THE ALGEBRAIC GEOMETRY ALGEBRAIC TOPOLOGY DUMP

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Contents

Abstract. Everything about Algebraic Geometry, Algebraic Topology

Part 1. Reading notes on Cox, Little, O'Shea's *Ideals, Varieties, and Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra*

1. Geometry, Algebra, and Algorithms

1.1. Polynomials and Affine Space.
fields are important is that linear algebra works over *any* field

Definition 1 (2). *set of all polynomials in $x_1, \dots, x_n$ with coefficients in k, denoted $k[x_1, \dots, x_n]$*

polynomial f *divides* polynomial g provided $g = fh$ for some $h \in k[x_1, \dots, x_n]$

$k[x_1, \dots, x_n]$ satisfies all field axioms except for existence of multiplicative inverses; commutative ring, $k[x_1, \dots, x_n]$ *polynomial ring*

Exercises for 1. **Exercise 1.** $\mathbb{F}_2$ commutative ring since it's an abelian group under addition, commutative in multiplication, and multiplicative identity exists, namely 1. It is a field since for $1 \neq 0$, the multiplicative identity is 1.

Exercise 2.

(a)
(b)
(c)

1.2. Affine Varieties.

1.3. Parametrizations of Affine Varieties.

1.4. Ideals.

1.5. Polynomials of One Variable.

2. Groebner Bases

2.1. Introduction.

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Part 2. Reading notes on Cox, Little, O'Shea's *Using Algebraic Geometry*

Using Algebraic Geometry. David A. Cox. John Little. Donal O'Shea. Second Edition. Springer. 2005. ISBN 0-387-20706-6 QA564.C6883 2004

7. Introduction

7.1. Polynomials and Ideals.
monomial

(1)
$$x_1^{\alpha_1} \dots x_n^{\alpha_n} \quad (1.1)$$

total degree of x^α is $\alpha_1 + \dots + \alpha_n \equiv |\alpha|$

field k, $k[x_1 \dots x_n]$ collection of all polynomials in $x_1 \dots x_n$ with coefficients k.

polynomials in $k[x_1 \dots x_n]$ can be added and multiplied as usual, so $k[x_1 \dots x_n]$ has structure of commutative ring (with identity)
however, only nonzero constant polynomials have multiplicative inverses in $k[x_1 \dots x_n]$, so $k[x_1 \dots x_n]$ not a field
however set of rational functions $\{f/g | f, g \in k[x_1 \dots x_n], \, g \neq 0\}$ is a field, denoted $k(x_1 \dots x_n)$

so

$$f = \sum_\alpha c_\alpha x^\alpha$$

where $c_\alpha \in k$
so

$$f \in k[x_1 \dots x_n] = \{f | f = \sum_\alpha c_\alpha x^\alpha, x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}, c_\alpha \in k\}$$

f homogeneous if all monomials have same total degrees
polynomial f is homogeneous if all monomials have the *same total degree*

Given a collection of polynomials $f_1 \dots f_s \in k[x_1 \dots x_n]$, we can consider all polynomials which can be built up from these by multiplication by arbitrary polynomials and by taking sums

Definition 2 (1.3). *Let $f_1 \dots f_s \in k[x_1 \dots x_n]$
Let $\langle f_1 \dots f_s \rangle = \{p_1 f_1 + \dots + p_s f_s | p_i \in k[x_1 \dots x_n]$ for $i = 1 \dots s\}$*

Exercise 1.
(a) $x^2 = x \cdot (x - y^2) + y \cdot (xy)$
(b)

$$p \cdot (x - y^2) = px - py^2$$

and for $pxy = (py)x$
(c)

$$p(y)(x - y^2) = p(y)x - p(y)y^2 \notin \langle x^2, xy \rangle$$

Exercise 2.

$$\sum_{i=1}^s p_i f_i + \sum_{j=1}^s q_j f_j = \sum_{i=1}^s (p_i + q_i) f_i, \quad p_i + q_i \in k[x_1 \dots x_n]$$

$\langle f_1 \dots f_s \rangle$ closed under sums in $k[x_1 \dots x_n]$

If $f \in \langle f_1 \dots f_s \rangle$,
$p \in k[x_1 \dots x_n]$

$$p \cdot f = p \sum_{i=1}^s q_j f_j = \sum_{i=1}^s p q_j f_j, \quad p q_j \in k[x_1 \dots x_n] \text{ so}$$
$$p \cdot f \in \langle f_1 \dots f_s \rangle$$

Done.

The 2 properties in Ex. 2 are defining properties of ideals in the ring $k[x_1 \dots x_n]$

Definition 3 (1.5). *Let $I \subset k[x_1 \dots x_n]$, $\quad I \neq \emptyset$
I ideal if*
(a) $f + g \in I, \quad \forall f, g \in I$
(b) $pf \in I, \quad \forall f \in I$, *arbitrary* $p \in k[x_1 \dots x_n]$

Thus $\langle f_1 \dots f_s \rangle$ is an ideal by Ex. 2.

we call it the ideal generated by $f_1 \dots f_s$.

Exercise 3. Suppose $\exists$ ideal J, $f_1 \dots f_s \in J$ s.t. $J \subset \langle f_1 \dots f_s \rangle$
if $f \in \langle f_1 \dots f_s \rangle$, $f = \sum_{i=1}^s p_i f_i, \quad p_i \in k[x_1 \dots x_n]$

$\forall i = 1 \dots s$, $p_i f_i \in J$ and so $\sum_{i=1}^s p_i f_i \in J$, by def. of J as an ideal.

$$\langle f_1 \dots f_s \rangle \subseteq J \qquad \Longrightarrow J = \langle f_1 \dots f_s \rangle$$

$\Longrightarrow \langle f_1 \dots f_s \rangle$ is smallest ideal in $k[x_1 \dots x_n]$ containing $f_1 \dots f_s$

Exercise 4. For $I = \langle f_1 \dots f_s \rangle$
$J = \langle g_1 \dots g_t \rangle$

$I = J$ iff $s = t$ and $\forall f \in I$, $f = \sum_{i=1}^t q_i g_i$ and if $0 = \sum_{i=1}^t q_i g_i$, $q_i = 0$, $\forall i = 1 \dots t$, and if $0 = \sum_{i=1}^s p_i f_i$, $p_i = 0$, $\forall i = 1 \dots s$

