SOLUTIONS TO *GEOMETRY,TOPOLOGY,PHYSICS* BY MIKIO NAKAHARA, 2003.

ERNEST YEUNG

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Solutions for *Geometry, Topology, and Physics.* Mikio Nakahara. Institute of Physics Publishing. 2003. ISBN 0 7503 0606 8

1. Quantum Physics

1.1. Analytical mechanics.

1.1.1. *Newtonian mechanics.*

1.1.2. *Lagrangian formalism.* $\mathcal{L}$ independent of coordinate q_k; q_k cyclic.

$q_k(t) \to q_k(t) + \delta q_k(t)$

$$S[q(t),\dot{q}(t)] = \int_{t_i}^{t_f} L(q,\dot{q})dt \qquad (1.3) \tag{1}$$

$$\delta S = \int_{t_i}^{t_f} \sum_k \delta q_k \left(\frac{\partial L}{\partial q_k} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} \right) + \sum_k \left[\delta_k \frac{\partial L}{\partial \dot{q}_k} \right]_{t_i}^{t_f} = 0$$

Note that

$$-\int \delta q_k \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} + \delta q_k \left. \frac{\partial L}{\partial \dot{q}_k}\right|_{t_i}^{t_f} = \int \delta \dot{q}_k \frac{\partial L}{\partial \dot{q}_k}$$

with $p_k = \frac{\partial L}{\partial \dot{q}_k}$.

$$\Longrightarrow \delta q_k(t_i) p^k(t_i) = \delta q_k(t_f) p^k(t_f) \tag{2}$$

since t_i, t_f arbitrary, $\delta q_k(t) p^k(t)$ independent of t and hence conserved.

Date: zima 2012.

1.1.3. *Hamiltonian formalism.* **Exercise 1.1.** $A = A(q,p), B(q,p)$ defined on phase space of $H = H(q,p)$

$$\begin{aligned}[A, c_1B_1 + c_2B_2] &= \partial_{q_k}A\partial_{p_k}(c_1B_1+c_2B_2) - \partial_{p_k}A\partial_{q_k}(c_1B_1+c_2B_2) = \\ &= c_1\partial_{q_k}A\partial_{p_k}B_1 - c_1\partial_{p_k}A\partial_{q_k}B_1 + c_2\partial_{q_k}A\partial_{p_k}B_2 - c_2\partial_{p_k}A\partial_{q_k}B_2 = \\ &= c_1[A,B_1] + c_2[A,B_2]\end{aligned}$$

$$[A,B] = \partial_{q_k}A\partial_{p_k}B - \partial_{p_k}A\partial_{q_k}B = -(\partial_{q_k}B\partial_{p_k}A - \partial_{p_k}B\partial_{q_k}A) = -[B,A]$$

$$\begin{aligned}[[A,B],C] = \partial_{q_k}(\partial_{q_l}A\partial_{p_l}B - \partial_{p_l}A\partial_{q_l}B)\partial_{p_k}C - \partial_{p_k}(\partial_{q_l}A\partial_{p_l}B - \partial_{p_l}A\partial_{q_l}B)\partial_{q_k}C = \\ = \partial^2_{q_kq_l}A\partial_{p_l}B\partial_{p_k}C - \partial^2_{q_kq_l}B\partial_{p_k}C\partial_{p_l}A + \\ + \partial^2_{q_kp_l}B\partial_{q_l}A\partial_{p_k}C - \partial^2_{q_kp_l}A\partial_{q_l}B\partial_{p_k}C + \\ + \partial^2_{p_kp_l}A\partial_{q_k}C\partial_{q_l}B - \partial^2_{p_kp_l}B\partial_{q_k}C\partial_{q_l}A + \\ + \partial^2_{q_lp_k}B\partial_{q_k}C\partial_{p_l}A - \partial^2_{q_lp_k}A\partial_{q_k}C\partial_{p_l}B \\ \Longrightarrow [[A,B],C] + [[C,A],B] + [[B,C],A] = 0\end{aligned}$$

(3)
$$\frac{dA}{dt} = \sum_k\left(\frac{dA}{dq_k}\frac{dq}{dt} + \frac{dA}{dp_k}\frac{dp_k}{dt}\right) = \sum_k\left(\frac{dA}{dq_k}\frac{\partial H}{\partial p_k} - \frac{dA}{dp_k}\frac{\partial H}{\partial q_k}\right) = [A,H] \qquad (1.23)$$

1.2. **Canonical quantization.**

1.2.1. *Hilbert space, bras, and kets.*

1.2.2. *Axioms of canonical quantization.* A3. Poisson bracket in classical mechanics is replaced by commutator.

(4)
$$[\widehat{A},\widehat{B}] \equiv \widehat{A}\widehat{B} - \widehat{B}\widehat{A} \qquad (1.31)$$

Problems. Problem 1.1. $H = \int d^nx\left[\frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 + \frac{1}{2}(\nabla\phi)^2 + V(\phi)\right]$

If ϕ time independent, $H[\phi] = H_1[\phi] + H_2[\phi]$

$$H_1[\phi] \equiv \frac{1}{2}\int d^nx(\nabla x)^2$$
$$H_2[\phi] \equiv \int d^nxV(\phi)$$

(1) $\phi(x) \to \phi(\lambda x)$

$$(\nabla\phi)^2 = \partial_j\phi\partial^j\phi = \lambda^{-2}\partial_i\phi\partial^i\phi = \lambda^{-2}(\nabla\phi)^2$$
$$\frac{\partial}{\partial x^i} = \frac{\partial y^i}{\partial x^i}\frac{\partial}{\partial y^i} = \lambda\frac{\partial}{\partial y^i}$$
$$d^ny = \lambda^nd^nx$$

(2) $H_1[\phi] \to \lambda^{n-2}H_1[\phi]$
$H_2[\phi] \to \lambda^nH_2[\phi]$

$$\Longrightarrow \partial_\lambda H = (n-2)H_1 + nH_2 = 0 \text{ or } (2-n)H_1 - nH_2 = 0$$

(3)

1.2.3. *Heisenberg equation, Heisenberg picture and Schrödinger picture.*

1.2.4. *Wavefunction.*

1.2.5. *Harmonic oscillator.*

1.3. **Path integral quantization of a Bose particle.**

1.3.1. *Path integral quantization.*

1.3.2. *Imaginary time and partition function.*

1.3.3. *Time-ordered product and generating functional.*

1.4. **Harmonic oscillator.**

1.4.1. *Transition amplitude.*

1.4.2. *Partition function.*

1.5. **Path integral quantization of a Fermi particle.**

1.6. **Quantization of a scalar field.**

1.7. **Quantization of a Dirac field.**

1.8. **Gauge theories.**

1.8.1. *Abelian gauge theories.*

(5) $$\nabla \cdot B = 0 \quad (1.241a)$$

(6) $$\frac{\partial B}{\partial t} + \nabla \times E = 0 \quad (1.241b)$$

(7) $$\nabla \cdot E = \rho \quad (1.241c)$$

(8) $$\frac{\partial E}{\partial t} - \nabla \times B = -j \quad (1.241d)$$

$$A_\mu = (-\phi, A)$$

$$\begin{aligned} B &= \nabla \times A \\ E &= \frac{\partial A}{\partial t} - \nabla\phi \qquad (1.242) \end{aligned}$$

Exercise 1.11.

$$\begin{gathered} \nabla \cdot B = 0 = \partial_i B_i = \partial_i (\nabla \times A)_i = \partial_i \epsilon_{ijk} \partial_j A_k = \epsilon_{ijk} \partial_i \partial_j A_k = \\ = \partial_1 \partial_2 A_3 - \partial_1 \partial_3 A_2 + \partial_2 \partial_3 A_1 - \partial_2 \partial_1 A_3 + \partial_3 \partial_1 A_2 - \partial_3 \partial_2 A_1 = \partial_1 F_{23} + \partial_2 F_{31} + \partial_3 F_{12} = 0 \end{gathered}$$

Watch out for convention.

$$\partial_\mu = \left(\frac{\partial}{\partial t}, \nabla \right)$$

$$\begin{gathered} \frac{\partial B}{\partial t} + \nabla \times E = \partial_0 \epsilon_{ijk} \partial_j A_k + \epsilon_{ijk} \partial_j E_k = \epsilon_{ijk} \partial_0 \partial_j A_k + \epsilon_{ijk} \partial_j \left(\partial_0 A_k - \partial_k \phi \right) = \epsilon_{ikj} \partial_0 \partial_k A_j + \epsilon_{ijk} \partial_j \partial_0 A_k + \epsilon_{ijk} \partial_j \partial_k A_0 = \\ = \epsilon_{ikj} \partial_0 \partial_k A_j + \epsilon_{ijk} \partial_j \partial_0 A_k + \epsilon_{ikj} \partial_j \partial_k A_0 = \end{gathered}$$

Fix i.

$$\Longrightarrow = \partial_0 \partial_k A_j - \partial_0 \partial_j A_k + \partial_j \partial_0 A_k - \partial_k \partial_0 A_j + \partial_j \partial_k A_0 - \partial_k \partial_j A_0$$

Exercise 1.12.

$$\begin{gathered} \mathcal{L} = (D^\mu \phi)^\dagger (D_\mu \phi) + m^2 \phi^\dagger \phi \\ \phi' = e^{-ie\alpha(x)} \phi \\ (\phi')^\dagger = \phi^\dagger e^{ie\alpha(x)} \\ A'_\mu = A_\mu - \partial_\mu \alpha(x) \qquad (1.257) \end{gathered}$$

Now easily

$$\begin{gathered} \phi^\dagger e^{ie\alpha(x)} e^{-ie\alpha(x)} \phi = \phi^\dagger \phi \\ D_\mu = \partial_\mu - ieA_\mu \\ D_\mu \phi = \partial_\mu \phi - ieA_\mu \phi \\ \partial_\mu (e^{-ie\alpha}\phi) - ie(A_\mu - \partial_\mu \alpha) e^{-ie\alpha} \phi = -e\partial_\mu \alpha \phi e^{-ie\alpha} + e^{-ie\alpha} \partial_\mu \phi - ieA_\mu e^{-ie\alpha} \phi + ie\partial_\mu \alpha e^{-ie\alpha} \phi = e^{-ie\alpha(x)} (D_\mu \phi) \\ D^\mu \phi = \partial^\mu \phi - ieA^\mu \phi \\ (D^\mu \phi)^\dagger = \partial^\mu \phi^\dagger + ieA^\mu \phi^\dagger \end{gathered}$$

$$(D^\mu \phi)^\dagger \to \partial^\mu (\phi^\dagger e^{ie\alpha}) + ie(A^\mu - \partial^\mu \alpha) \phi^\dagger e^{ie\alpha} = \partial^\mu \phi^\dagger e^{ie\alpha} + \phi^\dagger (ie\partial^\mu \alpha) e^{ie\alpha} + ieA^\mu \phi^\dagger e^{ie\alpha} - ie\partial^\mu \alpha \phi^\dagger e^{ie\alpha} = (D^\mu \phi)^\dagger e^{ie\alpha}$$

Clearly $\mathcal{L}$ is invariant.

1.8.2. *Non-Abelian gauge theories.*

1.8.3. *Higgs fields.*

1.9. **Magnetic monopoles.**

1.10. **Instantons.**

1.10.1. *Introduction.*

1.10.2. *The (anti-)self-dual solution.*

2. Mathematical Preliminaries

2.1. Maps.

2.1.1. *Definitions.* **Exercise 2.1.**
$D = [-\pi/2, \pi/2], \quad R = [-1.1] \quad f(x) = \sin(x)$ is bijective on D to R

Exercise 2.2.
$f : x \to x^2$, $g :\to \exp x$

$$f \circ g(x) = exp(2x) \qquad g \circ f(x) = \exp x^2$$

Exercise 2.3. Consider $f(x) = f(y)$

$$gf(x) = x = gf(y) = y \Longrightarrow x = y$$

f injective.

$$\forall\, x \in X,\, x = g \circ f(x) = g(f(x)) = g(y) \text{ since } f : X \to Y. \text{ so } \exists\, y \in Y, \text{ s.t. } g(y) = x$$

g surjective.

Exercise 2.4.

$$\begin{aligned} &a^{-1} : E^n \to E^n & R^{-1}(x) = R^T x R^{-1} R(x) = R^T R x = 1x = x \\ &a^{-1}(x) = x - a & RR^{-1}(x) = RR^T x = 1x = x \\ &a^{-1}a(x) = (x+a) - a = x & (R^T R)_{ij} = R^T_{ik} R_{kj} = R_{ki} R_{kj} = \delta_{ij} \\ &aa^{-1}(x) = (x-a) + a = x & (RR^T)_{ij} = R_{ik} R^T_{kj} = R_{ik} R_{jk} = R^T_{ki} R^T_{kj} = \delta_{ij} \end{aligned}$$

20120306 check above.

$$\begin{aligned} &(R,a)(x) = Rx + a \\ &(R,a)^{-1}(x) = R^T(x-a) \\ &(Ra)^{-1} Ra(x) = R^T(Rx + a - a) = x \\ &(Ra)(Ra)^{-1}(x) = R(R^T(x-a)) + a = x \end{aligned}$$

2.1.2. *Equivalence relation and equivalence class.* **Exercise 2.5.** $m \sim m$ since $\frac{m}{2} = \frac{m}{2}$
$m \sim n$. Then $n \sim m$ since remainder of n by 2 is the same as the remainder of m divided by 2.
$m \sim n$, $n \sim p$. $m \sim p$ since remainder of m by 2 is the same as the remainder of m by 2 which is the same as the remainder of p by 2.

Exercise 2.6. $H = \{\tau \in \mathcal{C} | Im\tau \geq 0\}$
$SL(2,\mathbb{Z}) \equiv \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} | a,b,c,d \in \mathbb{Z},\, ad - bc = 1\}$
$\tau \sim \tau'$ if $\exists\, A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$ s.t. $\tau' = \frac{a\tau + b}{c\tau + d}$

Note that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \tau \\ 1 \end{pmatrix} = \begin{pmatrix} a\tau + b \\ c\tau + d \end{pmatrix}$$

$$\tau \sim \tau \text{ since } \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \begin{pmatrix} \tau \\ 1 \end{pmatrix} = \begin{pmatrix} \tau \\ 1 \end{pmatrix}$$

If $\tau \sim \tau'$ s.t. $\tau' = \frac{a\tau+b}{c\tau+d}$. Consider $\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

$\tau' \sim \tau$ since

$$\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} \tau' \\ 1 \end{pmatrix} = \begin{pmatrix} d\tau' - b \\ -c\tau' + a \end{pmatrix} \qquad \frac{d\tau' - b}{-c\tau' + a} = \frac{d\left(\frac{a\tau+b}{c\tau+d}\right) - b}{-c\left(\frac{a\tau+b}{c\tau+d}\right) + a} = \frac{ad\tau - bc\tau}{ad - bc} = \tau$$

Given $\tau \sim \tau'$, $\tau' \sim \tau''$

$$\begin{aligned} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \tau \\ 1 \end{pmatrix} &= \begin{pmatrix} \tau' \\ 1 \end{pmatrix} \\ \begin{pmatrix} e & f \\ g & h \end{pmatrix} \begin{pmatrix} \tau' \\ 1 \end{pmatrix} &= \begin{pmatrix} \tau'' \\ 1 \end{pmatrix} \end{aligned} \qquad \Longrightarrow \begin{pmatrix} e & f \\ g & h \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \tau \\ 1 \end{pmatrix} = \begin{pmatrix} \tau'' \\ 1 \end{pmatrix} = \begin{pmatrix} ae + cf & be + df \\ ag + ch & bg + dh \end{pmatrix} \begin{pmatrix} \tau \\ 1 \end{pmatrix}$$

$$(ae + cf)(bg - dh) - (ag + ch)(be + df) = abeg + adeh + bcfg + cdfh - abeg - adfg - bceh - cdfh = 1$$

Example 2.6.
$g \sim g'$ if $\exists\, h \in H$ s.t. $g' = gh$
$[g] = \{gh | h \in H\} \equiv gH$ (left) coset.
quotient space $\equiv G/H$
if H normal subgroup of G, $ghg^{-1} \in H, \quad \forall\, g \in G$, $h \in H$, then G/H quotient group.
Since

$$(g')^{-1} h g' = h'' \in H \qquad \Longrightarrow hg' = g'h''$$

$$\Longrightarrow ghg'h' = gg'h''h' = gg'h'''$$

$[g][g'] = [gg']$ well-defined.

Note $[e]$, $[g]^{-1} = [g^{-1}]$

Exercise 2.7. $a, b \in G$ conjugate to each other, $a \simeq b$, if $\exists\, g \in G$ s.t. $b = gag^{-1}$ $a \simeq a$ since $a = eae^{-1} = a$
If $a \simeq b$, $g^{-1}bg = a$ and $g^{-1} \in G$, so $b \simeq a$.
If $a \simeq b$ and $b \simeq c$, then $c = hbh^{-1}$, so $c = hgag^{-1}h^{-1}$ and $(hg)^{-1} = g^{-1}h^{-1}$.

2.2. **Vector spaces.**

2.2.1. *Vectors and vector spaces.*

2.2.2. *Linear maps, images and kernels.*

Theorem 1 (2.1)**.** *If linear $f : V \to W$,*

$$dimV = dim(kerf) + dim(imf)$$

Proof. $\mathrm{ker} f$, $\mathrm{im} f$ vector spaces.

Let basis of $\mathrm{ker} f$ be $\{g_1 \dots g_r\}$
basis of $\mathrm{im} f$ be $\{h'_1 \dots h'_s\}$
$\forall\, i\, (1 \leq i \leq s)$, $h_i \in V$ s.t. $f(h_i) = h'_i$

Consider $\{g_1 \dots g_r, h_1 \dots h_s\}$

Let arbitrary $v \in V$
$f(v) \in \mathrm{im} f$, so $f(v) = c^i h'_i = c^i f(h_i)$
by linearity $f(v - c^i h_i) = 0$,
so $v - c^i h_i \in \mathrm{ker} f$
so $\forall$ arbitrary $v \in V$, v linear combination of $\{g_1 \dots g_r, h_1 \dots h_s\}$ $\square$

Exercise 2.8.

(1) If $x_1, x_2 \in kerf$, then for $x_1 + x_2$

$$\text{by linearity of } f,\ f(x_1+x_2) = f(x_1) + f(x_2) = 0 + 0 = 0 \quad \Longrightarrow x_1 + x_2 \in kerf$$

$$f(cx_1) = cf(x_1) = 0 \quad \Longrightarrow cx_1 \in kerf$$

Under closure of addition and multiplication, $kerf$ is a vector space.

If w_1, $w_2 \in imf$,

$$\begin{aligned} w_1 &= f(x_1) \\ w_2 &= f(x_2) \end{aligned} \text{ for some } x_1, x_2 \in V$$

$$w_1 + w_2 = f(x_1) + f(x_2) = f(x_1 + x_2) \in imf \text{ since } f \text{ linear and since } x_1 + x_2 \in V$$

$$cw_1 = cf(x_1) = f(cx_1) \in imf \text{ since } cx_1 \in V$$

(2) If $f : V \to V$ isomorphism, f bijective. $f(0) = 0$ always for a linear map. Consider $x \in kerf$. So $f(x) = 0$. $f(x) = f(0)$, so $x = 0$.
Since f linear, if $f(x) = f(y)$, $f(x) - f(y) = f(x - y) = 0$. Since $kerf = 0$, $x - y = 0$ so $x = y$. f bijective so f an isomorphism.

2.2.3. *Dual vector space.*

$$f(\mathbf{v}) = f_i \alpha^i (v^j \mathbf{e}_j) = f_i v^j \alpha^i (\mathbf{e}_j) = f_i v^i \qquad (2.12)$$

Use notation $\langle , \rangle : V^* \times V \to K$

$f : V \to W$
$g : W \to K$
$g \in W^*$
$g \circ f : V \to K$ (Note $g \circ f$ on V, key observation)
$g \circ f \equiv h \in V^*$

$$h(\mathbf{v}) \equiv g(f(\mathbf{v})) \quad \mathbf{v} \in V \qquad (2.13)$$

Given $g \in W^*$, $f : V \to W$ induces map $h \in V^*$
$f^* : W^* \to V^*$
$f^* : g \mapsto h = f^*(g) = g \circ f \qquad h$ is the pullback of g by f^*

Exercise 2.9.

Given $f_i = A_i^k e_k$,

$$\alpha^j f_i = A_i^k \alpha^j e_k = A_i^j$$
$$\beta^j A_j^i = \beta^j \alpha^i f_j = \alpha^i \Longrightarrow \alpha^i = \beta^j A_j^i$$

2.2.4. *Inner product and adjoint.* isomorphism $g : V \to V^*$, $g \in GL(m, K)$

$$g : v^j \to g_{ij} v^j \qquad (2.14)$$

$g(v_1, v_2) \equiv \langle gv_1, v_2 \rangle$

$$g(v_1, v_2) = v_1^i g_{ji} v_2^j \qquad (2.16)$$

$W = W(n, \mathbb{R})$, $\{f_\alpha\}$ basis $G : W \to W^*$

Given $f : V \to W$
adjoint of f, $\widetilde{f}$

$$G(w, fv) = g(v, \widetilde{f}w) \qquad (2.17)$$

where $v \in V$, $w \in W$

$$w^\alpha G_{\alpha\beta} f_i^\beta v^i = v^i g_{ij} \widetilde{f}_\alpha^j w^\alpha \qquad (2.18)$$

$G_{\alpha\beta} f_i^\beta = g_{ij} \widetilde{f}_\alpha^j$

$$\widetilde{f} = g^{-1} f^t G^t \qquad (2.19)$$

Exercise 2.10. (cf. wikipedia, "Rank") Consider a $m \times n$ matrix A with column rank A (maximum number of linearly independent column vectors of A).

dim. of column space of $A = r$. Then let $c_1 \dots c_r$ basis.
Place c_i's as column vectors to form $m \times r$ matrix $C = [c_1 \dots c_r]$

$\exists\, r\times m$ matrix R s.t. $A = CR$. $\quad r_{ij} \quad \begin{array}{l} i = 1\dots r \\ j = 1\dots m \end{array}$

$A = CR$, so $\forall$ row vector of A is a linear combination of row vectors of R, so row space of A contained in row space of R.

row rank $A \leq$ row rank R.

R has r rows, $a_{ij} = c_{ik}r_{kj} \quad j = 1\dots n$. row rank $R \leq r =$ column rank A
row rank $A \leq$ column rank A
row rank $A^T \leq$ column rank $A^T \implies$ row rank $A =$ column rank A or rank $(A) =$ rank (A^T)

Likewise for NfM by following the same arguments.

Exercise 2.11.
(a) $g(v_1, v_2) = \overline{v}_1^i g_{ij} v_2^j$

$$\begin{gathered} \overline{g(v_2,v_1)} = \overline{\overline{v}_2^i g_{ij} v_1^j} = v_2^i \overline{g}_{ij} \overline{v}_1^j = \overline{v}_1^j g_{ji} v_2^i = g(v_1, v_2) \\ g(v, \widetilde{f}w) = \overline{v}^i g_{ij} \widetilde{f}_{jk} w^k = \overline{G(w, fv)} = \overline{\overline{w}^\alpha G_{\alpha\beta} f_{\beta\gamma} v^\gamma} = w^\alpha \overline{G}_{\alpha\beta} \overline{f}_{\beta\gamma} \overline{v}^\gamma = \overline{G}_{k\beta} \overline{f}_{\beta i} \overline{v}^i w^k \\ \implies g_{ij}\widetilde{f}_{jk} = \overline{G}_{k\beta}\overline{f}_{\beta i} = f^\dagger_{i\beta} G^\dagger_{\beta k} \\ \implies \widetilde{f} = g^{-1} f^\dagger G^\dagger \end{gathered}$$

(b)

2.2.5. *Tensors.* tensor T of type (p,q) maps p dual vectors and q vectors to $\mathbb{R}$

$$T : \bigotimes^p V^* \bigotimes^q V \to \mathbb{R} \tag{9}$$

Exercise 2.12. $\begin{array}{l} f: V \to W, \text{ so } f(v) \in W. \\ f(v) = w \end{array}$
Then tensor identified with dual vector of W^*. Since $f : V$, Then $(1,1)$.

