nu}t^{\lambda_1\dots\lambda_p}_{\mu_1\dots\mu_{q-1}\kappa}$$

Clearly, ∇_X is uniquely fixed $\forall p\in M$ by choosing each of the $(\dim M)^3$ many connection coefficient functions Γ.

(b)
- $\nabla : \mathfrak{X}(M)\to\mathfrak{X}(M)$

  $\nabla : (p,q)$-tensor field $\mapsto (p,q)$-tensor field
- By definition, ∇ satisfies the Leibniz rule.
-
-
-

Exercise 2. : Practical rules for how ∇ acts Torsion-free covariant derivative boils down to a connection coefficient function Γ that is symmetric in the bottom indices.

- $$\nabla_X f = X(f) = X^i\frac{\partial f}{\partial x^i}$$
- $$(\nabla_X Y)^a = X^i\frac{\partial Y^a}{\partial x^i} + \Gamma^a_{jk}Y^jX^k$$
- $$(\nabla_X\omega)_a = X^i\frac{\partial\omega_a}{\partial x^j} - \Gamma^i_{ak}\omega_iX^k$$
- $$(\nabla_m T)^a_{\ bc} = \frac{\partial}{\partial x^m}(T^a_{\ bc}) + \Gamma^a_{\ im}T^i_{\ bc} - \Gamma^i_{\ bm}T^a_{\ ic} - \Gamma^j_{\ cm}T^a_{\ bj}$$
- $$(\nabla_{[m}A)_{n]} = (\nabla_m A)_n - (\nabla_n A)_m = \frac{\partial A_n}{\partial x^m} - \Gamma^i_{\ nm}A_i - \left(\frac{\partial A_m}{\partial x^n} - \Gamma^i_{\ mn}A_i\right) = \frac{\partial A_m}{\partial x^m} - \frac{\partial A_m}{\partial x^n}$$
- $$(\nabla_m\omega)_{nr} = \frac{\partial\omega_{nr}}{\partial x^m} - \Gamma^i_{\ nm}\omega_{ir} - \Gamma^i_{\ rm}\omega_{ni}$$

Exercise 3. : Connection coefficients

Question .

The connection coefficient functions Γ in chart $(U\cap V, y)$ is given, in terms of chart $(U\cap V, x)$ as follows:
Recall Eq. (65)

$$\Gamma^i_{\ jk}(y) = \frac{\partial y^i}{\partial x^q}\frac{\partial^2 x^q}{\partial y^j\partial y^k} + \frac{\partial y^i}{\partial x^q}\frac{\partial x^s}{\partial y^j}\frac{\partial x^p}{\partial y^k}\Gamma^q_{\ sp}(x)$$

21. Lecture 8: Parallel Transport & Curvature (International Winter School on Gravity and Light 2015)

21.1. Parallelity of vector fields.

Definition 75. (1) ***parallely transported*** *along smooth curve* $\gamma:\mathbb{R}\to M$ *if*

(66)
$$\boxed{\nabla_{v_\gamma}X = 0}$$

(2) *A slightly weaker condition is "parallel"*

$$(\nabla_{v_{\gamma,\gamma(\lambda)}}X)_{\gamma(\lambda)} = \mu(\lambda)X_{\gamma(\lambda)}$$

21.2. Autoparallely transported curves.

Definition 76. *curve* $\gamma:\mathbb{R}\to M$ *is called* ***autoparallely transported*** *if*

(67)
$$\nabla_{v_\gamma}v_\gamma \overset{!}{=} 0$$

21.3. Autoparallel equation.

$$\nabla_{v_\gamma}v_\gamma = 0$$

in summary:

(68)
$$\ddot{\gamma}^m_{(x)}(\lambda) + (\Gamma^m_{(x)})_{ab}(\gamma(\lambda))\dot{\gamma}^a_{(x)}(\lambda)\dot{\gamma}^b_{(x)}(\lambda) = 0$$

21.4. **Torsion.**

Definition 77. ***torsion*** *of a connection* ∇ *is the* $(1,2)$*-**tensor field***

(69) $$T(\omega, X, Y) := \omega(\nabla_X Y - \nabla_Y X - [X,Y])$$

(Inside a cloud)
$[X,Y]$ vector field defined by

$$[X,Y]f := X(Yf) - Y(Xf)$$

Proof. check T is C^∞-linear in each entry

$$T(\omega, fX, Y) = \omega(\nabla_{fX} Y - \nabla_Y (fX) - [fX,Y])$$

□

Definition 78. *A* $(M, \mathcal{O}, \mathcal{A}, \nabla)$ *is called torsion-free if* $T = 0$

In a chart

$$T^i{}_{ab} := T\left(dx^i, \frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^b}\right) = dx^i(\dots)$$
$$= \Gamma^i{}_{ab} - \Gamma^i{}_{ba} = 2\Gamma^i{}_{[ab]}$$

From now on, in these lectures, we only use torsion-free connections.

21.5. **4. Curvature.**

Definition 79. ***Riemann curvature*** *of a connection* ∇ *is the* $(1,3)$*-tensor field*

(70) $$Riem(\omega, Z, X, Y) := \omega(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z)$$

Proof. do it: C^∞-linear in each slot. □

Tutorials $\text{Riem}^i{}_{jab} = \dots$

TUTORIAL 8 PARALLEL TRANSPORT & CURVATURE

Exercise 1.

Exercise 2. : Where connection coefficients appear
It was suggested in the tutorial sheets and hinted in the lecture that the following should be committed to memory.

Question : Recall the autoparallel equation for a curve γ.

(a)
$$\nabla_{v_\gamma} v_\gamma = 0$$

(b)
$$\nabla_{v_\gamma} v_\gamma = \nabla_{\dot{\gamma} \frac{\partial}{\partial x^\mu}} v_\gamma = \dot{\gamma}^\nu \nabla_{\partial_\nu} v_\gamma = \dot{\gamma}^\nu \left[\frac{\partial v^\mu_\gamma}{\partial x^\nu} + \Gamma^\rho_{\mu\nu} v^\mu_\gamma \right] \frac{\partial}{\partial x^\rho} = \dot{\gamma}^\nu \left[\frac{\partial \dot{\gamma}^\rho}{\partial x^\nu} + \Gamma^\rho_{\mu\nu} \dot{\gamma}^\mu \right] \frac{\partial}{\partial x^\rho} = 0$$
$$\implies \boxed{\ddot{\gamma}^\rho + \Gamma^\rho_{\mu\nu} \dot{\gamma}^\mu \dot{\gamma}^\nu}$$

as, for example, for $F(x(t))$,

$$\frac{dF(x(t))}{dt} = \dot{x}\frac{\partial F}{\partial x} = \frac{d}{dt}F$$

so that

$$\dot{\gamma}^\nu \frac{\partial v^\mu_\gamma}{\partial x^\nu} = \frac{d}{d\lambda} v^\mu_\gamma = \frac{d^2}{d\lambda^2}\gamma^\mu$$

Question : Determine the coefficients of the Riemann tensor with respect to a chart (U, x).

Recall this manifestly covariant definition

$$\text{Riem}(\omega, Z, X, Y) = \omega(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z)$$

We want $R^i{}_{jab}$.
now

$$\nabla_X \nabla_Y Z = \nabla_X ((Y^\mu \frac{\partial}{\partial x^\mu} Z^\rho + \Gamma^\rho_{\mu\nu} Z^\mu Y^\nu)\frac{\partial}{\partial x^\rho}) = (X^\alpha \frac{\partial}{\partial x^\alpha}(Y^\mu \frac{\partial}{\partial x^\mu} Z^\rho + \Gamma^\rho_{\mu\nu} Z^\mu Y^\nu) + \Gamma^\rho_{\alpha\beta}(Y^\mu \frac{\partial}{\partial x^\mu} Z^\alpha + \Gamma^\alpha_{\mu\nu} Z^\mu Y^\nu) X^\beta)\frac{\partial}{\partial x^\rho}$$

For $X = \partial_a$, $Y = \partial_b$, $Z = \partial_j$, then the partial derivatives of the coefficients of the input vectors become zero.

$$\implies \nabla_{\partial_a} \nabla_{\partial_b} \partial_j = \frac{\partial}{\partial x^a}(\Gamma^i_{jb}) + \Gamma^i_{\alpha a}\Gamma^\alpha_{jb}$$

Now

$$[X,Y]^i = X^j \frac{\partial}{\partial x^j} Y^i - Y^j \frac{\partial X^i}{\partial x^j}$$

For coordinate vectors, $[\partial_i, \partial_j] = 0 \; \forall \, i, j = 0, 1 \dots d$.
Thus

$$\boxed{R^i{}_{jab} = \frac{\partial}{\partial x^a}\Gamma^i_{jb} - \frac{\partial}{\partial x^b}\Gamma^i_{ja} + \Gamma^i_{\alpha a}\Gamma^\alpha_{jb} - \Gamma^i_{\alpha b}\Gamma^\alpha_{ja}}$$

Question :Ric$(X,Y) :=$ Riem$^m{}_{amb} X^a Y^b$ define $(0,2)$-tensor?.

Yes, transforms as such:

EY developments. I roughly follow the spirit in Theodore Frankel's **The Geometry of Physics: An Introduction** Second Ed. 2003, Chapter 9 Covariant Differentiation and Curvature, Section 9.3b. The Covariant Differential of a Vector Field. P.S. EY : 20150320 I would like a copy of the Third Edition but I don't have the funds right now to purchase the third edition: go to my tilt crowdfunding campaign, `http://ernestyalumni.tilt.com`, and help with your financial support if you can or send me a message on my various channels and ernestyalumni gmail email address if you could help me get a hold of a digital or hard copy as a pro bono gift from the publisher or author.

The spirit of the development is the following:

> "How can we express connections and curvatures in terms of forms?" -Theodore Frankel.

From Lecture 7, connection ∇ on vector field Y, in the "direction" X,

$$\nabla_{\frac{\partial}{\partial x^k}} Y = \left(\frac{\partial Y^i}{\partial x^k} + \Gamma^i_{jk} Y^j \right) \frac{\partial}{\partial x^i}$$

Make the ansatz (approche, impostazione) that the connection ∇ acts on Y, the vector field, first:

$$\nabla Y(X) = \left(X^k \frac{\partial Y^i}{\partial x^k} + \Gamma^i_{jk} Y^j X^k \right) \frac{\partial}{\partial x^i} = X^k \left(\nabla_{\frac{\partial}{\partial x^k}} Y \right)^i \frac{\partial}{\partial x^i} = (\nabla_X Y)^i \frac{\partial}{\partial x^i} = \nabla_X Y$$

Now from Lecture 7, Definition for Γ,

$$dx^i \left(\nabla_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^j} \right) = \Gamma^i_{jk}$$

Make this ansatz (approche, impostazine)

$$\nabla \frac{\partial}{\partial x^j} = (\Gamma^i_{jk} dx^k) \otimes \frac{\partial}{\partial x^i} \in \Omega^1(M, TM) = T^*M \otimes TM$$

where $\Omega^1(M,TM) = T^*M \otimes TM$ is the set of all TM or vector-valued 1-forms on M, with the 1-form being the following:

$$\Gamma^i_{jk} dx^k = \Gamma^i{}_j \in \Omega^1(M) \qquad \begin{aligned} & i = 1 \dots \dim(M) \\ & j = 1 \dots \dim(M) \end{aligned}$$

So $\Gamma^i{}_j$ is a $\dim M \times \dim M$ matrix of 1-forms (EY !!!).

