SOLUTIONS TO INTRODUCTION TO SMOOTH MANIFOLDS BY JOHN M. LEE, 2012, SPRINGER.

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Contents

1. Smooth Manifolds

Topological Manifolds. M topological manifold of dim$n$, or topological n-manifold

- locally Euclidean, dim$n$ - $\forall\, p \in M$, $\exists$ neighborhood $U \equiv U_p$ s.t. $U_p \approx^{\text{homeo}}$ open $V \subset \mathbb{R}^n$

Exercise 1.1. Recall, M locally Euclidean $\dim n$ $\forall\, p \in M$, $\exists$ neighborhood homeomorphic to open subset.
open subset $\mathcal{O} \subseteq \mathbb{R}^n$ homeomorphic to open ball and $\mathcal{O}$ homeomorphic is $\mathbb{R}^n$ since $\mathbb{R}^n$ homeomorphic to open ball.
To see this explicitly, that open ball $B_\epsilon(x_0) \subseteq \mathbb{R}^n$ homeomorphic to $\mathbb{R}^n$

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Consider $\begin{aligned} &T : B_\epsilon(x_0) \to \mathbb{R}^n \\ &T(B_\epsilon(x_0)) = B_\epsilon(0) \\ &T(x) = x - x_0 \end{aligned}$
$T^{-1}(x) = x + x_0$. Clearly T homeomorphism.

Consider $\begin{aligned} &\lambda : \mathbb{R}^n \to \mathbb{R}^n \\ &\lambda(x) = \lambda x \\ &\lambda^{-1}(x) = \frac{1}{\lambda} x \end{aligned}$ for $\lambda > 0$. Clearly λ homeomorphism.
Consider $B \equiv B_1(0)$.

Consider $\begin{aligned} &g : \mathbb{R}^n \to \mathbb{R}^n \\ &g(x) = \frac{x}{1 + |x|} \end{aligned}$
g cont.

Let $\begin{aligned} &f : B \to \mathbb{R}^n \\ &f(x) = \frac{x}{1 - |x|} \end{aligned}$
How was f guessed at?
$|g(x)| = \left| \frac{x}{1+|x|} \right| = \frac{r}{1+r}$. Note $0 \leq |g(x)| < 1$
So $g(\mathbb{R}^n) = B$
For $|g(x)| = |y|$, $y \in B$, $|y|(1 + r) = r$, $r = \frac{|y|}{1-|y|}$
This is well-defined, since $0 \leq |y| < 1$ and $0 < 1 - |y| \leq 1$

$$gf(x) = \frac{\frac{x}{1-|x|}}{1 + \frac{|x|}{1-|x|}} = x$$
$$fg(x) = \frac{\frac{x}{1+|x|}}{1 - \frac{|x|}{1+|x|}} = x$$

f homeomorphism between B and $\mathbb{R}^n$. B and $\mathbb{R}^n$ homeomorphic. So an open ball in $\mathbf{R}^n$ is homeomorphic to $\mathbb{R}^n$

In practice, both the Hausdorff and second countability properties are usually easy to check, especially for spaces that are built out of other manifolds, because both properties are inherited by subspaces and products (Lemmas A.5 and A.8). In particular, it follows easily that any open subset of a topological n-manifold is itself a topological n-manifold (with the subspace topology, of course).

Coordinate Charts. chart on M, (U, φ) where open $U \subset M$ and
homeomorphism $\varphi : U \to \mathbb{R}^n$, $\varphi(U)$ open.

Examples of Topological Manifolds. Example 1.3. (Graphs of Continuous Functions)
Let open $U \subset \mathbb{R}^n$
Let $F : U \to \mathbb{R}^k$ cont.
graph of F: $\Gamma(F) = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^k | x \in U, y = F(x)\}$ with subspace topology.
$\pi_1 : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$ projection onto first factor.
$\varphi_k : \Gamma(F) \to U$ restriction of π_1 to $\Gamma(F)$
$\varphi_F(x, y) = x$, $(x, y) \in \Gamma(F)$

Example 1.4 (Spheres) $S^n = \{x \in \mathbb{R}^{n+1} | |x| = 1\}$
Hausdorff and second countable because it's topological subspace of $\mathbb{R}^n$

Example 1.5 (Projective Spaces)
$U_i \subset \mathbb{R}^{n+1} - 0$ where $x^i \neq 0$
$V_i = \pi(U_i)$
Let $a \in U_i$.

$$|x - a|^2 = (x^1 - a^1)^2 + \cdots + (x^i - a^i)^2 + \cdots + (x^{n+1} - a^{n+1})^2 < \frac{(n+1)\epsilon^2}{n+1} = \epsilon^2$$

$\forall\, a^i \in \mathbb{R}$, $\exists\, x^i$, s.t. $(x^i - a^i)^2 < \frac{\epsilon^2}{n+1}$, by choice of $0 < x^i < a^i + \frac{\epsilon}{\sqrt{n+1}}$ with $0 < x^i$ for $i = i$ index.
U_i indeed open set, *saturated open set*.
open $U_i \subset \mathbb{R}^{n+1} - 0$, $x^i \neq 0$
From Lemma A.10, recall (d) restriction of π to any saturated open or closed subset of X is a quotient map.
natural map $\pi : \mathbb{R}^{n+1} - 0 \to \mathbb{R}P^n$ given quotient topology.
By Tu, Prop. 7.14, $\sim$ on $\mathbb{R}^{n+1} - 0 \to \mathbb{R}P^n$ open equivalence relation.
$\Longrightarrow \pi|_{U_i}(U_i) = V_i$ open.

$$\begin{aligned} &\varphi_i : V_i \to \mathbb{R}^n \\ &\varphi_i[x^1 \dots x^{n+1}] = \left(\frac{x^1}{x^i} \dots \frac{x^{i-1}}{x^i}, \frac{x^{i+1}}{x^i} \dots \frac{x^{n+1}}{x^i} \right) \end{aligned}$$

φ_i well-defined since

$$\varphi_i[tx^1 \dots tx^{n+1}] = \left(\frac{tx^1}{tx^i} \dots \frac{t\widehat{x}^i}{tx^i} \dots \frac{tx^{n+1}}{tx^i} \right) = \left(\frac{x^1}{x^i} \dots \frac{\widehat{x}^i}{x^i} \dots \frac{x^{n+1}}{x^i} \right) = \varphi_i[x^1 \dots x^{n+1}]$$

φ_i cont. since $\varphi_i \pi$ cont.

$$\begin{tikzcd} U_i \subset \mathbb{R}^{n+1} - 0 \arrow[d, "\pi"'] \arrow[dr, "\varphi_i \pi"] & \\ V_i \subset \mathbb{R}P^n \arrow[r, "\varphi"'] & \mathbb{R}^n \end{tikzcd}$$

$$\begin{aligned} \varphi_i &: U_i \subset \mathbb{R}P^n \to \mathbb{R}^n \\ \varphi_i[x^1 \dots x^{n+1}] &= \left(\frac{x^1}{x^i} \dots \frac{\widehat{x}^i}{x^i} \dots \frac{x^{n+1}}{x^i} \right) \\ \varphi_i^{-1}(u^1 \dots u^n) &= [u^1 \dots u^{i-1}, 1, u^i \dots u^n] \\ \varphi_i^{-1}\varphi_i[x^1 \dots x^{n+1}] &= \left[\frac{x^1}{x^i} \dots \frac{x^{i-1}}{x^i}, 1, \frac{x^{i+1}}{x^i} \dots \frac{x^{n+1}}{x^i} \right] = [x^1 \dots x^{i-1}, x^i, x^{i+1} \dots x^{n+1}] \\ \varphi_i \varphi_i^{-1}(u^1 \dots u^n) &= (u^1 \dots u^{i-1}, u^i \dots u^n) \end{aligned}$$

cont. φ_i bijective, φ_i^{-1} cont. φ_i homeomorphism.
From a previous edition:

Exercise 1.2.

Let $\phi_t : \mathbb{R}^{n+1} \backslash \{0\} \to \mathbb{R}^{n+1} \backslash \{0\}$
$\quad \phi_t(x) = tx$
ϕ_t invertible, $\phi_t^{-1} = \phi_{\frac{1}{t}}$
ϕ_t, ϕ_t^{-1} C^1 (and C^∞), ϕ_t homeomorphism.

Let U open in $\mathbb{R}^{n+1} \backslash \{0\}$. Then $\phi_t(U)$ open in $\mathbb{R}^{n+1} \backslash \{0\}$.
Thus $\pi^{-1}([U]) = \bigcup_{t \in \mathbb{R}} \phi_t(U)$ open in $\mathbb{R}^{n+1} \backslash \{0\}$.
Thus $[U]$ open in $\mathbb{R}P^n$. $\sim$ open.

Note $\pi : \mathbb{R}^{n+1} \backslash \{0\} \to \mathbb{R}P^n$
$\quad \pi(x) = \dfrac{x}{\|x\|}$
$\mathbb{R}^n$ 2nd. countable, $\mathbb{R}P^n$ 2nd. countable.

Exercise 1.3.
S^n compact.
$\pi : \mathbb{R}^{n+1} \backslash \{0\} \to \mathbb{R}P^n$
$\pi(x) = [\frac{x}{\|x\|}]$
Let $x \in \mathbb{R}P^n$
$\quad y = \frac{x}{\|x\|} \in S^n$ and $\pi|_{S^n}(y) = [x]$
$\pi|_{S^n}$ surjective.

Exercise 1.6.

First, note that $\sim$ on $\mathbb{R}^{n+1} - 0$ in the definition of $\mathbb{R}P^n$ is an open $\sim$ i.e. open equivalence relation.
This is because of the following: $\forall\, U \subset \mathbb{R}^{n+1} - 0$,
$\pi^{-1}(\pi(U)) = \bigcup_{t \in \mathbb{R}} tU$, set of all pts. equivalent to some pt. of U.
multiplication by $t \in \mathbb{R}$ homeomorphism of $\mathbb{R}^{n+1} - 0$, so tU open $\forall\, t \in \mathbb{R}$.
$\pi^{-1}(\pi(U))$ open i.e. $\pi(U)$ open (for π is cont.).
Let $X = \mathbb{R}^{n+1} - 0$.
Consider $R = \{(x,y) \in X \times X | x \sim y \text{ or } y = tx \text{ for some } t \in \mathbb{R}\}$
$y = tx$ means $y_i = tx_i \quad \forall\, i = 0 \dots n$. Then $\frac{x_i}{y_i} = \frac{x_j}{y_j} \quad \forall\, i,j = 0 \dots n$. Hence $x_i y_j - y_i x_j = 0 \quad \forall\, i,j$.

Let $\begin{gathered} f : X \times X \to \mathbb{R} \\ f(x,y) = \sum_{i \neq j} (x_i y_j - y_i x_j)^2 \end{gathered}$

$$\begin{aligned} \frac{\partial f}{\partial x_i} &= \sum_{i \neq j} 2(x_i y_j - y_i x_j)(y_j - y_j) = 0 \\ \frac{\partial f}{\partial y_j} &= \sum_{j \neq i} 2(x_i y_j - y_i x_j)(x_i - x_i) = 0 \end{aligned}$$

Nevertheless, f is C^1 so f cont.
So $f^{-1}(0) = R$.
0 closed, so $f^{-1}(0) = R$ closed. By theorem, since $\sim$ open, $\mathbb{R}P^n = \mathbb{R}^{n+1} - 0/\sim$ Hausdorff.
cf. `http://math.stackexchange.com/questions/336272/the-real-projective-space-rpn-is-second-cour`
topological space is second countable if its topology has countable basis.
$\mathbb{R}^n$ second countable since $\mathcal{B} = \{B_r(q) | r, q \in \mathbb{Q}\}$ is a countable basis. $\forall\, x \in \mathbb{R}^n$

If X is second countable, with countable basis $\mathcal{B}$,

(1) If $Y \subseteq X$, Y also second countable with countable basis $\{B | B \in \mathcal{B}, Y \bigcap B \neq \varnothing\}$
(2) If $Z == X/\sim$, $\{\{[x] | x \in B\} | B \in \mathcal{B}\}$ is a countable basis for Z since $\mathcal{B}$ countable.

It is a basis since

$$\begin{array}{c} X = \bigcup_{B \in \mathcal{B}} B \\ \Big\downarrow \pi \\ Z = X/\sim = \pi(\bigcup_{B \in \mathcal{B}} B) = \bigcup_{B \in \mathcal{B}} \pi(B) = \bigcup_{B \in \mathcal{B}} \{[x] | x \in B\} \end{array}$$

Now let $Y = \mathbb{R}^n - 0$ and
$Z = \mathbb{R}P^n = \mathbb{R}^n - 0/\sim$

Exercise 1.7.
S^n compact so $S^n/\{\pm\}$ compact by Theorem, as $\pi_S(S^n) = S^n/\{\pm\}$, as π_S cont. surjective ($\forall\, [x] \in S^n/\{\pm\}$, $\exists\, x \in S^n$ s.t. $\pi_S(S^n) = [x]$)
g cont. bijective as defined above so since $g(S^n/\{\pm\}) = \mathbb{R}P^n$, $\mathbb{R}P^n$ compact.

Example 1.8 (Product Manifolds)

$$\begin{aligned} M_1 &= \bigcup_{\alpha \in \mathfrak{A}_1} U_\alpha^{(1)} \\ M_i &= \bigcup_{\alpha \in \mathfrak{A}_i} U_\alpha^{(i)} \end{aligned} \qquad M_1 \times \dots \times M_n = \bigcup_{\substack{\alpha_1 \in \mathfrak{A}_1 \\ \vdots \\ \alpha_n \in \mathfrak{A}_n}} U_{\alpha_1} \times \dots \times U_{\alpha_n} \qquad \text{(by def.)}$$

$\forall\, p = (p_1 \dots p_n) \in M_1 \times \dots M_n$, consider $p_i \in M_i$. Choose coordinate chart (U_{j_i}, φ_{j_i}), $\varphi_i(U_i) \subset \mathbb{R}^{n_i}$. Then,

Consider $\varphi : U_1 \times \dots \times U_n \to \mathbb{R}^{m_1} \times \dots \times \mathbb{R}^{m_n} = \mathbb{R}^{m_1 + \dots + m_n}$, $(U_1 \times \dots \times U_n, \varphi_1 \times \dots \times \varphi_n)$
$\varphi = \varphi_1 \times \dots \times \varphi_n$

φ also a homeomorphism. $\begin{gathered} \varphi\psi^{-1} \\ (\varphi_1 \times \dots \times \varphi_n) \circ (\psi_1 \times \dots \times \psi_n)^{-1} \end{gathered}$ also a diffeomorphism, cont. bijective and C^∞
$\{(U, \varphi) = (U_1 \times \dots \times U_n, \varphi_1 \times \dots \times \varphi_n) | (U_i, \varphi_i) \in \{(U_i, \varphi_i) | U_i \in M_i\}\}$ also an atlas.

Topological Properties of Manifolds.

Lemma 1 (1.10). $\forall$ *topological M, M has countable basis of precompact coordinate balls*

Proof. First consider M can be covered by single chart.
Suppose $\varphi : M \to \widehat{U} \subseteq \mathbb{R}^n$ global coordinate map.
Let $\mathcal{B} = \{B_r(x) |$ open $B_r(x) \subseteq \mathbb{R}^n$ s.t. $r \in \mathbb{Q}$, $x \in \mathbb{Q}$, i.e. x rational coordinates , $B_{r'}(x) \subseteq \widehat{U}$, for some $r' > r\}$
Clearly, $\forall\, B_r(x)$ precompact in $\widehat{U}$
$\mathcal{B}$ countable basis for topology of $\widehat{U}$
φ homeomorphism, it follows $\{\varphi^{-1}(B) | B \in \mathcal{B}\}$ countable basis for M
Let M arbitrary,
By def., $\forall\, p \in M$, $p \in$ domain U of a chart
Prop. A.16, $\forall$ open cover of second-countable space has countable subcover.
M covered by countably many charts $\{(U_i, \varphi_i)\}$
$\forall\, U_i$, U_i has countable basis of coordinate balls precompact in U_i
union of all these coordinates bases is countable basis for M.

If $V \subseteq U_i$ one of these balls,
then $\overline{V}$ compact in U_i. M Hausdorff, so $\overline{V}$ closed.
$\overline{V}$ in M is same as $\overline{V}$ in U_i, so V precompact in M. □

Connectivity.

Local Compactness and Paracompactness.

Fundamental Groups of Manifolds.

Smooth Structures. If open $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^m$,
$F : U \to V$ smooth (or C^∞) if $\forall$ cont. partial derivative of all orders exists.

F diffeomorphism if F *smooth*, bijective, and has a *smooth* inverse.
Diffeomorphism is a homeomorphism.

$(U, \varphi), (V, \psi)$ smoothly compatible if $UV = \varnothing$ or

$$\psi\varphi^{-1} : \varphi(UV) \to \psi(UV) \qquad \text{diffeomorphism}$$

atlas $\mathcal{A} \equiv \{(U, \varphi)\}$ s.t. $\bigcup U = M$. Smooth atlas if $\forall\, (U, \varphi), (V, \psi) \in \mathcal{A}$, $(U, \varphi), (V, \psi)$ smoothly compatible.

Smooth structure on topological n-manifold M is a maximal smooth atlas.

Smooth manifold $(M, \mathcal{A})$ where M topological manifold, $\mathcal{A}$ smooth structure on M.

Proposition 1 (1.17)**.** *Let M topological manifold.*
  (a) $\forall$ *smooth atlas for M is contained in ! maximal smooth atlas.*
  (b) *2 smooth atlases for M determine the same maximal smooth atlas iff union is smooth atlas.*

Proof. Let $\mathcal{A}$ smooth atlas for M
$\overline{\mathcal{A}} \equiv$ set of all charts that are smoothly compatible with every chart in $\mathcal{A}$
Want: $\overline{\mathcal{A}}$ smooth atlas, i.e. $\forall\, (U, \varphi), (V \psi) \in \overline{\mathcal{A}}$, $\psi\varphi^{-1} : \varphi(UV) \to \psi(UV)$ smooth.
Let $x = \varphi(p) \in \varphi(UV)$
$p \in M$, so $\exists$ some chart $(W, \theta) \in \mathcal{A}$ s.t. $p \in W$.
By given, $\theta\varphi^{-1}, \psi\theta^{-1}$ smooth where they're defined.
$p \in UVW$, so $\psi\varphi^{-1} = \psi\theta^{-1}\theta\varphi^{-1}$ smooth on x.
Thus $\psi\varphi^{-1}$ smooth in a neighborhood of each pt. in $\varphi(UV)$. Thus $\overline{\mathcal{A}}$ smooth atlas.
To check maximal,

□

Local Coordinate Representations.

Proposition 2 (1.19)**.** $\forall$ *smooth M has countable basis of regular coordinate balls*

Exercise 1.20.
smooth manifold M has smooth structure
Suppose single smooth chart φ has entire M as domain

$$\varphi : M \to \widehat{U} \subseteq \mathbb{R}^n$$

Let $\widehat{B} = \{\widehat{B}_r(x) \subseteq \mathbb{R}^n | r \in \mathbb{Q}, x \in \mathbb{Q}, \widehat{B}_{r'}(x) \subseteq \widehat{U}$ for some $r' > r\}$

$\forall \widehat{B}_r(x)$ precompact in $\widehat{U}$
$\widehat{\mathcal{B}}$ countable basis for topology of $\widehat{U}$
φ homeomorphism,

Let $\varphi^{-1}(\widehat{B}_r(0)) = B$
$\varphi^{-1}(\widehat{B}_{r'}(0)) = B'$
φ homeomorphism and since $\widehat{B}$ countable basis, $\{B\}$ countable basis of regular coordinate basis.

Suppose arbitrary smooth structure.
By def., $\forall p \in M$, p in some chart domain
Prop. A.16., $\forall$ open cover of second countable space has countable subcover
M covered by countably many charts $\{(U_i, \varphi_i)\}$

$\forall U_i$, U_i has countable basis of coordinate balls precompact in U_i
union of all these coordinate charts is countable basis for M.
If $V \subseteq U_i$, 1 of these balls,

$\varphi(V) = B_r(0)$
$\varphi(\overline{V}) = \overline{B}_r(0)$
and $\varphi(B') = B_{r'}(0)$, $r' > r$ for countable basis for U_i
So V regular coordinate ball.

Examples of Smooth Manifolds.

More Examples. **Example 1.25 (Spaces of Matrices)** Let $M(m \times n, \mathbb{R}) \equiv$ set of $m \times n$ matrices with real entries.
Example 1.26 (Open Submanifolds)
$\forall$ open subset $U \subseteq M$ is itself a dim$M$ manifold.
EY : $\forall$ *open subset $U \subseteq M$ is itself a dim$M$ manifold.*
Example 1.27 (The General Linear Group)
general linear group $GL(n, \mathbb{R}) = \{A | \det A \neq 0\}$
$\det : A \to \mathbb{R}$ is cont. (by def. of $\det A = \epsilon^{i_1 \dots i_n} a_{1 i_1} \dots a_{n i_n}$
$\det^{-1}(\mathbb{R} - 0)$ is open since $\mathbb{R} - 0$ open so $GL(n, \mathbb{R})$ open
$GL(n, \mathbb{R}) \subseteq M(n, \mathbb{R})$, $M(n, \mathbb{R})$ n^2-dim. vector space.
so $GL(n, \mathbb{R})$ smooth n^2-dim. manifold.
Example 1.28 (Matrices of Full Rank)
Suppose $m < n$

Let $M_n(m \times n, \mathbb{R}) \subseteq M(m \times n, \mathbb{R})$ with matrices of rank m
if $A \in M_m(m \times n, \mathbb{R})$,
$\text{rank} A = m$
means that A has some nonsingular $m \times m$ submatrix. (EY 20140205 ???)
Example 1.31 (Spheres)

$$\varphi_i^{\pm} : S^n \to B_1^n(0) \subset \mathbb{R}^n \quad (B_1^n(0) \text{ disk of radius } 1)$$
$$\varphi_i^{\pm}(x_1 \dots x_{n+1}) = (x_1 \dots \widehat{x}_i \dots x_{n+1}) = (y_1 \dots y_n)$$

Note $x_1^2 + \dots + x_i^2 + \dots + x_{n+1}^2 = 1$. $\quad x_i = \pm\sqrt{1 - (x_1^2 + \dots + \widehat{x}_i^2 + \dots + x_{n+1}^2)}$

$$(\varphi_i^{\pm})^{-1}(y_1 \dots y_n) = (y_1 \dots \pm\sqrt{1 - (y_1^2 + \dots + y_n^2)} \dots y_n) = (y_1 \dots y_{i-1}, \pm\sqrt{1 - |y|^2}, y_i \dots y_n)$$
$$\varphi_i^{\pm}(\varphi_j^{\pm})^{-1}(y_1 \dots y_n) = (y_1 \dots \widehat{y}_i \dots y_{j-1}, \pm\sqrt{1 - |y|^2}, y_j \dots y_n)$$
$$\varphi_i^{\pm}(\varphi_j^{\mp})^{-1}(y_1 \dots y_n) = (y_1 \dots \widehat{y}_i \dots y_{j-1}, \mp\sqrt{1 - |y|^2}, y_j \dots y_n)$$
$$\varphi_j^{\pm}(\varphi_i^{\mp})^{-1}\varphi_i^{\pm}(\varphi_j^{\pm})^{-1}(y_1 \dots y_n) = \varphi_j^{\pm}(y_1 \dots \pm y_i \dots y_{j-1}, \pm\sqrt{1 - |y|^2} \dots y_n) = (y_1 \dots \pm y_i \dots y_j \dots y_n)$$

This is symmetrical in i, j and so true if i, j reverse.

So $\varphi_i^{\pm}(\varphi_j^{\pm})^{-1}$ diff. and bijective. Likewise for $\varphi_i^{\pm}(\varphi_j^{\mp})^{-1}$.

So $\begin{matrix} \varphi_i^{\pm}(\varphi_j^{\pm})^{-1} \\ \varphi_i^{\pm}(\varphi_j^{\mp})^{-1} \end{matrix}$ diffeomorphisms.

Lemma 2 (1.35)**.** ***(Smooth Manifold Chart Lemma)*** *Let M be a set, suppose given $\{U_\alpha | U_\alpha \subset M\}$, given maps $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$ s.t.*

(i) *$\forall\, \alpha$, φ_α bijection between U_α and open $\varphi_\alpha(U_\alpha) \subseteq \mathbb{R}^n$*
(ii) *$\forall\, \alpha, \beta$, $\varphi_\alpha(U_\alpha \bigcap U_\beta)$, $\varphi_\beta(U_\alpha \bigcap U_\beta)$ open in $\mathbb{R}^n$*

Example 1.36 (Grassmann Manifolds)
$G_k(V) = \{S | S \subseteq V\}$ $\quad S$ k-dim. linear subspace of V
dim$V = n$, V vector space

if $V = \mathbb{R}^n$, $G_k(\mathbb{R}^n) \equiv G_{k,n} \equiv G(k,n)$ (notation) $\quad G_1(\mathbb{R}^{n+1}) = \mathbb{R}P^n$

Let $V = P \oplus Q$, $\begin{matrix} \text{dim}P = k \\ \text{dim}Q = n-k \end{matrix}$

linear $X : P \to Q$

$$\Gamma(X) = \{v + Xv | v \in P\}, \quad \Gamma(X) \subseteq V, \text{ dim}\Gamma(X) = k$$

$\Gamma(X) \bigcap Q = 0$ since $\forall\, w \in \Gamma(X)$, w has a P piece, and Q complementary to P

Converse: $\forall$ subspace $S \subseteq V$, s.t. $S \bigcap Q = 0$

let $\begin{matrix} \pi_P : V \to P \\ \pi_Q : V \to Q \end{matrix}$ $\quad$ projections by direct sum decomposition $V = P \oplus Q$

$\pi_P|_S : S \to P$ isomorphism
$\Longrightarrow X = (\pi_Q|_S) \cdot (\pi_P|_S)^{-1}$, $\; X : P \to Q$

Let $v \in P$. $v + Xv = v + \pi_Q|_S\,(\pi_P|_S)^{-1}v$. Let $v \in \pi_P|_S\,(S)$ $\quad \Gamma(X) = S$

Let $L(P;Q) = \{f |$ linear $f : P \to Q\}$, $\; L(P;Q)$ vector space
$U_Q \subseteq G_k(V)$, $\; U_Q = \{S | \text{dim}S = k,\ S \text{ subspace },\ S \bigcap Q = 0\}$
$\Gamma : L(P;Q) \to U_Q$
$X \mapsto \Gamma(X)$

Γ bijection by above
$\varphi = \Gamma^{-1} : U_Q \to L(P;Q)$

By choosing bases for P, Q, identify $L(P;Q)$ with $M((n-k)k;\mathbb{R})$ and hence with $\mathbb{R}^{k(n-k)}$

think of (U_Q, φ) as coordinate chart.
$\varphi(U_Q) = L(P;Q)$

Problems. **Problem 1.7.** (This was Problem 1.5 in previous editions)

$$S^n = \{(x_1 \dots x_{n+1}\} \in \mathbb{R}^{n+1} | \sum_{i=1}^{n+1} x_i^2 = 1\} \subset \mathbb{R}^{n+1}$$

Let $\begin{matrix} N = (0 \dots 0, 1) \\ S = (0 \dots 0, -1) \end{matrix}$ $\quad x \in S^n$.

(a) Consider $t(x - N) + N = tx + (1-t)N$ when $x_{n+1} = 0$

$$tx_{n+1} + (1-t) = 0 \text{ or } tx_{n+1} + -1 + t = 0$$
$$\Longrightarrow \frac{1}{1 - x_{n+1}} = t \quad \left(\text{or } \frac{1}{1 + x_{n+1}} \right)$$

$$\pi_1 : S^n - N \to \mathbb{R}^n$$
$$\pi_1(x_1 \dots x_{n+1}) = \left(\frac{x_1}{1 - x_{n+1}} \dots \frac{x_n}{1 - x_{n+1}}, 0 \right)$$
$$\pi_2 : S^n - S \to \mathbb{R}^n$$
$$\pi_2(x_1 \dots x_{n+1}) = \left(\frac{x_1}{1 + x_{n+1}} \dots \frac{x_n}{1 + x_{n+1}}, 0 \right)$$

Note that $-\pi_2(-x) = \pi_1$ and $\pi_1 \equiv \sigma$, $\pi_2 \equiv \widetilde{\sigma}$ in Massey's notation.

(b) Note, for $y_i = \frac{x_i}{1 - x_{n+1}}$

$$y_1^2 + \dots + y_n^2 = |y|^2 = \frac{1 - x_{n+1}^2}{(1 - x_{n+1})^2} = \frac{1 + x_{n+1}}{1 - x_{n+1}} \text{ or } x_{n+1} = \frac{|y|^2 - 1}{|y|^2 + 1}$$
$$x_i = y_i(1 - x_{n+1}) = \frac{2y_i}{1 + |y|^2}$$

$$\pi_1^{-1} : \mathbb{R}^n \to S^n - N$$
$$\pi_1^{-1}(y_1 \dots y_n) = \left(\frac{2y_1}{1 + |y|^2} \dots \frac{2y_n}{1 + |y|^2}, \frac{|y|^2 - 1}{|y|^2 + 1} \right)$$
$$\pi_2^{-1}(y_1 \dots y_n) = \left(\frac{2y_1}{1 + |y|^2} \dots \frac{2y_n}{1 + |y|^2}, \frac{1 - |y|^2}{|y|^2 + 1} \right)$$

π_1, π_2 diff., bijective, and $(S^n - N) \bigcup (S^n - S) = S^n$

(c) Computing the transition maps for the stereographic projections.

Consider $(S^n - N)(S^n - S) = S^n - N \bigcup S$

$$\pi_1 \pi_2^{-1}(y_1 \dots y_n) = \left(\frac{y_1}{|y|^2} \dots \frac{y_n}{|y|^2}, 0 \right)$$
$$\pi_2 \pi_1^{-1}(y_1 \dots y_n) = \left(\frac{y_1}{|y|^2} \dots \frac{y_n}{|y|^2}, 0 \right)$$

since, for example,

$$\frac{\frac{2y_i}{1+|y|^2}}{1 - \frac{1-|y|^2}{1+|y|^2}} = \frac{y_i}{|y|^2}$$

$\pi_1 \pi_2^{-1}$ bijective and C^∞, $\pi_1 \pi_2^{-1}$ diffeomorphism.

$\{(S^n - N, \pi_1), (S^n - S, \pi_2)\}$ C^∞ atlas or differentiable structure.

$$\partial_j \frac{y_i}{|y|^2} = \frac{-2y_i y_j}{(y_1^2 + \dots + y_n^2)^2}$$
$$\partial_j \frac{y_j}{|y|^2} = \frac{(y_1^2 + \dots + y_n^2) - 2y_j^2}{|y|^4} = \frac{y_1^2 + \dots + \widehat{y_j^2} + \dots + y_n^2 - y_j^2}{|y|^4}$$

$$\det(\partial_j \pi_1 \pi_2^{-1}(y)) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \partial_{\sigma_1} \frac{y_1}{|y|^2} \dots \partial_{\sigma_n} \frac{y_n}{|y|^2} = 0 \text{ only if } y = 0. \text{ But that's excluded}$$

(d) Consider $\begin{array}{l} \{(S^n \backslash, \pi_1), (S^n \backslash S, \pi_2)\} \\ \mathcal{A} = \{(U_i^\pm, \varphi_i^\pm)\} \end{array}$

Now

$$S^n \backslash N \bigcap U_i^+ = \begin{cases} U_i^+ & \text{if } i \neq n+1 \\ U_{n+1}^+ \backslash N & \text{if } i = n+1 \end{cases}$$
$$S^n \backslash N \bigcap U_i^- = U_i^-$$

$$\pi_1(\varphi_i^\pm)^{-1}(y_1 \dots y_n) = \pi_1(y_1 \dots y_{i-1}, \pm\sqrt{1 - |y|^2}, y_i \dots y_n) = \left(\frac{y_1}{1 - y_n} \dots \frac{y_{i-1}}{1 - y_n}, \frac{\pm\sqrt{1 - |y|^2}}{1 - y_n}, \frac{y_i}{1 - y_n} \dots \frac{y_{n-1}}{1 - y_n}, 0 \right)$$

Note $-1 < y_n < 1$ on $\varphi_i^\pm(S^n \backslash \bigcap U_i^\pm)$

$$\varphi_i^\pm \pi_1^{-1}(y_1 \dots y_n) = \varphi_i^\pm \left(\frac{2y_1}{1 + |y|^2} \dots \frac{2y_n}{1 + |y|^2}, \frac{|y|^2 - 1}{|y|^2 + 1} \right) = \left(\frac{2y_1}{1 + |y|^2} \dots \frac{2\widehat{y_i}}{1 + |y|^2} \dots \frac{2y_n}{1 + |y|^2}, \frac{|y|^2 - 1}{|y|^2 + 1} \right)$$

$\pi_{1,2}(\varphi_i^{\pm})^{-1}, \varphi_i^{\pm}\pi_{1,2}^{-1}$ are diffeomorphisms (bijective and differentiable).

So $\{(S^n \backslash N, \pi_1), (S^n \backslash S, \pi_2)\} \cup \mathcal{A}$ also a C^∞ atlas.

So $\{(S^n \backslash N, \pi_1), (S^n \backslash S, \pi_2)\}$, $\mathcal{A}$ equivalent.

2. Smooth Maps

Smooth Functions and Smooth Maps.

Definition 1. **smooth function** f, $f : M \to \mathbb{R}^k$, if $\forall\, p \in M$, $\exists$ smooth chart (U, φ), $U \ni p$, s.t. $f \circ \varphi^{-1}$ smooth on $\varphi(U) \subseteq \mathbb{R}^n$, $f \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}^k$

EY : 20150717 Recall, smooth chart just means that the chart belongs to maximal smooth atlas, and smooth in that the transition maps are smooth.

$$M \xrightarrow{f} \mathbb{R}^k$$

$$\begin{tikzcd}
U \arrow[r, "f"] \arrow[d, "\varphi"'] & f(U) \subset \mathbb{R}^k \\
\varphi(U) \subset \mathbb{R}^n \arrow[u, bend right, "\varphi^{-1}"'] \arrow[ur, "f \circ \varphi^{-1}"']
\end{tikzcd}$$

Exercise 2.3. Given $f : M \to \mathbb{R}^k$ smooth,
Consider $p \in M$. M smooth manifold, so $\exists$ chart (U, φ) s.t. $p \in U$
$\varphi : U \subset M \to \mathbb{R}^m$
φ homeomorphism, $\varphi(U)$ open in $\mathbb{R}^m$ $\varphi^{-1} : \varphi(U) \to M$
Consider another smooth chart (V, ψ), $p \in V$ so that $UV \neq \varnothing$

$$\begin{gathered}
f\psi^{-1} : \psi(UV) \to \mathbb{R}^k \\
f\psi^{-1}(y^1 \dots y^m) = (f\psi^{-1})_i(y^1 \dots y^m), \quad i = 1 \dots k \\
f\psi^{-1} = f\varphi^{-1}\varphi\psi^{-1} = (f\varphi^{-1})\varphi\psi^{-1}
\end{gathered}$$

$\varphi\psi^{-1}$ C^∞ (diffeomorphisms are smooth). $f\varphi^{-1}$ is smooth. $f\psi^{-1}$ also smooth.