2.3. **Topological spaces.**

2.3.1. *Definitions.* **Exercise 2.13.**
$\tau_{\mathbb{R}} = \{(a,b) | a, b \in \{\mathbb{R}, \pm\infty\}\}$

$\bigcap_{n=1}^\infty (a, b + \frac{1}{n}) = (a, b]$. Then $\{b\} \in \tau_{\mathbb{R}}$, $\forall\, b \in \mathbb{R}$.

So then $\forall$ subset $Y \subset X$ is open. Discrete topology.

2.3.2. *Continuous maps.* **Exercise 2.14.** If $\begin{array}{c} f: \mathbb{R} \to \mathbb{R} \\ f(x) = x^2 \end{array}$ then $(-\epsilon, +\epsilon) \mapsto [0, \epsilon^2]$

2.3.3. *Neighborhoods and Hausdorff spaces.* **Exercise 2.15.** Let $X = \{\text{John}, \text{Paul}, \text{Ringo}, \text{George}\}$.

$$\begin{aligned} U_0 &= \emptyset \\ U_1 &= \{\text{John}\} \\ U_2 &= \{\text{John}, \text{Paul}\} \\ U_3 &= X \end{aligned} \qquad \begin{aligned} U_1 \bigcup U_2 &= U_2 \\ U_1 U_2 &= U_1 \end{aligned}$$

Consider John and Ringo. Only neighborhood with open set is X for Ringo. Then $XU_{\text{John}} \neq \emptyset$

Exercise 2.16. $\forall\, a, b$, consider $\left(\frac{3a-b}{2}, \frac{a+b}{2}\right)$, $\left(\frac{b+a}{2}, \frac{3b-a}{2}\right)$

2.3.4. *Closed set.*

2.3.5. *Compactness.*

2.3.6. *Connectedness.*

2.4. **Homeomorphisms and topological invariants.**

2.4.1. *Homeomorphisms.*

2.4.2. *Topological invariants.* **Exercise 2.18.** $f : S^1 \to E$

$f(x,y) = (ax, by)$ $\quad ff^{-1} = f^{-1}f = 1$ bijective and cont.
$f^{-1}(x,y) = \left(\frac{x}{a}, \frac{y}{b}\right)$

2.4.3. *Homotopy type.*

2.4.4. *Euler characteristic: an example.*

3. Homology Groups

3.1. **Abelian groups.**

3.1.1. *Elementary group theory.* e.g. $f : \mathbb{Z} \to \mathbb{Z}_2 = \{0,1\}$
$f(2n) = 0$
$f(2n+1) = 1$
homomorphism $f(2m+1+2n) = f(2(m+n)+1) = 1 = 1+0 = f(2m+1) + f(2n)$
$k\mathbb{Z} \equiv \{kn|n \in \mathbb{Z}\}$, $k \in \mathbb{N}$ subgroup of $\mathbb{Z}$, $\mathbb{Z}_2 = \{0,1\}$ not a subgroup.

Let H subgroup of G, $\forall\, x,y \in G$, $x \sim y$ if $x - y \in H$
group operation in G/H naturally induced: $[x] + [y] = [x+y]$
G/H group since H always a normal subgroup of G.
$aH = Ha \,\forall\, a \in G$
if $aH = Ha \quad \forall\, a \in G$, normal indeed.
$aH = a + x - y = x - y + a = Ha$ (since G abelian)

If $H = G$, $0 - x \in G$, $\quad \forall\, x \in G$, $G/G = \{0\}$
If $H = \{0\}$, $G/H = G$ since $x - y = 0$ iff $x = y$

Ex. 3.1. Let us work out the quotient group $\mathbb{Z}/k\mathbb{Z}$.
$km - kn = k(m-n) \in k\mathbb{Z} \quad [km] = [kn]$

$\forall\, j, \quad 1 \leq j \leq k-1$, $(km+j) - (kn+j) = k(m-n) \in k\mathbb{Z}$. $[km+j] = [kn+j]$
$\forall\, j,l$, $0 \leq j,l \leq k-1$, $j \neq l$, $(km+j) - (kn+l) = k(m-n) + (j-l) \notin k\mathbb{Z}$. Never belong to the same equivalence class.

$\implies \mathbb{Z}.k\mathbb{Z} = \{[0], \ldots, [k-1]\}$.

Define isomorphism $\varphi : \mathbb{Z}/k\mathbb{Z} \to \mathbb{Z}_k$, then $\mathbb{Z}/k\mathbb{Z} \simeq \mathbb{Z}_k$
$\varphi([j]) = j$

Lemma 1 (3.1)**.** *Let $f : G_1 \to G_2$ homomorphism. Then*
(a) $\ker f = \{x | x \in G_1, f(x) =\}$ *subgroup of G. Note:* $\ker f$ ***normal subgroup*** *of G_1,*
$f(gxg^{-1}) = f(g)f(x)f(g^{-1}) = 1 \,\forall\, g \in G$, $x \in \ker f \implies \ker f$ *normal subgroup*
(b) $imf = \{x | x \in f(G_1) \subset G_2\}$ *subgroup of G_2*

Proof. (a) Let $x, y \in \ker f$
$xy \in \ker f$ since $f(xy) = f(x)f(y) = 1 \cdot 1 = 1$
Note $1 \in \ker f$ since $f(1) = f(1 \cdot 1) = f(1)f(1) \implies f(1) = 1$
$x^{-1} \in \ker f$ since $f(x^{-1} \cdot x) = f(x^{-1})f(x) = f(x^{-1})1 = f(1) = 1 \quad f(x^{-1}) = 1$

(b) Let $y_1 = f(x_1) \quad y_1, y_2 \in \mathrm{im}(f)$
$y_2 = f(x_2) \quad x_1, x_2 \in G_1$

$$y_1 y_2 = f(x_1)f(x_2) = f(x_1x_2) \quad x_1x_2 \in G_1 \quad y_1y_2 \in \mathrm{im} f$$

$1 \in \mathrm{im} f$ since $f(1) = 1$ (or $f(x_1) = f(x_1 \cdot 1) = f(x_1)f(1)$; so $f(1) = 1$ and $1 \in \mathrm{im} f$)

$1 = f(xx^{-1}) = f(x)f(x^{-1}) = yf(x^{-1}) \Longrightarrow f(x^{-1}) = y^{-1}$ and $y^{-1} \in \mathrm{im} f$ since $x^{-1} \in G$

□

Theorem 2 (3.1)**.** *(Fundamental Thm. of homomorphism)*

$$G_1/\ker f \simeq imf$$

Proof. By Lemma 3.1, both sides are groups. □

Define. $\varphi : G_1/\ker f \to \mathrm{im} f.$ φ well-defined since $\forall\, x' \in [x]$, $\exists\, h \in \ker f$ s.t. $\begin{matrix} x' = x + h \\ f(x') = f(x)f(h) = f(x) \end{matrix}$

$\varphi([x]) = f(x)$

$$\varphi([x] + [u]) = \varphi([x + y]) = f(x + y) = f(x) + f(y) = \varphi([x]) + \varphi([y])$$

φ 1-to-1: if $\varphi([x]) = \varphi([y])$, then $f(x) = f(y)$ or $f(x) - f(y) = f(x - y) = 0$. $x - y \in \ker f$ so $[x] = [y]$

φ onto: if $y \in \mathrm{im} f$, $\exists\, x \in G_1$ s.t. $f(x) = y = \varphi([x])$

3.1.2. *Finitely generated Abelian groups and free Abelian groups.*

Lemma 2 (3.2)**.** *Let G be free Abelian group of rank r,*

$G = \{n_1x_1 + \dots + n_rx_r | n_i \in \mathbb{Z},\, 1 \leq i \leq r,\, n_1x_1 + \dots + n_rx_r = 0$ *only if* $n_1 = \dots = n_r = 0\} \equiv$ *free Abelian group of rank* r

Let subgroup H. Choose p generators $x_1 \dots x_p$ of r generators of G so generate H.

$$H \simeq k_1\mathbb{Z} \oplus \dots \oplus k_p\mathbb{Z} \text{ and } H \text{ rank } p$$

Proof.

$$f : \underbrace{\mathbb{Z} \oplus \dots \oplus \mathbb{Z}}_{m} \to G$$

$$f(n_1 \dots n_m) = n_1x_1 + \dots + n_mx_m \quad \text{(surjective homomorphism)}$$

From Thm. 3.1. $\underbrace{\mathbb{Z} \oplus \dots \oplus \mathbb{Z}}_{m}/\ker f \simeq G$

$\ker f$ subgroup of $\underbrace{\mathbb{Z} \oplus \dots \oplus \mathbb{Z}}_{m}$, so Lemma 3.2, $\ker f \simeq k_1\mathbb{Z}\mathbb{Z} \oplus \dots \oplus k_p\mathbb{Z}$

$$G \simeq \underbrace{\mathbb{Z} \oplus \dots \oplus \mathbb{Z}}_{m}/\ker f \simeq \underbrace{\mathbb{Z} \oplus \dots \oplus \mathbb{Z}}_{m}/(k_1\mathbb{Z} \oplus \dots \oplus k_p\mathbb{Z}) \simeq \underbrace{\mathbb{Z} \oplus \dots \oplus \mathbb{Z}}_{m-p} \oplus Z_{k_1} \oplus \dots \oplus \mathbb{Z}_{k_p}$$

□

3.2. Simplexes and simplicial complexes.

3.2.1. *Simplexes.* number of q-faces in r-simplex is $\binom{r+1}{q+1}$

$r + 1 \quad p_0 \dots p_r$ pts., choose $p_{i_0} \dots p_{i_q}$ pts.

3.2.2. *Simplicial complexes and polyhedra.* Example 3.5. Fig. 3.5(b) not a triangulation of a cylinder.

$\sigma_2 = \langle p_0p_1p_2 \rangle$
$\sigma_2' = \langle p_2p_3p_0 \rangle$
$\sigma_2\sigma_2' \neq \emptyset$
$\sigma_2\sigma_2' = \langle p_0 \rangle \bigcup \langle p_2 \rangle \qquad \sigma_2\sigma_2'$ is not an actual face.

3.3. Homology groups of simplicial complexes.

3.3.1. *Oriented simplexes.*

3.3.2. *Chain group, Cycle group and boundary group.*

Definition 1 (3.2)**.** *r-chain group $C_r(K)$ of simplicial complex K is free Abelian group generated by oriented r-simplexes of K element of $C_r(K)$ is r-chain.*

Let I_r r- simplexes in K, $\sigma_{r,i}$ $(1 \leq i \leq I_r)$

(10) $$c = \sum_{i=1}^{I_r} c_i \sigma_{r,i} \quad c_i \in \mathbb{Z}, \text{ coefficients of } c \qquad (3.15)$$

addition of 2 r-chains $\begin{array}{l} c = \sum_i c_i \sigma_{r,i} \\ c' = \sum_i c'_i \sigma_{r,i} \end{array}$ $\qquad c + c' = \sum_i (c_i + c'_i)\sigma_{r,i} \qquad (3.16)$

inverse $-c = \sum_i (-c_i)\sigma_{r,i}$

(11) $$C_r(K) \simeq \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{I_r} \qquad \text{(3.17) free Abelian group of rank } I_r$$

0-simplex has no boundary: $\partial_0 p_0 = 0 \qquad (3.18)$

Fig. 3.7(a) oriented 1 simplex.

$$\partial_1(p_0p_1) + \partial_1(p_1p_2) = p_1 - p_0 + p_2 - p_1 = p_2 - p_0 = \partial_1(p_2p_0)$$

Fig. 3.7(b) triangle.

$$\partial_1(p_0p_1) + \partial_1(p_1p_2) + \partial_1(p_2p_0) = p_1 - p_0 + p_2 - p_1 + p_0 - p_2 = 0$$

Let $\sigma_r(p_0 \dots p_r)$ oriented r-simplex.

(12) $$\partial_r \sigma_r \equiv \sum_{i=0}^{r} (-1)^i (p_0 p_1 \dots \widehat{p}_i \dots p_r) \qquad (3.20)$$

$K \equiv n$-dim. simplicial complex.

chain complex $C(K)$.
i inclusion map $i : 0 \hookrightarrow C_n(K)$

(13) $$0 \xrightarrow{i} C_n(K) \xrightarrow{\partial_n} C_{n-1}(K) \xrightarrow{\partial_{n-1}} \dots \xrightarrow{\partial_2} C_1(K) \xrightarrow{\partial_1} C_0(K) \xrightarrow{\partial_0} 0 \qquad (3.23)$$

Definition 2 (3.3)**.** *If $c \in C_r(K)$, $\partial_r c = 0$, c r-cycle.*

$$\boxed{Z_r(K) = \{c | \partial_r c = 0\} = \ker \partial_r \equiv r - \textit{cycle group}}$$

if $r = 0$, $\partial_0 c = 0$, $Z_0(K) = C_0(K)$

Definition 3 (3.4)**.** *If $\exists\, d \in C_{r+1}(K)$, $c = \partial_{r+1} d$ $\qquad$ (3.25), c r-boundary*

$$\boxed{B_r(K) = im \partial_{r+1} \equiv r - \textit{boundary group} \qquad B_n(K) = 0}$$

Theorem 3 (3.3)**.**

(14) $$B_r(K) \subset Z_r(K) \quad (\subset C_r(K)) \qquad (3.27)$$

Proof.

$$\forall\, c \in B_r(K),\ \exists\, d \in c_{r+1}(K) \text{ s.t. } c = \partial_{r+1} d \qquad \partial_r c = \partial_r \partial_{r+1} d = 0 \,(\text{Lemma 3.3, } \partial^2 d = 0) \quad c \in Z_r(K)$$

□

3.3.3. *Homology groups.* **Exercise 3.1.** $K = \{p_0, p_1\}$. $I_r = I_0 = 2$ $0-$simplexes. $c = c_1 p_0 + c_2 p_1$ $0-$chains.

$$c_0(K) \simeq \mathbb{Z} \oplus \mathbb{Z}$$
$$\partial_0 c = 0 \quad \forall\, c \in C_0(K) \quad C_0(K) = Z_r(K)$$
$$B_0(K) = \mathrm{im}\partial_1 = 0$$
$$\Longrightarrow H_0(K) = \mathbb{Z} \oplus \mathbb{Z}$$

if $r \neq 0$, $Z_r(K) = 0$ since there are no other simplexes than $0-$simplexes. $H_r(K) = 0$

(15) $$\Longrightarrow H_r(K) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & (r = 0) \\ \{0\} & (r \neq 0) \end{cases} \qquad (3.34)$$

Ex. 3.7. $k = \{p_0, p_1, (p_0 p_1)\}$

$$C_0(K) = \{ip_0 + jp_1 | i, j \in \mathbb{Z}\}$$
$$C_1(K) = \{k(p_0 p_1) | k \in \mathbb{Z}$$

$B_1(K) = 0$, since $(p_0 p_1)$ not a boundary of any simplex in K i.e. $\exists\, d \in K$ s.t. $\partial_2 d = (p_0 p_1)$, since $\nexists$ 2-simplex

$$H_1(K) = Z_1(K)/B_1(K) = Z_1(K)$$

If $z = m(p_0 p1_) \in \mathbb{Z}_1(K)$,

$$\partial_1 z = m(p_1 - p_0) = 0 \Longrightarrow m = 0 \quad Z_1(K) = 0$$

(16) $$H_1(K) = 0 \qquad (3.35)$$

$Z_0(K) = C_0(K)$

Define surjective (onto) homomorphism. $f : Z_0(K) \to \mathbb{Z}$

$$f(ip_0 + jp_1) = i + j$$

$$\ker f = f^{-1}(0) = B_0(K)$$
$$\partial_1(k(p_0 p_1)) = kp_1 - kp_0 \text{ and } f(kp_1 - kp_0) = 0$$

Thm. 3.1. $Z_0(K)/\ker f \simeq \mathrm{im} f = \mathbb{Z}$.

(17) $$H_0(K) = Z_0(K)/B_0(K) \simeq \mathbb{Z} \qquad (3.30)$$

Ex. 3.8. Triangulation of S^1 (**triangulation of circle**)

$$K = \{p_0, p_1, p_2, (p_0 p_1), (p_1 p_2), (p_2 p_0)\}$$
$$B_1(K) = 0 \quad (\text{no 2-simplices in } K) \qquad H_1(K) = Z_1(K)/B_1(K) = Z_1(K)$$
$$\partial_1 z = \partial_1(i(p_0 p_1) + j(p_1 p_2) + k(p_2 p_0)) = i(p_1 - p_0 + j(p_2 - p_1) + k(p_0 - p_2) = (k - i)p_0 + (i - j)p_1 + (j - k)p_2 = 0$$
$$\Longrightarrow i = j = k$$
$$Z_1(K) = \{i((p_0 p_1) + (p_1 p_2) + (p_2 p_0)) | i \in \mathbb{Z}\} \simeq \mathbb{Z} \Longrightarrow H_1(K) = Z_1(K) \simeq \mathbb{Z} \qquad (3.37)$$

$$Z_0(K) = C_0(K)$$
$$B_0(K) = \{\partial_1[l(p_0 p_1) + m(p_1 p_2) + n(p_2 p_0)] | l, m, n \in \mathbb{Z}\} = \{(n - l)p_0 + (l - m)p_1 + (m - n)p_2 | l, m, n \in \mathbb{Z}\}$$

Exercise 3.2. Let $K = \{p_0, p_1, p_2, p_3, (p_0 p_1), (p_1 p_2), (p_2 p_3), (p_3 p_0)\}$
$B_1(K) = 0 \quad (KC_2(K) = \emptyset)$
$H_1(K) = Z_1(K)$

$$\partial_1(i(p_0 p_1) + j(p_1 p_2) + k(p_2 p_0) + l(p_3 p_0)) = (l - i)p_0 + (i - j)p_1 + (j - k)p_2 + (k - l)p_3$$

$\partial_1 c = 0$ if $i = l = j = k$, $Z_1(K) \simeq \mathbb{Z}$. $H_1(K) \simeq \mathbb{Z}$

$Z_0(K) = C_0(K) \quad (\partial_0 p_i = 0)$

Define surjective homomorphism $f : C_0(K) \mapsto \mathbb{Z}$

$$f(ip_0 + jp_1 + kp_2 + lp_3) = i + j + k + l$$

$$B_0(K) = \{ap_0 + bp_1 + cp_2 + dp_3 | d = -(a+b+c)\} \qquad \begin{aligned} a &= l - i \\ b &= i - j \\ c &= j - k \\ d &= k - l = -(a+b+c) \end{aligned}$$

$$\implies B_0(K) = \ker f$$

$$H_0(K) = Z_0(K)/B_0(K) = C_0(K)/\ker f \simeq \mathrm{im} f = \mathbb{Z}$$

Exercise 3.3. $B_2(K) = 0 \quad KC_2(K) = \emptyset$

$$\partial_2 c_2 = \partial_2(i(p_0p_1p_2) + j(p_0p_1p_3) + k(p_0p_2p_3) + l(p_1p_2p_3)) =$$

$$= (i+l)(p_1p_2) + (-i+k)(p_0p_2) + (i+j)(p_0p_1) + (-j+-k)(p_0p_3) + (j-l)(p_1p_3) + (k+l)(p_2p_3) = 0$$

$$\implies i = k = -l = -j \implies Z_2(K) \simeq \mathbb{Z} \text{ so } \boxed{H_2(K) \simeq \mathbb{Z}}$$

$$\partial_1 c_1 = \partial_1(a(p_0p_1) + b(p_0p_2) + c(p_0p_3) + d(p_1p_2) + e(p_1p_3) + f(p_2p_3)) =$$

$$= (-a-b-c)p_0 + (a-d-e)p_1 + (b+d-f)p_2 + (c+e+f)p_3 = 0$$

By linear algebra,

$$\begin{bmatrix} -1 & -1 & -1 & & & \\ 1 & & & -1 & -1 & \\ & 1 & & 1 & & -1 \\ & & 1 & & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix} = 0 \implies \begin{bmatrix} 1 & & & -1 & -1 & \\ & 1 & 1 & 1 & 1 & \\ & 1 & & 1 & & -1 \\ & & 1 & & 1 & 1 \end{bmatrix} \implies \begin{aligned} a &= d + e \\ b &= -d + f \\ c &= -e - f \end{aligned}$$

If

$$\begin{aligned} a &= i + j \\ b &= -i + k \\ c &= -j - k \\ d &= i + l \\ e &= j - l \\ f &= k + l \end{aligned}$$

Then $B_1(K) = Z_1(K)$. Then $Z_1(K)/B_1(K) = 0$ by algebra. $\boxed{H_1(K) = 0}$

Let $f : Z_0(K) \to \mathbb{Z}$

$f(ip_0 + jp_1 + kp_2 + lp_3) = i + j + k + l$

$\ker f = B_0(K)$ since if

$$\begin{aligned} i &= -(a+b+c) \\ j &= a - d - e \\ k &= b + d - f \\ l &= c + e + f \end{aligned}$$

then $i + j + k + l = 0$

$$Z_0(K)/\ker f \simeq \mathrm{im} f = \mathbb{Z} \implies \boxed{H_0 \simeq \mathbb{Z}}$$

3.3.4. *Computation of $H_0(K)$.*

Theorem 4 (3.5)**.** *Let K be* connected *simplicial complex. Then*

$$H_0(K) \simeq \mathbb{Z} \qquad (3.43) \tag{18}$$

3.3.5. *More homology computations.* Example 3.10. Fig. 3.8. triangulation of Möbius strip.
$B_2(K) = 0 \quad \partial_2 z = 0$ each $(p_0p_2), (p_1p_4), (p_2p_3), (p_4p_5), (p_3p_1), (p_5p_0)$ appear only once. $Z_2(K) = 0$

(19) $$H_2(K) = 0 \qquad (3.44)$$

$H_1(K) = Z_1(K)/B_1(K)$. $\ker \partial_1$ consists of closed loops. (closes on itself so $\partial_1 c_1 = 0$)

Note $z \sim z'$ if $z - z' = \partial_2 c_2 \in B_1(K)$
easily verify, $H_1(K)$ generated by just $[z]$. $H_1(K) = \{i[z] | i \in \mathbb{Z}\} \simeq \mathbb{Z} \qquad (3.45)$

Example 3.11. projective plane $\mathbb{R}P^2$

Example $H_2(K)$ from a slightly different view pt. Add all 2-simplexes in K with same coefficient

$$z \equiv \sum_{i=1}^{10} m\sigma_{2,i}, \qquad m \in \mathbb{Z}$$

Observe that each 1-simplex of K is a common face of exactly 2 2-simplices.

(20) $$\partial_2 z = 2m(p_3p_5) + 2m(p_5p_4) + 2m(p_4p_3) \qquad (3.47)$$

$Z_2(K) = 0$ since $\partial_2 z = 0$ if $m = 0$
Note, any 1-cycle homologous to a multiple of $z = (p_3p_5) + (p_5p_4) + (p_4p_3)$
even multiples of z is a boundary of 2-chain by Eq. 3.4.7.

(21) $$H_1(K) = \{[z] | [z] + [z] \sim [0]\} \simeq \mathbb{Z}_2 \qquad (3.48)$$

This example shows that a homology group is not necessarily free Abelian.

