THE DIFFERENTIAL GEOMETRY DIFFERENTIAL TOPOLOGY DUMP

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Contents

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Abstract. Everything about Differential Geometry, Differential Topology

Part 1. Combinatorics, Probability Theory

Theorem 1 (4.2. of Feller (1968) [1]). *Let $r_1, \dots r_k \in \mathbb{Z}$, s.t. $r_1 + r_2 + \dots + r_k = n;$, $r_i \geq 0$.*

Let

$$\frac{N!}{r_1! r_2! \dots r_k!} = \tag{1}$$

number of ways in which n elemnts can be divided into k ordered parts (partitioned into k subpopulations). cf. Eq. (4.7) of Feller (1968) [1].

Note that the order of the subpopulations is essential in the sense that $(r_1 = 2, r_2 = 3)$ and $(r_1 = 3, r_2 = 2)$ represent different partitions. However, no attention is paid to the order within the groups.

Proof.

$$\binom{n}{r_1}\binom{n-r_1}{r_2}\binom{n-r_1-r_2}{r_3}\dots\binom{n-r_1-\dots-r_{k-2}}{r_{k-1}} = \frac{n!}{r_1!r_2!\dots r_k!} \tag{2}$$

i.e. in order to effect the desired partition, we have to select r_1 elementsout of n, remaining $n - r_1$ elements select a second group of size r_2, etc. After forming the $(k-1)$st group there remains $n - r_1 - r_2 - \dots - r_{k-1} = r_k$ elements, and these form the last group. □

cf. pp. 37 of Feller (1968) [1] Examples. (g) Bridge. 32 cards are partitioned into 4 equal groups $\to 52!/(13!)^4$.

Probability each player has an ace (?).

The 4 aces can be ordered in $4! = 24$ ways, each order presents 1 possibility of giving 1 ace to each player.

Remaining 48 cards distributed $(48!)/(12!)^4$ ways.

$$\to p = 24 \frac{48!}{(12!)^4} / \frac{52!}{(13!)^4}$$

(h) A throw of 12 dice $\to 6^{12}$ different outcomes total. Event each face appears twice can occur in as many ways as 12 dice can be arranged in 6 groups of 2 each.

$$\frac{12!}{(2!)^6} / \frac{52!}{(13!)^4}$$

0.0.1. *Application to Occupancy Problems; binomial coefficients.* cf. Sec. 5 Application to Occupancy Problems of Feller (1968) [1].

Consider randomly placing r balls intos n cells.

Let r_k = occupancy number = number of balls in kth cell.

Every n-tuple of integers satisfying $r_1 + r_2 + \dots + r_n = r$; $r_k \geq 0$. describes a possible configuration of occupancy numbers.

With indistinguishable balls 2 distributions are distinguishable only if the corresponding n-tuples $(r_1, \dots r_n)$ are not identical.

(i) number of distinguishable distributions is

$$A_{r,n} = \binom{n+r-1}{r} = \binom{n+r-1}{n-1} \quad (3)$$

cf. Eq. (5.2) of Feller (1968) [1]

(ii) number of distinguishable distributions in which no cell remains empty is $\binom{r-1}{n-1}$.

Proof. Represent balls by stars, indicate n cells by n spaces between $n+1$ bars. e.g. $\begin{matrix} r = 8 \text{ balls} \\ n = 6 \text{ cells} \end{matrix}$.

$$\begin{matrix} 3 & 1 & 0 & 0 & 0 & 4 \\ | * * * & | * | & & | & | & | * * * * | \end{matrix}$$

Such a symbol necessarily starts and ends with a bar, but remaining $n-1$ bars and r starts appear in an arbitrary order. In this way, it becomes apparent that the number of distinguishable distributions equals the number of ways of selecting.

r places out of $n+r-1$, $\frac{(n+r-1)!}{(n-1)!r!} = \binom{n-1+r}{r}$

$$\begin{aligned} & ||||\dots|| && n+1 \text{ bars} \\ & * * * \cdots * * && r \text{ stars leave } r-1 \text{ spaces} \end{aligned}$$

Condition that no cell be empty imposes the restriction that no 2 bars be adjacent. r stars leave $r-1$ spaces of which $n-1$ are to be occupied by bars. Thus $\binom{r-1}{n-1}$ choices. □

Probability to obtain given occupancy numbers $r_1, \dots r_n = \frac{r!}{r_1!r_2!\dots r_n!}/n^r$, with $\begin{matrix} r \text{ balls} \\ n \text{ cells} \end{matrix}$ given by Thm. 4.2. of Feller (1968) [1], which is the Maxwell-Boltzmann distribution.

(a) Bose-Einstein and Fermi-Dirac statistics. Consider r indistinguishable particles, n cells, each particle assigned to 1 cell.
State of the system - random distribution of r particles in n cells.
If n cells distinguishable, n^r arrangements equiprobable $\to$ Maxwell-Boltzmann statistics.
Bose-Einstein statistics: only distinguishable arrangements are considered, and each assigned probability $\frac{1}{A_{r,n}}$

$$A_{r,n} = \binom{n+r-1}{r} = \binom{n-1+r}{n-1} \quad (4)$$

cf. Eq. 5.2 of Feller (1968) [1]
Fermi-Dirac statistics.
(1) impossible for 2 or more particles to be in the same cell. $\to r \leq n$.
(2) all distinguishable arrangements satisfying the first condition have equal probabilities.
$\to$ an arrangement is completely described by stating which of the n cells contain a particle
r particles $\to \binom{n}{r}$ ways r cells chosen.
Fermi-Dirac statistics, there are $\binom{n}{r}$ possible arrangements, prob. $1/\binom{n}{r}$.

pp. 39. Feller (1968) [1]. Consider cells themselves indistinguishable! Disregard order among occupancy numbers.
cf. Feller (1968) [1]

Part 2. Linear Algebra Review

cf. *Change of Basis*, of Appendix B of John Lee (2012) [3].

Exercise B.22. Suppose V, W, X finite-dim. vector spaces
$S : V \to W$, $T : W \to X$

(a) $\text{rank}S \leq \text{dim}V$ with $\text{rank}S = \text{dim}V$ iff S injective
(b) $\text{rank}S \leq \text{dim}W$ with $\text{rank}S = \text{dim}W$ iff S surjective
(c) if $\text{dim}V = \text{dim}W$ and S either injective or surjective, then S isomorphism
(d) $\text{rank}TS \leq \text{rank}S$ $\quad$ $\text{rank}TS = \text{rank}S$ iff $\text{im}S \bigcap \text{ker}T = 0$
(e) $\text{rank}TS \leq \text{rank}T$ $\quad$ $\text{rank}TS = \text{rank}T$ iff $\text{im}S + \text{ker}T = W$
(f) if S isomorphism, then $\text{rank}TS = \text{rank}T$
(g) if T isomorphism, then $\text{rank}TS = \text{rank}S$

EY : Exercise B.22(d) is useful for showing the chart and atlas of a Grassmannian manifold, found in the More examples, for smooth manifolds.

Proof. (a) Recall the **rank-nullity theorem**:

Theorem 2 (Rank-Nullity Theorem)**.**

$$dim(im(S)) + dim(ker(S)) = dimV \quad (5)$$

Now

$$\text{rank}(S) + \text{dim}(\text{ker}(S)) \equiv \text{dim}(\text{im}(S)) + \text{dim}(\text{ker}(S)) = \text{dim}V$$

$$\Longrightarrow \boxed{\text{rank}(S) \leq \text{dim}V}$$

If $\text{rank}(S) = \text{dim}V$,
then by rank-nullity theorem, $\text{dim}(\text{ker}(S)) = 0$, implying that $\text{ker}S = \{0\}$.
Suppose $v_1, v_2 \in V$ and that $S(v_1) = S(v_2)$. By linearity of S, $S(v_1) - S(v_2) = S(v_1 - v_2) = 0$, which implies, since $\text{ker}S = \{0\}$, that $v_1 - v_2 = 0$.
$\Longrightarrow v_1 = v_2$. Then by definition of injectivity, S injective.
If S injective, then $S(v) = 0$ implies $v = 0$. Then $\text{ker}S = \{0\}$. Then by rank-nullity theorem, $\text{rank}(S) = \text{dim}V$.

(b) $\forall\, w \in \text{im}(S)$, $w \in W$. Clearly $\text{rank}S \leq \text{dim}W$.
If S surjective, $\text{im}(S) = W$. Then $\text{dim}(\text{im}(S)) = \text{rank}S = \text{dim}W$.

If $\text{rank}S = \text{dim}W = m$, then $\text{im}(S)$ has basis $\{y_i\}_{i=1}^m$, $y_i \in \text{im}(S)$, so $\exists\, x_i \in V$, $i = 1 \dots m$ s.t. $S(x_i) = y_i$, with $\{S(x_i)\}_{i=1}^m$ linearly independent.
Since $\{S(x_i)\}_{i=1}^m$ linearly independent and $\text{dim}W = m$, $\{S(x_i)\}_{i=1}^m$ basis for W.
$\forall\, w \in W$, $w = \sum_{i=1}^m w^i S(x_i) = S(\sum_{i=1}^m w^i x_i)$. $\sum_{i=1}^m w^i x_i \in V$. S surjective.

(c)

(d) Now

$$\text{dim}V = \text{rank}TS + \text{nullity}TS$$

$$\text{dim}V = \text{rank}S + \text{nullity}S$$

$\text{ker}S \subseteq \text{ker}TS$, clearly, so $\text{nullity}S \leq \text{nullity}TS$

$$\Longrightarrow \boxed{\text{rank}TS \leq \text{rank}S}$$

If $\text{rank}TS = \text{rank}S$,
then $\text{nullity}S = \text{nullity}TS$
Suppose $w \in \text{Im}S \bigcap \text{ker}T$, $w \neq 0$
Then $\exists\, v \in S$, s.t. $w = S(v)$ and $T(w) = 0$
Then $T(w) = TS(v) = 0$. So $v \in \text{ker}TS$
$v \notin \text{ker}S$ since $w = S(v) \neq 0$
This implies $\text{nullity}TS > \text{nullity}S$. Contradiction.

$\Longrightarrow \mathrm{Im} S \bigcap \ker T = 0$

If $\mathrm{Im} S \bigcap \ker T = 0$,
Consider $v \in \ker TS$. Then $TS(v) = 0$.
. Then $S(v) \in \ker T$
$S(v) = 0$; otherwise, $S(v) \in \mathrm{Im} S$, contradicting given $\mathrm{Im} S \bigcap \ker T = 0$
$v \in \ker S$

$\ker TS \subseteq \ker S$
$\Longrightarrow \ker TS = \ker S$
So $\mathrm{nullity} TS = \mathrm{nullity} S$
$\Longrightarrow \mathrm{rank} TS = \mathrm{rank} S$

(e)
(f)
(g)

□

Part 3. Manifolds

1. Inverse Function Theorem

Shastri (2011) had a thorough and lucid and explicit explanation of the Inverse Function Theorem [5]. I will recap it here. The following is also a blend of Wienhard's Handout 4 https://web.math.princeton.edu/~wienhard/teaching/M327/handout4.pdf

Definition 1. *Let (X, a) metric space.*
contraction $\phi : X \to X$ *if* $\exists$ *constant* $0 < c < 1$ *s.t.* $\forall x, y \in X$

$$d(\phi(x), \phi(y)) \leq c d(x, y)$$

Theorem 3 (Contraction Mapping Principle)**.** *Let (X, d) complete metric space.*
Then $\forall$ contraction $\phi : X \to X$, $\exists ! y \in X$ s.t. $\phi(y) = y$, y fixed pt.

Proof. Recall def. of complete metric space X, X metric space s.t. $\forall$ Cauchy sequence in X is convergent in X (i.e. has limit in X).

$$\forall x_0 \in X, \text{ Define } \begin{aligned} & x_1 = \phi(x_0) \\ & x_2 = \phi(x_1) \\ & \vdots \\ & x_j = \phi(x_{j-1}) \\ & \vdots \\ & x_n = \phi(x_{n-1}) \end{aligned}$$

$$d(x_{n+1}, x_n) = d(\phi(x_n), \phi(x_{n-1})) \leq c d(x_n, x_{n-1}) \leq \dots \leq c^n d(x_1, x_0)$$

for some $0 < c < 1$.

$$d(x_m, x_n) \leq d(x_n, x_{n-1}) + d(x_{n-1}, x_m) \leq d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + \dots + d(x_{m+1}, x_m) \leq \sum_{k=n-1}^m c^k d(x_1, x_0)$$

Thus, $\forall \epsilon > 0$, $\exists n_0 > 0$, (n_0 large enough) s.t. $\forall m, n \in \mathbb{N}$ s.t. $n_0 < n < m$,

$$d(x_m, x_n) \leq \sum_{k=n-1}^m c^k d(x_1, x_0) < \epsilon d(x_1, x_0)$$

Thus, $\{x_n\}$ Cauchy sequence. Since X complete, $\exists$ limit pt. $y \in X$ of $\{x_n\}$.

$$\phi(y) = \phi(\lim_n x_n) = \lim_n \phi(x_n) = \lim_n x_{n+1} = y$$

Since by def. of y limit pt. of $\{x_n\}$, $\forall \epsilon > 0$, then $\{n | |x_n - y| \leq \epsilon, n \in \mathbb{N}\}$ is infinite.
Consider $\delta > \mathbb{N}$. Consider $\{n | |x_n - y| \leq \delta, n \in \mathbb{N}\}$
$\exists N_\delta \in \mathbb{N}$ s.t. $\forall n > N_\delta$, $|x_n - y| < \delta$; otherwise, $\forall N_\delta$, $\exists n > N_\delta$ s.t. $|x_n - y| \geq \delta$. Then $\{n | |x_n - y| \leq \delta, n \in \mathbb{N}\}$ finite. Contradiction.
ϕ cont. so by def. $\forall \epsilon > 0$, $\exists \delta > 0$ s.t. if $|x_n - y| < \delta$, then $|\phi(x_n) - \phi(y)| < \epsilon$.
Pick N_δ s.t. $\forall n > N_\delta$, $|x_n - y| < \delta$, and so $|\phi(x_n) - \phi(y)| < \epsilon$. There are infinitely many $\phi(x_n)$'s that satisfy this, and so $\phi(y)$ is a limit pt.

If $\exists y_1, y_2 \in X$ s.t. $\begin{aligned} \phi(y_1) &= y_1 \\ \phi(y_2) &= y_2 \end{aligned}$, then

$$d(y_1, y_2) = d(\phi(y_1), \phi(y_2)) \leq c d(y_1, y_2) \text{ with } c < 1$$

so $c = 1$ □

Theorem 4 (Inverse Function Theorem)**.** *Suppose open $U \subset \mathbb{R}^n$, let C^1 $f : U \to \mathbb{R}^n$, $x_0 \in U$ s.t. $Df(x_0)$ invertible.*
Then $\exists$ open neighborhoods $V \ni x_0$, $W \ni f(x_0)$ s.t. $V \subseteq U$ and $W \subseteq \mathbb{R}^n$, respectively, and s.t.

(i) *$f : V \to W$ bijection*
(ii) *$g = f^{-1} : V \to U$ differentiable, i.e. $g = f^{-1} : W \to V$ is C^1*
(iii) *$D(f^{-1})$ cont. on W.*
(iv) *$Dg(y) = (Df(g(y)))^{-1} \quad \forall y \in W$*

Also, notice that $f(g(y)) = y \, \forall y \in W$.

Proof. Consider $\widetilde{f}(x) = (Df(x_0))^{-1}(f(x + x_0) - f(x_0))$. Then
$\widetilde{f}(0) = 0$ and

$$D\widetilde{f} = (Df(x_0))^{-1}(Df(x + x_0) - 0)$$
$$D\widetilde{f}(0) = (Df(x_0))^{-1} Df(x_0) = 1$$

So let $\widetilde{f} \to f$ (notation) and so assume, without loss of generality, that $U \ni 0$, $f(0) = 0$, $Df(0) = 1$
Choose $0 < \epsilon \leq \frac{1}{2}$. Let $0 < \delta < 1$ s.t. open ball $V = B_\delta(0) \subseteq U$, and $\|Df(x) - 1\| < \epsilon$. $\forall x \in U$, since Df cont. at 0.
Let $W = f(V)$.

$\forall y \in W$, define $\phi_y : V \to \mathbb{R}^n$

$$\phi_y(x) = x + (y - f(x))$$
$$D(\phi_y)(x) = 1 + -Df(x) \quad \forall x \in V$$
$$\|D(\phi_y)(x)\| = \|1 - Df(x)\| \leq \epsilon < 1$$

$\forall x_1, x_2 \in V$, by mean value Thm. (not the equality that is only valid in 1-dim., but the inequality, that's valid for $\mathbb{R}^d$,

$$\|\phi_y(x_1) - \phi_y(x_2)\| \leq \|D(\phi_y)(x')\| \|x_1 - x_2\|$$

for some $x' = cx_2 + (1 - c)x_1$, $c \in [0, 1]$. V only needed to be convex set.

$$\Longrightarrow \|\phi_y(x_1) - \phi_y(x_2)\| \leq \epsilon \|x_1 - x_2\|$$

Then ϕ_y contraction mapping.

Suppose $f(x_1) = f(x_2) = y$, $x_1, x_2 \in V$.

$$\begin{gathered} \phi_y(x_1) = x_1 \\ \phi_y(x_2) = x_2 \\ \|\phi_y(x_1) - \phi_y(x_2)\| = \|x_1 - x_2\| \leq \epsilon \|x_1 - x_2\| \quad \forall \epsilon > 0 \Longrightarrow x_1 = x_2 \end{gathered}$$

$\Longrightarrow f|_U$ injective.

$W = f(V)$, so $f : V \to W$ surjective. f bijective.

Fix $y_0 \in W$, $y_0 = f(x_0)$, $x_0 \in V$.

Let $r > 0$ s.t. $B_r(x_0) \subset V$.

Consider $B_{r\epsilon}(y_0)$. If $y \in B_{r\epsilon}(y_0)$.

$$\begin{gathered} r\epsilon > \|y - y_0\| = \|y - f(x_0)\| = \|\phi_y(x_0) - x_0\| \text{ with} \\ \phi_y(x) = x + (y - f(x)) \end{gathered}$$

If $x \in B_r(x_0)$,

$$\|\phi_y(x) - x_0\| \leq \|\phi_y(x) - \phi_y(x_0)\| + \|\phi_y(x_0) - x_0\| \leq \epsilon\|x - x_0\| + r\epsilon < 2r\epsilon = r$$

Thus $\phi(B_r(x_0)) = B_r(x_0)$.

By contraction mapping principle, $\exists\, a \in B_r(x_0)$, s.t. $\phi_y(a) = a$. Then $\phi_y(a) = a + (y - f(a)) = a \Longrightarrow f(a) = y$.

$y \in f(V) = W$.

So $B_{r\epsilon}(y_0) \subset W$. W open.

Let $\text{Mat}(n,n) \equiv$ space of all $n \times n$ matrices; $\text{Mat}(n,n) = \mathbb{R}^{n^2}$.

□

There is a proof of the implicit function theorem and its various forms in Shastri (2011) [5], but I found Wienhard's Handout 4 for Math 327 to be clearer.[1]

Theorem 5 (Implicit Function Theorem)**.** *Let open* $U \subset \mathbb{R}^{m+n} \equiv \mathbb{R}^m \times \mathbb{R}^n$

C^1 $f : U \to \mathbb{R}^n$

$(a,b) \in U$ *s.t.* $f(a,b) = 0$ *and* $D_y f|_{(a,b)}$ *invertible.*

Then $\exists$ *open* $V \ni (a,b)$, $V \subset U$

$\exists$ *open neighborhood* $W \ni a$, $W \subseteq \mathbb{R}^m$

$\exists!$ C^1 $g : W \to \mathbb{R}^n$ *s.t.*

$$\{(x,y) \in V | f(x,y) = 0\} = \{(x, g(x)) | x \in W\}$$

Moreover,

$$dg_x = -\left.(d_y f)^{-1}\right|_{(x,g(x))} \left. d_x f\right|_{(x,g(x))}$$

and g *smooth if* f*.*

Proof. Define $F : U \to \mathbb{R}^{m+n}$

$$F(x,y) = (x, f(x,y))$$

Then $F(a,b) = (a,0)$ (given), and

$$DF = \begin{bmatrix} 1 & \\ \frac{\partial f^i(x,y)}{\partial x^j} & \frac{\partial f^i(x,y)}{\partial y^j} \end{bmatrix} \equiv \begin{bmatrix} 1 & \\ D_x f & D_y f \end{bmatrix}$$

$DF(a,b)$ invertible.

By inverse function theorem, since $DF(a,b)$ invertible at pt. (a,b),

$\exists$ open neighborhoods $\begin{aligned} & V \ni (a,b) \subseteq \mathbb{R}^m \times \mathbb{R}^n \\ & \widetilde{W} \ni (a,0) \subseteq \mathbb{R}^m \times \mathbb{R}^n \end{aligned}$ s.t. F diffeomorphism with $F^{-1} : \widetilde{W} \to V$.

Set $W = \{x \in \mathbb{R}^m | (x,0) \in \widetilde{W}\}$. Then $\pi_1(\widetilde{W}) = W$ open in $\mathbb{R}^m$.

Define $g : W \to \mathbb{R}^n$,

$$\begin{gathered} g(x) = \pi_2 \circ F^{-1}(x,0) \text{ or} \\ F^{-1}(x,0) = (h(x), g(x)) \end{gathered}$$

Now $FF^{-1}(x,0) = (x,0) = (h(x), f(h(x), g(x)))$ so $h(x) = x \,\forall\, x \in W$, $0 = f(x, g(x))$.

Then

$$\{(x,y) \in V | f(x,y) = 0\} = \{(x,y) \in V | F(x,y) = (x,0)\} = \{(x, g(x)) | x \in W, 0 = f(x, g(x))\}$$

Since π smooth and F^{-1} is C^1, g is C^1.

To reiterate, $f(x, g(x)) = 0$ on W.