Smooth Maps Between Manifolds.

Definition 2. Let M, N be smooth manifolds.

smooth map F, $F : M \to N$ if $\forall\, p \in M$, $\exists$ smooth charts $\begin{matrix} (U,\varphi) & U \ni p \\ (V,\psi) & V \ni F(p) \end{matrix}$

s.t. $F(U) \subseteq V$ and
$\psi \circ F \circ \varphi^{-1}$ smooth from $\varphi(U)$ to $\psi(V)$.

i.e.

$$M \xrightarrow{F} N$$

$$\begin{tikzcd}
U \arrow[r, "F"] \arrow[d, "\varphi"'] & F(U) \subseteq V \arrow[d, "\psi"] \\
\varphi(U) \subset \mathbb{R}^m \arrow[u, bend right, "\varphi^{-1}"'] \arrow[r, "\psi \circ F \circ \varphi^{-1}"] & \psi(V) \subseteq \mathbb{R}^n
\end{tikzcd}$$

Exercise 2.6.
smooth $F : M \to N$
(U, φ), (U', φ') smooth chart for M (V, ψ), (V', ψ') smooth chart for N

$$\psi' F(\varphi')^{-1} = \psi'\psi^{-1}\psi F \varphi^{-1}\varphi(\varphi')^{-1} = (\psi'\psi^{-1})(\psi F\varphi^{-1})(\varphi(\varphi')^{-1})$$

$\psi'\psi^{-1}$, $\varphi(\varphi')^{-1}$ aresmooth (in fact diffeomorphisms).
$\psi F \varphi^{-1}$ given to be smooth.
So $\psi' F(\varphi')^{-1}$ smooth.
Example 2.5. (**Smooth Maps**)

(a) inclusion $i : S^n \hookrightarrow \mathbb{R}^{n+1}$

(b)

$$\varphi_i^{\pm} : S^n \to B_1^n(0) \subset \mathbb{R}^n \quad (B_1^n(0) \text{ disk of radius 1; } \varphi_i^{\pm} \text{ is like a projection of a half hemisphere to a plane})$$
$$\varphi_i^{\pm}(x_1 \dots x_{n+1}) = (x_1 \dots \widehat{x}_i \dots x_{n+1})$$
$$(\varphi_i^{\pm})^{-1}(y_1 \dots y_n) = (y_1 \dots y_{i-1}, \pm\sqrt{1-|y|^2}, y_i \dots y_n) \quad |y|^2 = y_1^2 + \dots + y_n^2$$
$$\widehat{i} : \mathbb{R}^n \hookrightarrow \mathbb{R}^{n+1}$$
$$\widehat{i}(u_1 \dots u_n) = i(\varphi_i^{\pm})^{-1}(u^1 \dots u^n) = (u^1 \dots u^{i-1}, \pm\sqrt{1-|u|^2}, u_i \dots u_n) \quad |u|^2 = (u^1)^2 + \dots + (u^n)^2$$

$\widehat{i}$ is coordinate representation, and clearly, from the above formula, is smooth.

(c) $\pi : \mathbb{R}^{n+1} - 0 \to \mathbb{R}P^n$ smooth because

$$\widehat{\pi}(x^1 \dots x^{n+1}) = \varphi_i \pi(x^1 \dots x^{n+1}) = \varphi_i[x^1 \dots x^{n+1}] = \left(\frac{x^1}{x^i} \dots \frac{\widehat{x}^i}{x^i} \dots \frac{x^{n+1}}{x^i} \right)$$

$\varphi_i : U_i \to \mathbb{R}^n$

(d) $p : S^n \to \mathbb{R}P^n$ is restriction of $\pi : \mathbb{R}^{n+1} - 0 \to \mathbb{R}P^n$ to $S^n \subset \mathbb{R}^{n+1} - 0$. $\pi|_{S^n} = p$

$p = \pi i$ π, i smooth, so p smooth.

$p^{-1} = i^{-1}\pi^{-1}$. $|x|^2 = (x^1)^2 + \dots + (x^{n+1})^2$

On $\mathbb{R}P^n$,

$$pp^{-1}[x^1 \dots x^{n+1}] = pi^{-1}\pi^{-1}[x^1 \dots x^{n+1}] = \pi^{-1}\left(\frac{x^1}{|x|} \dots \frac{x^{n+1}}{|x|} \right) = \pi i \left(\frac{x^1}{|x|} \dots \frac{x^{n+1}}{|x|} \right) = \left[\frac{x^1}{|x|} \dots \frac{x^{n+1}}{|x|} \right] = [x^1 \dots x^{n+1}]$$
$$pp^{-1}[tx^1 \dots tx^{n+1}] = \pi^{-1}\left(\frac{tx^1}{|t||x|} \dots \frac{tx^{n+1}}{|t||x|} \right) = [x^1 \dots x^{n+1}]$$
$$p^{-1}p(x^1 \dots x^{n+1}) = p^{-1}[x^1 \dots x^{n+1}] = (x^1 \dots x^{n+1}, \text{ as } (x^1)^2 + \dots + (x^{n+1})^2 = 1$$

p bijective, smooth, and

$$\widehat{\pi}^{-1} = \pi^{-1}\varphi_i^{-1}(y^1 \dots y^n) = \pi^{-1}(y^1 \dots y^{i-1}, 1, y^i, y^i \dots y^n) = (y^1 \dots y^{-1}, 1, y^i \dots y^n) \text{ smooth}$$
$$\widehat{i}^{-1}(x^1 \dots x^{n+1}) = (\varphi_i^{\pm})(i^{-1}(x^1 \dots x^{n+1})) = (\varphi_i^{\pm})\left(\frac{x^1}{|x|} \dots \frac{x^{n+1}}{|x|} \right) = \left(\frac{x^1}{|x|} \dots \frac{\widehat{x}^i}{|x|} \dots \frac{x^{n+1}}{|x|} \right)$$

with $|x|^2 = (x^1)^2 + \dots + (x^{n+1})^2$

$\widehat{i}^{-1}$ smooth.

So $p^{-1} = i^{-1}\pi^{-1}$ smooth.

p diffeomorphism.

Diffeomorphisms.

Lie Groups. Lie group - smooth manifold G, with $\begin{array}{ll} m : G \times G \to G & i : G \to G \\ m(g,h) = gh & i(g) = g^{-1} \end{array}$ m, i smooth.

Exercise 2.10.

$$f : G \times G \to G$$
$$(g,h) \mapsto gh^{-1}$$
$$m(g,h) = f(g,h^{-1}) = gh \qquad (\exists\, h^{-1} \text{ since } G \text{ is a group})$$
$$f(e,g) = eg^{-1} = g^{-1} = i(g)$$

So m, i smooth as f is smooth.

Example 2.8 (Lie Group Homomorphisms)

(a)

(b)

(c)

(d)

(e)

(f) $C_g : G \to G$ conjugation.

$$C_g(h) = ghg^{-1}$$

C_g smooth because Lie group multiplication is smooth.

Suppose $hl = m$

$$C_g(hl) = ghlg^{-1} = ghg^{-1}glg^{-1} = C_g(m) = C_g(h)C_g(l)$$

Smooth Covering Maps.

Partitions of Unity.

Theorem 1 (2.23)**.** ***(Existence of Partitions of Unity)****. Suppose M is a smooth manifold with or without boundary, and $\chi = (X_\alpha)_{\alpha\in A}$ is any indexed open cover of M.*
Then $\exists$ smooth partition of unity subordinate to χ

Proof. □

Applications of Partitions of Unity.

Lemma 3 (2.26)**.** *Suppose M smooth manifold with or without boundary.*
closed $A \subseteq M$
$f : A \to \mathbb{R}^k$ smooth function.

$\forall$ open U, $U \supset A$,
$\exists$ smooth $\widetilde{f} : M \to \mathbb{R}^k$ s.t. $\widetilde{f}\big|_A = f$ and supp$\widetilde{f} \subseteq U$

Proof. Given $\begin{gathered} A \subseteq M \\ f : A \to \mathbb{R}^k \\ \forall\, U \supseteq A \end{gathered}$

$\forall\, p \in A$, choose neighborhood W_p of p, smooth $\widetilde{f_p} : W_p \to \mathbb{R}^k$ s.t. $\widetilde{f_p} = f$ on $W_p \bigcap A$

Replace W_p by $W_p \bigcap U$, so $W_p \subseteq U$
$\{W_p | p \in A\} \bigcup \{M \backslash A\}$ open cover of M
Let $\{\psi_p | p \in A\} \bigcup \{\psi_0\}$ smooth partition of unity subordinate to this cover, with supp$\psi_p \subseteq W_p$, supp$\psi_0 \subseteq M\backslash A$ (Thm.2.23, Existence of Partition of Unity)
$\forall\, p \in A$, $\psi_p \widetilde{f_p}$ smooth on W_p, and $\psi_p\widetilde{f_p}$ has smooth extension to all of M if $\psi_p \widetilde{f_p} = 0$ on $M\backslash$suppψ_p
on open $W_p \backslash$suppψ_p, they agree
define $\widetilde{f} : M \to \mathbb{R}^k$

$$\widetilde{f}(x) = \sum_{p\in A} \psi_p(x) \widetilde{f_p}(x)$$

$\{$supp$\psi_p\}$ locally finite, so $\sum_{p\in A} \psi_p \widetilde{f_p}(x)$ has only finite number of nonzero terms in neighborhood of $\forall\, x \in M$, so $\widetilde{f}(x)$ smooth
If $x \in A$, $\psi_0(x) = 0$, $\widetilde{f_p}(x) = f(x) \qquad \forall\, p$ s.t. $\psi_p(x) \neq 0$, so

$$\widetilde{f}(x) = \sum_{p\in A} \psi_p(x) f(x) = (\psi_0(x) + \sum_{p\in A} \psi_p(x)) f(x) = f(x)$$

so $\widetilde{f}$ extension of f.
Lemma 1.13(b), so supp$\widetilde{f} = \overline{\bigcup_{p\in A} \text{supp}\psi_p} = \bigcup_{p\in A}$ supp$\psi_p \subseteq U$

□

Problems. **Problem 2-11.**
G connected Lie group.
$U \subset G$ neighborhood of identity e.
Let $H \leq G$ subgroup generated by U. (cf. wikipedia - Generating set of a group U of H s.t. $\forall\, h \in H$, h = finite combination of u's $\in U$, u^{-1}'s)
$hU \subset H$. $\forall\, h \in H$
hU open neighborhood of h, since multiplication by h is cont. (U open, so $h^{-1}(hU)$ open)
So h open.
Let $g \in H^c$, $\left\{ {}^{U' = \{u^{-1} | u \in U\}} \right\}$. $i(u) = u^{-1}$. i inversion map, cont. $i(U) = i^{-1}(U) = U'$ open.
$gU' \subset H^c$ (otherwise $g \in H$, for if $h \in gU' \bigcap H$, $g \in hU \subset H$)
H^c open, so H closed.
H open and closed, so since G connected, $H = G$. U generates G.

3. Tangent Vectors

Tangent Vectors.

Geometric Tangent Vectors. Now, 1 thing that a Euclidean tangent vector provides is a means of taking "directional derivatives" of a function.

e.g. $\forall\, v_a \in \mathbb{R}^n_a$, v_a yields
$$\begin{aligned} &D_v|_a : C^\infty\mathbb{R}^n \to \mathbb{R} \\ &D_v|_a f = D_v f(a) = \left.\frac{d}{dt}\right|_{t=0} f(a+tv) \end{aligned} \qquad (3.1)$$
which takes the directional derivative in the direction v at a.
$D_v|_a$ linear and $D_v|_a (fg) = f(a)\, D_v|_a g + g(a)\, D_v|_a f$ (3.1)

$$\left.\frac{d}{dt}\right|_{t=0} f(a+tv) = v^i \frac{\partial f}{\partial x^i}(a)$$

$$D_v|_a f = v^i \frac{\partial f}{\partial x^i}(a)$$

If $v_a = e_j|_a$,

$$D_v|_a f = \frac{\partial f}{\partial x^i}(a)$$

derivation at a, a linear $X : C^\infty\mathbb{R}^n \to \mathbb{R}$, $a \in \mathbb{R}^n$

$$X(fg) = f(a)Xg + g(a)Xf$$

$T_a\mathbb{R}^n$ set of all derivations of $C^\infty\mathbb{R}^n$ at a. $T_a\mathbb{R}^n$ vector space.

Lemma 4 (3.1)**.** *$X(c) = 0$, 0 const., $X(fg) = 0$ if $f(a) = g(a) = 0$*

Proposition 3 (3.2)**.** *$\forall\, a \in \mathbb{R}^n$, map $v_a \mapsto D_v|_a$ isomorphism from $\mathbb{R}^n_a$ onto $T_a\mathbb{R}^n$*

Proof. $v_a \mapsto D_v|_a$ linear.

$$\begin{aligned} D_{bv+cw}|_a f = D_{bv+cw} f(a) = \left.\frac{d}{dt}\right|_{t=0} f(a+t(bv+cw)) = (bv^i + cw^i)\frac{\partial f}{\partial x^j}(a) = bv^j \frac{\partial f}{\partial x^j}(a) + cw^i \frac{\partial f}{\partial x^i}(a) = \\ = b\left.\frac{d}{dt}\right|_{t=0} f(a+tv) + c\left.\frac{d}{dt}\right|_{t=0} f(a+tw) = b\, D_v|_a f + c\, D_w|_a f = (bD_v|_a + c\, D_w|_a) f \end{aligned}$$

injective: $v_a \in \mathbb{R}^n_a$, write $v_a = v^i e_i|_a$
take f to be jth coordinate function $x^j : \mathbb{R}^n \to \mathbb{R}$, thought of as a smooth function on $\mathbb{R}^n$

$$0 = D_v|_a (x^j) = v^i \delta^j_i = v^j \quad \forall\, j$$

Then $v_a = 0$
surjective, let $X \in T_a\mathbb{R}^n$
define $v^i = X(x^i)$
We'll show $X = D_v|_a$, $v = v^i e_i$
Let f be any smooth function on $\mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}$.
By Taylor's formula with remainder (Thm. A.58), $\exists$ smooth $g_1 \dots g_n$ on $\mathbb{R}^n$ s.t. $g_i(a) = 0$

$$f(x) = f(a) + \sum_{i=1}^n \frac{\partial f}{\partial x^i}(a)(x^i - a^i) + \sum_{i=1}^n g_i(x)(x^i - a^i)$$

Recall Lemma 3.1, and note $x^i - a^i = 0$ if $x = a$

$$\begin{aligned} Xf = X(f(a)) + \sum_{i=1}^n X\left(\frac{\partial f}{\partial x^i}(a)(x^i - a^i)\right) + \sum_{i=1}^n X(g_i(x)(x^i - a^i)) = 0 + \sum_{i=1}^n \frac{\partial f}{\partial x^i}(a)\left(X(x^i) - X(a^i)\right) = \\ = \sum_{i=1}^n X(x^i)\frac{\partial f}{\partial x^j}(a) = v^i \frac{\partial f}{\partial x^i}(a) = D_v|_a f \end{aligned}$$

$$\Longrightarrow X = D_v|_a$$

□

Corollary 1 (3.3)**.** *$\forall\, a \in \mathbb{R}^n$, n derivatives. $\left.\frac{\partial}{\partial x^1}\right|_a \dots \left.\frac{\partial}{\partial x^n}\right|_a$ defined by $\left.\frac{\partial}{\partial x^i}\right|_a f = \frac{\partial f}{\partial x^i}(a)$ form a basis for $T_a\mathbb{R}^n$, $dimT_a\mathbb{R}^n = n$*

Proof. as above, $\forall\, X \in T_a\mathbb{R}^n$, X derivation. $Xf = v^i \left.\frac{\partial}{\partial x^i}\right|_a f$, so $\{\left.\frac{\partial}{\partial x^i}\right|_a\}$ spans $T_a\mathbb{R}^n$
for linear independence, $0 = v^i \left.\frac{\partial}{\partial x^i}\right|_a f$. Then $v^i = 0$, $\forall\, i$
Note $\left.\frac{\partial}{\partial x^i}\right|_a = D_{e_i}|_a$ with $e_i = \delta_i^{\ j} e_j$.

□

Tangent Vectors on a Manifold. linear $X : C^\infty M \to \mathbb{R}$ derivation at p if $X(fg) = f(p)Xg + g(p)Xf$. $\forall\, f, g \in C^\infty M$
tangent space T_pM = set of all derivations of $C^\infty M$

Exercise 3.1. Lemma 3.4

(a) f const. So let $f = 0$.

$$X(cf) = cX(f) = X(ff) = f(p)Xf + f(p)Xf = 2X(cf) \Longrightarrow X(f) = 0$$

(b) if $f(p) = g(p) = 0$, $X(fg) = 0$, by definition.

Pushforwards. Let smooth $F : M \to N$
$\forall\, p \in M$, define $F_* : T_pM \to T_{F(p)}N$, pushforward

$$(F_*X)(f) = X(fF)$$

$f \in C^\infty N$, $fF \in C^\infty M$, so $X(fF)$ makes sense.

$$(F_*X)(fg) = X((fg)F) = X((fF)(gF)) = fF(p)X(gF) + gF(p)X(fF) = f(F(p))(F_*X)g + g(F(p))(F_*X)(f)$$

Lemma 5 (3.5)**.** ***(Properties of Pushforwards)***

(a)

$$(F_*X)(af + bg) = X((af + bg)F) = aX(fF) + bX(gF) = aF_*X(f) + bF_*X(g)$$

F_*X linear

(b)

$$gf_* = g_*f_* : T_pM \to T_{gf(p)}P$$

Exercise 3.2.

(a)

(b) $\begin{aligned} f &: M \to N \\ g &: N \to P \\ gf &: M \to P \end{aligned}$ Consider $\begin{aligned} p &\in M, (U, \varphi) \subset M \\ q = f(p) &\in N, (V, \psi) \subset N \\ r = g(q) &\in P, (W, \chi) \subset P \end{aligned}$ $\quad \begin{aligned} V &\in T_pM \\ W &\in T_qN \\ X &\in T_{g(q)}P \end{aligned}$ $\quad \begin{aligned} V &= V^\alpha \frac{\partial}{\partial x^\alpha} \\ W &= W^\beta \frac{\partial}{\partial y^\beta} \\ X &= X^\gamma \frac{\partial}{\partial z^\gamma} \end{aligned}$

$$\begin{array}{lll} f_*V \in T_{f(p)}N & g_*W \in T_{g(q)}P & gf_*V \in T_{gf(p)}P \\ f_*V[h] = V[hf] & g_*W[k] = W[kg] & gf_*V[l] = V[lgf] \\ f_*V[h\psi^{-1}(y)] = V[hf\varphi^{-1}(x)] & g_*W[k\chi^{-1}(z)] = W[kg\psi^{-1}(y)] & gf_*V[l\chi^{-1}(z)] = V[lgf\varphi^{-1}(x)] \end{array}$$

$$\Longrightarrow g_*(f_*V)[l] = f_*V[lg] = V[lgf] = gf_*V[l]$$

In coordinates,

$$g_*(f_*V)[l\chi^{-1}(z)] = f_*V[lg\psi^{-1}(y)] = W^\alpha \frac{\partial}{\partial y^\alpha}[lg\psi^{-1}(y)] = V^\mu \frac{\partial y^\alpha}{\partial x^\mu} \frac{\partial}{\partial y^\alpha}[lg\psi^{-1}(y)] = V^\mu \frac{\partial(y^\alpha f\varphi^{-1}(x))}{\partial x^\mu} \frac{\partial}{\partial y^\alpha}[lg\psi^{-1}(y)]$$

$$gf_*V[l\chi^{-1}(z)] = V[lgf\varphi^{-1}(x)] = V^\alpha \frac{\partial}{\partial x^\alpha}(lgf\varphi^{-1}(x))$$

$$\Longrightarrow \frac{\partial}{\partial x^\mu}(lgf\varphi^{-1}(x)) = \frac{\partial(y^\alpha f\varphi^{-1}(x))}{\partial x^\mu} \frac{\partial}{\partial y^\alpha}[lg\psi^{-1}(y)]$$

Chain rule is reobtained.

Alternatively,

Let $\begin{aligned} F &: M \to N \\ G &: N \to P \end{aligned}$ $\quad p \in M$

$GF : M \to P$ $\quad . \; h \in C^\infty P$
$(GF)_* : T_pM \to T_{GF(p)}P$
Now consider

$$G_*(F_*X)(h) = F_*X(hG) = X(hGF) \Longrightarrow G_*F_* = (GF)_*$$

(c)

$$(1_M)_* X(f) = X(f1) = X(f)$$

so $(1_M)_* = 1_{T_pM}$

(d) Now

$$M \xrightarrow{F} N$$

$$T_pM \xrightarrow{F_*} T_{F(p)}N$$

cf. Tu, pp. 80, 8 Tangent Space, Corollary 8.7. If $F : M \to N$, $p \in M$, F diffeomorphism,

$$F_* : T_pM \to T_{F(p)}N \text{ isomorphism.}$$

Proof. To say that F is a diffeomorphism, means that it has a differentiable inverse $G : N \to M$ s.t.

$$\begin{aligned} GF &= 1_M & (GF)_* &= G_*F_* = (1_M)_* = 1_{T_pM} \\ FG &= 1_N & (FG)_* &= F_*G_* = (1_N)_* = 1_{T_{F(p)}N} \end{aligned}$$

So then F_*, G_* are isomorphisms, bijective homomorphism. □

identify T_pU with T_pM $\forall\, p \in U$. Since the action of a derivation on a function depends only on the values of the function in an arbitrary small neighborhood. In particular, this means that any tangent vector $X \in T_pM$ can be unambiguously applied to functions defined only in a neighborhood of p not necessarily on all of M (note partition of unity, bump functions).

Proposition 4 (3.7). *open submanifold $U \subset M$, inclusion $i : U \hookrightarrow M$. $\forall\, p \in U$, $i_* : T_pU \to T_pM$ isomorphism.*

Exercise 3.3. If $F : M \to N$ local diffeomorphism,
$\forall\, p \in M$, $\exists$ open $U \ni p$ s.t. $F(U)$ open in N and $F|_U : U \to F(U)$ diffeomorphism.
Consider $G : F(U) \to U$, G diff. (smooth) inverse of $F|_U$.
$F(p) \in$ open $F(U)$

$$\begin{aligned} (F|_U)_* &: T_pU \to T_{F(p)}F(U) \\ (G)_* &: T_{F(p)}F(U) \to T_pU \\ (FG)_* &= (1_{F(U)})_* = 1_{T_{F(P)}}F(U) = (F|_U)_*G_* \\ (GF)_* &= (1_U)_* = 1_{T_pU} = G_*(F|_U)_* \end{aligned}$$

Then $(F|_U)_*$, G_* are isomorphisms between $T_pU \to T_{F(p)}F(U)$. This must be true $\forall\, p \in M$, so $F_* : T_pM \to T_{F(p)}N$ isomorphism $\forall\, p \in M$

I think the idea for a local diffeomorphism is that "$F_* : T_pM \to T_{F(p)}F(M)$".

Computations in Coordinates.

Change of Coordinates.

The Tangent Bundle.

Proposition 5 (3.18). *TM has smooth structure making it $2n$-dim. smooth manifold.*
$\pi : TM \to M$ smooth

Proof. $\forall$ chart (U, φ) for M, $\varphi = (x^1 \dots x^n)$
Define $\widetilde{\varphi} : \pi^{-1}(U) \to \mathbb{R}^{2n}$

$$\widetilde{\varphi}\left(v^i \left.\frac{\partial}{\partial x^i}\right|_p \right) = (x^1(p) \dots x^n(p), v^1 \dots v^n)$$

$\widetilde{\varphi}(\pi^{-1}(U)) = \varphi(U) \times \mathbb{R}^n$, which is open
$\widetilde{\varphi}$ bijection since

$$\widetilde{\varphi}^{-1}(x^1 \dots x^n, v^1 \dots v^n) = v^i \left.\frac{\partial}{\partial x^i}\right|_{\varphi^{-1}(x)}$$

$$\widetilde{\varphi}\widetilde{\varphi}^{-1} = 1_{\mathbb{R}^{2n}},\ \widetilde{\varphi}^{-1}\widetilde{\varphi} = 1_{\pi^{-1}(U) \subset T_pM}$$

Suppose charts (U, φ) for M,
(V, ψ)

$(\pi^{-1}(U), \widetilde{\varphi})$ on TM ($\widetilde{\varphi}, \widetilde{\psi}$ homeomorphisms, cont. bijective, cont. inverse, π cont., $\pi^{-1}(U)$ open)
$(\pi^{-1}(V), \widetilde{\psi})$ $\pi^{-1}(V)$

$\widetilde{\varphi}(\pi^{-1}(U)\pi^{-1}(V)) = \varphi(UV) \times \mathbb{R}^n$ open in $\mathbb{R}^{2n}$
$\widetilde{\psi}(\pi^{-1}(U)\pi^{-1}(V)) = \psi(UV) \times \mathbb{R}^n$

$$\widetilde{\psi}\widetilde{\varphi}^{-1} : \varphi(UV) \times \mathbb{R}^n \to \psi(UV) \times \mathbb{R}^n$$

$$\widetilde{\psi}\widetilde{\varphi}^{-1}(x^1 \dots x^n, v^1 \dots v^n) = (y^1(x) \dots y^n(x), \frac{\partial y^1}{\partial x^j}(x)v^j \dots \frac{\partial y^n}{\partial x^j}(x)v^j)$$

$\widetilde{\psi}\widetilde{\varphi}^{-1}$ clearly smooth.
Choose countable cover $\{U_i\}$ of M by smooth coordinate domains.
$\{\pi^{-1}(U_i)\}$ countable cover of TM by coordinate domains.
fiber of $\pi : \pi^{-1}(\{p\})$ (fiber is like a preimage of a singleton set)
Consider $\widetilde{x}, \widetilde{y} \in \pi^{-1}(\{p\})$, then $\widetilde{x}, \widetilde{y} \in \widetilde{\varphi}$ (lie in 1 chart)

If $(p, X), (q, Y)$ lie in different fibers, $\exists$ disjoint smooth coordinate domains U, V for M (M Hausdorff) s.t. $p \in U$ and $q \in V$

$\pi^{-1}(U), \pi^{-1}(V)$ disjoint, smooth coordinate neighborhoods s.t. $\pi^{-1}(U) \ni (p, X)$
$\pi^{-1}(V) \ni (q, Y)$

$\pi(x, v) = x$, so π smooth.

□

The Tangent Space to a Manifold with Boundary. define pushforward by F at $p \in M$ to be linear $F_* : T_pM \to T_{F(p)}N$ defined by $(F_*X)f = X(fF)$

Lemma 6 (3.10)**.** *If M^n with boundary, $p \in \partial M$,*
then T_pM n-dim. vector space with basis $\left(\frac{\partial}{\partial x^1}\Big|_p \dots \frac{\partial}{\partial x^n}\Big|_p \right)$ in any smooth chart.

Proof. T_pM vector space with basis $\left(\frac{\partial}{\partial x^i}\Big|_p \right)$
$\forall$ smooth coordinate map φ, $\varphi_* : T_pM \to T_{\varphi(p)}\mathbb{H}^n$ isomorphism by the same argument as manifolds.
$\forall\, a \in \partial\mathbb{H}^n$, T_aH^n n-dim. and spanned by $\left(\frac{\partial}{\partial x^i}\Big|_p \right)$.
Consider inclusion $i : \mathbb{H}^n \hookrightarrow \mathbb{R}^n$. Show $i_* : T_a\mathbb{H}^n \to T_a\mathbb{R}^n$ isomorphism.
Suppose $i_*X = 0$.
Let smooth $f \in \mathbb{R}$ on neighborhood of a in $\mathbb{H}^n$
Let $\widetilde{f}$ extension of f to smooth function on an open subset of $\mathbb{R}^n$ (by extension lemma)

$$\widetilde{f} \circ i = f$$

$$Xf = X(\widetilde{f}i) = (i_*X)\widetilde{f} = 0$$

Then $X = 0$. So i_* injective.
If arbitrary $Y \in T_a\mathbb{R}^n$, define $X \in T_a\mathbb{H}^n$, by

$$Xf = Y\widetilde{f} \qquad Y^i \frac{\partial}{\partial x^i}\Big|_a \widetilde{f} = Y^i \frac{\partial \widetilde{f}}{\partial x^i}(a)$$

This is well-defined because by cont. the derivatives of $\widetilde{f}$ at a are determined by those of f in $\mathbb{H}^n$

$$X(fg) = Y(\widetilde{f}\widetilde{g}) = Y^i \frac{\partial}{\partial x^i}\Big|_a (\widetilde{f}\widetilde{g}) = Y^i \frac{\partial \widetilde{f}(a)}{\partial x^i}\widetilde{g}(a) + Y^i \widetilde{f}(a)\frac{\partial \widetilde{g}}{\partial x^i}(a) = \widetilde{g}(a)Y(\widetilde{f}) + \widetilde{f}(a)Y(\widetilde{g}) =$$
$$= g(a)Xf + f(a)Xg$$

X derivation at a. $Y = i_*X$, so i_* surjective.
i_* isomorphism.

□

Tangent Vectors to Curves. tangent vector to γ at $t_0 \in J \subset \mathbb{R}$

$$\gamma'(t_0) = \gamma_* \left(\frac{d}{dt}\Big|_{t_0} \right) \in T_{\gamma(t_0)}M$$

Tangent vectors act on functions by

$$\gamma'(t_0)f = \left(\gamma_* \frac{d}{dt}\Big|_{t_0} \right) f = \frac{d}{dt}\Big|_{t_0} (f\gamma) = \frac{d(f\gamma)}{dt}(t_0)$$

$\gamma : I \to M$
$\gamma_* : T_{t_0}\mathbb{R} \to T_pM$
$\dot{\gamma} : I \to T_pM$
For (U, x^i), $p \in U$

$$\dot{\gamma}(t_0)f = \gamma_*\left(\left.\frac{d}{dt}\right|_{t_0}\right)f = \left.\frac{d}{dt}\right|_{t_0}(f\gamma) = \left.\frac{d}{dt}\right|_{t_0}(f(x^i)^{-1}x^i\gamma) = \left.\frac{d}{dt}\right|_{t_0}(f(\gamma^i)(t)) = \left.\frac{\partial f}{\partial x^i}\right|_p \left.\frac{d\gamma^i}{dt}\right|_{t_0} = \dot{\gamma}^i \left.\frac{\partial f}{\partial x^i}\right|_p$$
$$\Longrightarrow \dot{\gamma}(t_0) = \dot{\gamma}^i(t_0)\left.\frac{\partial}{\partial x^i}\right|_p$$

Lemma 7 (3.11)**.** *Let $p \in M$. $\forall\, X \in T_pM$, X tangent vector is some smooth curve in M.*

Proof. Let (U, φ), $p \in U$, $X = X^i \left.\frac{\partial}{\partial x^i}\right|_p$
Define $\gamma : (-\epsilon, \epsilon) \to U$ by $\gamma(t) = (tX^1 \dots tX^n)$ i.e. $\gamma(t) = \varphi^{-1}(tX^1 \dots tX^n)$
$\gamma(0) = p$, $\gamma'(0) = X^i \left.\frac{\partial}{\partial x^i}\right|_{\gamma(0)} = X$

□

tangent vectors to curves behave well under composition with smooth maps.

Proposition 6 (3.12)**.** *(The tangent vector to a composite curve) Let smooth $F : M \to N$, smooth curve $\gamma : J \to M$ $\forall\, t_0 \in J$, tangent vector $F\gamma : J \to N$, $t = t_0$ given by*

$$(F\gamma)'(t_0) = F_*(\gamma'(t_0))$$

Proof.

$$(F\gamma)'(t_0) = (F\gamma)_* \left.\frac{d}{dt}\right|_{t_0} = F_*\gamma_* \left.\frac{d}{dt}\right|_{t_0} = F_*(\gamma'(t_0))$$

(use def. of tangent vector to a curve) □

Use it to compute pushforwards.
Suppose $F : M \to N$. $F_* = ?$
$\forall\, X \in T_pM$, choose smooth γ whose tangent vector at $t = 0$ is X,

$$F_*X = (F\gamma)'(0) \qquad (3.10)$$

Indeed, Lemma 3.11 $\gamma(0) = p$
$\dot{\gamma}(0) = X$

$$F_*(\gamma'(0)) = F_*X = (\dot{F\gamma})(0)$$

Alternative Definitions of the Tangent Space. smooth function element (f, U), open $U \subset M$, smooth $f : U \to \mathbb{R}$
$\forall\, p \in M$, $(f, U) \sim (g, V)$, if $f \equiv g$ on some neighborhood $W \ni p$

$$= \text{ germ of } f \text{ at } p$$
$$\{[(f, U)]\} \text{ at } p \ = C_p^\infty$$

C_p^∞ real vector space.

$$[(f, U)] + [(g, V)] = [(f + g, UV)]$$
$$c[(f, U)] = [(cf, U)]$$
$$[(f, U)][(g, V)] = [(fg, UV)]$$

Denote $[(f, U)] = [f]_p$
T_pM = set of all derivations, linear $X : C_p^\infty \to \mathbb{R}$ s.t.

$$X[fg]_p = f(p)X[g]_p + g(p)X[f]_p$$

By Prop. 3.6. $Xf = Xg$ if $f = g$ on some neighborhood W of p ($\psi \in C^\infty M$ smooth bump function with support needed).
This space is isomorphic to the tangent space as we've defined it (Prob. 3-7).

Problems. **Problem 3-1.** M connected.
$X \in T_pM$

$$F_*X(fg) = X((fg)F) = X((fF)(gF)) = f(F(p))X(gF) + g(F(p))X(fF) = 0$$

Let $g = f \in C^\infty M$. $f(F(p))X(fF) = 0$ $X(fF) = 0$. fF const. by properties of tangent vector X. f arbitrary, so F const.
Problem 3-3. M^m diffeomorphic to N^n by F.
T_pM isomorphic to $T_{F(p)}N$. Then $\dim(T_pM) = \dim(T_{F(p)}N)$ $m = n$

cf. Tu, Corollary 8.8. Indeed for $\begin{matrix} (U,\varphi), U \ni p \\ (V,\psi), V \ni F(p) \end{matrix}$ $\quad \begin{matrix} (\psi F \varphi^{-1}) : \mathbb{R}^m \to \mathbb{R}^n \text{ is a diffeomorphism} \\ (\psi F \varphi^{-1})_* : T_{\varphi(p)}\mathbb{R}^m \to T_{\psi(F(p))}\mathbb{R}^n \text{ is an isomorphism} \end{matrix}$
cf. wj32 has some good solutions specifically for Lee (2012) http://wj32.org/wp/wp-content/uploads/2012/12/Introduction-to-Smooth-Manifolds.pdf **Problem 3-4.**
Adapted from wj32 http://wj32.org/wp/wp-content/uploads/2012/12/Introduction-to-Smooth-Manifolds.pdf

Now clearly $\forall\, p \in S^1$ $\exists\, p \in S^1$, $\exists\, (U,\theta) \in \mathcal{A}_{S^1}$ s.t. $\begin{matrix} \theta : U \to \mathbb{R} \\ \theta(p) \in \mathbb{R} \end{matrix}$ $\quad U \ni p$. $T_pS^1 \ni \frac{\partial}{\partial \theta}\big|_p$.