Example 3.12. surface of the torus has no boundary, but it's not a boundary of some 3-chain.
Thus, $H_2(T^2)$ freely generated by 1 generator, surface itself, $H_2(T^2) \simeq \mathbb{Z}$
closed loops a, a' homologous since $a' - a = \partial d$ bounds shaded area Fig. 3.10 (could think of a running backwards)
See Figure 3.12, triangulation of the Klein bottle (clear from there).
inner 1 simplices cancel out to leave only outer 1-simplexes (1 side of the "square")

$$z = \sum m\sigma_{2,i}$$
$$\partial_2 z = -2ma$$
$$a = (p_0p_1) + (p_1p_2) + (p_2p_0)$$

$\partial_2 z = 0$ if $m = 0$

(22) $$H_2(K) = Z_2(K) \simeq 0 \qquad (3.50)$$
$$b = (p_0p_3) + (p_3p_4) + (p_4p_0)$$

1-cycle $\quad c_1 = ia + ib$
$\partial_1 c_1 = 0$ closed
Now $\partial_2 z = +2ma$ so

$$2ma \sim 0$$

Thus, $H_1(K)$ generated by 2 cycles a, b s.t. $a + a = 0$ (remember a is "glued backwards")

(23) $$H_1(K) = \{i[a] + j[b] | i, j \in \mathbb{Z}\} \simeq \mathbb{Z}_2 \oplus \mathbb{Z} \qquad (3.51)$$

3.4. **General properties of homology groups.**

3.4.1. *Connectedness and homology groups.*

Theorem 5 (3.6)**.** *Let $K = \bigcup_{k=1}^N K_i$, $K_iK_j \neq \emptyset$. K_i connected components. Then*

$$H_r(K) = \bigoplus_{i=1}^N H_r(K_i)$$

Proof. $C_r(K) = \bigoplus_{i=1}^N C_r(K_i)$ (rearrange cycles)
since $Z_r(K_i) \supset B_r(K_i)$, $H_r(K_i)$ well-defined. □

3.4.2. *Structure of homology groups.*

Definition 4 (3.6)**.** *Let K simplicial complex.*

$$b_r(K) \equiv \dim H_r(K;\mathbb{R}) \quad (ith\ Betti\ number) \qquad (3.56) \tag{24}$$

Theorem 6 (3.7)**.** *(The Euler-Poincarè theorem) Let K $n-$dim. simplicial complex, I_r number of r-simplexes in K.*

$$\chi(K) \equiv \sum_{i=0}^n (-1)^r I_r = \sum_{r=0}^n (-1)^r b_r(K) \qquad (3.57) \tag{25}$$

Problem 3.1.

Let S^2 with h handles and q holes $= X$

$$C_3(X) \bigcap X = \emptyset \text{ so } B_2(X) = 0 \quad Z_2(X) \simeq \mathbb{Z}\,(1 \text{ surface }) \boxed{H_2(X) \simeq \mathbb{Z} \quad b_2(X) = 1}$$

h handles. Think of T^2.
q holes. For orientable case, the first hole does not change the homology, i.e.

$$\partial c_1 = 0 \text{ but for } d = \sum_{\sigma_2 \in C_2(X)} m\sigma_2,\ \partial d = c_1 \quad c_1 = 0$$

$$\boxed{H_1(X) = \mathbb{Z}^{2h+q-1}}$$

The sphere is connected (and note compact). $\boxed{H_0(X) = \mathbb{Z}}$

Problem 3.2.

Each cross cap hole contributes to $H_1(X)$ a 1 cycle z_q s.t. $2z_q \sim 0$. Thus,

$$\boxed{H_1(X) = \mathbb{Z}_2^q}$$

Otherwise,

$$\boxed{\begin{array}{c} H_2(X) = \mathbb{Z} \\ H_0(X) = \mathbb{Z} \end{array}}$$

4. Homotopy Groups

5. Manifolds

5.1. **Manifolds.**

5.1.1. *Heuristic introduction.*

5.1.2. *Definitions.*

5.1.3. *Examples.* **Exercise 5.1.**

$$S^n = \{(x_1 \dots x_{n+1}\} \in \mathbb{R}^{n+1} | \sum_{i=1}^{n+1} x_i^2 = 1\} \subset \mathbb{R}^{n+1}$$

Let $N = (0 \dots 0, 1)$ $\qquad x \in S^n$. Consider $t(x - N) + N = tx + (1-t)N$ when $x_{n+1} = 0$
$S = (0 \dots 0, -1)$

$$tx_{n+1} + (1-t) = 0 \text{ or } tx_{n+1} + -1 + t = 0$$
$$\Longrightarrow \frac{1}{1 - x_{n+1}} = t \quad \left(\text{or } \frac{1}{1 + x_{n+1}} \right)$$

$$\pi_1 : S^n - N \to \mathbb{R}^n$$
$$\pi_1(x_1 \dots x_{n+1}) = \left(\frac{x_1}{1-x_{n+1}} \cdots \frac{x_n}{1-x_{n+1}}, 0 \right)$$
$$\pi_2 : S^n - S \to \mathbb{R}^n$$
$$\pi_2(x_1 \dots x_{n+1}) = \left(\frac{x_1}{1+x_{n+1}} \cdots \frac{x_n}{1+x_{n+1}}, 0 \right)$$

Note, for $y_i = \frac{x_i}{1-x_{n+1}}$

$$y_1^2 + \dots + y_n^2 = |y|^2 = \frac{1-x_{n+1}^2}{(1-x_{n+1})^2} = \frac{1+x_{n+1}}{1-x_{n+1}} \text{ or } x_{n+1} = \frac{|y|^2-1}{|y|^2+1}$$
$$x_i = y_i(1-x_{n+1}) = \frac{2y_i}{1+|y|^2}$$

$$\pi_1^{-1} : \mathbb{R}^n \to S^n - N$$
$$\pi_1^{-1}(y_1 \dots y_n) = \left(\frac{2y_1}{1+|y|^2} \cdots \frac{2y_n}{1+|y|^2}, \frac{|y|^2-1}{|y|^2+1} \right)$$
$$\pi_2^{-1}(y_1 \dots y_n) = \left(\frac{2y_1}{1+|y|^2} \cdots \frac{2y_n}{1+|y|^2}, \frac{1-|y|^2}{|y|^2+1} \right)$$

π_1, π_2 diff. injective, and $(S^n - N) \bigcup (S^n - S) = S^n$

Consider $(S^n - N)(S^n - S) = S^n - N \bigcup S$

$$\pi_1\pi_2^{-1}(y_1 \dots y_n) = \left(\frac{y_1}{|y|^2} \cdots \frac{y_n}{|y|^2}, 0 \right)$$
$$\pi_2\pi_1^{-1}(y_1 \dots y_n) = \left(\frac{y_1}{|y|^2} \cdots \frac{y_n}{|y|^2}, 0 \right)$$

since, for example,

$$\frac{\frac{2y_i}{1+|y|^2}}{1 - \frac{1-|y|^2}{1+|y|^2}} = \frac{y_i}{|y|^2}$$

$\pi_1\pi_2^{-1}$ bijective and C^∞, $\pi_1\pi_2^{-1}$ diffeomorphism.

$\{(S^n - N, \pi_1), (S^n - S, \pi_2)\}$ C^∞ atlas or differentiable structure.

5.2. **The calculus on manifolds.**

5.2.1. *Differentiable maps.* **Exercise 5.2.** If $f : M \to N$ smooth,

Consider $(U'_a, \varphi'_a) \in \mathcal{A}$ s.t. $f(U'_a) \subseteq V'_b$
$(V'_b, \psi'_b) \in \mathcal{B}$

Consider
$$\psi'_b f(\varphi'_a)^{-1} = \psi'_b(\psi_\beta^{-1}\psi_\beta) f(\varphi_\alpha^{-1}\varphi_\alpha)(\varphi'_a)^{-1} = (\psi'_b\psi_\beta^{-1})(\psi_\beta f \varphi_\alpha^{-1})(\varphi_\alpha\varphi'_a)^{-1}$$

$U_\alpha U'_a \neq \emptyset$ since $x \in U_\alpha, U'_a$
$V_\beta V'_b \neq \emptyset$ $f(x) \in V_\beta V'_\beta$

Then $(\psi'_b\psi_\beta^{-1}), (\varphi_\alpha\varphi'_a)^{-1}$, C^∞ (def. of atlas) and $(\psi_\beta f \varphi_\alpha^{-1})$ C^∞ (given).
So $\psi'_b f(\varphi'_a)^{-1}$ C^∞

5.2.2. *Vectors.* curve $c : I \to M$
$f : M \to \mathbb{R}$

$$\frac{d}{dt} f(c(t))|_{t=0} \qquad (5.18)$$

for $p = c(0) \in (U, \varphi) = (U, x^1 \dots x^n)$, $\quad fc = f\varphi^{-1}\varphi c$

$$\frac{d}{dt} f(c(t))|_{t=0} = \frac{\partial (f\varphi^{-1})}{\partial x^\mu}(x) \left. \frac{d(x^\mu(c))}{dt}(t) \right|_{t=0} = \frac{\partial f}{\partial x^\mu} \left. \frac{dx^\mu(c(t))}{dt} \right|_{t=0} = \frac{\partial f}{\partial x^\mu} \left. \frac{dc^\mu}{dt} \right|_{t=0} \qquad \text{(abuse of notation)}$$

Note that abuse of notation: $\frac{dx^\mu}{dt}(c(t))\big|_{t=0} \equiv \left. \frac{dx^\mu(\varphi(c(t)))}{dt} \right|_{t=0}$

$\left. \frac{df(c(t))}{dt} \right|_{t=0}$ obtained by differential operator X to f

$$X = X^\mu \left(\frac{\partial}{\partial x^\mu} \right) \qquad \left(X^\mu = \left. \frac{dx^\mu(c(t))}{dt} \right|_{t=0} \right) \qquad (5.20)$$

$$\left. \frac{df(c(t))}{dt} \right|_{t=0} = X^\mu \left(\frac{\partial f}{\partial x^\mu} \right) = X[f] \qquad (5.21)$$

Introduce equivalence class of curves in M.
curves $c_1(t) \sim c_2(t)$ if

(1) $c_1(0) = c_2(0) = p$
(2) $\left. \frac{dx^\mu(c_1(t))}{dt} \right|_{t=0} = \left. \frac{dx^\mu(c_2(t))}{dt} \right|_{t=0}$

Identify tangent vector X with $[c(t)]$.
T_pM, tangent space of M at p, all $[c]$ at $p \in M$
Use Sec. 2.2's theory of vector spaces to analyze T_pM

Recall curve $c(t) : \mathbb{R} \to M$,
$f : M \to \mathbb{R}$
Now tangent vector at $c(0)$ was, recall

$$\left. \frac{df(c(t))}{dt} \right|_{t=0} = \frac{\partial f}{\partial x^\mu} \left. \frac{dx^\mu(c(t))}{dt} \right|_{t=0} = \left. \frac{dx^\mu(c(t))}{dt} \right|_{t=0} \frac{\partial f}{\partial x^\mu}$$

Let $X^\mu = \left. \frac{dx^\mu(c(t))}{dt} \right|_{t=0}$

$$X[f] \equiv \left. \frac{df(c(t))}{dt} \right|_{t=0}$$

$$\implies X = x^\mu \frac{\partial}{\partial x^\mu}$$

So tangent vector X is spanned by $\frac{\partial}{\partial x^\mu}$
Suppose $a^\mu \frac{\partial}{\partial x^\mu} = 0$

$$a^\mu \frac{\partial}{\partial x^\mu} x^\nu = a^\mu \delta_\mu^{\ \nu} = a^\nu = 0$$

So $\{ \frac{\partial}{\partial x^\mu} \}$ linearly independent.
$\{ \frac{\partial}{\partial x^m} \}$ a basis.

(cf. wikipedia) Consider short $\varphi : U \to \mathbb{R}^n$, $p \in U$, define map
$$\begin{aligned} &(d\varphi)_p : T_xM \to \mathbb{R}^n \\ &(d\varphi)_p([c(0)]) = \frac{d}{dt}(\varphi(c(0))) \end{aligned}$$

Consider $c(0) = p = c(t) \quad \forall\, t$ s.t. $c(t) \in U$.
Surely $\varphi(c(t)) = \text{const.}$ (for φ is a well-defined function)
$\frac{d}{dt}\varphi(c(t)) = 0$. For $[c(0)] = [p]$, $(d\varphi)_p([p]) = 0 \quad \forall$ chart $\varphi \quad (d\varphi)_p$ injective.
Consider $a^i e_i \in \mathbb{R}^n$

$$a^i = \frac{d}{dt}(\varphi^i(c(0))) \equiv \left. \frac{d}{dt} x^i(c(t)) \right|_{t=0}$$

$f : M \to \mathbb{R} \in C^\infty(M)$ if $f\varphi^{-1} \in C^\infty$. $\forall$ chart $\varphi : U \to \mathbb{R}^n$

derivation at p is linear $D : C^\infty(M) \to \mathbb{R}$ (or $\begin{array}{l} X : C^\infty(M) \to \mathbb{R} \\ X[f] = \left.\dfrac{df(c(t))}{dt}\right|_{t=0} \end{array}$). These derivations form a vector space, tangent space T_pM.

$$\left.\frac{df(c(t))}{dt}\right|_{t=0} = \frac{\partial g}{\partial x^\mu}(x^\nu)\frac{dx^\mu(c(t))}{dt} = \frac{dx^\mu(c(t))}{dt}\frac{\partial}{\partial x^\mu}f$$

$g(x^\nu) = (f \cdot \varphi^{-1})(x^\nu)$

Consider $p \in U_iU_j$, $\begin{array}{l} x = \varphi_i(p) \\ y = \varphi_j(p) \end{array}$

$$X = X^\mu \frac{\partial}{\partial x^\mu} = Y^\nu \frac{\partial}{\partial y^\nu}$$

$$Y^\nu = X^\mu \frac{\partial y^\nu}{\partial x^\mu} \qquad (5.23)$$

5.2.3. *One-forms.* $\begin{array}{l} \omega \in T_p^*M \\ \omega : T_pM \to \mathbb{R} \end{array}$ **cotangent vector or 1-form**.

differential $df \in T_p^*M$ on $V \in T_pM$

$$\langle df, V\rangle = V[f] = V^\mu \frac{\partial f}{\partial x^\mu} \in \mathbb{R} \qquad (5.24)$$

arbitrary 1-form ω

$$\omega = \omega_\mu dx^\mu \qquad (5.26)$$

inner product defined

$$\langle\, ,\, \rangle : T_p^*M \times T_pM \to \mathbb{R}$$

$$\langle \omega, V\rangle = \omega_\mu V^\mu \langle dx^\mu, \frac{\partial}{\partial x^\nu}\rangle = \omega_\mu V^\mu \qquad (5.27)$$

For $p \in U_iU_j$, $\begin{array}{l} x = \varphi_i(p) \\ y = \varphi_j(p) \end{array}$

$\begin{array}{l} dy^\nu \in T_p^*M \text{ so} \\ dy^\nu = \omega_\mu dx^\mu \end{array}$ so

$$\langle dy^\nu, \frac{\partial}{\partial x^\gamma}\rangle = \delta^{\mu\gamma}\frac{\partial y^\nu}{\partial x^\mu} = \frac{\partial y^\nu}{\partial x^\gamma} = \omega_\mu \langle dx^\mu, \partial_{x^\gamma}\rangle = \omega_\gamma$$

$$\implies dy^\nu = \frac{\partial y^\nu}{\partial x^\mu}dx^\mu$$

$$\omega = \omega_\mu dx^\mu = \psi_\nu dy^\nu = \psi_\nu \frac{\partial y^\nu}{\partial x^\mu}dx^\mu \text{ or } \omega_\mu = \psi_\nu \frac{\partial y^\nu}{\partial x^\mu}$$

5.2.4. *Tensors.* $\tau^q_{r,p}(M)$ set of type (q, r) tensors at $p \in M$.
element of $\tau^q_{r,p}(M)$

$$T = T^{\mu_1 \dots \mu_q}{}_{\nu_1 \dots \nu_r} \frac{\partial}{\partial x^{\mu_1}} \dots \frac{\partial}{\partial x^{\mu_q}} dx^{\nu_1} \dots dx^{\nu_r} \qquad (5.29)$$

$T : \otimes^q T_p^*M \otimes^r T_pM \to \mathbb{R}$

Let $V_i = V_i^\mu \dfrac{\partial}{\partial x^\mu} \qquad (1 \leq i \leq r)$

$\omega_i = \omega_{i\mu}dx^\mu \qquad (1 \leq i \leq q)$

$$T(\omega_1 \dots \omega_q, V_1 \dots V_r) = T^{\mu_1 \dots \mu_q}{}_{\nu_1 \dots \nu_r}\omega_{1\mu_1} \dots \omega_{q\mu_q} V_1^{\nu_1} \dots V_r^{\nu_r}$$

$$T(\omega_1 \ldots a_i\omega_i + b_i\psi_i \ldots \omega_q, V_1 \ldots V_r) = T^{\mu_1\cdots\mu_q}{}_{\nu_1\ldots\nu_r}\omega_{1\mu_1}\ldots(q_i\omega_{i\mu_i} + b_i\psi_{i\mu_i})\ldots\omega_{q\mu_q}, V_1^{\nu_1}\ldots V_r^{\nu_r}) =$$
$$= a_iT^{\mu_1\cdots\mu_q}{}_{\nu_1\ldots\nu_r}\omega_{1\mu_1}\ldots\omega_{i\mu_i}\ldots\omega_{q\mu_q}, V_1^{\nu_1}\ldots V_r^{\nu_r} + b_iT^{\mu_1\cdots\mu_q}{}_{\nu_1\ldots\nu_r}\omega_{1\mu_1}\ldots\psi_{i\mu_i}\ldots\omega_{q\mu_q}, V_1^{\nu_1}\ldots V_r^{\nu_r} =$$
$$= a_iT(\omega_1 \ldots \omega_i \ldots \omega_q, V_1 \ldots V_r) + b_iT(\omega_1 \ldots \psi_i \ldots \omega_q, V_1 \ldots V_r)$$
$$T(\omega_1 \ldots \omega_q, V_1 \ldots c_jV_j + d_jW_j \ldots V_r) = T^{\mu_1\cdots\mu_q}{}_{\nu_1\ldots\nu_r}\omega_{1\mu_1}\ldots\omega_{q\mu_q}V_1^{\nu_1}\ldots(c_jV_j^{\nu_j} + d_jW_j^{\nu_j}\ldots V_r =$$
$$= c_jT^{\mu_1\cdots\mu_q}{}_{\nu_1\ldots\nu_r}\omega_{1\mu_1}\ldots\omega_{q\mu_q}V_1^{\nu_1}\ldots V_j^{\nu_j}\ldots V_r + d_jT^{\mu_1\cdots\mu_q}{}_{\nu_1\ldots\nu_r}\omega_{1\mu_1}\ldots\omega_{q\mu_q}V_1^{\nu_1}\ldots W_j^{\nu_j}\ldots V_r =$$
$$= c_jT(\omega_1 \ldots \omega_q, V_1 \ldots V_j \ldots V_r) + d_jT(\omega_1 \ldots \omega_q, V_1 \ldots W_j \ldots V_r)$$

5.2.5. *Tensor fields.* vector field - vector assigned smooth $\forall\, p \in M$
i.e. V vector field if $V[f] \in \mathcal{F}(M) \qquad \forall\, f \in \mathcal{F}(M)$

tensor field of type (q, r), $\tau^q_{r,p}(M) \quad \forall\, p \in M$
$\tau^0_1(M)$ set of dual vector fields $\equiv \Omega^1(M)$
$\tau^0_0(M) = \mathcal{F}(M)$

5.2.6. *Induced maps.* smooth $f : M \to N$

differential $f_* : T_pM \to T_{f(p)}N$
By def. of tangent vector as directional derivative along a curve,
if $g \in \mathcal{F}(N)$, $fg \in \mathcal{F}(M)$
vector $V \in T_pM$ acts on gf to give number $V[gf]$

define $f_*V \in T_{f(p)}N$
$$(f_*V)[g] = V[gf] \qquad (5.31)$$

or for $(U, \varphi) \subset M$
$(V, \psi) \subset N$
$$(f_*V)[g\psi^{-1}(y)] \equiv V[gf\varphi^{-1}(x)] \qquad (5.32)$$

Let $V = V^\mu \frac{\partial}{\partial x^\mu}$, $\quad f_*V = W^\alpha \frac{\partial}{\partial y^\alpha}$
$$W^\alpha \frac{\partial}{\partial y^\alpha}[g\psi^{-1}(y)] = V^\mu \frac{\partial}{\partial x^\mu}[gf\varphi^{-1}(x)]$$

With $y = \psi(f(p))$,
take $g = y^\alpha$
$$W^\alpha = V^\mu \frac{\partial y^\alpha}{\partial x^\mu} \qquad (5.33)$$

$f(p) = f\varphi^{-1}(x)$
$$V^\mu \frac{\partial}{\partial x^\mu}[y^\alpha f\varphi^{-1}(x)] = V^\mu \frac{\partial}{\partial x^\mu}[y^\alpha]$$

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Consider smooth $f : M \to N$
$f_* : T_pM \to T_{f(p)}N$
$$(U, \varphi) \subset M \qquad \varphi = x^\mu$$
$$(V, \psi) \subset N \qquad \psi = y^\nu$$

$$\psi f\varphi^{-1} : \varphi(U) \subseteq \mathbb{R}^m \to \mathbb{R}^n \qquad X \in T_pM \qquad X = X^\mu \frac{\partial}{\partial x^\mu}$$
$$\psi f\varphi^{-1} \equiv f^\nu(x^\mu) \qquad Y \in T_{f(p)}N \qquad Y = Y^\nu \frac{\partial}{\partial y^\nu}$$

$$f_*X \in T_{f(p)}N$$
$$f_*X = Y^\nu \frac{\partial}{\partial y^\nu}$$

$$g \in \mathcal{F}(N) \qquad g\psi^{-1} : \psi(N) \to \mathbb{R}$$
$$g : N \to \mathbb{R} \qquad g\psi^{-1} \equiv g(y^\nu)$$
$$gf : M \to \mathbb{R}$$
$$gf \in \mathcal{F}(M)$$

$$(f_*X)[g] \equiv X[gf]$$
$$(f_*X)[g] = Y^\nu \frac{\partial g}{\partial y^\nu}(y) = X^\mu \frac{\partial}{\partial x^\mu}[gf] = X^\mu \frac{\partial}{\partial x^\mu} g(f^\nu(x)) = X^\mu \frac{\partial g}{\partial y^\nu}(y) \frac{\partial f^\nu}{\partial x^\mu}(x)$$
$$\Longrightarrow \boxed{Y^\nu = X^\mu \frac{\partial f^\nu}{\partial x^\mu} = X^\mu \frac{\partial y^\nu}{\partial x^\mu}}$$

where

$$gf = g\psi^{-1}\psi f \varphi^{-1}$$
$$g\psi^{-1}\psi f\varphi^{-1} : \varphi(U) \to \mathbb{R}$$
$$g\psi^{-1}\psi f\varphi^{-1} = g(f^\nu(x))$$

Exercise 5.3.