Using chain rule while differentiating $f(x, g(x)) = 0$,

$$\begin{gathered} \partial_{x^j} f(x, g(x)) = \frac{\partial f(x, g(x))}{\partial x^k} \frac{\partial x^k}{\partial x^j} + \frac{\partial f(x, g(x))}{\partial y^k} \frac{\partial g^k(x)}{\partial x^j} = D_x f|_{(x,g(x))} + (D_y f)|_{(x,g(x))} \cdot (Dg)_x = 0 \text{ or} \\ (Dg)_x = -\left.(D_y f)\right|_{x,g(x)} \left. D_x f\right|_{(x,g(x))} \end{gathered}$$

□

2. Immersions

Definition 2 (Immersion)**.** *smooth* $f : M \to N$*, s.t.* $Df(p) : T_pM \to T_{f(p)}N$ *injective. Then* f ***immersion*** *at* p*.*

Absil, Mahony, and Sepulchre [7] pointed out that another definition for a *immersion* can utilize the theorem that rank of $Df \equiv DF = \dim T_pM$. Indeed, recall these facts from linear algebra:

for $T : V \to W$,

It's always true that $\begin{aligned} & \text{rank}T \leq V \text{ , and} \\ & \text{rank}T \leq W \end{aligned}$

$\text{rank}T = \dim V$ iff T injective.

$\text{rank}T = \dim W$ iff T surjective.

$$\begin{tikzcd} T_xM \arrow[r, "DF(x)"] & T_{F(x)}N = T_yN \\ x \in M \arrow[u] \arrow[r, mapsto, "F"] & y = F(x) \in N \arrow[u] \end{tikzcd}$$

$$M \xrightarrow{\quad F \quad} N$$

Now

$$\begin{gathered} \dim T_xM = \dim M \\ \dim T_{F(x)}N = \dim N \end{gathered}$$

And

$$\text{rank}(DF(x)) \equiv \text{rank of } F$$

I know that the notation above is confusing, but this is what all Differential Geometry books apparently mean when they say "rank of F".

[1] https://web.math.princeton.edu/~wienhard/teaching/M327/handout4.pdf

Now

$$\text{rank}(DF(x)) = \dim(\text{im}(DF(x))) = \dim T_x M \text{ iff } DF(x) \text{ injective}$$

If $\forall x \in M$, this is the case, then F an **immersion** .
Apply the rank-nullity theorem in this case:

$$\begin{gathered} \text{rank}(DF(x)) + \dim\ker(DF(x)) = \dim T_x M = \dim M \\ \implies \text{rank}(DF(x)) = \dim M \leq \dim T_{F(x)} N = \dim N \text{ or } \dim M \leq \dim N \end{gathered}$$

Now

$$\text{rank}(DF(x)) = \dim T_{F(x)} N \text{ iff } DF(x) \text{ surjective}$$

If $\forall x \in M$, this is the case, then F an **submersion** .

$$\text{rank}(DF(x)) = \dim T_{F(x)} N = \dim N \leq \dim M$$

Shastri (2011) has this as the "Injective Form of Implicit Function Theorem", Thm. 1.4.5, pp. 23 and Guillemin and Pollack (2010) has this as the "Local Immersion Theorem" on pp. 15, Section 3 "The Inverse Function Theorem and Immersions" [4].

Theorem 6 (Local immersion Theorem i.e. Injective Form of Implicit Function Theorem)**.** *Suppose $f : M \to N$ immersion at p, $q = f(p)$.*
Then $\exists$ local coordinates around p, q, x, y, respectively s.t. $f(x_1 \dots x_m) = (x_1 \dots x_m, 0 \dots 0)$.

Proof. Choose local parametrizations

$$\begin{tikzcd} U \subseteq M \arrow[r, "f"] \arrow[d, "\phi"] & N \supseteq V \arrow[d, "\psi"] \\ \phi(U) \arrow[r, "f"] & \psi(V) \end{tikzcd} \qquad \begin{aligned} \phi(p) &= x \\ \psi(q) &= y \end{aligned}$$

$D(\psi f \varphi^{-1}) \equiv Df$. $Df(p)$ injective (given f immersion). $Df(p) \in \text{Mat}(n, m)$
By change of basis in $\mathbb{R}^n$, assume $Df(p) = \begin{pmatrix} I_m \\ 0 \end{pmatrix}$.

Now define $G : \phi(U) \times \mathbb{R}^{n-m} \to \mathbb{R}^n$

$$G(x, z) = f(x) + (0, z)$$

Thus, $DG(x, z) = 1$ and for open $\phi(U) \times U_2$, $G(\phi(U) \times U_2)$ open.
By inverse function theorem, G local diffeomorphism of $\mathbb{R}^n$, at 0.
Now $f = G \circ \mathfrak{i}$, where $\mathfrak{i}$ is canonical immersion.

$$\begin{gathered} G(x, 0) = f(x) \\ \implies G^{-1}G(x, 0) = (x, 0) = G^{-1}f(x) \end{gathered}$$

Use $\psi \circ G$ as the local parametrization of N around pt. q. Shrink U, V so that

$$\begin{tikzcd} U \subseteq M \arrow[r, "f"] \arrow[d, "\phi"] & N \supseteq V \arrow[d, "\psi \circ G"] \\ \phi(U) \arrow[r, "\mathfrak{i}"] & \psi \circ G(V) \end{tikzcd}$$

□

Theorem 7 (Implicit Function Thm.)**.** *Let open subset $U \subseteq \mathbb{R}^n \times \mathbb{R}^d$, $(x, y) = (x^1 \dots x^n, y^1 \dots y^k)$ on U. Suppose smooth $\Phi : U \to \mathbb{R}^k$, $(a, b) \in U$, $c = \Phi(a, b)$*

If $k \times k$ matrix $\frac{\partial \Phi^i}{\partial y^j}(a, b)$ nonsingular, then $\exists$ neighborhoods $\begin{matrix} V_0 \subseteq \mathbb{R}^n \text{ of } a \\ W_0 \subseteq \mathbb{R}^k \text{ of } b \end{matrix}$ and smooth $F : V_0 \to W_0$ s.t.

$\Phi^{-1}(c) \bigcap (V_0 \times W_0)$ is graph of F, i.e.
$\Phi(x, y) = c$ for $(x, y) \in V_0 \times W_0$ iff $y = F(x)$.

3. Submersions; Rank Theorem

cf. pp. 20, Sec. 4 "Submersions", Ch. 1 of Guillemin and Pollack (2010) [4].
Consider $X, Y \in \mathbf{Man}$, s.t. $\dim X \geq \dim Y$.

Definition 3 (submersion)**.** *If $f : X \to Y$,*
if $Df_x \equiv df_x$ is surjective, *$f \equiv$ **submersion** at x.*

Recall that,

$$\begin{gathered} Df_x : T_x X \to T_{f(x)} Y \\ \dim T_x X \geq \dim T_{f(x)} Y \end{gathered}$$

$$\begin{aligned} \text{rank} Df_x &\leq \dim T_{f(x)} Y, \text{ in general, while} \\ \text{rank} Df_x &= \dim T_{f(x)} Y \text{ iff } Df_x \text{ surjective} \end{aligned}$$

Canonical submersion is standard projection:

If $\dim X = k$, $k \geq l$,
$\dim Y = l$

$$(a_1 \dots a_k) \mapsto (a_1 \dots a_l)$$

Theorem 8 (Local Submersion Theorem)**.** *Suppose $f : X \to Y$ submersion at x, and $y = f(x)$, Then $\exists$ local coordinates around x, y s.t.*

$$f(x_1 \dots x_k) = (x_1 \dots x_l)$$

i.e. f locally equivalent to canonical submersion near x

Proof. I'll have a side-by-side comparison of my notation and the 1 used in Guillemin and Pollack (2010) [4] where I can.
For charts $(U, \phi), (V, \psi)$ for X, Y, respectively, $y = f(x)$ for $x \in X$,

$$\begin{tikzcd} U \subseteq X \arrow[r, "f"] \arrow[d, "\phi"] & Y \supseteq V \arrow[d, "\psi \circ G"] \\ \mathbb{R}^k \arrow[r, "\mathfrak{i}"] & \mathbb{R}^l \end{tikzcd} \qquad \begin{tikzcd} x \arrow[r, mapsto, "f"] \arrow[d, mapsto, "\phi"] & f(x) = y \arrow[d, mapsto, "\psi"] \\ \phi(x) = (a^1 \dots a^k) \arrow[r, mapsto, "g"] & g(\phi(x)) = g(a^1 \dots a^k) = \psi(y) \end{tikzcd}$$

Dg_x surjective, so assume it's a $l \times k$ matrix $\begin{bmatrix} \mathbf{1}_l & 0 \end{bmatrix}$.
Define

$$\begin{aligned} &G : U \subset \mathbb{R}^k \to \mathbb{R}^k \\ &G(a) \equiv G(a^1 \dots a^k) := (g(a), a_{l+1}, \dots, a_k) \end{aligned} \tag{6}$$

Now

$$DG(a) = \begin{bmatrix} \mathbf{1}_l & 0 \\ & \mathbf{1}_{k-l} \end{bmatrix} = \mathbf{1}_k \tag{7}$$

so G local diffeomorphism (at 0).

So $\exists\ G^{-1}$ as local diffeomorphism of some U' of a into $U \subset \mathbb{R}^k$.

By construction,

$$g = \mathbb{P}_l \circ G \tag{8}$$

where $\mathbb{P}_l$ is the *canonical submersion*, the projection operator onto $\mathbb{R}^l$.

$$g \circ G^{-1} = \mathbb{P}_l$$

(since G diffeomorphism)

$$\begin{tikzcd} U \subseteq X \arrow[r, "f"] & V \subseteq Y \\ \mathbb{R}^k \arrow[u, "\phi^{-1} \circ G^{-1}"] \arrow[r, "\mathbb{P}_l"] & \mathbb{R}^l \arrow[u, "\psi^{-1}"'] \end{tikzcd}$$

for

$$\begin{tikzcd} \phi^{-1} \circ G^{-1}(a) \equiv \phi^{-1} \circ G^{-1}(a^1 \dots a^k) = x \arrow[r, mapsto, "f"] & f(x) = y = \psi^{-1}(a^1 \dots a^l) \\ (a^1 \dots a^k) \arrow[u, mapsto, "\phi^{-1} \circ G^{-1}"] \arrow[r, mapsto, "\mathbb{P}_l"] & (a^1 \dots a^l) \arrow[u, mapsto, "\psi^{-1}"'] \end{tikzcd}$$

$\Longrightarrow$

$\square$

"An obvious corollary worth noting is that if f is a submersion at x, then it is actually a submersion in a whole neighborhood of x." Guillemin and Pollack (2010) [4]

Suppose f submersion at $x \in f^{-1}(y)$.

By local submersion theorem

$$f(x_1 \dots x_k) = (x_1 \dots x_l)$$

Choose $y = (0, \dots, 0)$.

Then, near x, $f^{-1}(y) = \{(0, \dots 0, x_{l+1} \dots x_k)\}$ i.e. let $V \ni x$ neighborhood of x, define $(x_1 \dots x_k)$ on V.

Then $f^{-1}(y) \bigcap V = \{(0 \dots 0, x_{l+1}, \dots x_k) | x_1 = 0, \dots x_l = 0\}$.

Thus $x_{l+1}, \dots x_k$ form a coordinate system on open set $f^{-1}(y) \bigcap V \subseteq f^{-1}(y)$.

Indeed,

$$\begin{tikzcd} U \subseteq X \arrow[r, "f"] \arrow[d, "\phi"] & V \subseteq Y \arrow[d, "\psi"] \\ \mathbb{R}^k \arrow[r, "\mathbb{P}_l"] & \mathbb{R}^l \end{tikzcd} \qquad \begin{tikzcd} x \arrow[r, mapsto, "f"] \arrow[d, mapsto, "\phi"] & f(x) = y \arrow[d, mapsto, "\psi"] \\ \phi(x) = (x^1 \dots x^k) \arrow[r, mapsto, "\mathbb{P}_l"] & (x^1 \dots x^l) \end{tikzcd}$$

and now

$$\begin{tikzcd} f^{-1}(y) & y \arrow[l, mapsto, "f^{-1}"] \arrow[d, mapsto, "\psi"] \\ \{(0, \dots 0, x^1 \dots x^k)\} \arrow[u, mapsto, "\phi^{-1}"] & (0 \dots 0) \arrow[l, mapsto, "\mathbb{P}_l^{-1}"] \end{tikzcd}$$

3.1. **Rank Theorem.** Lee (2012) [3] in pp. 85, Ch. 4 Submersions, Immersions, and Embeddings, combines Theorems 6, 8 (local immersion and local submersion theorems, respectively) into the "Rank Theorem" (cf. Thm 4.12 "Rank Theorem" of Lee (2012)):

Theorem 9 (Rank Theorem)**.** *Suppose smooth manifolds M, N, $dim M = m$, $dim N = n$, smooth map $F : M \to N$, F has constant rank r.*
$\forall\, p \in M$, $\exists$ smooth charts (U, φ) for M, centered at p, (V, ψ) for N, centered at $F(p)$, s.t.

$$F(U) \subseteq V$$

in which F has coordinate representation of form

$$\widehat{F}(x^1 \dots x^r, x^{r+1} \dots x^m) = (x^1 \dots x^r, 0 \dots 0) \tag{9}$$

Particularly, if F smooth submersion,

$$\widehat{F}(x^1 \dots x^n, x^{n+1} \dots x^m) = (x^1 \dots x^n)$$

and if F smooth immersion

$$\widehat{F}(x^1 \dots x^m) = (x^1 \dots x^m, 0 \dots 0)$$

$$\begin{tikzcd} \mathbb{R}^m \arrow[r, "\psi F \varphi^{-1}"] & \mathbb{R}^n \\ U \subset M \arrow[u, "\varphi"] \arrow[r, "F"'] & V \subset N \arrow[u, "\psi"] \end{tikzcd}$$

Also remember that $DF(p) : T_p M \to T_{F(p)} N$

Proof. $DF(p)$ has rank r (given). Then $DF(p)$ is some $r \times r$ submatrix of a $n \times m$ matrix s.t. $\det DF(p)$ nonzero.

By change of basis in $\mathbb{R}^n$, or reordering coordinates, assume $DF(p)$ is upper left submatrix $\left(\frac{\partial F^i}{\partial x^j} \right) \quad \forall\, i, j = 1, \dots r$.

Relabel standard coordinate as

$$(x, y) = (x^1 \dots x^r, y^1 \dots y^{m-r}) \in \mathbb{R}^m$$
$$(v, w) = (v^1 \dots v^r, w^1 \dots w^{n-r}) \in \mathbb{R}^n$$

By initial translations of coordinates, assume without loss of generality $p = (0, 0)$, $F(p) = (0, 0)$

Suppose

$$F(x, y) = (Q(x, y), R(x, y))$$

for some smooth maps $Q : U \to \mathbb{R}^r$, $R : U \to \mathbb{R}^{n-r}$

Define

$$\varphi : U \to \mathbb{R}^m$$
$$\varphi(x, y) = (Q(x, y), y)$$

so

$$D\varphi(0,0) = \begin{pmatrix} \frac{\partial Q^i}{\partial x^j}(0,0) & \frac{\partial Q^i}{\partial y^j}(0,0) \\ 0 & \delta^i_j \end{pmatrix}$$

$D\varphi(0,0)$ nonsingular, since $\det \frac{\partial Q^i}{\partial x^j} \neq 0$ (by hypothesis).

By inverse function thm., $\exists$ connected neighborhoods U_0 of $(0,0)$, $\widetilde{U}_0$ of $\varphi(0,0) = (0,0)$ s.t.

$$\varphi : U_0 \to \widetilde{U}_0$$

is a diffeomorphism.

By shrinking $U_0, \widetilde{U}_0$, assume $\widetilde{U}_0$ open cube.

Write $\varphi^{-1}(x,y) = (A(x,y), B(x,y))$, for some smooth functions $\begin{aligned} A &: \widetilde{U}_0 \to \mathbb{R}^r , \\ B &: \widetilde{U}_0 \to \mathbb{R}^{m-r} \end{aligned}$

$$(x,y) = \varphi(A(x,y), B(x,y)) = (Q(A(x,y), B(x,y)), B(x,y))$$

$$\Longrightarrow \begin{aligned} & B(x,y) = y \\ & \varphi^{-1}(x,y) = (A(x,y), y) \end{aligned}$$

$$\varphi\varphi^{-1} = 1 \Longrightarrow x = Q(A(x,y), y)$$

Recall that we had hypotehsized that

$$F(x,y) = (Q(x,y), R(x,y))$$

Then

$$F \circ \varphi^{-1}(x,y) = F(A(x,y), y) = (Q(A(x,y), y), R(A(x,y), y)) = (x, R(A(x,y), y))$$

and so

$$F \circ \varphi^{-1}(x,y) = (x, \widetilde{R}(x,y))$$

where $\widetilde{R} : \widetilde{U}_0 \to \mathbb{R}^{n-r}$

$\widetilde{R}(x,y) = R(A(x,y), y)$

Compute

$$D(F \circ \varphi^{-1})(x,y) = \begin{pmatrix} \delta^i_j & 0 \\ \frac{\partial \widetilde{R}^i}{\partial x^j}(x,y) & \frac{\partial \widetilde{R}^i}{\partial y^j}(x,y) \end{pmatrix}$$

Since composing with a diffeomorphism doesn't change rank of map, $D(F \circ \varphi^{-1})$ has rank r everywhere in $\widetilde{U}_0$.

$\begin{pmatrix} \delta^i_j \\ \frac{\partial \widetilde{R}^i}{\partial x^j}(x,y) \end{pmatrix}$ $j = 1 \dots r$ are linearly independent, so $\frac{\partial \widetilde{R}^i}{\partial y^j}(x,y) = 0$ on $\widetilde{U}_0$, so $\widetilde{R}^i$ independent of y^j.

Let $S(x) = \widetilde{R}(x,0)$, then

$$F \circ \varphi^{-1}(x,y) = (x, S(x))$$

Let open $V_0 \subseteq V$, $(0,0) \in V$ be an open subset $V_0 = \{(v,w) \in V : (v,0) \in \widetilde{U}_0\}$.
Then V_0 is a neighborhood of $(0,0)$.

Because $\widetilde{U}_0$ is a cube, $F \circ \varphi^{-1}(x,y) = (x, S(x))$,

$$F \circ \varphi^{-1}(\widetilde{U}_0) \subseteq V_0$$

so $F(U_0) \subseteq V_0$.

Define $\psi : V_0 \to \mathbb{R}^n$

$\psi(v,w) = (v, w - S(v))$

Because $\psi^{-1}(s,t) = (s, t + S(s))$,
it is a diffeomorphism.

Thus (V_0, ψ) is a smooth chart.

$$\psi \circ F(\varphi^{-1}(x,y)) = \psi(x, S(x)) = (x, S(x) - S(x)) = (x, 0)$$

$\square$

Definition 4 (regular value)**.** *For smooth $f : X \to Y$, $X, Y \in$ **Man**,
$y \in Y$ is a **regular value** for f if $Df_x : T_xX \to T_yY$ surjective $\forall\, x$ s.t. $f(x) = y$.
$y \in Y$ **critical value** if y not a regular value of f.*

Absil, Mahony, and Sepulchre [7] pointed out that another definition for a *regular value* can utilize the theorem that rank of $Df \equiv DF = \dim T_pN = \dim N$, iff $DF(p)$ surjective, for $p \in M$, $F : M \to N$. Then
regular value $y \in N$, of F, if rank of $F \equiv \text{rank}(DF(x)) = \dim N$, $\forall\, x \in F^{-1}(y)$, for $F : M \to N$.

Theorem 10 (Preimage theorem)**.** *If y regular value of $f : X \to Y$,
$f^{-1}(y)$ is a submanifold of X, with $\dim f^{-1}(y) = \dim X - \dim Y$*

Proof. Given y is a regular value of $f : X \to Y$,
$\forall\, x \in f^{-1}(y)$, $Df_x : T_xX \to T_yY$ is surjective. By local submersion theorem,

$$f(x^1 \dots x^k) = (x^1 \dots x^l) = y$$

Since $x \in f^{-1}(y)$, $(x^1 \dots x^k) = (y^1 \dots y^l, x^{l+1} \dots x^k)$.

For this chart for (U, φ), $U \ni x$, consider $(U \cap f^{-1}(y), \psi)$ with $\psi(x) = (x^{l+1} \dots x^k) \quad \forall\, x \in U \cap f^{-1}(y)$.

$\forall\, f^{-1}(y)$ submanifold with $\dim f^{-1}(y) = k - l = \dim X - \dim Y$. $\square$

Examples for emphasis

If $\dim X > \dim Y$,

if $y \in Y$, regular value of $f : X \to Y$,

f submersion, $\forall\, x \in f^{-1}(y)$

If $\dim X = \dim Y$,

f local diffeomorphism $\forall\, x \in f^{-1}(y)$

If $\dim X < \dim Y$, $\forall\, y \in f(X)$ is a critical value.

Example: $O(n)$ as a submanifold of Mat(n,n)

Given $\text{Mat}(n,n) \equiv M(n) = \{n \times n \text{ matrices }\}$ is a manifold; in fact $\text{Mat}(n,n) \cong \mathbb{R}^{n^2}$,
Consider $O(n) = \{A \in \text{Mat}(n,n) | AA^T = 1\}$.

$$AA^T \in \text{Sym}(n) \equiv S(n) = \{S \in \text{Mat}(n,n) | S^T = S\} = \{ \text{ symmetric } n \times n \text{ matrices } \} \tag{10}$$

$\text{Sym}(n)$ submanifold of $\text{Mat}(n,n)$, $\text{Sym}(n)$ diffeomorphic to $\mathbb{R}^k$ (i.e. $\text{Sym}(n) \cong \mathbb{R}^k$), $k := \frac{n(n+1)}{2}$.

$$\begin{aligned} & f : \text{Mat}(n,n) \to \text{Sym}(n) \\ & f(A) = AA^T \end{aligned}$$

Notice f is smooth,

$$f^{-1}(1) = O(n)$$

$$Df_A(B) = \lim_{s \to 0} \frac{f(A + sB) - f(A)}{s} = \lim_{s \to 0} \frac{(A + sB)(A^T + sB^T) - AA^T}{s} = AB^T + BA^T$$

If $Df_A : T_A\text{Mat}(n,n) \to T_{f(A)}\text{Sym}(n)$ surjective when $A \in f^{-1}(1) = O(n)$ (???).