Definition 4 (1.6).

$$\sqrt{I} = \{g \in k[x_1 \dots x_n] | g^m \in I \textit{ for some } m \geq 1\}$$

e.g. $x + y \in \sqrt{\langle x^2 + 3xy, 3xy + y^2 \rangle}$
in $\mathbb{Q}[x,y]$ since

$$(x+y)^3 = x(x^2 + 3xy) + y(3xy + y^2) \in \langle x^2 + 3xy, 3xy + y^2 \rangle$$

- (Radical Ideal Property) $\forall$ ideal $I \subset k[x_1 \dots x_n]$, $\sqrt{I}$ ideal, $\sqrt{I} \supset I$
- **(Hilbert basis Thm.)** $\forall$ ideal $I \subset k[x_1 \dots x_n]$
  $\exists$ finite generating set,
  i.e. $\exists \{f_1 \dots f_2\} \subset k[x_1 \dots x_n]$ s.t. $I = \langle f_1 \dots f_s \rangle$
- (Division Algorithm in $k[x]$) $\forall f, g \in k[x]$ (EY : in 1 variable)
  $\forall f, g \in k[x]$ (in 1 variable)
  $f = qg + r$, $\exists!$ quotient q, $\exists$ remainder r

7.2.

7.3. **Gröbner Bases.**

Definition 5 (3.1). *Gröbner basis for* $I \equiv G = \{g_1 \dots g_k\} \subset I$ *s.t.* $\forall f \in I$, $LT(f)$ *divisible by* $LT(g_i)$ *for some* i

- (Uniqueness of Remainders) let ideal $I \subset k[x_1 \dots x_n]$
  division of $f \in k[x_1 \dots x_n]$ by Grö bner basis for I, produces $f = g + r$, $g \in I$, and no term in r divisible by any element of $\text{LT}(I)$

7.4. **Affine Varieties.** affine n-dim. space over k $\quad k^n = \{(a_1 \dots a_n) | a_1 \dots a_n \in k\}$
$\forall$ polynomial $f \in k[x_1 \dots x_n]$, $(a_1 \dots a_n) \in k^n$
$f : k^n \to k$
$f(a_1 \dots a_n)$ s.t. $x_i = a_i$ i.e.

if $f = \sum_\alpha c_\alpha x^\alpha$ for $c_\alpha \in k$, then
$f(a_1 \dots a_n) = \sum_\alpha c_\alpha a^\alpha \in k$, where $a^\alpha = a_1^{\alpha_1} \dots a_n^{\alpha_n}$

Definition 6 (4.1). *affine variety* $\mathbf{V}(f_1 \dots f_s) = \{(a_1 \dots a_n) | (a_1 \dots a_n) \in k^n, f_1(x_1 \dots x_n) = \dots = f_s(x_1 \dots x_n) = 0\}$ *subset* $V \subset k^n$ *is affine variety if* $V = V(f_1 \dots f_s)$ *for some* $\{f_i\}$, *polynomial* $f_i \in k[x_1 \dots x_n]$

- (Equal Ideals Have Equal Varieties) If $\langle f_1 \dots f_s \rangle = \langle g_1 \dots g_t \rangle$ in $k[x_1 \dots x_n]$, then $\mathbf{V}(f_1 \dots f_s) = \mathbf{V}(g_1 \dots g_t)$

so, recap
if $\langle f_1 \dots f_s \rangle = \langle g_1 \dots g_t \rangle$ in $k[x_1 \dots x_n]$,
then $V(f_1 \dots f_s) = V(g_1 \dots g_t)$

Recall Hilbert basis Thm. $\forall$ ideal $I \subset k[x_1 \dots x_n]$

$$I = \langle f_1 \dots f_s \rangle$$

$\Longrightarrow$ if $I = J$, then $V(I) = V(J)$
think of V defined by I, rather than $f_1 = \dots = f_s = 0$

Exercise 3.

Recall Def. 1.5 Let $I \subset k[x_1 \dots x_n]$

I ideal if $f + g \in I \quad \forall f, g \in I$
$pf \in I, \quad \forall f \in I$ arbitrary $p \in k[x_1 \dots x_n]$
Let $f, g \in I(V)$

$$(f+g)(a_1 \dots a_n) = f(a_1 \dots a_n) + g(a_1 \dots a_n) = 0 + 0 = 0 \qquad f + g \in I(V)$$
$$pf(a_1 \dots a_n) = p(a_1 \dots a_n) f(a_1 \dots a_n) = 0 \qquad pf \in I(V)$$

Then $I(V)$ an ideal.
$V = V(x^2)$ in $\mathbb{R}^2$
$I = \langle x^2 \rangle$ in $\mathbb{R}[x,y]$, $\quad I = \{px^2 | p \in k[x,y]\}$
$I \subset I(V)$, since $px^2 = 0$ for $x^2 = 0$, $(0, b)$, $\quad b \in \mathbb{R}$
But $p(x,y) = x \in I(V)$, as

$$I(V) = \{f \in k[x_1 \dots x_n] | f(a_1 \dots a_n) = 0, \forall (a_1 \dots a_n) \in V\}$$

$p(0,b) = x = 0$
But $x \notin I$

Exercise 4. $I \subset \sqrt{I}$
Recall Def. 1.6 $\sqrt{I} = \{g \in k[x_1 \dots x_n] | g^m \in I \text{ for some } m \geq 1\}$
$\forall f \in I$, $f = f^1$, $m = 1$, so $f \in \sqrt{I}$, $\quad I \subset \sqrt{I}$
Hilbert basis thm., $\forall$ ideal $I \subset k[x_1 \dots x_n]$ s.t. $I = \langle f_1 \dots f_s \rangle$
$\left\{ {}^{V(I) = \{(a_1 \dots a_n) | (a_1 \dots a_n) \in k^n, f_1(a_1 \dots a_n) = \dots = f_s(a_1 \dots a_n) = 0} \right\}$
$\mathbf{I}(\mathbf{V}(I)) = \{f \in k[x_1 \dots x_n] | f(a_1 \dots a_n) = 0 \quad \forall (a_1 \dots a_n) \in V(I)\}$
Let $g \in \sqrt{I}$, $\quad g^m \in I$, $\quad g^m = g^{m-1} g$
$g^m(a_1 \dots a_n) = 0 = g^{m-1}(a_1 \dots a_n) g(a_1 \dots a_n) = 0$. Then $g(a_1 \dots a_n) = 0$ or $g^{m-1}(a_1 \dots a_m) = 0$
as $g^m \in I$, and $V(I)$ is s.t. $f_1(a_1 \dots a_n) = \dots = f_s(a_1 \dots a_n) = 0$ for $I = \langle f_1 \dots f_s \rangle$