$$\begin{aligned} f &: M \to N \\ g &: N \to P \\ gf &: M \to P \end{aligned} \quad \text{Consider} \quad \begin{aligned} p \in M, (U,\varphi) &\subset M \\ q = f(p) \in N, (V,\psi) &\subset N \\ r = g(q) \in P, (W,\chi) &\subset P \end{aligned} \quad \begin{aligned} V &\in T_pM \\ W &\in T_qN \\ X &\in T_{g(q)}P \end{aligned} \quad \begin{aligned} V &= V^\alpha \frac{\partial}{\partial x^\alpha} \\ W &= W^\beta \frac{\partial}{\partial y^\beta} \\ X &= X^\gamma \frac{\partial}{\partial z^\gamma} \end{aligned}$$

$$\begin{aligned} &f_*V \in T_{f(p)}N && g_*W \in T_{g(q)}P && gf_*V \in T_{gf(p)}P \\ &f_*V[h] = V[hf] && g_*W[k] = W[kg] && gf_*V[l] = V[lgf] \\ &f_*V[h\psi^{-1}(y)] = V[hf\varphi^{-1}(x)] && g_*W[k\chi^{-1}(z)] = W[kg\psi^{-1}(y)] && gf_*V[l\chi^{-1}(z)] = V[lgf\varphi^{-1}(x)] \end{aligned}$$

$$\Longrightarrow g_*(f_*V)[l] = f_*V[lg] = V[lgf] = gf_*V[l]$$

In coordinates,

$$g_*(f_*V)[l\chi^{-1}(z)] = f_*V[lg\psi^{-1}(y)] = W^\alpha \frac{\partial}{\partial y^\alpha}[lg\psi^{-1}(y)] = V^\mu \frac{\partial y^\alpha}{\partial x^\mu}\frac{\partial}{\partial y^\alpha}[lg\psi^{-1}(y)] = V^\mu \frac{\partial(y^\alpha f\varphi^{-1}(x))}{\partial x^\mu}\frac{\partial}{\partial y^\alpha}[lg\psi^{-1}(y)]$$
$$gf_*V[l\chi^{-1}(z)] = V[lgf\varphi^{-1}(x)] = V^\alpha \frac{\partial}{\partial x^\alpha}(lgf\varphi^{-1}(x))$$
$$\Longrightarrow \frac{\partial}{\partial x^\mu}(lgf\varphi^{-1}(x)) = \frac{\partial(y^\alpha f\varphi^{-1}(x))}{\partial x^\mu}\frac{\partial}{\partial y^\alpha}[lg\psi^{-1}(y)]$$

Chain rule is reobtained.

$$\begin{aligned} &f : M \to N \\ &f^* : T^*_{f(p)}N \to T^*_pM \qquad \textbf{pull back} \qquad \langle f^*\omega, V\rangle = \langle \omega, f_*V\rangle \\ &\omega \in T^*_{f(p)}N \\ &V \in T_pM \end{aligned}$$

tensor of type $(0,r)$ $f^* : \tau^0_{r,f(p)}(N) \to \tau^0_{r,p}(M)$

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pullback

$$\begin{aligned} &f : M \to N && X \in T_pM \\ &f^* : T^*_{f(p)}N \to T^*_pM && X = X^\mu \frac{\partial}{\partial x^\mu} \\ &(U,\varphi) \subset M \quad \varphi = x^\mu && \omega \in T^*_{f(p)}N \\ &(V,\psi) \subset N \quad \psi = y^\nu && \omega = \omega_\alpha dy^\alpha \end{aligned}$$

$$f^*\omega \in T_p^*M \text{ so}$$
$$f^*\omega = \psi_\beta dx^\beta$$
$$\langle f^*\omega, X\rangle = \langle \psi_\beta dx^\beta, X^\mu \frac{\partial}{\partial x^\mu}\rangle = \psi_\mu X^\mu = \langle \omega, f_* X\rangle = \langle \omega_\alpha dy^\alpha, Y^\nu \frac{\partial}{\partial y^\nu}\rangle =$$
$$= \langle \omega_\alpha dy^\alpha, X^\mu \frac{\partial y^\nu}{\partial x^\mu}\frac{\partial}{\partial y^\nu}\rangle = \omega_\nu X^\mu \frac{\partial y^\nu}{\partial x^\mu}$$
$$\boxed{\psi_\mu = \omega_\nu \frac{\partial y^\nu}{\partial x^\mu} = \frac{\partial y^\nu}{\partial x^\mu}\omega_\nu}$$

Exercise 5.4.

$$\begin{aligned} \omega &= \omega_\alpha dy^\alpha \in T^*_{f(p)}N \qquad & V &\in T_pM \\ f^*\omega &= \xi_\mu dx^\mu \in T_p^*M \qquad & V &= V^\mu \frac{\partial}{\partial x^\mu} \end{aligned}$$
$$\langle f^*\omega, V\rangle = \langle \xi_\mu dx^\mu, V^\nu \frac{\partial}{\partial x^\nu}\rangle = \xi_\mu V^\mu =$$
$$= \langle \omega, f_*V\rangle = \langle \omega_\alpha dy^\alpha, W^\beta \frac{\partial}{\partial y^\beta}\rangle = \langle \omega_\alpha dy^\alpha, V^\mu \frac{\partial y^\beta}{dx^\mu}\frac{\partial}{\partial y^\beta}\rangle = \omega_\alpha V^\mu \frac{\partial y^\beta}{\partial x^\mu}\delta^\alpha_\beta = \omega_\alpha V^\mu \frac{\partial y^\alpha}{\partial x^\mu} =$$
$$\Longrightarrow \boxed{\xi_\mu = \omega_\alpha \frac{\partial y^\alpha}{\partial x^\mu}}$$

Exercise 5.5.

$$(gf)^* : T^*_{gf(p)}P \to T_p^*M$$
$$\langle (gf)^*\chi, V\rangle = \langle \chi, (gf)_*V\rangle = \langle \chi, g_*f_*V\rangle = \langle g^*\chi, f_*V\rangle = \langle f^*g^*\chi, V\rangle$$
$$\Longrightarrow (gf)^* = f^*g^*$$

with

$$\begin{aligned} g^* &: T^*_{g(q)}P \to T_q^*N \qquad & f^* &: T^*_{f(p)}N \to T_p^*M \\ \langle g^*\omega, W\rangle &= \langle \omega, g_*W\rangle \qquad & \langle f^*\nu, V\rangle &= \langle \nu, f_*V\rangle \end{aligned}$$

5.2.7. *Submanifolds.*

Definition 5 (5.3)**.** *(Immersion, submanifold, embedding)*
Let smooth $f : M \to N$
$dimM \leq dimN$

(a) *immersion f if $f_* : T_pM \to T_{f(p)}N$ injection (1-to-1) i.e.*
$$rankf_* = dimM$$
(b) *embedding f if f injection and immersion*
$f(M)$ submanifold of N.

5.3. **Flow and Lie derivatives.** Let vector field X in M.
integral curve $x(t)$ of X is a curve in M, tangent vector at $x(t)$ is $X|_x$

$$\frac{dx^\mu}{dt} = X^\mu(x(t))$$

where x^μ is μth component of $\varphi(x(t))$, $X = X^\mu \frac{\partial}{\partial x^\mu}$

Note the abus of notation: x used to denote a pt. in M as well as its coordinates.

Let $\sigma(t, x_0)$ integral curve of X which passes a pt. x_0 at $t = 0$.

(26) $$\frac{d}{dt}\sigma^\mu(t, x_0) = X^\mu(\sigma(t, x_0)) \qquad (5.40a)$$

(27) $$\sigma^\mu(0, x_0) = x_0^\mu \qquad (5.40b) \text{ initial condition}$$

$\sigma : \mathbb{R} \times M \to M$. Flow generated by $X \in \mathcal{X}(M)$ s.t.

$$\sigma(t, \sigma^\mu(s, x_0)) = \sigma(t + s, x_0)$$

The previous was true because of the following. From uniqueness of ODEs.

$$\frac{d}{dt}\sigma^{\mu}(t,\sigma^{\mu}(s,x_0)) = X^{\mu}(\sigma(t,\sigma^{\mu}(s,x_0)))$$
$$\sigma^{\mu}(0,\sigma(s,x_0)) = \sigma(s,x_0)$$
$$\frac{d}{dt}\sigma^{\mu}(t+s,x_0) = \frac{d}{d(t+s)}\sigma^{\mu}(t+s,x_0) = X^{\mu}(\sigma(t+s,x_0))$$
$$\sigma(0+s,x_0) = \sigma(s,x_0)$$

Theorem 7 (5.1)**.** *$\forall\, x \in M$, $\exists$ differentiable $\sigma : \mathbb{R}\times M \to M$ s.t.*

(i) $\sigma(0,x) = x$
(ii) $t \mapsto \sigma(t,x)$ *solution of (5.40a), (5.40b)*
(iii) $\sigma(t,\sigma^{\mu}(s,x)) = \sigma(t+s,x)$

Example 5.9. Let $M = \mathbb{R}^2$, $X((x,y)) = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$

$$X((x,y))[f] = -y\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y} = \frac{df(\sigma(t))}{dt} = \frac{dx^{\mu}(c(t))}{dt}\frac{\partial f}{\partial x^{\mu}}$$

$$\begin{aligned} \frac{dx}{dt} &= -y \\ \frac{dy}{dt} &= x \end{aligned} \qquad \frac{d^2y}{dt^2} = -y \Longrightarrow \begin{aligned} y &= Ac(t) = Bs(t) \\ x &= -As(t) + Bc(t) \end{aligned} \qquad \Longrightarrow \begin{aligned} A &= y \\ B &= x \end{aligned}$$

$$x = x\cos(t) - y\sin(t)$$
$$y = x\sin(t) + y\cos(t)$$

Exercise 5.7.

$$X = y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$$
$$X[f] = y\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y} = \frac{dx^{\mu}}{dt}\frac{\partial f}{\partial x^{\mu}}$$
$$\begin{aligned} \frac{dx}{dt} &= y \\ \frac{dy}{dt} &= x \end{aligned} \qquad \begin{aligned} \ddot{x} &= x \\ \ddot{y} &= y \end{aligned} \qquad \begin{aligned} x &= Ae^{t} + Be^{-t} \\ y &= Ae^{t} - Be^{-t} \end{aligned}$$

5.3.1. *One-parameter group of transformations.* For fixed $t \in \mathbb{R}$, flow $\sigma(t,x)$ is a diffeomorphism from M to M, $\sigma_t : M \to M$

σ_t made into a commutative group by

(i) $\sigma_t(\sigma_s(X)) = \sigma_{t+s}(X)$ i.e. $\sigma_t \circ \sigma_s = \sigma_{t+s}$
(ii) $\sigma_0 = 1$
(iii) $\sigma_{-t} = (\sigma_t)^{-1}$

Under action σ_{ϵ}, infinitesimal ϵ, from (5.40a), (5.40b)

$$\sigma^{\mu}_{\epsilon}(x) = \sigma^{\mu}(\epsilon,x) = x^{\mu} + \epsilon X^{\mu}(x)$$

So vector field X is the infinitesimal generator of transformation σ_{ϵ}

Given a vector field X, corresponding flow σ of referred to as exponentiation of X

$$\sigma^{\mu}(t,x) = \exp(tX)x^{\mu} \qquad (5.43)$$

Since

$$\sigma^{\mu}(t,x) = x^{\mu} + t\,\frac{d}{ds}\sigma^{\mu}(s,x)\Big|_{s=0} + \frac{t^2}{2!}\left(\frac{d}{ds}\right)^2\sigma^{\mu}(s,x)\Big|_{s=0} + \dots = \left[1 + t\frac{d}{ds} + \frac{t^2}{2!}\left(\frac{d}{ds}\right)^2 + \dots\right]\sigma^{\mu}(s,x)|_{s=0} =$$
$$= \exp\left(t\frac{d}{ds}\right)\sigma^{\mu}(s,x)|_{s=0} \qquad (5.44)$$

Properties

(i) $\sigma(0,x) = x = \exp(0X)x$ (5.45a)

(ii) $\frac{d\sigma(t,x)}{dt} = X\exp(tX)x = \frac{d}{dt}[\exp(tX)x]$ (5.45b)
(iii) $\sigma(t,\sigma(s,x)) = \sigma(t,\exp(sX)x) = \exp(tX)\exp(sX)x = \exp[(t+s)X]x = \sigma(t+s,x)$

5.3.2. *Lie derivatives.* Let $\sigma(t,x)$, $\tau(t,x)$ be 2 flows generated by vector fields X, Y

$$\frac{d\sigma^\mu(s,x)}{ds} = X^\mu(\sigma(s,x)) \qquad (5.46a)$$
$$\frac{d\tau^\mu(t,x)}{dt} = Y^\mu(\tau(t,x)) \qquad (5.46b)$$

evaluate change of Y along $\sigma(s,x)$
Compare Y at x and at $x' = \sigma_\epsilon(x)$ nearby
But components of Y at 2 pts. belong to different tangent spaces $T_pM, T_{\sigma_\epsilon(x)}M$
map $Y|_{\sigma_\epsilon(x)}$ to T_xM by

$$(\sigma_{-\epsilon})_* : T_{\sigma_\epsilon(x)}M \to T_xM$$

Lie derivative of vector field Y along flow σ of X

(28) $$\mathcal{L}_xY = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[(\sigma_{-\epsilon})_*\, Y|_{\sigma_\epsilon(x)} - Y|_x\right] \qquad (5.47)$$

$$(\sigma_{-\epsilon})_* : T_{\sigma_\epsilon(x)}M \to T_xM$$
$$((\sigma_{-\epsilon})_*Y)[g] = Y[g(\sigma_{-\epsilon})]$$

Exercise 5.8 .

$$(\sigma_\epsilon)_* : T_{\sigma_{-\epsilon}(x)}M \to T_xM$$
$$\mathcal{L}_XY = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[Y|_x - (\sigma_\epsilon)_*Y|_{\sigma_{-\epsilon}(x)}\right]$$
$$(\sigma_\epsilon)_* : T_xM \to T_{\sigma_\epsilon(x)}M$$
$$\mathcal{L}_XY = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[Y|_{\sigma_\epsilon(x)} - (\sigma_\epsilon)_*Y|_x\right]$$

Let (U,φ) be a chart with coordinates x

Let $X = X^\mu\dfrac{\partial}{\partial x^\mu}$
$Y = Y^\mu\dfrac{\partial}{\partial x^\mu}$

$$\sigma_\epsilon(x) = x^\mu + \epsilon X^\mu(x)$$
$$Y|_{\sigma_\epsilon(x)} = Y^\mu(x^\nu + \epsilon X^\nu(x))|_{x+\epsilon X} \simeq [Y^\mu(x) + \epsilon X^\nu(x)\partial_\nu Y^\mu(x)]\, e_\mu|_{x+\epsilon X}$$

$\{e_\mu = \frac{\partial}{\partial x^\mu}\}$ is the coordinate basis.
Consider Fig. 5.12. Note the need to "pullback" vector Y to x.
$(\sigma_{-\epsilon})_*$ at $\sigma_\epsilon(x)$ to x.
Recall that

$$\sigma_{-\epsilon} : M \to M$$
$$(\sigma_{-\epsilon})_* : T_{\sigma_\epsilon(x)}M \to T_xM$$
$$((\sigma_{-\epsilon})_*Y)[g] = Y[g(\sigma_{-\epsilon})]$$

$$Y|_{\sigma_\epsilon(x)} = Y^\mu(x^\nu + \epsilon X^\nu(x))\frac{\partial}{\partial x^\mu}\bigg|_{x+\epsilon X} \simeq [Y^\mu(x) + \epsilon X^\nu\partial_\nu Y^\mu(x)]\,\frac{\partial}{\partial x^\mu}\bigg|_{x+\epsilon X}$$

$$\text{Now }\ (\sigma_{-\epsilon})_*Y|_{\sigma_\epsilon(x)} = W^\alpha\frac{\partial}{\partial x^\alpha} = Y^\mu\frac{\partial}{\partial x^\mu}\sigma^\alpha_{-\epsilon}\frac{\partial}{\partial x^\alpha}$$
$$\implies W^\alpha = Y^\mu|_{\sigma_\epsilon(x)}\,\frac{\partial}{\partial x^\mu}\sigma^\alpha_{-\epsilon} =$$
$$= (Y^\mu(x) + \epsilon X^\nu\partial_\nu Y^\mu(x))\frac{\partial}{\partial x^\mu}(x^\alpha - \epsilon X^\alpha) = [Y^\mu(x) + \epsilon X^\nu\partial_\nu Y^\mu(x)](\delta_\mu^{\ \alpha} - \epsilon\partial_\mu X^\alpha) =$$
$$= Y^\alpha(x) + \epsilon X^\nu\partial_\nu Y^\alpha(x) - \epsilon Y^\mu\partial_\mu X^\alpha + \mathcal{O}(\epsilon^2)$$

So the first order term is

$$\epsilon(X^\nu \partial_\nu Y^\alpha(x) - Y^\nu \partial_\nu X^\alpha)$$

Exercise 5.9.

Given $X = X^\mu \dfrac{\partial}{\partial x^\mu}$

$Y = Y^\mu \dfrac{\partial}{\partial x^\mu}$

Lie bracket $[X,Y]$. $[X,Y]f = X[Y[f]] - Y[X[f]]$.

$$[X,Y]f = \left[X^\nu \frac{\partial Y^\mu}{\partial x^\nu} - Y^\nu \frac{\partial X^\mu}{\partial x^\nu} \right] \frac{\partial f}{\partial x^\mu}$$
$$\Longrightarrow \boxed{[X,Y] = \left[X^\mu \frac{\partial Y^\nu}{\partial x^\mu} - Y^\mu \frac{\partial X^\nu}{\partial x^\mu} \right] \frac{\partial}{\partial x^\nu}}$$

This is the **local form of the Lie bracket**

$\Longrightarrow \mathcal{L}_X Y = [X,Y]$ (5.49b). $[X,Y]$ indeed 1st.-order derivative and indeed a vector field.

Exercise 5.10.

(a) bilinearity

Want: $[X, c_1 Y_1 + c_2 Y_2] = c_1[X,Y_1] + c_2[X,Y_2]$
$[c_1 X_1 + c_2 X_2, Y] = c_1[X_1,Y] + c_2[X_2,Y]$

$$\begin{aligned} [X, c_1Y_1 + c_2Y_2] &= X^\mu \frac{\partial}{\partial x^\mu}(c_1Y_1 + c_2Y_2)^\nu - (c_1Y_1+c_2Y_2)^\mu \frac{\partial X^\nu}{\partial x^\mu} = \\ &= c_1 \left(X^\mu \frac{\partial Y_1^\nu}{\partial x^\mu} - Y_1^\mu \frac{\partial X^\nu}{\partial x^\mu} \right) + c_2 \left(X^\mu \frac{\partial Y_2^\nu}{\partial x^\mu} - Y_2^\mu \frac{\partial X^\nu}{\partial x^\mu} \right) = c_1[X,Y_1] + c_2[X,Y_2] \\ [c_1X_1 + c_2X_2, Y] &= (c_1X_1 + c_2X_2)^\mu \frac{\partial Y^\nu}{\partial x^\mu} - Y^\mu \frac{\partial}{\partial x^\mu}(c_1X_1+c_2X_2)^\nu = \\ &= c_1 \left(X_1^\mu \frac{\partial Y^\nu}{\partial x^\mu} - Y^\mu \frac{\partial X_1^\nu}{\partial x^\mu} \right) + c_2 \left(X_2^\mu \frac{\partial Y^\nu}{\partial x^\mu} - Y^\mu \frac{\partial X_2^\nu}{\partial x^\mu} \right) = c_1[X_1,Y] + c_2[X_2,Y] \end{aligned}$$

(b)

$$[Y,X] = Y^\mu \frac{\partial X^\nu}{\partial x^\mu} - X^\mu \frac{\partial Y^\nu}{\partial x^\mu} = -\left(X^\mu \frac{\partial Y^\nu}{\partial x^\mu} - Y^\mu \frac{\partial X^\nu}{\partial x^\mu} \right) = -[X,Y]$$

(c) Want:

$$[[X,Y],Z] + [[Z,X],Y] + [[Y,Z],X] = 0$$

Now

$$[[X,Y],Z]$$

$$\begin{aligned} [X,Y]^\mu \frac{\partial Z^\nu}{\partial x^\mu} - Z^\mu \frac{\partial}{\partial x^\mu}[X,Y]^\nu &= (X^a\partial_a Y^\mu - Y^a \partial_a X^\mu)\partial_\mu Z^\nu - Z^\mu(\partial_\mu X^a \partial_a Y^\nu + X^a \partial^2_{\mu a} Y^\nu - \partial_\mu Y^a \partial_a X^\nu - Y^a \partial^2_{\mu a} X^\nu) = \\ &= X^a \partial_a Y^\mu \partial_\mu Z^\nu - Y^a \partial_a X^\mu \partial_\mu Z^\nu - Z^\mu \partial_\mu X^a \partial_a Y^\nu - Z^\mu X^a \partial^2_{\mu a} Y^\nu + Z^\mu \partial_\mu Y^a \partial_a X^\nu + Z^\mu Y^a \partial^2_{\mu a} X^\nu \end{aligned}$$

Likewise,

$$[[Z,X],Y]^\nu = Z^a \partial_a X^\mu \partial_\mu Y^\nu - X^a \partial_a Z^\mu \partial_\mu Y^\nu - Y^\mu \partial_\mu Z^a \partial_a X^\nu - Y^\mu Z^a \partial^2_{\mu a} X^\nu + Y^\mu \partial_\mu X^a \partial_a Z^\nu + Y^\mu X^a \partial^2_{\mu a} Z^\nu$$
$$[[Y,Z],X]^\nu = Y^a \partial_a Z^\mu \partial_\mu X^\nu - Z^a \partial_a Y^\mu \partial_\mu X^\nu - X^\mu \partial_\mu Y^a \partial_a Z^\nu - X^\mu Y^a \partial^2_{\mu a} Z^\nu + X^\mu \partial_\mu Z^a \partial_a Y^\nu + X^\mu Z^a \partial^2_{\mu a} Y^\nu$$

All the 18 terms cancel.