Proposition 1. *If smooth $g_1 \dots g_l \in C^\infty(X)$ on X are independent $\forall\, x \in X$, s.t. $g_i(x) = 0$, $\forall\, i = 1 \dots l$,
then $Z = \{x \in X | g_1(x) = \dots = g_l(x) = 0\}$ = set of "common zeros" is a* submanifold *of X s.t. $\dim Z = \dim X - l$.*

Take note *that $g_1 \dots g_l$ are independent at x means, really, that $D(g_1)_x \dots D(g_l)_x$ are linearly independent on T_xX.*

Proof. Suppose smooth $g_1 \dots g_l \in C^\infty(X)$ on manifold X s.t. $\dim X = k \geq l$.

Consider $g = (g_1 \dots g_l) : X \to \mathbb{R}^l$, $Z \equiv g^{-1}(0)$.

Since $\forall g_i$ smooth, $D(g_i)_x : T_xX \to \mathbb{R}$ linear.

Now for

$$Dg_x = (D(g_1)_x \dots D(g_l)_x) : T_xX \to \mathbb{R}^l$$

By rank-nullity theorem (linear algebra), Dg_x surjective iff $\text{rank}Dg_x = l$ i.e. l functionals $D(g_1)_x \dots D(g_l)_x$ are linearly independent on T_xX.

"We express this condition by saying the l functions $g_1 \dots g_l$ are independent at x." (Guillemin and Pollack (2010) [4]) □

4. Submanifolds; immersed submanifold, embedded submanifolds, regular submanifolds

Definition 5 (Embedded Submanifold).

Recall immersion:
$F : M \to N$ immersion iff DF injective, i.e. iff $\text{rank}DF = \dim M$.

Consider manifolds $M \subseteq N$.

Consider inclusion map $i : M \to N$.

$$i : x \mapsto x$$

If i immersion, $Di(x) = \frac{\partial y^i}{\partial x^j} = \delta_j{}^i$ if $y^i = x^i$, $\forall i = 1, \dots \dim M$.

Definition 6 (immersed submanifold). ***immersed submanifold*** *$M \subseteq N$ if inclusion $i : M \to N$ is an immersion.*

cf. 3.3 Embedded Submanifolds of Absil, Mahony, and Sepulchre [7], also Ch. 5 Submanifolds, pp. 108, **Immersed Submanifolds** of John Lee (2012) [3].

Immersed submanifolds often arise as images of immersions.

Proposition 2 (Images of Immersions as submanifolds). *Suppose smooth manifold M,*
smooth manifold with or without boundaries N,
injective, smooth immersion $F : M \to N$ (F injective itself, not just immersion)
Let $S = F(M)$.
Then S has unique topology and smooth structure of smooth submanifolds of N s.t. $F : M \to S$ diffeomorphism.

cf. Prop. 5.18 of John Lee (2012) [3].

Proof. Define topology of S: set $U \subseteq S$ open iff $F^{-1}(U) \subseteq M$ open ($F^{-1}(U \cap V) = F^{-1}(U) \cap F^{-1}(V), F^{-1}(U \cup V) = F^{-1}(U) \cup F^{-1}(V)$).
Define smooth structure of S: $\{F(U), \varphi \circ F^{-1} | (U, \varphi) \in \text{atlas for } M$, i.e. (U, φ) any smooth chart of $M\}$.

"smooth compatibility condition":

$$(\varphi_2 \circ F^{-1})(\varphi_i F^{-1})^{-1} = \varphi_2 \circ F^{-1} F \varphi_1^{-1} = \varphi_2 \varphi_1^{-1}$$

since $\varphi_2\varphi_1^{-1}$ diffeomorphism ($\varphi_2\varphi_1^{-1}$ bijection and it and inverse is differentiable)

F diffeomorphism onto $F(M)$.

and these are the only topology and smooth structure on S with this property:

$$S \xrightarrow{F^{-1}} M \xrightarrow{F} N \qquad = \qquad S \hookrightarrow M$$

and F^{-1} diffeomorphism, F smooth immersion, so $i : S \to M$ smooth immersion. □

5. Integral Curves and Flows

cf. John Lee (2012) [3], Ch. 9, deals with time-dependent vector fields and I don't see other texts or references handling such an important, but overlooked, case.

6. Tensors

I'll go through Ch.7 *Tensors* of Jeffrey Lee (2009) [2].

Definition 7 (7.1[2]). *Let V, W be modules over commutative ring R, with unity.*
Then, algebraic W-valued tensor on V is multilinear map.

$$\tau : V_1 \times V_2 \times \dots \times V_m \to W \tag{11}$$

where $V_i = \{V, V^*\}$ $\quad \forall i = 1, 2, \dots m$.
If for r, s s.t. $r + s = m$, there are r $\; V_i = V^*$, $s\, V_i = V$, tensor is r-contravariant, s-covariant; also say tensor of total type $\binom{r}{s}$.

EY : 20170404 Note that

$$(\tau^{i\alpha}_\beta \frac{\partial}{\partial x^i} \text{ or } \tau^{i\alpha}_\beta e_i)(\omega_j dx^j \text{ or } \omega_j e^j \in V^*)$$
$$(\tau^\beta_{i\alpha} dx^i \text{ or } \tau^\beta_{i\alpha} e^i)(X^j \frac{\partial}{\partial x^j} \text{ or } X^j e_j \in V)$$

$\exists$ natural map $\begin{matrix} V \to V^{**}, \\ v \mapsto \widetilde{v} \end{matrix}$ $\widetilde{v} : \alpha \mapsto \alpha(v)$. If this map is an isomorphism, V is **reflexive** module, and identify V with V^{**}.

Exercise 7.5. Given vector bundle $\pi : E \to M$, open $U \subset M$, consider sections of π on U, i.e. cont. $s : U \to E$, where $(\pi \circ s)(u) = u, \;\; \forall u \in U$.

Consider $E^* \ni \omega = \omega_i e^i$.

$\forall s \in \Gamma(E)$, $\omega(s) = \omega_i (s(x))^i, \;\; \forall x \in U \subset M$. So define $\widetilde{s} : \omega, x \mapsto \omega(s(x)), \;\; \forall x \in U$.

If $\widetilde{s} = 0$, $\widetilde{s}(\omega, x) = \omega(s(x)) = 0 \qquad \forall \omega \in E^*, \forall x \in U$, and so $s = 0$. (Let $\omega_i = \delta_{iJ}$ for some J, and so $s^J(x) = 0 \qquad \forall J$).

$s = 0$. So $\ker(s \mapsto \widetilde{s}) = \{0\}$ (so condition for injectivity is fulfilled).

Since $\widetilde{s} : \omega, x \mapsto \omega(s(x)), \forall \omega \in E^*, \forall x \in U$, $s \mapsto \widetilde{s}$ is surjective.

$s \mapsto \widetilde{s}$ is an isomorphism so $\Gamma(E)$ is a *reflexive* module.

Proposition 3. *For R a ring (special case), $\exists$ module homomorphism:*

tensor product space $\to$ tensor, as a multilinear map, i.e. $\exists$

$$\begin{gathered} (\otimes_{i=1}^r V) \otimes (\otimes_{j=1}^s V^*) \to T^r{}_s(V; R) \\ u_1 \otimes \dots \otimes u_r \otimes \beta^1 \otimes \dots \otimes \beta^s \in (\otimes^r V) \otimes (\otimes^s V^*) \mapsto (\alpha^1 \dots \alpha^r, v_1 \dots v_s) \mapsto \alpha^1(u_1) \dots \alpha^r(u_r) \beta^1(v_1) \dots \beta^s(v_s) \end{gathered} \tag{12}$$

Indeed, consider

$$(\alpha^1 \dots \alpha^r, v_1 \dots v_s) \in \underbrace{V^* \times \dots \times V^*}_{r} \times \underbrace{V \times \dots \times V}_{s} \mapsto \alpha^1(u_1) \dots \alpha^r(u_r) \beta^1(v_1) \dots \beta^s(v_s)$$

and so for

$$\begin{aligned} \alpha^i &= \alpha^i_\mu e^\mu, \quad i = 1, 2, \dots r, \; \mu = 1, 2, \dots \dim V^* \qquad \alpha^i(u_i) = \alpha^i_\mu u^\mu_i \\ v_i &= v^\mu_i e_\mu, \quad i = 1, 2, \dots s, \; \mu = 1, 2, \dots \dim V \qquad \beta^i(v_i) = \beta^i_\mu v^\mu_i \end{aligned}$$

So that

$$\begin{aligned} \alpha^1(u_1) \dots \alpha^r(u_r) \beta^1(v_1) \dots \beta^s(v_s) &= \alpha^1_{\alpha_1} u_1^{\alpha_1} \dots \alpha^r_{\alpha_r} u_r^{\alpha_r} \beta^1_{\mu_1} v_1^{\mu_1} \dots \beta^s_{\mu_s} v_s^{\mu_s} = \\ &= (u_1^{\alpha_1} \dots u_r^{\alpha_r} \beta^1_{\mu_1} \dots \beta^s_{\mu_s})(\alpha^1_{\alpha_1} \dots \alpha^r_{\alpha_r} v_1^{\mu_1} \dots v_s^{\mu_s}) \end{aligned}$$

Identify $u_1 \otimes \dots \otimes u_r \otimes \beta^1 \otimes \dots \otimes \beta^s$ with this multiplinear map.

Proposition 4. *If V is finite-dim. vector space, or if $V = \Gamma(E)$, for vector bundle $E \to M$, map*

$$(\otimes_{i=1}^r V) \otimes (\otimes_{j=1}^s V^*) \to T^r{}_s(V; R) \tag{13}$$

is an isomorphism.

Definition 8. *tensor that can be written as*

$$u_1 \otimes \dots \otimes u_r \otimes \beta^1 \otimes \dots \otimes \beta^s \equiv u_1 \otimes \dots \otimes \beta^s \tag{14}$$

*is **simple** or **decomposable**.*

Now well that not *all* tensors are simple.

Definition 9 (7.7[2], tensor product)**.** $\forall S \in T^{r_1}_{s_1}(V)$, $\forall T \in T^{r_2}_{s_2}(V)$,
define tensor product

$$\begin{gathered} S \otimes T \in T^{r_1+r_2}_{s_1+s_2}(V) \\ S \otimes T(\theta^1 \dots \theta^{r_1+r_2}, v_1 \dots v_{s_1+s_2}) := S(\theta^1 \dots \theta^{r_1}, v_1 \dots v_{s_1}) T(\theta^{r_1+1} \dots \theta^{r_1+r_2}, v_{s_1+1} \dots v_{s_1+s_2}) \end{gathered} \tag{15}$$

Proposition 5 (7.8[2])**.**

$$\tau^{i_1 \dots i_r}{}_{j_1 \dots j_s} e_{i_1} \otimes \dots \otimes e_{i_r} \otimes e^{j_1} \otimes \dots \otimes e^{j_s} = \tau(e^{i_1} \dots e^{i_r}, e_{j_1} \dots e_{j_s}) e_{i_1} \otimes \dots \otimes e_{i_r} \otimes e^{j_1} \otimes \dots \otimes e^{j_s} = \tau$$

So $\{ e_{i_1} \otimes \dots \otimes e_{i_r} \otimes e^{j_1} \otimes \dots \otimes e^{j_s} | i_1 \dots i_r, j_1 \dots j_s \in 1 \dots n \}$ spans $T^r_s(V;R)$

Exercise 7.11. Let basis for V $e_1 \dots e_n$, corresponding dual basis for V^* $e^1 \dots e^n$
Let basis for V $\overline{e}_1 \dots \overline{e}_n$, corresponding dual basis for V^* $\overline{e}^1 \dots \overline{e}^n$
s.t.

$$\begin{gathered} \overline{e}_i = C^k_{\ i} e_k \\ \overline{e}^i = (C^{-1})^i_{\ k} e^k \end{gathered}$$

EY:20170404, keep in mind that

$$\begin{gathered} Ax = e_i A^i_{\ k} e^k (x^j e_j) = e_i A^i_{\ j} x^j = A^i_{\ j} x^j e_i \\ Ae_j = e_k A^k_{\ i} e^i(e_j) = A^k_{\ j} e_k = \overline{e}_j \\ \overline{\tau}^i_{\ jk} \overline{e}_i \otimes \overline{e}^j \otimes \overline{e}^k = \overline{\tau}^i_{\ jk} C^l_{\ i} e_l (C^{-1})^j_{\ m} e^m (C^{-1})^k_{\ n} e^n = \overline{\tau}^i_{\ jk} C^l_{\ i} (C^{-1})^j_{\ m} (C^{-1})^k_{\ n} = \tau^l_{\ mn} \\ \overline{\tau}^i_{\ jk} = C^c_{\ k} C^b_{\ j} (C^{-1})^i_{\ a} \tau^a_{\ bc} \end{gathered}$$

On Remark 7.13 of Jeffrey Lee (2009) [2]: first, egregious typo for $L(V,V)$; it shoudl be $L(V,W)$. Onward, for $L(V,W)$,
consider $W \otimes V^* \ni w \otimes \alpha$ s.t.

$$(w \otimes \alpha)(v) = \alpha(v) w \in W, \forall v \in V, \text{ so } w \otimes \alpha \in L(V,W)$$

Now consider (category of) left R-module,

$$_R\mathbf{Mod} \ni {}_{\mathrm{Mat}_{\mathbb{K}}(N,M)} \mathbb{K}^N \tag{16}$$

where

$$\begin{gathered} V = \mathbb{K}^N \\ W = \mathbb{K}^M \end{gathered}$$

For $A \in \mathrm{Mat}_{\mathbb{K}}(N,M)$, $x \in \mathbb{K}^N$,

$$\begin{gathered} e_i A^i_{\ ,\mu} e^\mu (x^\nu e_\nu) = Ax = e_i A^i_{\ \mu} x^\mu, \quad i = 1,2,\dots M, \; \mu = 1,2,\dots N \\ A \in \mathrm{Mat}_{\mathbb{K}}(N,M) \cong W \otimes V^* \cong L(V,W) \end{gathered}$$

Consider

$$\begin{gathered} \alpha \in (\mathbb{K}^N)^* = V^* \qquad \alpha = \alpha_\mu e^\mu \\ w \in \mathbb{K}^M = W \qquad w = w^i e_i \\ \alpha \otimes w = w \otimes \alpha = w^i \alpha_\mu e_i \otimes e^\mu \end{gathered}$$

(remember, isomoprhism between $\mathrm{Mat}_{\mathbb{K}}(N,M)$ and $W \otimes V^*$ guaranteed, if V, W are free R-modules, $R = \mathbb{K}$).

Let V, W be left R-modules, i.e. $V, W \in {}_R\mathbf{Mod}$.

$$V^* \in \mathbf{Mod}_R$$

For $V^* \otimes W \in \mathbf{Mod}_R \otimes {}_R\mathbf{Mod}$

$$\alpha \in V^*, w \in W$$

$$(\alpha \otimes w)(v) = \alpha(v) w, \text{ for } v \in V \in {}_R\mathbf{Mod}$$

But $(w \otimes \alpha)(v) = w\alpha(v)$.
Note $\alpha(v) \in R$.
Let V, W be right R-modules, i.e. $V, W \in \mathbf{Mod}_R$.

$$V^* \in {}_R\mathbf{Mod}$$

For $W \otimes V^* \in \mathbf{Mod}_R \otimes {}_R\mathbf{Mod}$.

$$\alpha \in V^*, \, w \in W$$

$$(v)(w \otimes \alpha) = w\alpha(v), \text{ with } \alpha(v) \in R, \, v \in V$$

So $W \otimes V^* \cong L(V,W)$, for $V, W \in \mathbf{Mod}_R$

Definition 10 (7.20[2], **contraction**)**.** *Let $(e_1, \dots e_n)$ basis for V, $(e^1 \dots e^n)$ dual basis.*
If $\tau \in T^r_s(V)$, then for $k \leq r$, $l \leq s$, define

$$\begin{gathered} C^k_l \tau \in T^{r-1}_{s-1}(V) \\ C^k_l \tau(\theta^1 \dots \theta^{r-1}, w_1 \dots w_{s-1}) := \\ \sum_{a=1}^n \tau(\theta^1 \dots \underbrace{e^a}_{kth \text{ position}} \dots \theta^{r-1}, w_1 \dots \underbrace{e_a}_{ith \text{ position}} \dots w_{s-1}) \end{gathered} \tag{17}$$

*C^k_l is called **contraction**, for some single $1 \leq k \leq r$, some single $1 \leq l \leq s$,*

$$C^k_l : T^r_s(V) \to T^{r-1}_{s-1}(V)$$

s.t.

$$(C^k_l \tau)^{i_1 \dots \widehat{i}_k \dots i_r}{}_{j_1 \dots \widehat{j}_l \dots j_s} := \tau^{i_1 \dots a \dots i_r}{}_{j_1 \dots a \dots j_s}$$

Universal mapping properties can be invoked to give a basis free definition of contraction (EY : 20170405???).
IN general,

$$\forall v_1 \dots v_s \in V, \forall \alpha^1 \dots \alpha^r \in V^*$$

so that

$$\begin{gathered} v_j = v^\mu_j e_\mu \quad j = 1 \dots s, \quad \mu = 1, \dots \dim V \\ \alpha^i = \alpha^i_\mu e^\mu \quad i = 1 \dots r, \quad \mu = 1 \dots \dim V^* \end{gathered}$$

then $\forall \tau \in T^r_s(V)$,

$$\begin{gathered} \tau(\alpha^1 \dots \alpha^r, v_1 \dots v_s) = \tau(\alpha^1_{\mu_1} e^{\mu_1} \dots \alpha^r_{\mu_r} e^{\mu_r}, v_1^{\nu_1} e_{\nu_1} \dots v_s^{\nu_s} e_{\nu_s}) = \\ = \alpha^1_{\mu_1} \dots \alpha^r_{\mu_r} v_1^{\nu_1} \dots v_s^{\nu_s} \tau(e^{\mu_1} \dots e^{\mu_r}, e_{\nu_1} \dots e_{\nu_s}) = \alpha^1_{\mu_1} \dots \alpha^r_{\mu_r} v_1^{\nu_1} \dots v_s^{\nu_s} \tau^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s} \end{gathered}$$

which is equivalent to

$$\begin{tikzcd}
\tau \in T^r_s(V) \arrow[r, mapsto, "\alpha^1 \otimes \dots \otimes \alpha^r \otimes v_1 \otimes \dots \otimes v_s \otimes"] & \alpha^1 \otimes \dots \otimes \alpha^r \otimes v_1 \otimes \dots \otimes v_s \otimes \tau \arrow[d, mapsto, "C^1_{s+1} C^2_{s+2} \dots C^r_{r+s} C^r_1 C^{r+1}_2 \dots C^{r+s}_s"'] \\
 & \tau(\alpha^1 \dots \alpha^r, v_1 \dots v_s) \in R
\end{tikzcd}$$

where I've tried to express the right-R-module, "right action" on $\alpha^1 \otimes \dots \otimes \alpha^r \otimes v_1 \otimes \dots \otimes v_s \in V^* \otimes \dots \otimes V$.

Conlon (2008) [16]

Part 4. Lie Groups, Lie Algebra

7. Lie Groups

- : Lie Groups
- : Groups
- : Ring
- : group algebra
- : Group Ring
- : Representation Theory
- : Modules
- : kG-modules

From Sec. 8.1 "Noncommutative Rings" of Rotman (2010) [9]:

Definition 11. *ring R - additive abelian group equipped with multiplication* $\begin{aligned} R \times R &\to R \\ (a,b) &\mapsto ab \end{aligned}$ *s.t.* $\forall\, a, b \in R$

(i) $a(bc) = (ab)c$
(ii) $a(b+c) = ab + ac$, $(b+c)a = ba + ca$
(iii) $\exists\, 1 \in R$ *s.t.* $\forall\, a \in R$, $1a = a = a1$

Example 8.1[9]

(ii) group algebra kG, k commutative ring, G group, "its additive abelian group is free k-module having basis labeled by elements of G,
i.e. $\forall\, a \in kG$, $a = \sum_{g \in G} a_g g$, $a_g \in k$, $\forall\, g \in G$, $a_g \neq 0$ for only finitely many $g \in G$.

define (ring) multiplication $\begin{aligned} kG \times kG &\to kG \\ ab &= ab \end{aligned}$ $\forall\, a, b \in kG$, $\begin{aligned} a &= \sum_{g \in G} a_g g \\ b &= \sum_{h \in G} b_h g \end{aligned}$ to be

$$\left(\sum_{g \in G} a_g g \right) \left(\sum_{h \in G} b_h h \right) = \sum_{z \in G} \left(\sum_{gh = z} a_g b_h \right) z$$

Definition 12. *Given R ring, left R-module is (additive) abelian group M equipped with scalar multiplication* $\begin{aligned} R \times M &\to M \\ (r,m) &\mapsto rm \end{aligned}$ *s.t.* $\forall\, m, m' \in M$, $\forall\, r, r', 1 \in R$

(i) $r(m + m') = rm + rm'$
(ii) $(r + r')m = rm + r'm$
(iii) $(rr')m = r(r'm)$
(iv) $1m = m$

EY : 20150922 Example : for kG-module V^σ, for $r \in kG$, so $r = \sum_{g \in G} a_g g$

$$\begin{aligned} R \times M &\to M \\ (r,m) &\mapsto rm \end{aligned} \Longrightarrow \begin{aligned} kG \times V &\to V \\ (r,v) &\mapsto tv \end{aligned}$$

For some representation $\sigma : G \to GL(V)$,

$$rv = \sum_{g \in G} a_g g \cdot v = \sum_{g \in G} a_g \sigma_g(v)$$

So a kG-module needs to be associated with some chosen representation.

Note for V as an additive abelian group, $\forall\, u, v, w \in V$,

$$\begin{aligned} & v + w = w + v, \ (u+v) + w = u + (v + w) \\ & v + 0 = v \quad \forall\, v \in V \text{ for } 0 \in V \\ & v + (-v) = 0 \quad \forall\, v \in V \end{aligned}$$

So a vector space can be an additive abelian group.