Thus

$$\nabla Y = (d(Y^i) + \Gamma^i_j Y^j) \otimes \frac{\partial}{\partial x^i}$$

So the connection is a (smooth) map from TM to the set of all vector-valued 1-forms on M, $\Omega^1(M,TM)$, and then, after "eating" a vector Y, yields the "covariant derivative":

$$\begin{aligned} & \nabla : TM \to \Omega^1(M,TM) = T^*M \otimes TM \\ & \nabla : Y \mapsto \nabla Y \\ & \nabla Y : TM \to TM \\ & \nabla Y(X) \mapsto \nabla Y(X) = \nabla_X(Y) \end{aligned}$$

Now

$$\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right] f = \frac{\partial}{\partial x^i} \left(\frac{\partial}{\partial x^j} \right) - \frac{\partial}{\partial x^j} \left(\frac{\partial}{\partial x^i} \right) = 0$$

(this is okay as on $p \in (U,x)$; x-coordinates on same chart (U,x))

EY : 20150320 My question is when is this nontrivial or nonvanishing (i.e. not equal to 0).

$$[e_a, e_b] = ?$$

for a frame (e_c) and would this be the difference between a tangent bundle TM vs. a (general) vector bundle?

Wikipedia helps here. cf. wikipedia, "Connection (vector bundle)"

$$\begin{aligned} \nabla : \Gamma(E) \to \Gamma(T^*M \otimes E) = \Omega^1(M,E) \\ \nabla e_a = \omega^c_{ab} f^b \otimes e_c \end{aligned}$$

$f^b \in T^*M$ (this is the dual basis for TM and, note, this is for the manifold, M

$$\begin{aligned} \nabla_{f_b} e_a = \omega^c_{ab} e_c \in E \\ \omega^c_a = \omega^c_{ab} f^b \in \Omega^1(M) \end{aligned}$$

is the connection 1-form, with $a, c = 1 \dots \dim V$. EY : 20150320 This V is a vector space living on each of the fibers of E. I know that $\Gamma(T^*M \otimes E)$ looks like it should take values in E, but it's meaning that it takes vector values of V. Correct me if I'm wrong: ernestyalumni at gmail and various social media.

Let $\sigma \in \Gamma(E)$, $\sigma = \sigma^a e_a$

$$\nabla \sigma = (d\sigma^c + \omega^c_{ab} \sigma^a f^b) \otimes e_c \text{ with}$$

$$d\sigma^c = \frac{\partial \sigma^c}{\partial x^b} f^b$$

$$\Longrightarrow \nabla_X \sigma = \left(X^b \frac{\partial \sigma^c}{\partial x^b} + \omega^c_{ab} \sigma^a X^b \right) e_c = X^b \left(\frac{\partial \sigma^c}{\partial x^b} + \omega^c_{ab} \sigma^a \right) e_c$$

22. Lecture 9: Newtonian spacetime is curved!

Axiom 1 (Newton I:). *A body on which* no *force acts moves uniformly along a straight line*

Axiom 2 (Newton II:). *Deviation of a body's motion from such uniform straight motion is effected by a force, reduced by a factor of the body's reciprocal mass.*

Remark:

(1) 1st axiom - in order to be relevant - must be read as a measurement prescription for the geometry of space ...
(2) Since gravity universally acts on every particle, in a universe with at least two particles, gravity must not be considered a force if Newton I is supposed to remain applicable.

22.1. Laplace's questions.

Laplace $*1749$
$\dagger 1827$

Q: "Can gravity be encoded in a curvature of space, such that its effects show if particles under the influence of (no other) force we postulated to more along straight lines in this curved space?"

Answer: No!

Proof. gravity is a force point of view

$$\begin{aligned} m\ddot{x}^\alpha(t) = F^\alpha(x(t)) \\ m\ddot{x}^\alpha(t) = \underbrace{mf^\alpha}_{F^\alpha}(x(t)) \end{aligned}$$

$-\partial_\alpha f^\alpha = 4\pi G \rho$ (Poisson)
ρ mass density of matter

$$m\ddot{x}^\alpha(t) = \underbrace{mf^\alpha}_{F^\alpha}(x(t))$$

True?

(EY : 20150330) You know this, $F = Gm_1 m_2 / r^2$

Yes

weak equivalence principle

$$\ddot{x}^\alpha(t) - f^\alpha(x(t)) = 0$$

Laplace asks: Is this ($\ddot{x}(t)$) of the form

$$\ddot{x}^\alpha(t) + \Gamma^\alpha_{\ \beta\gamma}(x(t))\dot{x}^\beta(t)\dot{x}^\gamma(t) = 0$$

Conclusion: One cannot find Γ s such that Newton's equation takes the form of an autoparallel.

□

Question (from audience) We can evaluate the autoparallel equation pointwise?! But at each point, we can set the Gammas to zero?!

Then, one should be able to write Newton's second law in the usual form?

Prof. Schuller: you (observer) fall with the mass (i.e. accelerated reference frame) and so you transform Γ's to be 0. The problem with this is if you do the same experiment in the North pole and fall with the body. If someone else at the South Pole does the same experiment at the same time, with that same transformation (of reference frames), the effect of gravity cannot be transformed out.

In a homogeneous gravitational field, you can possibly transform away gravity, $\Gamma = 0$. But in an inhomogeneous gravitational field, no.

22.2. **The full wisdom of Newton I.** use also the information from Newton's first law that particles (no force) move uniformly

introduce the appropriate setting to talk about the difference easily

insight: in spacetime $\boxed{\text{uniform \& straight motion}}$ is simply straight motion

So let's try in spacetime:

let $x : \mathbb{R} \to \mathbb{R}^3$

be a particle's trajectory in space $\longleftrightarrow$ worldline (history) of the particle

$$\begin{aligned} &X : \mathbb{R} \to \mathbb{R}^4 \\ &t \mapsto (t, x^1(t), x^2(t), x^3(t)) := \\ &:= (X^0(t), X^1(t), X^2(t), X^3(t)) \end{aligned}$$

That's all it takes:

Trivial rewritings:

$$\dot{X}^0 = 1$$

$$\Longrightarrow \boxed{\begin{aligned} &\ddot{X}^0 && = 0 \\ &\ddot{X}^\alpha - f^\alpha(X(t)) \cdot \dot{X}^0 \cdot \dot{X}^0 && = 0 \end{aligned}} \quad (\alpha = 1,2,3) \Longrightarrow \begin{gathered} a = 0,1,2,3 \\ \boxed{\ddot{X}^a + \Gamma^a_{\ bc}\dot{X}^b\dot{X}^c = 0} \\ \text{antoparallel eqn in spacetime} \end{gathered}$$

Yes, choosing $\Gamma^0_{\ ab} = 0$

$$\Gamma^\alpha_{\ \beta\gamma} = 0 = \Gamma^\alpha_{\ 0\beta} = \Gamma^\alpha_{\ \beta 0}$$

only: $\boxed{\Gamma^\alpha_{\ 00} \stackrel{!}{=} -f^\alpha}$

Question: Is this a coordinate-choice artifact?

No, since $R^\alpha_{\ 0\beta 0} = -\frac{\partial}{\partial x^\beta} f^\alpha$ (only non-vanishing components) (tidal force tensor, $-$ the Hessian of the force component)

Ricci tensor $\Longrightarrow R_{00} = R^m_{\ 0m0} = -\partial_\alpha f^\alpha = 4\pi G\rho$

Poisson: $-\partial_\alpha f^\alpha = 4\pi G \cdot \rho$

writing: $T_{00} = \frac{1}{2}s$

$$\Longrightarrow \boxed{R_{00} = 8\pi G T_{00}}$$

Einstein in 1912 $\boxed{\cancel{R_{ab} = 8\pi G T_{ab}}}$

Conclusion: Laplace's idea works in spacetime

Remark

$$\Gamma^\alpha_{\ 00} = -f^\alpha$$

$$R^\alpha_{\ \beta\gamma\delta} = 0 \qquad \alpha,\beta,\gamma,\delta = 1,2,3$$

$$\boxed{R_{00} = 4\pi G\rho}$$

Q: What about transformation behavior of LHS of

$$\underbrace{\underbrace{\ddot{x}^a + \Gamma^a_{\ bc}\dot{X}^b\dot{X}^c}_{(\nabla_{v_X} v_X)^a}}_{:=a^a \text{ "acceleration vector"}} = 0$$

22.3. **The foundations of the geometric formulation of Newton's axiom.** new start

Definition 80. *A **Newtonian spacetime** is a quintuple*

$$(M, \mathcal{O}, \mathcal{A}, \nabla, t)$$

where $(M, \mathcal{O}, \mathcal{A})$ 4-dim. smooth manifold

$$t : M \to \mathbb{R} \text{ smooth function}$$

(i) *"There is an absolute space"*

$$(dt)_p \neq 0 \qquad \forall\, p \in M$$

(ii) *"absolute time flows uniformly"*

$$\nabla dt \underbrace{=}_{\text{space of } (0,2)\text{-tensor fields}} 0 \qquad \text{everywhere}$$

∇dt is a $(0,2)$-tensor field

(iii) *add to axioms of Newtonian spacetime $\nabla = 0$ torsion free*

Definition 81. *absolute space at time τ*

$$S_\tau := \{p \in M | t(p) = \tau\}$$

$$\xrightarrow{dt \neq 0} M = \coprod S_\tau$$

Definition 82. *A vector $X \in T_pM$ is called*

(a) *future-directed if*

$$dt(X) > 0$$

(b) *spatial if*

$$dt(X) = 0$$

(c) *past-directed if*

$$dt(X) < 0$$

picture

Newton I: The worldline of a particle under the influence of no force (gravity isn't one, anyway) is a future-directed autoparallel i.e.

$$\nabla_{v_X} v_X = 0$$

$$dt(v_X) > 0$$

and (iii) ∇ is torsion-free.

Newton II:

$$\nabla_{v_X} v_X = \frac{F}{m} \Longleftrightarrow m \cdot \mathfrak{a} = F$$

where F is a spatial vector field:

$$dt(F) = 0$$

Convention: restrict attention to atlases $\mathcal{A}_{\text{stratefied}}$ whose charts $(\mathcal{U}, x)$ have the property

$$\begin{aligned} & x^0 : \mathcal{U} \to \mathbb{R} \\ & x^1 : \mathcal{U} \to \mathbb{R} \qquad x^0 = t|_{\mathcal{U}} \qquad \Longrightarrow \qquad 0 \overset{\text{"absolute time flows uniformly"}}{=} \nabla dt \\ & \vdots \quad \vdots \qquad\qquad\qquad\qquad 0 = \nabla_{\frac{\partial}{\partial x^a}} dx^0 = -\Gamma^0_{\ ba} \qquad a = 0,1,2,3 \\ & x^3 \end{aligned}$$

Let's evaluate in a chart $(\mathcal{U}, x)$ of a stratified atlas $\mathcal{A}_{\text{sheet}}$: Newton II:

$$\nabla_{v_X} v_X = \frac{F}{m}$$

in a chart.

$$(X^0)'' + \cancel{\Gamma^0_{\ cd}(X^a)'(X^b)'}^{\text{stratified atlas}} = 0$$

$$(X^\alpha)'' + \Gamma^\alpha_{\ \gamma\delta} X^{\gamma'} X^{\delta'} + \Gamma^\alpha_{\ 00} X^{0'} X^{0'} + 2\Gamma^\alpha_{\ \gamma 0} X^{\gamma'} X^{0'} = \frac{F^\alpha}{m} \qquad \alpha = 1,2,3$$

EY: 20160623: where the factor of 2 comes from torsion free ∇

$$\Longrightarrow (X^0)''(\lambda) = 0 \Longrightarrow X^0(\lambda) = a\lambda + b \quad \text{constants } a, b \text{ with}$$

$$X^0(\lambda) = (x^0 \circ X)(\lambda) \overset{\text{stratified}}{=} (t \circ X)(\lambda)$$

<u>convention</u> parametrize worldline by absolute time

$$\frac{d}{d\lambda} = a \frac{d}{dt}$$

$$a^2 \ddot{X}^\alpha + a^2 \Gamma^\alpha_{\ \gamma\delta} \dot{X}^\gamma \dot{X}^\delta + a^2 \Gamma^\alpha_{\ 00} \dot{X}^0 \dot{X}^0 + 2\Gamma^\alpha_{\ \gamma 0} \dot{X}^\gamma \dot{X}^0 = \frac{F^\alpha}{m}$$

$$\Longrightarrow \underbrace{\ddot{X}^\alpha + \Gamma^\alpha_{\ \gamma\delta} \dot{X}^\gamma \dot{X}^\delta + \Gamma^\alpha_{\ 00} \dot{X}^0 \dot{X}^0 + 2\Gamma^\alpha_{\ \gamma 0} \dot{X}^\gamma \dot{X}^0}_{a^\alpha} = \frac{1}{a^2} \frac{F^\alpha}{m}$$

23. Lecture 10: Metric Manifolds

cf. Lecture 10: Metric Manifolds (International Winter School on Gravity and Light 2015)

We establish a structure on a smooth manifold that allows one to assign vectors in each tangent space a length (and an angle between vectors in the same tangent space).