5.4. **Differential forms.** symmetry operation on tensor

$\omega \in \tau^0_{r,p}(M)$

(29) $$P\omega(v_1 \dots v_r) \equiv \omega(v_{P(1)} \dots v_{P(r)}) \qquad (5.59)$$

$v_i \in T_pM$, $P \in S_r$, symmetry group of order r

$$\omega(e_{\mu_1}\dots e_{\mu_r}) = \omega_{\mu_1\dots\mu_r}$$
$$P\omega(e_{\mu_1}\dots e_{\mu_r}) = \omega_{\mu_{P(1)}\dots\mu_{P(r)}}$$

symmetrizer $\mathcal{S}$, $\omega \in \tau^0_{r,p}(M)$

$$S_\omega = \frac{1}{r!}\sum_{P\in S_r} P\omega \qquad (5.60)$$

anti-symmetrizer $\mathcal{A}$

$$\mathcal{A}\omega = \frac{1}{r!}\sum_{P\in S_r}\text{sgn}(P)P\omega$$

5.4.1. *Definitions.*

Definition 6 (5.4)**.** *diff. form or r-form, totally antisymmetric tensor of type* $(0,r)$ *define wedge product* $\wedge$ *of r 1-forms by the totally antisymm. tensor product*

(30)
$$dx^{\mu_1}\wedge dx^{\mu_2}\wedge\dots\wedge dx^{\mu_r} = \sum_{P\in S_r} sgn(P) dx^{\mu} \qquad (5.62)$$

e.g.

$$dx^\mu\wedge dx^\nu = dx^\mu\otimes dx^\nu - dx^\nu\otimes dx^\mu$$
$$dx^\lambda\wedge dx^\mu\wedge dx^\nu = dx^\lambda dx^\mu dx^\nu + dx^\nu dx^\lambda dx^\mu + dx^\mu dx^\nu dx^\lambda - dx^\lambda dx^\nu dx^\mu - dx^\nu dx^\mu dx^\lambda - dx^\mu dx^\lambda dx^\nu$$

(i)
(ii)

$$dx^{\mu_1}\wedge\dots\wedge dx^{\mu_r} = \text{sgn}(P)dx^{\mu_{P(1)}}\wedge\dots\wedge dx^{\mu_{P(r)}}$$
$$\sum_{Q\in S_r}\text{sgn}(Q)dx^{\mu_{Q(1)}}dx^{\mu_{Q(2)}}\dots dx^{\mu_{Q(r)}} = \sum_{Q\in S_r}\text{sgn}(Q)(\text{sgn}(P))^2 dx^{\mu_{Q(P(1))}}\dots dx^{\mu_{Q(P(r))}} =$$
$$= (\text{sgn}P)\sum_{Q\in S_r}\text{sgn}(Q)\text{sgn}P dx^{\mu_{Q(P(1))}}\dots dx^{\mu_{Q(P(r))}}$$

vector space of r-forms at $p\in M$ by $\Omega^r_p(M)$
set of r-forms (5.62) forms basis of $\Omega^r_p(M)$

$$\omega\in\Omega^r_p(M)$$

(31)
$$\omega = \frac{1}{r!}\omega_{\mu_1\dots\mu_r}dx^{\mu_1}\wedge\dots\wedge dx^{\mu_r} \qquad (5.63)$$

$\omega_{\mu_1\dots\mu_r}$ totally antisymmetric, reflecting antisymmetry of basis
$\binom{m}{r}$ choices of $(\mu_1\dots\mu_r)$ out of $(1\dots n)$ in (5.62)

$$\text{dim}\Omega^r_p(M) = \binom{m}{r}$$

since $\binom{m}{r} = \binom{m}{m-r}$, $\Omega^r_p(M)\simeq\Omega^{m-r}_p(M)$

Let $\omega\in\Omega^q_p(M)$
$\xi\in\Omega^r_p(M)$
action of $(q+r)$-form $\omega\wedge\xi$ on $q+r$ vectors

(32)
$$(\omega\wedge\xi)(v_1\dots v_{q+r}) = \frac{1}{q!r!}\sum_{p\in S_{q+r}}\text{sgn}(P)\omega(v_{p(1)}\dots v_{p(q)})\xi(v_{p(q+1)}\dots v_{p(q+r)}) \qquad (5.65)$$

with this product, define

(33)
$$\Omega^*_p(M)\equiv\bigoplus_{k=0}^{m}\Omega^k_p(M) \qquad (5.66)$$

Exercise 5.13.

$$\begin{aligned} x &= r\cos\theta & r &= \sqrt{x^2+y^2} \\ y &= r\sin\theta & \theta &= \arctan\left(\frac{y}{x}\right) \end{aligned}$$

$$\partial_x r = \frac{x}{r} \qquad \partial_y r = \frac{y}{r}$$

$$\begin{aligned} \partial_r x &= c_\theta & \partial_\theta x &= -rs_\theta & \partial_x\theta &= \frac{\frac{-y}{x^2}}{1+\frac{y^2}{x^2}} = \frac{-y}{x^2+y^2} \\ \partial_r y &= s_\theta & \partial_\theta y &= rc_\theta & \partial_y\theta &= \frac{x}{x^2+y^2} \end{aligned}$$

$$dx\wedge dy = (c_\theta dr + -rs_\theta d\theta)\wedge(s_\theta dr + rc_\theta d\theta) = rc_\theta^2 dr\wedge d\theta + rs_\theta^2 dr\wedge d\theta = r dr\wedge d\theta$$

Exercise 5.14.

$\xi\wedge\xi = (v_1\wedge\dots\wedge v_q)\wedge(v_1\wedge\dots\wedge v_q) = (-1)^q v_1\wedge(v_1\wedge\dots\wedge v_q)\wedge(v_2\wedge\dots\wedge v_q) = (-1)^{q^2}\xi\wedge\xi = -\xi\wedge\xi$ if q odd

5.4.2. *Exterior derivatives.*

Definition 7 (5.5)**.** *exterior derivatives*

$$\begin{aligned} &d_r : \Omega^r(M) \to \Omega^{r+1}(M) \\ &\omega = \frac{1}{r!}\omega_{\mu_1\dots\mu_r} dx^{\mu_1}\wedge\dots\wedge dx^{\mu_r} \\ &d_r\omega = \frac{1}{r!}\left(\frac{\partial}{\partial x^\nu}\omega_{\mu_1\dots\mu_r}\right) dx^\nu\wedge dx^{\mu_1}\wedge\dots\wedge dx^{\mu_r} \end{aligned} \tag{5.68}$$

Example 5.10. in 3-dim.

$$\begin{aligned} \omega_0 &= f(x,y,z) \\ \omega_1 &= \omega_x(x,y,z)dx + \omega_y(x,y,z)dy + \omega_z(x,y,z)dz \\ \omega_2 &= \omega_{xy}(x,y,z)dx\wedge dy + \omega_{yz}(x,y,z)dy\wedge dz + \omega_{zx}(x,y,z)dz\wedge dx \\ \omega_3 &= \omega_{xyz}(x,y,z)dx\wedge dy\wedge dz \end{aligned}$$

Recall

$$\omega = \frac{1}{r!}\omega_{\mu_1\dots\mu_r} dx^{\mu_1}\wedge\dots\wedge dx^{\mu_r}$$

e.g.

$$\omega_{12}dx^1\wedge dx^2 + \omega_{13}dx^1\wedge dx^3 + \omega_{21}dx^2\wedge dx^1 + \dots = (\omega_{12}-\omega_{21})dx^1\wedge dx^2 + \dots$$

$\omega_{\mu_1\dots\mu_r}$ itself must be antisymmetrized.

$$d\omega = \frac{1}{r!}\left(\frac{\partial\omega_{\mu_1\dots\mu_r}}{\partial x^\nu}\right) dx^\nu\wedge dx^{\mu_1}\wedge\dots\wedge dx^{\mu_r}$$

(i)

$$d\omega_0 = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

(ii)

$$\begin{aligned} d\omega_1 &= \frac{1}{1!}\left(\frac{\partial\omega_x}{\partial y}dy\wedge dx + \frac{\partial\omega_x}{\partial z}dz\wedge dx + \frac{\partial\omega_y}{\partial x}dx\wedge dy + \frac{\partial\omega_y}{\partial z}dz\wedge dy + \frac{\partial\omega_z}{\partial x}dx\wedge dz + \frac{\partial\omega_z}{\partial y}dy\wedge dz\right) = \\ &= \left(\frac{\partial\omega_y}{\partial x} - \frac{\partial\omega_x}{\partial y}\right)dx\wedge dy + \left(\frac{\partial\omega_z}{\partial y} - \frac{\partial\omega_y}{\partial z}\right)dy\wedge dz + \left(\frac{\partial\omega_x}{\partial z} - \frac{\partial\omega_z}{\partial x}\right)dz\wedge dx \end{aligned}$$

(iii)

$$\begin{aligned} d\omega_2 &= \frac{1}{2!}\left(\frac{\partial\omega_{xy}}{\partial z}dz\wedge dx\wedge dy + \frac{\partial\omega_{yx}}{\partial z}dz\wedge dy\wedge dx + \frac{\partial\omega_{yz}}{\partial x}dx\wedge dy\wedge dz + \frac{\partial\omega_{zy}}{\partial x}dx\wedge dz\wedge dy + \dots\right) = \\ &= \left(\frac{\partial\omega_{yz}}{\partial x} + \frac{\partial\omega_{zx}}{\partial y} + \frac{\partial\omega_{xy}}{\partial z}\right)dx\wedge dy\wedge dz \end{aligned}$$

Exercise 5.15. 20130929

$\xi \in \Omega^q(M)$ $\qquad \xi \wedge \omega = \xi_{i_1 \dots i_q} \omega_{j_1 \dots j_r} dx^{i_1} \wedge \dots \wedge dx^{i_q} \wedge dx^{j_1} \wedge \dots \wedge dx^{j_r}$

$\omega \in \Omega^r(M)$

$$d(\xi \wedge \omega) = \frac{1}{(q+r)!} \frac{\partial(\xi_{i_1 \dots i_q} \omega_{j_1 \dots j_r})}{\partial x^\nu} dx^\nu \wedge dx^{i_1} \wedge \dots \wedge dx^{i_q} \wedge dx^{j_1} \wedge \dots \wedge dx^{j_r} =$$
$$= \frac{1}{(q+r)!} (\partial_\nu \xi_{i_1 \dots i_q} \omega_{j_1 \dots j_r} + \xi_{i_1 \dots i_q} \partial_\nu \omega_{j_1 \dots j_r}) dx^\nu \wedge dx^{i_1} \wedge \dots =$$
$$= \frac{1}{(q+r)!} \{ \partial_\nu \xi_{i_1 \dots i_q} \omega_{j_1 \dots j_r} dx^\nu \wedge dx^{i_1} \wedge \dots \wedge dx^{i_q} \wedge dx^{j_1} \wedge \dots \wedge dx^{j_r} +$$
$$+ \xi_{i_1 \dots i_q} \partial_\nu \omega_{j_1 \dots j_r} dx^{i_1} \wedge \dots \wedge dx^{i_q} \wedge dx^\nu \wedge dx^{j_1} \wedge \dots \wedge dx^{j_r} (-1)^q \}$$
$$d\xi \wedge \omega + (-1)^q \xi \wedge d\omega =$$
$$= \frac{1}{q!} \frac{\partial \xi_{i_1 \dots i_q}}{\partial x^\nu} dx^\nu \wedge dx^{i_1} \wedge \dots \wedge dx^{i_q} \wedge \omega_{j_1 \dots j_r} dx^{j_1} \wedge \dots \wedge dx^{j_r}$$

decomposable forms

$$\xi = f dx_{i_1} \wedge \dots \wedge dx_{i_q} = f dx_I$$
$$\omega = g dx_{j_1} \wedge \dots \wedge dx_{j_q} = g dx_I$$

$$d(\xi \wedge \omega) = d(f dx_I \wedge g dx_J) = d(fg) \wedge dx_2 \wedge dx_J = (f dg = g df) \wedge dx_2 \wedge dx_J =$$
$$= df \wedge dx_I \wedge g dx_J + (-1)^q f dx_I \wedge dg \wedge dx_J = d\xi \wedge \omega + \xi \wedge d\omega$$

$X = X^\mu \partial_{x^\mu} \in \mathcal{X}(M)$ $\qquad \omega = \omega_\mu dx^\mu \in \Omega^1(M)$

$Y = Y^\nu \partial_{x^\nu}$

$$X[\omega(Y)] - Y[\omega(X)] - \omega([X,Y]) = \frac{\partial \omega_\mu}{\partial x^\nu} (X^\nu Y^\mu - X^\mu Y^\nu)$$
$$[X,Y] = [X^\mu \partial_{x^\mu} Y^\nu - Y^\mu \partial_{x^\mu} X^\nu] \partial_{x^\nu}$$
$$\omega([X,Y]) = \omega_\nu (X^\mu \partial_\mu Y^\nu - Y^\mu \partial_\mu X^\nu)$$
$$X[\omega(Y)] = X^\mu \partial_\mu (\omega_\nu Y^\nu) = X^\mu \partial_\mu \omega_\nu Y^\nu + X^\mu \omega_\nu \partial_\mu Y^\nu$$
$$Y[\omega(X)] = Y^\mu \partial_\mu (\omega_\nu X^\nu) = Y^\mu \partial_\mu \omega_\nu X^\nu + Y^\mu \omega_\nu \partial_\mu X^\nu$$
$$\Longrightarrow X[\omega(Y)] - Y[\omega(X)] - \omega([X,Y]) = X^\mu \partial_\mu \omega_\nu Y^\nu - Y^\mu \partial_\mu \omega_\nu X^\nu = \partial_\mu \omega_\nu (X^\mu Y^\nu - Y^\mu X^\nu)$$

Suppose

$$d\omega(X_1 \dots X_{p+1}) = \sum_{i=1}^r (-1)^{i+1} X_i \omega(X_1 \dots \widehat{X}_i \dots X_{i+1}) + \sum_{i<j} (-1)^{i+1} \omega([X_i, X_j], X_1 \dots \widehat{X}_i \dots \widehat{X}_j \dots X_{i+1}) \qquad (5.71)$$

for r-form $\omega \in \Omega^r(M)$

$$\omega = \frac{1}{r!} \omega_{\mu_1 \dots \mu_r} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_r}$$
$$d\omega = \frac{1}{r!} \left(\frac{\partial}{\partial x^\nu} \omega_{\mu_1 \dots \mu_r} \right) dx^\nu \wedge dx^{\mu_1} \wedge \dots \wedge dx^{\mu_r}$$

5.4.3. *Interior product and Lie derivative of forms.* interior product $i_X : \Omega^r(M) \to \Omega^{r-1}(M)$ where $X \in \chi(M)$

Let $\omega \in \Omega^r(M)$, define

(34) $$i_X \omega(X_1 \dots X_{r-1}) \equiv \omega(X, X_1 \dots X_{r-1}) \qquad (5.78)$$

with

$$X = X^\mu \frac{\partial}{\partial x^\mu}$$
$$\omega = \frac{1}{r!} \omega_{\mu_1 \dots \mu_r} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_r}$$

(35)
$$i_X\omega = \frac{1}{(r-1)!}X^\nu\omega_{\nu\mu_2\dots\mu_r}dx^{\mu_2}\wedge\dots\wedge dx^{\mu_r} = \frac{1}{r!}\sum_{s=1}^r X^{\mu_s}\omega_{\mu_1\dots\mu_s\dots\mu_r}(-1)^{s-1}dx^{\mu_1}\wedge\dots\wedge\widehat{dx^{\mu_s}}\wedge\dots\wedge dx^{\mu_r} \qquad (5.79)$$

cf. wikipedia
$i_X : \Omega^p(M)\to\Omega^{p-1}(M)$ $\qquad$ i_X a map that sends a p-form ω to the $(p-1)$ form $i_X\omega$

$$(i_X\omega)(X_1\dots X_{p-1}) = \omega(X, X_1\dots X_{p-1})$$

α 1-form $i_X\alpha = \alpha(X)$

for $\beta\in\Omega^p(M)$
$\gamma\in\Omega^q(M)$

$$i_X(\beta\wedge\gamma) = (i_X\beta)\wedge\gamma + (-1)^p\beta\wedge(i_X\gamma)$$

$$i_X\omega = \frac{1}{(p-1)!}X^i\omega_{ii_2\dots i_p}dx^{i_2}\wedge\dots\wedge dx^{i_p} = \frac{1}{p!}\sum_{s=1}^p X^{i_s}\omega_{i_1\dots i_s\dots i_p}(-1)^{s-1}dx^{i_1}\wedge\dots\wedge\widehat{dx^{i_s}}\wedge\dots\wedge dx^{i_r}$$

$$\begin{aligned} i_{\partial_x}(dx\wedge dy) &= \frac{1}{(2-1)!}dy = dy \\ i_{\partial_x}(dy\wedge dz) &= 0 \\ i_{\partial_x}(dz\wedge dx) &= \delta^i{}_1\epsilon_{31}dx^3 = -dz \end{aligned}$$

Let ω 1-form. $i_X\omega = \omega(X)$

$$\begin{aligned} (di_X + i_Xd)\omega &= d(X^\mu\omega_\mu) + i_X[\frac{1}{2}(\partial_\mu\omega_\nu - \partial_\nu\omega_\mu)dx^\mu\wedge dx^\nu] = \\ &= (\omega_\mu\partial_\nu X^\mu + X^\mu\partial_\nu\omega_\mu)dx^\nu + \frac{1}{2}(X^\mu\partial_\mu\omega_\nu dx^\nu - X^\nu\partial_\nu\omega_\mu dx^\mu) \end{aligned}$$

using, recall,

$$i_X\omega = \frac{1}{(p-1)!}X^i\omega_{ii_2\dots i_p}dx^{i_2}\wedge\dots\wedge dx^{i_p}$$

Recall (5.55): $\mathcal{L}_X\omega = (X^\nu\partial_\nu\omega_\mu + \partial_\mu X^\nu\omega_\nu)dx^\mu$

(36)
$$\mathcal{L}_X\omega = (di_X + i_Xd)\omega \qquad (5.80)$$

for r-form, $\omega = \frac{1}{r!}\omega_{\mu_1\dots\mu_r}dx^{\mu_1}\wedge\dots\wedge dx^{\mu_r}$

(37)
$$\mathcal{L}_X\omega = \lim_{\epsilon\to 0}\frac{1}{\epsilon}((\sigma_\epsilon)^*\,\omega|_{\sigma_\epsilon(X)} - \omega|_X) = X^\nu\frac{1}{r!}\partial_\nu\omega_{\mu_1\dots\mu_r}dx^{\mu_1}\wedge\dots\wedge dx^{\mu_r} + \sum_{s=1}^r\partial_{\mu_s}X^\nu\frac{1}{r!}\omega_{\mu_1\dots(s\to\nu)\dots\mu_r}dx^{\mu_1}\wedge\dots\wedge dx^{\mu_r} \qquad (5.81)$$

Ex.5.12. (q^μ, p_μ) tangent bundle!!!
symplectic 2 form

(38)
$$\omega = dp_\mu\wedge dq^\mu \qquad (5.88)$$

1-form $\theta = q^\mu dp_\mu$ $\qquad$ or (???) $\theta = p_\mu dq^\mu$

$$\omega = d\theta$$

Given $f(q,p)$ in phase space
define Hamiltonian vector field

$$X_f = \frac{\partial f}{\partial p_\mu}\frac{\partial}{\partial q^\mu} - \frac{\partial f}{\partial q^\mu}\frac{\partial}{\partial p_\mu} \qquad (5.91)$$

$$i_{X_f}\omega = \frac{-\partial f}{\partial p_\mu}dp_\mu - \frac{\partial f}{\partial q^\mu}dq^\mu = -df$$

$$i_X\omega = \frac{1}{(p-1)!}X^i\omega_{ii_2\dots i_p}dx^{i_2}\wedge\cdots\wedge dx^{i_p}$$

Consider vector field generated by Hamiltonian

(39)
$$X_H = \frac{\partial H}{\partial p_\mu}\frac{\partial}{\partial q^\mu} - \frac{\partial H}{\partial q^\mu}\frac{\partial}{\partial p_\mu} \qquad (5.92)$$

Hamilton's eqns. of motion.

$$\begin{aligned} \dot{q}^\mu &= \frac{\partial H}{\partial p_\mu} \\ \dot{p}_\mu &= \frac{-\partial H}{\partial q^\mu} \end{aligned} \qquad (5.93) \qquad \text{(Hamilton's eqn. of motion)}$$

(40)
$$X_H = \dot{q}^\mu\frac{\partial}{\partial q^\mu} + \dot{p}_\mu\frac{\partial}{\partial p_\mu} = \frac{d}{dt} \qquad (5.94)$$

symplectic 2-form ω left-invariant along flow generated by X_H

$$\mathcal{L}_{X_H}\omega = d(i_{X_H}\omega) + i_{X_H}(d\omega) = d(i_{X_H}\omega) = -d^2H = 0$$

used $(di_X + i_Xd)\omega = \mathcal{L}_X\omega \qquad (5.82)$

Conversely, if X satisfies $\mathcal{L}_X\omega = 0$, $\exists$ Hamiltonian H s.t. Hamilton's eqn. of motion is satisfied along the flow generated by X

from $\mathcal{L}_X\omega = d(i_X\omega) = 0$ and hence by Poincaré's lemma, $\exists\, H(q,p)$ s.t.

$$i_X\omega = -dH$$

5.5. **Integration of differential forms.**

5.5.1. *Orientation.* integration of differential form over manifold M defined only when M is "orientable"

Let M connected m-dim. differential manifold

$\forall\, p \in M$, T_pM spanned by basis $\{\frac{\partial}{\partial x^\mu}\}$, x^μ local coordinate on chart $U_i \ni p$

Let U_j another chart s.t. $U_i \cap U_j \neq \emptyset$ with local cordinates y^α.

If $p \in U_i \cap U_j$, T_pM spanned by either $\{e_\mu\} = \{\frac{\partial}{\partial x^\mu}$, or $\{\widetilde{e}_\alpha\} = \{\frac{\partial}{\partial y^\alpha}\}$

(41)
$$\frac{\partial}{\partial y^\alpha} = \frac{\partial x^\mu}{\partial y^\alpha}\frac{\partial}{\partial x^\mu} \qquad (5.97)$$

If $J = \det\left(\frac{\partial x^\mu}{\partial y^\alpha}\right) > 0$ on $U_i \cap U_j$, $\{e_\mu\}$, $\{\widetilde{e}_\alpha\}$ define same orientation on $U_i \cap U_j$

If $J < 0$, opposite orientation.

Definition 8 (5.6)**.** *M connected manifold covered by $\{U_i\}$*

manifold M orientable if $\forall$ overlapping charts U_i, U_j, $\exists$ local coordinates $\begin{matrix} \{x^\mu\} \text{ for } U_i \\ \{y^\alpha\} \text{ for } U_j \end{matrix}$ *s.t.* $J = det\left(\frac{\partial x^\mu}{\partial y^\alpha}\right) > 0$

If M nonorientable, J can't be positive in all interactions of charts.

If m-dim. M orientable, $\exists\, m$-form ω s.t. $\omega \neq 0$

This m-form ω is volume element

2 vol. elements ω, ω' equivalent if $\exists$ strictly positive $h \in \mathcal{F}(M)$ s.t. $\omega = h\omega'$

take m-form

(42)
$$\omega = h(p)dx^1\wedge\cdots\wedge dx^m \qquad (5.98)$$

with positive-definite $h(p)$ on chart (U,φ), $x = \varphi(p)$

If M orientable, extend ω throughout M s.t. component h positive definite on chart U_i

If M orientable, ω vol. element.

Let $p \in U_i \cap U_j \neq \emptyset$, $\begin{matrix} x^\mu \text{ coordinates of } U_i \\ y^\alpha \text{ coordinates of } U_j \end{matrix}$

$$\omega = h(p)\frac{\partial x^1}{\partial y^{\mu_1}}dy^{\mu_1}\wedge\dots\wedge\frac{\partial x^m}{\partial y^{\mu_m}}dy^{\mu_m} = h(p)\det\left(\frac{\partial x^\mu}{\partial y^\nu}\right)dy^1\wedge\dots\wedge dy^m$$

$$\det\left(\frac{\partial x^i}{\partial y^j}\right) = \sum_{\sigma\in S_n}\mathrm{sgn}(\sigma)\frac{\partial x^1}{\partial y^{\sigma_1}}\dots\frac{\partial x^m}{\partial y^{\sigma_m}} = \sum_{i_1\dots i_m=1}^m \epsilon_{i_1\dots i_m}\frac{\partial x^1}{\partial y^{i_1}}\dots\frac{\partial x^m}{\partial y^{i_m}} = \epsilon_{i_1\dots i_m}\frac{\partial x^1}{\partial y^{i_1}}\dots\frac{\partial x^m}{\partial y^{i_m}}$$

$$\frac{\partial x^1}{\partial y^{i_1}}\dots\frac{\partial x^m}{\partial y^{i_m}}dy^{i_1}\wedge\dots\wedge dy^{i_m} = \frac{\partial x^1}{\partial y^{i_1}}\dots\frac{\partial x^m}{\partial y^{i_m}}\epsilon^{i_1\dots i_m}dy^1\wedge\dots\wedge dy^{i_m} = \det\left(\frac{\partial x^i}{\partial y^j}\right)dy^1\wedge\dots\wedge dy^{i_m}$$

5.5.2. *Integration of forms.* integration of function $f: M\to\mathbb{R}$ over oriented M
take vol. element ω
in coordinate neighborhood U_i, $\;x=\varphi(p)$, $p\in U_i$

(43) $$\int_{U_i} f\omega \equiv \int_{\varphi(U_i)} f(\varphi_i^{-1}(x))h(\varphi_i^{-1}(x))dx^1\dots dx^m \qquad (5.100)$$

Definition 9 (5.7)**.** *open covering $\{U_i\}$ of M s.t. $\forall\, p\in M$, p covered by a finite number of U_i. M paracompact.*

If diff. $\epsilon_i(p)$ s.t.
- (i) $0\le\epsilon_i(p)\le 1$
- (ii) $\epsilon_i(p)=0$ if $p\notin U_i$
- (iii) $\epsilon_1(p)+\epsilon_2(p)+\dots = 1 \qquad \forall p\in M$

$\{\epsilon(p)\}$ partition of unity subordinate to covering $\{U_i\}$
from (iii),

(44) $$f(p)=\sum_i f(p)\epsilon_i(p) = \sum_i f_i(p) \qquad (5.101)$$

$f_i(p)\equiv f(p)\epsilon_i(p)$, $\;f_i(p)=0 \;\;\forall\, p\notin U_i$
Hence, $\forall\, p\in M$, paracompactness ensures in (5.101), $f_i(p)<\infty$, finite, in sum $\sum_i$
define

(45) $$\int_M f\omega \equiv \sum_i\int_{U_i} f_i\omega \qquad (5.102)$$

Although a different atlas $\{(V_i,\psi_i)\}$ gives different coordinates and different partition of unity, integral defined by (5.102) same.