Note that

$$r(v+w) = \left(\sum_{g \in G} a_g g \right)(v+w) = \left(\sum_{g \in G} a_g \sigma_g \right)(v+w) = \sum_{g \in G} a_g \sigma_g(v) + \sum_{g \in G} a_g \sigma_g(w) = rv + rw$$

$$(r + r')v = \left(\sum_{g \in G} a_g g + b_g g \right) v = \sum_{g \in G} (a_g \sigma_g + b_g \sigma_g) v = \sum_{g \in G} a_g \sigma_g(v) + \sum_{g \in G} b_g \sigma_g(v) = rv + r'v$$

$$(rr')v = \left(\sum_{g \in G} a_g g \sum_{h \in G} b_h h \right) v = \left(\sum_{z \in G} \sum_{gh = z} a_g b_h z \right) v = \sum_{z \in G} \sum_{gh \in z} a_g b_h \sigma_z(v) = \sum_{g \in G} \sum_{h \in G} a_g b_h \sigma_g \sigma_h(v)$$

since $\sigma(gh) = \sigma(g)\sigma(h) = \sigma_g \sigma_h = \sigma_{gh}$ (σ homomorphism)

$1v = \sigma(1)v = 1v = v$

From Sec. 8.3 "Semisimple Ring" of Rotman (2010) [9]:

Definition 13. *k-representation of group G is homomorphism*

$$\sigma : G \to GL(V)$$

where V is vector field over field k

Proposition 6 (8.37 Rotman (2010)[9])**.** *$\forall\, k$-representation $\sigma : G \to GL(V)$ equips V with structure of left kG-module, denote module by V^σ.*
Conversely, $\forall$ left kG-module V determines k-representation $\sigma : G \to GL(V)$

Proof. Given $\begin{aligned} \sigma &: G \to GL(V), \\ \sigma_g =: \sigma(g) &: V \to V \end{aligned}$

define

$$kG \times V \to V$$
$$\left(\sum_{g\in G} a_g g \right) v = \sum_{g\in G} a_g \sigma_g(v)$$

$v, w \in V$

Let $r, r', 1 \in kG$

$r = \sum_{g\in G} a_g g$

$$r(v+w) = \left(\sum_{g\in G} a_g g \right)(v+w) = \left(\sum_{g\in G} a_g \sigma_g \right)(v+w) = \sum_{g\in G} a_g\sigma_g(v) + \sum_{g\in G} a_g \sigma_g(w) = rv + rw$$
$$(r+r')v = \left(\sum_{g\in G} a_g g + b_g g \right) v = \sum_{g\in G} (a_g\sigma_g + b_g \sigma_g) v = \sum_{g\in G} a_g \sigma_g(v) + \sum_{g\in G} b_g\sigma_g(v) = rv + r'v$$
$$(rr')v = \left(\sum_{g\in G} a_g g \sum_{h\in G} b_h h \right) v = \left(\sum_{z\in G}\sum_{gh=z} a_g b_h z \right) v = \sum_{z\in G}\sum_{gh\in z} a_g b_h \sigma_z(v) = \sum_{g\in G}\sum_{h\in G} a_g b_h \sigma_g \sigma_h(v)$$

since $\sigma(gh) = \sigma(g)\sigma(h) = \sigma_g\sigma_h = \sigma_{gh}$ (σ homomorphism)

$1v = \sigma(1)v = 1v = v$

Conversely, assume V left kG-module.

If $g \in G$, then $v \mapsto gv$ defines $T_g : V \to V$. T_g nonsingular since $\exists T_g^{-1} = T_{g^{-1}}$

Define $\sigma : G \to GL(V)$

$\sigma : g \mapsto T_g$

σ k-representation

$$\sigma(gh) = T_{gh} = T_g T_h = \sigma(g)\sigma(h)$$
$$\sigma(gh)(v) = T_{gh} v = ghv = T_g T_h v = \sigma(g)\sigma(h) v \quad \forall\, v \in V$$

□

Proposition 7. *Let group G, let $\sigma, \tau : G \to GL(V)$ be k-representations, field k. If V^σ, V^τ corresponding kG-modules in Prop. 6 (Prop. 8.37 in Rotman (2010) [9]), then $V^\sigma \simeq V^\tau$ as kG-modules iff $\exists$ nonsingular $\varphi : V \to V$ s.t.*

$$\varphi\tau(g) = \sigma(g)\varphi \quad \forall\, g \in G$$

Proof. If $\varphi : V^\tau \to V^\sigma$ kG-isomorphism, then $\varphi : V \to V$ isomorphism s.t.

$$\varphi(\sum a_g g v) = (\sum a_g)\varphi(v) \quad \forall\, v \in V, \forall\, g \in G$$

in V^τ, $kG \times V \to V$ $\quad$ in V^σ, $kG \times V \to V$ scalar multiplication

$gv = \tau(g)(v)$ $\quad$ $gv = \sigma(g)(v)$

$$\Longrightarrow \forall\, g \in G, v \in V, \quad \varphi(\tau(g)(v)) = \sigma(g)(\varphi(v))$$

I think

$$\varphi(gv) = \varphi(\tau(g)(v)) = g\varphi(v) = \sigma(g)\varphi(v)$$
$$\Longrightarrow \varphi\tau(g) = \sigma(g)\varphi \quad \forall\, g \in G$$

Conversely, if $\exists$ nonsingular $\varphi : V \to V$ s.t. $\varphi\tau(g) = \sigma(g)\varphi \quad \forall\, g \in G$

$$\varphi\tau(g)v = \varphi(\tau(g)v) = \sigma(g)\varphi(v) \quad \forall\, g \in G, \forall\, v \in V$$

Consider scalar multiplication

$$kG \times V \to V$$
$$\sum_{g\in G} a_g g(v) = \sum_{g\in G} a_g \tau_g(v)$$
$$\varphi\left(\sum_{g\in G} a_g \tau_g(v) \right) = \varphi\left(\sum_{g\in G} a_g \tau(g) v \right) = \sum_{g\in G} a_g \sigma(g)\sigma(g)\varphi(v) = \left(\sum_{g\in G} a_g g \right)\varphi(v)$$

□

Admittedly, after this exposition from Rotman (2010) [9], I still didn't understand how kG-modules relate to representation theory and group rings. I turned to Baker (2011) [10], which we'll do right now. Note that I found a lot of links to online resources on representation theory from Khovanov's webpage http://www.math.columbia.edu/~khovanov/resource/.

Note,

Definition 14. *vector subspace $W \subseteq V$ is called a G-submodule, G-subspace, EY : 20150922 "invariant" subspace? if $\forall\, g \in G$, for representation $\rho : G \to GL_k(V)$, $\rho_g(w) \in W$, $\forall\, w \in W, \forall\, g \in G$ i.e. closed under "action of elements of G" with $\rho_g =: \rho(g) : V \to V$*

Given basis $\mathbf{v} = \{v_1 \dots v_n\}$ for V, $\dim_k V = n$, $\forall\, g \in G$,

$$\rho_g v_j = \rho(g) v_j = r_{kj}(g) v_k$$

for, indeed,

$$\rho_g x^j v_j = \rho(g) x^j v_j = x^j \rho(g) v_j = x^j r_{kj}(g) v_k = r_{kj} x^j v_k$$

so that

$$\rho : G \to GL_k(V)$$
$$\rho(g) = [r_{ij}(g)]$$

Example 2.1 (Baker (2011) [10]): Let $\rho : G \to GL_k(V)$ where $\dim_k V = 1$

$$\forall\, v \in V,\, v \neq 0, \forall\, g \in G,\, \lambda_g \in k \text{ s.t. } g \cdot v = \rho_g(v) = \lambda_g v$$
$$\rho(hg) v = \rho_h \rho_g v = \lambda_{hg} v = \lambda_h \lambda_g v \Longrightarrow \lambda_{hg} = \lambda_h \lambda_g$$

$\Longrightarrow \exists$ homomorphism $\Lambda : G \to k^\times$

$\Lambda(g) = \lambda_g$

From Sec. 2.2 "G-homomorphisms and irreducible representations" of Baker (2011) [10], suppose $\begin{matrix} \rho : G \to GL_k(V) \\ \sigma : G \to GL_k(W) \end{matrix}$ are 2 representations

Many names for the same thing: G-equivalent, G-linear, G-homomorphism, EY : 20150922 kG-isomorphic?

If $\forall\, g \in G$,

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "\tau_g"'] & V \arrow[d, "\sigma_g"] \\ V \arrow[r, "\varphi"] & V \end{tikzcd} \iff V^\tau \overset{\varphi}{\simeq} V^\sigma$$

Indeed, define

$$\begin{aligned} &\varphi : V^\tau \to V^\sigma \\ &\varphi(v+w) = \varphi(v) + \varphi(w) \\ &\varphi(rv) = \varphi(\sum_{g\in G} a_g g \cdot v) = \varphi(\sum_{g\in G} a_g \tau_g(v)) = \sum_{g\in G} a_g \varphi(\tau_g(v)) = \sum_{g\in G} a_g \sigma_g \cdot \varphi(v) = r\varphi(v) \end{aligned}$$

EY : 20150922 So φ is a kG-isomorphism between left kG modules V^τ and V^σ if it's bijective and is "linear" in "scalars" $r \in kG$, i.e. $\varphi(rv) = r\varphi(v)$.

Define action of G on $\mathrm{Hom}_k(V,W)$ ($\mathrm{Hom}_k(V,W)$ is the vector space of k-linear transformations $V \to W$)

$$V \xrightarrow{f} W$$
$$v \overset{f}{\longmapsto} f(v)$$

$\forall f \in \mathrm{Hom}_k(V,W)$, $f : V \to W$

$f(v) \in W$

Consider

$$\begin{aligned} &G \times \mathrm{Hom}_k(V,W) \to \mathrm{Hom}_k(V,W) \\ &(g\cdot f) \mapsto (\sigma_g f) \circ \rho_{g^{-1}} \text{ i.e. } (g\cdot f)(v) = \sigma_g f(\rho_{g^{-1}} v) \quad (f \in \mathrm{Hom}_k(V,W)) \end{aligned}$$

Let $g, h \in G$,

$$(gh\cdot f)(v) = g \cdot \sigma_h f(\rho_{h^{-1}} v) = \sigma_g \sigma_h f \rho_{h^{-1}} \rho_{g^{-1}}(v) = (\sigma_{gh} f \rho_{(gh)^{-1}})(v)$$

Thus, $G \times \mathrm{Hom}_k(V,W) \to \mathrm{Hom}_k(V,W)$ is thus another G-representation of G.

$(g\cdot f) \mapsto (\sigma_g f) \circ \rho_{g^{-1}}$

For k-representation ρ, if the only G-subspaces of V are $\{0\}$, V, ρ **irreducible** or **simple**.

$$\begin{aligned} &\rho_g(\{0\}) = \{0\} \\ &\rho_g(V) = V \end{aligned}$$

given subrepresentation $W \subseteq V$, V/W admits linear action of G, $\overline{\rho}_W : G \to GL_k(V/W)$ quotient representation

$$\overline{\rho}_W(g)(v+W) = \rho(g)(v) + W$$

if $v' - v \in W$

$$\rho(g)(v') + W = \rho(g)(v + (v'-v)) + W = (\rho(g)(v) + \rho(g)(v'-v)) + W = \rho(g)(v) + W$$

Proposition 8 (2.7 Baker (2011)[10]). *if $f : V \to W$ G-homomorphism, then*

(a) *$\ker f$ is G-subspace of V*
(b) *$\mathrm{im} f$ is G-subspace of W*

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\rho_g"'] & W \arrow[d, "\sigma_g"] \\ V \arrow[r, "f"] & W \end{tikzcd}$$

Proof. Recall

(a) Let $v \in \ker f$. Then $\forall g \in G$,

$$f(\rho_g v) = \sigma_g f(v) = 0$$

so $\rho_g v \in \ker f$, $\forall g \in G$. So $\ker f$ is G-subspace of V

(b) Let $w \in \mathrm{im} f$. So $w = f(u)$ for some $u \in V$

$$\sigma_g w = \sigma_g f(u) = f(\rho_g u) \in \mathrm{im} f$$

So $\mathrm{im} f$ is G-subspace of W

□

Theorem 11 (Schur's Lemma). *Let $\begin{matrix} \rho : G \to GL_{\mathbb{C}}(V) \\ \sigma : G \to GL_{\mathbb{C}}(W) \end{matrix}$ be irreducible representations of G over field $k = \mathbb{C}$; let $f : V \to W$ be G-linear map.*

(a) *if $f \neq 0$, f isomorphism. True $\forall k$ field, not just $\mathbb{C}$*
(b) *if $V = W$, $\rho = \sigma$, then for some $\lambda \in \mathbb{C}$, f given by $f(v) = \lambda v$ ($v \in V$) (true for algebraically closed fields)*

Proof. (a) By Prop. 8, $\ker f \subseteq V$, $\mathrm{im} f \subseteq W$ are G-subspaces.
For ρ, only G-subspaces are 0 or V, so if $\ker f = V$, $f = 0$. If $\ker f = 0$, f injective.
For σ, only G-subspaces are 0 or V, so $\mathrm{im} f = 0$, $f = 0$. If $\mathrm{im} f = V$, f surjective.
$\Longrightarrow f$ isomorphism.

(b) Let $\lambda \in \mathbb{C}$ be an eigenvalue of f, $f(v_0) = \lambda v_0$ eigenvector, $v_0 \neq 0$.
Let linear $f_\lambda : V \to V$ s.t.

$$f_\lambda(v) = f(v) - \lambda v \quad (v \in V)$$

$\forall g \in G$

$$\rho_g f_\lambda(v) = \rho_g f(v) - \rho_g \lambda v = f(\rho_g v) - \lambda \rho_g v = f_\lambda(\rho_g v)$$

So f_λ is G-linear, for

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\rho_g"'] & V \arrow[d, "\rho_g"] \\ V \arrow[r, "f"] & V \end{tikzcd}$$

Since $f_\lambda(v_0) = 0$, by Prop. 8, $\ker f_\lambda = V$, (for $\ker f_\lambda \neq 0$ and so $\ker f_\lambda = V$)
By rank-nullity theorem, $\dim V = \dim\ker f_\lambda + \dim\mathrm{im} f_\lambda$.
So $\mathrm{im} f_\lambda = 0$, and so $f_\lambda(v) = 0$ ($\forall v \in V$) $\Longrightarrow f(v) = \lambda v$

□

Schur's lemma, at least the first part, implies that the left kG-modules associated with representations ρ, σ are kG-isomorphic, i.e.

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\rho_g"'] & W \arrow[d, "\sigma_g"] \\ V \arrow[r, "f"] & W \end{tikzcd} \iff V^\rho \overset{f}{\simeq} V^\sigma$$

with f being an isomorphism between V^ρ and V^σ s.t.

$$\begin{aligned} f(v+w) &= f(v) + f(w) \quad \forall\, v, w \in (V^\sigma, +) \\ f(rv) &= rf(v) \quad \forall\, r = \sum_{g\in G} a_g g \in kG \end{aligned}$$

Kosmann-Schwarzbach's **Groups and Symmetries**[11] is a very lucid text that's mathematically rigorous enough and practical for physicists. It's really good and very clear. Let's follow its development for $SU(2), SO(3), SL(2,\mathbb{C})$ and corresponding Lie algebras $\mathfrak{su}(2)$, $\mathfrak{so}(3)$, $\mathfrak{sl}(2,\mathbb{C})$.

From Chapter 2 "Representations of Finite Groups" of Kosmann-Schwarzbach (2010) [11]

Definition 15 (2.1 Kosmann-Schwarzbach (2010)[11])**.** *On $L^2(G)$, scalar product defined by*

$$\langle f_1 | f_2 \rangle = \frac{1}{|G|} \sum_{g\in G} \overline{f_1(g)} f_2(g)$$

$f_1, f_2 \in \mathcal{F}(G) \equiv \mathbb{C}[G]$ vector space of functions on G taking values on $\mathbb{C}$

Definition 16 (2.3 Kosmann-Schwarzbach (2010)[11])**.** *Let (E,ρ) be representation of G*

character of $\begin{aligned} &\rho \equiv \chi_\rho : G \to \mathbb{C} \\ &\chi_\rho(g) = tr(\rho(g)) = \sum_{i=1}^n (\rho(g))_{ii} \end{aligned}$

Note: equivalent representations have same character
each conjugacy class of G, function χ_p is constant

Looking at Def. 15

$$\langle \chi_{\rho_1} | \chi_{\rho_2} \rangle = \frac{1}{|G|} \sum_{g\in G} \chi_{\rho_1}(g^{-1}) \chi_{\rho_2(g)}$$

since $\overline{\chi_{\rho_1(g)}} = \chi_{\rho_1}(g^{-1})$ by unitarity of representation with respect to scalar product $\langle\, , \,\rangle$

Proposition 9 (2.7 Kosmann-Schwarzbach (2010)[11])**.** *Let* $\begin{matrix}(E_1,\rho_1)\\(E_2,\rho_2)\end{matrix}$ *be representations of G, let linear $u : E_1 \to E_2$.*

Then $\exists$ linear T_u s.t.

$$\begin{aligned} &T_u : E_1 \to E_2 \\ &T_u = \frac{1}{|G|} \sum_{g\in G} \rho_2(g) u \rho_1(g)^{-1} \end{aligned} \tag{18}$$

so that $\rho_2(g) T_u = T_u \rho_1(g) \quad \forall\, g \in G$

Proof.

$$\rho_2(g) T_u = \frac{1}{|G|} \sum_{h\in G} \rho_2(gh) u \rho_1(h^{-1}) = \frac{1}{|G|} \sum_{k\in G} \rho_2(k) u \rho_1(k^{-1}g) = T_u \rho_1(g)$$

$\square$

Thus, diagrammatically, we have that

$$E_1 \xrightarrow{u} E_2 \implies \begin{tikzcd} E_1 \arrow[r, "T_u"] \arrow[d, "\rho_1(g)"] & E_2 \arrow[d, "\rho_2(g)"] \\ E_1 \arrow[r, "T_u"] & E_2 \end{tikzcd}$$

From Definition 1.12 of Kosmann-Schwarzbach [11], "representations ρ_1 and ρ_2 are called **equivalent** if there is a bijective intertwining operator for ρ_1 and ρ_2." So I will interpret this as if an intertwining operator is not bijective, then the representations ρ_1, ρ_2 are not equivalent.

Proposition 10 (2.8 Kosmann-Schwarzbach (2010)[11])**.** *Let* $\begin{matrix}(E_1,\rho_1)\\(E_2,\rho_2)\end{matrix}$ *be irreducible representations of G, let linear $u : E_1 \to E_2$, define T_u by $T_u = \frac{1}{|G|}\sum_{g\in G} \rho_2(g) u \rho_1(g)^{-1}$ by Eq. 18.*

(i) *If ρ_1, ρ_2 inequivalent, then $T_u = 0$*
(ii) *If $E_1 = E_2 = E$ and $\rho_1 = \rho_2 = \rho$, then*

$$T_u = \frac{tr(u)}{dim E} 1_E$$

Proof. (i) if ρ_1, ρ_2 are inequivalent, by definition, T_u is not isomorphic. Then by Schur's lemma (first part), $T_u = 0$
(ii) By Schur's lemma, $T_u(v) = \lambda v \quad \forall\, v \in E = E_1 = E_2$. So $T_u = \lambda 1_E$. $\mathrm{tr} T_u = \lambda \mathrm{dim} E$ or $\lambda = \frac{\mathrm{tr} T_u}{\mathrm{dim} E}$. Thus, $T_u = \frac{\mathrm{tr} T_u}{\mathrm{dim} E} 1_E$ $\square$

Let $(e_1 \dots e_n)$ basis of E
$(f_1 \dots f_p)$ basis of F

$\forall\, u \in \mathcal{L}(E,F),$ $\begin{aligned} &u : E \to F \\ &u(x) = u(x^j e_j) = x^j u(e_j) = x^j u^i{}_j f_i \\ &u = u^i{}_j e^j \otimes f_i \end{aligned}$ for $\begin{aligned} x &= x^j e_j \in E \\ y &= y^i f_i \in F \end{aligned}$

For

$$\begin{aligned} &T : E^* \otimes F \to \mathcal{L}(E,F) \\ &T(\xi \otimes y) = u^i{}_j e^j \otimes f_i \text{ i.e. set } T(\xi\otimes y) \text{ to this } u \\ &T(\xi \otimes y) = T(\xi_l e^l \otimes y^k f_k) = \xi_l y^k T(e^l \otimes f_k) = (\xi_l y^k T^{li}_{kj}) e^j \otimes f_i \Longrightarrow \xi_l y^k T^{li}_{kj} = u^i{}_j \end{aligned}$$

Exercises. Exercises of Ch. 2 Representations of Finite Groups [11]

Exercise 2.6. [11] *The dual representation.*

Let (E,π) representation of group G.
$\forall\, g \in G$, $\xi \in E^*$, $x \in E$, set $\langle \pi^*(g)(\xi), x\rangle = \langle \xi, \pi(g^{-1})(x)\rangle$

(a) *dual* (or *contragredient*) of π, $\pi^* : G \to \text{End}(E^*)$, π^* is a representation, since

$$\begin{aligned} \langle \pi^*(gh)(\xi), x\rangle = \langle \xi, \pi((gh)^{-1})(x)\rangle = \langle \xi, \pi(h^{-1}g^{-1})(x)\rangle = \langle \xi, \pi(h^{-1})\pi(g^{-1})(x)\rangle = \langle \xi, \pi(h^{-1})(\pi(g^{-1})(x))\rangle = \\ = \langle \pi^*(h)(\xi), \pi(g^{-1})(x)\rangle = \langle \pi^*(g)\pi^*(h)(\xi), x\rangle \end{aligned}$$

since this is true, $\forall\, x \in E$, $\forall\, \xi \in E^*$, $\pi^*(gh) = \pi^*(g)\pi^*(h)$.
dual π^* of π is a representation.