- (Strong Nullstellensatz) if k algebraically closed (e.g. $\mathbb{C}$), I ideal in $k[x_1 \dots x_n]$, then
$$\mathbf{I}(\mathbf{V}(I) = \sqrt{I}$$
- (Ideal-variety correspondence) Let k arbitrary field
$$I \subset I(V(I))$$
$$V(I(V)) = V \quad \forall V$$

Additional Exercises for Sec.4. **Exercise 6.**

8. Solving Polynomial Equations

8.1.

8.2. **Finite-Dimensional Algebras.** Gröbner basis $G = \{g_1 \dots g_t\}$ of ideal $I \subset k[x_1 \dots x_n]$,
recall def.: Gröbner basis $G = \{g_1 \dots g_t\} \subset I$ of ideal I, $\quad \forall f \in I$, $\text{LT}(f)$ divisible by $\text{LT}(g_i)$ for some i
$f \in k[x_1 \dots x_n]$ divide by G produces $f = g + r$, $g \in I$, r not divisible by any $\text{LT}(I)$ uniqueness of r
$f \in k[x_1 \dots x_n]$ divide by G,
Recall from Ch. 1, divide $f \in k[x_1 \dots x_n]$ by G, the division algorithm yields

$$f = h_1 g_1 + \dots + h_t g_t + \overline{f}^G \qquad (2) \quad (2.1)$$

where remainder $\overline{f}^G$ is a linear combination of monomials $x^\alpha \notin \langle \text{LT}(I) \rangle$
since Gröbner basis, $f \in I$ iff $\overline{f}^G = 0$
$\forall f \in k[x_1 \dots x_n]$, we have coset $[f] = f + I = \{f + h | h \in I\}$ s.t. $[f] = [g]$ iff $f - g \in I$

We have a 1-to-1 correspondence

$$\text{remainders} \leftrightarrow \text{cosets}$$
$$\overline{f}^G \leftrightarrow [f]$$

algebraic

$$\overline{f}^G + \overline{g}^G \leftrightarrow [f] + [g]$$
$$\overline{\overline{f}^G \cdot \overline{g}^G} \leftrightarrow [f] \cdot [g]$$

$B = \{ x^\alpha | x^\alpha \notin \langle \text{LT}(I) \rangle \}$ is a basis of A, basis monomials, standard monomials

20141023 EY's take

$\forall [f] \in A = k[x_1 \dots x_n]/I, \quad [f] = p_i b_i; \quad b_i \in B = \{ x^\alpha | x^\alpha \notin \langle \text{LT}(I) \rangle \}$

For $I = \langle G \rangle$

e.g. $G = \{ x^2 + \frac{3}{2}xy + \frac{1}{2}y^2 - \frac{3}{2}x - \frac{3}{2}y, xy^2 - x, y^3 - y \}$

$\langle \text{LT}(I) \rangle = \langle x^2, xy^2, y^3 \rangle$

e.g. $B = \{ 1, x, y, xy, y^2 \}$

$[f] \cdot [g] = [fg]$

e.g. $f = x$, $g = xy$, $[fg] = [x^2 y]$

now $f = h_1 g_1 + \dots + h_t g_t + \overline{f}^G$

8.3.

8.4. **Solving Equations via Eigenvalues and Eigenvectors.**

9. Resultants

10. Computation in Local Rings

10.1. **Local Rings.**

Definition 7 (1.1)**.**

$$k[x_1 \dots x_n]_{\langle x_1 \dots x_n \rangle} \equiv \{ \frac{f}{g} | \textit{ rational functions } \frac{f}{g} \textit{ of } x_1 \dots x_n \textit{ with } g(p) \neq 0 \textit{ at } p \}$$

main properties of $k[x_1 \dots x_n]_{\langle x_1 \dots x_n \rangle}$

Proposition 1 (1.2)**.** *Let* $R = k[x_1 \dots x_n]_{\langle x_1 \dots x_n \rangle}$. *Then*

(a) R *subring of field of rational functions* $k(x_1 \dots x_n) \supset k[x_1 \dots x_n]$

(b) *Let* $M = \langle x_1 \dots x_n \rangle \subset R$ *(ideal generated by* $x_1 \dots X_n$ *in* R*)*

Then $\forall \frac{f}{g} \in R \backslash M$, $\frac{f}{g}$ *unit in* R *(i.e. multiplicative inverse in* R*)*

(c) M *maximal ideal in* R

Exercise 1. if $p = (a_1 \dots a_n) \in k^n$, $R = \{ \frac{f}{g} | f, g \in k[x_1 \dots x_n], g(p) \neq 0 \}$

(a) R subring of field of rational functions $k(x_1 \dots x_n)$

(b) Let M ideal generated by $x_1 - a_1 \dots x_n - a_n$ in R

Then $\forall \frac{f}{g} \in R \backslash M$, $\frac{f}{g}$ unit in R (i.e. multiplicative inverse in R)

(c) M maximal ideal in R

Proof. let $p = (a_1 \dots a_n) \in k^n$

let $g_1(p) \neq 0$, $g_2(p) \neq 0$

$$\frac{f_1}{g_1} + \frac{f_2}{g_2} = \frac{f_1 g_2 + f_2 g_1}{g_1 g_2} \qquad g_1(p) g_2(p) \neq 0 \text{ so } \frac{f_1}{g_1} + \frac{f_2}{g_2} \in R$$
$$\frac{f_1}{g_1} \cdot \frac{f_2}{g_2} = \frac{f_1 f_2}{g_1 g_2} \qquad g_1(p) g_2(p) \neq 0 \text{ so } \frac{f_1}{g_1} \frac{f_2}{g_2} \in R$$

$f = \frac{f}{1} \in R, \quad \forall f \in k[x_1 \dots x_n]$, so $k[x_1 \dots x_n] \subset R$ □