Ex. 5.13. S^1. $U_1 = S^1-\{(1,0)\}$ $\qquad \epsilon_1(\theta)=\sin^2\left(\frac{\theta}{2}\right) \qquad \epsilon_1+\epsilon_2=1$ on S^1
$U_2 = S^1-\{(-1,0)\} \qquad \epsilon_2(\theta)=\cos^2\left(\frac{\theta}{2}\right)$

$f=\cos^2\theta$

$$\int_0^{2\pi} d\theta\cos^2\theta = \pi$$
$$\int_S^1 d\theta\cos^2\theta = \int_0^{2\pi}d\theta\sin^2\frac{\theta}{2}\cos^2\theta + \int_{-\pi}^{\pi}d\theta\cos^2\frac{\theta}{2}\cos^2\theta = \frac{\pi}{2}+\frac{\pi}{2}=\pi$$

5.6. **Lie groups and Lie algebras.**

5.6.1. *Lie groups.* Take $x,y,z\in\mathbb{R}-0$ s.t. $xy=z \qquad xy=z \qquad \frac{\partial z}{\partial x}=y\neq 0$

Exercise 5.19.
- (a) $\mathbb{R}^+=\{x\in\mathbb{R}|x>0\} \qquad \frac{\partial}{\partial x}x^{-1}=-x^{-2}\neq 0$ diff.
- (b)

$$\partial_x z = \partial_x(x+y)=1$$
$$\partial_x(x^{-1})=\partial_x(-x)=-1$$

diff.

(c)

$$(a,b)+(x,y)=(a+x,b+y) \qquad Dg = Dg(x) = \begin{bmatrix} a & \\ & b \end{bmatrix}$$

$$(x,y)^{-1} = (-x,-y) \qquad D(x,y)^{-1} = \begin{bmatrix} -1 & \\ & -1 \end{bmatrix}$$

Lorentz group

$$O(1,3) = \{M \in GL(4,\mathbb{R}) | M\eta M^T = \eta\} \qquad \eta = \text{diag}(-1,1,1,1)$$

Exercise 5.20. 20130801

$$\text{det}M\eta M^T = (\text{det}M)^2\text{det}\eta = \text{det}\eta \qquad (\text{det}M)^2 = 1 \qquad \text{det}M = \pm 1$$

if $UMU^T = \text{diag}(\lambda_1,\lambda_2,\lambda_3,\lambda_5) = \lambda_1\lambda_2\lambda_3\lambda_4 = (\text{det}U)^2\text{det}M = (\pm1)(\text{det}U)^2$
$(0,0)$ entry of $M\eta M^T$

$$-m_0^2 + m_1^2 + m_2^2 + m_3^2 = 1$$

M unbounded so $O(1,3)$ noncompact

Theorem 8. *5.2 Every closed subgroup M of a Lie group G is a Lie subgroup*

e.g. $O(n)$, $SL(n,\mathbb{R})$, $SO(n)$ Lie subgroups of $GL(n,\mathbb{R})$

$SL(n,\mathbb{R})$ closed subgroup, consider $\begin{aligned} f &: GL(n,\mathbb{R}) \to \mathbb{R} \\ A &\mapsto \text{det}A \end{aligned}$

f cont. $\{1\}$ closed $f^{-1}(1) = SL(n,\mathbb{R})$ so $SL(n,\mathbb{R})$ closed. By Thm. 5.2., $SL(n,\mathbb{R})$ Lie subgroup.
Let G Lie group.
H Lie subgroup.
Define $g \sim g'$ if $\exists\, h \in H$ s.t. $g' = gh$

$$[g] = \{gh | h \in G\}$$

coset space G/H is a manifold (not necessarily a Lie group)

if H normal subgroup of G, i.e. $ghg^{-1} \in H$, $\forall \begin{array}{c} g \in G \\ h \in H \end{array}$, then G/H Lie groupp

take $[g],[g'] \in G/H$
Let $gh, g'h'$ representative of $[g],[g']$, resp.

$$[g][g'] = ghg'h' = gg'h''h' \in [gg']$$

$$g^{-1}h$$

$$[g][g^{-1}] = ghg^{-1}h' = gg^{-1}h''h' = eh''h' = h''h' \in [e]$$

5.6.2. *Lie algebras.* left-translation
$L_g : G \to G$
$L_g h = gh = x^{ik}(g)x^{kj}(h) = x^{ij}(gh)$
$L_e h = eh = x^{ik}(e)x^{kj}(h) = \delta^{ik}x^{kj}(h) = x^{ij}(h) = 1h = h$
$L_g e = ge = x^{ik}(g)x^{kj}(e) = x^{ik}(g)\delta^{kj} = x^{ik}(g) = g1 = g$
$L_{g*} : T_hG \to T_{gh}G$
Pushforward?
Recall local coordinate form of pushforward

$$X \equiv X_h \in T_hG$$

$$X_h = X_h^{ij} \left.\frac{\partial}{\partial x^{ij}}\right|_h$$

$$L_{g*}X_h \equiv L_{g*}X = X_h^{kl}\frac{\partial x^{ij}(gh)}{\partial x^{kl}(h)} \left.\frac{\partial}{\partial x^{ij}}\right|_{gh} = X_{gh}^{ij} \left.\frac{\partial}{\partial x^{ij}}\right|_{gh}$$

Note that $x^{ik}(g)x^{kj}(h) = x^{ij}(gh)$

$$\implies \frac{\partial x^{ij}(gh)}{\partial x^{kl}(h)} = x^{im}(g)\delta^{km}\delta^{jl} = x^{ik}(g)\delta^{jl} = \begin{cases} 0 & \text{if } j \neq l \\ x^{ik} & \text{if } j = l \end{cases}$$

so

$$L_{g*}X_h = X_h^{kl}\frac{\partial x^{ij}(gh)}{\partial x^{kl}(h)}\left.\frac{\partial}{\partial x^{ij}}\right|_{gh} = X_h^{kl}x^{ik}(g)\delta^{jl}\left.\frac{\partial}{\partial x^{ij}}\right|_{gh} = X_h^{kj}x^{ik}(g)\left.\frac{\partial}{\partial x^{ij}}\right|_{gh} = x^{ik}(g)X_h^{kj}\left.\frac{\partial}{\partial x^{ij}}\right|_{gh}$$

santiy check: $L_{e*}X_h = eX_h = X_h$

Consider left-invariant vector fields

$L_{g*}X = X \quad \forall g$ (cf. wikipedia)

$L_{g*}\left.X\right|_h = X_{gh}$ (cf. Nakahara)

Exercise 5.21.

(46)
$$L_{a^*}\left.X\right|_g = X^\mu(g)\frac{\partial x^\nu(ag)}{\partial x^\mu(g)}\left.\frac{\partial}{\partial x^\nu}\right|_{ag} = x^\nu(ag)\left.\frac{\partial}{\partial x^\nu}\right|_{ag} \qquad (5.110)$$

$$L_{a*}\left.X\right|_g = \left.X\right|_{ag} \qquad \begin{aligned} L_a g &= ag \\ y^i = y^i(x^j) &= a_{ij}x^j \\ \partial_j y^i &= a_{ij} \end{aligned}$$

$$d(y(x))X^\mu(g)\left.\frac{\partial}{\partial x^\mu}\right|_g = X^\mu(g)\left.\frac{\partial}{\partial x^\mu}\right|_g y^\nu$$

Recall that

$$\begin{aligned} V &= V^\mu\frac{\partial}{\partial x^\mu} \\ f_*V &= W^\alpha\frac{\partial}{\partial y^\alpha} \end{aligned} \qquad W^\alpha = V^\mu\frac{\partial y^\alpha}{\partial x^\mu}$$

$$L_{a*}\left.X\right|_g = X^\mu(g)\frac{\partial y^\nu(ag)}{\partial x^\mu}\left.\frac{\partial}{\partial y^\nu}\right|_{ag} = \left.X\right|_{ag} = Y^\nu\left.\frac{\partial}{\partial y^\nu}\right|_{ag}$$

$V \in T_eG$ defines unique left invariant vector field X_V

(47)
$$\left.X_V\right|_g = L_{g*}V \qquad g \in G \qquad (5.111)$$

$\eth \equiv$ set of left invariant vector fields on G

$T_eG \to \eth$ is an isomorphism

$V \mapsto X_V$

$\eth \subset \chi(G)$

Lie Bracket (Sec. 5.3) also defined on $\eth$

$$\begin{gathered} g, ag = L_a g \in G \\ X, Y \in \eth \end{gathered}$$

(48)
$$L_{a*}\left.[X,Y]\right|_g = [L_{a*}\left.X\right|_g, L_{a*}\left.Y\right|_g] = \left.[X,Y]\right|_{ag} \qquad (5.112)$$

so $[X,Y] \in \eth$

e.g. $GL(n,\mathbb{R})$ coordinates given by n^2 entries x^{ij} of the matrix.

$$g = \{x^{ij}(g)\} \quad a = \{x^{ij}(a)\} \in GL(n,\mathbb{R}) \qquad L_a g = ag = x^{ik}(a)x^{kj}(g)$$

take $V = V^{ij}\left.\frac{\partial}{\partial x^{ij}}\right|_e \in T_eG$

(49)
$$\begin{aligned} \left.X_V\right|_g = L_{g^*}V = V^{ij}\left.\frac{\partial}{\partial x^{ij}}\right|_e x^{kl}(g)x^{lm}(e)\left.\frac{\partial}{\partial x^{km}}\right|_g = v^{ij}x^{kl}(g)\delta_i^l\delta_j^m\left.\frac{\partial}{\partial x^{km}}\right|_g = V^{ij}x^{ki}(g)\left.\frac{\partial}{\partial x^{kj}}\right|_g = \\ = x^{ki}(g)V^{ij}\left.\frac{\partial}{\partial x^{kj}}\right|_g = (gV)^{kj}\left.\frac{\partial}{\partial x^{kj}}\right|_g \qquad (5.113) \end{aligned}$$

$$V = V^{ij} \left. \frac{\partial}{\partial x^{ij}} \right|_e$$

$$W = W^{ij} \left. \frac{\partial}{\partial x^{ij}} \right|_e$$

$$[X_V, X_W]|_g = x^{ki}(g) V^{ij} \left. \frac{\partial}{\partial x^{kj}} \right|_g x^{ca}(g) W^{ab} \left. \frac{\partial}{\partial x^{cb}} \right|_g - (V \leftrightarrow W) =$$

(50) $$= x^{ij}(g)[V^{jk}W^{kl} - W^{jk}V^{kl}] \left. \frac{\partial}{\partial x^{il}} \right|_g = (g[V,W[)^{ij} \left. \frac{\partial}{\partial x^{ij}} \right|_g \qquad (5.114)$$

(51) $$\Longrightarrow L_{g*}V = gV \qquad (5.115)$$

(52) $$[X_V, X_W]|_g = L_{g*}[V,W] = g[V,W] \qquad (5.116)$$

Definition 10 (5.11)**.** *ð ≡ set of left-invariant vector fields with Lie bracket* $[\,,\,] : ð \times ð \to ð$ *Lie algebra of Lie group G*

20141117 EY recap: Recapping,

$$L_a : G \to G \qquad R_a : G \to G$$
$$L_a g = ag \qquad R_a g = ga$$

L_a, R_a are diffeomorphisms; indeed $L_{a^{-1}} = (L_a)^{-1}$, $R_{a^{-1}} = (R_a)^{-1}$ $\quad \forall\, a \in G$

$\forall\, X_g = T_g G$,

locally $X_g = X^i_g \frac{\partial}{\partial g^i}$

$L_{a*} : T_g G \to T_{ag} G$

locally, $L_{a*}X_g = X^i_g \frac{\partial (ag)^j}{\partial g^i} \frac{\partial}{\partial (ag)^j}$ (because that's what pushforwards do)

Note the abuse of notation above.

if X_g is **left-invariant**,

$$L_{a*}X_g = X_{ag}$$

this implies

$$X^i_g \frac{\partial (ag)^j}{\partial g^i} = X^j_{ag}$$

Now isomorphisms can be shown so that

$$T_1 G = \mathfrak{g} = \{X | X \in TG, \forall\, a, g \in G, L_{a*}X_g = X_{ag}\} \text{ i.e. set of left-invariant vector fields}$$

indeed, for instance, $\forall\, V \in T_1 G$, locally $V = V^i \left. \frac{\partial}{\partial x^i} \right|_1$,

$$L_{g*}V = X$$
$$L_{a*}X = L_{a*}L_{g*}V = L_{ag*}V = X_{ag}$$

uniqueness can be shown in either cases, note.

Specialize to the case of $G = GL(n)$

$$(L_a g)^{ij} = a^{ik} g^{kj}$$

$$X_g = X^{ij}_g \frac{\partial}{\partial g^{ij}}$$

$$L_{a*}X_g = X^{ij}_g \frac{\partial (ag)^{kl}}{\partial g^{ij}} \frac{\partial}{\partial (ag)^{kl}} = X^{ij}_g a^{ki} \frac{\partial}{\partial (ag)^{kj}} = (aX_g)^{kj} \frac{\partial}{\partial (ag)^{kj}}$$

where I used the following calculation:

$$(ag)^{kl} = a^{km} g^{ml} \Longrightarrow \frac{\partial (ag)^{kl}}{\partial g^{ij}} = a^{ki} \delta^{jl}$$

If X_g left invariant,

$$aX_g = X_{ag}$$

---

e.g. $\mathbf{so}(n)$ Lie algebra of $SO(n)$
e.g. 5.15

(a) $G = \mathbb{R}$ define $La : x \mapsto x + a$, left invariance field $X = \frac{\partial}{\partial x}$

$$L_{a*} \left. X\right|_x = \frac{\partial(a+x)}{\partial x} \frac{\partial}{\partial(a+x)} = \frac{\partial}{\partial(x+a)} = \left. X\right|_{x+a}$$

$X = \frac{\partial}{\partial \theta}$ unique left vector field on $G = SO(2) = \{e^{i\theta} | 0 \leq \theta \leq 2\pi\}$

$$\frac{\partial(\phi+\theta)}{\partial\theta} \frac{\partial}{\partial(\phi+\theta)} = \frac{\partial}{\partial(\phi+\theta)}$$

(b) Let $\mathbf{gl}(n,\mathbb{R})$ curves $\begin{matrix} c : (-\epsilon,\epsilon) \\ c(0) = 1 \end{matrix} \to GL(n,\mathbb{R})$ $c(s) = 1 + sA + \mathcal{O}(s^2)$ near $s = 0$, A $n \times n$ matrix of real entries

5.6.3. *The one-parameter subgroup.*

5.6.4. *Frames and structure equation.* $\{V_1 \dots V_n\}$ basis of $T_1 G = \mathfrak{g}$

(53) $$[X_a, X_b] = c^c_{ab} X_c \qquad (5.133)$$

From local form of the Lie bracket, cf. Exercise 5.9

$$[X,Y] = \left(X^\mu \frac{\partial Y^\nu}{\partial x^\mu} - Y^\mu \frac{\partial X^\nu}{\partial x^\mu} \right) \frac{\partial}{\partial x^\nu}$$

$$[V_a, V_b] = \left(V^i_a \frac{\partial V^j_b}{\partial x^i} - V^i_b \frac{\partial V^j_a}{\partial x^i} \right) \frac{\partial}{\partial x^j} = c^c_{ab} V_c = c^c_{ab} X^j_c \frac{\partial}{\partial x^j}$$

dual basis to $\{X_a\}$, $\{\theta^a\}$ s.t. $\langle \theta^a, X_b \rangle = \delta^a_{\ b}$
dual basis satisfies **Maurer-Cartan's structure equation**

$$d\theta^a(X_b, X_c) = X_b \delta^a_{\ c} - X_c \delta^a_{\ b} - \theta^a([X_b, X_c]) = X_b \delta^a_{\ c} - X_c \delta^a_{\ b} - \theta^a(c^d_{bc} X_d) = X_b \delta^a_{\ c} - X_c \delta^a_{\ b} - c^a_{bc} = -c^a_{bc}$$

where I used $d\omega(X,Y) = X\omega(Y) - Y\omega(X) - \omega([X,Y])$, cf. wikipedia "exterior derivative", etc.

(54) $$d\theta^a = \frac{-1}{2} c^a_{bc} \theta^b \wedge \theta^c \qquad (5.136)$$

define Lie algebra valued 1-form $\theta : T_g G \to T_1 G$, **canonical 1-form** or **Maurer-Cartan form** on G

$$\theta \in \Omega^1(G; \mathfrak{g})$$

(55) $$\theta : X \mapsto (L_{g^{-1}})_* X = (L_g)^{-1}_* X \text{ where } X \in T_g G \qquad (5.137)$$

Theorem 9 (5.3)**.** (a) *canonical 1-form is $\theta = V_a \otimes \theta^a$, where $\{V_a\}$ basis of $T_1 G = \mathfrak{g}$, $\{\theta^a\}$ dual basis of $T^*_g G$*
EY : 20141117 here's where Nakahara has a mistake, θ isn't at e but at g

(b) *where*
$d\theta = V_a \otimes d\theta^a$ *and*

(56) $$[\theta \wedge \theta] \equiv [V_a, V_b] \otimes \theta^a \wedge \theta^b \qquad (5.140)$$

Proof. (a) $\forall Y = Y^a X_a \in T_g G$

$$\theta(Y) = (L_{g^{-1}})_* Y = (L_g)^{-1}_* Y = Y^a (L_{g^{-1}})_* X_a = Y^a (L_{g^{-1}})_* (L_g)_* V_a = Y^a V_a$$

On the other hand

$$(V_a \otimes \theta^a)(Y) = V_a \otimes \theta^a(Y^b X_b) = V_a(Y^b(\theta^a(X_b))) = Y^a V_a$$

EY : 20141117 note that Nakahara, I believe, made a mistake with thinking θ^a is a dual basis at e, not g

Thus

$$\boxed{\theta = V_a \otimes \theta^a}$$

(b) Now $[V_a, V_b] = c^c_{ab} V_c$

$$d\theta + \frac{1}{2}[\theta \wedge \theta] = 0$$
$$\frac{1}{2}[\theta \wedge \theta] = \frac{1}{2}[V_a, V_b] \otimes \theta^a \wedge \theta^b = \frac{1}{2} V_c \otimes c^c_{ab} \theta^a \wedge \theta^b$$
$$d\theta = V_c \otimes d\theta^c = V_c \otimes \frac{-1}{2} c^c_{ab} \theta^a \wedge \theta^b$$
$$\Longrightarrow \boxed{d\theta + \frac{1}{2}[\theta \wedge \theta] = 0}$$

□

20141117 EY's recap:
c^c_{ab} independent of $g \in G$
$\forall\, g \in G$, $\forall\, \theta^a$ in dual basis for $T^*_g G$ (also $\mathfrak{g}^*$), $\theta^a \in T^*_g G$,
$\forall\, d\theta^a \in \Omega^2_g(G)$, then

$$d\theta^a = \frac{-1}{2} c^a_{bc} \theta^b \wedge \theta^c$$

is satisfied, **Maurer-Cartan's structure equation**.

5.7. **The action of Lie groups on manifolds.**

5.7.1. *Definitions.*

5.7.2. *Orbits and isotropy groups.*

5.7.3. *Induced vector fields.*

5.7.4. *The adjoint representation.*

6. de Rham Cohomology Groups

r-form ω in $\mathbb{R}^r$

$$\omega = a(x) dx^1 \wedge dx^2 \wedge \cdots \wedge dx^r$$

define integration of ω over $\overline{\sigma}_r$

(57) $$\int_{\overline{\sigma}_r} \omega \equiv \int_{\overline{\sigma}_r} a(x) dx^1 dx^2 \dots dx^r \qquad (6.2)$$

$$\int_{\overline{\sigma}_2} \omega = \int_{\overline{\sigma}_2} dxdy = \int_0^1 dx \int_0^{1-x} dy = \frac{1}{2}$$
$$\int_0^1 dx \int_0^{1-x} dy \int_0^{1-y-x} dz = \int_0^1 dx \int_0^{1-x} dy (1-y-x) = \frac{1}{6}$$

Let smooth $f : \sigma_r \to M$
$s_r = f(\sigma_r) \subset M$ (singular) r-simplex in M
define integration of r-form ω over r-chain in M

(58) $$\int_{s_r} \omega = \int_{\overline{\sigma}_r} f^* \omega \qquad (6.6)$$

general r-chain $c = \sum_i a_i s_{r,i} \in C_r(M)$

(59) $$\int_c \omega = \sum_i a_i \int_{s_{r,i}} \omega \qquad (6.7)$$

6.1. **Stokes' theorem.**

Theorem 10 (Stokes' thm.)**.** $\omega \in \Omega^{r-1}(M)$
$c \in C_r(M)$

then

(60) $$\int_c d\omega = \int_{\partial c} \omega \qquad (6.8)$$

Proof. c linear combination of r-simplexes
suffices to prove (6.8) for r-simplex s_r in M

Let $f : \overline{\sigma}_r \to M$ s.t. $f(\overline{\sigma}_r) = s_r$

$$\int_{s_r} d\omega = \int_{\overline{\sigma}_r} f^*(d\omega) = \int_{\overline{\sigma}_r} d(f^*\omega)$$

using (5.75)

Also we have

$$\int_{\partial s_r} \omega = \int_{\partial \overline{\sigma}_r} f^*\omega$$

□

6.1.1. *Preliminary consideration.*

6.1.2. *Stokes' theorem.*

6.2. **de Rham cohomology groups.**

7. Riemannian Geometry

7.1. **Riemannian manifolds and pseudo-Riemannian manifolds.**

7.1.1. *Metric tensor.*

Definition 11 (7.1)**.** (i) $g_p(U,V) = g_p(V,U)$

Since $g \in \tau_2^0(M)$ (2 covariant indices, type (0,2) tensor)

Recall from Ch. 5, 5.2.3, 1-forms, $df \in T_p^*M$ on $V \in T_pM$ defined.

$$\langle df, V \rangle = V[f] = V^\mu \frac{\partial f}{\partial x^\mu} \in \mathbb{R}$$

If $\exists$ metric g

$$g_p : T_pM \otimes T_pM \to \mathbb{R}$$

define

$$\begin{aligned} g_p(U, \ ) :& T_pM \to \mathbb{R} \\ & V \mapsto g_p(U,V) \end{aligned}$$

Then $g_p(U, \ )$ identified with 1-form $\omega_U \in T_p^*M$

Similarly, $\omega \in T_p^*M$ induces $V_\omega \in T_pM$ by $\langle \omega, U \rangle = g(V_\omega, U)$

Thus g_p isomorphism between T_pM and T_p^*M

Consider $V = v^\alpha \dfrac{\partial}{\partial x^\alpha}$ (since $v \in T_pM$)

$\omega = \omega_\alpha dx^\alpha$ (since $\omega \in T_p^*M$)