(b) Consider $G \times \mathcal{L}(E,F) \to \mathcal{L}(E,F)$.

$$g \cdot u = \rho(g) \circ u \circ \pi(g^{-1})$$

Define

$$\begin{aligned} &\sigma : G \to \text{End}(\mathcal{L}(E,F)) \\ &\sigma(g) : \mathcal{L}(E,F) \to \mathcal{L}(E,F) \\ &\sigma(g)(u) = \rho(g) \circ u \circ \pi(g^{-1}) \end{aligned}$$

Let $(e_1 \dots e_n)$ be a basis of E. Let $\xi = \xi_i e^i \in E^*$, $x = x^j e_j \in E$.
Consider the isomorphism $T : E^* \otimes F \to \mathcal{L}(E,F)$ defined as[2]

$$\begin{aligned} &T : E^* \otimes F \to \mathcal{L}(E,F) = \text{Hom}(E,F) \\ &\xi \otimes y \mapsto (x \mapsto \xi(x)y) \end{aligned}$$

Choose bases $\begin{matrix} (e_1 \dots e_n) \text{ of } E \\ (e^1 \dots e^n) \text{ of } E^* \\ (f_1 \dots f_p) \text{ of } F \end{matrix}$. Then

$$\begin{aligned} &T(e^j \otimes f_i)(x) = T(e^j \otimes f_i)(x^k e_k) = \delta^j_{\ k} x^k f_i = x^j f_i \\ &T(e^j \otimes f_i)(e_k) = \delta^j_{\ k} f_i \end{aligned}$$

Consider

$$\begin{aligned} &u \in \mathcal{L}(E,F) \\ &u : E \to F \\ &u(x) = u(x^j e_j) = x^j u(e_j) = x^j u^i_{\ j} f_i \\ &u(e_j) = u^i_{\ j} f_i \text{ i.e. } u : e_j \to u^i_{\ j} f_i \end{aligned}$$

Then $\forall u \in \mathcal{L}(E,F)$,

$$T(u^i_{\ j} e^j \otimes f_i)(e_k) = u^i_{\ j}\delta^j_{\ k} f_i = u^i_{\ k} f_i = u(e_k) \Longrightarrow u = T(u^i_{\ j} e^j \otimes f_i)$$

so T is surjective.
With $T(\xi \otimes y) = T(\xi' \otimes y')$,

$$\begin{aligned} T(\xi \otimes y)(x) &= T(\xi' \otimes y')(x) \\ \xi(x)y = \xi'(x)y' &\Longrightarrow \xi(x)y - \xi'(x)y' = 0 \end{aligned}$$

which implies that $\xi \otimes y = \xi' \otimes y'$. So T is injective. Or, one could consider that $T^{-1} : \mathcal{L}(E,F) \to E^* \otimes F$, $T^{-1} : u \mapsto u^i_{\ j} e^j \otimes f_i$, which is the inverse of T.

Remark 1.

$$\begin{aligned} &E^* \otimes F \overset{T}{\simeq} \mathcal{L}(E,F) = \mathit{Hom}(E,F) \\ &(\xi, y) \mapsto (x \mapsto \xi(x)y) \end{aligned}$$

and so $(e^j \otimes f_i) \mapsto (x \mapsto e^j(x) f_i = x^j f_i)$
So $E^* \otimes F$ *is isomorphic to* $\mathcal{L}(E,F) = \mathit{Hom}(E,F)$

For representation π,

$$\begin{aligned} &\pi : G \to \text{End}(E) \\ &\pi(g) : E \to E \\ &\pi(g)(x) = \pi(g)(x^j e_j) = x^j \pi(g)(e_j) = x^j \pi(g)^i_{\ j} e_i = (\pi(g)^i_{\ j} x^j e_i \end{aligned}$$

Consider this matrix formulation:

$$\begin{aligned} \pi^*(g)(\xi) = \pi^*(g)(\xi_i e^i) = \xi_i \pi^*(g)(e^i) = \xi_i (\pi^*(g))^i_{\ j} e^j \\ \Longrightarrow \langle \pi^*(g)(\xi), x\rangle = \xi_i (\pi^*(g))^i_{\ j} x^j \end{aligned}$$

and

$$\langle \xi, \pi(g^{-1})(x)\rangle = \xi_i \pi(g^{-1})^i_{\ j} x^j$$

so that

$$\langle \pi^*(g)(\xi), x\rangle = \langle \xi, \pi(g^{-1})(x)\rangle \Longrightarrow \pi(g^{-1})^i_{\ j} = (\pi^*(g))^i_{\ j}$$

Thus, given a choice of basis for E, the *dual* of π, $\pi^*(g)^i_{\ j}$, and $\pi(g^{-1})^i_{\ j}$ are formally equal.
So for a choice of basis of E and of F,

$$(\pi^* \otimes \rho)(g)(\xi, y) = (\pi^*(g) \otimes \rho(g))(\xi, y) = \pi^*(g)\xi \otimes \rho(g)y = \xi_l \pi(g^{-1})^l_{\ j} e^j \otimes \rho(g)^i_{\ k} y^k f_i = \rho(g)^i_{\ k} y^k \xi_l \pi(g^{-1})^l_{\ j} e^j \otimes f_i$$

Applying T,

$$T(\pi^* \otimes \rho)(g)(\xi, \rho) = \rho(g)^i_{\ k} y^k \xi_l \pi(g^{-1})^l_{\ j} = \rho(g) T(\xi, y) \pi(g^{-1})$$

$$\begin{tikzcd} E^* \otimes F \arrow[r, "T"] \arrow[d, "(\pi^* \otimes \rho)(g)"'] & \mathcal{L}(E,F) \arrow[d, "\sigma(g)"] \\ E^* \otimes F \arrow[r, "T"'] & \mathcal{L}(E,F) \end{tikzcd} \qquad \begin{tikzcd} (\xi, y) \arrow[r, mapsto, "T"] \arrow[d, mapsto, "(\pi^* \otimes \rho)(g)"'] & (x \mapsto \xi(x)y) = y^i \xi_j \arrow[d, mapsto, "\sigma(g)"] \\ \pi^*(g)(\xi) \otimes \rho(g)y \arrow[r, mapsto, "T"'] & \rho(g) y^i \xi_j \pi(g^{-1}) = \rho(g) T(\xi, y) \pi(g^{-1}) \end{tikzcd}$$

Thus
Thus, representation $\sigma(g)$ is equivalent to representation $(\pi^* \otimes \rho)$, a tensor product of representations.

Exercise 2.15. *Representation of* $GL(2,\mathbb{C})$ *on the polynomials of degree 2*

Let group G, let representation ρ of G on $V = \mathbb{C}^n$, i.e. $\rho : G \to \text{End}(V)$
Let $P^{(k)}(V)$ vector space of complex polynomials on V that are homogeneous of degree k.
For $f \in P^{(k)}(V)$, the general form is

$$f = \sum_{\substack{i_1 + i_2 + \dots + i_n = k \\ 0 \leq i_j \leq k}} a_{i_1 i_2 \dots i_d} x_1^{i_1} x_2^{i_2} \dots x_n^{i_n}$$

[2] *Mathematics stackexchage* Isomorphism between Hom and tensor product [duplicate] http://math.stackexchange.com/questions/428185/isomorphism-between-hom-and-tensor-product http://math.stackexchange.com/questions/57189/understanding-isomorphic-equivalences-of-tensor-product

Given

$$\binom{n+k}{k} = \binom{k-1}{k-1} + \binom{k}{k-1} + \dots + \binom{n+k-1}{k-1} = \sum_{i=0}^n \binom{k-1+i}{k-1}$$

$\binom{k+n-1}{n-1}$ is number of monomials of degree k.

So $\dim P^{(k)}(V) = \binom{k+n-1}{n-1}$. This is a very lucid and elementary exposition on the basics of polynomials which I found was useful for the basic facts I forgot[3].

So we have the graded algebra

$$P(V) = \bigoplus_{k=0}^{\infty} P^{(k)}(V)$$

$$\begin{aligned} \rho^{(k)} &: G \to \mathrm{End}(P^{(k)}(V)) \\ \rho^{(k)}(g) &: P^{(k)}(V) \to P^{(k)}(V) \\ \rho^{(k)}(g)(f) &= f \circ \rho(g^{-1}) \end{aligned}$$

This is a representation of G since

(a)

$$\left. \begin{aligned} \rho^{(k)}(gh)(f) &= f\circ \rho((gh)^{-1}) = f \circ \rho(h^{-1}g^{-1}) = f\circ \rho(h^{-1}\rho(g^{-1}) \\ \rho^{(k)}(g)\rho^{(k)}(h)(f) &= \rho^{(k)}(g)(f\circ \rho(h^{-1})) = f\circ\rho(h^{-1})\circ\rho(g^{-1}) \end{aligned} \right. \implies \rho^{(k)}(gh) = \rho^{(k)}(g)\rho^{(k)}(h)$$

(b) Choose basis $(e_1 \dots e_n)$ of V, $x = x^j e_j \in V$, $\rho : G \to \mathrm{End}(V)$, and so $\rho(g)(x) = \rho(g)(x^j e_j) = x^j \rho(g)(e_j) = x^j (\rho(g))^i{}_j e_i$.
With $\xi(e_i) = \xi_i \implies \langle \xi, \rho(g^{-1})x\rangle = \xi_i x^j (\rho(g^{-1}))^i{}_j$
$\forall\, \xi \in V^*$, $\xi = \xi_i e^i$,

$$\rho^*(g)(\xi) = \rho^*(g)(\xi_i e^i) = \xi_i \rho^*(g)^i{}_j e^j$$

$$\implies \langle \rho^*(g)(\xi), x\rangle = \xi_i x^j (\rho^*(g))^i{}_j \implies (\rho^*(g))^i{}_j = (\rho(g^{-1}))^i{}_j$$

So $\forall\, f \in P^{(1)}(V)$, $x\in V$, $\rho(g^{-1})x = x^j(\rho(g^{-1}))^i{}_j e_i$. So $f\circ \rho(g^{-1})(x) = \sum_{i=1}^n a_i (\rho(g^{-1}))^i{}_j x^j = \sum_{i=1}^n a_i (\rho^*(g))^i{}_j x^j$
$\implies \rho^{(1)}(g)(f) = f \circ \rho^*(g)$

(c) Suppose $G = GL(2,\mathbb{C})$, $V = \mathbb{C}^2$, ρ fundamental representation $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $g^{-1} = \frac{1}{\det g}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ for $\det g = ad - bc$.
Let $k=2$, $\dim P^{(2)}(\mathbb{C}^2) = \binom{2+2-1}{2-1} = \binom{3}{1} = 3$
$\forall\, f \in P^{(2)}(\mathbb{C}^2)$, $f(x,y) = Ax^2 + 2Bxy + Cy^2$
Let

$$P^{(2)}(\mathbb{C}^2) \to \mathbb{C}^3$$

$$f(x,y) = Ax^2 + 2Bxy + Cy^2 \mapsto \begin{pmatrix} A \\ B \\ C \end{pmatrix} \in \mathbb{C}^3$$

Call this transformation T, $T : P^{(2)}(\mathbb{C}^2) \to \mathbb{C}^3$.
$\forall \begin{pmatrix} A \\ B \\ C \end{pmatrix} \in \mathbb{C}^3$, $f(x,y) = Ax^2 + 2Bxy + Cy^2$ and $Tf(x,y) = \begin{pmatrix} A \\ B \\ C \end{pmatrix}$. T surjective.
Suppose $Tf(x,y) = Tf'(x,y)$,

$$\begin{aligned} &\implies Ax^2 + 2Bxy + Cy^2 = A'x^2 + 2B'xy + C'y^2 \\ &\implies (A-A')x^2 + 2(B-B')xy + (C-C')y^2 = 0 \end{aligned}$$

Then since the monomials form a basis, and its basis elements are independent (by definition), then $A = A'$, $B = B'$, $C = C'$. T injective. So T is bijective, an isomorphism.

[3]Polynomials. Math 4800/6080 Project Course http://www.math.utah.edu/~bertram/4800/PolyIntroduction.pdf

(This is all in `groups.sage`)

```
sage: P2CC.<x,y> = PolynomialRing(CC,2) # this declares a PolynomialRing of field of complex numbers,
# of order 2 (i.e. only 2 variables for a polynomial, such as x, y)
sage: A = var('A')
sage: assume(A,''complex'')
sage: B = var('B')
sage: assume(B,''complex'')
sage: C = var('C')
sage: assume(C,''complex'')
sage: f(x,y) = A*x**2 +2*B*x*y + C*y**2

sage: a = var('a')
sage: assume(a,''complex'')
sage: b = var('b')
sage: assume(b,''complex'')
sage: c = var('c')
sage: assume(c,''complex'')
sage: d = var('d')
sage: assume(d,''complex'')
sage: g = Matrix([[a,b],[c,d]] )
sage: X = Matrix([[x],[y]])
sage: f( (g.inverse()*X)[0,0], (g.inverse()*X)[1,0] ).expand()
sage: f( (g.inverse()*X)[0,0], (g.inverse()*X)[1,0] ).expand().coefficient(x^2).full_simplify()
(C*c^2 - 2*B*c*d + A*d^2)/(b^2*c^2 - 2*a*b*c*d + a^2*d^2)
sage: f( (g.inverse()*X)[0,0], (g.inverse()*X)[1,0] ).expand().coefficient(x*y).full_simplify()
-2*(C*a*c + A*b*d - (b*c + a*d)*B)/(b^2*c^2 - 2*a*b*c*d + a^2*d^2)
sage: f( (g.inverse()*X)[0,0], (g.inverse()*X)[1,0] ).expand().coefficient(y^2).full_simplify()
(C*a^2 - 2*B*a*b + A*b^2)/(b^2*c^2 - 2*a*b*c*d + a^2*d^2)
```

So

$$\rho^{(2)}(g)(f)(x,y) = f\circ\rho(g^{-1})(x,y) =$$

$$= \frac{Cc^2 - 2Bcd + Ad^2}{(ad-bc)^2}x^2 + -2\frac{(Cac + Abd - (bc+ad)B)}{(ad-bc)^2}xy + \frac{Ca^2 - 2Bab + Ab^2}{(ad-bc)^2}y^2$$

So define $\widetilde{\rho} : G \to \mathrm{End}(\mathbb{C}^3)$. $\widetilde{\rho}$ is a representation, for

$$\forall v = \begin{pmatrix} A \\ B \\ C \end{pmatrix} \in \mathbb{C}^3, \quad \widetilde{\rho}(gh)(v) = T\circ f \circ \rho((gh)^{-1}) = T \circ f\circ \rho(h^{-1}g^{-1}) = T\circ f\circ \rho(h^{-1})\rho(g^{-1})$$

$$\text{Now } \widetilde{\rho}(h)(v) = T\circ f\circ\rho(h^{-1})$$

$$\implies \widetilde{\rho}(g)\widetilde{\rho}(h)(v) = T\circ(f\circ\rho(h^{-1}))\circ\rho(g^{-1}) = T\circ f\circ\rho(h^{-1})\rho(g^{-1}) \text{ and so}$$

$$\widetilde{\rho}(gh) = \widetilde{\rho}(g)\widetilde{\rho}(h)$$

And so

$$\widetilde{\rho}^*(g)(v) = Tf\rho(g^{-1})$$

and consider this commutation diagram, that (helped me at least and) clarifies the relationships:

$$\begin{tikzcd} P^{(2)}(\mathbb{C}^2) \arrow[r, "T"] \arrow[d, "\rho^{(2)}(g)"'] & \mathbb{C}^3 \arrow[d, "\widetilde{\rho}(g)"] \\ P^{(2)}(\mathbb{C}^2) \arrow[r, "T"'] & \mathbb{C}^3 \end{tikzcd} \qquad \begin{tikzcd} f \arrow[r, mapsto, "T"] \arrow[d, mapsto, "\rho^{(2)}(g)"'] & \begin{pmatrix} A \\ B \\ C \end{pmatrix} \arrow[d, mapsto, "\widetilde{\rho}(g)"] \\ f \circ \rho(g^{-1}) \arrow[r, mapsto, "T"'] & \begin{pmatrix} D \\ E \\ F \end{pmatrix} \end{tikzcd}$$

with

$$\begin{pmatrix} D \\ E \\ F \end{pmatrix} = \begin{pmatrix} \frac{Cc^2-2Bcd+Ad^2}{(ad-bc)^2} \\ -2\frac{(Cac+Abd-(bc+ad)B)}{(ad-bc)^2} \\ \frac{Ca^2-2Bab+Ab^2}{(ad-bc)^2} \end{pmatrix}$$

Now define the dual $\widetilde{\rho}^*$ as such:

$$\begin{gathered} \widetilde{\rho}^*(g) : (\mathbb{C}^3)^* \to (\mathbb{C}^3)^* \\ \widetilde{\rho}^*(g) = \widetilde{\rho}(g^{-1}) \\ \forall\, \xi \in (\mathbb{C}^3)^* \\ \widetilde{\rho}^*(g)\xi = \xi_i(\widetilde{\rho}^*(g))^i{}_j e^j = \xi_i(\widetilde{\rho}(g^{-1}))^i{}_j e^j \end{gathered}$$

So for $v = \begin{pmatrix} A \\ B \\ C \end{pmatrix} \in \mathbb{C}^3$, $f = T^{-1}v = Ax^2 + 2Bxy + Cy^2 \in P^2(\mathbb{C}^2)$,

$$\widetilde{\rho}(g^{-1})(v) = T \circ (f\rho(g)) = \begin{bmatrix} Aa^2 + 2Bac + Cc^2 \\ Aab + Bbc + Bad + Ccd \\ Ab^2 + 2Bbd + Cd^2 \end{bmatrix}$$

which was found using Sage Math:

```
sage: f((g*X)[0,0],(g*X)[1,0])
(a*x + b*y)^2*A + 2*(a*x + b*y)*(c*x + d*y)*B + (c*x + d*y)^2*C
sage: f((g*X)[0,0],(g*X)[1,0]).expand()
A*a^2*x^2 + 2*B*a*c*x^2 + C*c^2*x^2 + 2*A*a*b*x*y + 2*B*b*c*x*y + 2*B*a*d*x*y + 2*C*c*d*x*y + A*b^2*y^2 + 2*B*b*d*y^2 + C*d^2*y^2
sage: f((g*X)[0,0],(g*X)[1,0]).expand().coefficient(x^2)
A*a^2 + 2*B*a*c + C*c^2
sage: f((g*X)[0,0],(g*X)[1,0]).expand().coefficient(x*y)
2*A*a*b + 2*B*b*c + 2*B*a*d + 2*C*c*d
sage: f((g*X)[0,0],(g*X)[1,0]).expand().coefficient(y^2)
A*b^2 + 2*B*b*d + C*d^2
```

or

```
sage: T( f((g*X)[0,0],(g*X)[1,0]).expand() )
[A*a^2 + 2*B*a*c + C*c^2,
2*A*a*b + 2*B*b*c + 2*B*a*d + 2*C*c*d,
A*b^2 + 2*B*b*d + C*d^2]
```

So then

$$\widetilde{\rho}(g^{-1}) = \begin{bmatrix} a^2 & 2ac & c^2 \\ 2ab & 2(ad+bc) & 2cd \\ b^2 & 2bd & d^2 \end{bmatrix}$$

So then

$$\widetilde{\rho}^*(g) = \begin{bmatrix} a^2 & 2ac & c^2 \\ 2ab & 2(ad+bc) & 2cd \\ b^2 & 2bd & d^2 \end{bmatrix}$$

and operate on row vectors $\xi \in (\mathbb{C}^3)^*$ with $\widetilde{\rho}^*(g)$ from the row vector's right.

More: Let $G = SU(2)$. Then $U = e^{i\phi}\begin{bmatrix} a & b \\ -\overline{b} & \overline{a} \end{bmatrix}$

$$\begin{aligned} & \widetilde{\rho} : SU(2) \to \mathrm{End}(\mathbb{C}^3) \\ & \widetilde{\rho}(U) : \mathbb{C}^3 \to \mathbb{C}^3 \\ & \widetilde{\rho}(U)(v) = e^{-2i\varphi}\begin{bmatrix} A\overline{a}^2 + 2B\overline{a}\overline{b} + C\overline{b}^2 \\ -A\overline{a}b + B + Ca\overline{b} \\ Ab^2 - 2Bab + Ca^2 \end{bmatrix} \\ & \Longrightarrow \widetilde{\rho}(U) = e^{-2i\varphi}\begin{bmatrix} -\overline{a}^2 & 2\overline{a}\overline{b} & \overline{b}^2 \\ -\overline{a}b & 1 & a\overline{b} \\ b^2 & -2ab & a^2 \end{bmatrix} \end{aligned}$$

cf. Ch. 5 Lie Groups of Jeffrey Lee (2009) [2]

Definition 17 (Lie Group)**.** ***Lie Group*** $G :=$ *smooth manifold* G *is a* ***Lie Group*** *if* G *is a group (abstract group), s.t.*

multiplication map $\mu : G \times G \to G$
$$\mu(g,h) = gh$$

inverse map $inv : G \to G$
$$inv(g) = g^{-1}$$
are C^∞ *maps.*

If group is abelian, use additive notation $g + h$ *for group operation.*

Definition 18 ($GL(n,\mathbb{R})$)**.** $GL(n,\mathbb{R}) :=$ *group of all invertible real* $n \times n$ *matrices.*
global chart on $GL(n,\mathbb{R}) = \{x^i_j\}$*,* n^2 *functions* x^i_j*, where if* $A \in GL(n,\mathbb{R})$*, then* $x^i_j(A)$ *is* ij*th entry of* A*.*

Claim: $GL(n,\mathbb{R})$ is a Lie group.

Proof. multiplication is clearly smooth: $(AB)_{ij} = A_{ik}B_{kj}$,

$$\frac{\partial}{\partial x^l_m}(x^i_k(A)x^k_j(B)) = \delta^i_l\delta^m_k x^k_j(B) + x^i_k(A)\delta^k_l\delta^m_j$$

inversion map; appeal to formula for A^{-1}, $A^{-1} = \mathrm{adj}(A)/\det(A)$, $\mathrm{adj}(A) \equiv$ adjoint matrix (whose entries are cofactors).
$\Longrightarrow A^{-1}$ depends smoothly on entries of A.

Similarly, $GL(n,\mathbb{C})$, group of invertible $n \times n$ complex matrices, is a Lie group. □

Exercise 5.5. Let subgroup H of G, consider cosets gH, $g \in G$.
Recall G is disjoint union of cosets of H.
Claim: if H open, so are all its cosets. And H closed.

Proof. cf. stackexchange: Open subgroups of a topological group are closed

$gH = \{gh | h \in H\}$ is an open neighborhood of g (since $1 \in H$, and mapping $h \mapsto gh$ sends open sets to open sets, since its inverse, $gh \mapsto h$, is C^∞ (so continuous)).

$$gH \to H \qquad H \to gH$$
$$gh \xrightarrow{g^{-1}} h = \mu(g^{-1}, gh) \qquad h \xrightarrow{g} gh = \mu(g, h)$$

Then $\forall$ coset gH, gH is open.
Suppose $g' \in H^c \equiv G - H \equiv G \backslash H$.
Consider $h \in H$, if $g'h \in H$, then $g' = (g'h)h^{-1} \in H$ (recall $h^{-1} \in H$, and H is a subgroup).
Contradiction.
$\Longrightarrow \forall g' \in H^c$, $\exists$ open neighborhood $g'H \subset H^c$, so H^c open (by definition). Then H closed.