EY : 20141027, to recap,

Let $V = k^n$

Let $p = (a_1 \dots a_n)$

single pt. $\{p\}$ is (an example of) a variety

$I(\{p\}) = \{ x_1 - a_1 \dots x_n - a_n \rangle \subset k[x_1 \dots x_n]$

$R \equiv k[x_1 \dots x_n]_{\langle x_1 - a_1 \dots x_n - a_n \rangle}$

$$R = \{ \frac{f}{g} | \text{ rational function } \frac{f}{g} \text{ of } x_1 \dots x_n, \, g(p) \neq 0, \, p = (a_1 \dots a_n) \, \}$$

Prop. 1.2. properties

(a) R subring of field of rational functions $k(x_1 \dots x_n) \qquad k(x_1 \dots x_n) \subset R$

(b) $M = \langle x_1 \dots a_1 \dots x_n - a_n \rangle \subset R$. ideal generated by $x_1 - a_1 \dots x_n - a_n$

Then $\forall \frac{f}{g} \in R \backslash M$, $\frac{f}{g}$ unit in R ($\exists$ multiplicative inverse in R)

(c) M maximal ideal in R.

in R we allow denominators that are not elements of this ideal $I(\{p\})$

Definition 8 (1.3)**.** *local ring is a ring that has exactly 1 maximal ideal*

Proposition 2 (1.4)**.** *ring* R *with proper ideal* $M \subset R$ *is local ring if* $\forall \frac{f}{g} \in R \backslash M$ *is unit in* R

localization Ex. 8, Ex. 9

parametrization

Exercise 2.

$$x = x(t) = \frac{-2t^2}{1 + t^2}$$
$$y = y(t) = \frac{2t}{1 + t^2}$$

$k[t]_{\langle t \rangle} \qquad \frac{-2t^2}{1+t^2}$ rational function of t. $1 + t^2 \neq 0$

if $k = \mathbb{C}$ or $\mathbb{R}$

Consider set of convergent power series in n variables

$$k\{x_1 \dots x_n\} = \{ \sum_{\alpha \in \mathbb{Z}^n_{\geq 0}} c_\alpha x^\alpha | c_\alpha \in k, \text{ series converges in some open } U \ni 0 \in k^n \, \} \tag{3}$$
(1.5)

Consider set $k[[x_1 \dots x_n]]$ of formal power series

$$k[[x_1 \dots x_n]] = \{ \sum_{\alpha \in \mathbb{Z}^n_{\geq 0}} c_\alpha x^\alpha | c_\alpha \in k \} \text{ series need not converge} \tag{4}$$
(1.6)

variety V

$k[x_1 \dots x_n]/\mathbf{I}(V) \qquad$ variety V

10.2. **Multiplicities and Milnor Numbers.** if I ideal in $k[x_1 \dots x_n]$, then denote $Ik[x_1 \dots x_n]_{\langle x_1 \dots x_n \rangle}$ ideal generated by I in larger ring $k[x_1 \dots x_n]_{\langle x_1 \dots x_n \rangle}$

Definition 9 (2.1)**.** *Let I 0-dim. ideal in $k[x_1 \dots x_n]$, so $V(I)$ consists of finitely many pts. in k^n.*
Assume $(0 \dots 0) \in V(I)$
multiplicity of $(0 \dots 0) \in V(I)$ is

$$dim_k k[x_1 \dots x_n]_{\langle x_1 \dots x_n \rangle} / Ik[x_! \dots x_n]_{\langle x_1 \dots x_n \rangle}$$

generally, if $p = (a_1 \dots a_n) \in V(I)$
multiplicity of p, $m(p) = \dim k[x_1 \dots x_n]_M / Ik[x_1 \dots x_n]_M$

$$\dim k[x_1 \dots x_n]_M / Ik[x_1 \dots x_n]_M$$

localizing $k[x_1 \dots x_n]$ at maximal ideal $M = I(\{p\}) = \langle x_1 - a_1 \dots x_n - a_n \rangle$

11.

12.

13. Polytopes, Resultants, and Equations

14. Polyhedral Regions and Polynomials

14.1. **Integer Programming.** Prop. 1.12.

Suppose 2 customers A, B ship to same location
A: ship 400 kg pallet taking up $2\, m^3$ volume
B: ship 500 kg pallet taking up $3\, m^3$ volume

shipping firm trucks carry up to 3700 kg, up to $20\, m^3$

B's product more perishable, paying \$ 15 per pallet

A pays \$ 11 per pallet
How many pallets from A, B each in truck to maximize revenues?

(5) $$\begin{aligned} & 4A + 5B \leq 37 \\ (1.1) \quad & 2A + 3B \leq 20 \\ & A, B \in \mathbb{Z}^*_{\geq 0} \end{aligned}$$

maximize $11A + 15B$

integer programming.
max. or min. value of some linear function

$$l(A_1 \dots A_n) = \sum_{i=1}^n c_i A_i$$

on set $(A_1 \dots A_n) \in \mathbb{Z}^n_{\geq 0}$ s.t.
3. Finally, by introducing additional variables; rewrite linear constraint inequalities as equalities. The new variables are called "slack variables"

(6) $$(1.4) \qquad a_{ij} A_j = b_i, \quad A_j \in \mathbb{Z}_{\geq 0}$$

introduce indeterminate z_i, $\forall$ equation in (1.4)

$$z_i^{a_{ij} A_j} = z_i^{b_i}$$

m constraints

$$\prod_{i=1}^m z_i^{a_{ij} A_j} = \prod_{i=1}^m z_i^{b_i} = \left(\prod_{i=1}^m z_i^{a_{ij}} \right)^{A_j}$$

Proposition 3 (1.6)**.** *Let k field, define $\varphi : k[w_1 \dots w_n] \to k[z_1 \dots z_m]$ by*

$$\varphi(w_j) = \prod_{i=1}^m z_i^{a_{ij}} \qquad \forall\, j = 1 \dots n$$

and

$$\varphi(g(w_1 \dots w_n)) = g(\varphi(w_1) \dots \varphi(w_n))$$

$\forall$ *general polynomial $g \in k[w_1 \dots w_n]$*
Then $(A_1 \dots A_n)$ integer pt. in feasible region iff $\varphi : w_1^{A_1} \dots w_n^{A_n} \mapsto z_1^{b_1} \dots z_m^{b_m}$

