ANALYSIS DUMP

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Abstract. Everything about Analysis, real analysis, complex analysis, functional analysis, Fourier series, Fourier transforms, Fourier analysis

Part 1. Fourier Analysis

1. Fourier transform

cf. Ch. IX of Reed and Simon [1], from pp. 318

Definition 1 (Schwartz space)**.** *Showing Reed and Simon [1]'s notation and wikipedia's notation (that'll be used here), respectively*

$$\mathcal{S}(\mathbb{R}^n) \equiv S(\mathbb{R}^n) = \text{ Schwartz space of } C^\infty \text{ functions of rapid decrease, i.e.}$$

$$S(\mathbb{R}^n) = \{ f \in C^\infty(\mathbb{R}^n) | \|f\|_{\alpha,\beta} < \infty, \forall \, \alpha, \beta \in \mathbb{Z}^n_+ \} \tag{1}$$

where α, β are multiindices, $C^\infty(\mathbb{R}^n)$ is set of smooth functions from $\mathbb{R}^n$ to $\mathbb{C}$, and

$$\|f\|_{\alpha,\beta} = \sup_{\mathbf{x}\in\mathbb{R}^n} |\mathbf{x}^\alpha D^\beta f(\mathbf{x})| \tag{2}$$

cf. wikipedia definition of Schwartz space

cf. IX.1 The Fourier transform on $S(\mathbb{R}^n)$ and $S'(\mathbb{R}^n)$, convolutions of Reed and Simon [1]

Definition 2. *Suppose $f \in S(\mathbb{R}^n)$,*
***Fourier transform** of f, $\widehat{f}$, give by*

$$\widehat{f}(\lambda) = \frac{1}{(2\pi)^{n/2)}} \int_{\mathbb{R}^n} e^{-i\mathbf{x}\cdot\lambda} f(\mathbf{x}) d\mathbf{x} \tag{3}$$

where $\mathbf{x}\cdot\lambda = \sum_{i=1}^n x_i \lambda_i$

Inverse Fourier transform of f, $\check{f}$,

$$\check{f}(\lambda) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i\mathbf{x}\cdot\lambda} f(\mathbf{x}) d\mathbf{x}$$

Reed and Simon [1] mentions this notation and I will use it more here

$$\widehat{f} \equiv \mathcal{F} f$$

Standard multiindex notation:

$$\alpha = \langle \alpha_1, \dots \alpha_n \rangle$$

n-tuple of nonnegative integers, $\alpha \in \mathbb{Z}^n_+$
$I^n_+ \equiv$ collection of all multiindices
Define:

$$|\alpha| := \sum_{i=1}^n \alpha_i$$

$$\mathbf{x}^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}$$

$$D^\alpha := \frac{\partial^{|n|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_n^{\alpha_n}} x^2 := \sum_{i=1}^n x_i^2$$

Lemma 1. *$\widehat{\ }, \check{\ } \equiv \mathcal{F}, \mathcal{F}^{-1}$ are cont. linear transformations of $S(\mathbb{R}^n)$ into $S(\mathbb{R}^n)$.*
Furthermore, if $\alpha, \beta \in I^n_+$, then

$$((i\lambda)^\alpha D^\beta \widehat{f})(\lambda) = D^\alpha(\widehat{(-ix)^\beta f(x)}) \tag{4}$$

cf. (IX.1) of Reed and Simon [1].

Proof. $\widehat{\ } \equiv \mathcal{F}$ clearly linear (since $\int$ linear),
Since

$$(\lambda^\alpha D^\beta \widehat{f})(\lambda) = \lambda^\alpha D^\beta \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\mathbf{x}\cdot\lambda} f(\mathbf{x}) d\mathbf{x} = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \lambda^a (-i\mathbf{x})^\beta e^{-i\lambda\cdot\mathbf{x}} f(\mathbf{x}) d\mathbf{x} =$$

$$= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \frac{1}{(-i)^a} (D^a_x e^{-i\lambda\cdot\mathbf{x}})(-i\mathbf{x})^\beta f(\mathbf{x}) d\mathbf{x} = 0 + \frac{(-i)^\alpha}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\lambda\cdot\mathbf{x}} D^\alpha_x ((-i\mathbf{x})^\beta f(\mathbf{x})) d\mathbf{x}$$

Last step is just integration by parts and using given $\|f\|_{\alpha,\beta} < \infty$ property.

Date: 20 Nov 2017.
Key words and phrases. Analysis, Functional Analysis.