□

cf. Thm. 5.6 in Jeffrey Lee (2009) [2].

Theorem 12. *If G connected Lie group, U neighborhood of identity element e, then U generates the group, i.e. $\forall g \in G$, g is a product of elements of U.*

Proof. Note $V = \text{inv}(U) \cap U$ is an open neighborhood of e. Note $\text{inv}(V) = V$. $\text{inv}(V) \equiv V^{-1} = \{V^{-1} | v \in V\}$. We say that V is *symmetric*.

Claim: V generates G.
$\forall$ open W_1, open $W_2 \subset G$,
$W_1 W_2 = \{w_1 w_2 | w_1 \in W_1, w_2 \in W_2\}$ is an open set being a union of open sets $\bigcup_{g \in W_1} gW_2$.
Thus, inductively defined sets

$$V^n = VV^{n-1}, \quad n = 1, 2, 3, \dots$$

are open.

$$e \in V \subset V^2 \subset \dots V^n \subset \dots$$

It's easy to check that each V^n is symmetric.

$$\text{inv}(V) = V$$
$$\text{inv}(V^2) = \text{inv}(\bigcup_{v \in V} vV) = V\text{inv}(V) = V = V^2$$
$$\text{inv}(V^{n+1}) = \text{inv}\left(\bigcup_{v \in V} vV^n\right) = V\text{inv}(V^n) = VV^n = V^{n+1}$$

so $V^\infty := \bigcup_{n=1}^\infty V^n$ is symmetric.
V^∞ closed under inversion, also multiplication. Thus V^∞ is an open subgroup.
From Exercise 5.5, Jeffrey Lee (2009) [2], i.e. Exercise 7, V^∞ also closed, since G is connected, $V^\infty = G$. (a topological space X is **connected** iff the only open and closed (clopen) sets are $\emptyset$ and X). □

Definition 19. *Identity component of G, G_0.*
$G_0 :=$ connected component of Lie group G that contains identity;
G_0 is a Lie group, and is generated by any open neighborhood of the identity.

Definition 20. *For Lie group G, fixed element $g \in G$,*
left translation (by g) $L_g : G \to G$, $L_g x = gx$, $\quad \forall x \in G$
right translation (by g) $R_g : G \to G$, $R_g x = xg$, $\quad \forall x \in G$

L_g, R_g are diffeomorphisms with $L_g^{-1} = L_{g^{-1}}$, $R_g^{-1} = R_{g^{-1}}$.

Definition 21 (Product Lie group)**.** *If G, H are Lie groups, then product manifold $G \times H$ is a Lie group, where multiplication*

$$(g_1, h_1) \cdot (g_2, h_2) = (g_1 g_2, h_1 h_2)$$

*Lie group $G \times H$ is called **product Lie group***

e.g. product group $S^1 \times S^1 \equiv$ 2-torus group.
Generally, higher torus groups $T^n = S^1 \times \dots \times S^1$ (n factors).

Definition 22 (Lie subgroup of G, H)**.** *Let H be an abstract subgroup of Lie group G.*
*If H is a Lie group s.t. inclusion map $i : H \to G \equiv H \hookleftarrow G$ is an immersion, then H is a **Lie subgroup** of G.*

Recall $i : H \to G$ immersion iff Di injective, i.e. iff $\text{rank}Di = \text{dim}H$
cf. Prop. 5.9 in Jeffrey Lee (2009) [2].

Proposition 11. *If H abstract subgroup of Lie group G, that's also a regular submanifold $\equiv$ embedded submanifold, then H closed Lie subgroup.*

Recall that

embedded submanifold $\equiv$ regular submanifold

Each name is used frequently and we shouldn't be biased against one or the other; we'll have to refer to both, to emphasize they're *exactly the same*.
embedded submanifold $\equiv$ regular submanifold is an immersed submanifold s.t. inclusion map i is a topological embedding, i.e. embedded submanifold $\equiv$ regular submanifold $S \subset M$,
immersed submanifold S if $i : S \to M \equiv S \hookrightarrow M$ is an immersion, i.e. Di injective, i.e. $\text{rank}Di \equiv \text{dim}S$.
topological embedding := homeomorphism onto its image, i.e.
injective cont. map $f : X \to Y$, X, Y topological spaces, is a **topological embedding** if f is a homeomorphism between X and $f(X)$.
f homeomorphism is a bijection, continuous, and f^{-1} continuous.

e.g. $\forall$ embedding $f : M \to N$, $f(M) \subset N$ naturally has the structure of an embedding submanifold $\equiv$ regular submanifold.
Useful, intrinsic definition of **embedded submanifold** $\equiv$ regular submanifold.
Let manifold M, $\text{dim}M = n$, let $k \in \mathbb{Z}^+$, s.t. $0 \leq k \leq n$.
A k-dim. embedded submanifold $\equiv$ regular submanifold S is subset $S \subset M$ s.t. $\forall p \in S$, $\exists$ chart $(U \subset M, \varphi : U \to \mathbb{R}^n \ni 0)$, s.t. $\varphi(S \cap U)$ is the intersection of a k-dim. plane with $\varphi(U)$.
(pairs $(S \cap U, \varphi|_{S \cap U})$ form an atlas for differential structure on S).
Proof 1:

Proof. H subgroup of G, so
multiplication map $H \times H \to H$
inversion map $H \to H$
are restrictions of multiplication and inversion maps on G.

Since H regular submanifold, maps are smooth.
Recall H regular submanifold iff H immersive submanifold (i.e. $H \hookleftarrow G$ is an immersion) and H topological subspace of G, i.e. submanifold topology on H is same as subspace topology.

Claim: H closed.
Let $x_0 \in \overline{H}$
Let (U, x) be a chart adapted to H, whose domain contains e.
Let

$$\delta : G \times G \to G$$
$$\delta(g_1, g_2) = g_1^{-1} g_2$$

Choose open set V s.t. $e \in V \subset \overline{V} \subset U$.
By continuity map δ, find open neighborhood O of identity e s.t. $O \times O \subset \delta^{-1}(V)$
If $\{h_i\}$ sequence in H converging to $x_0 \in \overline{H}$, then $x_0^{-1} h_i \to e$ and $x_0^{-1} h_i \in O$ for all sufficiently large i.

Since $h_j^{-1}h_i = (x_0^{-1}h_j)^{-1}x_0^{-1}h_i$, $h_j^{-1}h_i \in V$ for sufficiently large i, j.

For any sufficiently large fixed j,

$$\lim_{i\to 0} h_j^{-1}h_i = h_j^{-1}x_0 \in \overline{V} \subset U$$

Since U is domain of a single-slice chart, $U \subset H$ closed in U.

Thus, since $\forall\, h_j^{-1}h_i \in U\cap H$, $h_j^{-1}x_0 \in U \cap H \subset H$, $\quad \forall$ sufficiently large j.
$\Longrightarrow x_0 \in H$, and since x_0 arbitrary, done.

□

Proof 2:
cf. 9.2 The Closed Subgroup Theorem I of 427 Notes[4]

Proof. **Claim**: Since H is an embedded submanifold $\equiv$ regular submanifold, $\exists$ neighborhood U of 1, $1 \in G$, s.t. $U \cap H$ closed in U.

Let $x_0 \in \overline{H}$, $\overline{H} \equiv$ closure of x_0.
Then $x_0U^{-1} \subseteq G$ is a neighborhood of x_0 in G (since $1 \in U^{-1}$, $x_01 = x_0 \in x_0U^{-1}$)

$$\Longrightarrow x_0U^{-1} \cap H \neq \emptyset$$

$\forall\, x \in x_0U^{-1}\cap H$, $x = x_0U^{-1}$ for some $u \in U$. Thus, $x^{-1}x_0 = u \in U$.

Now
$L_{x^{-1}} : G \to G$ is a homeomorphism, so $L_{x^{-1}}(H) = H$. By continuity, $L_{x^{-1}}(\overline{H}) = \overline{H}$. Thus $x^{-1}x_0 \in \overline{H}$.
Claim: $x^{-1}x_0 \in H \cap U$.
Since $x^{-1}x_0 \in \overline{H}\cap U$, $\exists$ sequence $\{h_i\} \subset H \cap U$ s.t. $h_0 \to x^{-1}x_0$.
But recall $H \cap U$ closed in U, so $x^{-1}x_0 \in H \cap U$.

$$\Longrightarrow x_0 \in xH = H, \quad \overline{H} \subseteq H$$

Thus H closed.

Claim: If H abstract subgroup of Lie group G, that's also an embedded submanifold $\equiv$ regular submanifold, then H is a Lie subgroup.

Recall that by definition, Lie group has group multiplication and inverse map to be C^∞. Then, just show group multiplication is C^∞, first.

Since G is a Lie group, then

$$\begin{aligned} &\mu : G\times G \to G \\ &\mu(x,y) = xy \end{aligned}$$

is C^∞ (by definition).

Then $\mu : G\times G \to G$ cont.
Consider subgroup $H \subseteq G$ and $\mu : H \times H \to H$.
Since $H\times H \subseteq G\times G$, $\forall\,(x,y) \in H\times H$ (fix $(x,y) \in H \times H$), $\forall$ neighborhood V of $\mu(x,y) = xy$, $V \subset G$, $\exists$ neighborhood U of (x,y) s.t. $\mu(U) \subseteq V$ (by $\mu : G\times G \to G$ cont.).

Since H embedded submanifold $\equiv$ regular submanifold of G,
$\exists$ neighborhood $V' \subseteq V$ of $xy \in G$, coordinate map $\varphi : V' \to \mathbb{R}^n$ ($n = \dim G$) s.t.

$$\varphi(H\cap V') = \varphi(V') \cap (\mathbb{R}^k \times \{0\})$$

where $k = \dim H$
(since H is a k-dim. embedded submanifold $\equiv$ regular submanifold, $H \subseteq G$, s.t. $\forall\, p \in H$, $\exists$ chart $(V \subset G, \varphi : U \to \mathbb{R}^n \ni 0)$, s.t. $\varphi(U\cap V) = \varphi(V) \cap (\mathbb{R}^k \times \{0\})$).
Now

$$\varphi\circ\mu : \mu^{-1}(V')\cap U \to \mathbb{R}^n \text{ is } C^\infty, \text{ and } \varphi\circ\mu(\mu^{-1}(V')\cap U) \subseteq \mathbb{R}^k \times \{0\}$$

Let projection $\pi : \mathbb{R}^n \to \mathbb{R}^k$ be the standard projection,

$$\pi\circ\varphi\circ\mu : \mu^{-1}(V')\cap U \to \mathbb{R}^k \text{ is } C^\infty$$

$\Longrightarrow \mu$ is C^∞ □

From Chapter 4 "Lie Groups and Lie Algebras" of Kosmann-Schwarzbach (2010) [11]
While Proposition 2.6 of Kosmann-Schwarzbach (2010) [11] states that

$$\det(\exp(X)) = \exp{(\mathrm{tr} X)}$$

here are some other resources online that gave further discussion on the characteristic polynomial, $\det(A - \lambda 1)$ and the different terms of it, called Newton identities:

- http://scipp.ucsc.edu/~haber/ph116A/charpoly_11.pdf
- http://math.stackexchange.com/questions/1126114/how-to-find-this-lie-algebra-proof-that-mathfraksl-is-trace
- http://mathoverflow.net/questions/131746/derivative-of-a-determinant-of-a-matrix-field

Theorem 13 (5.1 [11])**.** *Consider* $\mathfrak{g} = \{X = \gamma'(0) | \gamma : 1 \to G \text{ of class } C^1, \gamma(0) = 1\}$
Let Lie group G

(i) $\mathfrak{g}$ *vector subspace of* $\mathfrak{gl}(n,\mathbb{R})$
(ii) $X \in \mathfrak{g}$ *iff* $\forall\, t \in \mathbb{R}$, $\exp{(tX)} \in G$
(iii) *if* $X \in \mathfrak{g}$, *if* $g \in G$, *then* $gXg^{-1} \in \mathfrak{g}$
(iv) $\mathfrak{g}$ *closed under matrix commutator, i.e. if* $X, Y \in \mathfrak{g}$, $[X,Y] \in \mathfrak{g}$

Proof. (i)
(ii) If $\exp{(tX)} \in G$, then $X\ \frac{d}{dt}\exp{(tX)}\big|_{t=0} \in \mathfrak{g}$ (by def.)
If $X \in \mathfrak{g}$, then by def., $X = \frac{d}{dt}\gamma(t)\big|_{t=0}$ with $\gamma(t) \in G$.
Now Taylor expand; $\forall\, k \in \mathbb{Z}^+$

$$\gamma\left(\frac{t}{k}\right) = 1 + \frac{t}{k}X + O\left(\frac{1}{k^2}\right) = \exp\left(\frac{t}{k}X + O\left(\frac{1}{k^2}\right)\right)$$

$$\Longrightarrow \left(\gamma\left(\frac{t}{k}\right)\right)^k = \exp{(tX)}$$

$$\gamma\left(\frac{t}{k}\right) \in G \quad \forall\, k \in \mathbb{Z}^+$$

G closed subgroup, so $\lim_{k\to\infty}(\gamma\left(\frac{t}{k}\right))^k = \exp{(tX)} \in G$
(iii)
(iv)

□

[4] https://faculty.math.illinois.edu/~lerman/519/s12/427notes.pdf

Definition 23 (Lie algebra)**.** *Lie algebra $\mathfrak{g}$, tangent space to G at 1, i.e. $\mathfrak{g} := T_1 G$ is called* Lie algebra *of Lie group G.*

$$\mathfrak{g} := \{ X = \gamma'(0) | \gamma : 1 \to G \text{ of class } C^1 , \gamma(0) = 1 \} = T_1 G$$

This is based on Proposition 5.3 of Kosmann-Schwarzbach (2010) [11].
For Lie group

$$U(n) = \{ U \in GL(n,\mathbb{C}) | UU^\dagger = 1 \}$$

If $X \in \mathfrak{u}(n)$, then $\exp{(tX)} \in U(n)$. Then

$$\exp{(tX)}\exp{(tX)}^\dagger = (1 + tX + O(t^2))(1 + tX^\dagger + O(t^2)) = 1 + t(X + X^\dagger) + O(t^2) = 1 \forall\, t \in \mathbb{R} \Longrightarrow X + X^\dagger = 0$$

i.e. $X \in \mathfrak{u}(n)$ is an anti-Hermitian complex $n \times n$ matrix.
$\mathfrak{u}(n) = \{ X \in \mathfrak{gl}(n,\mathbb{C}) | X + X^\dagger = 0 \}$
Physicists: $X = iA$ and so $A - A^\dagger$. $A \in \mathfrak{u}(n)$ is a Hermitian complex $n \times n$ matrix.
$\mathfrak{u}(n) = \{ A \in \mathfrak{gl}(n,\mathbb{C}) | A - A^\dagger = 0 \}$
Regardless, $\dim_{\mathbb{R}} \mathfrak{u}(n) = n^2 = 2n^2 - n^2$
For Lie group

$$SU(n) = \{ U \in GL(n,\mathbb{C}) | UU^\dagger = 1, \det U = 1 \}$$

Then

$$\mathfrak{su}(n) = \{ X \in \mathfrak{gl}(n,\mathbb{C}) | X + X^\dagger = 1, \operatorname{tr} X = 0 \}$$

is the Lie algebra of traceless anti-Hermitian complex $n \times n$ matrices, and that

$$\dim_{\mathbb{R}} \mathfrak{su}(n) = n^2 - 1$$

In summary,

$$\begin{array}{ccc} \mathfrak{u}(n) = \{ X \in \mathfrak{gl}(n,\mathbb{C}) | X + X^\dagger = 0 \} & & \mathfrak{su}(n) = \{ X \in \mathfrak{gl}(n,\mathbb{C}) | X + X^\dagger = 0, \operatorname{tr} X = 0 \} \\ \Big\downarrow \exp{(tX)} & & \Big\downarrow \exp{(tX)} \\ U(n) = \{ U \in GL(n,\mathbb{C}) | UU^\dagger = 1 \} & & SU(n) = \{ U \in GL(n,\mathbb{C}) | UU^\dagger = 1, \det U = 1 \} \end{array}$$

$$\dim_{\mathbb{R}} \mathfrak{u}(n) = n^2 \qquad \dim_{\mathbb{R}} \mathfrak{su}(n) = n^2 - 1$$

From Chapter 5 "Lie Groups $SU(2)$ and $SO(3)$" of Kosmann-Schwarzbach (2010) [11],

7.0.1. *Bases of $\mathfrak{su}(2)$, Subsection 1.1 of Chapter 5of Kosmann-Schwarzbach (2010)* [11]. Recall that

$$\begin{array}{c} \mathfrak{su}(n) = \{ X \in \mathfrak{gl}(n,\mathbb{C}) | X + X^\dagger = 0, \operatorname{tr} X = 0 \} \\ \Big\downarrow \exp{(tX)} \\ SU(n) = \{ U \in GL(n,\mathbb{C}) | UU^\dagger = 1, \det U = 1 \} \end{array}$$

$$\dim_{\mathbb{R}} \mathfrak{su}(n) = n^2 - 1$$

and so

$$\begin{array}{c} \mathfrak{su}(2) = \{ X \in \mathfrak{gl}(2,\mathbb{C}) | X + X^\dagger = 0, \operatorname{tr} X = 0 \} \\ \Big\downarrow \exp{(tX)} \\ SU(2) = \{ U \in GL(n,\mathbb{C}) | UU^\dagger = 1, \det U = 1 \} \end{array}$$

$$\dim_{\mathbb{R}} \mathfrak{su}(2) = 3$$

Also, recall that $\mathfrak{g} \subseteq \mathfrak{gl}(n,\mathbb{C})$ is a vector subspace (13) and that
$X \in \mathfrak{g}$ iff $\forall t \in \mathbb{R}$, $\exp{(tX)} \in G$.
if $X \in \mathfrak{g}$, if $g \in G$, then $gXg^{-1} \in \mathfrak{g}$

$\mathfrak{g}$ closed under $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$
$(X,Y) \mapsto [X,Y]$
and so with $\mathfrak{g}$ as a vector space, we can have a choice of bases.

(a) $\xi_1 = \frac{i}{2}\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$
$\xi_2 = \frac{1}{2}\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$
$\xi_3 = \frac{i}{2}\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$
satisfying

$$[\xi_k, \xi_l] = \epsilon_{klm}\xi_m$$

(b) *Physics*
$\sigma_1 = -2i\xi_1 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$
$\sigma_2 = 2i\xi_2 = \begin{pmatrix} & -i \\ i & \end{pmatrix}$
$\sigma_3 = -2i\xi_3 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$
satisfying

$$[\sigma_k, \sigma_l] = 2i\epsilon_{klm}\sigma_m$$

EY : 20151001 Sage Math 6.8 doesn't run on Mac OSX El Capitan: I suspect that it's because in Mac OSX El Capitan, `/usr` cannot be modified anymore, even in an Administrator account. The TUG group for MacTeX had a clear, thorough, and useful (i.e. copy UNIX commands, paste, and run examples) explanation of what was going on:
http://tug.org/mactex/elcapitan.html
So keep in mind that my code for Sage Math is for Sage Math 6.8 that doesn't run on Mac OSX El Capitan. I'll also use `sympy` in Python as an alternative and in parallel.
One can check in `sympy` the traceless anti-Hermitian (or Hermitian) property of the bases and Pauli matrices, and the commutation relations (see `groups.py`):

```
import itertools
from itertools import product, permutations
```

```
import sympy
from sympy import I, LeviCivita
from sympy import Rational as Rat

from sympy.physics.matrices import msigma # <class 'sympy.matrices.dense.MutableDenseMatrix'>

def commute(A,B):
"""
commute = commute(A,B)
commute takes the commutator of A and B
"""
return (A*B - B*A)

def xi(i):
"""
xi = xi(i)
xi is a function that returns the independent basis for
Lie algebra su(2)\equiv su(2,\mathbb{C}) of Lie group SU(2) of
traceless anti-Hermitian matrices, based on msigma of sympy
cf. http://docs.sympy.org/dev/_modules/sympy/physics/matrices.html#msigma
"""
if i not in [1,2,3]:
raise IndexError("Invalid Pauli index")
elif i==1:
return I/Rat(2)*msigma(1)
elif i==2:
return -I/Rat(2)*msigma(2)
elif i==3:
return I/Rat(2)*msigma(3)
```

```
## check anti-Hermitian property and commutation relations with xi
# xi is indeed anti-Hermitian
xi(1) == -xi(1).adjoint() # True
xi(2) == -xi(2).adjoint() # True
xi(3) == -xi(3).adjoint() # True

# xi obeys the commutation relations

for i,j in product([1,2,3],repeat=2): print i,j

for i,j in product([1,2,3],repeat=2): print i,j, "\t Commutator: ", commute(xi(i),xi(j))

## check traceless Hermitian property and commutation relations with Pauli matrices
# Pauli matrices i.e. msigam is indeed traceless Hermitian

msigma(1) == msigma(1).adjoint() # True
msigma(2) == msigma(2).adjoint() # True
msigma(3) == msigma(3).adjoint() # True

msigma(1).trace() == 0 # True
msigma(2).trace() == 0 # True
msigma(3).trace() == 0 # True

# Pauli matrices obey commutation relation
print "For Pauli matrices, the commutation relations are :\n"
for i,j in product([1,2,3],repeat=2): print i,j, "\t Commutator: ", commute(msigma(i),msigma(j))

for i,j,k in permutations([1,2,3],3): print "Commute: ", i,j,k, msigma(i), msigma(j), \
": and is 2*i of ", msigma(k), commute(msigma(i),msigma(j)) == 2*I*msigma(k)*LeviCivita(i,j,k)
```

And finally the traceless property of the Pauli matrices:

```
>>> msigma(1).trace()
0
>>> msigma(2).trace()
0
>>> msigma(3).trace()
0
```

7.1. **Spin.** Let's follow the development by Baez and Muniain (1994) on pp. 175 of the Section II.1 "Lie Groups", the second (II) chapter on "Symmetry" [8].