From this structure, one can then define a notion of length of a curve.
Then we can look at shortest curves.

Requiring then that the shortest curves coincide with the straightest curves (wrt ∇) will result in ∇ being determined by the metric structure.

$$g \overset{\text{straight=short}}{\underset{T=0}{\rightsquigarrow}} \nabla \rightsquigarrow \text{Riem}$$

23.1. Metrics.

Definition 83. *A metric g on a smooth manifold $(M, \mathcal{O}, \mathcal{A})$ is a $(0,2)$-tensor field satisfying*

(i) *<u>symmetry</u> $g(X,Y) = g(Y,X) \quad \forall X, Y$ vector fields*
(ii) *<u>non-degeneracy</u>: the <u>musical map</u>*

$$\begin{aligned} \text{"flat" } \flat : \Gamma(TM) & \to \Gamma(T^*M) \\ X & \mapsto \flat(X) \end{aligned}$$

where $\flat(X)(Y) := g(X,Y)$
$\flat(X) \in \Gamma(T^*M)$
In thought bubble: $\flat(X) = g(X, \cdot)$
... is a C^∞-isomorphism in other words, it is invertible.

<u>Remark</u>: $(\flat(X))_a$ or
X_a
$(\flat(X))_a := g_{am} X^m$
Thought bubble: $\flat^{-1} = \sharp$
$\flat^{-1}(\omega)^a := g^{am}\omega_m$
$\flat^{-1}(\omega)^a := (g^{\text{"}-1\text{"}})^{am}\omega_m \Longrightarrow$ not needed. (all of this is not needed)

Definition 84. *The $(2,0)$-tensor field $g^{\text{"}-1\text{"}}$ with respect to a metric g is the symmetric*

$$\begin{aligned} g^{\text{"}-1\text{"}} : \Gamma(T^*M) \times \Gamma(T^*M) & \to C^\infty(M) \\ (\omega, \sigma) \mapsto \omega(\flat^{-1}(\sigma)) & \qquad \flat^{-1}(\sigma) \in \Gamma(TM)) \end{aligned}$$

<u>chart</u>: $g_{ab} = g_{ba}$
$(g^{-1})^{am} g_{mb} = \delta^a_b$

<u>Example</u>: $(S^2, \mathcal{O}, \mathcal{A})$
chart $(\mathcal{U}, x)$
$\varphi \in (0, 2\pi)$
$\theta \in (0, \pi)$
<u>define</u> the metric

$$g_{ij}(x^{-1}(\theta, \varphi)) = \begin{bmatrix} R^2 & 0 \\ 0 & R^2 \sin^2\theta \end{bmatrix}_{ij}$$

$R \in \mathbb{R}^+$
"the metric of the <u>round sphere of radius R</u>"

$$A^a{}_m v^m = \lambda v^a \qquad \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

23.2. **Signature.** Linear algebra:

$$g_{am} v^m = \lambda \cdot v^a ? \rightsquigarrow \begin{pmatrix} 1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & 1 & & & & & & \\ & & & -1 & & & & & \\ & & & & \ddots & & & & \\ & & & & & -1 & & & \\ & & & & & & 0 & & \\ & & & & & & & \ddots & \\ & & & & & & & & 0 \end{pmatrix}$$

$(1,1)$ tensor has eigenvalues
$(0,2)$ has signature (p,q) (well-defined)

$$\left.\begin{matrix} (+++) \\ (++-) \\ (+--) \\ (---) \end{matrix}\right\} d+1 \text{ if } p+q = \dim V$$

Definition 85. *A metric is called*
Riemannian *if its signature is* $(++\cdots+)$
Lorentzian *if* $(+-\cdots-)$

23.3. **Length of a curve.** Let γ be a smooth curve.
Then we know its veloctiy $v_{\gamma,\gamma(\lambda)}$ at each $\gamma(\lambda) \in M$.

Definition 86. *On a Riemannian metric manifold* $M, \mathcal{O}, \mathcal{A}, g)$*, the* ***speed*** *of a curve at* $\gamma(\lambda)$ *is the number*

$$(\sqrt{g(v_\gamma, v_\gamma)})_{\gamma(\lambda)} = s(\lambda)$$

F. Schuller: "I feel the need for speed." -Top Gun.
(I feel the need for speed, then I feel the need for a metric)
Aside: $[v^a] = \frac{1}{T}$
$[g_{ab}] = L^2$
$[\sqrt{g_{ab} v^a v^b}] = \sqrt{\frac{L^2}{T^2}} = \frac{L}{T}$

Definition 87. *Let* $\gamma : (0,1) \to M$ *a smooth curve.*
Then the ***length of*** γ *is the number*

$$\mathbb{R} \ni L[\gamma] := \int_0^1 d\lambda s(\lambda) = \int_0^1 d\lambda \sqrt{(g(v_\gamma, v_\gamma))_{\gamma(\lambda)}}$$

F. Schuller: "velocity is more fundamental than speed, speed is more fundamental than length"
Example: reconsider the round sphere of radius R
Consider its equator:
$\theta(\lambda) := (x^1 \circ \gamma)(\lambda) = \frac{\pi}{2}$
$\varphi(\lambda) := (x^2 \circ \gamma)(\lambda) = 2\pi\lambda^3$
$\theta'(\lambda) = 0$
$\varphi'(\lambda) = 6\pi\lambda^2$

on the same chart $g_{ij} = \begin{bmatrix} R^2 & \\ & R^2 \sin^2\theta \end{bmatrix}$
F.Schuller: do everything in this chart

$$L[\gamma] = \int_0^1 d\lambda \sqrt{g_{ij}(x^{-1}(\theta(\lambda), \varphi(\lambda)))(x^i \circ \gamma)'(\lambda)(x^j \circ \gamma)'(\lambda)} = \int_0^1 d\lambda \sqrt{R^2 \cdot 0 + R^2 \sin^2(\theta(\lambda)) 36\pi^2\lambda^4} =$$

$$= 6\pi R \int_0^1 d\lambda \lambda^2 = 6\pi R [\frac{1}{3}\lambda^3]_0^1 = 2\pi R$$

Theorem 25. $\gamma : (0,1) \to M$ *and*
$\sigma : (0,1) \to (0,1)$ *smooth bijective and* *increasing* *"reparametrization"*

$$L[\gamma] = L[\gamma \circ \sigma]$$

Proof. $\Longrightarrow$ Tutorials $\square$

23.4. **Geodesics.**

Definition 88. *A curve* $\gamma : (0,1) \to M$ *is called a* ***geodesic*** *on a Riemannian manifold* $(M, \mathcal{O}, \mathcal{A}, g)$ *if its a* stationary *curve with respect to a length functional* L*.*

Thought bubble: in classical mechanics, deform the curve a little, ϵ times this deformation, to first order, it agrees with $L[\gamma]$

Theorem 26. γ *geodesic iff it satisfies the Euler-Lagrange equations for the Lagrangian*

$$\begin{aligned} \mathcal{L} : & TM \to \mathbb{R} \\ & X \mapsto \sqrt{g(X,X)} \end{aligned}$$

In a chart, the Euler Lagrange equations take the form:

$$\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^m} \right)^{\cdot} - \frac{\partial \mathcal{L}}{\partial x^m} = 0$$

F.Schuller: this is a chart dependent formulation
here:

$$\mathcal{L}(\gamma^i, \dot{\gamma}^i) = \sqrt{g_{ij}(\gamma(\lambda))\dot{\gamma}^i(\lambda)\dot{\gamma}^j(\lambda)}$$

Euler-Lagrange equations:

$$\frac{\partial \mathcal{L}}{\partial \dot{\gamma}^m} = \frac{1}{\sqrt{\cdots}} g_{mj}(\gamma(\lambda))\dot{\gamma}^j(\lambda)$$

$$\left(\frac{\partial \mathcal{L}}{\partial \dot{\gamma}^m} \right)^{\cdot} = \left(\frac{1}{\sqrt{\cdots}} \right)^{\cdot} g_{mj}(\gamma(\lambda)) \cdot \dot{\gamma}^j(\lambda) + \frac{1}{\sqrt{\cdots}} \left(g_{mj}(\gamma(\lambda))\ddot{\gamma}^j(\lambda) + \dot{\gamma}^s (\partial_s g_{mj})\dot{\gamma}^j(\lambda) \right)$$

Thought bubble: reparametrize $g(\dot{\gamma}, \dot{\gamma}) = 1$ (it's a condition on my reparametrization)
By a clever choice of reparametrization $(\frac{1}{\sqrt{\cdots}})^{\cdot} = 0$

$$\frac{\partial \mathcal{L}}{\partial \gamma^m} = \frac{1}{2\sqrt{\cdots}} \partial_m g_{ij}(\gamma(\lambda))\dot{\gamma}^i(\lambda)\dot{\gamma}^j(\lambda)$$

putting this together as Euler-Lagrange equations:

$$g_{mj}\ddot{\gamma}^j + \partial_s g_{mj}\dot{\gamma}^s\dot{\gamma}^j - \frac{1}{2}\partial_m g_{ij}\dot{\gamma}^i\dot{\gamma}^j = 0$$

Multiply on both sides $(g^{-1})^{qm}$

$$\ddot{\gamma}^q + (g^{-1})^{qm}(\partial_i g_{mj} - \frac{1}{2}\partial_m g_{ij})\dot{\gamma}^i\dot{\gamma}^j = 0$$

$$\boxed{\ddot{\gamma}^q + (g^{-1})^{qm}\frac{1}{2}(\partial_i g_{mj} + \partial_j g_{mi} - \partial_m g_{ij})\dot{\gamma}^i\dot{\gamma}^j = 0}$$

geodesic equation for γ in a chart.

$$\boxed{(g^{-1})^{qm}\frac{1}{2}(\partial_i g_{mj} + \partial_j g_{mi} - \partial_m g_{ij}) =: \Gamma^q_{\ ij}(\gamma(\lambda))}$$

Thought bubble: $\left(\frac{\partial \mathcal{L}}{\partial \xi_x^{a+\dim M}}\right)^{\cdot}_{\sigma(x)} - \left(\frac{\partial \mathcal{L}}{\partial x i_x^a}\right)_{\sigma(x)} = 0$

Definition 89. *"Christoffel symbol" ${}^{L.C.}\Gamma$ are the connection coefficient functions of the so-called Levi-Civita connection ${}^{L.C.}\nabla$*

We usually make this choice of ∇ if g is given.
$(M, \mathcal{O}, \mathcal{A}, g) \to (M, \mathcal{O}, \mathcal{A}, g, {}^{\text{L.C.}}\nabla)$
abstract way: $\nabla g = 0$ and $T = 0$ (torsion)
$\implies \nabla = {}^{\text{L.C.}}\nabla$