Define $F : S^1 \times \mathbb{R} \to TS^1$ s.t.

$$(p,r) \mapsto r \left.\frac{\partial}{\partial \theta}\right|_p \text{ for } U \ni p, \left.\frac{\partial}{\partial \theta}\right|_p \in T_pU$$

Consider smooth chart (U,θ), $U \ni p$, $\begin{matrix} \widehat{\theta} : U \times \mathbb{R} \to \mathbb{R} \times \mathbb{R} \\ \widehat{\theta}(p,r) = (\theta(p), r) \end{matrix}$
The strategy is to think of the map from $\mathbb{R}^2$ to $\mathbb{R}^2$. So consider

$$F = F \circ \widehat{\theta}^{-1}\widehat{\theta}$$

Consider smooth chart $\xi_\theta : TU \to \mathbb{R}^2$

$$\xi_\theta : X \mapsto ((\theta \circ \pi)(X), (d\theta)_{\pi(X)}X)$$

$$F \circ \widehat{\theta}^{-1}(\theta, r) = r \left.\frac{\partial}{\partial \theta}\right|_p$$

$$\xi_\theta \circ F \circ \widehat{\theta}^{-1}(\theta, r) = \left(\theta \circ \pi \left(r \frac{\partial}{\partial \theta} \right)_p, (d\theta)_{\pi(r \frac{\partial}{\partial \theta}|_p)} \left(r \left.\frac{\partial}{\partial \theta}\right|_p \right) \right) = (\theta(p), r)$$

$F^{-1} : X_p \in T_pS^1 \mapsto (p,r)$ where $X_p = r\frac{\partial}{\partial \theta}\big|_p$
$\xi_\theta \circ F \circ \widehat{\theta}^{-1}$ clearly smooth and bijective. F diffeomorphism.

4. Submersions, Immersions, and Embeddings

rank - dim. of its image

smooth immersions - whose differentials are injective everywhere
smooth embeddings - injective smooth immersions that are also homeomorphisms onto their images

Maps of Constant Rank. Suppose smooth manifolds M, N with or without boundary.
rank of F at p - given smooth $F : M \to N$, $p \in M$,
rank of linear $dF_p : T_pM \to T_{F(p)}N$, i.e. rank of Jacobian of F or dim. of Im$dF_p \subseteq T_{F(p)}N$
constant rank - if F has same rank r at any pt.
smooth $F : M \to N$ smooth submersion if F_* surjective at every pt. $\iff$ rank$F$ = dim$N$ $\quad (\dim M \geq \dim N)$
smooth immersion if F_* injective at every pt. $\iff$ rank$F$ = dim$M$ $\quad (\dim M \leq \dim N)$
EY : 20150717 I get confused between the rank of F and the rank of DF. I'm going to rewrite the above in my notation:

$$M \xrightarrow{F} N \qquad \begin{tikzcd} T_pM \arrow[r, "DF_p"] & T_{F(p)}N \\ p \arrow[u, mapsto] \arrow[r, mapsto, "F"] & F(p) \arrow[u, mapsto] \end{tikzcd}$$

Let $\dim M = \dim T_p M = m$

$\dim N = \dim T_{F(p)} N = n$

Now $DF_p : T_p M \to \text{im}(DF_p) \subseteq T_{F(p)} N$
Let $r = \text{rank} DF_p = \text{dimim}(DF_p)$.
$r \leq n$ (clearly, since $\text{im}(DF_p) \subseteq T_{F(p)} N$)
Recall nullity-rank theorem: For linear $T : V \to T(V) = \text{im}T$,

$\text{dimim}T + \ker T = \dim V$

$\Longrightarrow \text{dimim}T = \dim V - \ker T \leq \dim V$.

$\Longrightarrow r \leq m$

Definition 3. For smooth $F : M \to N$
smooth submersion F if dF surjective $\Longleftrightarrow \text{rank} DF_p = \dim T_{F(p)} N$ i.e. $r = n$
smooth immersion F if dF injective $\Longleftrightarrow \text{rank} DF_p = \dim T_p M$ i.e. $r = m$

Exercise 4.4. cf. http://www.math.ucla.edu/~iacoley/hw/diffhwfall/HW%202.pdf For $q = F(p)$,
If DF_p, DG_q surjective, $DG_q \circ DF_p = D(G \circ F)_p$ surjective. $G \circ F$ smooth submersion.
If DF_p, DG_q injective, $DG_q \circ DF_p = D(G \circ F)_p$ injective. $G \circ F$ smooth immersion.

It'd be instructive to view this as a commutative diagram. For

$$\begin{tikzcd}
T_p M \arrow[r, "DF_p"] \arrow[rr, bend left, "D(G \circ F) = DG_q \circ DF_p"] \arrow[d] & T_{F(p)} M \arrow[r, "DG_q"] \arrow[d] & T_{G(q)} P = T_{G \circ F(p)} P \arrow[d] \\
p \arrow[r, "F"] \arrow[rr, bend right, "G \circ F"'] & F(p) \arrow[r, "G"] & G(q)
\end{tikzcd}$$

$F : M \to N$
$G : N \to P$
$p \in M$
$q = F(p) \in N$
$G(q) \in P$

The Inverse Function Theorem and Its Friends.

Theorem 2 (7.6)**.** ***(Inverse Function Theorem)**. Suppose open $U, V \subset \mathbb{R}^n$, smooth $F : U \to V$*

If $DF(p)$ nonsingular, $p \in U$, $\exists$ connected neighborhood $U_0 \subset U \ni p$

$V_0 \subset V \ni F(p)$

s.t. $F|_{U_0} : U_0 \to V_0$ diffeomorphism.

Let X metric space. $G : X \to X$ contraction if $\exists \lambda < 1$ s.t. $d(G(x), G(y)) \leq \lambda d(x, y)$, $\forall x, y \in X$.
Clearly, $\forall$ contraction is cont.

Lemma 8 (7.7)**.** *(Contraction Lemma) Let X complete metric space*
$\forall$ contraction $G : X \to X$, $\exists!$ fixed pt., i.e. $x \in X$ s.t. $G(x) = x$

Theorem 3 (7.9)**.** *(Implicit Function Theorem) Let open $U \subset \mathbb{R}^n \times \mathbb{R}^k$, $(x, y) = (x^1 \dots x^n, y^1 \dots y^k)$ coordinates on U.*

Suppose $\Phi : U \to \mathbb{R}^k$ smooth. $(a, b) \in U$, $c = \Phi(a, b)$
If $k \times k$ matrix

$$\frac{\partial \Phi^i}{\partial y^j}(a, b)$$

nonsingular,

then $\exists$ neighborhoods $V_0 \subset \mathbb{R}^n$,

$W_0 \subset \mathbb{R}^k$

smooth $F : V_0 \to W_0$ *s.t.*
$(\Phi^{-1}(c))V_0 \times W_0$ *is the graph of* F*, i.e.* $\Phi(x,y) = c$*,* $\forall\ (x,y) \in V_0 \times W_0$ *iff* $y = F(x)$

Embeddings.

Definition 4. smooth embedding of M into N, F, if
smooth immersion $F : M \to N$ and
F topological embedding i.e. F homeomorphism onto its image $F(M) \subseteq N$ in subspace topology.

Exercise 4.16.

Let $F : M \to N$.
$G : N \to P$
F, G smooth immersions so $G \circ F$ smooth immersion (cf. Exercise 4.4, idea is composition of DF, DG is injective).
Now $(G \circ F)(M) = G(F(M))$
F, G bijective onto $F(M), G(N)$. G bijective on $F(M) \subseteq N$ onto $G(F(M))$. Then $G \circ F$ bijective on M onto $G(F(M))$
F, G cont., so $G \circ F$ cont.
F^{-1}, G^{-1} cont., so $(G \circ F)^{-1} = F^{-1} \circ G^{-1}$ cont.
$G \circ F$ homeomorphism onto $G \circ F(M) \subseteq P$.
So $G \circ F$ is a smooth embedding.

Proposition 7 (4.22)**.** *Suppose smooth manifolds* M, N*, with or without boundaries, and injective smooth immersion* $F : M \to N$
If any of the following holds, then F *is a smooth embedding.*
(a) F *open or closed map*
(b) F *proper map*
(c) M *compact*
(d) M *has empty boundary and dim*M = *dim*N

5. Submanifolds

Examples of Embedded Submanifolds.

Lemma 9 (8.6)**.** *(**Graphs as Submanifolds**). If open* $U \subset \mathbb{R}^n$*, smooth* $F : U \to \mathbb{R}^k$*, then graph of* F *is an embedded* n*-dim. submanifold of* $\mathbb{R}^{n+k}$

Proof. Define $\varphi : U \times \mathbb{R}^k \to U \times \mathbb{R}^k$
$\varphi(x,y) = (x, y - F(x))$
φ clearly smooth.
φ diffeomorphism because its inverse can be written explicitly

$$\varphi^{-1}(u,v) = (u, v + F(u))$$

$\varphi(\Gamma(F))$ is the slice $\{(u,v) : v = 0\}$ of $U \times \mathbb{R}^k$, so graph $\Gamma(F)$ is an embedded submanifold. □

Level Sets.

Immersed Submanifolds.

Definition 5. immersed submanifold of M, S, is $S \subseteq M$, with topology (not necessarily subspace topology) with respect to which it's a topological manifold (without boundary), and
smooth structure with respect to inclusion map $i : S \hookrightarrow M$ is smooth immersion (recall $Di \equiv i_*$ injective $\Longleftrightarrow$ rankDi = dim$S$

codim$S$ = dim$M$ − dim$S$

smooth hypersurface is immersed submanifold of codimension 1.

6. The Cotangent Bundle

Covectors. V finite-dim. vector space
covector on V - real-valued linear functional on V, i.e. linear map $\omega : V \to \mathbb{R}$
Exercise 6.1. Suppose $\sum a_i \epsilon^i = A = 0$. Consider arbitrary $v = v^i e_i$. $A(v) = \sum a_i \epsilon^i(v^j e_j) = \sum a_i v^i = 0$
v arbitrary so let $v = \delta^j_k e_j$. $\forall\ i$, $a_i = 0$. linearly independent.

Consider linear map $\omega : V \to \mathbb{R}$, a covector.

$$\omega(v^i e_i) = v^i(\omega(e_i)) = k \in \mathbb{R}$$
$$\omega(e_i) = \omega_j \delta^j_i = \omega_j \epsilon^j(e_i)$$

So ω spanned by $\omega_j \epsilon^j$. Done.

Exercise 6.2. $X \in V$

$$(A^*\omega)(aX + bY) = \omega(A(aX + bY)) = a\omega(AX) + b\omega(AY) \in \mathbb{R} \text{ since } \omega : W^* \to \mathbb{R}$$

linear map $A^*\omega : V \to \mathbb{R}$ is a functional.

$$A^*(a\omega + b\nu)(X) = (a\omega + b\nu)(AX) = a\omega(AX) + b\nu(AX) \in \mathbb{R}$$

Exercise 6.3.

(a) $X \xrightarrow{A} Y \xrightarrow{B} Z$

$X^* \xleftarrow[A^*]{} Y^* \xleftarrow[B^*]{} Z^*$

$(BA)^* : Z^* \to X^*$

$$((BA)^*\zeta)(x) = \zeta(BAx) = (\zeta B)(Ax) = (B^*\zeta)(Ax) = (A^*B^*)\zeta(x)$$

(b)

$$(Id_V)^*(\nu(x)) = \nu(1x) = \nu(x)$$

Tangent Covectors on Manifolds.

The Cotangent Bundle.

Proposition 8 (6.5)**.** *Let M smooth manifold, $T^*M = \coprod_{p \in M} T^*_p M$*

with $\pi : T^*M \to M$
$\omega \in T^*_p M \to p$
natural vector space structure on each fiber,
$\exists\,!$ *smooth manifold structure making $T * M$ rank-n vector bundle over M,*
s.t. all coordinate covectors are smooth local sections

The Differential of a Function.

7. Lie Groups

Basic Definitions. **Lie group** smooth manifold G s.t. multiplication map $\begin{array}{l} m : G \times G \to G \\ m(g,h) = gh \end{array}$

inversion map $\begin{array}{l} i : G \to G \\ i(g) = g^{-1} \end{array}$ smooth.

Proposition 9 (7.1)**.** *If $(g,h) \mapsto gh^{-1}$ smooth, G Lie group*

Exercise 7.2.

Proof. $\forall\, g, h \in G$, $gh^2 \in G$ since G group
$(gh^2, h) \mapsto gh$ smooth. Define $m(g,h) = (gh^2, h) \mapsto gh$. So m smooth.
$1 \in G$ since G group. $(1, g) \mapsto g^{-1}$ smooth so $i(g) = g^{-1}$, defined this way, smooth. ☐

Example 7.3 (Lie Groups).

(a) $A \in GL(n, \mathbb{R})$

$$(AB)_{ij} = A_{ik}B_{kj} \qquad \frac{\partial (AB)_{ij}}{\partial A_{lm}} = \delta_{il}\delta_{km}B_{kj} = \delta_{il}B_{mj}$$
$$\frac{\partial (AB)_{ij}}{\partial B_{lm}} = A_{ik}\delta_{lk}\delta_{mj} = A_{il}\delta_{mj}$$
$$(A^{-1})_{ij} = \frac{1}{\det(A)}\text{adj}(A)_{ij} = \frac{1}{\det(A)}C^T_{ij} = \frac{1}{\det(A)}(-1)^{i+j}\det A_{ji}$$

AB, A^{-1} smooth functions of the entires of A_{ij}, B_{kl}, A_{ij} respectively.

(b)

(c)

Lie Group Homomorphisms. Example 7.4 (Lie Group Homomorphisms)

(a)

(b)

(c)

(d)

(e)

(f) **conjugation** by g

$$C_g : G \to G$$
$$C_g(h) = ghg^{-1}$$

$H \subseteq G$ **normal** if $C_g(H) = H, \ \forall g \in G$

Theorem 4 (7.5)**.** *Every Lie group homomorphism has constant rank.*

8. Vector Fields

Exercise 4.1. Consider $\begin{aligned} 1 : \mathbb{R}^n &\to \mathbb{R}^n, \\ x^i(x) &= x^i \end{aligned}$ smooth structure on $\mathbb{R}^n$, that's open.

Consider $\begin{aligned} &F : T\mathbb{R}^n \to \mathbb{R}^{2n} \\ &F(x^1 \dots x^n, v^1 \dots v^n) = (x^1 \dots x^n, v^1 \dots v^n) \end{aligned}$

$F = F^{-1}$, so clearly $F = 1_{T\mathbb{R}^n}$ is cont., bijective, and it's inverse cont. and smooth. F diffeomorphism.

Exercise 4.2. $F : M \to N$. Consider (3.6)

$$\begin{aligned} &(U, \varphi) \subset M \qquad \varphi = (x^1 \dots x^m) \qquad X = X^i \frac{\partial}{\partial x^i} \\ &(V, \psi) \subset N \qquad \psi = (y^1 \dots y^n) \qquad Y = Y^j \frac{\partial}{\partial y^j} \end{aligned}$$

$$(F_* X)(f) = Y^j \frac{\partial}{\partial y^j} f = X(fF) = X^i \frac{\partial}{\partial x^i} fF = X^i \frac{\partial (f\psi^{-1})}{\partial y^j} \frac{\partial}{\partial x^i}(\psi F^j \varphi^{-1})(\varphi(p)) = X^i \frac{\partial F^j}{\partial x^i}(p)\frac{\partial f}{\partial y^j}$$

where

$$fF = f\psi^{-1}\psi F \varphi^{-1}\varphi \Longrightarrow fF(p) = (f\psi^{-1})(y)(\psi F\varphi^{-1})(\varphi(p))$$

(a serious case of abuse of notation)

For $F_* X$,

$$Y^j = X^i \frac{\partial F^j}{\partial x^i}$$

$$F_* \left. \frac{\partial}{\partial x^i} \right|_p = F_* \frac{\partial}{\partial x^i} = \delta_i^{\ k} \frac{\partial F^j}{\partial x^k}\frac{\partial}{\partial y^j} = \frac{\partial F^j}{\partial x^i}\frac{\partial}{\partial y^j} = \frac{\partial F^j}{\partial x^i}(p) \left. \frac{\partial}{\partial y^j} \right|_{F(p)}$$

with $X^k = \delta_i^{\ k}$

$$F_* : TM \to TN$$
$$F_*(x^1 \dots x^n, v^1 \dots v^n) = (y^1(x) \dots y^n(x), v^i \frac{\partial F^1}{\partial x^i} \dots v^i \frac{\partial F^n}{\partial x^i})$$

Clearly F_* smooth since F smooth.

Lemma 10 (4.8)**.** *Suppose smooth $F : M \to N$,* $\begin{matrix} Y \in \tau(M) \\ Z \in \tau(N) \end{matrix}$
Y, Z, F-related iff $\forall$ smooth $\mathbb{R}$-valued f on open $V \subset N$,

$$Y(fF) = (Zf)F \qquad (4.4)$$

Proof. $\forall\, p \in M$, $\forall$ smooth $\mathbb{R}$-valued f, f defined near $F(p)$

$$Y(fF)(p) = Y_p(fF) = (F_* Y_p)f \qquad (F_*Y)f = Y(fF)$$
$$(Zf)F(p) = (Zf)(F(p)) = Z_{F(p)}f$$

if $(Zf)F(p) = Y(fF)(p) = Zf = (F_*Y)f$

$$(Zf)F = Y(fF) \Longleftrightarrow Z = F_*Y \text{ i.e. iff } Y, Z \quad F\text{-related}$$

□

Vector Fields on a Manifold with Boundary.

Lie Brackets.

Lemma 11 (4.12)**.** *Lie bracket of smooth vector fields V, W,* $\begin{matrix} [V,W] : C^\infty M \to C^\infty M \\ [V,W]f = VWf - WVf \end{matrix}$ *is a smooth vector fields.*

Proof. By Prop. 4.7. (M smooth, map $\mathcal{Y} : C^\infty M \to C^\infty M$ is a derivation iff $\mathcal{Y}f = Yf$, Y some smooth vector field $Y \in \tau(M)$.
Suffices to show $[V, W]$ derivation of $C^\infty M$

$$\begin{gathered} (fg) = V(W(fg)) - W(V(fg)) = V(fWg + gWf) - W(fVg + gVf) = \\ = VfWg + fVWg + VgWf + gVWf - WfVg - fWVg - WgVf - gWVf = \\ = fVWg + gVWf - fWVg - gWVf = f[V,W]g + g[V,W]f \end{gathered}$$

□

extremely useful coordinate formula for Lie bracket

Lemma 12 (4.13)**.** *Let* $\begin{matrix} V = V^i \dfrac{\partial}{\partial x^i} \\ W = W^j \dfrac{\partial}{\partial x^j} \end{matrix}$

$$[V,W] = \left(V^i \frac{\partial W^j}{\partial x^i} - W^i \frac{\partial V^j}{\partial x^i} \right) \frac{\partial}{\partial x^j} \qquad (45)$$
$$[V,W] = (VW^j - WV^j)\frac{\partial}{\partial x^j} \qquad (46)$$

Proof. $[V,W]$ smooth vector field already, its values are determined locally $([V,W]f)|_U = [V,W](f|_U)$
It suffices to compute in a single smooth chart.

$$\begin{gathered} f = V^i \frac{\partial}{\partial x^i}\left(W^j \frac{\partial f}{\partial x^j} \right) - W^j \frac{\partial}{\partial x^j}\left(V^i \frac{\partial f}{\partial x^i} \right) = V^i \frac{\partial W^i}{\partial x^i}\frac{\partial f}{\partial x^j} + V^i W^j \frac{\partial^2 f}{\partial x^i \partial x^j} - W^j \frac{\partial V^i}{\partial x^j}\frac{\partial f}{\partial x^i} - W^j V^i \frac{\partial^2 f}{\partial x^j \partial x^i} = \\ = \left(V^i \frac{\partial W^j}{\partial x^i} - W^i \frac{\partial V^j}{\partial x^i} \right) \frac{\partial f}{\partial x^j} \end{gathered}$$

□

Exercise 4.6.
For Lemma 4.15 (Properties of the Lie Bracket), part (d),
the point is to use the derivative properties of the vector fields.

$$\begin{gathered} [fV, gW]h = fV(gWh) - gW(fVh) = (fVg)(Wh) + g(fV(Wh)) - (gWf)(Vh) - f(gW)(Vh) = \\ = g(fVW)h - (fgWV)h + f(Vg)Wh - g(Wf)Vh \end{gathered}$$

Proposition 10 (4.16)**.** *(Naturality of the Lie Bracket) Let smooth $F : M \to N$,* $\begin{matrix} V_1, V_2 \in \tau(M) \\ W_1, W_2 \in \tau(N) \end{matrix}$, *V_i F-related to W_i, $i = 1, 2$.*

Then $[V_1, V_2]$ F-related to $[W_1, W_2]$

Proof. Use Lemma 4.8, and given V_i, F-related to W_i

$$\begin{aligned} V_1V_2(fF) &= V_1(W_2f)F = W_1W_2fF \\ V_2V_1(fF) &= (W_2W_1f)F \end{aligned} \qquad \Longrightarrow [V_1,V_2](fF) = ([W_1,W_2]f)F$$

So $[V_1,V_2]$, F-related to $[W_1,W_2]$ $\square$

Corollary 2 (4.17)**.** *Suppose $F: M \to N$ diffeomorphism, $V_1, V_2 \in \tau(M)$*
Then $F_*[V_1,V_2] = [F_*V_1, F_*V_2]$

Proof. F diffeomorphism. Then Lemma 4.9, $\exists$ push-forward (or alternatively, by Prop. 4.16, $W_i = F_*V_i$ i.e. F-related).

$$F_*[V_1,V_2] = [W_1,W_2] = [F_*V_1, F_*V_2]$$

$\square$

The Lie Algebra of a Lie Group.

$$L_g = mi_g$$

$$G \xrightarrow{i_g} G \times G \xrightarrow[m]{} G$$

$i_g(h) = (g,h)$, m is multiplication, follows L_g smooth.
L_g diffeomorphism of G, since $L_{g^{-1}}$ smooth inverse.
$\forall$ 2 pts. $g_1, g_2 \in G$, $\exists!\, L_{g_2g_1^{-1}}$ s.t. $L_{g_2g_1^{-1}}g_1 = g_2$ many important properties of Lie groups follow from $L_{g_2g_1^{-1}}$ as diffeomorphism.
vector field X on G *left invariant* if

(1) $$(L_g)_*X_{g'} = X_{gg'} \qquad \forall\, g, g' \in G \qquad (4.8)$$

L_g diffeomorphism.

$$(L_g)_*(aX + bY) = a(L_g)_*X + b(L_g)_*Y$$

set of all smooth left-invariant vector fields on G is a linear subspace $\tau(M)$, *and* closed under Lie bracket.

Lemma 13 (4.18)**.** *Let G Lie group, suppose X, Y smooth left-invariant vector fields on G*
Then $[X,Y]$ also left invariant.

Proof. Given $(L_g)_*X = X$ by def. of left-invariance. $\square$
$(L_g)_*Y = Y$

Vector Fields on Manifolds.

Lemma 14 (8.6)**.** ***(Extension Lemma for Vector Fields)***
M smooth manifold with or without boundary
$A \subseteq M$ closed subset.
Suppose X smooth vector field along A.
Give open $U \supset A$, $\exists$ smooth global vector field $\widetilde{X}$ on M s.t. $\widetilde{X}\big|_A = X$ and supp$\widetilde{X} \subseteq U$

Exercise 8.9.

(a) $\forall\, p \in M$, $\quad X_p = X^i(p)\dfrac{\partial}{\partial x^i}$

$$Y_p = Y^i(p)\frac{\partial}{\partial x^i}$$

$f, g \in C^\infty(M)$

$$(fX)_p = f(p)X_p = f(p)X^i(p)\frac{\partial}{\partial x^i}$$

$$(gY)_p = g(p)Y_p = g(p)Y^i(p)\frac{\partial}{\partial x^i}$$

$$(fX+gY)_p = f(p)X_p + g(p)Y_p = f(p)X^i(p)\frac{\partial}{\partial x^i} + g(p)Y^i(p)\frac{\partial}{\partial x^i} = (f(p)X^i(p) + g(p)Y^i(p))\frac{\partial}{\partial x^i}$$

$f(p)X^i(p) + g(p)Y^i(p)$ smooth so $(fX+gY)_p$ smooth.

(b) Let $g = f$

$$(fX + fY)_p = f(p)X_p + f(p)Y_p = f(p)X^i(p)\frac{\partial}{\partial x^i} + f(p)Y^i(p)\frac{\partial}{\partial x^i} = f(p)(X^i(p) + Y^i(p))\frac{\partial}{\partial x^i} = (f(X+Y))_p$$

Let $Y = X$ so $\forall\, p$,

$$(fX + gX)_p = f(p)X_p + g(p)X_p = f(p)X^i(p)\frac{\partial}{\partial x^i} + g(p)X^i(p)\frac{\partial}{\partial x^i} = (f(p) + g(p))X^i(p)\frac{\partial}{\partial x^i} = ((f+g)X)_p$$

$$(g(fX))_p = g(p)(fX)_p = g(p)f(p)X_p = ((gf)X)_p$$

Let $f = 1$, $g = 0$, $1X = X$

Local and Global Frames.

Vector Fields as Derivations of $C^\infty(M)$. if $X \in \mathfrak{X}(M)$, smooth f defined on open $U \subseteq M$, obtain

$$\begin{aligned} &Xf : U \to \mathbb{R} \\ &(Xf)(p) = X_pF \end{aligned}$$

From J. Lee: (Be careful not to confuse the notations fX and Xf: the former is the smooth *vector field* on U obtained by multiplying X by f, while the latter is the real-valued *function* on U obtained by applying the vector field X to the smooth function f)

Proposition 11 (8.14)**.** $X : M \to TM$
equivalent

(a) X *smooth*
(b) $\forall\, f \in C^\infty(M)$*,* Xf *smooth on* M
(c) $\forall$ *open* $U \subseteq M$*,* $\forall\, f \in C^\infty(M)$*,* $Xf \in C^\infty(U)$

Proof. (a) $\Longrightarrow$ (b), assume X smooth,
let $f \in C^\infty(M)$
M manifold, $\forall\, p \in M$, choose smooth x^i on open $U \ni p$
Then $\forall\, x \in U$,

$$Xf(x) = \left(X^i(x)\frac{\partial}{\partial x^i}\bigg|_x \right) f = X^i(x)\frac{\partial f}{\partial x^i}(x)$$

X^i smooth on U by Prop. 8.1, Xf smooth in U □

Vector Fields and Smooth Maps.

Proposition 12 (8.16)**.** *Suppose smooth* $F : M \to N$*,* $\begin{aligned} &X \in \mathfrak{X}(M) \\ &Y \in \mathfrak{X}(N) \end{aligned}$

Then X, Y F*-related iff* $\forall$ *smooth* h*, defined on open* $V \subset N$

$$X(hF) = (Yh)F$$

Proof. $\forall\, p \in M$, $\forall$ smooth h defined on open $V \ni F(p)$

$$\begin{aligned} X(hF)(p) &= X_p(hF) = dF_p(X_p)h \\ (Yh)F(p) &= Yh(F(p)) = Y_{F(p)}h \end{aligned}$$

$X(hF) = (Yh)F \qquad \forall\, h \in C^\infty(N)$ iff $dF_p(X_p) = Y_{F(p)} \quad \forall\, p$

□

Proposition 13 (8.19)**.** *smooth* M, N*, diffeomorphism* $F : M \to N$
$\forall\, X \in \mathfrak{X}(M)$*,* $\exists!$ *smooth vector field on* N F*-related to* X

Proof. $\forall\, p \in M$, $F(p) = q \in N$

define Y by

$$\begin{aligned} &Y_q = dF_{F^{-1}(q)}(X_{F^{-1}(q)}) = dF_p(X_p) \\ \Longrightarrow &Y_{F(p)} = dF_p(X_p) \end{aligned}$$

$Y : N \to TN$

$$Y = N \xrightarrow{F^{-1}} M \xrightarrow{X} TM \xrightarrow{dF} TN$$

$Y = dF \circ X \circ F^{-1}$
dF, X, F^{-1} smooth. Y smooth. □

pushforward of X by F, denote F_*X

(2) $$(F_*X)_q = dF_{F^{-1}(q)}(X_{F^{-1}(q)}) \qquad (8.7)$$

$(F_*X)_q = dF_p(X_p)$

Corollary 3 (8.21). *Suppose diffeomorphism $F: M \to N$, $X \in \mathfrak{X}(M)$*
$\forall\, h \in C^\infty(N)$

$$((F_*X)h) \circ F = X(h \circ F)$$

Vector Fields and Submanifolds.

Lie Brackets.

Proposition 14 (8.26). ***(Coordinate Formula for the Lie Bracket)***

(3) $$[X,Y] = \left(X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i} \right) \frac{\partial}{\partial x^j} \qquad (8.8)$$

The Lie Algebra of a Lie Group. Recall that G acts smoothly and transitively on itself by left translation:

$$L_g(h) = gh$$

X on G **left-invariant** if

(4) $$d(L_g)_{g'}(X_{g'}) = X_{gg'} \quad \forall\, g, g' \in G \qquad (8.12)$$

L_g diffeomorphism, so

$$(L_g)_* X = X \qquad \forall\, g \in G$$

Example 8.36 (Lie Algebras)

(a)
(b)
(c)
(d)
(e)
(f) $\forall$ vector V becomes Lie algebra if $[,] = 0$
such a Lie algebra is **abelian**

Lie$G$ Lie algebra of all smooth left-invariant vector fields on Lie Group G **Lie algebra of** G

Theorem 5 (8.37). $\epsilon : Lie(G) \to T_eG$
$\epsilon(X) = X_e$
ϵ vector space isomorphism

Proof. If $\epsilon(X) = X_e = 0$ for some $X \in \text{Lie}(G)$
left invariant $d(L_g)_{g'}(X_{g'}) = X_{gg'}$

$$d(L_g)_e(X_e) = X_g = 0 \quad \forall\, g \in G, \text{ so } X = 0$$

ϵ injective
Let $V \in T_eG$ arbitrary.
define (rough) vector field v^L on G by

(5) $$v^L\big|_g = d(L_g)_e(v) \qquad (8.13)$$

□

Example 8.40

(a) $L_b(x) = b + x \qquad bx = b + x \qquad y = x + b$

$d(L_g) = 1$
$X_x = X^i \frac{\partial}{\partial x^i}$
$d(L_b)_x X_x = 1 X_x = X_x = \widetilde{X}^i \frac{\partial}{\partial (x+b)} = \widetilde{X}^i(x+b) \frac{\partial}{\partial x} = X^i(x) \frac{\partial}{\partial x^i}$
X^i constants

$[X,Y] = 0$ if X, Y constants.
lie algebra of $\mathbb{R}^n$ abelian (cf. Example 8.36, (f))

(b)
(c)

Proposition 15 (8.41)**.** ***(Lie Algebra of the General Linear Group)***

(6)
$$Lie(GL(n,\mathbb{R})) \to T_{1_n}GL(n,\mathbb{R}) \to \mathfrak{gl}(n,\mathbb{R}) \qquad (8.14)$$

is isomorphism

Proof. global coordinates ${X^i}_j$ on $GL(n,\mathbb{R})$
natural isomorphism

$$T_1GL(n,\mathbb{R}) \longleftrightarrow \mathfrak{gl}(n,\mathbb{R})$$
$${A^i}_j \left.\frac{\partial}{\partial {X^i}_j}\right|_{1_n} \longleftrightarrow ({A^i}_j)$$

Recall

$$d(L_g)_{g'}(X_{g'}) = X_{gg'}$$

Recall Lie algebra of all smooth left invariant vector fields on G
Recall (8.13)

(7)
$$v|_g = d(L_g)_e(v) \qquad (8.13)$$

L_X is restriction to $GL(n,\mathbb{R})$ of linear map $A \mapsto XA$ on $\mathfrak{gl}(n,\mathbb{R})$

$$L_Xg = Xg = {X^i}_k g^k_j$$
$$L_X1 = X1 = {X^i}_k\delta^k_j = {X^i}_j = X$$

${X^i}_j$ global coordinates on $GL(n,\mathbb{R})$, so

$$\left.\frac{\partial}{\partial {X^i}_j}\right|_1 = \left.\frac{\partial}{\partial {X^i}_j}\right|_X$$
$$DL_X = \frac{\partial}{\partial A^k_j}{(XA)^i}_j = \frac{\partial}{\partial A^k_l}{X^i}_m A^m_j = {X^i}_m\delta^m_k{\delta^l}_j = {X^i}_k{\delta^i}_j$$
$$(DL_X)_1(A) = ({(DL_X)^i}_j{}^l{}_k A^k_l \left.\frac{\partial}{\partial {X^i}_j}\right|_X = ({X^i}_k{\delta^l}_j A^k{}_l) \left.\frac{\partial}{\partial {X^i}_j}\right|_X = ({X^i}_k A^k_j) \left.\frac{\partial}{\partial {X^i}_j}\right|_X$$

□

Problems. Problem 8-1.
$\forall\, p \in A$, choose neighborhood W_p of p, smooth $\widetilde{X} : A \to TM$ s.t. $\widetilde{X} = X$ on $W_p \bigcap A$
Replace W_p by $W_p \bigcap U$, so $W_p \subseteq U$
$\{W_p | p \in A\} \bigcup \{M\backslash A\}$ open cover of M
Let $\{\psi_p | p \in A\} \bigcup \{\psi_0\}$ smooth partition of unity subordinate to this cover, with supp$\psi_p \subseteq W_p$, supp$\psi_0 \subseteq M\backslash A$
$\forall\, p \in A$, (U_p, x^i) smooth coordinate chart

$$X_p = X^i(p)\left.\frac{\partial}{\partial x^i}\right|_p$$
$$X^i : U_p \to \mathbb{R}$$

$\psi_p\widetilde{X}^i(p)$ smooth on W_p
$\psi_p\widetilde{X}^i(p)$ has smooth extension to all of M if $\psi_p\widetilde{X}^i(p) = 0$ on $M\backslash$suppψ_p

on open $W_p\backslash$suppψ_p, they agree
define $\widetilde{X}^i : M \to \mathbb{R}$

$$\widetilde{X}^i(x) = \sum_{p\in A}\psi_p(x)\widetilde{X}^i(p)$$

$\{$supp$\psi_p\}$ locally finite, so $\sum_{p\in A}\psi_p(x)\widetilde{X}^i(p)$ has only finite number of nonzero terms in neighborhood of $\forall\, x \in M$, so $\widetilde{X}^i(x)$ smooth
If $x \in A$, $\psi_0(x) = 0$, $\widetilde{X}^i(x) = X^i(x) \qquad \forall\, p$ s.t. $\psi_p(x) \neq 0$, so

$$\widetilde{X}^i(x) = \sum_{p\in A}\psi_p(x)X^i(x) = (\psi_0(x) + \sum_{p\in A}\psi_p(x))X^i(x) = X^i(x)$$

so $\widetilde{X}^i$ extension of X

Problem 8-2. EULER'S HOMOGENEOUS FUNCTION THEOREM $y = \lambda x$

$$\frac{\partial f}{\partial y^i}\frac{\partial y^i}{\partial\lambda} = x^i\frac{\partial f}{\partial y^i} = x^i\frac{\partial f}{\partial(\lambda x^i)} = \frac{d}{d\lambda}f(y) = \frac{d}{d\lambda}f(\lambda x) = \frac{d}{d\lambda}(\lambda^c f(x)) = c\lambda^{c-1}f(x)$$