Exercise 3.
Now

$$\varphi(w_j) = \prod_{i=1}^m z_i^{a_{ij}}$$
$$z_i^{a_{ij} A_j} = z_i^{b_i}$$

If $(A_1 \dots A_n)$ an integer pt. in feasible region, $a_{ij} A_j = b_i$

$$z_i^{a_{ij} A_j} = z_i^{b_i} = \prod_{j=1}^n z_i^{a_{ij} A_j} \Longrightarrow \prod_{j=1}^n \prod_{i=1}^m (z_i^{a_{ij}})^{A_j} = \prod_{i=1}^m z_i^{b_i} = \prod_{j=1}^n \varphi(w_j)^{A_j} = \prod_{j=1}^n \varphi(w_j)^{A_j} = \varphi \left(\prod_{j=1}^n w_j^{A_j} \right) = \prod_{i=1}^m z_i^{b_i}$$

since $\varphi(g(w_1 \dots w_n)) = g(\varphi(w_1) \dots \varphi(w_n))$

If $\varphi : \prod_{j=1}^n w_j^{A_j} \mapsto \prod_{i=1}^m z_i^{b_i}$

$$\varphi \left(\prod_{j=1}^n w_j^{A_j} \right) = \prod_{j=1}^n (\varphi(w_j))^{A_j} = \prod_{i=1}^m z_i^{b_i} = \prod_{j=1}^n \left(\prod_{i=1}^m z_i^{a_{ij}} \right)^{A_j} \Longrightarrow \prod_{j=1}^n z_i^{a_{ij} A_j} = z_i^{b_i}$$

or $a_{ij} A_j = b_i$. So $(A_1 \dots A_n)$ integer pt.

Exercise 4.

$$\prod_{i=1}^m z_i^{b_i} = \prod_{i=1}^m \prod_{j=1}^n z_i^{a_{ij} A_j} = \prod_{j=1}^n \left(\prod_{i=1}^m z_i^{a_{ij}} \right)^{A_j} = \prod_{j=1}^n \varphi(w_j)^{A_j} = \varphi \left(\prod_{j=1}^n w_j^{A_j} \right)$$

So if given $(b_1 \dots b_m) \in \mathbb{Z}^m$, and for a given a_{ij}, $a_{ij} A_j = b_i$

For $m \leq n$, then a_{ij} is surjective, so $\exists\, A_j$ s.t. $\prod_{i=1}^m z_i^{b_i} = \varphi \left(\prod_{j=1}^n w_j^{A_j} \right)$

Proposition 4 (1.8)**.** *Suppose $f_1 \dots f_n \in k[z_1 \dots z_m]$ given*
Fix monomial order in $k[z_1 \dots z_n, w_1 \dots w_n]$ with elimination property:
$\forall$ monomial containing 1 of z_i greater than any monomial containing only w_j

Let $\mathcal{G}$ Gröbner basis for ideal
$$I = \langle f_1 - w_1 \dots f_n - w_n \rangle \subset k[z_1 \dots z_m, w_1 \dots w_n]$$
$\forall f \in k[z_1 \dots z_m]$, let $\overline{f}^{\mathcal{G}}$ be remainder on division of f by $\mathcal{G}$
Then

(a) *polynomial f s.t. $f \in k[f_1 \dots f_n]$ iff $g = \overline{f}^{\mathcal{G}} \in k[w_1 \dots w_n]$*

(b) *if $f \in k[f_1 \dots f_n]$ as in part (a),*
$g = \overline{f}^{\mathcal{G}} \in k[w_1 \dots w_n]$
then $f = g(f_1 \dots f_n)$, giving an expression for f as polynomial in f_j

(c) *if $\forall f_i, f$ monomials, $f \in k[f_1 \dots f_n]$,*
then g also a monomial.

14.2. Integer Programming and Combinatorics.

15. Algebraic Coding Theory

16. The Berlekamp-Massey-Sakata Decoding Algorithm

Gröbner Bases, Martin R. Albrecht of the DTU Crypto Group

Part 3. Algebraic Topology

cf. Bredon (1997) [3]

17. Simplicial Complexes

cf. pp. 245, from Sec. 21 Simplicial Complexes of Ch. 4 Homology Theory in Bredon (1997) [3]
$\mathbf{v}_0, \dots \mathbf{v}_n \in \mathbb{R}^\infty$, "affinely independent" if they span an affine n-plane, i.e.

$$\text{if } \left(\sum_{i=0}^n \lambda_i \mathbf{v}_i = 0, \sum_{i=0}^n \lambda_i = 0 \right), \text{ then } \implies \forall \lambda_i = 0$$

If not, then, e.g. $\lambda_0 \neq 0$, assume $\lambda_0 = -1$, and solve the equations to get

$$\mathbf{v}_0 = \sum_{i=1}^n \lambda_i \mathbf{v}_i$$
$$\sum_{i=1}^n \lambda_i = 1$$

i.e. $\mathbf{v}_0$ is in affine space spanned by $\mathbf{v}_1 \dots \mathbf{v}_n$.
If $\mathbf{v}_0, \dots \mathbf{v}_n$ affinely independent, then

$$\sigma = (\mathbf{v}_0, \dots \mathbf{v}_n) = \{ \sum_{i=0}^n \lambda_i \mathbf{v}_i | \sum_{i=0}^n \lambda_i = 1, \lambda_i \geq 0 \} \tag{7}$$

is "affine simplex" spanned by $\mathbf{v}_i$; also convex hull of $\mathbf{v}_i$.
$\forall k \leq n$, k-face of σ is any affine simplex of form $(\mathbf{v}_{i_1}, \dots \mathbf{v}_{i_k})$, where vertices all distinct, so are affinely independent.