Definition 90. (a) *The Riemann-Christoffel curvature is defined by*

$$R_{abcd} := g_{am}R^m_{\ bcd}$$

(b) *Ricci:* $R_{ab} = R^m_{\ amb}$
Thought bubble: with a metric, ${}^{L.C.}\nabla$
(c) *(Ricci) scalar curvature:*

$$R = g^{ab}R_{ab}$$

Thought bubble: ${}^{L.C.}\nabla$

Definition 91. *Einstein curvature* $(M, \mathcal{O}, \mathcal{A}, g)$

$$G_{ab} := R_{ab} - \frac{1}{2}g_{ab}R$$

Convention: $g^{ab} := (g^{\text{“}-1\text{''}})^{ab}$
F. Schuller: these indices are not being pulled up, because what would you pull them up with
(student) Question: Does the Einstein curvature yield new information?
Answer:

$$g^{ab}G_{ab} = R_{ab}g^{ab} - \frac{1}{2}g_{ab}g^{ab}R = R - \delta^a_a R = R - \frac{1}{2}\dim M\, R = (1 - \frac{d}{2})R$$

Tutorial 9: Metric manifolds. Exercise 3: Levi-Civita Connection. Suppose torsion-free $T = 0$ and metric-compatible connection $\nabla g = 0$

Question Recall $T = 0$ on a chart.

$$\boxed{\Gamma^c_{ba} = \frac{1}{2}(g^{-1})^{cm}\left(\frac{\partial g_{bm}}{\partial x^a} + \frac{\partial g_{ma}}{\partial x^b} - \frac{\partial g_{ab}}{\partial x^m}\right)}$$

or

$$\Gamma^a_{bc} = \frac{1}{2}(g^{-1})^{am}\left(\frac{\partial g_{bm}}{\partial x^c} + \frac{\partial g_{mc}}{\partial x^b} - \frac{\partial g_{bc}}{\partial x^m}\right)$$

24. Symmetry

EY : 20150321 This lecture tremendously and lucidly clarified, for me at least, what a symmetry of the Lie algebra is, and in comparing structures $(M, \mathcal{O}, \mathcal{A})$ vs. $(M, \mathcal{O}, \mathcal{A}, \nabla)$, clarified differences, and asking about differences is a good way to learn, the difference between $\mathcal{L}$ and ∇, respectively.

Feeling that the round sphere

$$(S^2, \mathcal{O}, \mathcal{A}, g^{\text{round}})$$

has rotational symmetry, while
the potato

$$(S^2, \mathcal{O}, \mathcal{A}, g^{\text{potato}})$$

does not.

24.1.

24.2. Important

24.3. Flow of a complete vector field.
Let $(M, \mathcal{O}, \mathcal{A})$ smooth X vector field on M

Definition 92. *A curve $\gamma : I \subseteq \mathbb{R} \to M$ is called an integral curve of X if*

$$v_{\gamma, \gamma(\lambda)} = X_{\gamma(\lambda)}$$

Definition 93. *A vector filed X is **complete** if all integral curves have $I = \mathbb{R}$ EY: 20150321 (i.e. domain is all of $\mathbb{R}$)*

Ex. minute 48:30 EY : reall good explanation by F.P.Schuller; take a pt. out for an incomplete vector field.

Theorem 27. *compactly supported smooth vector field is complete.*

Definition 94. *The flow of a complete vector field X is a 1-parameter family*

$$h^X = \mathbb{R} \times M \to M$$

where $\gamma_p : \mathbb{R} \to M$ is the integral curve of X with
$\gamma(0) = p$
Then for fixed $\lambda \in \mathbb{R}$

$$h^X_\lambda : M \to M \textit{ smooth}$$

picture $h^X_\lambda(S) \neq S($ *if* $X \neq 0)$

24.4. Lie subalgebras of the Lie algebra $(\Gamma(TM), [\cdot, \cdot])$ of vector fields.

(a) $\Gamma(TM) = \{$ set of all vector fields $\}$ $C^\infty(M)$-module $= \mathbb{R}$-vector space

$$\implies [X, Y] \in \Gamma(TM) \qquad [X, Y]f := X(Yf) - Y(Xf)$$

(i) $[X, Y] = -[Y, X]$
(ii) $[\lambda X + Z, Y] = \lambda[X, Y] + [Z, Y]$
(iii) $[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0$
$(\Gamma(TM), [\cdot, \cdot])$ Lie algebra

(b) Let $X_1 \dots X_s$ for s (many) vector fields on M, such that

Tutorial 11 Symmetry. Exercise 1. : True or false?

(a)
-
- $\phi^* : T^*N \to T * M$ i.e. $\phi^*\nu(X) = \nu(\phi_* X)$ for smooth $\phi : M \to N$, so the pullback of a covector $\nu \in T^*N$ maps to a covector in $T * M$.
-
-
-
-

(b)

(c)

Exercise 2. : Pull-back and push-forward

Question . Let's check this locally

$$\phi^*(df)(X) = (df)(\phi_* X) = (df)(X^i \frac{\partial y^j}{\partial x^i}\frac{\partial}{\partial y^j}) = X^i \frac{\partial y^j}{\partial x^i}\frac{\partial f}{\partial y^j} \text{ where } \quad \phi_* X = X^i \frac{\partial y^j}{\partial x^i}\frac{\partial}{\partial y^j}$$

$$d(\phi^* f)(X) = d(f(\phi))(X) = \frac{\partial f}{\partial y^j}\frac{\partial y^j}{\partial x^i} dx^i(X) = X^i \frac{\partial y^j}{\partial x^i}\frac{\partial f}{\partial y^j}$$

So

$$\boxed{\phi^*(df) = d(\phi^* f)} \qquad \forall\, p \in M,\ \forall\, X \in \mathfrak{X}(M)$$

The big idea is that this is a showing of the **naturality** of the pullback ϕ^* with d, i.e. that this commutes:

$$\begin{tikzcd} \Omega^1(M) & \Omega^1(N) \arrow[l, "\phi^*"'] \\ C^\infty(M) \arrow[u, "d"] & C^\infty(N) \arrow[l, "\phi^*"] \arrow[u, "d"'] \end{tikzcd}$$

Question .

$$(\phi_*)^a_{\ b} := (dy^a)(\phi_*(\frac{\partial}{\partial x^b}))$$

Let $g \in C^\infty(N)$

$$\phi_*\left(\frac{\partial}{\partial x^b}\right) g = \frac{\partial x^b}{g}\phi(p) = \frac{\partial}{\partial x^b} g\phi x^{-1} x(p) = \frac{\partial}{\partial x^b}(gyy^{-1}\phi x^{-1})(x) =$$

$$= \frac{\partial}{\partial x^b}(gy^{-1}(y\phi x^{-1}(x(p)))) = \left.\frac{\partial g}{\partial y}^b\right|_y \left.\frac{\partial y^a}{\partial x^b}\right|_x = \frac{\partial y^a}{\partial x^b}\frac{\partial g}{\partial y^a}$$

Then

$$\phi_*\left(\frac{\partial}{\partial x^b}\right) = \frac{\partial y^a}{\partial x^b}\frac{\partial}{\partial y^a}$$

and so

$$(\phi_*)^a_{\ b} = \frac{\partial y^a}{\partial x^b}$$

Question .

Exercise 3. :Lie derivative-the pedestrian way

Question . While it is true that $\forall\, p \in S^2$, for $x(p) = (\theta, \varphi)$, and $(yix^{-1})(\theta,\varphi) = (y^1, y^2, y^3) \in \mathbb{R}^3$ and that, at this point p, $(y^1)^2/a^2 + (y^2)^2/b^2 + (y^3)^2/c^3 = 1$, this doesn't imply (EY: 20150321 I think) that, globally, it's an ellipsoid (yet). In the familiar charts given,
spherical chart $(U, x) \in \mathcal{A}$ and
$(\mathbb{R}^3, y = \mathrm{id}_{\mathbb{R}^3}) \in \mathcal{B}$
it looks like an ellipsoid, but change to another choice of charts, and it could look something very different.

Question .

Equip $(\mathbb{R}^3, \mathcal{O}_{\mathrm{st}}, \mathcal{B})$ with the Euclidean metric g, and pullback g.
Note that the pullback of the inclusion from $\mathbb{R}^3$ onto S^2 for the Euclidean metric is the following:

$$i^* g\left(\frac{\partial}{\partial\theta^i}, \frac{\partial}{\partial\theta^j}\right) = g\left(i_*\frac{\partial}{\partial\theta^i}, i_*\frac{\partial}{\partial\theta^j}\right) = g\left(\frac{\partial x^a}{\partial\theta^i}\frac{\partial}{\partial x^a}, \frac{\partial x^b}{\partial\theta^j}\frac{\partial}{\partial x^b}\right) = g_{ab}\frac{\partial x^a}{\partial\theta^i}\frac{\partial x^b}{\partial\theta^j}$$

With $g_{ab} = \delta_{ab}$, the usual Euclidean metric, this becomes the following:

$$g_{ij}^{\text{ellipsoid}} = \frac{\partial x^a}{\partial\theta^i}\frac{\partial x^a}{\partial\theta^j}$$

At this point, one should get smart (we are in the 21st century) and use some sort of CAS (Computer Algebra System). I like Sage Math (version 6.4 as of 20150322). I also like the Sage Manifolds package for Sage Math.

I like Sage Math for the following reasons:

- Open source, so it's open and freely available to anyone, which fits into my principle of making online education open and freely available to anyone, anytime
- Sage Math structures everything in terms of Category Theory and Categories and Morphisms naturally correspond to Classes and Class methods or functions in Object-Oriented Programming in Python and they've written it that way

and I like Sage Manifolds for roughly the same reasons, as manifolds are fit into a category theory framework that's written into the Python code. e.g.

```
sage: S2 = Manifold(2, 'S^2', r'\mathbb{S}^2', start_index=1) ; print S2
sage: print S2
2-dimensional manifold 'S^2'
sage: type(S2)
<class 'sage.geometry.manifolds.manifold.Manifold_with_category'>
```

With code (I've provided for convenience; you can make your own as I wrote it based upon to example of S^2 on the sage-manifolds documentation website page), load it and do the following:

cf. https://github.com/ernestyalumni/diffgeo-by-sagemnfd/blob/master/S2.sage
http://sagemanifolds.obspm.fr/examples.html

```
sage: load("S2.sage")
sage: U_ep = S2.open_subset('U_{ep}')
sage: eps.<the,phi> = U_ep.chart()
sage: a = var(\a")
sage: b = var(\b")
sage: c = var("c")
sage: inclus = S2.diff_mapping(R3, {(eps, cart): [ a*cos(phi)*sin(the), b*sin(phi)*sin(the),c*cos(the) ]} , name="inc",latex_name=r'\mathcal{i}')
sage: inclus.pullback(h).display()
inc_*(h) = (c^2*sin(the)^2 + (a^2*cos(phi)^2 + b^2*sin(phi)^2)*cos(the)^2) dthe*dthe - (a^2 - b^2)*cos(phi)*cos(the)*sin(phi)*sin(the) dthe*dphi
- (a^2 - b^2)*cos(phi)*cos(the)*sin(phi)*sin(the) dphi*dthe + (b^2*cos(phi)^2 + a^2*sin(phi)^2)*sin(the)^2 dphi*dphi
sage: inclus.pullback(h)[2,2].expr()
(b^2*cos(phi)^2 + a^2*sin(phi)^2)*sin(the)^2
```

A new open subset U_{ep} was declared in S^2, a new chart $(U_{\text{ep}}, (\theta, \phi))$ was declared, the constants, a, b, c, were declared, and the inclusion map given in the problem

$$y \circ \mathrm{i} \circ x^{-1} : (\theta, \phi) \mapsto (a \cos\phi \sin\theta, b \sin\phi \sin\theta, c \cos\theta)$$

Then the pullback of the inclusion map $\rangle$ was done on the Euclidean metric h, defined earlier in the file

`S2.sage`

. Then one can access the components of this metric and do, for example,

`simplify_full(),full_simplify(), reduce_trig()`

on the expression.