$\lambda = 1$

$$\boxed{x^i \frac{\partial f}{\partial x^i} = V_x f(x) = cf(x)}$$

Problem 8-29.

$$\mathfrak{o}(n) = \{ A \in \mathfrak{gl}(n, \mathbb{R}) | A^T + A = 0 \}$$
$$\mathfrak{o}(3) = \{ A \in \mathfrak{gl}(3, \mathbb{R}) | A^T + A = 0 \}$$
$$\left\{ \begin{matrix} \mathfrak{su}(n) = \{ A \in \mathfrak{gl}(n, \mathbb{C}) | A^* + A = 0, \text{tr}A = 0 \} \\ \end{matrix} \right\}$$
$$\left\{ \begin{matrix} \mathfrak{su}(2) = \{ A \in \mathfrak{gl}(2, \mathbb{C}) | A^* + A = 0, \text{tr}A = 0 \} \\ \end{matrix} \right\}$$

$\forall\, A \in \mathfrak{su}(2)$, A is of the form, for $a, b, c \in \mathbb{R}$,

$$A = \begin{pmatrix} ia & b + ic \\ -b + ic & -ia \end{pmatrix}$$

$\forall\, B \in \mathfrak{o}(3)$, B is of the form

$$B = \begin{pmatrix} & a & b \\ -a & & c \\ -b & -c & \end{pmatrix}$$

Then the following identification, F is clearly an isomorphism, as its 1-to-1 and onto:

$$F : \mathfrak{su}(2) \to \mathfrak{o}(3)$$
$$F \begin{pmatrix} ia & b + ic \\ -b + ic & -ia \end{pmatrix} = \begin{pmatrix} & a & b \\ -a & & c \\ -b & -c & \end{pmatrix}$$

9. Integral Curves and Flows

Integral Curves. If smooth curve $\begin{aligned} c : & I \to M \\ c(t) & = p \end{aligned}$

$\forall\, t \in I$, $c' \equiv c'(t) \in T_{c(t)}M \equiv T_p M$

If X vector field on M,
integral curve of X is differentiable $c : I \to M$ s.t.
$\forall\, t \in I$, $c'(t) \equiv \dot{c} = X_{c(t)} = X_p$

EY

$$\begin{aligned} & c : I \to M \\ & c(t) = p \\ & \varphi c(t) = \varphi(p) \Longrightarrow \begin{aligned} c^i(t) & = x^i(t) \\ \dot{c}^i(t) & = \dot{x}^i(t) \end{aligned} \end{aligned}$$

$$X_p = X^i(p) \frac{\partial}{\partial x^i} \equiv X^i \frac{\partial}{\partial x^i} = \dot{c}^i \frac{\partial}{\partial x^i}$$
$$f : M \to \mathbb{R}$$
$$X_p f = X_p f(p) = X_p f \varphi^{-1} \varphi(p) = X_(f \varphi^{-1})(x^j) = X^i \frac{\partial f}{\partial x^i}(x^j) \equiv X^i(p) \frac{\partial f}{\partial x^i}(x^j) = \dot{c}^i \left. \frac{\partial f}{\partial x^i} \right|_p = \frac{d}{dt}(f \circ c)(t)$$

Example 9.1. (Integral Curves)

(a) Let $X = \frac{\partial}{\partial x}$, $(x, y) \in \mathbb{R}^2$

$$c(t) = (x(t), y(t)) = x\partial_x + y\partial_y$$
$$c' = \dot{x}\partial_x + \dot{y}\partial_y = \partial_y \Longrightarrow \dot{y} = 0 \qquad y = b$$
$$\dot{x} = 1 \qquad x = a + t$$
$$c = (a + t, b)$$

(b) $X = x\partial_x - y\partial_x$. Comparing the components of these vectors, we see that this is equivalent to

$$\begin{array}{l} \dot{x} = -y \\ \dot{y} = x \end{array} \Longrightarrow \begin{array}{l} y = a\sin t + b\cos t \\ x = a\cos t - b\sin t \end{array}$$

Proposition 16 (9.2)**.** *Let smooth vector field X on smooth M,*
$\forall\, p \in M$, $\exists\, \epsilon > 0$, $\exists$ smooth $c : (-\epsilon, \epsilon) \to M$ i.e. integral curve of X starting at p

Proof. existence from Thm. D.1 □

Lemma 15 (9.3)**.** *(Rescaling Lemma) $\widetilde{c}(t) = c(at)$ integral curve of aX, where $\widetilde{I} = \{t | at \in I\}$*

Proof. Let smooth f defined in neighborhood of $\widetilde{c}(t_0)$
e.g. of rescaling - $2t = 2 \cdot 1 = 2 \quad a = 2$

$$\widetilde{c}(t) = c(at) = c(\tau) = p \in M$$

$$\dot{\widetilde{c}}(t)f = \frac{d}{dt}(f \circ \widetilde{c})(t) = \frac{d}{dt}(f\varphi^{-1})(\varphi\widetilde{c}(t)) = \frac{d}{dt}(f\varphi^{-1})(\varphi c(at)) = \frac{d}{dt}(f\varphi^{-1})(c^i(at)) = \left.\frac{\partial f}{\partial x^i}\right|_p \frac{dc^i}{d\tau}(\tau)a = a\frac{d}{d\tau}(f \circ c)(\tau) = aX_pf$$

□

Lemma 16 (9.4)**.** *(Translation Lemma)*

$$\begin{aligned} \widetilde{I} &= \{t | t + a \in I\} \\ \widetilde{c}(t) &= c(t + a) \end{aligned}$$

Exercise 9.5.

Proof.

$$\widetilde{c}(t) = c(t + a) = c(\tau) = p \in M$$

$$\dot{\widetilde{c}}(t)f = \frac{d}{dt}(f \circ \widetilde{c})(t) = \frac{d}{dt}f(\widetilde{c}(t)) = \frac{d}{dt}f(c(t + a)) = \left.\frac{\partial f}{\partial x^i}\right|_p \frac{dc^i(\tau)}{d\tau} = \dot{c}^i(\tau)\left.\frac{\partial f}{\partial x^i}\right|_p = X_p^i \left.\frac{\partial f}{\partial x^i}\right|_p = X_pf$$

□

Proposition 17 (9.6)**.** *(Naturality of Integral curves) Suppose smooth $F : M \to N$*

Then $\begin{array}{l} X \in \mathfrak{X}(M) \\ Y \in \mathfrak{X}(N) \end{array}$ *F-related iff F takes integral curves of X to integral curves of Y*

Proof. Recall

$$X, Y\ F-\text{related means } dF(X) = Y$$

Let $\gamma = Fc$

$$\dot{\gamma} = \frac{d}{dt}(F \circ c)(t) = (dF)(\dot{c}) = dF(X) = Y$$

$\Longrightarrow \gamma$ integral curve of Y

if $\gamma = Fc$ integral curve of Y, $\dot{\gamma} = Y$. $\quad q = F(p)$. $p = c(t)$

$$Yg = Y_qg = Y^j \left.\frac{\partial g}{\partial y^j}\right|_q = \dot{\gamma}^j(t) \left.\frac{\partial g}{\partial y^j}\right|_{F(p)} = \frac{d}{dt}(F \circ c) \left.\frac{\partial g}{\partial y^j}\right|_{F(p)} = \frac{\partial y^j}{\partial x^k}\dot{c}^k(t) \left.\frac{\partial g}{\partial y^j}\right|_q = \frac{\partial y^j}{\partial x^k}X_p^k \left.\frac{\partial g}{\partial y^j}\right|_q = (F_*X)g = dF(X)g$$
$$\Longrightarrow dF(X) = Y$$

□

Flows. Let $X \in \mathfrak{X}(M)$
Suppose $\forall\, p \in M$, $\exists!$ integral curve starting at p, $\phi^{(p)} : \mathbb{R} \to M$

$\forall\, t \in \mathbb{R}$, define $\begin{array}{l} \phi_t : M \to M \\ \phi_t(p) = \phi^{(p)}(t) \end{array}$

$$\theta_0(p) = \theta^{(p)}(0) = p$$

EY: ϕ_t pushes p to $\phi^{(p)}(t)$ over time interval t

translation lemma implies $t \mapsto \phi^{(p)}(t + s)$ is integral curve of X starting at $q = \phi^{(p)}(s)$

assuming uniqueness of integral curves, $\phi^{(p)}(t) = \phi^{(p)}(t+s)$, so

$$\begin{aligned}\phi_t \circ \phi_s(p) &= \phi_{t+s}(p) \\ \phi_0(p) &= \phi^{(p)}(0) = p\end{aligned}$$

$\Longrightarrow \phi : \mathbb{R}\times M \to M$ is an action of additive group $\mathbb{R}$ on M.

define *global flow* on M (1-parameter group action) - cont. left $\mathbb{R}$-action on M, i.e.
cont. $\phi : \mathbb{R}\times M \to M$ s.t. $\forall\, s,t \in \mathbb{R}$, $\forall\, pM$

(8) $$\phi(t,\phi(s,p)) = \phi(t+s,p) \quad \phi(0,p) = p \qquad (9.2)$$

given global flow ϕ

$\forall\, t\in\mathbb{R}$, define cont. $\begin{aligned}\phi_t &: M \to M \\ \phi_t(p) &\to \phi(t,p)\end{aligned}$

$$\xrightarrow{(9.2)} \begin{aligned}\phi_t \cdot \phi_s &= \phi_{t+s} \\ \phi_0 &= 1_M\end{aligned}$$

$\phi_t : M \to M$ homeomorphism; if flow smooth, ϕ_t diffeomorphism.

$\forall\, p \in M$, define $\begin{aligned}\phi^{(p)} &: \mathbb{R} \to M \\ \phi^{(p)}(t) &= \phi(t,p)\end{aligned}$

$\phi^{(p)}$ is orbit of p under group action.
smooth global flow $\theta : \mathbb{R}\times M \to M$
$\forall\, p\in M$, define $V_p \in T_pM$

$$V_p = (\theta^{(p)})'(0)$$

Proposition 18 (9.7). *Let smooth global flow $\phi : \mathbb{R}\times M\to M$ on smooth M*

infinitesimal generator X of ϕ $\begin{aligned}&p \mapsto X_p \\ &X_p = \dot{\phi}^{(p)}(0)\end{aligned}$ *is smooth vector field on M, and $\forall\, \phi^{(p)}$, $\phi^{(p)}$ integral curve of X*

Proof. Show X smooth. Use Prop. 8.14,
f smooth on open $U \subseteq M$, $f : U \to \mathbb{R}$

$$Xf(p) = X_pf = \dot{\phi}^{(p)}(0)f \equiv (\dot{\phi}^{(p)}(0))[f] = \frac{d}{dt}(f\circ\phi^{(p)})\Big|_{t=0} = \frac{\partial}{\partial t}\,(f\circ\phi(t,p))|_{t=0}$$

$f\circ\phi(t,p) = f(\phi(t,p))$ smooth function of (t,p) by composition, so $\partial_t(f\circ\phi)$ smooth. So Xf smooth, so X smooth.

Let $q = \phi^{(p)}(a) = \phi_a(p)$

(9) $$\phi^{(q)}(t) = \phi_t(q) = \phi_t(\phi_a(p)) = \phi_{t+a}(p) = \phi^{(p)}(t+a) \qquad (9.4)$$

(10) $$X_qf = \dot{\phi}^{(q)}(0)f = \dot{\phi}^{(q)}(0)[f] = \frac{d}{dt}\,(f\circ\phi^{(q)}(t))\big|_{t=0} = \frac{d}{dt}\,(f\circ\phi^{(p)}(t+a))\big|_{t=0} = \dot{\phi}^{(p)}(a)f = X_{\phi^{(p)}(a)}f \qquad (9.5)$$

□

So by def., $\phi^{(p)}(t)$ integral curve of X

The Fundamental Theorem on Flows. flow domain for M is open $\mathcal{D}\subseteq\mathbb{R}\times M$ s.t. $\forall\, p\in M$, $\mathcal{D}^{(p)} = \{t\in\mathbb{R} | (t,p)\in\mathcal{D}\}$ is an open interval containing 0.

flow on M is cont. $\phi : \mathcal{D}\to M$ s.t. group laws :

(11) $$\forall\, p \in M,\ \phi(0,p) = p \qquad (9.6)$$

(12) $\quad \begin{array}{l} \forall\, s \in \mathcal{D}^{(p)} \\ \forall\, t \in \mathcal{D}^{(\phi(s,p))} \end{array}$ s.t. $s + t \in \mathcal{D}^{(p)}$, $\quad \phi(t, \phi(s,p)) = \phi(t+s, p)$ (9.7)

Proposition 19 (9.11)**.** *If $\phi : \mathcal{D} \to M$ smooth flow,*
then infinitesimal generator X of ϕ smooth vector field and $\forall\, \phi^{(p)}$ integral curve of X

Proof. Recall that

infinitesimal generator X of ϕ, $\begin{array}{l} p \mapsto X_p \\ X_p = \dot{\phi}(0) \end{array}$ now on open $\mathcal{D}$, $\mathcal{D} \subseteq \mathbb{R} \times M$ s.t. $\forall\, p \in M$, $\mathcal{D}^{(p)} = \{ t \in \mathbb{R} | (t,p) \in \mathcal{D} \}$ open,

given ϕ smooth flow,
$\phi(t,q)$ defined and smooth $\forall\, (t,q)$ sufficiently close to $(0,p)$ since $\mathcal{D}$ open. With f smooth on open U in this open neighborhood of $(0,p)$,

$$\Longrightarrow Xf(p) = \frac{\partial}{\partial t}(f \circ \phi(t,p))\Big|_{t=0}$$

$f\phi$ smooth (by composition), so $\partial_t f \phi$ smooth, so X smooth itself around $\forall\, p \in M$.

Suppose $t \in \mathcal{D}^{(p)}$
$\mathcal{D}^{(p)}$, $\mathcal{D}^{(\phi_t(p))} = \mathcal{D}^{(q)}$ open (by def.)
$\phi_{\Delta t}\phi_t(p) = \phi_{\Delta t + t}(p)$ by def. of flow.

□

$$\begin{tikzcd}
\mathfrak{X}(M) & & \\
M \arrow[u] \arrow[drr, "\phi_t"] & & \\
\mathcal{D} \arrow[u] \arrow[d] \arrow[rr, "\phi"'] & & M \arrow[uull, "\left.\frac{\partial}{\partial t}\right|_{t=0} = \left.\frac{d}{dt}\right|_{t=0}"'] \\
\mathcal{D}^{(p)} \arrow[urr, "\phi^{(p)}"'] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
X_p & & \\
p \arrow[u, mapsto] \arrow[drr, mapsto, "\phi_t"] & & \\
(t,p) \arrow[u, mapsto] \arrow[d, mapsto] \arrow[rr, mapsto, "\phi"'] & & \phi(t,p) = \phi^{(p)}(t) = \phi_t(p) \arrow[uull, mapsto, "\left.\frac{\partial}{\partial t}\right|_{t=0} = \left.\frac{d}{dt}\right|_{t=0}"'] \\
t \arrow[urr, mapsto, "\phi^{(p)}"'] & &
\end{tikzcd}$$

Theorem 6 (9.12)**.** *(Fundamental Theorem on Flows) Let smooth vector field X on smooth manifold M.*

$\exists\,!$ *smooth maximal flow $\phi : \mathcal{D} \to M$ whose infinitesimal generator is X (recall $\begin{array}{l} p \mapsto X_p \\ X_p = \dot{\phi}(0) \end{array}$) s.t.*

(a) *$\forall\, p \in M$, curve $\phi^{(p)} : \mathcal{D}^{(p)} \to M$ is unique maximal integral curve of X starting at p.*
(b) *If $s \in \mathcal{D}^{(p)}$, then $\mathcal{D}^{(\phi(s,p))}$ is interval $\mathcal{D}^{(p)} - s = \{ t - s | t \in \mathcal{D}^{(p)} \}$*
(c) *$\forall\, t \in \mathbb{R}$, M_t open in M, and $\phi_t : M_t \to M_{-t}$ diffeomorphism with inverse ϕ_{-t}*

Proof. From Proposition 9.2 ($\forall\, p \in M$, $\exists\, \epsilon > 0$, $\exists$ smooth $c : (-\epsilon, \epsilon) \to M$, i.e. integral curve X starting at p)

Suppose $c, \widetilde{c} : I \to M$ 2 integral curves of X, open I s.t. $c(t_0) = \widetilde{c}(t_0)$ for some $t_0 \in I$
Let $S = \{ t | t \in I,$ s.t. $c(t) = \widetilde{c}(t) \}$
Clearly $S \neq \varnothing$ since $c(t_0) = \widetilde{c}(t_0)$ (hypothesis)
S closed in I by continuity (of $c, \widetilde{c}$)
Suppose $t_1 \in S$
$c(t_1) = \widetilde{c}(t_1) = p$ Then in smooth coordinate neighborhood around $p = c(t_1)$, $c, \widetilde{c}$ both solutions to same ODE with same initial conditions $c(t_1) = \widetilde{c}(t_1) = p$

By uniqueness part of Thm. D.1, $c \equiv \widetilde{c}$ on interval containing t_1
$\Longrightarrow S$ open in I.
Since I connected, $S = I$ (S clopen)
$c = \widetilde{c} \quad \forall\, t \in I$

Thus, $\forall\, c, \widetilde{c}$ that agrees at 1 pt. agree on common domain.

$\forall\, p \in M$, let $\mathcal{D}^{(p)} = \bigcup_\alpha I_\alpha$, open $I_\alpha \subseteq \mathbb{R}$ s.t. $0 \in I_\alpha$, and integral curve $\begin{gathered} c_\alpha : I_\alpha \to M \\ c_\alpha(0) = p \end{gathered}$ starting at p is defined.

define $\begin{aligned} &\phi^{(p)} : \mathcal{D}^{(p)} \to M \\ &\phi^{(p)}(t) = c(t) \end{aligned}$ where c is any integral curve s.t. $c(0) = p$ and c defined on I_α s.t. $0, t \in I_\alpha$.

since all integral curves agree at t by argument above, $\phi^{(p)}$ well-defined
and is obviously unique maximal integral curve starting at p.

Let $\mathcal{D} = \{(t,p) \in \mathbb{R} \times M | t \in \mathcal{D}^{(p)}\}$

define $\begin{aligned} &\phi : \mathcal{D} \to M \\ &\phi(t,p) = \phi^{(p)}(t) \equiv \phi_t(p) \end{aligned}$ (notation for last statement)

By def. ϕ satisfies (a): $\forall\, p \in M$, $\exists !$ maximal integral curve of X, $\phi^{(p)}$, starting at p.

Fix $p \in M$, $s \in \mathcal{D}^{(p)}$
write $q = \phi(s,p) = \phi^{(p)}(s)$

define $\begin{aligned} &\widetilde{c} : \mathcal{D}^{(p)} - s \to M \\ &\widetilde{c}(t) = \phi^{(p)}(t+s) \text{ s.t. } \widetilde{c}(0) = \phi^{(p)}(s) = q \end{aligned}$

By translation lemma (9.4), $\begin{gathered} \widetilde{c}(t) = c(t+s) \\ \widetilde{I} = \{t | t + s \in I\} \end{gathered}$ e.g. $\begin{gathered} I = (-2,6),\ s = 1 \\ \widetilde{I} = (-3,5) \end{gathered}$ $\widetilde{c}(t)$ also integral curve of X.

By uniqueness of ODE solutions,
$\widetilde{c}$ agrees with $\phi^{(q)}$ on their common domain,
equivalent to second group law (9.7)

$$\widetilde{c}(t) = \phi^{(p)}(t+s) = \phi(t+s,p) = \phi^{(q)}(t) = \phi(t,q) = \phi(t,\phi(s,p))$$

□

Lemma 17 (9.19)**.** ***(Escape Lemma)*** *Suppose smooth M, $V \in \mathfrak{X}(M)$.*
If $\gamma : J \to M$ maximal integral curve of V s.t. domain J has finite least upper bound b,
then $\forall\, t_0 \in J$, $\gamma([t_0, b))$ not contained in any compact subset of M

Proof. See Problem 9-6. Solution there. □

Flowouts. Suppose smooth M, $S \subseteq M$ embedded k-dim. submanifold.
smooth $V \in \mathfrak{X}(M)$ s.t. V nowhere tangent to S.
Let $\theta : \mathcal{D} \to M$ be flow of V
Let $\mathcal{O} = (\mathbb{R} \times S) \bigcap \mathcal{D}$
$\Phi = \theta|_{\mathcal{O}}$

(a) $\Phi : \mathcal{O} \to M$ immersion
(b) $\frac{\partial}{\partial t} \in \mathfrak{X}(\mathcal{O})$ is Φ-related to V
(c) $\exists$ smooth $\delta > 0$, $\delta : S \to \mathbb{R}$ s.t.
$\Phi|_{\mathcal{O}_\delta}$ injective, where $\mathcal{O}_\delta \subseteq \mathcal{O}$ flow domain.

(13) $$\mathcal{O}_\delta = \{(t,p) \in \mathcal{O} | |t| < \delta(p)\} \qquad (9.9)$$

Thus $\Phi(\mathcal{O}_\delta)$ immersed submanifold of M containing S. V tangent to $\Phi(\mathcal{O}_\delta)$
(d) If S codim. 1, $\Phi|_{\mathcal{O}_\delta}$ diffeomorphism onto open submanifold of M

Flows and Flowouts on Manifolds with Boundary.

Lie Derivatives.

(14)
$$D_v W(p) = \frac{d}{dt}\Big|_{t=0} W_{p+tv} = \lim_{t\to 0} \frac{W_{p+tv} - W_p}{t} \qquad (9.15)$$

$$D_v W(p) = D_v W^i(p)\ \frac{\partial}{\partial x^i}\Big|_p$$

Lie derivative of W with respect to V

(15)
$$(\mathcal{L}_V W)_p = \frac{d}{dt}\Big|_{t=0} d(\theta_{-t})_{\theta_t(p)}(W_{\theta_t(p)}) = \lim_{t\to 0} \frac{d(\theta_{-t})_{\theta_t(p)}(W_{\theta_t(p)}) - W_p}{t} \qquad (9.16)$$

$$\begin{tikzcd}
\mathfrak{X}(M) & \mathfrak{X}(M) \arrow[l, "d(\phi_{-t}) = (\phi_{-t})_*"'] \\
\mathfrak{X}(M) & \\
M \arrow[u] \arrow[r, "\phi_t"] & M \arrow[uu] \arrow[l, bend left, "\phi_{-t}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
d(\phi_{-t})_{\phi_t(p)}(W_{\phi_t(p)}) & W_{\phi_t(p)} \arrow[l, mapsto, "d(\phi_{-t}) = (\phi_{-t})_*"'] \\
W_p & \\
p \arrow[u, mapsto] \arrow[r, mapsto, "\phi_t"] & \phi_t(p) \arrow[uu, mapsto] \arrow[l, bend left, mapsto, "\phi_{-t}"]
\end{tikzcd}$$

Lemma 18 (9.36)**.** *Suppose smooth M with or without ∂, $V, W \in \mathfrak{X}(M)$*
If $\partial M \neq \varnothing$, assume v trangent to ∂M
Then $\exists$ smooth $(\mathcal{L}_V W)_p \quad \forall\, p \in M$

Proof. $\forall\, (t,x) \in J_0 \times U_0$,
matrix $d(\theta_{-t})_{\theta_t(x)} : T_{\theta_t(x)}M \to T_x M$

$$\left(\frac{\partial \theta^i}{\partial x^j}(-t, \theta(t,x)) \right)$$

Therefore,

$$d(\theta_{-t})_{\theta_t(x)}(W_{\theta_t(x)}) = \frac{\partial \theta^i}{\partial x^j}(-t, \theta(t,x)) W^j(\theta(t,x))\ \frac{\partial}{\partial x^i}\Big|_x$$

□

Exercise 9.37.
Given $V = v^i \frac{\partial}{\partial x^i}$ with constant coefficients (i.e. v^i constant)

$$\begin{aligned}
&\dot{\theta}^{(x)}(t) = V \\
&\dot{\theta}^i(t) = v^i \qquad\qquad \Longrightarrow \qquad \frac{\partial \theta^i}{\partial x^j} = \delta^i{}_j \\
&\theta^i(t) = v^i t + x^i
\end{aligned}$$

$$\frac{d}{dt} W^i(\theta(t,x)) = \frac{\partial W^i}{\partial y^j}(v^j)$$

From the proof of Lemma 9.36,

$$d(\theta_{-t})_{\theta_t(x)}(W_{\theta_t(x)}) = \frac{\partial \theta^i}{\partial x^j}(-t, \theta(t,x)) W^j(\theta(t,x))\ \frac{\partial}{\partial x^i}\Big|_x =$$
$$= \delta^i{}_j W^j(\theta(t,x))\ \frac{\partial}{\partial x^i}\Big|_x = W^i(\theta(t,x))\ \frac{\partial}{\partial x^i}\Big|_x$$

From (9.16)

$$(\mathcal{L}_V W)_p = \frac{d}{dt}\Big|_{t=0} d(\theta_{-t})_{\theta_t(p)}(W_{\theta_t(p)}) = v^j \frac{\partial W^i}{\partial x^j}\ \frac{\partial}{\partial x^i}\Big|_p = D_V W^i(p)\ \frac{\partial}{\partial x^i}\Big|_p = D_V W(p)$$

Theorem 7 (9.38)**.** *If smooth M, and $V, W \in \mathfrak{X}(M)$,*

$$\mathcal{L}_V W = [V, W]$$

Corollary 4 (9.39)**.** (a)
(b)
(c)

(d)
(e)

Exercise 9.40.

(a)

$$\mathcal{L}_V W = [V, W] = -[W, V] = \mathcal{L}_W V$$

(b)
(c)
(d)
(e)

Prop. 9.41 is about derivative of $d(\theta_{-t})_{\theta_t(p)}(W_{\theta_t(p)})$ at other times

Proposition 20 (9.41)**.** *Suppose smooth M with or without ∂ and $V, W \in \mathfrak{X}(M)$*
If $\partial M \neq \emptyset$, assume V tangent to ∂M
Let θ flow of V.

$\forall\, (t_0, p)$ *in domain of* θ

$$\left.\frac{d}{dt}\right|_{t=t_0} d(\theta_{-t})_{\theta_t(p)}(W_{\theta_t(p)}) = d(\theta_{-t_0})((\mathcal{L}_V W)_{\theta_{t_0}(p)})$$

Commuting Vector Fields.

Time-Dependent Vector Fields. Let smooth manifold M
time-dependent vector field on M, V
cont. $V : J \times M \to TM$, interval $J \subseteq \mathbb{R}$
s.t.

$$V(t,p) \in T_p M \quad \forall\, (t,p) \in J \times M$$

i.e. $\forall\, t \in J$,
$\begin{aligned} V_t &: M \to TM \\ V_t(p) &= V(t,p) \end{aligned}$ is a vector field on M
EY : 20150226
time-dependent vector field on M, V is

$$\begin{gathered} \text{cont. } V : J \times M \to TM \qquad \text{interval } J \subseteq \mathbb{R} \\ V(t,p) \in T_p M \quad \forall\, (t,p) \in J \times M \end{gathered}$$

i.e. $\forall\, t \in J$

$$\begin{aligned} V_t &: M \to TM \\ V_t(p) &= V(t,p) \in \mathfrak{X}(M) \end{aligned}$$

integral curve of V is diff. $\begin{aligned} \gamma &: J_0 \to M \\ \dot{\gamma}(t) &= V(t, \gamma(t)) \quad \forall\, t \in J_0 \end{aligned}$ where $J_0 \subset J$ s.t.
$\forall\, X \in \mathfrak{X}(M)$, determines time dependent vector field $V : \mathbb{R} \times M \to TM$ by

$$V(t,p) = X_p$$

$$
\begin{tikzcd}
 & \mathfrak{X}(\mathbb{R}\times M) = T\mathbb{R}\oplus TM & TM \arrow[l, "\oplus T\mathbb{R}"'] \\
J\times M \arrow[ur, "\left.\frac{\partial}{\partial t}\right|_{t=t_0}"] \arrow[r, "t=t_0"] & \mathbb{R}\times M \arrow[u, "\widetilde{V}"] \arrow[ur, "V"] \arrow[r] & M \arrow[u, "X"'] \arrow[d, "\psi_{tt_0}"'] \\
 & \mathcal{E}\subseteq J\times J\times M \arrow[ul, "\widetilde{\theta}"] \arrow[u] \arrow[ur] \arrow[r, "\psi"'] \arrow[d] & M \\
 & \mathcal{E}^{(t_0,p)} \arrow[ur, "\psi^{(t_0,p)}"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & \left.\frac{\partial}{\partial t}\right|_{t=t_0}\widetilde{\theta} = \widetilde{V}_{(t_0,p)} = \left(\left.\frac{\partial}{\partial s}\right|_{t_0}, V(t_0,p)\right) & V(t_0,p) = X_p \arrow[l, mapsto, "+\left.\frac{\partial}{\partial s}\right|_{t_0}"'] \\
\widetilde{\theta}(t,(t_0,p)) = (\alpha(t,(t_0,p)),\beta(t,(t_0,p))) \arrow[ur, mapsto, "\left.\frac{\partial}{\partial t}\right|_{t=t_0}"] \arrow[r, mapsto, "t=t_0"] & (t_0,p) \arrow[u, mapsto, "\widetilde{V}"] \arrow[ur, mapsto, "V"] \arrow[r, mapsto] & p \arrow[u, mapsto, "X"'] \arrow[d, mapsto, "\psi_{tt_0}"'] \\
 & (t,t_0,p) \arrow[ul, mapsto, "\widetilde{\theta}"] \arrow[u, mapsto] \arrow[ur, mapsto] \arrow[r, mapsto, "\psi"'] \arrow[d, mapsto] & \psi(t,t_0,p) = \psi^{(t_0,p)}(t) \\
 & t \arrow[ur, mapsto, "\psi^{(t_0,p)}"'] &
\end{tikzcd}
$$

with

$$
\left.\frac{\partial}{\partial t}\right|_{t=t_0}\widetilde{\theta}(t,(t_0,p)) = \left.\left(\frac{\partial\alpha}{\partial t}(t,(t_0,p)), \frac{\partial\beta}{\partial t}(t,(t_0,p))\right)\right|_{t=t_0} = (1, V(t_0,p))
$$

EY : 20150725 I don't like Lee's choice of notation. Let me rewrite the above diagrams:

$$
\begin{tikzcd}
 & \mathfrak{X}(\mathbb{R}\times M) = T\mathbb{R}\oplus TM & TM \arrow[l, "\oplus T\mathbb{R}"'] \\
J\times M \arrow[ur, "\left.\frac{\partial}{\partial s}\right|_{s=0}"] \arrow[r, "s=0"] & \mathbb{R}\times M \arrow[u, "\widetilde{V}"] \arrow[ur, "V"] \arrow[r] & M \arrow[u, "V_t"'] \arrow[d, "\psi_{st}"'] \\
 & \mathcal{E}\subseteq J\times J\times M \arrow[ul, "\widetilde{\theta}"] \arrow[u] \arrow[ur] \arrow[r, "\psi"'] \arrow[d] & M \\
 & \mathcal{E}^{(t,p)} \arrow[ur, "\psi^{(t,p)}"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & \frac{\partial}{\partial s}\big|_{s=0}\widetilde{\theta} = \widetilde{V}_{(t,p)} = \left(\frac{\partial}{\partial t}\big|_{s=0}, V(t,p)\right) & V(t,p) = V_t(p) \arrow[l, mapsto, "+\frac{\partial}{\partial t}\big|_{s=0}"'] \\
\widetilde{\theta}(s,(t,p)) = (\alpha(s,(t,p)),\beta(s,(t,p))) \arrow[ur, mapsto, "\frac{\partial}{\partial s}\big|_{s=0}"] \arrow[r, mapsto, "s=0"] & (t,p) \arrow[u, mapsto, "\widetilde{V}"] \arrow[ur, mapsto, "V"] \arrow[r, mapsto] & p \arrow[u, mapsto, "V_t"'] \arrow[d, mapsto, "\psi_{st}"'] \\
 & (s,t,p) \arrow[ul, mapsto, "\widetilde{\theta}"] \arrow[u, mapsto] \arrow[ur, mapsto] \arrow[r, mapsto, "\psi"'] \arrow[d, mapsto] & \psi(s,t,p) = \psi^{(t,p)}(s) \\
 & s \arrow[ur, mapsto, "\psi^{(t,p)}"'] &
\end{tikzcd}
$$

with

$$\frac{\partial}{\partial s}\bigg|_{s=0}\widetilde{\theta}(s,(t,p)) = \left(\frac{\partial\alpha}{\partial s}(s,(t,p)), \frac{\partial\beta}{\partial s}(s,(t,p))\right)\bigg|_{s=0} = (1, V(t,p))$$

Theorem 8 (9.48)**.** ***(Fundamental Theorem on Time-Dependent Flows)***
Let M smooth manifold
open $J \subseteq \mathbb{R}$
$V : J \times M \to TM$ smooth time-dependent vector field on M
$\exists$ open $\mathcal{E} \subseteq J \times J \times M$, smooth $\psi : \mathcal{E} \to M$ called time-dependent flow of V s.t.