For arbitrary $U \in T_pM$

$$g_p(V,U) = g_p(v^\beta \frac{\partial}{\partial x^\beta}, U) = v^\beta g_p\left(\frac{\partial}{\partial x^\beta}, U \right) = \omega \cdot U = \omega_\alpha dx^\alpha(U)$$

Let $U = U^\lambda \frac{\partial}{\partial x^\lambda}$

$$\Longrightarrow v^\beta U^\lambda g_p\left(\frac{\partial}{\partial x^\beta},\frac{\partial}{\partial x^\lambda}\right)=\omega_\alpha dx^\alpha U^\lambda\frac{\partial}{\partial x^\lambda}=\omega_\alpha U^\lambda dx^\alpha\left(\frac{\partial}{\partial x^\lambda}\right)=\omega_\alpha U^\lambda\frac{\partial x^\alpha}{\partial x^\lambda}=\omega_\alpha U^\alpha=$$
$$=v^\beta U^\lambda g_{\beta\lambda}=v^\beta g_{\beta\alpha}U^\alpha$$
$$\Longrightarrow \omega_\alpha=g_{\alpha\beta}v^\beta$$

$$g_p=g_{\mu\nu}(p)dx^\mu\otimes dx^\nu \qquad (7.1a)$$
$$g_{\mu\nu}(p)=g_p\left(\frac{\partial}{\partial x^\mu},\frac{\partial}{\partial x^\nu}\right)=g_{\nu\mu}(p)\quad (p\in M)\qquad (7.1b)$$

since $(g_{\mu\nu})$ has maximal rank (if $g_p(U,U)=0$, $U=0$, so kernel is 0), $g_{\mu\nu}$ has inverse $g^{\mu\nu}$
isomorphism between T_pM and T_p^*M expressed as $\omega_\mu=g_{\mu\nu}U^\nu$, $U^\mu=g^{\mu\nu}\omega_\nu$ (7.2)
Take an infinitesimal displacement $dx^\mu\frac{\partial}{\partial x^\mu}\in T_pM$

$$ds^2=g\left(dx^\mu\frac{\partial}{\partial x^\mu},dx^\nu\frac{\partial}{\partial x^\nu}\right)=g_{\mu\nu}dx^\mu dx^\nu \qquad (7.3)$$

Exercise 7.1.

$$\begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & & \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & & \\ & & 1 & \\ & & & 1\end{pmatrix}\begin{pmatrix} & 1 & & \\ 1 & & & \\ & & 1 & \\ & & & 1\end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & & \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & & \\ & & 1 & \\ & & & 1\end{pmatrix}=\begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1\end{pmatrix}$$

Consider the light-cone basis

$$\{e_+,e_-,e_2,e_3\}$$
$$e_\pm\equiv\frac{e_1\pm e_0}{\sqrt{2}}$$
$$g(e_\pm,e_\pm)=\frac{1}{2}g(e_1\pm e_0,e_1\pm e_0)=\frac{1}{2}(1+(-1))=0 \qquad g_{++}=g_{--}=0$$
$$g(e_\pm,e_\mp)=\frac{1}{2}g(e_1\pm e_0,e_1\mp e_0)=\frac{1}{2}(1-(-1))=1 \qquad g_{+-}=g_{-+}=1$$

Indeed the light cone metric is $\begin{pmatrix} & 1 & & \\ 1 & & & \\ & & 1 & \\ & & & 1\end{pmatrix}$

$$\omega_\mu=g_{\mu\nu}U^\nu$$
$$\omega_+=V^-$$
$$\omega_-=V^+$$
$$\omega_2=V^2$$
$$\omega_3=V^3$$

7.1.2. *Induced metric.* Let M be m-dim. submanifold of n-dim. Riemannian manifold N with metric g_N.
If $f:M\to N$ embedding which induces the submanifold structure of M. (Sec. 5.2).
(recall, smooth $f:M\to N$, $\dim M\leq N$, f immersion if $f_*:T_pM\to T_{f(p)}N$ injection, so rank $f_*=\dim M$. f embedding if f immersion and f injection. Also $f(M)$ submanifold of N).
- pullback f^* induces natural metric $g_M=f^*g_N$ on M

(61) $$g_{M\mu\nu}(x)=g_{N\alpha\beta}\frac{\partial f^\alpha}{\partial x^\mu}\frac{\partial f^\beta}{\partial x^\nu}\qquad (7.5)$$

Recall the pullback: $f^*:T^*_{f(p)}N\to T^*_pM$ $\qquad \langle f^*\omega,V\rangle=\langle\omega,f_*V\rangle$
$\omega\in T^*_{f(p)}N$ $\qquad$ and
$V\in T_pM$ $\qquad f_*V[g]=V[gf],\quad g\in\mathcal{F}(N)$

Now $\begin{aligned} g_N &: T_{f(p)}N \otimes T_{f(p)}N \to \mathbb{R} \\ g_N(U,\ ) &\in T^*_{f(p)}N \end{aligned}$

$$\langle f^* g_N(U,\ ), V\rangle = \langle g_N(U,\ ), f_* V\rangle = \langle g_N(U,\ ), V^\mu \frac{\partial f^\nu}{\partial x^\mu}\frac{\partial}{\partial y^\nu}\rangle$$

For

$$f_*V[g] = V[gf] = V^\mu \frac{\partial}{\partial x^\mu}[gf] = V^\mu \frac{\partial g}{\partial y^\nu}\frac{\partial f^\nu}{\partial x^\mu} = V^\mu \frac{\partial f^\nu}{\partial x^\mu}\frac{\partial g}{\partial y^\nu}$$

with $gf(x^\mu)$ so then (20121026)

$$g_N\left(U, V^\mu \frac{\partial}{\partial y^\nu}\right)\frac{\partial f^\nu}{\partial x^\mu} = g_N(U, V^\nu \frac{\partial}{\partial y^\beta})\frac{\partial f^\beta}{\partial x^\nu}$$
$$\Longrightarrow g_N(f(x))\frac{\partial f^\alpha}{\partial x^\mu}\frac{\partial f^\beta}{\partial x^\nu}$$

with $U \in T_{f(p)}N$

For example,

$$f : (\theta, \phi) \mapsto (s_\theta c_\varphi, s_\theta s_\varphi, c_\theta)$$
$$\nabla f = \begin{vmatrix} c_\theta c_\varphi & -s_\theta s_\varphi \\ c_\theta s_\varphi & s_\theta c_\varphi \\ -s_\theta & 0 \end{vmatrix}$$
$$g_{\mu\nu} dx^\mu \otimes dx^\nu = \delta_{\alpha\beta}\frac{\partial f^\alpha}{\partial x^\mu}\frac{\partial f^\beta}{\partial x^\nu} dx^\mu \otimes dx^\nu = d\theta \otimes d\theta + s_\theta^2 d\varphi \otimes d\varphi$$

Exercise 7.2. Let $f : T^2 \to \mathbb{R}^3$. Embedding of torus into $(\mathbb{R}^3, \delta)$ defined by

$$f : (\theta, \varphi) \mapsto ((R + \cos\theta)\cos\varphi, (R + rc_\theta)s_\varphi, rs_\theta), \quad R > r$$
$$\nabla f = \begin{vmatrix} -rs_\theta c_\varphi & -(R + rc_\theta)s_\varphi \\ -rs_\theta s_\varphi & (R + rc_\theta)c_\varphi \\ rc_\theta & 0 \end{vmatrix}$$
$$g_{\mu\nu} dx^\mu \otimes dx^\nu = \delta_{\alpha\beta}\frac{\partial f^\alpha}{\partial x^\mu}\frac{\partial f^\beta}{\partial x^\nu} dx^\mu \otimes dx^\nu = r^2 d\theta \otimes d\theta + (R + rc_\theta)^2 d\varphi \otimes d\varphi = g_{\theta\theta} d\theta \otimes d\theta + g_{\varphi\varphi} d\varphi \otimes d\varphi$$

7.2. **Parallel transport, connection and covariant derivative.**

7.2.1. *Heuristic introduction.*

7.2.2. *Affine connections.*

$$\nabla_X(fY) = X[f]Y + f\nabla_X Y \qquad (7.13d)$$

$$\nabla_\nu e_\mu = \nabla_{e_\nu} e_\mu = e_\lambda \Gamma^\lambda_{\ \nu\mu} \qquad (7.14)$$

$$\nabla_V W = V^\mu \nabla_\mu (W^\nu \partial_\nu) = V^\mu(\partial_\mu W^\nu \partial_\nu + W^\nu \nabla_\mu \partial_\nu) = V^\mu((\partial_\mu W^\nu)\partial_\nu + W^\nu \Gamma^\lambda_{\ \mu\nu}\partial_\lambda) = V^\mu(\partial_\mu W^\lambda + W^\nu \Gamma^\lambda_{\ \mu\nu})\partial_\lambda = V^\mu \nabla_\mu W^\lambda \partial_\lambda$$

$$\nabla_\mu(W^\lambda \partial_\lambda) = (\partial_\mu W^\lambda)\partial_\lambda + W^\lambda(\nabla_\mu \partial_\lambda) = \partial_\mu W^\lambda \partial_\lambda + W^\lambda \Gamma^a_{\ \mu\lambda}\partial_a = (\partial_\mu W^\lambda + W^b \Gamma^\lambda_{\ \mu b})\partial_\lambda$$

7.2.3. *Parallel transport and geodesics.*

$$\nabla_V X = 0 \qquad (7.18a)$$

X parallel transported along $c(t)$ where $V = \frac{d}{dt} = \frac{dx^\mu(c(t))}{dt}\partial_\mu\Big|_{c(t)}$

Now

$$\dot{X} = \partial_\mu X \dot{c}^\mu = \partial_\mu X V^\mu$$

where $X = X(c(t))$. So

$$\nabla_V X = V^\mu(\partial_\mu X^\lambda + X^\nu \Gamma^\lambda_{\ \mu\nu})\partial_\lambda = 0$$
$$\Longrightarrow V^\mu \partial_\mu X^\lambda + \Gamma^\lambda_{\ \mu\nu} V^\mu X^\nu = 0 \text{ or } \dot{X} + \Gamma^\lambda_{\ \mu\nu}\frac{dx^\mu(c(t))}{dt} X^\nu = 0$$

If

$$\nabla_V V = 0 \qquad (7.19a)$$

tangent vector $V(t)$ itself is parallel transported along $c(t)$. $c(t)$ *geodesic.*

Exercise 7.3. Recall what left-invariant means. A left-invariant vector field is such that

$$L_{a*}\, X|_g = X|_{ag}$$

Recall

$$\frac{d^2x^\mu}{dt^2} + \Gamma^\mu_{\ \nu\lambda}\frac{dx^\nu}{dt}\frac{dx^\lambda}{dt} = 0 \qquad (7.19b)$$

affine reparametrization

$$t \to at + b \qquad (a, b \in \mathbb{R})$$

Recall

$$\frac{dx^\mu}{dt} \to \frac{dt}{dt'}\frac{dx^\mu}{dt}$$

In this case, $\frac{dt}{dt'} = \frac{1}{a}$. So under this affine reparametrization

$$\frac{d^2x^\mu}{dt^2} = \frac{d}{dt}\left(\frac{dx^\mu}{dt}\right) \to \frac{1}{a}\frac{d}{dt}\left(\frac{dx^\mu}{dt'}\right) = \frac{1}{a^2}\frac{d^2x^\mu}{dt^2}$$

So

$$\frac{d^2x^\mu}{dt^2} + \Gamma^\mu_{\ \nu\lambda}\frac{dx^\nu}{dt}\frac{dx^\lambda}{dt} \to \frac{1}{a^2}\frac{d^2x^\mu}{dt^2} + \Gamma^\mu_{\ \nu\lambda}\frac{1}{a}\frac{dx^\nu}{dt}\frac{1}{a}\frac{dx^\lambda}{dt} = \frac{1}{a^2}\left(\frac{d^2x^\mu}{dt^2} + \Gamma^\mu_{\ \nu\lambda}\frac{dx^\nu}{dt}\frac{dx^\lambda}{dt}\right) = 0$$

7.2.4. *The covariant derivative of tensor fields.* Define

$$\nabla_X f = X[f] \qquad (7.21)$$

$$\nabla_X(fY) = X[f]Y + f\nabla_X Y \qquad (7.13d)$$
$$\nabla_X(fY) = (\nabla_X f)Y + f\nabla_X Y \qquad (7.13d')$$

Require this

(62) $$\nabla_X(T_1 \otimes T_2) = (\nabla_X T_1)\otimes T_2 + T_1 \otimes (\nabla_X Y_2) \qquad (7.22)$$

$\langle \omega, Y\rangle \in \mathcal{C}^\infty M$, $Y \in \mathcal{X}(M)$

$$X[\langle\omega, Y\rangle] = \nabla_X[\langle\omega, Y\rangle] = \langle\nabla_X\omega, Y\rangle + \langle\omega, \nabla_X Y\rangle$$

$$\langle\nabla_X\omega, Y\rangle = (\nabla_X\omega)_i Y^i$$
$$\langle\omega, \nabla_X Y\rangle = \omega_i X^\mu(\partial_\mu Y^i + Y^\nu\Gamma^i_{\mu\nu})$$
$$X[\langle\omega, Y\rangle] = X(\omega_i Y^i) = X^i\partial_j\omega_i Y^i + X^j\omega_i\partial_j Y^i$$

$$X^j\partial_j\omega_i Y^i + X^j\omega_i\partial_j Y^i = (\nabla_X\omega)_i Y^i + \omega_i X^\mu(\partial_\mu Y^i) + \omega_j X^\mu Y^i\Gamma^j_{\mu i}$$
$$(\nabla_X\omega)_\nu = X^\mu\partial_\mu\omega_\nu - X^\mu\Gamma^\lambda_{\mu\nu}\omega_\lambda \qquad (7.23)$$

$X = \partial_\mu$
$X^\nu = \delta^\nu_{\ \mu}$

$$(\nabla_\mu\omega)_\nu = \partial_\mu\omega_\nu - \Gamma^\lambda_{\mu\nu}\omega_\lambda \qquad (7.24)$$

Recall $\nabla_\nu e_\mu \equiv \nabla_{e_\nu} e_\mu = e_\lambda\Gamma^\lambda_{\nu\mu}$ (7.14)

$$\omega = \delta_j^{\ i} dx^j = dx^i$$
$$(\nabla_\mu dx^i)_\nu = \partial_\mu\delta_\nu^{\ i} - \Gamma^\lambda_{\mu\nu}\delta_\lambda^{\ i} = -\Gamma^i_{\mu\nu}$$
$$\nabla_\mu dx^\nu = -\Gamma^\nu_{\mu\lambda}dx^\lambda$$

$$\nabla_\nu t^{\lambda_1\dots\lambda_p}_{\mu_1\dots\mu_q} = \partial_\nu t^{\lambda_1\dots\lambda_p}_{\mu_1\dots\mu_q} + \Gamma^{\lambda_1}_{\nu\kappa}t^{\kappa\nu_2\dots\lambda_p}_{\mu_1\dots\mu_q} + \dots + \Gamma^{\lambda_p}_{\nu\kappa}t^{\lambda_1\dots\lambda_{p-1}\kappa}_{\mu_1\dots\mu_q} - \Gamma^\kappa_{\nu\mu_1}t^{\lambda_1\dots\lambda_p}_{\kappa\mu_2\dots\mu_q} - \dots - \Gamma^\kappa_{\nu\mu_q}t^{\lambda_1\dots\lambda_p}_{\mu_1\dots\mu_{q-1}\kappa} \qquad (7.26)$$

Metric tensor. $g : TM \times_M TM \to \mathbb{R}$ restriction to a fiber, $g_p : T_pM \times T_pM \to \mathbb{R}$

Exercise 7.4.

$$\nabla_\lambda g_{\mu\nu} = \partial_\lambda g_{\mu\nu} - \Gamma^\kappa_{\lambda\mu} g_{\kappa\nu} - \Gamma^\kappa_{\lambda\nu} g_{\mu\kappa}$$

Notice that $g(X) : \mathcal{X}(M) \to C^\infty(M)$
so $g(X)$ a 1-form.

as a 1-form

$$\nabla_a(g(e_i)) = \nabla_a(g_{ik}\sigma^k) = (\nabla_a g_{ik})\sigma^k + g_{ik}\nabla_a\sigma^k = (\nabla_a g_{ik})\sigma^k + g_{ik}(-\omega^k_{aj}\sigma^j)$$

as tensor product, using rules.

$$\nabla_a(g(e_i)) = (\nabla_a g)e_i + g\nabla_a e_i = (\nabla_a g)e_i + g\omega^j_{ai}e_j$$

subtract the 2 facts above.

$$(\nabla_a g)e_i = (\nabla_a g_{ik})\sigma^k - g_{ij}\omega^j_{ak}\sigma^k - g_{jk}\omega^j_{ai}\sigma^k$$
$$\Longrightarrow \boxed{(\nabla_a g)_{ij} = (\nabla_a g_{ij}) - g_{ik}\omega^k_{aj} - g_{kj}\omega^k_{ai}}$$

7.2.5. *The transformation properties of connection coefficients.* another chart (U, ψ), or $UV \neq \emptyset$. $y = \psi(p)$
Let $\{\partial_{y_a}\}$
$\partial_{y^a} = \delta^b{}_a \partial_{y_b}$
Recall

$$\nabla_V X = V^\mu(\partial_\mu X^\lambda + X^\nu \Gamma^\lambda_{\mu\nu})\partial_\lambda = 0$$
$$\nabla_\nu \partial_\mu = \Gamma^\lambda_{\nu\mu}\partial_\lambda \qquad (7.14)$$
$$\nabla_a \partial_b = \widetilde{\Gamma}^c_{ab}\partial_c \qquad (7.28)$$

$\partial_a = \frac{\partial x^\nu}{\partial y^\mu}\partial_\nu$

$$\nabla_a\partial_b = \nabla_a\left(\frac{\partial x^\mu}{\partial y^b}\partial_\mu\right) = \frac{\partial^2 x^\mu}{\partial y^a \partial y^b}\partial_\mu + \frac{\partial x^\mu}{\partial y^b}\nabla_a\partial_\mu = \frac{\partial^2 x^\mu}{\partial y^a \partial y^b}\partial_\mu + \frac{\partial x^\mu}{\partial y^b}\left(\frac{\partial x^\nu}{\partial y^a}\Gamma^\lambda_{\nu\mu}\partial_\lambda\right)$$
$$\nabla_a\partial_\mu = \frac{\partial x^\nu}{\partial y^a}(\partial_\nu \delta_\mu{}^\nu + \delta_\mu{}^\rho \Gamma^\lambda_{\nu\rho})\partial_\lambda = \frac{\partial x^\nu}{\partial y^a}\Gamma^\lambda_{\nu\mu}\partial_\lambda$$
$$\frac{\partial^2 x^\nu}{\partial y^a \partial y^b}\partial_\nu + \frac{\partial x^\mu}{\partial y^b}\frac{\partial x^\lambda}{\partial y^a}\Gamma^\nu_{\lambda\mu}\partial_\nu = \widetilde{\Gamma}^c_{ab}\frac{\partial x^\nu}{\partial y^c}\partial_\nu$$
$$\widetilde{\Gamma}^c_{ab} = \frac{\partial x^\lambda}{\partial y^a}\frac{\partial x^\mu}{\partial y^b}\frac{\partial y^c}{\partial x^\nu}\Gamma^\nu_{\lambda\mu} + \frac{\partial y^c}{\partial x^\nu}\frac{\partial^2 x^\nu}{\partial y^a \partial y^b} \qquad (7.29)$$

7.2.6. *The metric connection.*

7.3. **Curvature and torsion.**

7.3.1. *Definitions.*

7.3.2. *Geometrical meaning of the Riemann tensor and the torsion tensor.*

7.3.3. *The Ricci tensor and the scalar curvature.*

7.4. **Levi-Civita connections.**

7.4.1. *The fundamental theorem.*

7.4.2. *The Levi-Civita connection in the classical geometry of surfaces.*

7.4.3. *Geodesics.*

7.4.4. *The normal coordinate system.*

7.4.5. *Riemann curvature tensor with Levi-Civita connection.*

7.5. **Holonomy.**

7.6. **Isometries and conformal transformations.**

7.6.1. *Isometries.*

7.6.2. *Conformal transformations.*

7.7. **Killing vector fields and conformal Killing vector fields.**

7.7.1. *Killing vector fields.*

7.7.2. *Conformal Killing vector fields.*

7.8. **Non-coordinate bases.**

7.9. **Differential forms and Hodge theory.**

8. Complex Manifolds

8.1. **Complex manifolds.**

8.2. **Calculus on complex manifolds.**

9. Fibre bundles

9.1. **Tangent bundles.** $\pi^{-1}(p) = T_pM$, fibre at p

section of TM, $s : M \to TM$ s.t. $\pi \circ s = 1_M$
local section $s_i : U_i \to TU_i$ on chart U_i.

$$\Longrightarrow \begin{aligned} & s(p) = X \equiv X|_p \equiv U \in TM \\ & \pi(u) = p \end{aligned}$$

9.2. **Fibre bundles.**

9.2.1. *Definitions.*

Definition 12 (9.1)**.** *(diff.) fibre bundle* (E, π, M, F, G)

(i) *diff. manifold* E *total space.*
(ii) *diff. manifold* M *base space.*
(iii) *diff. manifold* F *fibre (or typical fibre)*
(iv) *surjection* $\pi : E \to M$ *projection.*
$\pi^{-1}(p) = F_p \simeq F$ *fibre at* p.
(v) *Lie group* G *structure group. Left action on* F.
(vi) *open cover* $\{U_i\}$ *of* M, *diffeomorphism* $\Phi_i : U_i \times F \to \pi^{-1}(U_i)$ *s.t.*

$$\pi \Phi_i(p, f) = p$$

Φ_i *local trivialization since* $\Phi_i^{-1} : \pi^{-1}(U_i)$ *onto direct product* $U_i \times F$
(vii) $\Phi_i(p, f) = \Phi_{i,p}(f)$, $\Phi_{i,p} : F \to F_p$ *diffeomorphism.*
on $U_iU_j \neq \emptyset$, $t_{ij}(p) \equiv \Phi_{i,p}^{-1}\Phi_{j,p} : F \to F$, *require* $t_{ij}(p) \in G$
Then Φ_i, Φ_j *related by smooth* $t_{ij} : U_iU_j \to G$ *as*

$$\Phi_j(p, f) = \Phi_i(p, t_{ij}(p)f) \qquad (9.4)$$

t_{ij} *transition functions.*

coordinate bundle $(E, \pi, M, F, G, \{U_i\}, \{\Phi_i\})$, $\{U_i\}$ specified covering of M.
Take chart U_i of M.
$\pi^{-1}(U_i)$ diffeomorphic to $U_i \times F$.
$\Phi_i^{-1} : \pi^{-1}(U_i) \to U_i \times F$ diffeomorphism.
Let u s.t. $\pi(u) = p \in U_iU_j$.

$$\begin{aligned} \Phi_i^{-1}(u) &= (p, f_i) \\ \Phi_j^{-1}(u) &= (p, f_j) \end{aligned}$$

$\exists\, t_{ij} : U_iU_j \to G$, s.t. $f_i = t_{ij}(p)f_j$
possible set of transition functions is far from unique.
Let $\{\phi_i\}, \{\widetilde{\phi}_i\}$ be 2 sets of local trivializations giving rise to the same fiber bundle.

$$t_{ij}(p) = \phi_{i,p}^{-1}\phi_{j,p} \qquad (9.7a)$$

$$\widetilde{t}_{ij}(p) = \widetilde{\phi}_{i,p}^{-1}\widetilde{\phi}_{j,p} \qquad (9.7b)$$

define $g_i(p) : F \to F \qquad \forall\, p \in M$

$g_i(p) \equiv \phi_{i,p}^{-1} \circ \widetilde{\phi}_{i,p}$

require $g_i(p)$ homeomorphism s.t. $g_i(p) \in G$.