In Python, I could easily do this, and give an answer quick in LaTeX:

```
sage: for i in range(1,3):
....:     for j in range(1,3):
....:         print inclus.pullback(h)[i,j].expr()
....:         latex(inclus.pullback(h)[i,j].expr() )
....:
c^2*sin(the)^2 + (a^2*cos(phi)^2 + b^2*sin(phi)^2)*cos(the)^2
```

(EY: I'll suppress the LaTeX output but this sage math function gives you LaTeX code)

and so

$$\boxed{\begin{gathered} i^* g = c^2 \sin(the)^2 + \left(a^2 \cos(\phi)^2 + b^2 \sin(\phi)^2\right) \cos(the)^2 \, d\theta \otimes d\theta + \\ -2\left(a^2 - b^2\right) \cos(\phi) \cos(the) \sin(\phi) \sin(the) \, d\theta \otimes d\phi + \\ + \left(b^2 \cos(\phi)^2 + a^2 \sin(\phi)^2\right) \sin(the)^2 \, d\phi \otimes d\phi \end{gathered}}$$

Question .

```
sage: polar_vees = eps.frame()
sage: X_1 = - sin(phi) * polar_vees[1] - cot( the ) * cos(phi) * polar_vees[2]
sage: X_2 = cos( phi ) * polar_vees[1] - cot( the ) * sin( phi) * polar_vees[2]
sage: X_3 = polar_vees[2]
sage: X_2.lie_der(X_1).display()
(cos(the)^2 - 1)/sin(the)^2 d/dphi
sage: X_3.lie_der(X_1).display()
cos(phi) d/dthe - cos(the)*sin(phi)/sin(the) d/dphi
sage: X_3.lie_der(X_2).display()
sin(phi) d/dthe + cos(phi)*cos(the)/sin(the) d/dphi
```

Indeed, one can check on a scalar field $f_{\text{eps}} \in C^\infty(S^2)$:

```
sage: f_eps = S2.scalar_field({eps: function('f', the, phi ) }, name='f' )
sage: (X_1( X_2(f_eps)) - X_2(X_1(f_eps) ) ).display()
U_{ep} --> R
(the, phi) |--> -D[1](f)(the, phi)
sage: X_2.lie_der(X_1) == -X_3
True
sage: X_3.lie_der(X_1) == X_2
True
sage: X_3.lie_der(X_2) == -X_1
True
```

$$\implies \boxed{[X_i, X_j] = -\epsilon_{ijk} X_k}$$

So $\text{span}_{\mathbb{R}}\{X_1, X_2, X_3\}$ equipped with $[\,,\,]$ constitute a Lie subalgebra on S^2 (It's closed under $[\,,\,]$

25. Integration

25.1.

25.2.

25.3. **Volume forms.**

Definition 95. *On a smooth manifold* $(M, \mathcal{O}, \mathcal{A})$
a $(0, dim M)$*-tensor field* Ω *is called a* <u>*volume form*</u> *if*

(a) Ω *vanishes nowhere (i.e.* $\Omega \neq 0 \; \forall \, p \in M$*)*
(b) *totally antisymmetric*

$$\Omega(\dots, \underbrace{X}_{ith}, \dots, \underbrace{Y}_{jth} \dots) = -\Omega(\dots, \underbrace{Y}_{ith}, \dots, \underbrace{X}_{jth} \dots)$$

In a chart:

$$\Omega_{i_1 \dots i_d} = \Omega_{[i_1 \dots i_d]}$$

<u>Example</u> $(M, \mathcal{O}, \mathcal{A}, g)$ metric manifold
construct volume form Ω from g
In <u>any</u> chart: (U, x)

$$\Omega_{i_1 \dots i_d} := \sqrt{\det(g_{ij}(x))} \epsilon_{i_1 \dots i_d}$$

where **Levi-Civita symbol** $\epsilon_{i_1 \dots i_d}$ is <u>defined</u> as $\epsilon_{123 \dots d} = +1$
$\epsilon_{1 \dots d} = \epsilon_{[i_1 \dots i_d]}$

Proof. (well-defined) Check: What happens under a change of charts

$$\begin{aligned} \Omega(y)_{i_1 \dots i_d} &= \sqrt{\det(g(y)_{ij})} \epsilon_{i_1 \dots i_d} = \\ &= \sqrt{\det(g_{mn}(x) \frac{\partial x^m}{\partial y^i} \frac{\partial x^n}{\partial y^j})} \frac{\partial y^{m_1}}{\partial x^{i_1}} \dots \frac{\partial y^{m_d}}{\partial x^{i_d}} \epsilon_{[m_1 \dots m_d]} = \\ &= \sqrt{|\det g_{ij}(x)|} \left| \det \left(\frac{\partial x}{\partial y} \right) \right| \det \left(\frac{\partial y}{\partial x} \right) \epsilon_{i_1 \dots i_d} = \sqrt{\det g_{ij}(x)} \epsilon_{i_1 \dots i_d} \text{sgn} \left(\det \left(\frac{\partial x}{\partial y} \right) \right) \end{aligned}$$

□

EY : 20150323

Consider the following:

$$\Omega(y)(Y_{(1)}\dots Y_{(d)}) = \Omega(y)_{i_1\dots i_d} Y^{i_1}_{(1)}\dots Y^{i_d}_{(d)} =$$
$$= \sqrt{\det(g_{ij}(y))}\epsilon_{i_1\dots i_d} Y^{i_1}_{(1)}\dots Y^{i_d}_{(d)} =$$
$$= \sqrt{\det(g_{mn}(x))\frac{\partial x^m}{\partial y^i}\frac{\partial x^n}{\partial y^j}}\epsilon_{i_1\dots i_d}\frac{\partial y^{i_1}}{\partial x^{m_1}}\dots\frac{\partial y^{i_d}}{\partial x^{m_d}} X^{m_1}\dots X^{m_d} =$$
$$= \sqrt{\det(g_{mn}(x))\frac{\partial x^m}{\partial y^i}\frac{\partial x^n}{\partial y^j}}\det\left(\frac{\partial y}{\partial x}\right)\epsilon_{m_1\dots m_d} X^{m_1}\dots X^{m_d} =$$
$$= \sqrt{\det(g_{mn}(x))}\left|\det\left(\frac{\partial x}{\partial y}\right)\right|\det\left(\frac{\partial y}{\partial x}\right)\epsilon_{m_1\dots m_d} X^{m_1}\dots X^{m_d} =$$
$$= \sqrt{\det(g_{mn}(x))}\epsilon_{m_1\dots m_d}\mathrm{sgn}\left(\det\left(\frac{\partial x}{\partial y}\right)\right) X^{m_1}\dots X^{m_d} = \mathrm{sgn}(\det\left(\frac{\partial x}{\partial y}\right))\Omega_{m_1\dots m_d}(x) X^{m_1}\dots X^{m_d}$$

If $\det\left(\frac{\partial y}{\partial x}\right) > 0$,

$$\Omega(y)(Y_{(1)}\dots Y_{(d)}) = \Omega(x)(X_{(1)}\dots X_{(d)})$$

This works also if Levi-Civita symbol $\epsilon_{i_1\dots i_d}$ doesn't change at all under a change of charts. (around 42:43 https://youtu.be/2XpnbvPy-Zg)

Alright, let's require,
restrict the smooth atlas $\mathcal{A}$
to a subatlas ($\mathcal{A}^\uparrow$ still an atlas)

$$\mathcal{A}^\uparrow \subseteq \mathcal{A}$$

s.t. $\forall (U,x),(V,y)$ have chart transition maps $y\circ x^{-1}$
$x \circ y^{-1}$

s.t. $\det\left(\frac{\partial y}{\partial x}\right) > 0$
such $\mathcal{A}^\uparrow$ called an **oriented** atlas

$$(M,\mathcal{O},\mathcal{A},g) \Longrightarrow (M,\mathcal{O},\mathcal{A}^\uparrow,g)$$

Note: associated bundles.
Note also: $\det\left(\frac{\partial y^b}{\partial x^a}\right) = \det(\partial_a(y^b x^{-1}))$ $\frac{\partial y^b}{\partial x^a}$ is an endomorphism on vector space V.

$\varphi : V \to V$
$\det\varphi$ independent of choice of basis
g is a $(0,2)$ tensor field, not endomorphism (not independent of choice of basis) $\sqrt{|\det(g_{ij}(y))|}$

Definition 96. *Ω be a volume form on $(M,\mathcal{O},\mathcal{A}^\uparrow)$ and consider chart (U,x)*

Definition 97. $\omega_{(X)} := \Omega_{i_1\dots i_d}\epsilon^{i_1\dots i_d}$ *same way* $\epsilon^{12\dots d} = +1$
$\epsilon^{[\dots]}$

one can show

$$\boxed{\omega_{(y)} = \det\left(\frac{\partial x}{\partial y}\right)\omega_{(x)}} \qquad \textit{scalar density}$$

25.4. Integration on <u>one</u> chart domain U.

Definition 98.

$$\boxed{\int_U f :\overset{(U,y)}{=} \int_{y(U)} d^d\beta\,\omega_{(y)}(y^{-1}(\beta))f_{(y)}(\beta)} \tag{71}$$

Proof. : Check that it's (well-defined), how it changes under change of charts

$$\int_U f :\overset{(U,y)}{=} \int_{y(U)} d^d\beta\,\omega_{(y)}(y^{-1}(\beta))f_{(y)}(\beta) = \underset{(U,y)}{=} \int_{x(U)}\int d^d\alpha\left|\det\left(\frac{\partial y}{\partial x}\right)\right| f_{(x)}(\alpha)\omega_{(x)}(x^{-1}(\alpha)\det\left(\frac{\partial x}{\partial y}\right) =$$
$$= \int_{x(U)} d^d\alpha\,\omega_{(x)}(x^{-1}(x))f_{(x)}(\alpha)$$

□

On an oriented metric manifold $(M,\mathcal{O},\mathcal{A}^\uparrow,g)$

$$\int_U f := \int_{x(U)} d^d\alpha \underbrace{\sqrt{\det(g_{ij}(x))(x^{-1}(\alpha))}}_{\sqrt{g}} f_{(x)}(\alpha)$$

25.5. Integration on the entire manifold.

26. Lecture 13: Relativistic spacetime

Recall, from Lecture 9, the definition of Newtonian spacetime

$$(M,\mathcal{O},\mathcal{A},\nabla,t) \quad \begin{array}{l} \nabla \text{ torsion free} \\ t\in C^\infty(M) \\ dt \neq 0 \\ \nabla dt = 0 \quad \text{(uniform time)}\end{array}$$

and the definition of relativistic spacetime (before Lecture)

$$(M,\mathcal{O},\mathcal{A}^\uparrow,\nabla,g,T) \quad \begin{array}{l} \nabla \text{ torsion-free} \\ g \text{ Lorentzian metric}(+---) \\ T \text{ time-orientation}\end{array}$$

26.1. Time orientation.

Definition 99. *$(M,\mathcal{O},\mathcal{A}^\uparrow,g)$ a Lorentzian manifold. Then a time-orientation is given by a vector field T that*
(i) *does **not** vanish anywhere*
(ii) $g(T,T) > 0$

Newtonian vs. relativistic
Newtonian
X was called future-directed if

$$dt(X) > 0$$

$\forall p \in M$, take half plane, half space of T_pM
also stratified atlas so make planes of constant t straight
relativistic

half cone $\forall p, q \in M$, half-cone $\subseteq T_pM$

This definition of spacetime
Question
I see how the cone structure arises from the new metric. I don't understand however, how the T, the time orientation, comes in

Answer
$(M, \mathcal{O}, \mathcal{A}, g)$ $g \xleftarrow{(} + - - -)$
requiring $g(X,X) > 0$, select cones
T chooses which cone

This definition of spacetime has been made to enable the following physical postulates:

(P1) The worldline γ of a massive particle satisfies
 (i) $g_{\gamma(\lambda)}(v_{\gamma,\gamma(lambda)}, v_{\gamma,\gamma(\lambda)}) > 0$
 (ii) $g_{\gamma(\lambda)}(T, v_{\gamma,\gamma(\lambda)}) > 0$