(a) $\forall\, t_0 \in J, \quad \forall\, p \in M,$
open $\mathcal{E}^{(t_0,p)} = \{t \in J | (t, t_0, p) \in \mathcal{E}_0\}$ *s.t.* $t_0 \in \mathcal{E}^{(t_0,p)}$

smooth curve $\psi^{(t_0,p)} : \mathcal{E}^{(t_0,p)} \to M$
$\psi^{(t_0,p)}(t) = \psi(t, t_0, p)$
is unique maximal integral curve of V with $\psi^{(t_0,p)}(t_0) = p$

(b) *If* $t_1 \in \mathcal{E}^{(t_0,p)}$
$q = \psi^{(t_0,p)}(t_1)$

then $\mathcal{E}^{(t_1,q)} = \mathcal{E}^{(t_0,p)}$ *and*
$\psi^{(t_1,q)} = \psi^{(t_0,p)}$

(c) $\forall\, (t_1, t_0) \in J \times J$
$M_{t_1,t_0} = \{p \in M | (t_1, t_0, p) \in \mathcal{E}\}$ *open in M and*

$\psi_{t_1t_0} : M_{t_1t_0} \to M$ *is a diffeomorphism from $M_{t_1t_0}$ onto $M_{t_0t_1}$ with inverse $\psi_{t_0t_1}$*
$\psi_{t_1t_0}(p) = \psi(t_1, t_0, p)$

(d) *If $p \in M_{t_1t_0}$, $\psi_{t_1t_0}(p) \in M_{t_0t_1}$,*
then $p \in M_{t_2t_0}$ and

(16) $$\psi_{t_2t_1}\psi_{t_1t_0}(p) = \psi_{t_2t_0}(p) \qquad (9.18)$$

Proof. Consider smooth vector field $\widetilde{V} \in \mathfrak{X}(J \times M)$ defined by

$$\widetilde{V}_{(s,p)} = \left(\frac{\partial}{\partial s}\bigg|_s, V(s,p)\right)$$

identify $T_{(s,p)}(J \times M)$ with $T_sJ \oplus T_pM$ (Prop. 3.14)

Let $\widetilde{\theta} : \widetilde{\mathcal{D}} \to J \times M$ flow of $\widetilde{V}$
$\widetilde{\theta}(t,(s,p)) = (\alpha(t,(s,p)), \beta(t,(s,p)))$

then $\alpha : \widetilde{D} \to J$ s.t.
$\beta : \widetilde{D} \to M$

$$\begin{aligned} \frac{\partial \alpha}{\partial t}(t,(s,p)) &= 1 & \alpha(0,(s,p)) &= s \\ \frac{\partial \beta}{\partial t}(t,(s,p)) &= V(\alpha(t,(s,p)),\beta(t,(s,p))) & \beta(0,(s,p)) &= p \end{aligned}$$

$\implies \alpha(t,(s,p)) = t+s$ so

(17)
$$\frac{\partial \beta}{\partial t}(t,(s,p)) = V(t+s, \beta(t,(s,p))) \qquad (9.19)$$

□

Let $\mathcal{E} \subseteq \mathbb{R}\times J \times M$ defined
$\mathcal{E} = \{ (t,t_0,p) | (t-t_0,(t_0,p)) \in \widetilde{\mathcal{D}} \}$
$\mathcal{E}$ open because $\widetilde{\mathcal{D}}$ is.
since $\alpha : \widetilde{D} \to J$, if $(t,t_0,p) \in \mathcal{E}$, then $t = \alpha(t-t_0,(t_0,p)) \in J$, implies $\mathcal{E} \subseteq J \times J \times M$
$\mathcal{E}$ open so $M_{t_1t_0} = \{ p \in M | (t_1,t_0,p) \in \mathcal{E}\}$ open.

define $\psi : \mathcal{E} \to M$
$\psi(t,t_0,p) = \beta(t-t_0,(t_0,p))$
EY : 20150725 Remark: Out of the proof immediately above, there are a number of takeaways that really *should* be mentioned. Let's collect the facts:

$$\begin{aligned} & \widetilde{\theta}(s,(t,p)) = (\alpha(s,(t,p)),\beta(s,(t,p))) := \widetilde{\theta}^{(t,p)}(s) \\ & \widetilde{\theta}(0,(t,p)) = (\alpha(0,(t,p)),\beta(0,(t,p))) = (t,p) \\ & \frac{\partial \alpha}{\partial s}(s,(t,p)) = 1 \\ & \frac{\partial \beta}{\partial s}(s,(t,p)) = V(\alpha(s,(t,p)),\beta(s,(t,p))) \text{ so} \\ & \alpha(s,(t,p)) = s+t \\ & \frac{\partial \beta}{\partial s}(s,(t,p)) = V(s+t,\beta(s,(t,p))) \\ & \left.\frac{\partial}{\partial s}\right|_{s=0} \widetilde{\theta}(s,(t,p)) = \left.\left(\frac{\partial \alpha}{\partial s}(s,(t,p)), \frac{\partial \beta}{\partial s}(s,(t,p)) \right)\right|_{s=0} = (1, V(t,p)) = \frac{d\widetilde{\theta}^{(t,p)}}{ds}(s=0) := \widetilde{V}_{(t,p)} \end{aligned}$$

Also, we can write the flow $\widetilde{\theta}_s$ as

$$\widetilde{\theta}^{(t,p)}(s) = \widetilde{\theta}(s,(t,p)) = \widetilde{\theta}_s(t,p) = (\alpha(s,(t,p)),\beta(s,(t,p))) = (s+t,\beta(s,(t,p)))$$

Now consider the Lie derivative:

$$\begin{tikzcd}
\mathfrak{X}(\mathbb{R}\times M) & \mathfrak{X}(\mathbb{R}\times M) \arrow[l, "d(\widetilde{\theta}_{-s}) = (\widetilde{\theta}_{-s})_*"'] \\
\mathfrak{X}(\mathbb{R}\times M) & \\
J\times M \arrow[u] \arrow[r, "\widetilde{\theta}_s"] & J\times M \arrow[uu] \arrow[l, bend left, "\widetilde{\theta}_{-s}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
d(\widetilde{\theta}_{-s})_{\widetilde{\theta}_s(t,p)}(W_{\widetilde{\theta}_s(t,p)}) & W_{\widetilde{\theta}_s(t,p)} \arrow[l, mapsto, "d(\widetilde{\theta}_{-s}) = (\widetilde{\theta}_{-s})_*"'] \\
W_{(t,p)} & \\
(t,p) \arrow[u, mapsto] \arrow[r, mapsto, "\widetilde{\theta}_s"] & \widetilde{\theta}_s(t,p) \arrow[uu, mapsto] \arrow[l, mapsto, bend left, "\widetilde{\theta}_{-s}"]
\end{tikzcd}$$

with $\widetilde{\theta}$ being the flow of $\widetilde{V}$. Let's define the Lie derivative:

(18)
$$\mathcal{L}_{\widetilde{V}}W = (\mathcal{L}_{\widetilde{V}}W)_{(t,p)} = \left.\frac{d}{ds}\right|_{s=0} (d\widetilde{\theta}_{-s})_{\widetilde{\theta}_s(t,p)}(W_{\widetilde{\theta}_s(t,p)}) = \lim_{s\to 0} \frac{(d\widetilde{\theta}_{-s})_{\widetilde{\theta}_s(t,p)}(W_{\widetilde{\theta}_s(t,p)}) - W_{\widetilde{\theta}_s(t,p)}}{s}$$

Use Case 1 of the proof of Lee's Theorem 9.38, for showing $\mathcal{L}_V W = [V, W]$.
Let open neighborhood $U \subseteq J \times M$, with $(t,p) \in U$. On open U, choose smooth coordinates (t, u^i) on U. By Theorem 9.22, that at a regular point $p \in M$, $\exists (u^i)$ coordinates s.t. $V_p = \frac{\partial}{\partial u^1}$, then consider

$$\widetilde{V} = \frac{\partial}{\partial t} + \frac{\partial}{\partial u^1} \in \mathfrak{X}(\mathbb{R} \times M)$$

with $V(t)(p) = \frac{\partial}{\partial u^1} \in \mathfrak{X}(M)$. (Remember, $V(t)$ is a vector-field that is time-dependent, but is on M. I will use this as a justification for using Thm. 9.22).
Now the flow $\widetilde{\theta}_s$ takes on these forms:

$$\begin{gathered} \widetilde{\theta}^{(t,p)}(s) = \widetilde{\theta}(s,(t,p)) = \widetilde{\theta}_s(t,p) = \\ = (\alpha(s,(t,p)), \beta(s,(t,p))) = (s+t, \beta(s,(t,p))) \end{gathered}$$

Given these conditions, that
$\beta(0,(t,p)) = p = (u^1, u^2, \dots u^n)$ and

$$\left.\frac{\partial \beta}{\partial s}(s,(t,p))\right|_{s=0} = V(t,p) = \frac{\partial}{\partial u^1} = \left.\frac{d}{ds}\beta^{(t,p)}(s)\right|_{s=0}$$

then a β that satisfies these conditions above is

$$\beta(s,(t,p)) = \beta_s(t,p) = (u^1 + s, u^2 \dots u^n)$$

so that we can conclude that

$$\widetilde{\theta}_s(t,p) = (t+s, u^1+s, u^2, \dots, u^n)$$

For fixed s, then

$$d(\widetilde{\theta}_{-s})_{\widetilde{\theta}_s(t,p)} = 1_{T_{\widetilde{\theta}_s(t,p)}(\mathbb{R}\times M)}$$

so that

$$\begin{gathered} d(\widetilde{\theta}_{-s})_{\widetilde{\theta}_s(t,p)}(W_{\widetilde{\theta}_s(t,p)}) = d(\widetilde{\theta}_{-s})_{\widetilde{\theta}_s(t,p)} \cdot W^j(t+s, u^1+s, u^2 \dots u^n) \left.\frac{\partial}{\partial u^j}\right|_{\widetilde{\theta}_s(t,p)} = W^j(t+s, u^1+s, u^2 \dots u^n) \left.\frac{\partial}{\partial u^j}\right|_{(t,p)} \\ \Longrightarrow \left.\frac{d}{ds}\right|_{s=0} W^j(t+s, u^1+s, u^2 \dots u^n) \left.\frac{\partial}{\partial u^j}\right|_{(t,p)} = \left(\frac{\partial}{\partial t} W^j(t, u^1 \dots u^n) + \frac{\partial}{\partial u^1} W^j(t, u^1 \dots u^n)\right) \left.\frac{\partial}{\partial u^j}\right|_{(t,p)} \end{gathered}$$

Thus, we can conclude that

$$\boxed{\mathcal{L}_{\widetilde{V}} W = \mathcal{L}_{\frac{\partial}{\partial t} + V} W = \left(\mathcal{L}_{\frac{\partial}{\partial t}V} W\right)_{(t,p)} = \left(\left(\frac{\partial}{\partial t} + V\right) W^j\right) \left.\frac{\partial}{\partial x^j}\right|_{(t,p)}} \qquad (19)$$

First-Order Partial Differential Equations.

Problems. Problem 9-21. Note that from wikipedia,
ambient isotopy
Let N, M manifolds,
g, h embeddings of N in M
cont. map $F : M \times [0,1] \to M$ s.t.
$F : g \mapsto h$
if $F_0 = 1$
F_t homeomorphism, $F_t : M \to M$
$F_1 : g \mapsto h$

smooth isotopy of M is smooth $H : M \times J \to M$, $J \subseteq R$ interval s.t.

$\forall\, t \in J,\ H_t : M \to M$ is a diffeomorphism.
$H_t(p) = H(p,t)$
Suppose open interval $J \subseteq \mathbb{R}$
smooth isotopy $H : M \times J \to M$

EY : 20140206
By definition

$\forall\, t,\ H_t : M \to M$
$H_t(p) = H(p,t)$

Then $DH_t = (H_t)_*$

$$DH_t : TM \to TM$$

I tried,
for some time t, consider $H(x^i(p), t) = y^j(x^i, t)$

$$\frac{\partial H(p,t)}{\partial t} = \frac{\partial y^j(x^i,t)}{\partial t}$$

Consider integral curve $x = x(t)$ s.t. $\dot{x} = X(t)$

$$\dot{y} = \frac{dy}{dt} = \frac{d}{dt} y(x(t), t) = \frac{\partial y^j}{\partial x^i} \dot{x}^i = (DH_t)X$$
$$(H_t)_* : TM \to TM$$
$$DH_t : X \mapsto \dot{y} = Y$$

$\psi(t, t_0, p) = H_t \circ H_{t_0}^{-1}(p)$ domain $J \times J \times M$??

10. Vector Bundles

Vector Bundles. **(real) vector bundle of rank k over M** is topological space E,
surjective cont. $\pi : E \to M$ s.t.

(i) $\forall\, p \in M$, $E_p = \pi^{-1}(p)$ endowed with structure of k-dim. real vector space
(ii) $\forall\, p \in M$,
$\exists$ neighborhood $U \ni p$
$\exists$ homeomorphism $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ (local trivialization of E over U)

s.t.
- $\pi_U \circ \Phi = \pi$ (where $\pi_U : U \times \mathbb{R}^k \to U$)
- $\forall\, q \in U$, $\Phi|_{E_q}$ is vector space isomorphism from E_q to $\{q\} \times \mathbb{R}^k \cong \mathbb{R}^k$

if M, E smooth manifolds with or without boundary, π smooth, local trivializations Φ can be chosen to be diffeomorphisms,
E **smooth vector bundle**,
$\forall\, \Phi$ that's a diffeomorphism onto its image a **smooth local trivialization**

(real) line bundle - rank 1 vector bundle
complex vector bundle - $\mathbb{R}^k$ replaced by $\mathbb{C}^k$

Exercise 10.1. Suppose E smooth vector bundle over M.
π surjective by def.
EY : 20140204
submersion at $p \in M$, F, if for differentiable manifolds M, N, differentiable $F : M \to N$,
differential $DF_p : T_pM \to T_{F(p)}N$
is surjective

By def., $\forall\, p \in M$, $\exists\, U, \Phi$
Φ vector space isomorphism and can be chosen to be diffeomorphism
Φ diffeomorphism if Φ bijection, Φ^{-1} differentiable

$\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$
$D\Phi : T\pi^{-1}(U) \to T(U \times \mathbb{R}^k) = TU \times T\mathbb{R}^k$

$\pi_U : U \times \mathbb{R}^k \to U$
$D\pi_U : T(U \times \mathbb{R}^k) \to TU$
$\pi : \pi^{-1}(U) \to U$
$\pi = \pi_U \circ \Phi$

So by chain rule,

$$D\pi = D\pi_U D\Phi$$
$$D\pi : T\pi^{-1}(U) \to TU$$

$D\pi_U$ clearly a surjection.
$D\Phi$ bijection.
So $D\pi$ a surjection.

---

11. The Cotangent Bundle

Covectors. **Exercise 11.2.** For linear $\omega : V \to \mathbb{R}$, $\omega \in V^*$

$$\omega(E_j) = \omega(\delta^i_j E_i) = \delta^i_j \omega(E_i) = \omega(E_i)\epsilon^i(E_j)$$
$$\Longrightarrow \omega = \omega(E_i)\epsilon^i$$

So ω spanned by ϵ^i, $\quad i = 1 \dots n$
Suppose $0 = \omega_i \epsilon^i$. Then $\forall\, x \in V$,

$$\omega_i \epsilon^i (x^j E_j) = \omega_i x^j \delta^i_j = \omega_i x^i = \omega(E_i) x^i = \omega(x) = 0$$

Then $\omega = 0$ since $\omega(x) = 0$, $\forall\, x \in V$. So ϵ^i form a linear basis of V^*.
By theorem, $\dim V^* = \dim V$

12. Lie Group Actions

12.1. **Group Actions.** action G on M cont. if $\begin{aligned} G \times M &\to M \\ (g,p) &\mapsto gp \end{aligned}$ or $\begin{aligned} M \times G &\to M \\ (p,g) &\mapsto pg \end{aligned}$ cont.
For cont. action, θ_g homeomorphism, since $\exists$ cont. inverse $\theta_{g^{-1}}$. $\theta_g : M \to M$
orbit
$\forall\, p \in M$, orbit of $p = Gp = \{gp | g \in G\}$
action θ transitive if $\forall\, p, q \in M$, $\exists\, g$ s.t. $gp = q$, i.e. $Gp = M$
isotropy group of p, $G_p = \{g \in G | gp = p\}$
action θ is free if the only element of G that fixes any element of M is e,
i.e. if $g \cdot p = p$, $p \in M$, $g = e$.
i.e. $G_p = 1 \quad \forall\, p \in M$
Example 9.1 (Lie Group Actions)

(a) trivial action of G on M is $gp = p \quad \forall\, g \in G, \quad G_p = G$
(b) action of $GL(n,\mathbb{R})$ on $\mathbb{R}^n$, $(A,x) \mapsto Ax$; $x \in \mathbb{R}^n$ column matrix.
Ax smooth, because components of Ax depend polynomially on matrix entries of A and components of x.

Exactly only 2 orbits : $0, \mathbb{R}^n - 0 \quad (\forall\, y = 0,\, y \in \mathbb{R}^n, \begin{aligned} Ax &= y \\ x &= A^{-1}y \end{aligned})$
(c) $O(n) \times \mathbb{R}^n \to \mathbb{R}^n$
orbits : $0, S^{n-1}(R), \quad \forall\, R > 0$. To show this, complete $\frac{v}{|v|}, \frac{v'}{|v'|}, \quad \forall\, v, v' \neq 0$, to orthonormal bases. Let A, A' columsn be these orthonormal bases
$$A' A^{-1}(v) = v'$$
(d) $O(n) \times S^{n-1} \to S^{n-1}$. $O(n)$ transitive action of $O(n)$ on S^{n-1}.
action $O(n) \times S^{n-1} \to S^{n-1}$ smooth by Corollary 8.25, S^{n-1} embedded submanifold of $\mathbb{R}^n$

Representations. If G Lie group,
(finite-dim.) representation of G is a Lie group homomorphism

$$\rho : G \to GL(V)$$

$\forall$ representation ρ yields or smooth left action of G on V,

$$g \cdot v = \rho(g) v, \qquad \forall\, g \in G,\, v \in V$$

Equivariant Maps.

Proper Actions.

Quotient of Manifolds by Group Actions. Suppose Lie group G acts on manifold M
$p \sim q$ if $\exists\, g \in G$ s.t. $gp = q \quad$ equivalence classes are exactly the orbits of G in M
M/G set of orbits, with quotient topology, orbit space of the action

Multilinear Algebra. $F : V_1 \times \cdots \times V_k \to W$ multilinear if $\forall\, i$, linear in each variable, $F(v_1 \dots av_1 + a'v_i' \dots v_k) = aF(v_1 \dots v_i \dots v_k) + a'F(v_1 \dots v_i' \dots v_k)$
multilinear function of 2 variables is bilinear.

$L(V_1, \dots V_k; W)$ - set of all multilinear maps from $V_1 \times \cdots \times V_k$ to W

$\{T : \underbrace{V \times \cdots \times V}_{k\text{times}} \to \mathbb{R}\} = T^k(V)$

$S \in T^k(V)$, $T \in T^l(V)$
tensor product $S \otimes T : V \times \cdots \times V \to \mathbb{R}$, covariant $(k+l)$-tensor

$$S \otimes T(x_1 \dots x_{k+l}) = S(x_1 \dots x_k) T(x_{k+1} \dots x_{k+l})$$

Exercise 12.3.

$$F(v_1 \dots av_i + bw_i \dots v_k)G(v_{k+1} \dots v_{k+l}) = aF(v_1 \dots v_i \dots v_k)G + bF(v_1 \dots w_i \dots v_k)G =$$
$$= aF \otimes G(v_1 \dots v_i \dots v_k \dots v_{k+l}) + bF \otimes G(v_1 \dots w_i \dots v_k \dots v_{k+l}) = F \otimes G(v_1 \dots av_i + bw_i \dots v_k, v_{k+1} \dots v_{k+l})$$
$$F(v_1 \dots v_k)G(v_{k+1} \dots av_{k+i} + bw_{k+i} \dots v_{k+l}) =$$
$$= F(v_1 \dots v_k)(aG(v_{k+1} \dots v_{k+i}, v_{k+i+1} \dots v_{k+l}) + bG(v_{k+1} \dots w_{k+i} v_{k+i+1} \dots v_{k+l}) =$$
$$= aF \otimes G(v_{k+1} \dots v_{k+i}, v_{k+i+1} \dots v_{k+l}) + bF \otimes G(v_1 \dots w_{k+i}, v_{k+i+1} \dots v_{k+l}) =$$
$$= F \otimes G(v_1 \dots v_k, v_{k+1} \dots av_{k+i} + bw_{k+i} \dots v_{k+i+1} \dots v_{k+l})$$

$$(F \otimes G) \otimes H = (F \otimes G)(x_1 \dots x_{k+l} H(x_{k+l+1} \dots x_{k+l+m}) = F(x_1 \dots x_k)G(x_{k+1} \dots x_{k+l})H(x_{k+l+1} \dots x_{k+l+m}) =$$
$$= F(x_1 \dots x_k)(G \otimes H)(x_{k+1} \dots x_{k+l+m}) = F \otimes (G \otimes H)(x_1 \dots x_{k+l+m})$$

Proposition 21 (12.4)**.** *(A Basis for the Space of Multilinear Functions) Let V real vector space of dim. n, (E_i) any basis for V, ϵ^i dual basis.*
set of all k-tensors of form $\epsilon^{i_1} \otimes \cdots \otimes \epsilon^{i_k}$, $\ 1 \le i_1 \dots i_k \le n$ basis for $T^k(V)$, dim. n^k

Proof. Let $\mathcal{B} = \{\epsilon^{i_1} \otimes \cdots \otimes \epsilon^{i_k} | 1 \le i_1 \dots i_k \le n\}$

Suppose arbitrary $T \in T^k(V)$
Define $T_{i_1 \dots i_k} = T(E_{i_1} \dots E_{i_k})$

$$T_{i_1 \dots i_k} \epsilon^{i_1} \otimes \cdots \otimes \epsilon^{i_k}(E_{j_1} \dots E_{j_k}) \underbrace{=}_{\text{(by definition)}} T_{i_1 \dots i_k} \epsilon^{i_1}(E_{j_1}) \dots \epsilon^{i_k}(E_{j_k}) = T_{i_1 \dots i_k} \delta^{i_1}_{j_1} \dots \delta^{i_k}_{j_k} = T_{j_1 \dots j_k} = T(E_{j_1} \dots E_{j_k})$$

T spanned by $\mathcal{B}$

□

Abstract Tensor Products of Vector Spaces. free vector space on S, $\mathbb{R}\langle S \rangle = \{\mathcal{F}\}$
finite formal linear combination - function $\mathcal{F} : S \to \mathbb{R}$ s.t. $\mathcal{F}(s) = 0$ for all but finite many $s \in S$
$\forall\, \mathcal{F} \in \mathbb{R}\langle S \rangle$, $\mathcal{F} = \sum_{i=1}^m a_i x_i$, $\ x_1 \dots x_m \in S$ s.t. $\mathcal{F}(x_i) \ne 0$, $a_i = \mathcal{F}(x_i)$

Exercise 12.6. (Characteristic Property of Free Vector Spaces)

$$F : S \to W$$
$$x \mapsto w \in W$$

Consider $w \in W$ and $w = \sum c_\alpha w_\alpha$, $w_\alpha = F(x_\alpha)$; $x_\alpha \in S$, $w_\alpha \in W$.
Consider $\overline{F} : \mathbb{R}\langle S \rangle \to W$, $\ \sum_{i=1}^m c_i x_i \mapsto \sum_{I=1}^m c_i F(x_i) = \sum_{i=1}^m c_i w_i \in W$

Let $w = v$,
$$w = \sum_{I=1}^m c_i w_i = \sum_{i=1}^m c_i F(x_i)$$
$$v = \sum_{i=1}^n b_i v_i = \sum_{i=1}^n b_i F(y_i)$$

$w - v = 0$ so for a vector space, this implies $w_i = v_i$, $m = n$,

$$\sum_{i=1}^m (c_i - b_i) w_i = 0, \qquad c_i = b_i$$

$\mathcal{R} \equiv$ subspace of free vector space $\mathbb{R}\langle V \times W\rangle$ spanned by

(20)
$$\begin{gathered} a(v,w) - a(v,w) \\ a(v,w) - (v,aw) \\ (v,w) + (v',w) - (v+v',w) \\ (v,w) + (v,w') - (v,w+w') \end{gathered} \tag{12.4}$$

tensor product of V, W

$$V \otimes W = \mathbb{R}\langle V \times W\rangle / \mathcal{R}$$

equivalence class of element (v,w) if $v \otimes w \in V \otimes W$

Proposition 22 (12.7)**.** *(Characteristic Property of the Tensor Product Space) If bilinear $A : V \times W \to X$, $\exists !$, $\widetilde{A} : V \otimes W \to X$, any vector space X s.t.*

$$\begin{tikzcd} V \times W \arrow[r, "A"] \arrow[d, "\pi"'] & X \\ V \otimes W \arrow[ur, "\widetilde{A}"'] & \end{tikzcd} \tag{12.6}$$

$\pi(v,w) = V \otimes W$

Proof. By characteristic property of free vector space, $A : V \times W \to X$ extends uniquely to linear $\overline{A} : \mathbb{R}\langle V \times W\rangle \to X$

$$\overline{A}(v,w) = A(v,w) \text{ if } (v,w) \in V \times W \subset \mathbb{R}\langle V \times W\rangle$$

A bilinear

$$\begin{aligned} \overline{A}(av,w) = A(av,w) = aA(v,w) = a\overline{A}(v,w) &= \overline{A}(a(v,w)) \\ \overline{A}(v,aw) = A(v,aw) = aA(v,w) &= \overline{A}(a(v,w)) \\ \overline{A}(v+v',w) = A(v+v',w) = A(v,w) + A(v',w) = \overline{A}(v,w) &= \overline{A}(a(v',w)) \end{aligned}$$

□

Likewise for (12.4)
subspace $\mathcal{R} \subset \ker\overline{A}$
$\therefore \overline{A}$ descends to linear $\widetilde{A} : V \otimes W = \mathbb{R}\langle V \times W\rangle/\mathcal{R} \to X$ s.t.
$\widetilde{A} \circ \pi = \overline{A}$, $\pi : \mathbb{R}\langle V \times W\rangle \to V \otimes W$
uniqueness from $\forall\, v \otimes w \in V \otimes W$, $v \otimes w$ = linear combination of $v \otimes w$
$\widetilde{A}$ uniquely determined on $\widetilde{A}(v \otimes w) = \overline{A}(v,w) = A(v,w)$

Proposition 23 (11.4)**.** ***(Other Properties of Tensor Products****). Let V, W, and X be finite-dimensional real vector spaces*

(a) *$V^* \otimes W^*$ canonically isomorphic to $B(V,W)$, bilinear maps from $V \times W$ into $\mathbb{R}$*

(b) *if (E_i) basis for V, (F_j) basis for W, then $\{E_i \otimes F_j\}$ basis is basis for $V \otimes W$, $\therefore dim(V \otimes W) = dimV dimW$*

(c) $\exists!$ *isomorphism* $\begin{aligned} V \otimes (W \otimes X) &\to (V \otimes W) \otimes X \\ v \otimes (w \otimes x) &\mapsto (v \otimes w) \otimes x \end{aligned}$

Proof. (a) canonical isomorphism (basis independence) construction between $V^* \otimes W^*$ and $B(V,W)$ (space of bilinear maps)

Define $\Phi : V^* \times W^* \to B(V,W)$
$\Phi(\omega,\eta)(v,w) = \omega(v)\eta(\omega)$
Φ bilinear (easy to check).
Prop. 11.3 $\forall$ bilinear $A : V \times W \to X$, $\exists!\widetilde{A} : V \otimes W \to X$, any vector space X s.t. $\widetilde{A}\pi = A$

$$\begin{tikzcd} V^* \times W^* \arrow[r, "\Phi"] \arrow[d, "\pi"'] & X \\ V^* \otimes W^* \arrow[ur, "\widetilde{\Phi}"'] & \end{tikzcd}$$

i.e. descends uniquely to linear $\widetilde{\Phi} : V^* \otimes W^* \to B(V,W)$

Let (e_i) be bases for V, (ϵ^i) dual basis
(f_j) W (ϕ^j)

since $V^* \otimes W^*$ spanned by elements of the form $\omega \otimes \eta$, $\omega \in V^*$
$\eta \in W^*$

$\forall\, \tau \in V^* \otimes W^*$, $\tau = \tau_{ij}\epsilon^i \otimes \varphi^j$

Define $\Psi : B(V,W) \to V^* \otimes W^*$
$\Phi(b) = b(e_k, f_l)\epsilon^k \otimes \varphi^l$

$$\Psi\widetilde{\Phi}(\tau) = \widetilde{\Phi}(\tau)(e_k, f_l)\epsilon^k \otimes \varphi^l = \tau_{ij}\widetilde{\Phi}(\epsilon^i \otimes \varphi^j)(e_k, f_l)\epsilon^k \otimes \varphi^l = \tau_{ij}\Phi(\epsilon^i, \varphi^j)(e_k, f_l)\epsilon^k \otimes \varphi^l =$$
$$= \tau_{ij}\epsilon^i(e_k)\varphi^j(f_l)\epsilon^k \otimes \varphi^l = \tau_{kl}\epsilon^k \otimes \varphi^l = \tau$$

For $v \in V$
$w \in W$

$$\widetilde{\Phi} \circ \Psi(b)(v,w) = \widetilde{\Phi}b(e_j, f_k)\epsilon^j \otimes \varphi^k(v,w) = b(e_j, f_k)\widetilde{\Phi}(\epsilon^j \otimes \varphi^k)(v,w) =$$
$$= b(e_j, f_k)\Phi(e^i, \varphi^k)(v,w) = b(e_j, f_k)e^i(v)\varphi^k(w) = b(e_j, f_k)v^i\varphi^k = b(v,w)$$

(b) Given $\{e_i | i \in I\} = \mathcal{B}_U$
$\{f_j | j \in J\} = \mathcal{B}_U$

By the bilinearity of tensor product: $a_i e_i \otimes b_j f_j = a_i b_j e_i \otimes f_j$

Consider dual basis elements $e_k^*(e_i) = \delta_{ik}$ and
$f_l^*(f_j) = \delta_{jl}$

$U \times V \to K$
$(u,v) \mapsto e_k^*(u) \cdot f_l^*(v)$

induces $U \otimes V \to K$
$u \otimes v \mapsto e_k^*(u) \cdot f_l^*(v)$

$$e_i \otimes f_j \mapsto \delta_{ik}\delta_{jl}$$
$$c_{ij} e_i \otimes f_j = 0 = c_{ij}\delta_{ik}\delta_{jl} = c_{kl} = 0 \qquad \forall\, k,l \text{ so } e_i \otimes f_j \text{ form a basis}$$

□

Corollary 5 (11.5)**.** *V finite-dim. real vector space, space $T^k(V)$ of covariant k-tensors on V canonically isomorphic to k-fold tensor product $V^* \otimes \cdots \otimes V^*$*

Exercise 11.3. Prove Corollary 11.5.
It's enough to consider the basis (good strategy).
$T^k(V)$ basis $\mathcal{B} = \{\epsilon^{i_1} \otimes \cdots \otimes \epsilon^{i_k} | 1 \le i_1 \dots i_k \le n\}$ (Prop. 11.2) dimn^k
Use Prop. 11.4(b). Surely V^* finite-dim. real vector space as well, on its own, even though it's a dual basis.

Prop.11.4(b) if (E_i) basis for V , then $\{E_i \otimes E_j\}$ basis for $V \otimes W$ and $\dim(V \otimes W) = \dim V \dim W$
(E_j) basis for W

basis for $V^* \otimes \cdots \otimes V^* = \{\epsilon^{i_1} \otimes \cdots \otimes \epsilon^{i_k} | 1 \le i_1 \dots i_k \le n\}$, $\dim(V^* \otimes \cdots \otimes V^*) = n^k$
dimensions are same. isomorphic.

Lemma 19 (11.7)**.** *Let smooth M, suppose* $\sigma \in \mathcal{T}^k(M)$ $f \in C^\infty(M)$
$\tau \in \mathcal{T}^l(M)$

Then $f\sigma$, $\sigma \otimes \tau$ also smooth tensor fields whose

$$(f\sigma)_{i_1\dots i_k} = f\sigma_{i_1\dots\sigma_k}$$
$$(\sigma\otimes\tau)_{i_1\dots i_{k+l}} = \sigma_{i_1\dots i_k}\tau_{i_{k+1}\dots i_{k+l}}$$

Exercise 11.7.
Prove Lemma 11.7. Note $T^0M = T_0M = M\times\mathbb{R}$

$$f\sigma(p, e^{(1)}_{i_1},\dots,e^{(k)}_{i_k}) = f(p)\sigma(e^{(1)}_{i_1}\dots e^{(k)}_{i_k}) = f(p)\sigma_{i_1\dots i_k} = (f\sigma)_{i_1\dots i_k}$$

Suppose smooth $F: M\to N$
$\forall$ smooth covariant k-tensor field σ on N,
define k-tensor field $F^*\sigma$ on M by
$$(F^*\sigma)_p = F^*(\sigma_{F(p)})$$
explicitly, if $X_1\dots X_k\in T_pM$, then
$$(F^*\sigma)_p(X_1\dots X_k) = \sigma_{F(g)}(F_*X_1\dots F_*X_k)$$

Proposition 24 (11.9)**.** *(The properties of Tensor Field Pullbacks) Suppose smooth* $\begin{array}{lll} F: M\to N, & \sigma\in\mathcal{T}^k(N), & f\in C^\infty(N) \\ G: N\to P & \tau\in\mathcal{T}^l(N) & \end{array}$

(a) $F^*(f\sigma) = (f\circ F)F^*\sigma$
(b) $F^*(\sigma\otimes\tau) = F^*\sigma\otimes F^*\tau$
(c) $F^*\sigma$ *smooth tensor field*
(d) $F^*:\mathcal{T}^k(N)\to\mathcal{T}^k(M)$ *linear over* $\mathbb{R}$
(e) $(GF)^* = F^*G^*$
(f) $(Id_N)^*\sigma = \sigma$

Exercise 11.9. Prove Prop. 11.9

Corollary 6 (11.10)**.** *Let smooth* $F: M\to N$, $\sigma\in\mathcal{T}^k(N)$
If $p\in M$, *smooth coordinates* (y^j) *for* N *on neighborhood of* $F(p)$, *then* $F^*\sigma$ *near* p
$$F^*(\sigma_{j_1\dots j_k}dy^{j_1}\otimes\cdots\otimes dy^{j_k}) = (\sigma_{j_1\dots j_k}\circ F)d(y^{j_1}\circ F)\otimes\cdots\otimes d(y^{j_k}\circ F)$$

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However, in the special case of a diffeomorphism, tensor fields of any variance can be pushed forward and pulled back at will (see Problem 11-6)

Symmetric Tensors. 20130919

Exercise 11.10.

$$T_{i_1\dots i_k} = T(E_{i_1}\dots E_{i_k}) = T(E_{i_1}\dots E_{i_s}\dots E_{i_r}\dots E_{i_k}) = T_{i_1\dots i_s\dots i_r\dots i_k} \qquad r<s$$

set of symmetric covariant k-tensors on V by $\sum^k(V)$

define ${}^\sigma T^(X_1\dots X_k)$=$T(X_{\sigma(1)}\dots X_{\sigma(k)})$

define Sym$T = \frac{1}{k!}\sum_{\sigma\in S_k}{}^\sigma T$

If $S\in\sum^k(V)$, define $ST = \text{Sym}(S\otimes T)$
$T\in\sum^l(V)$

$$ST(X_1\dots X_{k+l}) = \frac{1}{(k+l)!}\sum_{\sigma\in S_{k+l}}S(X_{\sigma(1)}\dots X_{\sigma(k)})T(X_{\sigma(k+1)},\dots,X_{\sigma(k+l)})$$

Proposition 25 (12.15)**.** *(Properties of the Symmetric Product)*

(a)
(b) *if* ω,η *covectors,* $\omega\eta = \frac{1}{2}(\omega\otimes\eta+\eta\otimes\omega)$

20130919 **Exercise 12.16.** Prove Proposition 12.15

(a)

(b)

$$\omega\eta(e_i,e_j) = \frac{1}{2}(\omega(e_i)\eta(e_j) + \omega(e_j)\eta(e_i)) = \frac{1}{2}(\omega(e_i)\eta(e_j) + \eta(e_i)\omega(e_j)) = \frac{1}{2}(\omega\otimes\eta + \eta\otimes\omega)(e_i,e_j)$$

direct application of definition of $ST \equiv \text{Sym}(S\otimes T)$ and $S\otimes T(X_1\dots X_{k+l}) = S(X_1\dots X_k)T(X_{k+1}\dots X_{k+l})$ definition of tensor product.