Definition 10. *(geometric) simplicial complex $K :=$ collection of affine simplices s.t.*

(1) *$\sigma \in K \implies$ any face of $\sigma \in K$; and*
(2) *$\sigma, \tau \in K \implies \sigma \bigcap \tau$ is a face of both σ and τ, or $\sigma \bigcap \tau = \emptyset$*

If K simplicial complex, $|K| = \bigcup \{ \sigma | \sigma \in K \} \equiv$ "polyhedron" of K

Definition 11 (Def. 21.2 of Bredon (1997) [3])**.** *polyhedron := space X if $\exists$ homeomorphism $h : |K| \xrightarrow{\approx} X$ for some simplicial complex K. h, K is triangulation of X; (map h, complex K)*

Let K finite simplicial complex.
Choose ordering of vertices $\mathbf{v}_0, \mathbf{v}_1 \dots$ of K.
If $\sigma = (\mathbf{v}_{\sigma_0}, \dots \mathbf{v}_{\sigma_n})$ is simplex of K, where $\sigma_0 < \dots < \sigma_n$, then
let $f_\sigma : \Delta_n \to |K|$ be
$$f_\sigma = [\mathbf{v}_{\sigma_b}, \dots \mathbf{v}_{\sigma_n}]$$
in notation of Def. 1.2. Bredon (1997) [3].
Then this gives CW-complex structure on $|K|$ with f_σ as characteristic maps.

Part 4. Graphs, Finite Graphs

18. Graphs, Finite Graphs, Trees

Serre (1980) [4]
cf. Chapter I. Trees and Amalgams, Section 1 Amalgams, Subsection 1.1 Direct limits of Serre (1980) [4]
Let $(G_i)_{i \in I}$, family of groups.
$\forall$ pair (i, j), let $F_{ij} =$ set of homomorphisms of G_i into G_j
Want: group $G = \varinjlim G_i$ and
$$\{ f_i | f_i : G_i \to G \} \text{s.t. } f_j \circ f = f_i \quad \forall f \in F_{ij}$$
group G and family $\{ f_i \}$ universal in that
(*) if H group, if $\{ h_i | h_i : G_i \to H; h_j \circ f = h_i \quad \forall f \in F_{ij} \}$,
then $\exists ! h : G \to H$ s.t. $h_i = h \circ f_i$
i.e. $\text{Hom}(G, H) \simeq \varprojlim \text{Hom}(G_i, H)$, the inverse limit being taken relative to F_{ij}.
i.e. G direct limit of G_i relative to the F_{ij}.

Proposition 5. *$\exists !$ pair G, family $(f_i)_{i \in I}$, i.e. (pair consisting of $G, (f_i)_{i \in I}$, unique up to unique isomorphism.*

Proof. Define G by generators and relations.
Take generating family to be disjoint union of those for G_i.
relations - xyz^{-1} where $x, y, z \in G_i$, $z = xy \in G_i$
xy^{-1} where $x \in G_i$, $y \in G_j$, $y = f(x)$ for at least $f \in F_{ij}$.
Thus, existence of $G, \{ f_i \}$.
G represents functor $H \mapsto \varprojlim \text{Hom}(G_i, H)$.
Thus, uniqueness (also from universal property). □

e.g. groups A, G_1, G_2, homomorphisms $f_1 : A \to G_1$.
$f_2 : A \to G_2$
G obtained by amalgamating A in G_1, G_2 by $f_1, f_2 \equiv G_1 *_A G_2$.
1 can have $G = \{1\}$, even though f_1, f_2 non-trivial.
Application: (Van Kampen Thm.)
Let topological space X be covered by open U_1, U_2.
Suppose $U_1, U_2, U_{12} = U_1 \bigcap U_2$ arcwise connected.
Let basept. $x \in U_{12}$.
Then $\pi_1(X; x)$ obtained by taking 3 groups
$$\pi_1(U_1; x), \pi_1(U_2; x), \pi_1(U_{12}; x)$$

and amalagamating them according to homomorphism

$$\pi_1(U_{12};x) \to \pi_1(U_1;x)$$
$$\pi_1(U_{12};x) \to \pi_1(U_2;x)$$

Exercise 1. Let homomorphisms $f_1 : A \to G_1$ amalgam $G = G_1 *_A G_2$.
$f_2 : A \to G_2$

Define subgroups A^n, G_1^n, G_2^n, of A, G_1, G_2 recursively by

$$A^1 = \{1\}$$
$$G_1^1 = \{1\}$$
$$G_2^1 = \{1\}$$

A^n = subgroup of A generated by $f_1^{-1}(G_1^{n-1})$ and $f_2^{-1}(G_2^{n-1})$

$$G_1^n = \text{subgroup of } G_i \text{ generated by } f_i(A^n)$$

Let A^∞, G_i^∞ be unions of A^n, G_i^n resp.

Show that f_i defines injection $A/A^\infty \to G_i/G_i^\infty$.

So the amalgamation is $G \simeq G_1/G_1^\infty *_{A/A^\infty} G_2/G_2^\infty$.

Take the first induction case (for intuition about the solution).

$$A^2 = \langle f_1^{-1}(G_1^1), f_2^{-1}(G_2^1)\rangle = \langle f_1^{-1}(\{1\}), f_2^{-1}(\{1\})\rangle$$
$$G_i^2 = f_i(A^2)$$

Let $f_i(a) = f_i(b) \in G_i/G_i^\infty$; $a, b \in A/A^\infty$.

Then since $f_i(a), f_i(b) \in G_i/G_i^\infty$, $f_i(a), f_i(b) \in \{gG_i^\infty | g \in G_i\}$ (quotient is defined to be the set of all left cosets of G_i^∞, which has to be a normal subgroup for G_i/G_i^∞ to be a quotient group).

Since $a, b \in A/A^\infty$, suppose we take $a, b \in A$.

And suppose we take

$$f_i(a) = f_i(a)G_i^\infty = f_i(a)f_i(A^{n_a}) = f_i(aA^{n_a})$$
$$f_i(b) = f_i(b)G_i^\infty = f_i(b)f_i(A^{n_b}) = f_i(bA^{n_b})$$

Taking f_i^{-1} (recall for group homomorphisms, they map inverse of element of 1st. group to inverse of image of this element).

$aA^{n_a} = bA^{n_b} \in A/A^\infty$ (This is okay as we've "quotiented out A^∞; so indeed, they're equal)

cf. Subsection 1.2 Structure of amalgams of Serre (1980) [4]

Suppose given group A, family of groups $(G_i)_{i\in I}$, and, $\forall\, i \in I$, injective homomorphism $A \to G_i$.