(P2) Worldlines of massless particles satisfy
 (i) $g_{\gamma(\lambda)}(v_{\gamma,\gamma(\lambda)}, v_{\gamma,\gamma(\lambda)}) = 0$
 (ii) $g_{\gamma(\lambda)}(T, v_{\gamma,\gamma(\lambda)}) > 0$
 picture: spacetime:

Answer (to a question) T is a smooth vector field, T determines future vs. past, "general relativity: we have such a time orientation; smoothness makes it less arbitrary than it seems" -FSchuller,
Claim: 9/10 of a metric are determined by the cone
spacetime determined by distribution, only one-tenth error

26.2. Observers. $(M, \mathcal{O}, \mathcal{A}^{\uparrow}, \nabla, g, T)$

Definition 100. *An observer is a worldline γ with*
$$g(v_\gamma, v_\gamma) > 0$$
$$g(T, v_\gamma) > 0$$
together with a choice of basis
$$v_{\gamma,\gamma(\lambda)} \equiv e_0(\lambda), e_1(\lambda), e_2(\lambda), e_3(\lambda)$$
of each $T_{\gamma(\lambda)}M$ where the observer worldline passes, if $g(e_a(\lambda), e_b(\lambda)) = \eta_{ab} = \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix}_{ab}$

precise: observer = smooth curve in the frame bundle LM over M

26.2.1. *Two physical postulates.*

(P3) A **clock** carried by a specific observer (γ, e) will measure a **time**
$$\tau := \int_{\lambda_0}^{\lambda_1} d\lambda \sqrt{g_{\gamma(\lambda)}(v_{\gamma,\gamma(\lambda)}, v_{\gamma,\gamma(\lambda)})}$$
between the two "events"
$$\gamma(\lambda_0) \qquad \text{"start the clock"}$$
and
$$\gamma(\lambda_1) \qquad \text{"stop the clock"}$$
Compare with Newtonian spacetime:
$$t(p) = 7$$
Thought bubble: proper time/eigentime τ

Application/Example.
$$\begin{aligned} & M = \mathbb{R}^4 \\ & \mathcal{O} = \mathcal{O}_{\text{st}} \\ & \mathcal{A} \ni (\mathbb{R}^4, \text{id}_{\mathbb{R}^4}) \\ & g : g_{(x)ij} = \eta_{ij} \quad ; \quad T^i_{(x)} = (1,0,0,0)^i \end{aligned}$$
$$\implies \Gamma^i_{(x)\ jk} = 0 \text{ everywhere}$$

$\implies (M, \mathcal{O}, \mathcal{A}^{\uparrow}, g, T, \nabla)$ Riemm $= 0$
$\implies$ spacetime is flat
This situation is called special relativity.
Consider two observers:
$$\gamma : (0,1) \to M$$
$$\gamma^i_{(x)} = (\lambda, 0, 0, 0)^i$$
$$\delta : (0,1) \to M$$
$$\alpha \in (0,1) : \delta^i_{(x)} = \begin{cases} (\lambda, \alpha\lambda, 0, 0)^i & \lambda \leq \frac{1}{2} \\ (\lambda, (1-\lambda)\alpha, 0, 0)^i & \lambda > \frac{1}{2} \end{cases}$$
let's calculate:
$$\tau_\gamma := \int_0^1 \sqrt{g_{(x)ij}\dot{\gamma}^i_{(x)}\dot{\gamma}^j_{(x)}} = \int_0^1 d\lambda 1 = 1$$
$$\tau_\delta := \int_0^{1/2} d\lambda \sqrt{1-\alpha^2} + \int_{1/2}^1 \sqrt{1^2 - (-\alpha)^2} = \int_0^1 \sqrt{1-\alpha^2} = \sqrt{1-\alpha^2}$$
Note: piecewise integration
Taking the clock postulate (P3) seriously, one better come up with a realistic clock design that supports the postulate.
idea.
2 little mirrors

(P4) Postulate
Let (γ, e) be an observer, and
δ be a *massive* particle worldline that is parametrized s.t. $g(v_\gamma, v_\gamma) = 1$ (for parametrization/normalization convenience)
Suppose the observer and the particle *meet* somewhere (in spacetime)
$$\delta(\tau_2) = p = \gamma(\tau_1)$$
This observer measures the 3-velocity (spatial velocity) of this particle as
$$v_\delta : \epsilon^\alpha(v_{\delta,\delta(\tau_2)})e_\alpha \qquad \alpha = 1,2,3 \tag{72}$$
where $\epsilon^0, \boxed{\epsilon^1, \epsilon^2, \epsilon^3}$ is the unique dual basis of $e_0, \boxed{e_1, e_2, e_3}$

EY:20150407
There might be a major correction to Eq. (72) from the Tutorial 14 : Relativistic spacetime, matter, and Gravitation, see the second exercise, Exercise 2, third question:
$$v := \frac{\epsilon^\alpha(v_\delta)}{\epsilon^0(v_\delta)} e_\alpha \tag{73}$$

Consequence: An observer (γ, e) will extract quantities measurable in his laboratory from objective spacetime quantities always like that.
Ex: F Faraday $(0,2)$-tensor of electromagnetism:

$$F(e_a, e_b) = F_{ab} = \begin{bmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_3 & B_2 & -B_1 & 0 \end{bmatrix}$$

observer frame e_a, e_b
$E_\alpha := F(e_0, e_\alpha)$
$B^\gamma := F(e_\alpha, e_\rho)\epsilon^{\alpha\beta\gamma}$ where $\epsilon^{123} = +1$ totally antisymmetric

26.3. **Role of the Lorentz transformations.** Lorentz transformations emerge as follows:
Let (γ, e) and $(\widetilde{\gamma}, \widetilde{e})$ be observers with $\gamma(\tau_1) = \widetilde{\gamma}(\tau_2)$
(for simplicity $\gamma(0) = \widetilde{\gamma}(0)$)
Now

$$\begin{aligned} e_0, \dots, e_1 \qquad & \text{at } \tau = 0 \\ \text{and } \widetilde{e}_0, \dots, \widetilde{e}_1 \qquad & \text{at } \tau = 0 \end{aligned}$$

both bases for the same $T_{\gamma(0)}M$
Thus: $\widetilde{e}_a = \Lambda^b{}_a e_b \qquad \Lambda \in GL(4)$
Now:

$$\begin{aligned} \eta_{ab} = g(\widetilde{e}_a, \widetilde{e}_b) = g(\Lambda^m{}_a e_m, \Lambda^n{}_b e_n) = \\ = \Lambda^m{}_a \Lambda^n{}_b \underbrace{g(e_m, e_n)}_{\eta_{mn}} \end{aligned}$$

i.e. $\Lambda \in O(1,3)$
Result: Lorentz transformations relate the *frames* of *any two observers* at the same point.
"$\widetilde{x}^\mu - \Lambda^\mu{}_\nu x^\nu$" is utter nonsense

Tutorial. I didn't see a tutorial video for this lecture, but I saw that the Tutorial sheet number 14 had the relevant topics. Go there.

27. Lecture 14: Matter

two types of matter
point matter
field matter
point matter
massive point particle
more of a phenomenological importance
field matter
electromagnetic field
more fundamental from the GR point of view
both classical matter types

27.1. **Point matter.** Our postulates (P1) and (P2) already constrain the possible particle worldlines.
But what is their precise law of motion, possibly in the presence of "forces",

(a) without external forces

$$S_{\text{massive}}[\gamma] := m \int d\lambda \sqrt{g_{\gamma(\lambda)}(v_{\gamma,\gamma(\lambda)}, v_{\gamma,\gamma(\lambda)})}$$

with:

$$g_{\gamma(\lambda)}(T_{\gamma(\lambda)}, v_{\gamma,\gamma(\lambda)}) > 0$$

dynamical law Euler-Lagrange equation

similarly

$$S_{\text{massless}}[\gamma, \mu] = \int d\lambda \mu g(v_{\gamma,\gamma(\lambda)}, v_{\gamma,\gamma(\lambda)})$$

$$\begin{aligned} \delta_\mu \qquad & g(v_{\gamma,\gamma(\lambda)}, v_{\gamma,\gamma(\lambda)}) = 0 \\ \delta_\gamma \qquad & \text{e.o.m.} \end{aligned}$$

Reason for describing equations of motion by actions is that composite systems have an action that is the sum of the actions of the parts of that system, possibly including "interaction terms."
Example.

$$S[\gamma] + S[\delta] + S_{\text{int}}[\gamma, \delta]$$

(b) presence of external forces
or rather presence of fields to which a particle "couples"
Example

$$S[\gamma; A] = \int d\lambda m \sqrt{g_{\gamma(\lambda)}(v_{\gamma,\gamma(\lambda)}, v_{\gamma,\gamma(\lambda)})} + qA(v_{\gamma,\gamma(\lambda)})$$

where A is a **covector field** on M. A fixed (e.g. the electromagnetic potential)
Consider Euler-Lagrange eqns. $L_{\text{int}} = qA_{(x)}\dot{\gamma}^m_{(x)}$

$$m(\nabla_{v_\gamma} v_\gamma)_a + \underbrace{\left(\frac{\partial L_{\text{int}}}{\partial \dot{\gamma}^m_{(x)}}\right)^{\cdot} - \frac{\partial L_{\text{int}}}{\partial \gamma^m_{(x)}}}_{*} = 0 \Longrightarrow \boxed{m(\nabla_{v_\gamma} v_\gamma)^a = \underbrace{-qF^a{}_m \dot{\gamma}^m}_{\text{Lorentz force on a charged particle in an electromagnetic field}}}$$

$$\frac{\partial L}{\partial \dot{\gamma}^a} = qA_{(x)a}, \qquad \left(\frac{\partial L}{\partial \dot{\gamma}^m}\right)^{\cdot} = q \cdot \frac{\partial}{\partial x^m}(A_{(x)m}) \cdot \dot{\gamma}^m_{(x)}$$

$$\frac{\partial L}{\partial \gamma^a} = q \cdot \frac{\partial}{\partial x^a}(A_{(x)m})\dot{\gamma}^m$$

$$* = q\left(\frac{\partial A_a}{\partial x^m} - \frac{\partial A_m}{\partial x^a}\right)\dot{\gamma}^m_{(x)} \quad = q \cdot F_{(x)am}\dot{\gamma}^m_{(x)}$$

$F \leftarrow$ Faraday

$$S[\gamma] = \int (m\sqrt{g(v_\gamma, v_\gamma)} + qA(v_\gamma))d\lambda$$

27.2. **Field matter.**

Definition 101. *Classical (non-quantum) field matter is any tensor field on spacetime where equations of motion derive from an action.*

Example:

$$S_{\text{Maxwell}}[A] = \frac{1}{4}\int_M d^4x \sqrt{-g} F_{ab} F_{cd} g^{ac} g^{bd}$$

A $(0,1)$-tensor field
$=$ thought cloud: for simplicity one chart covers all of M
$-$ for $\sqrt{-g}$ $(+---)$

$F_{ab} := 2\partial_{[a}A_{b]} = 2(\nabla_{[a}A)_{b]}$
Euler-Lagrange equations for fields

$$0 = \frac{\partial \mathcal{L}}{\partial A_m} - \frac{\partial}{\partial x^s}\left(\frac{\partial \mathcal{L}}{\partial \partial_s A_m}\right) + \frac{\partial}{\partial x^s}\frac{\partial}{\partial x^t}\frac{\partial^2 \mathcal{L}}{\partial \partial_t \partial_s A_m}$$

Example . . .

$$(\nabla_{\frac{\partial}{\partial x^m}} F)^{ma} = j^a$$

inhomogeneous Maxwell

thought bubble $j = qv_\gamma$

$$\partial_{[a} F_{b]} - ()$$

homogeneous Maxwell

Other example well-liked by textbooks

$$S_{\text{Klein-Gordon}}[\phi] := \int_M d^4x \sqrt{-g}[g^{ab}(\partial_a \phi)(\partial_b \phi) - m^2\phi^2]$$

ϕ $(0,0)$-tensor field

27.3. **Energy-momentum tensor of matter fields.** At some point, we want to write down an action for the metric tensor field itself.