Alternating Tensors.

Lie Derivatives of Tensor Fields.

Lemma 20 (12.30)**.** *smooth M, V, A, $\exists$ (12.8) $\forall\, p\in M$, and defines $\mathcal{L}_V A$ as smooth tensor field on M*

Exercise 12.31.
Suppose smooth M, smooth V, smooth covariant tensor A
$I = (i_1\dots i_k)$
$i_i = 1\dots \text{dim}M$

$\theta_t(p) = y$ (notation)

$$A = A(y) = A(\theta_t(p)) = A_I(y)dy^I = A_I(\theta_t(p))dy^I$$

with A_I smooth function of $\theta_t(p) = y$

$$v_1 = \delta^{j_1}_{i_1}\frac{\partial}{\partial x^{j_1}} \qquad v^{j_1}_{(1)} = \delta^{j_1}_{i_1}$$

$$d(\theta_t)^*_p(A_{\theta_t(p)})\frac{\partial}{\partial x^I} = A_{\theta_t(p)}(d(\theta_t)_p\frac{\partial}{\partial x^I})$$

$d(\theta_t)_p = \frac{\partial y^i}{\partial x^j}$

$$d(\theta_t)_p v_1 = \frac{\partial y^{i_1}}{\partial x^{j_1}} v^{j_1}_{(1)}\frac{\partial}{\partial y^{i_1}}$$

$$\Longrightarrow \frac{\partial y^{k_1}}{\partial x^{j_1}}\delta^{j_1}_{i_1}\frac{\partial}{\partial y^{k_1}} = \frac{\partial y^{j_1}}{\partial x^{i_1}}\frac{\partial}{\partial y^{j_1}}$$

$\frac{\partial}{\partial x^I} = \frac{\partial}{\partial x^{i_1}}\otimes\dots\otimes\frac{\partial}{\partial x^{i_k}}$

$$d(\theta_t)_p\frac{\partial}{\partial x^I} = d(\theta_t)_p v_1\dots d(\theta_t)_p v_k = \frac{\partial y^J}{\partial x^I}\frac{\partial}{\partial y^J}$$

$$A_{\theta_t(p)}(d(\theta_t)_p(v_1)\dots d(\theta_t)_p(v_k)) = A_J(\theta_t(p))\frac{\partial y^J}{\partial x^I}$$

$$d(\theta_t)^*_p(A_{\theta_t(p)}) = A^*_J dx^J$$

$$d(\theta_t)^*_p(A_{\theta_t(p)})\frac{\partial}{\partial x^I} = A^*_I = A_J\frac{\partial y^J}{\partial x^I}$$

$$\Longrightarrow (\mathcal{L}_V A)_p = \left.\frac{d}{dt}\right|_{t=0}(\theta^*_t A)_p = \left.\frac{d}{dt}\right|_{t=0} A_J\left.\frac{\partial\theta^J(t,x)}{\partial x^I}\right|_p dx^I$$

$A_J = A_J(\theta_t(p)) = A_J(\theta(t,x))$
θ smooth in t, x, so A_J smooth in t and x

so since $\left.(\mathcal{L}_V A)_I\right|_p$ smooth $\forall\, I\in\{(i_1\dots i_k)|i_i = 1\dots\text{dim}M,\ i = 1\dots k\}$,
$\left.(\mathcal{L}_V A)\right|_p$ smooth tensor field on M

Proposition 26 (12.32)**.** (a) $\mathcal{L}_V f = Vf$
(b) $\mathcal{L}_V(fA) = (\mathcal{L}_V f)A + f\mathcal{L}_V A$
(c) $\mathcal{L}_V(A\otimes B) = (\mathcal{L}_V A)\otimes B + A\otimes\mathcal{L}_V B$
(d) *If $X_1\dots X_k$ smooth vector fields, A smooth k-tensor field,*

(21) $$\mathcal{L}_V(A(X_1\dots X_k)) = (\mathcal{L}_V A)(X_1\dots X_k) + A(\mathcal{L}_V X_1\dots X_k) + \dots + A(X_1\dots\mathcal{L}_V X_k) \qquad (12.9)$$

Corollary 7 (12.33)**.**

(22) $$(\mathcal{L}_V A)(X_1\dots X_k) = V(A(X_1\dots X_k)) - A([V,X_1],X_2\dots X_k) - \dots - A(X_1\dots X_{k-1},[V,X_k]) \qquad (12.10)$$

Riemannian Manifolds. Riemannian metric on M - smooth symmetric 2-tensor field positive definite at each pt.

Riemannian manifold - pair (M, g)

If g on M, then $\forall\, p \in M$, g_p inner product on T_pM. Because of this, we will often use the notation $\langle X, Y\rangle_g$ to denote

$$g_p(X,Y) \in \mathbb{R} \quad \forall\, X, Y \in T_pM$$

$\forall$ smooth, local coordinates (x^i), write Riemannian metric

$$g = g_{ij} dx^i \otimes dx^j$$

where g_{ij} symmetric positive definite matrix of smooth functions. $g_{ij} = g_{ji}$

$$g = g_{ij}dx^i \otimes dx^j = \frac{1}{2}(g_{ij}dx^i \otimes dx^j + g_{ji}dx^i \otimes dx^j) = \frac{1}{2}(g_{ij}dx^i \otimes dx^j + g_{ij}dx^j \otimes dx^i) =$$

$$= g_{ij}dx^i dx^j \quad \text{(by Prop. 12.15(b))} \quad \text{Notice that } dx^i dx^j \text{ is symmetrized!}$$

Example 13.1 (The Euclidean Metric) Euclidean metric on $\mathbb{R}^n$, defined in standard coordinates

$$g = \delta_{ij} dx^i dx^j$$

It is common to use the abbreviation ω^2 for the symmetric product of a tensor ω with itself, so the Euclidean metric can also be writen

$$\overline{g} = (dx^1)^2 + \cdots + (dx^n)^2$$

Applied to $v, w \in T_p\mathbb{R}^n$

$$\overline{g}_p(v,w) = \delta_{ij}v^i w^j = \sum_{i=1}^n v^i w^i = v \cdot w$$

under coordinate change, use Corollary 11.10

Proposition 27 (13.3)**.** ***(Existence of Riemannian Metrics)***
$\forall$ smooth manifold M, M with or without ∂M, $\exists$ Riemannian metric g

Proof. Choose covering of M by smooth coordinate charts $(U_\alpha, \varphi_\alpha)$
$\overline{g}$ Euclidean metric
$\forall\, U_\alpha$, $\exists$ Riemannian metric $g_\alpha = \varphi_\alpha^* \overline{g}$

$$\varphi_\alpha : U_\alpha \to \mathbb{R}^n$$

Let $\{\psi_\alpha\}$ smooth partition of unity subordinate to cover $\{U_\alpha\}$
Define $g = \sum_\alpha \psi_\alpha g_\alpha$
s.t. $\forall\, g_\alpha$, $\psi_\alpha g_\alpha = 0$ outside suppψ_α
By local finiteness, $\exists$ only finitely many $\psi_\alpha g_\alpha \neq 0$ in neighborhood of each pt.
so $g = \sum_\alpha \psi_\alpha g_\alpha$ defines a smooth tensor field

□

defined on Riemannian manifold (M, g)

- length or norm of $X \in T_pM$ defined

$$|X|_g = \langle X, X\rangle_g^{1/2} = g_p(X,X)^{1/2}$$

- angle between $X, Y \in T_pM$, $X, Y \neq 0$ is unique $\theta \in [0, \pi]$ satisfying

$$\cos\theta = \frac{\langle X, Y\rangle_g}{|X|_g |Y|_g}$$

- $X, Y \in T_pM$ orthogonal if $\langle X, Y\rangle_g = 0$
- If $\gamma : [a,b] \to M$ piecewise smooth curve segment, length of γ is

$$L_g(\gamma) = \int_a^b |\gamma'(t)|_g dt$$

Just as we did in Chapter 8 for $\mathbb{R}^n$
define orthonormal frame for M to be local frame $(E_1 \dots E_n)$ defined on some open subset $U \subset M$ s.t.
$(E_1|_p \dots E_n|_p)$ orthonormal basis for $T_pM \qquad \forall\, p \in U$, or equivalently s.t. $\langle E_i, E_j\rangle_g = \delta_{ij}$

Example 13.14. coordinate frame $\left(\frac{\partial}{\partial x^i}\right)$ global orthonormal frame on $\mathbb{R}^n$

Corollary 8 (13.8)**.** *(Existence of Local Orthonormal Frames). Let (M,g) Riemannian manifold. $\forall\, p \in M$, $\exists$ smooth orthonormal frame on neighborhood of p.*

Observe Corollary 13.8. doesn't show that $\exists$ smooth coordinates near p for which coordinate frame is orthonormal.

Pullback Metrics. Suppose $\begin{matrix} (M,g) \\ (\widetilde{M},\widetilde{g}) \end{matrix}$ Riemannian manifolds.

isometry - smooth $F : M \to \widetilde{M}$ if F diffeomorphism s.t. $F^*\widetilde{g} = g$
if $\exists$ isometry F, $M, \widetilde{M}$ isometric.
F local isometry is $\forall\, p \in M$, $\exists$ neighborhood U s.t. $F|_U$ isometry of U onto open $\widetilde{U} \subset M$
g on M flat if $\forall\, p \in M$, $\exists$ neighborhood $U \subset M$ s.t. $(U, g|_U)$ isometric to open $\widetilde{U} \subset \mathbb{R}^n$ with Euclidean metric.

Prob. 11-14 shows $\exists$ only if metric flat.

Riemannian Submanifolds.

Riemannian Submanifolds. $S \subset M$
define $g|_S = i^* g$, for $i : S \hookrightarrow M$

$$(g|_S)(X,Y) = i^*g(X,Y) = g(i_*X, i_*Y) = g(X,Y)$$

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in general, $S \subset M$

$F : S \to M$ e.g. $F(u^1, u^2) = (x^1, x^2, x^3)$
$\qquad F(u^1 \dots u^s) = (x^1 \dots x^m)$ s.t. $s \leq m$ (for this case)

Note

$$x^i = x^i(u^1 \dots u^s) \text{ or e.g. } x^2 = x^2(u^1, u^2)$$

Recall these facts about pullbacks and pushforwards.

$F^* : T_{F(p)}M \to T_pS$ (pullback!)
$F_* : T_pS \to T_{F(p)}M$ (push forward; remember we can only pushforward if F diffeomorphism, i.e. F, F^{-1} diff. and F bijective)
$F^* : \tau^2(M) \to \tau^2(S)$ (can always pullback tensors; in this case (rank 2))

Consider charts (U,u), $p \in U \subset S$, (V,x), $F(p) \in V \subset M$, $V \subset F(U)$
For $f : M \to \mathbb{R}$, i.e. $f \in \mathcal{C}^\infty(M)$
$fF : S \to \mathbb{R}$ i.e. $fF \in \mathcal{C}^\infty(S)$

$$fF = f(x^i)^{-1}x^iF(u^j)^{-1}u^j = (f(x^i)^{-1})(x^iF(u^j)^{-1})u^j = f(x^i(u^j))$$

Consider $\overline{g}(x_i, x_j)$
$\qquad \overline{g} = \delta_{ij}dx^idx^j$ ($\overline{g}$ as a tensor (rank 2) in its local coordinate form, with coordinates y^i. So $\overline{g}$ is (like, or is) a Euclidean metric)

$$F_*E_i(f) = E_i(fF) = \omega^k_{(i)}\frac{\partial}{\partial x^k}f = \frac{\partial}{\partial u^i}f(F(u)) = \frac{\partial f}{\partial x^k}\frac{\partial x^k}{\partial u^i} \qquad \omega^k_{(i)} = \frac{\partial x^k}{\partial u^i}$$

By definition, for
$F^*\overline{g}(x^{(i)}, x^{(j)})$ on by (notation) $F^*g(A,B)$, $A, B \in T_pS$
$F^*\overline{g}(E_i, E_j) = \overline{g}(F_*E_i, F_*E_j)$

$$F^*\overline{g}(E_i,E_j) = \mathring{g}(E_i,E_j) = \mathring{g}_{ij} = \overline{g}(F_*E_i, F_*E_j) = \overline{g}\left(\frac{\partial x^k}{\partial u^i}\frac{\partial}{\partial x^k}, \frac{\partial x^l}{\partial u^j}\frac{\partial}{\partial x^l}\right) =$$
$$= \overline{g}_{kl}\frac{\partial x^k}{\partial u^i}\frac{\partial x^l}{\partial u^j}$$

Formula for pullback of metric on M to metric on S i.e. formula for metric on S

$$\boxed{(F^*\overline{g})_{ij} = \mathring{g}_{ij} = \overline{g}_{kl}\frac{\partial x^k}{\partial u^i}\frac{\partial x^l}{\partial u^j}}$$

For $\overline{g}_{kl} = \delta_{kl}$ (Euclidean metric)

$$\overset{\circ}{g}_{ij} = \frac{\partial x^k}{\partial u^i}\frac{\partial x^k}{\partial u^j} = \left(\frac{\partial x^i}{\partial u^k}\right)^T \frac{\partial x^k}{\partial u^j} = (D_u x)^T (D_u x) \equiv (D_u x)^2$$

$\overset{\circ}{g}_{ij}$ is just the square of the Jacobian (The square of the Jacobian is the metric in S^1).
Then you could get the matrix form of the metric.

Example 13.16
$\overset{\circ}{g} = \overline{g}|_{S^n}$ $\quad S^n \hookrightarrow \mathbb{R}^{n+1}$ round metric (or standard metric) on sphere.
It's usually easiest to compute the induced metric on a Riemannian submanifold in terms of local parametrizations (see Chapter 5)
Example 13.17 **(Induced Metrics in Graph Coordinates.)**
Let open $U \subset \mathbb{R}^n$
$M \subset \mathbb{R}^{n+1}$ graph of smooth $f : U \to \mathbb{R}$

Then $X : U \to \mathbb{R}^{n+1}$
$X(u^1 \dots u^n) = (u^1 \dots u^n, f(u))$ smooth (global) parametrization of M
induced metric on M,
$\overline{g} = \delta_{ij} dy^i dy^j$ (note y^i local coordinates on $\mathbb{R}^{n+1}$)
Recall Prop.11.9. $F^*(\sigma \otimes \tau) = F^*\sigma \otimes F^*\tau$
Corollary 11.10. $F : M \to N$,
$F^*(\sigma_{j_1 \dots j_k} dy^{j_1} \otimes \dots \otimes dy^{j_k}) = (\sigma_{j_1 \dots j_k} \circ F) d(y^j F) \otimes \dots \otimes d(y^{j_k} F)$

$$X^* \overline{g} = (\delta_{ij} \circ X) d(y^i X) d(y^j X) = (du^1)^2 + \dots + (du^n)^2 + (df)^2$$
$$X^* \overline{g}_p (E_i, E_j) = \overline{g}_{X(p)} (X_* E_i, X_* E_j)$$

The Normal Bundle. Suppose (M, g), Riemannian submanifold $S \subset M$
$\forall\, p \in S$, vector $N \in T_p N$ normal to S if N orthogonal to $T_p S$ with respect to g
$N_p S \subset T_p M$, $N_p S$ = all vectors normal to S at $p = \{N | \langle N, X \rangle_g = 0,\ \forall\, X \in T_p S\}$ normal space to S at p

The Riemannian Distance Function. Exercise 13.23.

$$L_g(\gamma) = \int_a^b |\gamma'(t)|_g dt = \int_a^c |\gamma'(t)|_g dt + \int_c^b |\gamma'(t)|_g dt = L_g(\gamma|_{[a,c]}) + L_g(\gamma|_{[c,b]})$$

Exercise 13.24.
On every coordinate patch, consider on some interval $I \subset \mathbb{R}$ parametrizing curve γ on M and $\widetilde{\gamma}$ on $\widetilde{M}$ in the same way, and that $F^*\widetilde{\gamma} = \widetilde{\gamma}$

$$L_{\widetilde{g}}(F \circ \gamma) = \int_I |F\gamma|_{\widetilde{g}} ds = \int_I (\widetilde{g}(F\dot{\gamma}(t), F\dot{\gamma}(t)))^{1/2} ds = \int_I \left(\widetilde{g}(\dot{\gamma}^i \frac{\partial y^j}{\partial x^i}\frac{\partial}{\partial y^j}, \dot{\gamma}^k \frac{\partial y^l}{\partial x^k}\frac{\partial}{\partial y^l}) \right)^{1/2} ds = \int_I (\dot{\gamma}^i \dot{\gamma}^k)^{1/2} \left(\widetilde{g}_{jl} \frac{\partial y^j}{\partial x^i}\frac{\partial y^l}{\partial x^i} \right)^{1/2} ds =$$
$$= \int_I (F^*\widetilde{g}(\dot{\gamma}(t), \dot{\gamma}(t)))^{1/2} dt = \int_I (g(\dot{\gamma}(t), \dot{\gamma}(t)))^{1/2} = L_g(\gamma)$$

Rather, think in terms of a coordinate-free manner.

$$L_g(\gamma) = \int_a^b |\gamma(t)|_g dt = \int_a^b (g(\dot{\gamma}(t), \dot{\gamma}(t)))^{1/2} dt = \int_a^b dt (F^*\widetilde{g}(\dot{\gamma}(t), \dot{\gamma}(t)))^{1/2} = \int_a^b dt (\widetilde{g}(F_*\dot{\gamma}(t), F_*\dot{\gamma}(t)))^{1/2} =$$
$$= \int_a^b |F\dot{\gamma}(t)|_{\widetilde{g}} dt = L_{\widetilde{g}}(F\gamma)$$

Proposition 28 (13.25)**.** ***(Parameter independence of Length)***
Let (M, g), $\gamma : [a, b] \to M$ piecewise smooth curve segment
If $\widetilde{\gamma}$ any reparametrization of γ, then $L_g(\widetilde{\gamma}) = L_g(\gamma)$

Proof. Suppose γ smooth.
$\varphi : [c, d] \to [a, b]$ diffeomorphism s.t. $\widetilde{\gamma} = \gamma \circ \varphi$
φ diffeomorphism implies $\varphi' > 0$ or $\varphi' < 0$ everywhere.

(Recall diffeomorphism (cf. wikipedia) *differentiable*, bijective, inverse *differentiable*; so DF, Jacobian *matrix*, bijective, F differentiable, so it can't be 0 at any pt. (linear algebra, need $\exists$ inverse))

Assume $\varphi' > 0$

$$L_g(\widetilde{\gamma}) = \int_c^d |\widetilde{\gamma}'(t)|_g dt = \int_c^d \left| \frac{d}{dt}(\gamma \circ \varphi) \right|_g dt = \int_c^d |\gamma'(\varphi(t))\dot{\varphi}|_g dt = \int_a^b |\gamma'(\varphi(t))|_g \dot{\varphi} dt = \int_a^b |\gamma'(s)|_g =$$
$$= \int_a^b |\gamma'(s)|_g ds = L_g(\gamma)$$

where second-to-last equality follows from change of variables formula for ordinary integrals.

□

If (M, g) connected Riemannian manifold
$d_g(p, q)$ (Riemannian) distance between p, q - infinum of $L_g(\gamma)$ over all piecewise smooth curve segments γ from p to q.

The key is the following technical lemma, which shows that any Riemannian metric is locally comparable to Euclidean metric in coordinates.

Lemma 21 (13.28)**.** *Let g on open $U \subset \mathbb{R}^n$*

For compact $K \subset U$, $\exists$ constants c, C s.t. $\forall\, x \in K$
$\forall\, v \in T_x\mathbb{R}^n$

$$c|v|_{\overline{g}} \leq |v|_g \leq C|v|_{\overline{g}}$$

Theorem 9 (13.29)**.** *(Riemannian Manifolds as Metric Spaces)*
Let connected (M, g)
with $d_g(p, q)$; M metric space whose metric topology same as original manifold topology.

local orthonormal frame $(E_1 \dots E_n)$ for M on open $U \subset M$ is adapted to S if first k vectors $(E_1|_p \dots E_k|_p)$ span $T_pS \quad \forall\, p \in S$.
follows $(E_{k+1}|_p \dots E_n|_p)$ span N_pS

Prop. 11.24 proved exactly some way as counterpart for submanifolds of $\mathbb{R}^n$ (Prop. 10.17)

Proposition 29 (11.24)**.** *(Existence of Adapted Orthonormal Fames) Let $S \subset M$ embedded Riemannian submanifold*
$\forall\, p \in S$, $\exists$ smooth adapted orthonormal frame on neighborhood $U \ni p \subset M$

Recall $F : M \to N$ immersion if DF injective everywhere, F embedding if F injective (homeomorphism onto its image) and F immersion.
normal bundle to S

$$NS = \coprod_{p \in S} N_pS$$

The Tangent-Cotangent Isomorphism. EY 20140521, Below, in between the lines, are my notes off the previous edition. It's frustrating to not be able to obtain instantly the most up-to-date edition automatically, online, available freely for download. Notation had changed. It's important to me to keep up-to-date with the latest notation; it's not trivial (cf. Zee, A.; Srednicki's QFT vs. previous QFT notation)

Given (M, g) define bundle map $\widetilde{g} : TM \to T^*M$
$\forall\, p \in M$, $\forall\, X_p \in T_pM$, $\widetilde{g}(X_p) \in T_p^*M$ be covector defined $\widetilde{g}(X_p)(Y_p) = g_p(X_p, Y_p) \quad \forall\, Y_p \in T_pM$

To see this is a smooth bundle map, consider its action on smooth vector fields:

$$\widetilde{g}(X)(Y) = g(X, Y) \quad \forall\, X, Y \in \tau(M)$$

Because $\widetilde{g}(X)(Y)$ linear over $C^\infty(M)$ as a function of Y,
from Prob. 6-8, $\widetilde{g}(X)$ smooth covector field.
because $\widetilde{g}(X)$ linear over $C^\infty(M)$ as a function of X, $\widetilde{g}$ smooth bundle map by def. by Prop. 5.16.

Use same symbol: pointwise bundle map $\widetilde{g} : TM \to T^*M$
linear map on sections $\widetilde{g} : \mathcal{T}(M) \to \mathcal{T}^*(M)$

$\widetilde{g}$ injective: $\widetilde{g}(X_p) = 0$ implies $0 = \widetilde{g}(X_p)(X_p) = \langle X_p, X_p \rangle_g$ so $X_p = 0$
By dim., $\widetilde{g}$ bijective, so it's a bundle isomorphism (Prob. 5-9)

If X, Y smooth vector fields,

$$\widetilde{g}(X)(Y) = g_{ij}X^iY^j$$
$$\widetilde{g}(X) = g_{ij}X^i dy^j$$

customary to denote $X_j = g_{ij}X^i$ so $\widetilde{g}(X) = X_j dy^j$

$\widetilde{g}^{-1} : T_p^*M \to T_pM$ is inverse of (g_{ij}) (Because (g_{ij}) matrix of the isomorphism $\widetilde{g}$, it is invertible $\forall\, p$)
let (g^{ij}) inverse of $g_{ij}(p)$ so $g^{ij}g_{jk} = g_{kj}g^{ji} = \delta^i_k$
Thus for covector field $\omega \in \mathcal{T}^*M$,

$$\widetilde{g}^{-1}(\omega) = \omega^i \frac{\partial}{\partial x^i}, \quad \omega^i = g^{ij}\omega_j$$

ω^i is a vector, which we visualize as a (sharp) arrow, while X_j covector, which we visualize by means of its (flat) level sets.

$\forall$ smooth f on (M, g), $f \in \mathbb{R}$, define vector field $\quad \text{grad} f = \widetilde{g}^{-1}(df)$

$\forall\, X \in \mathcal{T}(M)$

$$\langle \text{grad} f, X\rangle_g = \widetilde{g}(\text{grad} f)(X) = df(X) = Xf$$

thus $\langle \text{grad} f, X\rangle_g = Xf \quad \forall\, X \in \chi(M)$
or equivalently $\langle \text{grad} f, \cdot\rangle_g = df$

$$\text{grad} f = g^{ij}\frac{\partial f}{\partial x^i}\frac{\partial}{\partial x^j}$$

---

bundle isomorphism $\widehat{g} : TM \to T^*M$

$\forall\, p \in M \qquad \widehat{g}(v) \in T_p^*M$ defined by
$\forall\, v \in T_pM \qquad \widehat{g}(v)(w) = g_p(v, w) \quad \forall\, w \in T_pM$

$$\widehat{g}(X)(Y) = g(X, Y) \qquad \forall\, X, Y \in \mathfrak{X}(M)$$

$\widehat{g}(X)(Y)$ linear over $C^\infty(M)$ as a function of Y, Lemma 12.24 $\Longrightarrow \widehat{g}(X)$ smooth covector field
$g = g_{ij}dx^i dx^j$

$$\widehat{g}(X)(Y) = g_{ij}X^iY^j \Longrightarrow \widehat{g}(X) = g_{ij}X^i dx^j = X_j dx^j \text{ where } X_j = g_{ij}X^i$$

$X^\flat = \widehat{g}(X)$
Now

$$\widehat{g}^{-1} : T_p^*M \to T_pM$$

$\forall$ covector field $\omega \in \mathfrak{X}^*(M)$

$$\widehat{g}^{-1}(\omega) = \omega^i \frac{\partial x^i}{,} \quad \omega^i = g^{ij}\omega_j$$

$omega^\sharp = \widehat{g}^{-1}(\omega)$
gradient of f by $\text{grad} f = (df)^\sharp = \widehat{g}^{-1}(df)$
$\forall\, X \in \mathfrak{X}(M)$

$$\langle \text{grad} f, X\rangle_g = \widehat{g}(\text{grad} f)(X) = df(X) = Xf$$

or
$\langle \text{grad} f, \cdot\rangle_g = df$
$\text{grad} f = g^{ij}\frac{\partial f}{\partial x^i}\frac{\partial}{\partial x^j} \qquad$ so $\text{grad} f$ is smooth

Problems. Problem 11-1. Recall $\forall$ bilinear $A : V \times W \to Y$, $\exists!$ linear $\widetilde{A} : Z \to Y$ s.t.

$$\begin{tikzcd} V \times W \arrow[r, "\widetilde{\pi}"] \arrow[d, "\otimes"'] & Z \\ V \otimes W \arrow[ur, "\exists!\pi"'] & \end{tikzcd}$$

is the universal property s.t. $\pi\otimes = \widetilde{\pi}$
Suppose bilinear $\widetilde{\pi} : V \times W \to Z$ s.t.,
$\forall$ bilinear $A : V \times W \to Y$, $\exists!$ linear $\widetilde{A} : Z \to Y$ s.t.

$$\begin{tikzcd} V \times W \arrow[r, "A"] \arrow[d, "\widetilde{Z}"'] & Y \\ Z \arrow[ur, "\widetilde{A}"'] & \end{tikzcd}$$

$$\begin{tikzcd} V \times W \arrow[r, "\widetilde{\pi}"] \arrow[d, "\otimes"'] & Z \arrow[dl, shift left, "\lambda"'] \\ V \otimes W \arrow[ur, shift right, "\widetilde{\Phi}"'] & \end{tikzcd}$$

Consider
$\Phi\otimes = \widetilde{\pi}$ (by universal property)

$$\Phi(v \otimes w) = \widetilde{\pi}(v, w)$$

$\exists !$ linear $\lambda : Z \to V \otimes W$

$$\lambda \circ \widetilde{\pi} = \otimes$$
$$\lambda \otimes \widetilde{\pi}(v, w) = v \otimes w$$

$$\Phi\lambda(\widetilde{\pi}(v, w)) = \Phi(v \otimes w) = \widetilde{\pi}(v, w)$$
$$\Phi\lambda = \mathrm{id}_Z$$
$$\lambda\Phi(v \otimes w) = \lambda\widetilde{\pi}(v, w) = v \otimes w$$
$$\lambda\Phi = \mathrm{id}_{V \otimes W}$$

So Φ is an isomorphism between $Z, V \otimes W$. As λ is unique, so is Φ

Problem 11-2.

tensor product of U, V is vector space $U \otimes V$ with bilinear map $\begin{aligned} \otimes : & U \times V \to U \otimes V \\ & (u, v) \mapsto u \otimes v \end{aligned}$

with universal property with any vector space W.
$K \cong k1$
By bilinearity,

$$\begin{tikzcd} U \otimes K \arrow[r] & U \otimes 1 \arrow[r] & U \end{tikzcd}$$

$$\begin{tikzcd} u \otimes k \arrow[r, mapsto] & ku \otimes 1 & ku \arrow[l, mapsto, "q"'] \end{tikzcd}$$

$$\begin{tikzcd} u \otimes k \arrow[r, mapsto] & ku \otimes 1 & ku \arrow[l, mapsto, "q"'] \end{tikzcd}$$

$q : U \to U \otimes 1$
$u \mapsto u \otimes 1$
then $U \otimes K \simeq U$

$$\begin{tikzcd} U^* \otimes V \arrow[r] \arrow[d] & \mathrm{Hom}(U, V) \\ \otimes \arrow[ur, dotted] & \end{tikzcd}$$

Problem 11-3. $\begin{aligned} & U^* \times V \to \mathrm{Hom}(U, V) \\ & (u^*, v) \to (u \to u^*(u), v) \end{aligned}$ induces a natural homomorphism (injective), $U^* \otimes V \to \mathrm{Hom}(U, V)$

If $\dim U, \dim V < \infty$,

for arbitrary $v_i \in V$, $\displaystyle\sum_i e_i^* \otimes v_i \in U^* \otimes V$

$$e_j \to e_i^*(e_j)v_i = v_j$$

$\sum_i e_i^* \otimes v_i$ corresponds to homomorphism $U \to V$ mapping $e_i \to v_i$
$U^* \otimes V \to \mathrm{Hom}(U, V)$ surjective.
$\dim U^* \otimes V = \dim U^* \dim V$
isomorphism by dim. reason.

15. Differential Forms

The Geometry of Volume Measurement.

The Algebra of Alternating Tensors.

$$\mathrm{Alt}: T^k(V) \to \Lambda^k(V)$$
$$(\mathrm{Alt}T)(X_1 \dots X_k) = \frac{1}{k!}\sum_{\sigma \in S_k} (\mathrm{sgn}\sigma) T(X_{\sigma(1)} \dots X_{\sigma(k)})$$

Exercise 12.2. If T alternating, by Exercise 12.1, $T(X_{\sigma(1)} \dots X_{\sigma(k)}) = (\mathrm{sgn}\sigma)T(X_1 \dots X_k)$

$$\frac{1}{k!}\sum_{\sigma \in S_k}(\mathrm{sgn}\sigma)T(X_{\sigma(1)}\dots T_{\sigma(k)}) = \frac{1}{k!}\sum_{\sigma\in S_k}(\mathrm{sgn}\sigma)(\mathrm{sgn}\sigma)T(X_1\dots X_k) = \frac{1}{k!}\sum_{\sigma\in S_k}T(X_1\dots X_k) = T(X_1 \dots X_k) \Longrightarrow \mathrm{Alt}T = T$$

Elementary Alternating Tensors.

$$I = (i_1 \dots i_k)$$
$$I_\sigma = (i_{\sigma(1)} \dots i_{\sigma(k)}) \qquad \sigma \in S_k$$

$$\delta_I{}^J = \begin{cases} \mathrm{sgn}\sigma & \text{if } I \text{ and } J \text{ has no repeated indices and } J = I_\sigma \text{ for some } \sigma \in S_k \\ 0 & \text{if } I \text{ and } J \text{ has repeated index or } J \neq I_\sigma \quad \forall\, \sigma \in S_k \end{cases}$$

Let V vector space, $(\epsilon^1 \dots \epsilon^n)$ basis for V^*
define covariant k-tensor ϵ^I by

$$\epsilon^I(x_1 \dots x_k) = \det\begin{pmatrix} \epsilon^{i_1}(X_1) & \dots & \epsilon^{i_k}(X_k) \\ \vdots & & \vdots \\ \epsilon^{i_k}(X_1) & \dots & \epsilon^{i_k}(X_k)\end{pmatrix} = \det\begin{pmatrix} X_1^{i_1} & \dots & X_k^{i_k} \\ \vdots & & \vdots \\ X_1^{i_k} & \dots & X_k^{i_k}\end{pmatrix}$$

denote sum over only increasing multi-indice

$$\sum_I{}' T_I \epsilon^I = \sum_{\{I: 1 \le i_1 < \dots < i_k \le n\}} T_I \epsilon^I$$

Proposition 30 (12.5)**.** *for $k \le n$*
$\{\epsilon = \epsilon^I : I$ an increasing multi-index of length $k\}$ is a basis for $\Lambda^k(V)$

$$\Longrightarrow \dim\Lambda^k(V) = \binom{n}{k}$$

Lemma 22. *12.6 $\omega \in \Lambda^n(V)$*
If linear $T: V \to V$, $X_1 \dots X_n \in V$

(23)
$$\omega(TX_1 \dots TX_n) = \det T \omega(X_1 \dots X_n) \qquad (12.2)$$

Proof. By Prop. 12.5, $\mathcal{E} = \{\epsilon^I | I$ increasing multi-index of length $k\}$ basis for $\Lambda^k(V)$

$$\omega = c\epsilon^{1\dots n} \qquad \binom{n}{n} = 1$$

By multilinearity of $\omega(TX_1 \dots TX_n)$ and $(\det T)\omega(X_1 \dots X_n)$, it suffices to verify it in special case

$$X_i = E_i, \quad i = 1 \dots n$$

□

$$\det T \omega(X_1 \dots X_n) = \det T c \epsilon^{1\dots n}(E_1 \dots E_n) = c\det T$$
$$\omega(TE_1 \dots TE_n) = c\epsilon^{1\dots n}(T_1 \dots T_n) = c\det(\epsilon^j(T_i)) = c\det(\epsilon^j(T_i{}^k e_k)) = c\det T_i{}^j$$
$$\text{where, recall } \epsilon^I(X_1 \dots X_k) = \det X_j{}^{i_k}$$

The Wedge Product.

Lemma 23 (14.10)**.**

(24)
$$\epsilon^I \wedge \epsilon^J = \epsilon^{IJ}$$

Differential Forms on Manifolds. Recall T^kT^*M bundle of covariant k-tensors on M
alternating tensors $\Lambda^kT^*M \subset T^kT^*M$

$$\Lambda^kT^*M = \coprod_{p\in M} \Lambda^k(T_p^*M)$$

Exercise 14.14.
Λ^kT^*M smooth subbundle of T^kT^*M
EY : 20140501 EY? ???