Let $V=\mathbb{C}^2$, $G=SU(2)$. Then consider the graded algebra of polynomials on $V=\mathbb{C}^2 \ni (x,y)$

$$P(V) = \bigoplus_{k=0}^{\infty} P^{(k)}(V) = \bigoplus_{\substack{j=0 \\ 2j\in\mathbb{Z}}}^{\infty} P^{(2j)}(V) = \bigoplus_{\substack{j=0 \\ j\in\mathbb{Z}}}^{\infty} P^{(2j)}(V) \oplus \bigoplus_{\substack{j=1/2 \\ 2j \text{ odd}}}^{\infty} P^{(2j)}(V)$$

$$P^{(2j)}(V) \equiv \text{ vector space of complex polynomials of degree } 2j$$

and recall this representation on $P^{(2j)}(V)$

$$\begin{aligned} & \rho^{(2j)} : G \to \text{End}(P^{(2j)}(V)) \\ & \rho^{(2j)} : P^{(2j)}(V) \to P^{(2j)}(V) \\ & \rho^{(2j)}(g)(f) = f\circ \rho(g^{-1}) \text{ where } \rho \text{ is the fundamental representation of } G=SU(2) \\ & \rho^{(2j)}(g)(f)(v) = f\circ\rho(g^{-1})(v) \quad \forall\, f\in P^{(2j)}(V), \forall\, v\in V=\mathbb{C}^2 \end{aligned}$$

Note,
$\dim P^{(2j)} = \binom{2j+2-1}{2-1} = 2j+1$

Exercise 21. [8] *spin-0* Consider the trivial representation τ:

$$\begin{tikzcd} \mathbb{C} \arrow[r, "T"] \arrow[d, "\tau(g)"'] & P^{(0)}(V) \arrow[d, "\rho^{(0)}(g)"] \\ \mathbb{C} \arrow[r, "T"] & P^{(0)}(V) \end{tikzcd}$$

$$\begin{aligned} & \tau : G\to \text{End}(\mathbb{C}) \\ & \tau(g) : \mathbb{C}\to\mathbb{C} \\ & \tau(g) = 1_{\mathbb{C}} \end{aligned}$$

Clearly, $P^{(0)}(V)=\mathbb{C}$, since $P^{(0)}(V)$ consists of polynomials of constants in $\mathbb{C}$.

Consider $c_0\in\mathbb{C}$, $f=k_0\in P^{(0)}(V)$
$\rho^{(0)}(g)(f) = f\circ\rho(g^{-1}) = k_0$
$\Longrightarrow \rho^0(g)T(c_0) = T\circ\tau(g)c_0 = T(c_0)$. Let $T=1_{\mathbb{C}} = 1_{P^0(V)}$

So $\rho^{(0)}(g)=\tau(g)=1$. $T=1$. So representations $\rho^{(0)}$ and trivial representation τ on G are equivalent.

Exercise 22. [8] *spin-$\frac{1}{2}$* For spin-$\frac{1}{2}$, $j=\frac{1}{2}$, $2j=1$.

$\forall\, f\in P^{(1)}(V)$, $V=\mathbb{C}^2$. So in general form, $f(x,y)=ax+by\in P^{(1)}(V)$, $\begin{pmatrix} x \\ y \end{pmatrix}\in V=\mathbb{C}^2$

Recall the fundamental representation
$$\begin{aligned} & \rho : G\to GL(2,\mathbb{C}) \equiv GL(\mathbb{C}^2) \\ & \rho(g) : \mathbb{C}^2 \to \mathbb{C}^2 \\ & \rho(g) = g \end{aligned}$$

So consider T such that

$$\begin{tikzcd} \mathbb{C}^2 \arrow[r, "T"] \arrow[d, "\rho(g)"'] & P^{(1)}(V) \arrow[d, "\rho^{(1)}(g)"] \\ \mathbb{C}^2 \arrow[r, "T"] & P^{(1)}(V) \end{tikzcd}$$

Consider $\forall\, v \in \mathbb{C}^2$, $v = \begin{pmatrix} x \\ y \end{pmatrix}$, then

$$\rho(g)v = gv = \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix}$$

```
sage: g*X
[a*x + b*y]
[c*x + d*y]
```

For notation, let $U \in G = SU(2)$ s.t. $UU^\dagger = 1$.
Consider $(\rho^{(2j)}(U)(f))(x) = f(U^{-1}x)$, $\forall\, x \in \mathbb{C}^2$.

Choose $f(x,y) = x$. So for $f(x,y) = Ax + By$, $A = 1, B = 0$. Choose $U = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$ so $U^{-1} = \begin{pmatrix} \bar{a} & -b \\ \bar{b} & a \end{pmatrix}$. Then $U^{-1}x = \begin{pmatrix} \bar{a}x - by \\ \bar{b}x + ay \end{pmatrix}$

So

$$(\rho^{(1)}(U)(f))(x) = f(U^{-1}x) = \bar{a}x - by$$
$$(\rho^{(1)}(U)(f))(x) = f(U^{-1}x) = \bar{b}x + ay \text{ for } f(x,y) = y$$

Let $f(x,y) = Ax + By$

$$(\rho^{(1)}(U)(f))(x) = f(U^{-1}x) = (A\bar{a} + B\bar{b})x + (Ba - Ab)y = (\bar{a}x - by)A + (\bar{b}x + ay)B = (A\bar{a} + B\bar{b})x + (Ba - Ab)y$$

which was calculated with the assistance of Sage Math:

```
sage: U_try1 = Matrix( [[a.conjugate(),-b],[b.conjugate(),a ] ] )
sage: f1( U_try1*X).coefficient(x)
A*conjugate(a) + B*conjugate(b)
sage: f1( U_try1*X).coefficient(y)
B*a - A*b
```

Treating $P^{(1)}(\mathbb{C}^2)$ as a vector space, in its matrix formulation, then $f(x,y) = Ax + By \in P^{(1)}(\mathbb{C}^2)$ is treated as $\begin{bmatrix} A \\ B \end{bmatrix}$, then $(\rho^{(1)}(U)f)$ is

$$\Longrightarrow \begin{bmatrix} \bar{a} & \bar{b} \\ -b & a \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} A\bar{a} + B\bar{b} \\ -Ab + Ba \end{bmatrix}$$

so conclude in general that $\rho^{(1)}(U) = (U^\dagger)^T$.

Now, as Kosmann-Schwarzbach (2010) [11] says, on pp. 13, Chapter 2 Representations of Finite Groups, "Two representations (E_1, ρ_1) and (E_2, ρ_2) are equivalent if and only if there is a basis B_1 of E_1 and a basis B_2 of E_2 such that for every $g \in G$, the matrix of $\rho_1(g)$ in the basis B_1 is equal to the matrix of $\rho_2(g)$ in the basis B_2. In particular, if the representations (E_1, ρ_1) and (E_2, ρ_2) are equivalent, then E_1 is isomorphic to E_2." So we need a change of basis between $\rho(U) = U$ and $\rho^{(1)}(U)$. What's the linear transformation T s.t.

$$T^{-1}\rho^{(1)}(U)T = U?$$

By intuition,

$$T = \sigma_x \sigma_z \equiv \sigma_1 \sigma_3$$

where σ_i's are Pauli matrices.

Indeed,

```
sage: Paulimat[3] * Paulimat[1]*U_try*Paulimat[1] * Paulimat[3]
[conjugate(a) conjugate(b)]
[          -b            a]
```

Then $\rho^{(1)}(U) \circ T = TU$, so this $T = \sigma_1\sigma_3$ is an "intertwining operator" between $\rho^{(1)}(U)$ and fundamental representation $\rho(U) = U$, with $T = \begin{bmatrix} & -1 \\ 1 & \end{bmatrix}$, and $T^{-1} = \begin{bmatrix} & 1 \\ -1 & \end{bmatrix}$.

T is an isomorphism between $\mathbb{C}^2$ and $P^{(1)}(\mathbb{C}^2)$. So fundamental representation ρ of $G = SU(2)$ is equivalent to $\rho^{(1)}(U)$ on $P^{(1)}(\mathbb{C}^2)$.

Exercise 23. [8] (Also from Exercise 2.6 of Kosmann-Schwarzbach (201) [11])

Let (E, π) representation of group G.
$\forall\, g \in G$, $\xi \in E^*$, $x \in E$, set $\langle \pi^*(g)(\xi), x\rangle = \langle \xi, \pi(g^{-1})(x)\rangle$
dual (or *contragredient*) of π, $\pi^* : G \to \text{End}(E^*)$, π^* is a representation, since

$$\begin{aligned} \langle \pi^*(gh)(\xi), x\rangle = \langle \xi, \pi((gh)^{-1})(x)\rangle = \langle \xi, \pi(h^{-1}g^{-1})(x)\rangle = \langle \xi, \pi(h^{-1})\pi(g^{-1})(x)\rangle = \langle \xi, \pi(h^{-1})(\pi(g^{-1})(x))\rangle = \\ = \langle \pi^*(h)(\xi), \pi(g^{-1})(x)\rangle = \langle \pi^*(g)\pi^*(h)(\xi), x\rangle \end{aligned}$$

since this is true, $\forall\, x \in E$, $\forall\, \xi \in E^*$, $\pi^*(gh) = \pi^*(g)\pi^*(h)$.
dual π^* of π is a representation.

7.2. **Adjoint Representation.** I will first follow Sec. 7.3 The Adjoint Representation of Ch. 4 Lie Groups and Lie Algebras of Kosmann-Schwarzbach (201) [11]).

The *conjugation action* $\mathcal{C}_g : G \to G$ is defined as

$$\mathcal{C}_g : G \to G$$
$$\mathcal{C}_g : h \mapsto ghg^{-1}$$

So

$$\mathcal{C} : G \to \text{Aut}(G)$$
$$\mathcal{C}g = \mathcal{C}_g$$

Now define the *adjoint action* of g as the differential or push forward of $\mathcal{C}_g$:

$$\text{Ad}_g := D_1\mathcal{C}_g \equiv (\mathcal{C}_g)_{*1} \equiv (\mathcal{C}_g)_*|_{g=1} \qquad \text{(adjoint action of } g)$$

Now $\text{Ad}_g : \mathfrak{g} \to \mathfrak{g}$, so $\text{Ad} : G \to \text{End}(\mathfrak{g})$

$$\text{Ad}(g) \equiv \text{Ad}_g$$

Note $\mathcal{C}_{gg'} = \mathcal{C}_g\mathcal{C}_{g'} \equiv \mathcal{C}(gg') = \mathcal{C}(g) \circ \mathcal{C}(g')$ and so

$$\xrightarrow{D_1} \text{Ad}_{gg'} = \text{Ad}_g \circ \text{Ad}_{g'}$$

Kosmann-Schwarzbach (201) [11]) claims, because $\text{Ad}_g = 1_{\mathfrak{g}}$ when $g = 1$,

$\text{Ad} : G \to GL(\mathfrak{g})$ is a representation of G on $\mathfrak{g}$. (EY : 20160505 ???)
$\text{Ad} : g \mapsto \text{Ad}_g$

Definition 24. *representation Ad of G on $V = \mathfrak{g}$ is called adjoint representation of Lie group G.*

Denote adjoint representation of Lie algebra $\mathfrak{g}$, ad.
By definition, $\text{Ad}_{\exp(tX)} = \exp{(t\text{ad}_X)}$
cf. Prop. 7.8 of Kosmann-Schwarzbach (201) [11])

Proposition 12. (1) *Let A invertible matrix, $A \in$ Lie group G.*
Let X matrix s.t. $X \in \mathfrak{g}$. Then

$$Ad_A(X) = AXA^{-1}$$

(2) *Let $X,Y\in\mathfrak{g}$. Then*
$$ad_X(Y)=[X,Y]$$
(3) *Let $X,Y\in\mathfrak{g}$. Then*
$$ad_{[X,Y]}=[ad_X,ad_Y]$$

Proof. (1) By def., $\forall\,B\in G$, $\mathcal{C}_A(B)=ABA^{-1}$, and thus
$$\mathrm{Ad}_A(X)=\left.\frac{d}{dt}A\exp{(tX)}A^{-1}\right|_{t=0}=AXA^{-1}$$
(2)
$$\mathrm{ad}_X(Y)=\left.\frac{d}{dt}\mathrm{Ad}_{\exp{(tX)}}(Y)\right|_{t=0}=\left.\frac{d}{dt}\exp{(tX)}Y\exp{(tX)}\right|_{t=0}=$$
$$=XY-YX=[X,Y]$$
(3) Use Jacobi identity:
$$[A,[B,C]]+[B,[C,A]]+[C,[A,B]]=0\text{ or}$$
$$[[A,B],C]=[A,[B,C]]-[B,[A,C]]$$
$$\mathrm{ad}_{[X,Y]}C=[[X,Y],C]=[X,[Y,C]]-[Y,[X,C]]=[X,\mathrm{ad}_YC]-[Y,\mathrm{ad}_XC]\text{ and that}$$
$$\mathrm{ad}_X\mathrm{ad}_YC=[X,[Y,C]]\Longrightarrow\mathrm{ad}_{[X,Y]}C=[\mathrm{ad}_X,\mathrm{ad}_Y]C$$

□

Part 5. Cohomology; Stoke's Theorem

8. Stoke's Theorem

Theorem 14 (Stoke's Theorem)**.** *Let M be oriented, smooth n-manifold with boundary, let ω be a compactly supported smooth $(n-1)$-form on M, or if $\omega\in A^{n-1}_c(M)$, Then*

(19)
$$\int_M d\omega=\int_{\partial M}\omega$$

If $\partial M=\emptyset$, then $\int_{\partial M}\omega=0$
$\int_{\partial M}\omega$ interpreted as $\int_{\partial M}i^*_{\partial M}\omega=\int_{\partial M}i^*\omega$ so

(20)
$$\int_M d\omega=\int_{\partial M}i^*(\omega)$$

where inclusion $i:\partial M\hookrightarrow M$

Proof. Begin with very special case:
Suppose $M=\mathbb{H}^n$ (upper half space), $\partial M=\mathbb{R}^{n-1}$
ω has compact support, so $\exists\,R>0$ s.t. $\mathrm{supp}\omega\subseteq$ rectangle $A=[-R,R]\times\dots\times[-R,R]\times[0,R]$.
$\forall\,\omega\in A^{n-1}_c(\mathbb{H}^n)$

(21)
$$\omega=\sum_{j=1}^n(-1)^{j-1}f_jdx^1\wedge\dots\wedge\widehat{dx}^j\wedge\dots\wedge dx^n\equiv\sum_{i=1}^n\omega_idx^1\wedge\dots\wedge\widehat{dx}^i\wedge\dots\wedge dx^n$$

with Conlon (2008) [16] and John Lee (2012) [3]'s notation, respectively, and where f_j has compact support.
$$i^*\omega=(f_1\circ i)dx^2\wedge\dots\wedge dx^n\in A^{n-1}_c(\partial\mathbb{H}^n)$$

$$d\omega=\sum_{i=1}^n d\omega_i\wedge dx^1\wedge\dots\wedge\widehat{dx}^i\wedge\dots\wedge dx^n=\sum_{i,j=1}^n\frac{\partial\omega_i}{\partial x^j}dx^j\wedge dx^1\wedge\dots\wedge\widehat{dx}^i\wedge\dots\wedge dx^n=$$
$$=\sum_{i=1}^n(-1)^{i-1}\frac{\partial\omega_i}{\partial x^i}dx^1\wedge\dots\wedge dx^n$$
i.e. (for another notation)
$$d\omega=\left(\sum_{j=1}^n\frac{\partial f_j}{\partial x^j}\right)dx^1\wedge\dots\wedge dx^n\in A^n_c(\mathbb{H}^n)$$
$$d\omega=\left(\sum_{j=1}^n\frac{\partial f_j}{\partial x^j}\right)dx^1\wedge\dots\wedge dx^n\in A^n_c(\mathbb{H}^n)$$
$$\int_{\mathbb{H}^n}d\omega=\sum_{i=1}^n(-1)^{i-1}\int_A\frac{\partial\omega_i}{\partial x^i}dx^1\wedge\dots\wedge dx^n=\sum_{i=1}^n(-1)^{i-1}\int_0^R\int_{-R}^R\dots\int_{-R}^R dx^1\dots dx^n\frac{\partial\omega_i}{\partial x^i}(x)$$
We can change order of integration in each term so to do x^i integration first.
By fundamental thm. of calculus, terms for which $i\neq n$ reduce to
$$\sum_{i=1}^{n-1}(-1)^{i-1}\int_0^R\int_{-R}^R\dots\int_{-R}^R\frac{\partial\omega_i}{\partial x^i}(x)dx^1\dots dx^n=\sum_{i=1}^{n-1}(-1)^{i-1}\int_0^R\int_{-R}^R\dots\int_{-R}^R\frac{\partial\omega_i}{\partial x^i}(x)dx^idx^1\dots\widehat{dx}^i\dots dx^n=$$
$$=\sum_{i=1}^{n-1}(-1)^{i-1}\int_0^R\int_{-R}^R\dots\int_{-R}^R[\omega_i(x)]^{x^i=R}_{x^i=-R}dx^1\dots\widehat{dx}^i\dots dx^n=0$$
because we've chosen R large enough that $\omega=0$ when $x^i=\pm R$.

□

Part 6. Prástaro

Prástaro (1996) [12]

8.0.1. *Affine Spaces.* cf. Sec. 1.2 - *Affine Spaces* of Prástaro (1996) [12]

Definition 25 (affine space)**.**

(22)
$$\begin{gathered}\textit{affine space}\qquad(M,\mathbf{M},\alpha)\\ \textit{with}\\ M\equiv\ \textit{set (set of pts.)}\\ \mathbf{M}\equiv\ \textit{vector space (space of free vectors)}\\ \alpha\equiv\mathbf{M}\times M\to M\equiv\ \textit{translation operator}\\ \alpha:(v,p)\mapsto p'\equiv p+v\end{gathered}$$

*Note: α is a **transitive** action and without fixed pts. (free).*
i.e. $\forall\,p\in M$,

$\forall$ pt. $O\in M$, $\alpha:(v,O)\mapsto O'\equiv O+v$, $\alpha(\cdot,O)\equiv\alpha_O\equiv\alpha(O)$. $\alpha_O(v)=O'=O+\mathbf{v}$ $\quad\forall\,O'\in M$, $\exists\,\mathbf{v}\in\mathbf{M}$ s.t. $O'=O+\mathbf{v}$ $\Longrightarrow M\equiv\mathbf{M}$.
$\forall\,(O,\{e_i\})_{1\leq i\leq n}$, where $\{e_i\}$ basis of $\mathbf{M}$, $M\equiv\mathbf{M}=\mathbb{R}^n$ so isomorphism $M\simeq\mathbb{R}^n$

i.e. α is **without fixed pts.**, meaning,
Given pointed space (M,O), where base pt. $O\in M$, we can associate $\forall\,p\in M$, vector $\mathbf{x}\in\mathbf{M}$, by 1-to-1 mapping $M\to\mathbf{M}$.

So for

$$\begin{aligned} &\alpha : \mathbf{M} \times M \to M \\ &\alpha(\mathbf{x}, p) = p' = p + \mathbf{x} \end{aligned}$$

Consider

$$\alpha(\mathbf{x}, O) = p = \alpha_O(\mathbf{x}) = p \Longrightarrow \exists\, \alpha_O^{-1}(p) = \mathbf{x} \in \mathbf{M}$$

(1) tangent space of M in $p \in M$ is vector space $T_pM \equiv (\mathbf{M}, p) \cong M$
(2) If $\mathbf{M}$ Euclidean space, affine space $(M, \mathbf{M}, \alpha)$ is Euclidean
(3) Call dim. of affine space $(M, \mathbf{M}, \alpha)$, dim. of $\mathbf{M} \equiv \text{dim}\mathbf{M}$

$\{\mathbf{e}_i\}$ basis of $\mathbf{M}$

Definition 26. $(O, \{e_i\}) \equiv$ *affine frame.*
$\forall$ *affine frame* $(O, \{e_i\})$, $\exists$ *coordinate system* $x^\alpha : M \to \mathbb{R}$,
where $x^\alpha(p)$ *is* α*th component, in basis* $\{e_i\}$, *of vector* $p - O$

Proposition 13 (1.6, Prástaro (1996) [12]). $\forall\, O \in M$, *we have canonical identification* $M \equiv \mathbf{M}$, *since*

$$\begin{aligned} &\alpha_O^{-1} : M \to \mathbf{M} && \alpha_O : \mathbf{M} \to M \\ &\alpha_O^{-1}(p) = \mathbf{x} && \alpha_O : \mathbf{x} = \alpha(\mathbf{x}, O) = p \end{aligned}$$

Furthermore,
$\forall$ ***affine frame*** $(O, \{\mathbf{e}_i\})_{1 \leq i \leq d}$, *where* $\{\mathbf{e}_i\}$ *basis of* $\mathbf{M}$,
$\exists$ *isomorphism* $M \cong \mathbb{R}^d$,
Then, $\forall\, (O, \{\mathbf{e}_i\})_{1 \leq i \leq d}$,
$\exists$ *coordinate system* $x^\alpha : M \to \mathbb{R}$,
where $x^\alpha(p) = \alpha$*th component, in basis* $\{\mathbf{e}_i\}$, *of vector* $p - O$.