But then, this action $S_{\text{grav}}[g]$ will be added to any $S_{\text{matter}}[A, \phi, \dots]$ in order to describe the total system.

$$S_{\text{total}}[g, A] = S_{\text{grav}}[g] + S_{\text{Maxwell}}[A, g]$$

$$\begin{aligned} \delta A \quad & :\Longrightarrow \text{ Maxwell's equations} \\ \delta g_{ab} \quad & : \boxed{\frac{1}{16\pi G} G^{ab}} + (-2T^{ab}) = 0 \end{aligned}$$

G Newton's constant

$$G^{ab} = 8\pi G_N T^{ab}$$

Definition 102. *$S_{matter}[\Phi, g]$ is a matter action, the* ***so-called energy-momentum tensor*** *is*

$$T^{ab} := \frac{-2}{\sqrt{-g}} \left(\frac{\partial \mathcal{L}_{matter}}{\partial g_{ab}} - \partial_s \frac{\partial \mathcal{L}_{matter}}{\partial \partial_s g_{ab}} + \dots \right)$$

$-$ of $\frac{-2}{\sqrt{g}}$ is Schrödinger minus (EY : 20150408 F.Schuller's joke? but wise)

choose all sign conventions s.t.

$$T(\epsilon^0, \epsilon^0) > 0$$

Example: For S_{Maxwell}:

$$T_{ab} = F_{am}F_{bn}g^{mn} - \frac{1}{4}F_{mn}F^{mn}g_{ab}$$

$T_{ab} \equiv T_{\text{Maxwell}ab}$

$$T(e_0, e_0) = \underline{E}^2 + \underline{B}^2$$
$$T(e_0, e_\alpha) = (E \times B)_\alpha$$

Fact: One often does not specify the fundamental action for some matter, but one is rather satisfied to assume certain properties / forms of

$$T_{ab}$$

Example Cosmology: (homogeneous & isotropic)

perfect fluid

of pressure p and density ρ modelled by

$$T^{ab} = (\rho + p)u^a u^b - pg^{ab}$$

radiative fluid

What is a fluid of photons:

$$T^{ab}_{\text{Maxwell}} g_{ab} = 0$$

observe:
$$\begin{aligned} & T^{ab}_{\text{p.f.}} g_{ab} \overset{!}{=} 0 \\ & = (\rho + p)u^a u^b g_{ab} - p \underbrace{g^{ab}g_{ab}}_{4} \end{aligned}$$

$$\leftrightarrow \rho_p 0 4p = 0$$
$$\rho = 3p$$

$p = \frac{1}{3}\rho$

Reconvene at 3 pm? (EY : 20150409 I sent a Facebook (FB) message to the International Winter School on Gravity and Light: there was no missing video; it continues on Lecture 15 immediately)

Tutorial 14: Relativistic Spacetime, Matter and Gravitation. **Exercise 2: Lorentz force law.**

Question electromagnetic potential.

28. Lecture 15: Einstein gravity

Recall that in Newtonian spacetime, we were able to reformulate the Poisson law $\Delta\phi = 4\pi G_N \rho$ in terms of the Newtonian spacetime curvature as

$$R_{00} = 4\pi G_N \rho$$

R_{00} with respect to ∇_{Newton}

G_N = Newtonian gravitational constant

This prompted Einstein to postulate < 1915 that the relativistic field equations for the Lorentzian metric g of (relativistic) spacetime

$$R_{ab} = 8\pi G_N T_{ab}$$

However, this equation suffers from a problem

LHS: $(\nabla_a R)^{ab} \neq 0$

generically

RHS:

$$(\nabla_a T)^{ab} = 0$$

thought bubble: = formulated from an action

Einstein tried to argue this problem away.

Nevertheless, the equations cannot be upheld.

28.1. **Hilbert.** Hilbert was a specialist for variational principles.

To find the appropriate left hand side of the gravitational field equations, Hibert suggested to start from an action

$$S_{\text{Hilbert}}[g] = \int_M \sqrt{-g} R_{ab} g^{ab}$$

thought bubble = "simplest action"

aim: varying this w.r.t. metric g_{ab} will result in some tensor

$$G^{ab} = 0$$

28.2. Variation of S_{Hilbert}.

$$0 \overset{!}{=} \underbrace{\delta}_{g_i} S_{\text{Hilbert}}[g] = \int_M [\underbrace{\delta\sqrt{-g}g^{ab}R_{ab}}_{1} + \underbrace{\sqrt{-g}\delta g^{ab}R_{ab}}_{2} + \underbrace{\sqrt{-g}g^{ab}\delta R_{ab}}_{3}]$$

$$\text{and } 1 : \delta\sqrt{-g} = \frac{-(\det g)g^{mn}\delta g_{mn}}{2\sqrt{-g}} = \frac{1}{2}\sqrt{-g}g^{mn}\delta g_{mn}$$

thought bubble

$$\delta\det(g) = \det(g)g^{mn}\delta g_{mn}$$
e.g. from
$$\det(g) = \exp \operatorname{trln} g$$

ad 2: $g^{ab}g_{bc} = \delta^a_c$

$$\Longrightarrow (\delta g^{ab})g_{bc} + g^{ab}(\delta g_{bc}) = 0$$
$$\Longrightarrow \delta g^{ab} = -g^{am}g^{bn}\delta g_{mn}$$

ad 3:

$$\Delta R_{ab} \underbrace{=}_{\text{normal coords at point}} \delta\partial_b\Gamma^m_{\ am} - \delta\partial_m\Gamma^m_{\ ab} + \Gamma\Gamma - \Gamma\Gamma =$$
$$= \partial_b\delta\Gamma^m_{\ am} - \partial_m\delta\Gamma^m_{\ ab} =$$
$$= \nabla_b(\delta\Gamma)^m_{\ am} - \nabla_m(\delta\Gamma)^m_{\ ab}$$
$$\Longrightarrow \sqrt{-g}g^{ab}\delta R_{ab} = \sqrt{-g}$$

"if you formulate the variation properly, you'll see the variation δ commute with ∂_b" EY : 20150408 I think one uses the integration at the bounds, integration by parts trick

$\Gamma^i_{(x)\ jk} - \widetilde{\Gamma}^i_{(x)\ jk}$ are the components of a $(1,2)$-tensor.

Notation: $(\nabla_b A)^i_{\ g} =: A^i_{\ j;b}$

$$\Longrightarrow \sqrt{-g}g^{ab}\delta R_{ab}$$
$$\underbrace{=}_{\nabla g=0} \sqrt{-g}(g^{ab}\delta\Gamma^m_{\ am})_{;b} - \sqrt{-g}(g^{ab}\delta\Gamma^m_{\ ab})_{;m} = \sqrt{-g}A^b_{\ ;b} - \sqrt{-g}B^m_{\ ,m}$$

Question: Why is the difference of coefficients a tensor?

Answer:

$$\Gamma^i_{(y)\ jk} = \frac{\partial y^i}{\partial x^m}\frac{\partial x^m}{\partial y^j}\frac{\partial x^q}{\partial y^k}\Gamma^m_{(x)\ \ ,nq} + \frac{\partial y^i}{\partial x^m}\frac{\partial^2 x^m}{\partial y^j\partial y^k}$$

Collecting terms, one obtains

$$0 \overset{!}{=} \delta S_{\text{Hilbert}} = \int_M [\frac{1}{2}\sqrt{-g}g^{mn}\delta g_{mn}g^{ab}R_{ab} - \sqrt{-g}g^{am}g^{bn}\delta g_{mn}R_{ab} + \underbrace{(\sqrt{-g}A^a)_{,a}}_{\text{surface}} - \underbrace{(\sqrt{-g}B^b)_{,b}}_{\text{surface term}}]$$

$$= \int_M \sqrt{-g}\delta \underbrace{g_{mn}}_{\text{arbitrary variation}} [\frac{1}{2}g^{mn}R - R^{mn}] \Longrightarrow G^{mn} = R^{mn} - \frac{1}{2}g^{mn}R$$

Hence Hilbert, from this "mathematical" argument, concluded that one may take

$$\boxed{R_{ab} - \frac{1}{2}g_{ab}R = 8\pi G_N T_{ab}}$$
Einstein equations
$$S_{E-H}[g] = \int_M \sqrt{-g}R$$

28.3. 3. Solution of the $\nabla_a T^{ab} = 0$ issue. One can show ($\to$ Tutorials) that the Einstein curvature

$$G_{ab} = R_{ab} - \frac{1}{2}g_{ab}R$$

satisfy the so-called contracted differential Bianchi identity

$$(\nabla_a G)^{ab} = 0$$

28.4. Variants of the field equations.

(a) a simple rewriting:

$$R_{ab} - \frac{1}{2}g_{ab}R = 8\pi G_N T_{ab} = T_{ab}$$

$G_N = \frac{1}{8\pi}$

Contract on both sides g^{ab}

$$R_{ab} - \frac{1}{2}g_{ab}R = T_{ab}||g^{ab}$$
$$R - 2R = T := T_{ab}g^{ab}$$
$$\Longrightarrow R = -T$$
$$\Longrightarrow R_{ab} + \frac{1}{2}g_{ab}T = T_{ab}$$
$$\Longleftrightarrow R_{ab} = (T_{ab} - \frac{1}{2}Tg_{ab}) =: \widehat{T}_{ab}$$
$$\boxed{R_{ab} = \widehat{T}_{ab}}$$

(b)

$$S_{E-H}[g] := \int_M \sqrt{-g}(R + 2\Lambda)$$

thought bubble: Λ cosmological constant

History:
1915: $\Lambda < 0$ (Einstein) in order to get a non-expanding universe
>1915: $\Lambda = 0$ Hubble
today $\Lambda > 0$ to account for an accelerated expansion
$\Lambda \neq 0$ can be interpreted as a contribution
$-\frac{1}{2}\Lambda g$ to the energy-momentum
"dark energy"
Question: surface terms scalar?
Answer: for a careful treatment of the surface terms which we discarded, see, e.g. E. Poisson, "A relativist's toolkit" C.U.P. "excellent book"
Question: What is a constant on a manifold?
Answer: $\int\sqrt{-g}\Lambda = \Lambda\int\sqrt{-g}1$
[back to dark energy]
[Weinberg, QCD, calculated]
idea: 1 could arise as the vacuum energy of the standard model fields
$\Lambda_{\text{calculated}} = 10^{120} \times \Lambda_{\text{obs}}$
"worst prediction of physics"
Tutorials: check that

- Schwarzschild metric (1916)
- FRW metric
- pp-wave metric
- Reisner-Nordstrom

$\implies$ are solutions to Einstein's equations

in high school
$m\ddot{x} + m\omega^2 x^2 = 0$
$x(t) = \cos{(\omega t)}$
ET: [elementary tutorials]
study motion of particles & observers in Schwarzscheld S.T.
Satellite: Marcus C. Werner
Gravitational lensing
odd number of pictures Morse theory (EY:20150408 Morse Theory !!!)
Domenico Giulini
Hamiltonian form Canonical Formulations
Key to Quantum Gravity

Tutorial 13 Schwarzschild Spacetime

EY : 20150408 I'm not sure which tutorial follows which lecture at this point.

The tutorial video is excellent itself. Here, I want to encourage the use of CAS to do calculations. There are many out there. Again, I'm partial to the Sage Manifolds package for Sage Math which are both open-source and based on Python. I'll use that here.

Exercise 1. Geodesics in a Schwarzschild spacetime

Question Write down the Lagrangian.

Load "Schwarzschild.sage" in Sage Math, which will always be available freely here https://github.com/ernestyalumni/diffgeo-by-sagemnfd/blob/master/Schwarzschild.sage:

```
sage: load("Schwarzschild.sage")
4-dimensional manifold 'M'
open subset 'U_sph' of the 4-dimensional manifold 'M'
Levi-Civita connection 'nabla_g' associated with the Lorentzian metric 'g' on the 4-dimensional manifold 'M'
```

and so on.