---

denote section

$$\Omega^k(M) = \Gamma(\Lambda^kT^*M)$$

in any smooth chart, k form ω can be written locally as

$$\omega = \sum_I{}' \omega_I dx^{i_1} \wedge \dots \wedge dx^{i_k} = \sum_I{}' \omega_I dx^I$$

By Lemma 14.7(c)

$$dx^{i_1} \wedge \dots \wedge dx^{i_k} \left(\frac{\partial}{\partial x^{j_1}} \dots \frac{\partial}{\partial x^{j_k}} \right) = \delta^I{}_J$$

component functions ω^I of ω determined by

$$\omega^I = \omega \left(\frac{\partial}{\partial x^{i_1}} \dots \frac{\partial}{\partial x^{i_k}} \right)$$

$$(F^*\omega)(v_1 \dots v_k) = \omega(dF(v_1) \dots dF(v_k))$$

$$e_i \in TM$$
$$f_j \in TN$$
$$F(x) = F^j(x) = y^j(x^i)$$

$$\begin{aligned} F_*(v) &= w = w^j f_j \\ F_*(v) &= vF = v^i \frac{\partial F^j}{\partial x^i} f_j \end{aligned} \qquad \Longrightarrow w^j = v^i \frac{\partial^j F}{\partial x^i}$$

$$(F^*\omega)(v_1 \dots v_k) = \omega(F_*v_1 \dots F * v_k) = \omega(v_1F \dots v_kF)$$
$$(F^*\omega) = (F^*\omega)_{\underline{I}} dx^{\underline{I}}$$
$$(F^*\omega)e_I = (F^*\omega)_{\underline{I}}$$
$$\omega(dF(e_I)) = \omega(F_*e_I) = \omega_{\underline{J}}(DF)^{\underline{J}}_{\underline{I}}$$

$$F_*e_I = (F_*e_I)^J f_J = (DF)^J{}_I f_J$$
$$(F^*\omega)_{\underline{I}} = \omega_{\underline{J}}(DF)^{\underline{J}}_{\underline{I}}$$

Lemma 24 (14.16)**.** (a) *$F^* : \Omega^k(N) \to \Omega^k(M)$ linear over $\mathbb{R}$*
(b) *$F^*(\omega \wedge \eta) = (F^*\omega) \wedge (F^*\eta)$*
(c) *in any smooth chart*

$$F^*(\omega_{\underline{I}} dy^{\underline{I}}) = (\omega_{\underline{I}}F)(dy^{\underline{I}}F)$$

Exercise 14.17.

$$\omega, \eta \in \Omega^k(N) \qquad \begin{aligned} F^*\omega &= \alpha \\ F^*\eta &= \beta \end{aligned}$$

$$(F^*\omega) = (F^*\omega)_{\underline{I}} e^{\underline{I}} = \alpha_{\underline{I}} e^{\underline{I}}$$
$$(F^*\eta) = (F^*\eta)_{\underline{J}} e^{\underline{J}} = \beta_{\underline{J}} e^{\underline{J}}$$

(a)

$$\alpha + \beta = F^*\omega + F^*\eta = (\alpha_{\underline{I}} + \beta_{\underline{I}})dx^{\underline{I}} = (\omega(dF(e_{\underline{I}})) + \eta(dF(e_{\underline{I}})))dx^{\underline{I}} = (\omega + \eta)dF(e_I)dx^{\underline{I}} = F^*(\omega + \eta)(e_{\underline{I}})dx^{\underline{I}} =$$
$$= F^*(\omega + \eta)$$
$$F^*c\omega = c\omega DF = cF^*\omega$$

(b)

$$\alpha \wedge \eta = \alpha_{\underline{I}} \beta_{\underline{J}} e^{\underline{I}} \wedge e^{\underline{J}}$$
$$\alpha \wedge \eta = (\alpha \wedge \eta)_{\underline{K}} e^{\underline{K}}$$
$$(\alpha \wedge \eta)_{\underline{K}} = \alpha_{\underline{I}} \beta_{\underline{J}} \delta^{\underline{IJ}}_{\underline{K}}$$
$$\omega \wedge \eta = \omega_{\underline{I}} f^{\underline{I}} \wedge \eta_{\underline{J}} f^{\underline{J}} = \omega_{\underline{I}} \eta_{\underline{J}} f^{\underline{I}} \wedge f^{\underline{J}}$$

$$F^*(\omega \wedge \eta) e_{\underline{K}} = (\omega \wedge \eta) DF e_{\wedge K} = \omega_{\underline{I}} \eta_{\underline{J}} f^{\underline{I}} \wedge f^{\underline{J}} DF^K_{\underline{K}} f_L = \omega_{\underline{I}} \eta_{\underline{J}} DF^L_{\underline{K}} \delta^{\underline{IJ}}_{\underline{L}} = \omega_{\underline{I}} \eta_{\underline{J}} DF^{\underline{IJ}}_{\underline{K}}$$

In the last equality, $\underline{IJ}$ in $DF^{\underline{IJ}}_{\underline{K}}$ is some permutation of $\underline{IJ}$

$$(F^*\omega) \wedge (F^*\eta) e_{\underline{K}} = \omega_{\underline{J}} DF^{\underline{J}}_{\underline{I}} \eta_{\underline{L}} DF^{\underline{L}}_{\underline{M}} (e^{\underline{I}} \wedge e^{\underline{M}}) e_{\underline{K}} = \omega_{\underline{J}} \eta_{\underline{L}} DF^{\underline{J}}_{\underline{I}} DF^{\underline{L}}_{\underline{M}} \delta^{\underline{IM}}_{\underline{K}} = \omega_{\underline{I}} \eta_{\underline{J}} DF^{\underline{I}}_{\underline{L}} DF^{\underline{J}}_{\underline{M}} \delta^{\underline{LM}}_{\underline{K}}$$

(c)

Exterior Derivatives. $\forall$ manifold, $\exists$ differential operator

$$d : \mathcal{A}^k(M) \to \mathcal{A}^{k+1}(M)$$

s.t. $d(d\omega) = 0 \quad \forall \omega$
necessary: if smooth k-form $\omega = d\eta$, some $(k-1)$ form η, then $d\omega = 0$
in coordinates

(25) $$d\left(\sum_J{}' \omega_J dx^J \right) = \sum_J{}' d\omega_J \wedge dx^J \qquad (12.15)$$

(26) $$d\left(\sum_J{}' \omega_J dx^{j_1} \wedge \dots \wedge dx^{j_k} \right) = \sum_J{}' \sum_i \frac{\partial \omega_J}{\partial x^i} dx^i \wedge dx^{j_1} \wedge \dots \wedge dx^{j_k} \qquad (12.16)$$

Theorem 10 (12.14)**.** *(The Exterior Derivative)*
$\forall$ *smooth* M, $\exists !$ *linear* $d : \mathcal{A}^k(M) \to \mathcal{A}^{k+1}(M)$ *s.t.*

(i) *if f smooth, $f \in \mathbb{R}$ (0-form), then differential df, defined*

$$df(X) = Xf$$

(ii) *if $\omega \in \mathcal{A}^k(M)$, then $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$*
$\eta \in \mathcal{A}^l(M)$
(iii) $d^2 = 0$

(a) $\forall$ *smooth coordinate chart, d given by (12.15)*
(b) *d local; if $\omega = \omega'$ on open $U \subset M$, then $d\omega = d\omega'$ on U*
(c) *d commutes with restriction if $U \subset M$ any open set*

(27) $$d(\omega|_U) = d(\omega)|_U \qquad (12.17)$$

Proof. Suppose M covered by a single chart.
define d by (12.15)

$$d\left(\sum_J{}' \omega_J dx^J \right) = \sum_J{}' d\omega_J \wedge dx^J \qquad (12.15)$$

$$df(X) = df(X^i e_i) = X^i f(e_i)$$

d linear, (i) satisfied.

Consider $\omega = f dx^I$
$\eta = g dx^J$

$$d(\omega \wedge \eta) = d((f dx^I) \wedge (g dx^J)) = d(fg dx^I \wedge dx^J) = (g df + f dg) \wedge dx^I \wedge dx^J =$$
$$= (df \wedge dx^I) \wedge (g dx^J) + (-1)^k (f dx^I) \wedge (dg \wedge dx^J) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$$

(ii) proved.
0-form

$$d(df) = d\left(\frac{\partial f}{\partial x^j}dx^j\right) = \frac{\partial^2 f}{\partial x^i \partial x^j}dx^i \wedge dx^j = \sum_{i<j} \partial^2_{ij} f dx^i \wedge dx^j + \sum_{j<k} \partial^2_{ij} f dx^j \wedge dx^i =$$

$$= \sum_{i<j}\left(\frac{\partial^2 f}{\partial x^i \partial x^j} - \frac{\partial^2 f}{\partial x^j \partial x^i}\right)dx^i \wedge dx^j = 0$$

for k-form, use $k = 0$ case, (ii)

$$d(d\omega) = d\left(\sum_J{}' d\omega_J \wedge dx^{j_1} \wedge \dots \wedge dx^{j_k}\right) =$$

$$= \sum_J{}' d(d\omega_J) \wedge dx^{j_1} \wedge \dots \wedge dx^{j_k} + \sum_J{}' \sum_{i=1}^k (-1)^i d\omega_J \wedge dx^{j_1} \wedge \dots \wedge d(dx^{j_i}) \wedge \dots \wedge dx^{j_k} = 0$$

from (12.17), $d(\omega|_U) = (d\omega)|_U$

$$(d_U\omega)|_{UU'} = d_{UU'}\omega = (d_{U'}\omega)|_{UU'}$$

Suppose $\exists$ another operator $\widetilde{d} : \mathcal{A}^k(M) \to \mathcal{A}^{k+1}(M)$
$\eta = \omega - \omega'$, let $p \in U$.
Let $\varphi \in \mathcal{C}^\infty(M)$ smooth bump function, $\varphi = 1$ in neighborhood of p, supported in U.
Then $\varphi\eta = 0$ in M, so

$$0 = \widetilde{d}(\varphi\eta)_p = d\varphi_p \wedge \eta_p + \varphi(p)\widetilde{d}\eta_p = \widetilde{d}\eta_p$$

because $\varphi \equiv 1$ in neighborhood of p
p arbitrary, so $d\eta = 0$ on U. $d\omega = d\omega'$ (locality)

□

Antiderivation of degree $g \in \mathbb{Z}$ on $\mathbb{Z}$-graded $\mathbb{R}$-algebra $A = \bigoplus_{k\in\mathbb{Z}} A_k$ in $\mathbb{R}$-linear $D : A \to A$

$$D(A_k) = A_{k+g}$$

s.t.

$$D(a_k a_l) = (Da_k)a_l + (-1)^k a_k(Da_l) \qquad a_k \in \mathcal{A}_k,\ a_l \in A_l$$

Example 12.15.

$$\omega = Pdx + Qdy + Rdz$$

Recall

$$d\left(\sum_J{}' \omega_J dx^J\right) = \sum_J{}' d\omega_J \wedge dx^J$$

$$d\omega = dP \wedge dx + dQ \wedge dy + dR \wedge dz =$$

$$= \left(\frac{\partial P}{\partial x}dx + \frac{\partial P}{\partial y}dy + \frac{\partial P}{\partial z}dz\right) \wedge dx + \left(\frac{\partial Q}{\partial x}dx + \frac{\partial Q}{\partial y}dy + \frac{\partial Q}{\partial dz}dz\right) \wedge dy + \left(\frac{\partial R}{\partial x}dx + \frac{\partial R}{\partial y}dy + \frac{\partial R}{\partial z}dz\right) \wedge dz$$

$$= \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx \wedge dy + \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right)dx \wedge dz + \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)dy \wedge dz$$

An Invariant Formula for the Exterior Derivative.

Proposition 31 (14.29 (Exterior Derivative of a 1-Form))**.**

$$d\omega(X,Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y]) \tag{28}$$

Proof. $\forall\, \omega \in \Omega^1(M)$, $\omega = udv$ for some $u, v \in C^\infty(M)$

$$\begin{aligned}
&d\omega(X,Y) = d(udv)(X,Y) = du \wedge dv(X,Y) = du(X)dv(Y) - dv(X)du(Y) = X(u)Y(v) - X(v)Y(u)\\
&X(udv(Y)) = X(uY(v)) = X(u)Y(v) + uXY(v)\\
&Y(udv(X)) = Y(uX(v)) = Y(u)X(v) + uYX(v)\\
&udv([X,Y]) = uXYv - uYXv\\
&\implies d\omega(X,Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y])
\end{aligned}$$

□

Proposition 32 (14.30)**.** *Let smooth manifold M of $dim M = n$,*
Let (E_i) smooth local frame for M, let (ϵ^i) dual coframe.
Let $d\epsilon^i = \sum_{j<k} b^i_{jk}\epsilon^j \wedge \epsilon^k$
$\quad [E_j, E_k] = c^i_{jk}E_i$
Then $b^i_{jk} = -c^i_{jk}$

Proof is Exercise 14.31.

Exercise 14.31.

Proof. Assume $j < k$ without loss of generality.

$$d\epsilon^i(E_j, E_k) = \sum_{j'<k'} b^i_{j'k'}(\delta^{j'}_j \delta^{k'}_k - \delta^{j'}_k \delta^{k'}_j = b^i_{jk} = E_j \delta^i_k - E_k \delta^i_j - c^i_{jk}$$
$$\implies b^i_{jk} = -c^i_{jk}$$

□

Lie Derivatives of Differential Forms.

Proposition 33 (14.33)**.** *Suppose M smooth manifold, $V \in \mathfrak{X}(M)$, $\omega, \eta \in \Omega^*(M)$*

$$\mathcal{L}_V(\omega \wedge \eta) = (\mathcal{L}_V \omega) \wedge \eta + \omega \wedge (\mathcal{L}_V \eta)$$

Theorem 11 (14.35)**.** *(Cartan's Magic Formula)*

$$\mathcal{L}_V \omega = V \lrcorner (d\omega) + d(V \lrcorner \omega) =$$

EY

$$= i_V(d\omega) + d(i_V \omega)$$

Corollary 9 (14.36)**.** *(The Lie Derivative Commutes with d)*

$$\mathcal{L}_V(d\omega) = d(\mathcal{L}_V \omega)$$

16. Orientations

Orientations of Vector Spaces. **Exercise 13.1.**

$$(E_1 \dots E_n) \sim (E_1 \dots E_n)$$
$$E_i = \delta_i^{\ j} E_j$$
$$\det(\delta_i^{\ j}) = 1$$

If $E_i = B_i^{\ j} \widetilde{E}_j$, $\det B_i^{\ j} > 0$,
$\det(BB^{-1}) = \det B \det B^{-1} = 1$. $\det B^{-1} > 0$ since $\det B > 0$.
If $E_i = B_i^{\ j} \widetilde{E}_j$

$$\widetilde{\widetilde{E}}_i = A_i^{\ j} E_j$$
$$\widetilde{\widetilde{E}}_i = A_i^{\ k} B_k^{\ j} \widetilde{E}_j \qquad \det AB = \det A \det B > 0$$

So it's an equivalence relation and since $\det B \neq 0$, $\det B > 0$ or $\det B < 0$ and as above there are only 2 equivalence classes; $\det A \gtrless 0$

Orientations of Manifolds. lcaol frame (E_i) for M is (positively) oriented if $(E_1|_p \dots E_n|_p)$ positively oriented basis for T_pM. $\forall\, p \in U$
cont. if $\forall\, p \in M$, $p \in$ domain of oriented local frame.
orientation of M is cont., pointwise orientation.
M orinetable if $\exists$ orientation to M

Exercise 13.2. connected domain U. Consider some $\Omega \in \Lambda^m(V)$, consider

$$U \to \mathbb{R}$$
$$p \mapsto (E_1|_p \dots E_n|_p) \mapsto \Omega(E_1|_p \dots E_n|_p) = \det(\epsilon^i(E_j)\big|_p)$$

det cont. function s.t. $\det = \begin{cases} +1 \\ -1 \end{cases}$ on U. U connected so $\forall$ cont. functions from X to $\{0, 1\}$ or, the same, $\{1, -1\}$, constant.
So $\Omega(E_1|_p \dots E_n|_p) = \det(\epsilon^i(E_j)\big|_p)$ constant on U, otherwise U separated.

Proposition 34 (13.4)**.** *$\dim M = m \geq 1$*
$\forall$ m-form Ω on M, $\Omega \neq 0$ determines a unique orientation of M s.t. Ω positively oriented $\forall\, p \in M$,
Conversely, if M given orientation, then $\exists$ smooth m-form Ω on M that's positively oriented $\forall\, p \in M$

$F : M \to N$ local diffeomorphism.
$\forall\, p \in M$, $\exists\, (U, \varphi)$ chart and consider $U_F \subset U$ s.t. $F(U_F)$ open, $F|_{U_F} : U_F \to F(U_F)$ diffeomorphism.
For $F(p) \in N$, $\exists\, (V, \psi)$ chart and consider $F(U_F) \cap V$
$\psi F \varphi^{-1}(x^1 \dots x^m) = F^j(x^1 \dots x^m) \qquad \det(\partial_i F^j) > 0$ suppose.

The Orientation Covering.

Orientations of Hypersurfaces. interior multiplication or contraction with X

$$i_X\omega(Y_1\dots Y_{k-1}) = \omega(X, Y_1\dots Y_{k-1})$$

$i_X\omega$ obtained by inserting X into the first slot.
Notation $X \lrcorner\, \omega = i_X\omega$
Suppose M smooth manifold
$S \subset M$ submanifolds (immersed or embedded)
vector field along S is cont. $N : S \to TM$ s.t. $N_p \in T_pM \quad \forall\, p \in S$
(Note difference between vector field along S and vector field on S, s.t. $N_p \in T_pS\;\forall\, p$)
$N_p \in T_pM$, $p \in S$ transverse to S if T_pM spanned by N_p, T_pS
Similarly, vector field N along S transverse to S if N_p transverse to S, $\;\forall\, p \in S$

Proposition 35 (15.21, 13.12 in previous version)**.** *Suppose M oriented smooth m-manifold*
S immersed hypersurface in M.
N transverse vector field along S.
Then S has unique orientation s.t. $\forall\, p \in S$, $(E_1\dots E_{n-1})$ oriented basis for T_pS iff $(N_p, E_1\dots E_{n-1})$ oriented basis for T_pM
If Ω orientation form for M,
then $(N \lrcorner\, \Omega)|_S \equiv i_N\Omega|_S$ orientation form for S with respect to this orientation.

Recall that smooth hypersurface S is $S \subseteq M$ equipped with $i : S \hookrightarrow M$, smooth immersion, i.e. Di injective.
Now orientation form $\omega = dN_p \wedge dE_1 \wedge \dots \wedge dE_{n-1}$, then
$i_{N_p}\omega = dE_1 \wedge \dots \wedge dE_{n-1}$

$$i_S^*(i_{N_p}\omega) = i_S^*(dE_1 \wedge \dots \wedge dE_{n-1}) = i_S^*(dE_1) \wedge \dots \wedge i_S^*(dE_{n-1}) = d(i_S^*E_1) \wedge \dots \wedge d(i_S^*E_{n-1})$$

Proof. Let Ω
$\omega = (N \lrcorner\, \Omega)|_S$ $m-1$ form.
$(E_1\dots E_{n-1})$ basis for T_pS
N transverse to S implies $(N_p, E_1\dots E_{n-1})$ basis for T_pM.
Ω orientation form so Ω nonvanishing.

$$\omega_p(E_1\dots E_{n-1}) = \Omega_p(N_p, E_1\dots E_{n-1}) \neq 0$$

since $\omega_p(E_1\dots E_{n-1}) > 0$ iff $\Omega_p(N_p, E_1\dots E_n) > 0$,
orientation determined by ω is the 1 defined in the statement of the proposition.

□

17. Integration on Manifolds

Integration of Functions on Riemannian Manifolds.

Proposition 36 (16.28)**.** *(M, g) oriented Riemannian manifold with or without ∂*
Suppose f compactly supported cont. on M, $f \in \mathbb{R}$, $f \geq 0$
Then $\int_M f dV_g \geq 0$
$\int_M f dV_g = 0$ iff $f = 0$

Proof. If f supported in
domain of single oriented smooth chart (U, φ)

By Prop. 15.31

$$\int_M f dV_g = \int_{\varphi(U)} f(x)\sqrt{\det(g_{ij})}dx^1\dots dx^n \geq 0$$

□

Exercise 16.29. given oriented Riemannian manifold (M, g)
compact supported cont. $f : M \to \mathbb{R}$

Then if f supported in
domain of single oriented smooth chart (U, φ)

$$\left|\int_M f dV_g\right| = \left|\int_{\varphi(U)} f(x)\sqrt{\det(g_{ij})}dx^1\dots dx^n\right| \geq \int_{\varphi(U)} |f(x)|\sqrt{\det(g_{ij})}dx^1\dots dx^n = \int_M |f| dV_g$$

where inequality above is from some thm. in calculus.

The Divergence Theorem.

(29)
$$\begin{aligned} * &: C^\infty(M) \to \Omega^n(N) \\ * f &= f dV_g \end{aligned} \qquad (16.10)$$

smooth bundle isomorphism
EY : 20140501 smooth bundle isomorphism?
smooth bundle isomorphism

$$\begin{aligned} &\beta : \mathfrak{X}(M) \to \Omega^{n-1}(M) \\ &\beta(X) = X \lrcorner dV_g \end{aligned}$$

technical lemma

Lemma 25 (16.30)**.** *(M,g) oriented Riemannian manifold with or without ∂*
Suppose $S \subseteq M$ immersed hypersurface with orientation by
unit normal vector field N and
$\widetilde{g}$ induced metric on S

If X any vector field along S,

(30)
$$i_S^*(\beta(X)) = \langle X, N \rangle_g dV_{\widetilde{g}} \qquad (16.12)$$

Proof. Define vector fields X^T, $X^\perp$ along S

$$\begin{aligned} X^\perp &= \langle X, N \rangle_g N \\ X^T &= X - X^\perp \end{aligned}$$

$$\beta(X) = X \lrcorner dV_g = X^\perp \lrcorner dV_g + X^T \lrcorner dV_g$$

pull back to S
Prop. 15.32

$$i_S^*(X^\perp \lrcorner dV_g) = \langle X, N \rangle_g i_S^*(N \lrcorner dV_g) = \langle X, N \rangle_g dV_{\widetilde{g}}$$

If $X_1 \dots X_{n-1}$ any vectors tangent to S

$$(X^T \lrcorner dV_g)(X_1 \dots X_{n-1}) = dV_g(X^T, X_1 \dots X_{n-1}) = 0$$

□

18. De Rham Cohomology

$d\omega = 0$ closed
$\omega = d\eta$ exact.

Prop. 6.24 smooth 1-form conservative iff exact.

The de Rham Cohomology Groups. closed 1-form

(31)
$$\omega = \frac{xdy - ydx}{x^2 + y^2} \qquad (15.1)$$

Suppose

$$\begin{aligned} x &= r\cos\theta \qquad & dx &= c_\theta dr + -rs_\theta d\theta \\ y &= rs_\theta \qquad & dy &= s_\theta dr + rc_\theta d\theta \end{aligned}$$

$$d\alpha = \omega = \frac{xdy - ydx}{x^2 + y^2} = \frac{x}{x^2+y^2} - \frac{y}{x^2+y^2} dx = \frac{1}{r} c_\theta (s_\theta dr + rc_\theta d\theta) - \frac{s_\theta}{r}(c_\theta dr - rs_\theta d\theta) = d\theta$$

M smooth manifold
$d : \mathcal{A}^p(M) \to \mathcal{A}^{p+1}(M)$ linear, ker, im linear subspaces.
$\mathcal{Z}^p(M) = \ker[d : \mathcal{A}^p(M) \to \mathcal{A}^{p+1}(M)] = \{$ closed p-forms on M $\}$
$\mathcal{B}^p(M) = \mathrm{im}[d : \mathcal{A}^{p-1}(M) \to \mathcal{A}^p(M)] = \{$ exact p-forms on M $\}$
By convention, $\mathcal{A}^p(M)$ zero vector space, $p < 0$ or $p > n = \dim M$
e.g.

$$\begin{aligned} \mathcal{B}^0(M) &= 0 \\ \mathcal{Z}^n(M) &= \mathcal{A}^n(M) \end{aligned}$$

$d^2 = 0$ so
$\forall$ exact form closed. $\mathcal{B}^p(M) \subset \mathcal{Z}^p(m)$

$$H^p_{dR}(M) = \frac{\mathcal{Z}^p(M)}{\mathcal{B}^p(M)}$$

Homotopy Invariance.

Lemma 26 (15.4)**.** *(Existence of a Homotopy Operator) $\forall$ smooth manifold M, $\exists$ homotopy operator ${}^*_0$, ${}^*_1$*

Proof. $\forall\, p$, define linear $h : \mathcal{A}^p(M \times I) \to \mathcal{A}^{p-1}(M)$ s.t.

(32) $$h(d\omega) + d(h\omega) = {}^*_1\omega - {}^*_0\omega \qquad (15.5)$$

define $h\omega = \int_0^1 \left(\frac{\partial}{\partial t} \lrcorner \omega\right) dt$
$h\omega$ $(p-1)$ form on M whose action on $X_1 \dots X_{p-1} \in T_qM$ is

$$(h\omega)_q(X_1 \dots X_{p-1}) = \int_0^1 \left(\frac{\partial}{\partial t} \lrcorner \omega(q,t)\right)(X_1 \dots X_{p-1}) dt = \int_0^1 \omega_{(q,t)}\left(\frac{\partial}{\partial t}, X_1 \dots X_{p-1}\right) dt$$

choose smooth local coordinates (x^i) on M.

Consider separately $\omega = f(x,t) dt \wedge dx^{i_1} \wedge \dots \wedge dx^{i_{p-1}}$
$\omega = f(x,t) dx^{i_1} \wedge \dots \wedge dx^{i_p}$

□

19. Distributions and Foliations

Distributions and Involutivity. distribution on M of rank k is rank-k subbundle of TM, **smooth distribution** if it's smooth subbundle
Often rank-k distribution described by specifying $\forall\, p \in M$ linear subspace $D_p \subseteq T_pM$ of $\dim D_p = k$,
$D = \bigcup_{p \in M} D_p$
Lemma 10.32, local frame criterion for subbundles, that D smooth distribution iff $\forall\, p \in M$, $\exists$ open $U \ni p$ on which $\exists$ smooth vector fields $X_1 \dots X_k : U \to TM$ s.t. $X_1|_q \dots X_k|_q$ is basis for D_q $\forall\, q \in U$

Integral Manifolds and Involutivity. Suppose smooth distribution $D \subseteq TM$
integral manifold of D : immersed submanifold $N \neq \varnothing$, $N \subseteq M$ if $T_pN = D_p$ $\forall\, p \in N$
Example 19.1 (Distributions and Integral Manifolds)

(a)
(b)
(c)
(d)

D **involutive** if $\forall$ pair of smooth local sections of D (i.e. smooth vector fields X, Y defined on open subset of M s.t. $X_p, Y_p \in D_p$ $\forall\, p$)

integrable : smooth distribution D on M integrable if $\forall\, p \in M$, p in integral manifold of D, i.e.
$T_pM = D_p$

Proposition 37 (19.3)**.** *$\forall$ integrable distribution is involutive.*

Proof. Let $D \subseteq TM$ is integrable distribution.
suppose smooth local sections of D, X, Y on some open $U \subseteq M$.
$\forall\, p \in U$, let N integral manifold of D, $N \ni p$
X, Y sections of D, so X, Y tangent to N
By Corollary 8.32, $[X, Y]$ also tangent to N, so $[X, Y]_p \in D_p$ □

Involutivity and Differential Forms.

Lemma 27 (19.5)**.** ***(1-form Criterion for Smooth Distributions)*** *Suppose smooth n-dim. manifold M, distribution $D \subseteq TM$, rank k D smooth iff $\forall\, p \in M$, $\exists$ neighborhood U on which $\exists$ smooth 1-forms $\omega^1 \dots \omega^{n-k}$ s.t. $\forall\, q \in U$,*

(33) $$D_q = ker\ \omega^1\big|_q \bigcap \cdots \bigcap ker \omega^{n-k}\big|_q \qquad (19.1)$$

Proof. By Prop. 10.15, complete forms $\omega^1 \dots \omega^{n-k}$ to smooth coframe $(\omega^1 \dots \omega^n)$ $\forall\, p$
if $(E_1 \dots E_n)$ dual frame, easy to sheet that D locally spanned by $E_{n-k+1}, \dots, E_n$, so smooth by local frame criterion.
Converse, suppose D smooth.
$\forall$ open $U \ni p \in M$, $\exists$ smooth vector fields $Y_1 \dots Y_k$ spanning D.

By Prop. 16.5, complete $Y_1 \dots Y_k$ to smooth local frame $(Y_1 \dots Y_n)$ for M in open $U \ni p$
with dual coframe $(\epsilon^1 \dots \epsilon^n)$, it follows easily that D characterized locally by $D_q = \ker \epsilon^{k+1}|_q \cap \dots \cap \ker \epsilon^n|_q$

□

if D rank-k distribution on smooth n-manifold M, any
$n-k$ linearly independent 1-forms $\omega^1 \dots \omega^{n-k}$ on open $U \subseteq M$ s.t. (19.1)

$$D_q = \ker\omega^1|_q \bigcap \dots \bigcap \ker\omega^{n-k}|_q = \{ X | X = X^i X_i,\, i = 1 \dots k,\, \omega^1(X) = 0 \} \bigcap \dots \bigcap \{ X | \omega^{n-k}(X) = 0 \}$$

$\forall\, q \in U$ are **local defining forms** for D

Proposition 38 (19.8)**.** ***(Local Coframe Criterion for Involutivity)*** *Let D smooth distribution of rank k on smooth n-manifold M let $\omega^1 \dots \omega^{n-k}$ smooth defining forms for D on open $U \subseteq M$.*
The following are equivalent:

(a) *D is involutive on U*
(b) *$d\omega^1 \dots d\omega^{n-k}$ annihilate D*
(c) *$\exists$ smooth 1-forms $\{ \alpha^i_j | i,j = 1 \dots n-k \}$ s.t.*

$$d\omega^i = \sum_{j=1}^{n-k} \omega^j \wedge \alpha^i_j \qquad \forall\, i = 1 \dots n-k$$

Exercise 19.9. Prove the preceding proposition, 19.8.

Proof. (a) $\Longrightarrow$ (b)
On open $U \subseteq M$, $\forall\, q \in U$, ω^i smooth defining form for D, $i = 1 \dots k$, and $\omega^i(X) = 0 \quad \forall\, X \in D_q$
Then $d\omega^i$ also annihilates D on U (Thm. 19.7 1-form Criterion for Involutivity (19.3))
$d\omega^1 \dots d\omega^{n-k}$ annihilate D
(b) $\Longrightarrow$ (c)
$d\omega^i \in \Omega^2_q(M)$, $\forall\, q \in U$
By Lemma 19.6, smooth p-form η on U annihilates D iff η ofform $\eta = \sum_{i=1}^{n-k} \omega^i \wedge \beta^i$, for $(p-1)$ forms $\beta^1 \dots \beta^{n-k}$ on U
$d\omega^i$ annihilates D
$\Longrightarrow d\omega^i = \sum_{j=1}^{n-k} \omega^j \wedge \beta^i_j \qquad \beta^i_j$ smooth 1-forms on U, $i,j = 1 \dots n-k$
(c) $\Longrightarrow$ (a)
Use Thm. 19.7 Proof

$$\omega^i([X,Y]) = X(\omega^i(Y)) - Y(\omega^i(X)) - d\omega^i(X,Y) = 0 - 0 - d\omega^i(X,Y)$$
$$d\omega^i(X,Y) - \sum_{j=1}^{n-k} \omega^j \wedge \alpha^i_j(X,Y) = \sum_{j=1}^{n-k} \omega^j(X)\alpha^i_j(Y) - \alpha^i_j \omega^j(Y) = 0 - 0 = 0$$

where I used this local formula:

$$(\alpha \wedge \beta)_p(v,w) = \alpha_p(v)\beta_p(w) - \alpha_p(w)\beta_p(v)$$

$\omega^i([X,Y]) = 0$ so $[X,Y] \in \ker\omega^i \qquad \forall\, i = 1 \dots n-k$

□

Problems. **Problem 19-3.**
Let $\omega \in \Omega^1(M)$
integrating factor μ for $\omega \equiv \mu \in C^\infty(M)$, $\mu > 0$, and $\mu\omega$ exact on U, i.e. $\mu\omega = df$, for some $f \in \mathcal{C}^\infty(M)$

(a) If $\omega \neq 0$ on U,
Suppose ω admits an integrating factor μ.

$$d\omega \wedge \omega = d\left(\frac{df}{\mu} \right) \wedge \frac{df}{\mu} = \left(\frac{d^2 f}{\mu} + -\frac{df}{\mu^2} \frac{\partial \mu}{\partial x^i} dx^i \wedge df \right) \wedge \frac{df}{\mu} = 0$$

as $d^2 f = 0$ and $df \wedge df = 0$
If $d\omega \wedge \omega = 0$, consider $\mu \in \mathcal{C}^\infty(M)$ s.t. $\mu > 0$ (i.e. positive) on open $U \subseteq M$ (build it up with partitions of unity if need to).
Now, using the formula for exterior differentiation,

$$d(\mu\omega) = d\mu \wedge \omega + (-1)^0 \mu d\omega$$

so that

$$d(\mu\omega) \wedge \omega = d\mu \wedge \omega \wedge \omega + \mu d\omega \wedge \omega = 0 + \mu d\omega \wedge \omega = 0 + 0 = 0$$

ω nonzero, so $d(\mu\omega) = 0$. EY : 20150221 I'm not sure about this statement. Surely, locally,

$$d(\mu\omega) \wedge \omega = \frac{1}{2}(d(\mu\omega))_{ij} \omega_k dx^i \wedge dx^j \wedge dx^k = d(\mu\omega)_{\underline{I}} \omega_k dx^{\underline{I}} \wedge dx^k$$

with $\underline{I} = (i_1, i_2)$ and $i_1 < i_2$.

By considering every $k \neq \underline{I}$, then I think one can conclude, component by component, that $d(\mu\omega) = 0$.
Then, consider a compact submanifold B, $\dim B = 3$ that is a submersion of U. Then use Stoke's theorem in the following:

$$\int_B d(\mu\omega) = \int_{\partial B} \mu\omega = 0 \Longrightarrow \mu\omega = df$$

So

$$\boxed{\begin{array}{c} \text{If } \omega \neq 0 \text{ on } U \\ \omega \text{ admits an integrating factor } \mu \text{ iff } d\omega \wedge \omega = 0 \end{array}}$$

EY 20150221 : I didn't use Frobenius' theorem for the converse. Should I have?

(b) If $\dim M = 2$, $d\omega \wedge \omega = 0$ (immediately)
Then ω admits an integrating factor by the above solution.