Conclude $\|\widehat{f}\|_{\alpha,\beta} = \sup_\lambda |\lambda^\alpha (D^\beta \widehat{f})(\lambda)| \leq \frac{1}{(2\pi)^{n/2}} \int |D^\alpha_{\mathbf{x}} (\mathbf{x}^\beta f)| d\mathbf{x} < \infty$. □

Thus,

$$\mathcal{F} : S(\mathbb{R}^n) \to S(\mathbb{R}^n)$$

and

$$((i\lambda)^\alpha D^\beta \widehat{f})(\lambda) = D^\alpha (\widehat{(-ix)^\beta f(x)})$$

If k large enough, $\int (1+x^2)^{-k} d\mathbf{x} < \infty$ (Clearly $\int \frac{1}{(1+x^2)} = \arctan x \xrightarrow{\infty, -\infty} \frac{\pi}{2} - (-\frac{\pi}{2}) = \pi < \infty$), so

$$\|\widehat{f}\|_{\alpha,\beta} \leq \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \frac{(1+x^2)^{-k}}{(1+x^2)^{-k}} |D^\alpha_x (\mathbf{x}^\beta f)| d\mathbf{x} \leq \frac{1}{(2\pi)^{n/2}} (\int_{\mathbb{R}^n} (1+x^2)^{-k} d\mathbf{x}) \sup_{\mathbf{x}} |(1+x^2)^k D^\alpha_x (\mathbf{x}^\beta f)|$$

By Leibniz rule, $(fg)^{(n)}(x) = \sum_{k=0}^n \binom{n}{k} f^{(n-k)}(\mathbf{x}) g^{(k)}(\mathbf{x})$, $\exists$ constants c_j, multiindices $\alpha_j \beta_j \in I^n_+$, s.t.

$$\|\widehat{f}\|_{\alpha,\beta} \leq \sum_{i=1}^M c_j \|f\|_{\alpha_j,\beta_j}$$

where $\|f\|_{\alpha_j,\beta_j} = \sup_{\mathbf{x}\in\mathbb{R}^n} |\mathbf{x}^\alpha D^\beta f(\mathbf{x})|$, which we recall, was used.

Thus $\|\widehat{f}\|_{\alpha,\beta}$ bounded, and by, as Reed and Simon [1] said, Thm. V.4, therefore cont. But I think that reference is incorrect. I looked up possible theorems online, and possibly it's,
since $\widehat{f}$ bounded and has closed graph $(\lambda, \widehat{f}(\lambda))$, then f cont.

Likewise for $\check{f}$

□

cf. Thm. IX.1. of Reed and Simon (1980)[1]

Theorem 1 ((Fourier inverse thm.))**.** *Fourier transform $\mathcal{F}$ is linear, bicont., bijection: $\mathcal{F} : S(\mathbb{R}^n) \to S(\mathbb{R}^n)$, and $\mathcal{F}^{-1} = \check{}$.*

Proof. Prove $\mathcal{F}\mathcal{F}^{-1} f = \mathcal{F}^{-1}\mathcal{F} f = f$ for f contained in dense set $C^\infty(\mathbb{R}^n)$.

Let C_ϵ be cube of volume $\left(\frac{2}{\epsilon}\right)^n$ centered at $0 \in \mathbb{R}^n$.

Choose ϵ small enough s.t. support of f is contained in C_ϵ.

Let $K_\epsilon := \{\mathbf{k} \in \mathbb{R}^n | \forall k_i/\pi\epsilon \text{is an integer } \}$, then

$$f(x) = \sum_{\mathbf{k}\in K_\epsilon} ((\frac{1}{2}\epsilon)^{n/2} e^{i\mathbf{k}\cdot\mathbf{x}}, f)(\frac{1}{2}\epsilon)^{n/2} e^{-i\mathbf{k}\cdot\mathbf{x}}$$

where $(\cdot,\cdot)$ is the inner product.

The expression immediately above for $f(x)$ is just the Fourier series of f, which converges uniformly in C_ϵ, to f, since f cont. diff. (Thm. II.8 of Reed and Simon (1980) [1]). Recall this theorem says:
Suppose $f(x)$ periodic of period 2π and is cont. diff. Then functions $\sum_{-M}^M c_n e^{inx} \xrightarrow{M\to\infty} f(x)$ uniformly converges.

$$f(x) = \sum_{\mathbf{k}\in K_\epsilon} \frac{\widehat{f}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}}}{(2\pi)^{n/2}} (\pi\epsilon)^n \tag{5}$$

cf. (IX.2) of Reed and Simon (1980)[1].

Since $\mathbb{R}^n$ is the disjoint union of cubes of volume $(\pi\epsilon)^n$ centered around pts. in K_ϵ, (indeed, $K_\epsilon = \{\mathbf{k} \in \mathbb{R}^n | k_i/\pi\epsilon \in \mathbb{Z} \forall i = 1, 2, \dots n\}$) then

$$\sum_{\mathbf{k}\in K_\epsilon} \frac{\widehat{f}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}}}{(2\pi)^{n/2}} (\pi\epsilon)^n$$

is just Riemann sum for integral of function

$$\widehat{f}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} / (2\pi)^{n/2}$$

By lemma, $\widehat{f}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} \in S(\mathbb{R}^n)$, so Riemann sums converge to integral. Thus

$$\mathcal{F}^{-1}\mathcal{F} f = f$$

References

[1] Michael Reed and Barry Simon. **Functional Analysis (Methods of Modern Mathematical Physics, Vol. 1)**. Academic Press. 1980.