Look at the code and I had defined the Lagrangian to be

```
L
```

. To get out the coefficients of L of the components of the tangent vectors to the curve, i.e. t', r', θ', ϕ', denoted

```
tp,rp,thp,php
```

in my .sage file, do the following:

```
sage: L.expr().coefficients(tp)[1][0].factor().full_simplify()
(2*G_N*M_0 - r)/r
sage: L.expr().coefficients(rp)[1][0].factor().full_simplify()
-r/(2*G_N*M_0 - r)
sage: L.expr().coefficients(php)[1][0].factor().full_simplify()
r^2
sage: L.expr().coefficients(thp)[1][0].factor().full_simplify()
r^2*sin(th)^2
```

Question There are 4 Euler-Lagrange equations for this Lagrangian. Derive the one with respect to the function $t(\lambda)$!.

```
sage: L.expr().diff(t)
0
```

This confirms that $\frac{\partial L}{\partial t} = 0$

For $\frac{d}{d\lambda}\frac{\partial L}{\partial t'}$, then one needs to consider this particular workaround for Sage Math (computer technicality). One takes derivatives with respect to declared variables (declared with var) and then substitute in functions that are dependent upon λ, and then take the derivative with respect to the parameter λ. This does that:

```
sage: L.expr().diff( thp ).factor().subs( r == gamma1 ).subs( thp == gamma3.diff( tau ) ).subs( th == gamma3 ).diff(tau)\
....: .factor()
2*(2*cos(gamma3(tau))*gamma1(tau)*D[0](gamma3)(tau)^2 + 2*sin(gamma3(tau))*D[0](gamma1)(tau)*D[0](gamma3)(tau)
+ gamma1(tau)*sin(gamma3(tau))*D[0, 0](gamma3)(tau))*gamma1(tau)*sin(gamma3(tau))
```

Question Show that the Lie derivative of g with respect to the vector fields $K_t := \frac{\partial}{\partial t}$.

The first line defines the vector field by accessing the frame defined on a chart with spherical coordinates and getting the time vector. The second line is the Lie derivative of g with respect to this vector field.

```
sage: K_t = espher[0]
sage: g.lie_der(K_t).display() # 0, as desired
0
```

EY : 20150410 My question is this: $\forall\, X \in \Gamma(TM)$ i.e. X is a vector field on M, or, specifically, a section of the tangent bundle, then does

$$\mathcal{L}_X g = 0$$

instantly mean that X is a symmetry for (M,g)? $\mathcal{L}_X g$ is interpreted geometrically as how g changes along the flow generated by X, and if it equals 0, then g doesn't change.

29.

30.

31. Canonical Formulation of GR I

Dynamical and Hailtonian formulation of General Relativity.
Purpose

(1) formulate and solve initial-value problems
(2) integrate Einstein's Equations by numerical codes
(3) characterize degrees of freedom
(4) characterize isolated systems, associated symmetry groups and conserved quantities, like Energy/Mass, Momenta (linear and angular), Poincaré charges
(5) starting point for "canonical quantization" program.

How. We will rewrite Einstein's Eq. in form of a *constrained Hamiltonian system.*

$$\underbrace{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R}_{G_{\mu\nu}} + \underbrace{\Lambda}_{\text{kosm.}const.} g_{\mu\nu} = \underbrace{k}_{\frac{8\pi G}{c^4}} T_{\mu\nu}$$

$(-+++)$

$$T^{\mu\nu} = \begin{pmatrix} W & \frac{1}{c} S^m \\ c g^m & \mathbf{t}^{mn} \end{pmatrix}$$

W = Energy density (1 component)
g^m = Momentum density, (3 components)
S^m = Energy current-density (3 components)
$\mathbf{t}^{mn}$ = Momentum current-density (6 components)

$$T^{\mu\nu} = T^{\nu\mu} \Longrightarrow S^m = c^2 g^m$$

10 independent komp. (components)
Phys. dim. $[T^{\mu\nu}] = \frac{J}{m^3}$
$[G^{\mu\nu}] = \frac{1}{m^2}$

$$k = \frac{\text{curvature}}{\text{Energy} \cdot \text{density}}$$

$$[k] = \frac{1}{m^2}/\frac{J}{m^3},\ {}^2k \quad \frac{\text{Curvature}}{\text{mass density}} = \left(\frac{1}{1.5\,\text{AU}}\right)^2 / \text{ Density of water}$$

$$= \left(\frac{1}{10\,km}\right)^2 / \text{ Nuclear density in core of neutron star } \simeq 5\cdot 10^{17}\,kg/m^3$$

If "Ein" for Einstein Tensor, $G_{\mu\nu} = \text{Ein}\left(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu}\right)$

$$\text{Ein}(v,w) = \frac{1}{4}[\text{Ein}(v+w,v+w) - \text{Ein}(v-w,v-w)$$

$$\text{Ein}(w,w) = -g(w,w)\sum_{\perp w}\text{Sec}$$

where $\perp w$ is take the sum over any triple of mutually perp. 2-planes in $\perp w$
$\text{Sec}(\text{Span}\{v,w\}) = \frac{\text{Riem}(v,w,v,w)}{[g(v,w)]^2 - g(v,v)g(w,w)}$
"sectional curvature"
Identity: $\nabla_\mu G^{\mu\nu} = 0$ (follows from twice-contracted II. Bianchi Identity
$\sum_{\lambda\mu\nu \text{ cycl}} \nabla_\lambda R_{\alpha\beta\mu\nu} = 0$)

$$\underbrace{\partial_0 G^{0\nu}}_{\text{contains at most 1st time der.}} \qquad + \qquad \underbrace{\partial_k G^{k\nu} + \Gamma G + \Gamma G \equiv 0}_{\text{contains at most 2nd. time derivatives}}$$

$\Longrightarrow$ 4 out of 10 Einstein Eq. do not evolve the fields but rather constrain the initial data. The space-space components (6 Eqns.) are the evolution Eqns.
10 Einstein Eq. - 4 constraints (underdetermined elliptic type)
\ - 6 evolution equations (undetermined hyperbolic type)

32.

33.

34.

35. Lecture 22: Black Holes

Only depends on Lectures 1-15, so does lecture on "Wednesday"
Schwarzschild solution also vacuum solution (from tutorial EY : oh no, must do tutorial)
Study the Schwarzschild as a vacuum solution of the Einstein equation:
$m = G_N M$ where M is the "mass"

$$g = \left(1 - \frac{2m}{r}\right) dt\otimes dt - \frac{1}{1-\frac{2m}{r}} dr\otimes dr - r^2(d\theta\otimes d\theta + \sin^2\theta d\varphi\otimes d\varphi)$$

in the so-called Schwarzschild coordinates

t	r	θ	φ
$(-\infty,\infty)$	$(0,\infty)$	$(0,\pi)$	$(0,2\pi)$

What staring at this metric for a while, two questions naturally pose themselves:

(i) What exactly happens $r = 2m$?

t	r	θ	φ
$(-\infty,\infty)$	$(0,2m)\cup(2m,\infty)$	$(0,\pi)$	$(0,2\pi)$

(ii) Is there anything (in the real world) beyond $t \to -\infty$?
$t \to +\infty$

idea: Map of Linz, blown up
Insight into these two issues is afforded by stopping to stare.
Look at *geodesic* of g, instead.

35.1. **Radial null geodesics.** null - $g(v_\gamma, v_\gamma) = 0$
Consider null geodesic in "Schd"

$$S[\gamma] = \int d\lambda \left[\left(1-\frac{2m}{r}\right)\dot t^2 - \left(1-\frac{2m}{r}\right)^{-1}\dot r^2 - r^2(\dot\theta^2 + \sin^2\theta\dot\varphi^2)\right]$$

with $[\dots] = 0$
and one has, in particular, the t-eqn. of motion:

$$\left(\left(1-\frac{2m}{r}\right)\dot t\right)^{\cdot} = 0$$

$\Longrightarrow$

$$\boxed{\left(1-\frac{2m}{r}\right)\dot t = k} = \text{ const.}$$

Consider radial null geodesics
$\theta \overset{!}{=}$ const. $\qquad \varphi =$ const.
From $\square$ and $\square$

$$\Longrightarrow \dot r^2 = k^2 \leftrightarrow \dot r = \pm k$$

$$\Longrightarrow r(\lambda) = \pm k\cdot\lambda$$

Hence, we may consider

$$\widetilde{t}(r) := t(\pm k\lambda)$$

Case A: $\oplus$

$$\frac{d\widetilde{t}}{dr} = \frac{\dot{\widetilde{t}}}{\dot r} = \frac{k}{\left(1-\frac{2m}{r}\right)k} = \frac{r}{r-2m}$$

$$\Longrightarrow \widetilde{t}_+(r) = r + 2m\ln|r-2m|$$

(**outgoing** null geodesics)
Case b. $\pm$ (Circle around $-$, consider $-$):

$$\widetilde{t}_-(r) = -r - 2m\ln|r-2m|$$

(**ingoing** null geodesics)
Picture

35.2. **Eddington-Finkelstein.** Brilliantly simple idea:

change (on the domain of the Schwarzschild coordinates) to different coordinates, s.t. in those new coordinates, *ingoing* null geodesics appear as straight lines, of slope -1

This is achieved by

$$\bar{t}(t,r,\theta,\varphi) := t + 2m\ln|r-2m|$$

Recall: ingoing null geodesic has

$$\widetilde{t}(r) = -(r + 2m\ln|r-2m|) \qquad (Schdcoords)$$

$$\begin{aligned} \Longleftrightarrow \bar{t} - 2m\ln|r-2m| &= -r - 2m\ln|r-2m| + \text{ const.} \\ \therefore \bar{t} &= -r + \text{ const.} \end{aligned}$$

(Picture)

outgoing null geodesics

$$\bar{t} = r + 4m\ln|r-2m| + \text{ const.}$$

Consider the new chart (V,g) while (U,x) was the Schd chart.

$$\underbrace{U}_{\text{Schd}} \bigcup \{\text{ horizon }\} = V$$

"chart image of the horizon"

Now calculate the *Schd metric* g w.r.t. Eddington-Finkelstein coords.

$$\begin{aligned} \bar{t}(t,r,\theta,\varphi) &= t + 2m\ln|r-2m| \\ \bar{r}(t,r,\theta,\varphi) &= r \\ \bar{\theta}(t,r,\theta,\varphi) &= \theta \\ \bar{\varphi}(t,r,\theta,\varphi) &= \varphi \end{aligned}$$

EY : 20150422 I would suggest that after seeing this, one would calculate the metric by your favorite CAS. I like the Sage Manifolds package for Sage Math.

Schwarzschild_BH.sage on github

Schwarzschild_BH.sage on Patreon

Schwarzschild_BH.sage on Google Drive

```
sage: load(``Schwarzschild_BH.sage'')
4-dimensional manifold 'M'
expr = expr.simplify_radical()
Levi-Civita connection 'nabla_g' associated with the Lorentzian metric 'g' on the 4-dimensional manifold 'M'
Launched png viewer for Graphics object consisting of 4 graphics primitives
```

Then calculate the Schwarzschild metric g but in Eddington-Finkelstein coordinates. Keep in mind to calculate the set of coordinates that uses $\bar{t}$, not $\widetilde{t}$:

```
sage: gI.display()
gI = (2*m - r)/r dt*dt - r/(2*m - r) dr*dr + r^2 dth*dth + r^2*sin(th)^2 dph*dph
sage: gI.display( X_EF_I_null.frame())
gI = (2*m - r)/r dtbar*dtbar + 2*m/r dtbar*dr + 2*m/r dr*dtbar + (2*m + r)/r dr*dr + r^2 dth*dth + r^2*sin(th)^2 dph*dph
```

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