$*_A G_i \equiv$ direct limit (cf. no. 1.1) of family (A, G_i) with respect to these homomorphisms, call it *sum* (in category theory sense, i.e. product) of G_i with A amalgamated.

e.g. $A = \{1\}$,

$*G_i \equiv$ free product of G_i.

18.0.1. *reduced word.* $\forall\, i \in I$, choose set S_i of right coset representations of G_i modulo A,

assume $1 \in S_i$,

$(a, s) \mapsto as$ is bijection of $A \times S_i$ onto G_i,

$A \times (S_i - \{1\}) \to G_i - A$ (onto)

Let $\mathbf{i} = (i_1 \dots i_n)$, $n \geq 0$, $i_j \in I$, s.t.

$$i_m \neq i_{m+1} \text{ for } 1 \leq m \leq n-1 \tag{8}$$

cf. (T) of Serre (1980) [4].

So *reduced word* m is defined as

$$m = (a; s_1 \dots s_n)$$

where $a \in A, s_1 \in S_{i_1} \dots s_n \in S_{i_n}$, and $s - j \neq 1\, \forall\, j$.

$f \equiv$ canonical homomorphism of A into group $G = *_A G_i$

$f_i \equiv$ canonical homomorphism of G_i into group $G = *_A G_i$

EY : 20170611 (Further explanations, basic examples, from me):

Given $A, \{G_i\}_{i\in I}$, injective (group) homomorphisms $\{f_i : A \to G_i\}_i$.

$G_i \backslash f_i(A) = \{f_i(A)g | g \in G_i\}$.

Right coset representation of $f_i(A)g \mapsto g$.

e.g. A, G_1, G_2, $f_1 : A \to G_1$.
$f_2 : A \to G_2$

$$G_1 \backslash f_1(A) = \{f_1(A)g | g \in G_1\}$$
$$G_2 \backslash f_2(A) = \{f_2(A)g | g \in G_2\}$$

$\mathbf{i} = (i_1 \dots i_n)$, $i_j \in I$, $i_m \neq i_{m+1}$ for $1 \leq m \leq n-1$.

Consider $(1212 \dots 12)$

$m = (a; f_1 g_2 f_3 g_4 \dots f_{2n-1}, g_{2n})$ where f's $\in S_1 \subset G_1$, g's $\in S_2 \subset G_2$.

and so

Definition 12 (reduced word)**.** ***reduced word*** *of type* $\mathbf{i}$, m,

$$m = (a; s_1 \dots s_n) \tag{9}$$

where $a \in A, s_1 \in S_{i_1}, \dots s_n \in S_{i_n}$, $s_j \neq 1 \quad \forall\, j$,
$\mathbf{i} = (i_1 \dots i_n)$, $i_j \in I$, *s.t.* $i_m \neq i_{m+1}$ *for* $1 \leq m \leq n-1$,
with $S_i = \{g | g \in f_i(A)g \in f_i(A)G_i\}$

Theorem 1 (1 of Serre (1980) [4])**.** $\forall\, g \in G$, $\exists$ *sequence* $\mathbf{i}$ *s.t.* $i_m \neq i_{m+1}$ *for* $1 \leq m \leq n-1$ *and reduced word*

$$m = (a; s_1 \dots s_n)$$

of type $\mathbf{i}$ *s.t.*

$$g = f(a) f_{i_1}(s_1) \dots f_{i_n}(s_n)$$

Furthermore, $\mathbf{i}$ and m unique.

Remark. Thm. 1 implies $f; f_i$ injective.

Then identify A and G_i with images $f(A), f_i(G_i)$ in G, and reduced decomposition (*) of $g \in G$

$$g = as_1 \dots s_n, \quad a \in A,\ s_1 \in S_{i_1} - \{1\} \dots s_n \in S_{i_n} - \{1\}$$

Likewise, $G_i \bigcap G_j = A$ if $i \neq j$.

In particular, $S_i - \{1\}$ pairwise disjoint in G.

Proof. Let $X_i \equiv$set of reduced words of type $\mathbf{i}$, $X = \coprod X_i$.

Make G act on X.

In view of universal property of G, sufficient to make $\forall\, i, G_i$ act,

check action induced on A doesn't depend on i

Suppose then that $i \in I$, and let Y_i = set of reduced words of form $(1; s_1 \dots s_n)$, with $i_1 \neq i$.

EY : 20170611

Recall that

$$S_i = \{g | g \in f_i(A)g \in f_i(A)G_i\}$$
$$A \times S_i \to G_i \text{ onto}$$
$$A \times (S_i - \{1\}) \to G_i - A \text{ onto}$$
$$(a, s) \mapsto as \text{ bijection}$$

Let Y_i = set of reduced words of form $(1; s_1 \dots s_n) = \{(1; s_1 \dots s_n) | 1 \in A; s_1 \in S_{i_1} \dots s_n \in S_{i_n}; \mathbf{i} = (i_1 \dots i_n), i_j \in I$ s.t. $i_m \neq i_{m+1}$ for $1 \leq m \leq n-1\}$.

$$A \times Y_i \to X = \coprod_i X_i$$
$$(a, (1; s_1 \dots s_n)) \mapsto (a; s_1 \dots s_n)$$
$$A \times \{S_i - \{1\}\} \times Y_i \to X$$
$$((a,s),(1;s_1 \dots s_n)) \mapsto (a; s, s_1 \dots s_n)$$

and remember that X_i = set of reduced words of type $\mathbf{i}$.

It's clear that this yields a bijection $A \times Y_i \bigcup A \times (S_i - \{1\}) \times Y_i \to X$.

Let $x \in X$. Then $x \in X_\mathbf{i}$ for some $\mathbf{i}$. So x is a reduced word of type $\mathbf{i}$: $x = (a; s_1 \dots s_n)$. Then clearly $x = (a; s_1 \dots s_n) \mapsto (a, (1; s_1 \dots s_n)) \in A \times Y_i$. □

cf. pp. 13, Sec. 2. Trees, 2.1 Graphs of Serre (1980) [4]

Definition 13 (1. of Serre (1980) [4]). ***graph*** *$\Gamma = (X, Y, Y \to X \times X, Y \to Y)$, where* set $X =$ vert Γ
set $Y =$ edge Γ

$$Y \to X \times X$$
$$y \mapsto (o(y), t(y))$$
$$Y \to Y$$
$$y \mapsto \overline{y}$$

s.t. $\forall y \in Y$, $\overline{\overline{y}} = y$, $\overline{y} \neq y$, $o(y) = t(\overline{y})$.

vertex $P \in X$ of Γ.