20. The Exponential Map

20.1. One-Parameter Subgroups and the Exponential Map.

20.1.1. *One-Parameter Subgroups.*

Theorem 12 (20.1)**.** ***(Characterizations of One-Parameter Subgroups)***

Proposition 39 (20.8)**.** ***(Properties of Exponential Map)*** *Let* $\begin{array}{ll} G & \text{Lie group} \\ \mathfrak{g} & \text{Lie algebra} \end{array}$

(a) $\exp : \mathfrak{g} \to g$ *smooth*
(b) $\forall\, X \in \mathfrak{g}$, $s, t \in \mathbb{R}$, $\exp{(s+t)X} = \exp{sX} \exp{tX}$
(c) $\forall\, X \in \mathfrak{g}$, $(\exp X)^{-1} = \exp{(-X)}$
(d) $\forall\, X \in \mathfrak{g}$, $n \in \mathbb{Z}$, $(\exp X)^n = \exp{(nX)}$
(e) *differential* $(d\exp)_0 : T_0\mathfrak{g} \to T_e G$ *is identity, under canonical identifications of both* $T_0\mathfrak{g}$ *and* $T_e G$ *with* $\mathfrak{g}$ *itself.*
(f) $\exp$ *restricts to diffeomorphism from some neighborhood of* 0 *in* $\mathfrak{g}$ *to neighborhood of* e *in* G.
(g) *if* $\Phi : G \to H$ *lie group homomorphism, following commutes*

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\Phi_*"] \arrow[d, "\exp"'] & \mathfrak{h} \arrow[d, "\exp"] \\ G \arrow[r, "\Phi"] & H \end{tikzcd}$$

(h) *flow* θ *of left-invariant vector field* X, $\theta_t = R_{\exp tX}$

Proof. (a)
(b)
(c)
(d)

(e) Let $X \in \mathfrak{g}$ arbitrary, let $\begin{array}{l} \sigma : \mathbb{R} \to \mathfrak{g} \\ \sigma(t) = tX \\ \dot{\sigma}(0) = X \end{array}$

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\exp"] \arrow[d] & G \arrow[d, "(d\exp)_0"] \\ T_0\mathfrak{g} \arrow[r] & T_e G \end{tikzcd}$$

$$(d\exp)_0(X) = (d\exp)_0(\dot{\sigma}(0)) = (\exp \circ \sigma)'(0) = (\exp{(tX)})'(0) = \left.\frac{d}{dt}\right|_{t=0} \exp{(tX)} = X$$

(f) $(d\exp)_0 = 1$ and so by inverse function thm., $\exists\, (d\exp)_0^{-1} = 1^{-1} = 1$, and so $\begin{array}{ccc} U \ni 0 & \xrightarrow{\simeq} & V \ni e \\ U \subseteq \mathfrak{g} & & V \subseteq G \end{array}$

$\square$

21. Quotient Manifolds

Problems. Problem 21-6.
Suppose Lie group G acts smoothly, freely, and properly on smooth manifold M

22. Symplectic Manifolds

22.1. Symplectic Tensors. Exercise 22.1.

http://math.stackexchange.com/questions/342267/non-degenerate-bilinear-forms-and-invertible-matrices (shout out to Branimir Cacic for the answer) gave me a hint at how to approach this exercise, even though the original question was for symmetric bilinear forms.
Let $\{e_1 \dots e_n\}$ be (some) basis of V
Let $\{f^1 \dots f^n\}$ be dual basis of V s.t. $f^i(e_j) = \delta^i_{\ j}$
Now

$$\widehat{\omega}(e_i) = (\widehat{\omega}(e_i))_j f^j$$
$$\widehat{\omega}(e_i) e_j = (\widehat{\omega}(e_i))_j = i_{e_i}\omega(e_j) = \omega(e_i, e_j) \equiv \omega_{ij}$$

so $\omega_{ij} = (\widehat{\omega}(e_i))_j$ (i.e. matrix ω_{ij} is precisely $(\widehat{\omega}(e_i))_j$.
If ω_{ij} nonsingular, i.e. $\exists\, \omega^{-1}_{ki}$ s.t. $\omega^{-1}_{ki}\omega_{ij} = \omega_{ki}\omega^{-1}_{kj} = \delta_{kj}$ (by def.)
If $\widehat{\omega}$ invertible, $\widehat{\omega}^{-1}\widehat{\omega}(v) = v$

$$\widehat{\omega}^{-1}\widehat{\omega}(e_i) = \widehat{\omega}^{-1}(\widehat{\omega}(e_i))_j f^j = (\widehat{\omega}(e_i))_j \widehat{\omega}^{-1}(f^j) = (\widehat{\omega}(e_i))_j (\widehat{\omega}^{-1}(f^j))^k e_k = e_i$$
$$\Longrightarrow (\widehat{\omega}(e_i))_j (\widehat{\omega}^{-1}(f^j))^k = \delta_i^{\ k}$$

So if ω_{ij} nonsingular, $(\widehat{\omega}^{-1}(f^j))^k$ exists and $(\widehat{\omega}(f^j))^k = \omega^{-1}_{jk}$
if $\widehat{\omega}$ invertible, ω^{-1}_{jk} exists and is given by $\omega^{-1}_{jk} = (\widehat{\omega}(f^j))^k$
So the (a) $\Longleftrightarrow$(c) part of the exercise is done.
Show (a) $\Longleftrightarrow$(b) and we're done.
If $\widehat{\omega} : V \to V^*$ linear isomorphism,

$$\ker\widehat{\omega} = 0$$

Suppose $\nexists\, w \in V$ s.t. $\omega(v, w) \neq 0$ (proof by contradiction strategy)
Then $\forall\, w \in V$, $\omega(v, w) = 0$
$\omega(v, w) = 0 = \widehat{\omega}(v)(w) \quad \forall\, w \in V$.
Then $v = 0$. Contradiction.
(b) $\Longrightarrow$ (a): if $\forall\, v \neq 0$, $\exists\, w \in V$ s.t. $\omega(v, w) \neq 0$
then if $\forall\, w \in V$, $\omega(v, w) = 0$, then $v = 0$
$\omega(v, w) = \widehat{\omega}(v)(w) = 0$ implies $v = 0$, $\forall\, w \in V$.
Then $\ker\widehat{\omega} = 0$. So $\widehat{\omega}$ linear isomorphism. So ω nondegenerate.

Exercise 22.4. There was a proof of this in Konstantin Athanassopoulos, **Notes on Symplectic Geometry**, Iraklion, 2013 http://www.math.uoc.gr/~athanako/symplectic.pdf
Recall that
$S^\perp = \{v \in V | \omega(v, w) = 0 \quad \forall\, w \in S\}$
$(S^\perp)^\perp = \{u \in V | \omega(u, v) = 0 \quad \forall\, v \in S^\perp\}$
Let $s \in S$. $\omega(s, v) = 0 \quad \forall\, v \in S^\perp$, by def. of $S^\perp$

$s \in (S^\perp)^\perp$
$\Longrightarrow S \subseteq (S^\perp)^\perp$
Then $\dim S \leq \dim (S^\perp)^\perp$ with equality iff $S = (S^\perp)^\perp$
Now by Lemma 22.3,
$\dim S + \dim S^\perp = \dim V$, $\forall$ linear subspace $S \subseteq V$
$\dim S^\perp + \dim (S^\perp)^\perp = \dim V = \dim S + \dim S^\perp \Longrightarrow \dim S = \dim (S^\perp)^\perp$

$\dim S \leq \dim (S^\perp)^\perp$ with equality iff $S = (S^\perp)^\perp$

22.2. Symplectic Structures on Manifolds. Exercise 22.10.

$F : N \to M$ smooth immersion. Recall definition: F_* injective. Recall Appendix B, Exercise B.22 (EY: 20150512 This exercise is **very useful**; I can't emphasize that enough). F_* injective so $\operatorname{rank} F_* = \dim N$. (implying $\ker F_* = 0$).
Recall that F isotropic if

$(F_*)_p(T_pN) \subseteq T_{F(p)}M$ isotropic, i.e.

$$(F_*)_p(T_pN) \subseteq ((F_*)_p(T_pN))^\perp$$

Consider $X, Y \in T_pN$, with X, Y nonzero. Then, as F_* injective, $Z, W \in ((F_*)_p(T_pN))$ nonzero, for $\begin{aligned} Z &= (F_*)_pX \\ W &= (F_*)_pY \end{aligned}$

Suppose $\omega(Z, W) = 0$, $\forall\, W \in (F_*)_p(T_pN)$. $Z \in ((F_*)_p(T_pN))^\perp$.

$$\omega(Z, W) = \omega((F_*)_pX, (F_*)_pY) = (F^*)_p\omega(X, Y) = 0$$

If F isotropic, then this is the case $\forall\, Z \in ((F_*)_p(T_pN)) \subseteq ((F_*)_p(T_pN))^\perp$.
Then since $(F^*)_p\omega(X, Y) = 0 \quad \forall\, p \in N$, $\forall\, X, Y \in T_pN$, then $F^*\omega = 0$.
If F symplectic, $(F_*)_p(T_pN) \cap ((F_*)_p(T_pN))^\perp = 0$
Likewise, for the reverse.
If F symplectic,

$$(F_*)_p(T_pN) \cap ((F_*)_p(T_pN))^\perp = 0$$

For $X, Y \in T_pN$, suppose

$$F_p^*\omega(X, Y) = \omega((F_*)_pX, (F_*)_pY) = 0$$

This implies $(F_*)_pX \in (F_*)_p(T_pN) \cap ((F_*)_p(T_pN))^\perp$
Then
since F_* immersion, $X, Y = 0$. So $F^*\omega$ is nondegenerate and so is a symplectic form.

22.2.1. *the Canonical Symplectic Form on the Cotangent Bundle.*

The most important symplectic manifolds are total spaces of cotangent bundles, which carry canonical symplectic structures that we now define.

22.3. **The Darboux Theorem.**

22.4. **Hamiltonian Vector Fields.** **Hamiltonian vector field** of f

$$X_f = \widehat{\omega}^{-1}(df)$$

Hamiltonian vector field of f in Darboux coordinates:

$$X_f = \sum_{i=1}^n \left(\frac{\partial f}{\partial y^i} \frac{\partial}{\partial x^i} - \frac{\partial f}{\partial x^i} \frac{\partial}{\partial y^i} \right) \tag{34}$$

(22.9)
smooth $X \in \mathfrak{X}(M)$ **symplectic** if ω invariant under flow of X, i.e. $\mathcal{L}_X\omega = 0$

22.4.1. *Poisson Brackets.* $f \in C^\infty(M)$ **conserved quantity** if f constant on every integral curve of X_H.
smooth $V \in \mathfrak{X}(M)$ **infinitesimal symmetry** of (M, ω, H) if ω, H invariant under flow of V, i.e. EY (20150521)

$$\mathcal{L}_V\omega = 0 \quad \mathcal{L}_VH = 0$$

Proposition 40 (22.21)**.** *Let (M, ω, H) Hamiltonian system*

(a) *$f \in C^\infty(M)$ conserved quantity iff $\left\{ \substack{\{f,H\} \\ =0} \right\}$*
(b) *infinitesimal symmetries of (M, ω, H) are precisely symplectic fields V s.t. $VH = 0$*
(c) *if θ flow of infinitesimal symmetry and γ trajectory of system*

Proof. This is the solution to Problem 22-18.

(a) if $f \in C^\infty(M)$ conserved quantity, by def. f constant on every integral curve of X_H

$$\{f, H\} = \frac{\partial f}{\partial x^i}\frac{\partial H}{\partial y^i} - \frac{\partial f}{\partial y^i}\frac{\partial H}{\partial x^i} = X_Hf = 0$$

for

$$X_H = \frac{\partial H}{\partial y^i}\frac{\partial}{\partial x^i} - \frac{\partial H}{\partial x^i}\frac{\partial}{\partial y^i}$$

likewise, if $\{f, H\} = 0$, then $X_Hf = 0$, $X_Hf = \mathcal{L}_{X_H}f = 0$, so f constant on flow of X_H

(b) Recall smooth $V \in \mathfrak{X}(M)$ infinitesimal symmetry of (M, ω, H) if ω, H invariant under flow of V, i.e.

$$\mathcal{L}_V\omega = 0 \quad \mathcal{L}_VH = 0$$

smooth $V \in \mathfrak{X}(M)$ symplectic if ω invariant under flow of V, i.e. $\mathcal{L}_V\omega = 0$

$$\mathcal{L}_VH = VH = 0$$

(c) EY : 20150521 I'm not sure how to go about this because what is a trajectory?
$\gamma : I \to M$
θ flow of an infinitesimal symmetry, so (collecting facts)

$$\begin{aligned} &\mathcal{L}_{\dot{\theta}}\omega = di_{\dot{\theta}}\omega + i_{\dot{\theta}}d\omega = di_{\dot{\theta}}\omega \quad (\omega \text{ closed so } i_{\dot{\theta}}d\omega) \\ &\dot{\theta}H = 0 \end{aligned}$$

Now $\theta_s \circ \gamma : I \to M$

$$\frac{d}{dt}(\theta_s \circ \gamma)(t) = (D\theta_s)(\gamma(t))\dot{\gamma}(t) = V_{s,\gamma(t)}\dot{\gamma}(t)$$

□

Problem 22.1.

Proof. (a) If S symplectic, $S \cap S^{\perp} = 0$. $S = (S^{\perp})^{\perp}$ so $(S^{\perp})^{\perp} \cap S^{\perp} = 0$. $S^{\perp}$ symplectic.
If $S^{\perp}$ symplectic, $S^{\perp} \cap (S^{\perp})^{\perp} = 0$. $S = (S^{\perp})^{\perp}$ so $(S^{\perp})^{\perp} = S \cap S^{\perp} = 0$. S symplectic.

(b) Suppose for $s \in S \cap S^{\perp}$, $s \neq 0$. Then as $s \in S^{\perp}$, $\omega(s,w) = 0 \;\forall\; w \in S^{\perp}$.
Then $\omega(s,s) = 0$. But ω nondegenerate so $s = 0$. Contradiction.
Suppose S symplectic. For $\omega|_S(s,t) = \omega(s,t) = 0$, for some $s \in S$, $\forall\, t \in S$, then $S \cap S^{\perp} = 0$ implies that $s,t = 0$. Then $\omega|_S$ nondegenerate.

(c) If S isotropic, $S \subseteq S^{\perp}$ so that $\omega(s,t) = 0 \quad \forall\, t \in S$ (def. of $S^{\perp}$). $\omega|_S = 0$ as $\omega(s,t) = 0, \quad \forall\, s,t \in S$.
If $\omega|_S = 0$, then $\forall\, s,t \in S$, $\omega(s,t) = 0 \;\forall\, t \in S$. By def. of $S^{\perp}$, $S \subseteq S^{\perp}$.

(d) if S coisotropic, $S \supseteq S^{\perp}$. $S^{\perp} \subseteq S = (S^{\perp})^{\perp}$. Then $S^{\perp}$ isotropic.
If $S^{\perp}$ isotropic, $S^{\perp} \subseteq (S^{\perp})^{\perp} = S$, so S coisotropic.

(e) If S Lagrangian, $\forall\, s \in S$, $s \in S^{\perp}$, so that $\omega(s,t) = 0 \quad \forall\, t \in S$. Then $\omega|_S = 0$, (i.e. identically 0).

$\dim S + \dim S^{\perp} = 2\dim S = \dim V$ by Lemma 22.3, so $\dim S = \frac{1}{2}\dim V$.
If $\dim S = \frac{1}{2}\dim V$, $\dim S^{\perp} = \frac{1}{2}\dim V = \dim S$. $\omega|_S = 0$, so S isotropic, i.e. $S \subseteq S^{\perp}$. $\dim S \leq \dim S^{\perp}$, with equality iff $S = S^{\perp}$.

□

Problem 22.-17. Given Hamiltonian system (T^*Q, ω, E).
Recall that

$$\begin{aligned} q(t) &= (q_1^1(t), q_1^2(t), q_1^3(t) \dots q_n^1(t), q_n^2(t), q_n^3(t)) = 0 \\ &= (q^1(t) \dots q^{3n}(t)) \end{aligned}$$

Now $p(t) = (p_1^1, p_1^2, p_1^3 \dots p_n^1, p_n^2, p_n^3)$ and
$p_i(t) = M_{ij}\dot{q}^j(t)$ with
M_{ij} $3n \times 3n$ diagonal matrix $(m_1, m_1, m_1, m_2, m_2, m_2 \dots m_n, m_n, m_n)$
Now $E \in C^{\infty}(T^*Q)$ where

$$E(q,p) = V(q) + K(p) = V(q) + \frac{1}{2}M^{ij}p_i p_j$$

(a) Let $\mathbf{u} = (u^1, u^2, u^3)$

$$\begin{aligned} &P : T^*Q \to \mathbb{R} \\ &P(q,p) = \mathbf{u}\cdot\mathbf{p}_1 + \mathbf{u}\cdot\mathbf{p}_2 \\ &\qquad = u^1 p_1^1 + u^2 p_1^2 + u^3 p_1^3 + u^1 p_2^1 + u^2 p_2^2 + u^3 p_2^3 \end{aligned}$$

Recall Prop. 22.21, Let (M, ω, H) Hamiltonian system
(a) $f \in C^{\infty}(M)$ conserved quantity iff $\{f, H\} = 0$
Now in Darboux coordinates,

$$\{f,g\} = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}\frac{\partial g}{\partial y^i} - \frac{\partial f}{\partial y^i}\frac{\partial g}{\partial x^i}$$

(22.16)
For

$$E = V(|\mathbf{q}_2 - \mathbf{q}_1|) + \frac{1}{2}M^{ij}p_i p_j$$

note that

$$V(|\mathbf{q}_2 - \mathbf{q}_1|) = V(r) \text{ with}$$
$$r = \sqrt{(q_2^1 - q_1^1)^2 + \dots + (q_2^3 - q_1^3)^2}$$
$$\frac{\partial V}{\partial q_i^j} = \frac{\partial V}{\partial r}\frac{1}{2}\frac{1}{r}(2)(q_2^j - q_1^j)(-1)^i = \frac{\partial V}{\partial r}\frac{1}{r}(q_2^j - q_1^j)(-1)^i$$

then

$$\frac{\partial E}{\partial p_i^j} = \frac{p_i^j}{m_i}$$
$$\frac{\partial E}{\partial q_i^j} = \frac{\partial V}{\partial r}\frac{1}{r}(-1)^i(q_2^j - q_1^j)$$

with $r = |\mathbf{q}_2 - \mathbf{q}_1| = \sqrt{(q_2^1 - q_1^1)^2 + \dots + (q_2^3 - q_1^3)^2}$

$$P = \mathbf{u}\cdot(\mathbf{p}_1 + \mathbf{p}_2) = u^i p_1^i + u^i p_2^i$$
$$\frac{\partial P}{\partial p_i^j} = u^j$$
$$\frac{\partial P}{\partial q} = 0$$
$$\{P, E\} = 0 - u^j \frac{\partial V}{\partial r}\frac{1}{r}(-1)^i(q_2^j - q_1^j) = -\mathbf{u}\cdot(\mathbf{q}_2 - \mathbf{q}_1)\frac{\partial V}{\partial r}\frac{1}{r}(-1) + -\mathbf{u}\cdot(\mathbf{q}_2 - \mathbf{q}_1)\frac{\partial V}{\partial r}\frac{1}{r} = 0$$

(b)

$$L(q,p) = q_1^1 p_1^2 - q_1^2 p_1^1 + q_2^1 p_2^2 - q_2^2 p_2^1$$
$$\frac{\partial L}{\partial q_i^j} = p_i^k \epsilon^{jk}$$
$$\frac{\partial L}{\partial p_i^k} = q_i^j \epsilon^{jk}$$
$$\{L, E\} = p_i^k \epsilon^{jk}\frac{p_i^j}{m_i} - q_i^k \epsilon^{kj}\frac{\partial V}{\partial r}\frac{1}{r}(-1)^i(q_2^j - q_1^j) = \frac{p_i^2 p_i^1}{m_i} - \frac{p_i^1 p_i^2}{m_i} - q_i^k \epsilon^{kj}\frac{\partial V}{\partial r}\frac{1}{r}(-1)^i(q_2^j - q_1^j) =$$
$$= 0 - \frac{\partial V}{\partial r}\frac{1}{r}(-q_1^2(q_2^1 - q_1^1)(-1) + q_1^1(q_2^2 - q_1^2)(-1) - q_2^2(q_2^1 - q_1^1) + q_2^1(q_2^2 - q_1^2)) = 0$$

Problem 22-18.

(a) if $f \in C^\infty(M)$ conserved quantity, by def. f constant on every integral curve of X_H

$$\{f, H\} = \frac{\partial f}{\partial x^i}\frac{\partial H}{\partial y^i} - \frac{\partial f}{\partial y^i}\frac{\partial H}{\partial x^i} = X_H f = 0$$

for

$$X_H = \frac{\partial H}{\partial y^i}\frac{\partial}{\partial x^i} - \frac{\partial H}{\partial x^i}\frac{\partial}{\partial y^i}$$

likewise, if $\{f, H\} = 0$, then $X_H f = 0$, $X_H f = \mathcal{L}_{X_H} f = 0$, so f constant on flow of X_H

(b) Recall smooth $V \in \mathfrak{X}(M)$ infinitesimal symmetry of (M, ω, H) if ω, H invariant under flow of V, i.e.

$$\mathcal{L}_V \omega = 0 \quad \mathcal{L}_V H = 0$$

smooth $V \in \mathfrak{X}(M)$ symplectic if ω invariant under flow of V, i.e. $\mathcal{L}_V \omega = 0$

$$\mathcal{L}_V H = VH = 0$$

(c) EY : 20150521 I'm not sure how to go about this because what is a trajectory?

$\gamma : I \to M$

θ flow of an infinitesimal symmetry, so (collecting facts)

$$\mathcal{L}_{\dot{\theta}}\omega = di_{\dot{\theta}}\omega + i_{\dot{\theta}}d\omega = di_{\dot{\theta}}\omega \quad (\omega \text{ closed so } i_{\dot{\theta}}d\omega)$$
$$\dot{\theta}H = 0$$

Now $\theta_s \circ \gamma : I \to M$

$$\frac{d}{dt}(\theta_s \circ \gamma)(t) = (D\theta_s)(\gamma(t))\dot{\gamma}(t) = V_{s,\gamma(t)}\dot{\gamma}(t)$$

Appendix A. Review of Topology

Topological Spaces.

- **neighborhood of** p open subset $\mathcal{O}$ containing p

Bases and Countability. Suppose X topological space.
basis for topology of X

$$\mathcal{B} = \{B | \text{ open } B \subseteq X, \forall \text{ open } \mathcal{O} \subset X, \mathcal{O} = \bigcup_{\alpha} B_{\alpha}\}$$

neighborhood basis at p $\mathcal{B}_p = \{\text{neighborhoods } B_p \text{ of } p \,|\forall\, \mathcal{O} \text{ of } p, B_p \subset \mathcal{O},\ B_p \in \mathcal{B}_p \text{ for at least 1 }\}$
X **first countable** - if $\exists$ countable neighborhood basis $\forall\, p$
second countable if $\exists$ countable basis

Proposition 41 (A.16)**.** *Let X second countable topological space.*
$\forall$ open cover of X has countable subcover

Proof. X second countable. $\exists$ countable basis $\mathcal{B}$ for X
Let $\mathcal{U}$ arbitrary open cover of X.
Let $\mathcal{B}' \subseteq \mathcal{B}$ s.t. $\mathcal{B}' = \{B | B \in \mathcal{B}, B \subseteq U \text{ for some } U \in \mathcal{U}\}$

$\forall\, B \in \mathcal{B}'$, choose particular $U_B \in \mathcal{U}$ containing B

$\{U_B | B \in \mathcal{B}'\}$ countable

$\forall\, x \in X$, $\exists\, V \in \mathcal{U}$, $x \in V$
$\mathcal{B}$ basis, s.t. $\exists\, B \in \mathcal{B}$ s.t. $x \in B \subseteq V$
then $B \in \mathcal{B}'$, so $x \in B \subseteq U_B$ □

Subspaces, Products, Disjoint Unions, and Quotients.

Review of Topology

Topological Spaces. Let X set
topology on X = collection $\tau = \{U | U \subseteq X\}$, U called open subsets s.t.

(i) $X, \varnothing$ open
(ii) $\bigcup_{\alpha} U_{\alpha}$ is open, $\forall\, U_{\alpha}$ open subset
(iii) $\bigcap_{i=1}^{n} U_i$ is open

(X, τ) topological space
Example A.5. (Metric Spaces) metric space = set M with metric $d : M \times M \to \mathbb{R}$ s.t. $\forall\, x, y, z \in M$

(i) Positivity $d(x, y) \geq 0$, $d(x, y) = 0$ iff $x = y$
(ii) Symmetry $d(x, y) = d(y, x)$
(iii) Triangle inequality $d(x, z) \leq d(x, y) + d(y, z)$

if M metric space, $x \in M$, $r > 0$
open ball of radius r **around** x

$$B_r(x) = \{y \in M | d(x, y) < r\}$$

closed ball of radius r

$$\underline{B}_r(x) = \{y \in M | d(x, y) \leq r\}$$

metric topology on M defined by declaring $S \subseteq M$ open iff $\forall\, x \in S$, $\exists\, r > 0$ s.t. $B_r(x) \subseteq S$

Appendix B. Review of Linear Algebra

B.1. **Linear Maps. Exercise B.1.**

(a)
(b)
(c)
(d) **Want**: if $(v_1 \dots v_k)$ linearly dependent k-tuple in V, $v_1 \neq 0$,
then some $v_i = \sum_{j=1}^{i-1} c^j v_j$

Proof. $(v_1 \dots v_k)$ linearly dependent, so if $\sum_{i=1}^k a^i v_i = 0$, a^i not all equal to 0.

Suppose for fixed i, $2 \leq i \leq k$, $a^{i+1} = \dots = a^k = 0$.
Indeed, suppose for $\sum_{i=1}^k a^i v_i = 0$, $a^2 = \dots = a^k = 0$. $a^1 v_1 = 0$, $v_1 \neq 0$, $a^1 = 0$. Then $(v_1 \dots v_k)$ linearly independent. Contradiction.
if $i = 2$,
$a^1 v_1 + a^2 v_2 = 0$
$\implies v_2 = \frac{-a^1}{a^2} v_1$

if $i = k$, $v_k = \frac{-\sum_{i=1}^{k-1} a^i v_i}{a^k}$

So in general, $v_i = \frac{-\sum_{j=1}^{i-1} a^j v_j}{a^i}$ □

Exercise B.9. given $(E_1 \dots E_n)$ basis for V
$\exists \{i_1 \dots i_k\} \subset \{1 \dots n\}$ s.t.
span$(E_{i_1} \dots E_{i_k})$ is complement to S

Hence $\forall$ subspace $S \subseteq V$, $\exists$ complementary subspace T in V, so $V = S \oplus T$

Proof. $\forall$ subspace S is itself a vector space, closed under addition and multiplication.
Hence S has basis $(F_1 \dots F_m)$ with dim$S = m$

Consider ordered $(m+n)$-tuple

$$(F_1 \dots F_m, E_1 \dots E_n)$$

$(F_1 \dots F_m, E_1 \dots E_n)$ linearly dependent in V, by linear algebra.
For $j_1 \in \{1 \dots n\}$, E_{j_1} linear combination of previous vectors (cf. Exercise B.1(d))
eliminate E_{j_1} : $(F_1 \dots F_m, E_1 \dots \widehat{E}_{j_1} \dots E_n)$.

Repeat, until there are $n - m$ E basis vectors left, labeled $i_1 \dots i_{n-m}$ (hence **use Exercise B.1(d)** many and enough times, m times)

$$\implies (F_1 \dots F_m, E_{i_1} \dots E_{i_{n-m}})$$

By linear algebra, $(F_1 \dots F_m, E_{i_1} \dots E_{i_{n-m}})$ a basis for V, linearly independent. The procedure wouldn't have "overshot" by a Thm. (see Apostol's **Calculus** Vol. 2, first few chapters, linear algebra part)

$\forall\, v \in V$, $v = a^i F_i + b^{i_j} E_{i_j}$ with $a^i F_i \in S$. Then $b^{i_j} E_{i_j} \in T$.
since $V = S \oplus T$, T complement to S
$\implies$ given fixed basis of V, $(E_1 \dots E_n)$, subspace $S \subseteq V$, S having basis $(F_1 \dots F_m)$, S has complementary subspace in V, T,
s.t. basis of T is $(E_{i_1} \dots E_{i_{n-m}})$ and $V = S \oplus T$

□

Exercise B.13. Suppose $\exists$ linear $T : V \to W$ s.t. $T(E_i) = w_i, \quad i = 1 \dots n$
Suppose $\exists$ linear $T' : V \to W$ s.t. $T'(E_i) = w_i, \quad i = 1 \dots n$
Let $x \in V$, so $x = x^i E_i$ (the key idea is that with a basis, the vector space is completely determined, vectors in the vector space are spanned by the basis elements)

$$(T - T')(x) \equiv T(x) - T'(x) = x^i w_i - x^i w_i = 0$$

$T(x) = T'(x) \quad \forall\, x \in V$
so $T = T'$. T unique.
T exists by construction.

---

affine subspace of V parallel to S, linear subspace $S \subseteq V$, $v + S = \{v + w | w \in S\}$, some fixed $v \in V$

affine map $F : V \to W$ if $F(v) = w + Tv$ for some $T : V \to W$, some fixed $w \in W$

Exercise B.16.
Let $a, b \in \mathbb{C}$, $x, y \in F(V)$

Now

$$F(V) = \{y | y = w + Tv = F(v),\ v \in V,\ \text{fixed } w \in W,\ \text{some } T\}$$

B.1.1. *Change of Basis.* **Exercise B.22.** Suppose V, W, X finite-dim. vector spaces
$S : V \to W, \ \ T : W \to X$

(a) $\text{rank}S \leq \text{dim}V$ $\quad$ $\text{rank}S = \text{dim}V$ iff S injective
(b) $\text{rank}S \leq \text{dim}W$ $\quad$ $\text{rank}S = \text{dim}W$ iff S surjective
(c) if $\text{dim}V = \text{dim}W$ and S either injective or surjective, then S isomorphism
(d) $\text{rank}TS \leq \text{rank}S$ $\quad$ $\text{rank}TS = \text{rank}S$ iff $\text{im}S \bigcap \text{ker}T = 0$
(e) $\text{rank}TS \leq \text{rank}T$ $\quad$ $\text{rank}TS = \text{rank}T$ iff $\text{im}S + \text{ker}T = W$
(f) if S isomorphism, then $\text{rank}TS = \text{rank}T$
(g) if T isomorphism, then $\text{rank}TS = \text{rank}S$

EY : Exercise B.22(d) is useful for showing the chart and atlas of a Grassmannian manifold, found in the More examples, for smooth manifolds.

Proof. (a)
(b)
(c)
(d) Now

$$\text{dim}V = \text{rank}TS + \text{nullity}TS$$
$$\text{dim}V = \text{rank}S + \text{nullity}S$$

$\text{ker}S \subseteq \text{ker}TS$, clearly, so $\text{nullity}S \leq \text{nullity}TS$

$$\Longrightarrow \boxed{\text{rank}TS \leq \text{rank}S}$$

If $\text{rank}TS = \text{rank}S$,
then $\text{nullity}S = \text{nullity}TS$
Suppose $w \in \text{Im}S \bigcap \text{ker}T$, $w \neq 0$
Then $\exists\, v \in S$, s.t. $w = S(v)$ and $T(w) = 0$
Then $T(w) = TS(v) = 0$. So $v \in \text{ker}TS$
$v \notin \text{ker}S$ since $w = S(v) \neq 0$
This implies $\text{nullity}TS > \text{nullity}S$. Contradiction.
$\Longrightarrow \text{Im}S \bigcap \text{ker}T = 0$

If $\text{Im}S \bigcap \text{ker}T = 0$,
Consider $v \in \text{ker}TS$. Then $TS(v) = 0$.
. Then $S(v) \in \text{ker}T$
$S(v) = 0$; otherwise, $S(v) \in \text{Im}S$, contradicting given $\text{Im}S \bigcap \text{ker}T = 0$
$v \in \text{ker}S$

$\text{ker}TS \subseteq \text{ker}S$
$\Longrightarrow \text{ker}TS = \text{ker}S$
So $\text{nullity}TS = \text{nullity}S$
$\Longrightarrow \text{rank}TS = \text{rank}S$

(e)
(f)
(g)

□

Inner Products and Norms.

Norms. If V real vector space,
norm on V, $v \mapsto |v| \in \mathbb{R}$ s.t.

(i) POSITIVITY $|v| \geq 0, \ \ \forall\, v \in V$, $|v| = 0$ iff $v = 0$
(ii) HOMOGENEITY $|cv| = |c||v| \quad \forall\, c \in \mathbb{R}, v \in V$
(iii) TRIANGLE INEQUALITY $|v + w| \leq |v| + |w|, \ \ \forall\, v, w \in V$

2 norms $|\cdot|_1, |\cdot|_2$ on vector space V equivalent if $\exists$ constants $c, C > 0$ s.t.

$$c|v|_1 \leq |v|_2 \leq C|v|_1 \qquad \forall\, v \in V$$

Exercise B.49. $\forall\, x \in V$,
Consider $B_r(x) = \{ y \in V || y - x|_2 < r \}$
Note $y - x \in V$ as V is a vector space

$$c|y-x|_1 \leq |y-x|_2 \leq C|y-x|_1$$
$$c|y-x|_1 \leq |y-x_2| < r$$
$$|y-x|_1 < \frac{r}{c}$$

Now suppose $S \subseteq V$ open in $|\cdot|_2$

But S also open in $|\cdot|_1$ as $\exists \frac{r}{c'} > 0$ s.t. $B_{\frac{r}{c'}}(x) \subseteq S$ for the exact same pts. as S so $|\cdot|_1, |\cdot|_2$ equivalent norms yield the same metric topology.

EY : 20141220

Direct Products and Direct Sums.

Appendix C. Review of Calculus

Total and Partial Derivatives.

Proposition 42 (C.3)**.** *(The Chain Rule for Total Derivatives)*
Suppose V, W, X finite-dim. vector spaces

open $U \subseteq V$
$\widetilde{U} \subseteq W$

maps $F : U \to \widetilde{U}$
$G : \widetilde{U} \to X$

if F diff. at $a \in U$, G diff. at $F(a) \in \widetilde{U}$,
then GF diff. at a

$$D(GF)(a) = DG(F(a)) \circ DF(a)$$

Proposition 43 (C.4)**.** *Suppose* $\begin{matrix} U \subseteq \mathbb{R}^n, \\ V \subseteq \mathbb{R}^m \end{matrix}$ $F : U \to V$ *diffeomorphism*

Then $m = n$ and
$\forall\, a \in U$, $DF(a)$ invertible, with

$$DF(a)^{-1} = D(F^{-1})(F(a))$$

Proof. $F^{-1}F = 1_U$
Chain rule implies $\forall\, a \in U$,

(35) $$1_{\mathbb{R}^n} = D(1_U)(a) = D(F^{-1}F)(a) = DF^{-1}(F(a))DF(a) \qquad (C.5)$$

Similarly $FF^{-1} = 1_V$ implies

$$DF(a)DF^{-1}(F(a)) = 1_{\mathbb{R}^m}$$

so thus

$$DF(a)^{-1} = D(F^{-1})(F(a))$$

□