9.2.2. *Reconstruction of fibre bundles.*

9.2.3. *Bundle maps.*

9.2.4. *Equivalent bundles.*

9.2.5. *Pullback bundles.*

9.2.6. *Homotopy axiom.*

9.3. **Vector bundles.**

9.3.1. *Definitions and examples.*

9.3.2. *Frames.*

9.3.3. *Cotangent bundles and dual bundles.*

9.3.4. *Sections of vector bundles.*

9.3.5. *The product bundle and Whitney sum bundle.*

9.3.6. *Tensor product bundles.*

9.4. **Principal bundles.**

9.4.1. *Definitions.* $\Phi_i : U_i \times G \to \pi^{-1}(U_i)$ local trivialization
right action
$\Phi_i^{-1}(ua) = (p, g_i a)$
$ua = \Phi_i(p, g_i a)$
$\forall\, a \in G$, $u \in \pi^{-1}(p)$

Since the right action commutes with the left action
EY : ??? Since the right action commutes with the left action ???
Theodore Frankel says: cf. pp. 455, Proof of Theorem 17.8,

$$g_j a = \tau_{ji}(p) g_i a = \tau_{ji} g_i a$$

i.e.

$$ua = \Phi_j(p, g_j a) = \Phi_j(p, \tau_{ji}(p) g_i a) = \Phi_i(p, g_i a)$$

if $p \in U_i \bigcap U_j$

$$ua = \Phi_j(p, g_j a) = \Phi_j(p, \tau_{ji}(p) g_i a) = \Phi_i(p, g_i a)$$

since $\tau_{ji} = \Phi_j^{-1} \Phi_i$

Thus right multiplication defined without reference to local trivializations. Notation of this : i.e. $P \times G \to P$ or $(u, a) \mapsto ua$
$\pi(ua) = \pi(u)$
EY :

$$\boxed{\pi(ua) = \pi(u)}$$

right action of G on $\pi^{-1}(p)$ transitive since G acts on G transitively on the right and $F_p = \pi^{-1}(p)$ diffeomorphic to G

cf. wikipedia Group action, types of action
action of G on X,
transitive if $X \neq \emptyset$, $\forall\, x, y \in X$, $\exists\, g \in G$ s.t. $gx = y$
free if given $g, h \in G$, $\exists\, x \in X$, with $gx = hx$ implies $g = h$ i.e. if $gx = x$, (i.e. if g has at least 1 fixed pt.), then $g = e$

For $u \in \pi^{-1}(p)$, $p \in U_i$, $\exists!$, $g_U \in G$ s.t. $u = s_i(p)g_u$
define $\Phi_i^{-1}(u) = (p, g_u)$

$t_{ji} = \Phi_j^{-1}\Phi_i(p) \equiv \Phi_j^{-1}\Phi_i$

$$(p, e) = \Phi_i^{-1}(s_i(p)) = \Phi_i^{-1}(u)$$
$$ug = \Phi_i(p, e)g = \Phi_i(p, g)$$

Example 9.7
Let P be principal bundle with fiber $U(1) = S^1$, base space S^2
This principal bundle represents the topological setting of the **magnetic monopole**.

$G = U(1) = S^1$
$M = S^2$

$$\Phi_N : U_N \times S^1 \to \pi^{-1}(U_N)$$
$$\Phi_S : U_S \times S^1 \to \pi^{-1}(U_S)$$

Let $u \in \pi^{-1}(x)$

$$u = \Phi_N(x, e^{i\alpha_N}) = \Phi_S(x, e^{i\alpha_S})$$

$$\tau_{NS} = \Phi_N^{-1}\Phi_S$$
$$\tau_{NS}g_S = g_N$$

then $\tau_{NS} = e^{i(\alpha_N - \alpha_S)}$
Now $\tau_{NS} : U_S \times G \to U_N \times G$
But on equator, S^1
uniquely define $\tau_{NS}(p)$ on equator

$$\tau_{NS}(p) = e^{in\phi} \quad n \in \mathbb{Z}$$

EY : why like this? but surely

$$n\phi = \alpha_N - \alpha_S$$

(from EY)
Note that

$$e^{in\phi} \to e^{in(\phi + 2\pi m)} = e^{in\phi}e^{i\pi 2mn}$$

surely (EY)
Example 9.8. If we identify all the infinite points of the Euclidean space $\mathbb{R}^m$, the 1-point compactification $S^m = \mathbb{R}^m \bigcup \{\infty\}$ is obtained.
If a trivial G bundle is defined over $\mathbb{R}^m$ we shall have a new G bundle over S^m after compactification, which is not necessarily trivial.

$$A = \begin{pmatrix} u & v \\ -v^* & u^* \end{pmatrix}$$

$u = t + iz$
$v = y + ix$

$$\pi_3(SU(2)) \cong \pi_3(S^3) \cong \mathbb{Z}$$
$$\pi_3(S^3) = \{[f] | f : S^3 \to X = S^3\}$$

$\pi_3(S^3)$ classifies maps from S^3 to $SU(2) \cong S^3$

(63)
$$f : S^3 \to S^3 \cong SU(2)$$
$$f(x, y, z, t) \mapsto \begin{pmatrix} t + iz & y + ix \\ -y + ix & t - iz \end{pmatrix} = t1 + i(x\sigma_x + y\sigma_y + z\sigma_z) \qquad (9.47)$$

$p = (x, y, z, t) \in U_N \bigcap U_S$
$R = (x^2 + y^2 + z^2 + t^2)^{1/2}$

$\tau_{NS}(p) = \frac{1}{R}(t1 + ix^i\sigma_i)$

$$\Phi_N^{-1}(u) = (p, g_N)$$
$$\Phi_S^{-1}(u) = (p, g_S)$$

where $g_N, g_S \in SU(2)$

9.4.2. *Associated bundles.* principal fiber bundle $P(M, G)$
Let G act on manifold F on left

define action of $g \in G$ on $P \times F$ by

$$(u, f) \mapsto (ug, g^{-1}f), \qquad \begin{array}{l} u \in P \\ f \in F \end{array}$$

associated fiber bundle (E, π, M, G, F, P) is equivalence class $P \times F/G$ which

$$(u, f) \sim (ug, g^{-1}f)$$

e.g.
Consider F k-dim. vector space V
ρ k-dim. representation of G
$P \times \rho V$

$$(u, v) \sim (ug, \rho(g)^{-1}v) \text{ of } P \times V, \qquad \begin{array}{l} u \in P \\ g \in G \\ v \in V \end{array}$$

e.g. $P(M, GL(k, \mathbb{R}))$
associated vector bundle over M with fiber $\mathbb{R}^k$

$E = P \times_\rho V \qquad \pi_E : E \to M$
$\qquad\qquad\qquad \pi_E(u, v) = \pi(u)$

$\pi(u) = \pi(ug)$ implies

$$\pi_E(ug, \rho(g)^{-1}v) = \pi(ug) = \pi_E(u, v)$$

local trivialization $\Phi_i : U_i \times V \to \pi_E^{-1}(U_i)$
transition function of E given by $\rho(t_{ij}(p))$ where $t_{ij}(p)$ is that of p

9.4.3. *Triviality of bundles.*

Problems.

10. Connections on Fibre Bundles

10.1. **Connections on principal bundles.**

10.1.1.

10.1.2.

10.1.3. *The local connection form and gauge potential.* Let $\{U_i\}$ open covering of M
σ_i local section defined on each U_i

introduce Lie algebra-valued 1-form $\mathcal{A}_i$ on U_i

(64)
$$\mathcal{A}_i \equiv \sigma_i^* \omega \in \mathfrak{g} \otimes \Omega^1(U_i) \qquad (10.6)$$

$\omega \in \mathfrak{g} \otimes T^*P$

Remember $\omega \in \mathfrak{g} \otimes T^*P$
$\pi\sigma = 1$
$\pi\sigma(p) = p, \quad p \in U \subset M$

given Lie algebra valued 1-form A_i on U_i, we can reconstruct connection 1-form ω s.t. $\sigma_i^*\omega = A_i$ i.e.

Theorem 11 (10.1)**.** *given $\mathfrak{g}$-valued 1-form A_i on U_i, $A_i \in \mathfrak{g} \otimes \Omega^1(U_i)$,*
local section $\sigma_i : U_i \to \pi^{-1}(U_i)$

$\exists$ *connection 1-form ω s.t. $A_i = \sigma_i^*\omega$*

Proof. define $\mathfrak{g}$ valued 1-form ω on P

(65)
$$\omega_i \equiv g_i^{-1}\pi^* A_i g_i + g_i^{-1} d_P g_i \qquad (10.7)$$

d_P exterior derivative on P

g_i canonical local trivialization $\Phi_i^{-1}(u) = (p, g_i) \qquad u = \sigma_i(p) g_i$

for $X \in T_pM$,

$$\begin{aligned} \sigma_i^*\omega_i(X) = \omega_i(\sigma_{i*}X) = g_i^{-1}\pi^* A_i g_i(\sigma_{i*}X) + g_i^{-1} d_P g_i(\sigma_{i*}X) = \\ = \pi^* A_i(\sigma_{i*}X) + d_P g_i(\sigma_{i*}X) = A_i(\pi_*\sigma_{i*}X) + d_P g_i(\sigma_{i*}X) \end{aligned}$$

□

10.1.4. *Horizontal lift and parallel transport.*

Theorem 12 (10.2)**.** *Let $\gamma : [0,1] \to M$, $u_0 \in \pi^{-1}(\gamma(0))$*
Then $\exists!$ horizontal lift $\widetilde{\gamma}(t)$ in P s.t. $\widetilde{\gamma}(0) = u_0$

(66)
$$g_i(\gamma(t)) = g_i(t) = \mathcal{P}\exp\left(-\int_0^t A_{i\mu} \frac{dx^\mu}{dt} dt \right) = \mathcal{P}\exp\left(-\int_{\gamma(0)}^{\gamma(t)} A_{i\mu}(\gamma(t)) dx^\mu \right) \qquad (10.14)$$

10.2. **Holonomy.**

10.2.1. *Definitions.* loop $\gamma \subset M$ defines *transformation* $\tau_\gamma : \pi^{-1}(p) \to \pi^{-1}(p)$ on fiber following from (10.18), EY : ??? , transformation compatible with right action

(67)
$$R_g \Gamma(\widetilde{\gamma}) = \Gamma(\widetilde{\gamma}) R_g \qquad (10.18)$$

(68)
$$\tau_\gamma(ug) = \tau_\gamma(u) g \qquad (10.21)$$

$$C_p(M) = \{\gamma : [0,1] \to M | \gamma(0) = \gamma(1) = p\}$$

subgroup of structure group G, **holonomy group** at u is

(69)
$$\Phi_u \equiv \{ g \in G | \tau_\gamma(u) = ug, \, \gamma \in C_p(M)\} \qquad (10.22)$$

Exercise 10.6.

solution already found! Recall

(70) $$g_i(\gamma(t)) = g_i(t) = \mathcal{P}\exp\left(-\int_0^t A_{i\mu}\frac{dx^\mu}{dt}dt\right) = \mathcal{P}\exp\left(-\int_{\gamma(0)}^{\gamma(t)} A_{i\mu}(\gamma(t))dx^\mu\right) \qquad (10.14)$$

$$g_i(\gamma(1)) = g_i(1) = \mathcal{P}\exp\left(-\int_0^1 A_{i\mu}\frac{dx^\mu}{dt}dt\right) = \mathcal{P}\exp\left(-\int_{\gamma(0)}^{\gamma(1)} A_{i\mu}(\gamma(t))dx^\mu\right) = \mathcal{P}\exp\left(-\oint A_{i\mu}dx^\mu\right)$$

(71) $$g_\gamma = \mathcal{P}\exp\left(-\oint_\gamma A_{i\mu}dx^\mu\right) \qquad (10.26)$$

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11. Characteristic Classes

e.g. cf. Sec. 10.5, $SU(2)$ bundle over S^4 classified by $\pi_3(SU(2)) \cong \mathbb{Z}$
number $n \in \mathbb{Z}$ tells us how transition functions twist local pieces of the bundle when glued together.
cf. Thm. 10.7, $\pi_3(SU(2))$ evaluated by integrating $\mathrm{tr}F^2 \in H^4(S^4)$ over S^4

11.1. Invariant polynomials and the Chern-Weil homomorphism.

11.1.1. *Invariant polynomials.* $M(k,\mathbb{C})$ - set of complex $k\times k$ matrices

Let $S^r(M(k,\mathbb{C}))$ denote vector space of symmetric r-linear $\mathbb{C}$-valued functions on $M(k,\mathbb{C})$, i.e.
$\widetilde{P} : \bigotimes^r M(k,\mathbb{C}) \to \mathbb{C}$
$\widetilde{P} \in S^k(M(k,\mathbb{C}))$

(72) $$\widetilde{P}(a_1\dots a_i\dots a_j\dots a_r) = \widetilde{P}(a_1\dots a_j\dots a_i\dots a_r) \qquad 1\le i,j\le r \qquad (11.1)$$
where $a_p \in GL(k,\mathbb{C})$

$\widetilde{P} \in S^r(\mathfrak{g})$ invariant if $\forall\, g\in G$, $A_i \in \mathfrak{g}$,

(73) $$\widetilde{P}(\mathrm{Ad}_g A_1 \dots \mathrm{Ad}_g A_r) = \widetilde{P}(A_1\dots A_r) \qquad (11.3)$$

where

$$\mathrm{Ad}_g A_i = g^{-1}A_i g$$

e.g.

(74) $$\widetilde{P}(A_1\dots A_r) = \mathrm{str}(A_1\dots A_r) \equiv \frac{1}{r!}\sum_P \mathrm{tr}(A_{P(1)}\dots A_{P(r)})$$

invariant polynomial P of degree r

(75) $$P(A) \equiv \widetilde{P}(\underbrace{A\dots A}_{r}) \qquad A\in\mathfrak{g}, \quad \widetilde{P}\in I^r(G) \qquad (11.6)$$

Conversely P defines invariant and symmetric r-linear form $\widetilde{P}$ by

$$P(t_1A_1+\dots+t_rA_r)$$

$$\widetilde{P}(t_1A_1+\dots+t_rA_r\dots t_1A_1+\dots+t_rA_r) \Longrightarrow t_1\dots t_r\widetilde{P}(A_1\dots A_r) \text{ term}$$

$\frac{1}{r!}\widetilde{P}(A_1\dots A_r)$ term the polarization of P

In the previous chapter, introduced local gauge potential A, field strength F on a principal bundle.
We have shown these geometrical objects describe the associated vector bundles as well.
Since the set of connections A_i describes the twisting of a fiber bundle, the non-triviality of a principal bundle is equally shared by associated vector bundle
In fact, if (10.57) employed as definition of local connection in a vector bundle, it can be defined even without reference to the principal bundle with which it is originally associated
Later we encounter situations in which use of vector bundles is essential (the Whitney sum bundle, the splitting principle, etc.)

Theorem 13 (11.1)**.** *(Chern-Weil theorem)*
Let invariant polynomial P

(a) $dP(F) = 0$
(b) F, F' *curvature 2-forms corresponding to different connections A, A'. Then $P(F') - P(F)$ exact.*

Proof. (a) Consider invariant polynomial $P_r(F)$ homogeneous of degree r, since any invariant polynomial can be decomposed into homogeneous polynomials

$$\Omega_i = X_i \eta_i$$
$$d\Omega_i = X_i d\eta_i$$

(76) $$\sum_{i=1}^{r} (-1)^{p(p_1 + \dots + p_i)} \widetilde{P}_r(\Omega_1 \dots [\Omega_i, A] \dots \Omega_r) = 0 \qquad (11.12)$$

(b) Let A, A' 2 connections on E, F, F' respective field strength

Define interpolating gauge potential A_t

(77) $$A_t \equiv A + t\theta \qquad \theta \equiv (A' - A) \qquad 0 \leq t \leq 1 \qquad (11.15)$$

$$\begin{aligned} F_1 &= F + D\theta + \theta^2 \\ F_0 &= F \\ F' &= dA' + (A')^2 \\ F &= dA + A^2 \end{aligned} \qquad F_1 - F_0 = D\theta + \theta^2 = d\theta + [A, \theta] + \theta^2 = F' - F$$

$$F' - F = d\theta + (A')^2 - A^2$$
$$D\theta = d\theta + [A, \theta] = d\theta + A \wedge \theta + \theta \wedge A$$
$$A \wedge \theta + \theta \wedge A = A \wedge (A' - A) + (A' - A) \wedge A = A \wedge A' - A^2 + A' \wedge A - A^2$$
$$\theta^2 = (A' - A) \wedge (A' - A) = (A')^2 - A' \wedge A - A \wedge A' + A^2$$

(78) $$P_r(F') - P_r(F) = P_r(F_1) - P_r(F_0) = \int_0^1 dt \frac{d}{dt} P_r(F_t) = r \int_0^1 dt \widetilde{P}_r \left(\frac{dF_t}{dt}, F_t \dots F_t \right) \qquad (11.17)$$

(79) $$\frac{d}{dt} P_r(F_t) = r \widetilde{P}_r(D\theta, F_t \dots F_t) + 2rt \widetilde{P}_r(\theta^2, F_t \dots F_t) \qquad (11.18)$$

Use (11.12) and $\Omega_1 = A = \theta \qquad p_1 = 1$
$\Omega_2 = \dots = \Omega_m = F_t \qquad p_2 = \dots = p_r = 2$

$$\sum_{i=1}^{r} (-1)^{p(p_1 + \dots + p_i)} \widetilde{p}_r(\Omega_1 \dots [\Omega_i, A] \dots \Omega_r) = 0 \Longrightarrow \widetilde{P}_r([\theta, A], F_t \dots F_t) + (r-1) \widetilde{P}_r(\theta, [F_t, \theta], F_t \dots F_t) = 0$$

From (11.18), (11.19) and the previous identity, we obtain

$$\frac{d}{dt} P_r(F_t) = r d[\widetilde{P}_r(\theta, F_t \dots F_t)]$$

$$P_r(F') - P_r(F) = \int_0^1 dt \frac{d}{dt} P_r(F_t) = \int_0^1 dt r d[\widetilde{P}_r(\theta, F_t \dots F_t)]$$

(80) $$P_r(F') - P_r(F) = d \left[r \int_0^1 \widetilde{P}_r(A' - A, F_t \dots F_t) dt \right] \qquad (11.20)$$

$P_r(F')$ differs from $P_r(F)$ by an exact form.

□

transgression $TP_r(A', A)$ of P_r

(81)
$$TP_r(A', A) \equiv r\int_0^1 dt\widetilde{P}_r(A' \dots A, F_t \dots F_t) \qquad (11.21)$$

$\widetilde{P}_r$ is polarization of P.

Let $\dim M = m$
$P_m(F')$ differs from $P_m(F)$ by an exact form, integrals over M without boundary, should be same

(82)
$$\int_M P_m(F') - \int_M P_m(F) = \int_M dTP_m(A', A) = \int_{\partial M} P_m(A', A) = 0 \qquad (11.22)$$

As proved, invariant polynomial closed and in general nontrivial.
Accordingly, defines cohomology class of M

Thm. 11.1(b) (F, F' curvature 2-forms corresponding to different connections A, A', difference $P(F') - P(F)$ is exact) ensures that this cohomology class is independent of the gauge potential chosen.
The cohomology class thus defines is called the **characteristic class**.
The characteristic class defined by an invariant polynomial P is denoted by $\chi_E(P)$ where E is a fiber bundle on which connections and curvatures are defined.

Remark: Since a principal bundle and its associated bundles share the same gauge potentials and field strengths, the Chern-Weil theorem applies equally to both bundles. Accordingly, E can be either a principal bundle or a vector bundle.

11.2. **Chern classes.** `http://www.johno.dk/mathematics/fiberbundlestryk.pdf`

Definition 13 (9.7)**.** characteristic class c *(with* $\mathbb{R}$ *coefficients) for principal* G*-bundle associates to every principal* G*-bundle* (E, π, M) *cohomology class* $c(E) \in H^*_{dR}(M)$ *s.t.* $\forall$ *bundle map* $(\overline{f}, f)$*:*$(E', \pi', M') \to (E, \pi, M)$ *(EY* $\overline{f}$ *is just the induced* f *when you go up to* E *level, vs.* M *to* M' *level) we have*

$$c(E') = f^*(c(E))$$

if $c(E) \in H^l_{dR}(M)$ *then* c *has degree* l

11.3.

11.4.

11.5. **Chern-Simons form.**

11.5.1. *Definition.* Let $P_j(F)$ be an arbitrary $2j$-form characteristic class.
$P_j(F)$ closed, so by Poincaré's lemma, it's locally exact.

(83)
$$P_j(F) = dQ_{2j-1}(A, F) \qquad (11.100)$$

$$Q_{2j-1}(A, F) \in \mathfrak{g} \otimes \Omega^{2j-1}(M)$$

This can't be true globally.
otherwise, if $P_j = dQ_{2j-1}$ globally on manifold M with no boundary,

$$\int_M P_{m/2} = \int_M dQ_{m-1} = \int_{\partial M} Q_{m-1} = 0$$

$m = \dim M$
$2j-1$ from $Q_{2j-1}(A, F)$ is the Chern-Simons form of $P_j(F)$
From Pf. of Thm. 11.2(b),
Q is given by the transgression of P_j,

(84)
$$Q_{2j-1}(A, F) = TP_j(A, 0) = j\int_0^1 \widetilde{P}_j(A, F_t \dots F_t)dt \qquad (11.101)$$

where $\widetilde{P}_j$ polarization of P_j, $F = dA + A^2$
set $A' = F' = 0$

of course $A' = 0$ only on local chart over which bundle is trivial.

Suppose $\dim M = m = 2l$ s.t. $\partial M \neq \emptyset$. By Stoke's

(85) $$\int_M P_l(F) = \int_M dQ_{m-1}(A,F) = \int_{\partial M} Q_{m-1}(A,F) \qquad (11.102)$$

$\int_M P_l(F) \in \mathbb{Z}$ and so does $\int_{\partial M} Q_{m-1}(A,F)$

Thus Q_{m-1} is a characteristic class in its own right and describes the topology of boundary ∂M

11.5.2. *The Chern-Simons form of the Chern character.* Chern character $\text{ch}_j(F)$
connection A_t which interpolates between 0 and A,

(86) $$A_t = tA \qquad (11.103)$$

the corresponding curvature is

(87) $$F_t = A_t^2 + dA_t = tdA + t^2A^2 = tF + (t^2 - t)A^2 \qquad (11.104)$$

(11.21)

(88) $$Q_{2j-1}(A,F) = \frac{1}{(j-1)!}\left(\frac{i}{2\pi}\right)^j \int_0^1 dt \text{str}(A, F_t^{j-1})$$

(89) $$Q_1(A,F) = \frac{i}{2\pi}\int_0^1 dt \text{str}(A, F_t^0) = \frac{i}{2\pi}\text{tr}A \qquad (11.106a)$$

(90) $$\begin{aligned} Q_3(A,F) &= \left(\frac{i}{2\pi}\right)^2 \int_0^1 dt\text{str}(A, F_t) = \left(\frac{i}{2\pi}\right)^2 \int_0^1 dt\text{str}(A, tF + (t^2-t)A^2) = \\ &= \left(\frac{i}{2\pi}\right)^2 \int_0^1 dt\text{str}(A, tdA + tA^2 + (t^2-t)A^2) = \frac{1}{2}\left(\frac{i}{2\pi}\right)^2 \text{tr}(AdA + \frac{2}{3}A^3) \end{aligned}$$

12. Index Theorems

13. Anomalies in Gauge Field Theories

14. Bosonic String Theory