(oriented) edge $y \in Y$, $\overline{y} \equiv$ inverse edge.

origin of $y :=$ vertex $o(y) = t(\overline{y})$.

terminus of $y :=$ vertex $t(y) = o(\overline{y})$

extremities of $y := \{o(y), t(y)\}$

*If 2 vertices **adjacent**, they're extremities of some edge.*

orientation of graph $\Gamma = Y_+ \subset Y =$ edge Γ s.t. $Y = Y_+ \coprod \overline{Y}_+$. It always exists.

oriented graph defined, up to isomorphism, by giving 2 sets X, Y_+ and $Y+ \to X \times X$.

corresponding set of edges is $Y = Y_+ \coprod \overline{Y}_+$ where $\overline{Y}_+ \equiv$ copy of Y_+

18.0.2. *Realization of a Graph.* cf. Realization of a Graph in Serre (1980) [4].

Let graph Γ, $X = \text{vert}\Gamma$, $Y = \text{edge}\Gamma$.

topological space $T = X \coprod Y \times [0,1]$, where X, Y provided with discrete topology.

Let R be finest equivalence relation on T for which

$$\begin{aligned} &(y,t) \equiv (\overline{y}, 1-t) \\ &(y,0) \equiv o(y) \qquad \forall y \in Y, \forall t \in [0,1] \\ &(y,1) \equiv t(y) \end{aligned} \tag{10}$$

quotient space $\text{real}(\Gamma) = T/R$ is *realization* of graph Γ. (realization is a functor which commutes with direct limits).

Let $n \in \mathbb{Z}^+$. Consider oriented graph of $n+1$ vertices $0, 1, \dots n$,

Definition 14. *path (of length n) in graph Γ is morphism c of $Path_n$ into Γ*

orientation given by n edges $[i, i+1]$, $0 \leq i < n$, $o([i,i+1]) = i$
$t([i,i+1]) = i+1$

For $n \geq 1$,

$(y_1 \dots y_n)$ sequence of edges $y_i = c([i-1, i])$ s.t.

$$t(y_i) = o(y_{i+1}), \qquad 1 \leq i < n \text{ determine } c$$

If $P_i = c(i)$,

c is a path from P_0 to P_n, and P_0 and P_n are *extremities of the path c*.

pair of form $(y_i, y_{i+1}) = (y_i, \overline{y}_i)$ in path is **backtracking**.

path (of length $n-2$), from P_0 to P_n given (for $n > 2$) by $(y_1 \dots y_{i-1}, y_{i+2} \dots y_n)$

If $\exists$ path from P to Q in Γ, $\exists$ one without backtracking (by induction)

direct limit $\text{Path}_\infty = \varinjlim \text{Path}_n$ provides notion of infinite path.

$\text{Path}_\infty \ni$ infinite sequence $(y_1, y_2, \dots)$ of edges s.t. $t(y_i) = o(y_{i+1}) \quad \forall i \geq 1$.

Definition 15 (connected graph; Def. 3 of Serre (1980) [4]). *graph connected if $\forall$ 2 vertices, 2 vertices are extremities of at least 1 path.*

maximal connected subgraphs (under relation of inclusion) are connected components *of graph.*

18.0.3. *Circuits.* Let $n \in \mathbb{Z}^+$, $n \geq 1$.

Consider

set of vertices $\mathbb{Z}/n\mathbb{Z}$, orientation given by n edges $[i, i+1]$, $(i \in \mathbb{Z}/n\mathbb{Z})$ with $o([i,i+1]) = i$
$t([i,i+1]) = i+1$

Definition 16 (circuit; Def. 4 of Serre (1980) [4]). *circuit (length n) in graph is subgraph isormorphic to $Circ_n$.*

i.e. subgraph = path $(y_1 \dots y_n)$, without backtracking, s.t. $P_i = t(y_i)$, $(1 \leq i \leq n)$ distinct, s.t. $P_n = o(y_1)$

$n = 1$ case: Circ_1, $\mathbb{Z}/\mathbb{Z} = \{0\}$, 1 edge, $[0,1]$, $0 \in \mathbb{Z}/1\mathbb{Z}$, $o([0,1]) = 0$
$t([0,1]) = 1$

Note Circ_1 has automorphism of order 2, which changes its orientation, i.e.

$\exists$ automorphism $\sigma \in \text{Aut}(\text{Circ}_1)$ s.t. $|\sigma| = 2$, i.e. $\sigma^2 = 1$.

loop := circuit of length 1; so loop $\in \overline{\text{Circ}_1}$.

path (y_1), $P_1 = t(y_1) = o(y_1)$.

$n = 2$ case: Circ_2, $\mathbb{Z}/2\mathbb{Z} = \{0, 1\}$, 2 edges $[0,1], [1,2]$,

path (y_1, y_2), $(1 \leq i \leq 2)$, $P_1 = t(y_1)$
$P_2 = t(y_2) = o(y_1)$

18.1. **Combinatorial graphs.** Let $(X, S) \equiv$ simplicial complex of dim. ≤ 1, with

$X \equiv$ set

$S \equiv$ set of subsets of X with 1 or 2 elements, containing all the 1-element subsets.

associates with it a graph $\Gamma = (X, \{(P,Q)\})$.

X is its set of vertices.

edges $= \{(P,Q) \in X \times X\}$ s.t. $P \neq Q$, $\{P,Q\} \in S$, with $\overline{(P,Q)} = (Q,P)$

$$o(P,Q) = P$$
$$t(P,Q) = Q$$

In this graph, 2 edges with same origin and same terminus are equal. This is equivalent to (see following Def.)

Definition 17 (combinatorial; Def. 5 of Serre (1980) [4]). *graph is combinatorial if it has no circuit of length* ≤ 2

Conversely, it's easy to see that
every combinatorial graph Γ derived (up to isomorphism) by construction above from simplicial complex (X, S), where
$X = \text{vert}\Gamma$
$S =$ set of subset $\{P, Q\}$ of X s.t. P and Q either adjacent or equal.

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