Theorem 15 (1.4 Prástaro (1996) [12]). *Let* $(x^\alpha), (\overline{a}^\alpha)$ *2 coordinate systems correspond to affine frames* $(O, \{e_i\})$, $(\overline{O}, \{\overline{e}_i\})$, *respectively.*

$$\overline{x}^\alpha = A^\alpha_{\ \beta} x^\beta + y^\alpha \tag{23}$$

where

$$y^\alpha \in \mathbb{R}^n, \qquad A^\alpha_{\ \beta} \in GL(n; \mathbb{R})$$

Definition 27 (1.10 Prástaro (1996) [12]).

$$A(n) \equiv Gl(n, \mathbb{R}) \times \mathbb{R}^n \tag{24}$$

affine group of dim. n

Theorem 16 (1.5). *symmetry group of* n*-dim. affine space, called affine group* $A(M)$ *of* M. $\exists$ *isomoprhism,*

$$A(M) \simeq A(n), \qquad f \mapsto (f^\alpha_{\ \beta}, y^\alpha)\,; \qquad f^\alpha \equiv x^\alpha \circ f = f^\alpha_{\ \beta} x^\beta + y^\alpha \tag{25}$$

cf. Eq. 1.4 Prástaro (1996) [12]

Definition 28 (metric). *Let smooth manifold* M, $dimM = n$, $\forall\, p \in M$, $\exists$ *vector space* T_pM, *and so for*

$$\begin{aligned} &g_p (T_pM)^2 \to \mathbb{R} \\ &g_p : (X_p, Y_p) \mapsto g_p(X_p, Y_p) \in \mathbb{R} \end{aligned} \tag{26}$$

with g_p *being bilinear, symmetric (in* X_p, Y_p*), nondegenerate (i.e. if* $g_p(X_p, Y_p) = 0$*, then* X_p *or* $Y_p = 0$*)*
Note that

$$g \in \Gamma((TM \otimes TM)^*)$$

and that for $X = X^i \dfrac{\partial}{\partial x^i}$ *so*

$$Y = Y^i \frac{\partial}{\partial x^i}$$

$$g(X, Y) = g_{ij} X^i Y^j$$

Now for

$$\begin{aligned} &F : M \to N && DF \equiv F_* : T_pM \to T_{F(p)}N \\ &F : x \mapsto y = y(x) && DF : X_p \mapsto (DF)(X^j \frac{\partial}{\partial x^j}) = X^j \frac{\partial y^i}{\partial x^j} \frac{\partial}{\partial y^i} \end{aligned}$$

$$(F^* g')(X, Y) = (F^* g')(X^i \frac{\partial}{\partial x^i}, Y^j \frac{\partial}{\partial x^j}) = (F^* g')_{ij} X^i Y^j = g'(F_* X, F_* Y) = \\ =$$

Part 7. Holonomy

Definition 29 (Conlon, 10.1.2). *If* $X, Y \in \mathfrak{X}(M)$, $M \subset \mathbb{R}^m$, ***Levi-Civita connection*** *on* $M \subset \mathbb{R}^m$

$$\begin{aligned} &\nabla : \mathfrak{X}(M) : \mathfrak{X}(M) \to \mathfrak{X}(M) \\ &\nabla_X Y := p(D_X Y) \end{aligned} \tag{27}$$

with

$$D_X Y := \sum_{j=1}^m X(Y^j) \frac{\partial}{\partial x^j} = \sum_{i,j=1}^m X^i \frac{\partial Y^j}{\partial x^i} \frac{\partial}{\partial x^j} \qquad \begin{aligned} &\forall\, X = \sum_{i=1}^m X^i \frac{\partial}{\partial x^i}, \\ &\forall\, Y = \sum_{i=1}^m Y^i \frac{\partial}{\partial x^i} \end{aligned}$$

$$\nabla_{fX} Y = f(D_{fX} Y) = p(f D_X Y) = f p D_X Y = f \nabla_X Y$$

$$\nabla_X f Y = p(D_X f Y) = p \left(\sum_{i,j=1}^m \left(X^i f \frac{\partial Y^j}{dx^i} + X^i Y^j \frac{\partial f}{\partial x^i} \right) \frac{\partial}{\partial x^j} \right) = f \nabla_X Y + p \sum_{j=1}^m X(f) Y^j \frac{\partial}{\partial x^j} = f \nabla_X Y + X(f) p(Y)$$

Definition 30 (Conlon, 10.1.4; Christoffel symbols).

$$\begin{aligned} &\nabla_{\frac{\partial}{\partial x^i}} \tfrac{\partial}{\partial x^j} = \Gamma^k_{ij} \tfrac{\partial}{\partial x^k} && \textit{(Conlon's notation)} \\ &\nabla_{\frac{\partial}{\partial x^j}} \tfrac{\partial}{\partial x^i} = \Gamma^k_{ij} \tfrac{\partial}{\partial x^k} && \textit{(F. Schuller's notation)} \end{aligned} \tag{28}$$

Definition 31 (torsion).

$$\begin{aligned} &T : \mathfrak{X}(M) \in \mathfrak{X}(M) \to \mathfrak{X}(M) \\ &T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] \end{aligned} \tag{29}$$

If $T = 0$, ∇ *torsion-free or symmetric.*

$$\begin{aligned} T(fX, Y) &= f \nabla_X Y - (f \nabla_Y X + Y(f) X) - \{(fXY - (Y(f) X + fYX)\} = fT(X, Y) \\ T(X, fY) &= f \nabla_X Y + X(f) Y - f \nabla_Y X - \{((X(f) Y + fXY) - fYX\} = fT(X, Y) \end{aligned}$$

Thus, $T(X, Y)$ $C^\infty(M)$-bilinear.
$T \in \tau^2_1(M)$.
$T(v, w) \in T_xM$ defined, $\forall\, v, w \in T_xM$, $\forall\, x \in M$.

Thus, torsion is a **tensor**.

Exercise 10.1.7 Conlon (2008)[16] . .
If $T(X,Y)=0$,
$$T(e_i,e_j) = \Gamma^k_{ji}e_k - \Gamma^k_{ij}e_k - 0 = 0 \Longrightarrow \Gamma^k_{ji}=\Gamma^k_{ij}$$
If $\Gamma^k_{ij}=\Gamma^k_{ji}$, $T(e_i,e_j)=0$.

Exercise 10.1.8, Conlon (2008)[16].
If $M\subset \mathbb{R}^m$ smoothly embedded submanifold, $\forall\, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^i} \in T_xM$, spanning T_xM, consider $\frac{\partial}{\partial x^j} = X^k_j \frac{\partial}{\partial \widetilde{x}^k}$, $\frac{\partial}{\partial x^i} = X_i^k(\widetilde{x})\frac{\partial}{\partial\widetilde{x}^k}$

$$\nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i} = pD_{X^k_j \frac{\partial}{\partial\widetilde{x}^k}} X^l_i \frac{\partial}{\partial \widetilde{x}^l} = p\left(X^k_j \frac{\partial X^l_i}{\partial\widetilde{x}^k}\frac{\partial}{\partial\widetilde{x}^l}\right) = X^k_j p \left(\frac{\partial X^l_i}{\partial \widetilde{x}^k}\frac{\partial}{\partial\widetilde{x}^l}\right)$$
$$\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j} = X^k_i p\left(\frac{\partial X^l_j}{\partial\widetilde{x}^k}\frac{\partial}{\partial\widetilde{x}^l}\right)$$

If $X\in\mathfrak{X}(M)$, smooth $s:[a,b]\to M$,
then $\forall\, s(t)$,
$$X'_{s(t)} = \nabla_{\dot{s}(t)}X \in T_{s(t)}M$$
In fact, it's often natural to consider fields $X_{s(t)}$ along s, parametrized by parameter t, allowing
$$X_{s(t_1)} \neq X_{s(t_2)}$$
each of $s(t_1)=s(t_2)$.

Definition 32 (10.1.9)**.** *Let smooth $s:[a,b]\to M$.*
Vector field along s is smooth $v:[a,b]\to TM$ s.t.

$$\begin{tikzcd}
 & TM \arrow[d, "\pi"] \\
{[a,b]} \arrow[ur, mapsto, "v"] \arrow[r, mapsto, "s"] & M
\end{tikzcd}$$

commutes.
Note that $v\in\mathfrak{X}(s)\subset\mathfrak{X}(M)$

e.g. $(Y|s)(t) = Y_{s(t)}$, restriction of $Y\in\mathfrak{X}(M)$ to s.
e.g. $\dot{s}(t)\in\mathfrak{X}(M)$.
$\forall\, v,w\in\mathfrak{X}(s)$, $v+w\in\mathfrak{X}(s)$,
$$(fv+gv)(t) := (f(s(t))+g(s(t)))v(t) = f(s(t))v(t)+g(s(t))v(t) = (f+g)v(t)$$
Likewise,
$$f(v+w) = fv+fw$$
$\mathfrak{X}(s)$ is a real vector space and $C^\infty[a,b]$-module.

Definition 33 (10.1.10)**.** *Let conection ∇ on M.*
Associated covariant derivative *is operator*
$$\frac{\nabla}{dt}\mathfrak{X}(s)\to\mathfrak{X}(s)$$
$\forall$ *smooth s on M, s.t.*

(1) $\frac{\nabla}{dt}$ $\mathbb{R}$*-linear*
(2) $\left(\frac{\nabla}{dt}\right)(fv) = \frac{df}{dt}v + f\frac{\nabla}{dt}v$, $\forall\, f\in C^\infty[a,b]$, $\forall\, v\in\mathfrak{X}(s)$
(3) *If $Y\in\mathfrak{X}(M)$, then*
$$\frac{\nabla}{dt}(Y|s)(t) = \nabla_{\dot{s}(t)}Y \in T_{s(t)}M, \quad a\leq t\leq b$$

Theorem 17 (Conlon Thm. 10.1.11[16])**.** $\forall$ *connection ∇ on M, $\exists!$ associated covariant derivative $\frac{\nabla}{dt}$*

Proof. Consider arbitrary coordinate chart $(U,x^1\dots x^n)$.
Consider smooth curve $s:[a,b]\to U$.
Let $v\in\mathfrak{X}(s)$, $v(t) = v^i(t)\frac{\partial}{\partial x^i}$; $\dot{s}(t) = s^j\frac{\partial}{\partial x^j}$.
$$\frac{\nabla v}{dt} = \frac{dv^i(t)}{dt}\frac{\partial}{\partial x^i} + v^i(t)\frac{\nabla}{dt}\frac{\partial}{\partial x^i} = \frac{dv^i}{dt}\frac{\partial}{\partial x^i} + v^i\nabla_{\dot{s}(t)}\frac{\partial}{\partial x^i} = \dot{v}^i\frac{\partial}{\partial x^i} + v^i\dot{s}^j\Gamma^k_{ij}\frac{\partial}{\partial x^k} = \left(\dot{v}^k + v^i\dot{s}^j\Gamma^k_{ij}\right)\frac{\partial}{\partial x^k}$$
This is an explicit, local formula in terms of connection, proving uniqueness.
Existence: $\forall$ coordinate chart $(U,x^1\dots x^n)$, $\left(\dot{v}^k + v^i\dot{s}^j\Gamma^k_{ij}\right)\frac{\partial}{\partial x^k} =: \frac{\nabla v}{dt}$.
$$\frac{\nabla}{dt}(fv) = \dot{f}v^k + f\dot{v}^k + fv^i\dot{s}^j = \dot{f}v + f\frac{\nabla v}{dt}$$
If f constant, then $\frac{\nabla}{dt}$ is $\mathbb{R}$-linear. $\square$

Definition 34 (10.1.12 Conlon (2008)[16])**.** *Let (M,∇). Let $v\in\mathfrak{X}(s)$ for smooth $s:[a,b]\to M$.*
If $\frac{\nabla v}{dt}\equiv 0$ on s, then v is ***parallel*** *along s.*

Theorem 18 (10.1.13)**.** *Let (M,∇), smooth $s:[a,b]\to M$, $c\in[a,b]$, $v_0\in T_{s(c)}M$.*
Then $\exists!$ parallel field $v\in\mathfrak{X}(s)$ s.t. $v(c)=v_0$.
v parallel transport along s.

Proof.
$$\begin{aligned}
\dot{s}(t) &= \dot{s}^j(t)e_j \\
v(t) &= v^i(t)e_i \\
v_0 &= a^ie_i
\end{aligned}$$
$$0 = \left(\frac{dv^k}{dt}(t) + v^i(t)\dot{s}^j(t)\Gamma^k_{ij}(s(t))\right)e_k$$
or equivalently

(30)
$$\frac{dv^k}{dt} = -v^i\dot{s}^j\Gamma^k_{ij}, \qquad 1\leq k\leq n \qquad (10.1)$$

with initial conditions $v^k(c) = a^k$, $1\leq k\leq n$.
By existence and uniqueness of solutions of O.D.E.
$\exists\,\epsilon>0$ s.t. $\exists!$ solutions $v^k(t)$. For $c-\epsilon<t<c+\epsilon$.
In fact, these ODEs being linear in v^k, by ODE theory (Appendix C, Thm. C.4.1).
$\nexists$ restriction on ϵ, so $\exists!\,v^k(t)\quad\forall\, t\in[a,b]$, $1\leq k\leq n$ $\square$

8.0.2. *Principal bundle, vector bundle case for parallel transport.* Recall the 2 different forms or viewpoints for Lie-algebra valued 1-forms, or vector-valued 1-forms, or sections of 1-form-valued endomorphisms:

$$\omega^k_{\ i\mu}dx^\mu \equiv \omega^k_{\ i} \in \Omega^1(M, \mathfrak{gl}(n,\mathbb{F})) = \Gamma(\mathfrak{gl}(n, \mathbb{R}\otimes T^*M|_U))$$

for $i,k = 1\dots n = \text{dim}E$.

$\mu = 1\dots d = \text{dim}E$

Now

$$D_X\mu = X^\mu D_{\frac{\partial}{\partial x^\mu}}\mu = X^\mu\left[\left(\frac{\partial}{\partial x^\mu}\mu^k\right)e_k + \mu^i\omega^k_{\ i\mu}e_k\right] = \left(X(\mu^k) + \mu^i\omega^k_{\ i}(X)\right)e_k = \left(d\mu^k(X) + \mu^i\omega^k_{\ i}(X)\right)e_k$$

So then define

$$\begin{aligned} &D : \Gamma(E) \to \Gamma(E)\otimes\Gamma(T^*M) \\ &D\mu = D(\mu^i e_i) = e_k(d\mu^k + \mu^i\omega^k_{\ i}) \equiv (d+A)\mu \end{aligned} \tag{31}$$

Also, D can be defined for this case:

$$D : \Gamma(\text{End}(E)) \to \Gamma(\text{End}E)\otimes\Gamma(T^*M)$$

Let $\sigma = \sigma^i_{\ j}e_i\otimes e^j \in \Gamma(\text{End}(E))$

$$\begin{aligned} D\sigma &= D(\sigma^i_{\ j}e_i)\otimes e^j + \sigma^i_{\ j}e_i\otimes D^*e^j = \left(d\sigma^k_{\ j} + \sigma^i A^k_{\ i}\right)e_k\otimes e^j + \sigma^i_{\ j}e_i\otimes(A^*)^{\ j}_k e^k = \\ &= (d\sigma^k_{\ j} + \sigma^i_{\ j}A^k_{\ i})e_k\otimes e^j + \sigma^k_{\ i}e_j\otimes(-A^i_{\ j})e^j = (d\sigma^k_{\ j} + [A,\sigma]^k_{\ j})e_k\otimes e^j \end{aligned} \tag{32}$$

cf. Def. 4.1.4 of Jost (2011), pp. 138.

For $\mu\in\Gamma(E)$, smooth $s:[a,b]\to M$, $X(t) = \dot{s}(t)$,

$$D_{\dot{s}(t)}\mu = \dot{s}^\mu D_{\frac{\partial}{\partial x^\mu}}\mu = \dot{s}^\mu\left[\frac{\partial\mu^k}{\partial x^\mu}e_k + \mu^i\omega^k_{\ i\mu}e_k\right] = \left[\dot{s}^\mu\frac{\partial\mu^k}{\partial x^\mu} + \dot{s}^\mu\mu^i\omega^k_{\ i\mu}\right]e_k = \frac{d}{dt}\mu(s(t)) + \mu^i\dot{s}^\mu\omega^k_{\ i\mu}e_k \tag{33}$$

Let $D_{\dot{s}(t)}\mu = 0$. Then,

$$\frac{d}{dt}\mu(s(t)) = -\mu^i\dot{s}^\mu\omega^k_{\ i\mu}e_k \tag{34}$$

Recall, given vector bundle $E\xrightarrow{\pi}N$, given $\varphi : M\to N$, then pullback

$$\varphi^*E \to M \tag{35}$$

i.e.

$$\begin{tikzcd} \varphi^*E \arrow[d, "\psi"] & E \arrow[l, "\varphi^*"'] \arrow[d, "\pi"] & (\varphi^*E)_x = E_{\varphi(x)} \\ M \arrow[r, "\varphi"] & N & x\in M \arrow[u, mapsto] \end{tikzcd}$$

i.e. if $s\in\Gamma(E)$,

$$\varphi^*s = s\circ\varphi \in \Gamma(\varphi^*E)$$

Thus,

$$\begin{tikzcd} \gamma^*E \arrow[d] & E \arrow[l, "\gamma^*"'] \arrow[d, "\pi"] & (\varphi^*E)_c = E_{\gamma(c)} \\ {[a,b]} \arrow[r, "\gamma"] & M & c\in[a,b] \arrow[u, mapsto] \end{tikzcd}$$

For

$$\begin{gathered} \dot{v}^k = -v^i\dot{s}^j\Gamma^k_{ij} \\ v^k(c) = v^k_0 \qquad 1\leq k\leq m \\ \dot{v} = -v^i\dot{s}^j\Gamma_{ij} \\ (v\dot{+}w) = -(v^i+w^i)\dot{s}^j\Gamma_{ij}(v+w)(c) = v(c)+w(c) = v_0+w_0 \end{gathered}$$

so $v+w\in\mathfrak{X}(s)$ is parallel transport of v_0+w_0.

Likewise, $\forall\, a\in\mathbb{F}$, $av\in\mathfrak{X}(s)$ is the parallel transport of av_0.

$$\dot{\mu}^k = -\mu^i\dot{s}^\mu\omega^k_{\ i\mu} = -\mu^i\omega^k_{\ i}(\dot{s}^\mu)$$

Suppose γ^*E trivialized over $[a,b]$.

Closed interval is contractible, so this is always possible.

For chart (U,φ),

$$\begin{tikzcd} \gamma^*E \arrow[d] & E \arrow[l, "\gamma^*"'] \arrow[d, "\pi"] & E|_U \arrow[r, "\psi"] & U\times V \\ {[a,b]} \arrow[r, "\gamma"] & M & U\subset M \arrow[u, "\pi^{-1}"] \arrow[ur] & \end{tikzcd}$$

Consider

$$\begin{gathered} \varphi : [a,b]\times V \to \gamma^*E \\ \varphi(t,\cdot) = \gamma^*\circ\psi^{-1}(\gamma(t),\cdot) \end{gathered}$$

$\forall\,\mu\in\Gamma(E|_{x\in M})$,

$\mu = \mu^i e_i$.

$\varphi(t,e_i) = \epsilon_i$ is a basis for γ^*E.

$\forall\,\sigma\in\Gamma(\gamma^*E)$,

$$\begin{gathered} \sigma = \sigma^i\epsilon_i, \quad \sigma^i : [a,b]\to\mathbb{F} \\ \nabla_{\frac{\partial}{\partial x^\mu}}\sigma = \frac{\partial\sigma^k}{\partial x^\mu}\epsilon_k + \omega^k_{\ j\mu}\sigma^j\epsilon_k = \left(\frac{\partial\sigma^k}{\partial x^\mu} + \omega^k_{\ ,j\mu}\sigma^j\right)\epsilon_k \\ \nabla\sigma = \epsilon_k\otimes(d\sigma^k + \omega^k_{\ j\mu}dx^\mu\sigma^j) = \epsilon_k\otimes(d\sigma^k + \omega^k_{\ j}\sigma^j) \\ \nabla_{\frac{d}{dt}}\sigma = \epsilon_k\otimes\left(\frac{d\sigma^k}{dt} + \omega^k_{\ j\mu}\dot{x}^\mu\sigma^j\right) \end{gathered}$$

Now

$$\frac{d}{dt} = \dot{x}^\nu\frac{\partial}{\partial x^\nu}$$

Then σ parallel along γ if

$$\frac{d\sigma^k}{dt} + \omega^k_{\ j\mu}\dot{x}^\mu\sigma^j = 0$$

Definition 35 (3.1.4 [17])**.** *Parallel transport along γ is*

$$P_\gamma : E_{\gamma(a)} \to E_{\gamma(b)} \qquad (36)$$
$$P_\gamma(v) \mapsto \sigma(b)$$

where $\sigma \in \Gamma(\gamma^* E)$, σ unique and s.t. $\sigma(a) = v$.

Lemma 1 (10.1.16[16])**.** *holonomy*

$$h_s : T_x M \to T_{x_0} M$$

if ∇ around piecewise smooth loop s is a linear transformation.

Lemma 2 (10.1.18 Conlon (2008)[16])**.** *Let piecewise smooth loop $s : [a,b] \to M$ at x_0.*
Let weak reparametrization $\widetilde{s} = s \circ r : [c,d] \to M$.
If reparametrization is orientation-preserving, then $h_{\widetilde{s}} = h_s$,
If reparametrization is orientation-reversing, then $h_{\widetilde{s}} = h_s^{-1}$,

Proof. Without loss of generality, assume smooth s, r

$$\widetilde{s}(\tau) = s(r(\tau))$$
$$\widetilde{v}(\tau) = v(r(\tau))$$

$$\widetilde{u}^j(\tau) = \frac{dt}{d\tau}(\tau) u^j(r(\tau))$$
$$\frac{d\widetilde{v}^k}{d\tau}(\tau) = \frac{dr}{d\tau}(\tau)\frac{dv^k}{dt}(r(\tau))$$
$$\frac{d\widetilde{v}^k}{d\tau} = -\widetilde{v}^i \widetilde{u}^j \Gamma^k_{ij}$$

since

$$\frac{dv^k}{dt} = -v^i u^j \Gamma^k_{ij}; \qquad 1 \leq k \leq n$$
$$v^k(c) = a^k; \qquad 1 \leq k \leq a$$
$$\frac{dr}{d\tau}\frac{dv^k}{dt} = -v^i \frac{dr}{d\tau} u^j \Gamma^k_{\ ij} = \frac{d\widetilde{v}^k}{d\tau} = -\widetilde{v}^i \widetilde{u}^j \Gamma^k_{\ ij}$$

Thus, if $r(c) = a$, $r(d) = b$

$$h_{\widetilde{s}}(v_0) = \widetilde{v}(d) = v(b) = h_s(v_0)$$

If $r(c) = a$, $r(d) = b$, then

$$\widetilde{v}(c) = v(b) = h_s(v_0)$$

and

$$h_{\widetilde{s}}(h_s(v_0)) = h_{\widetilde{s}}(v(b)) = \widetilde{v}(d) = v(a) = v_0$$

At this point, I will switch to my notation because it clarified to me, at least, what was going on, in that a holonomy h_s is *invariant* under orientation-preserving reparametrization, and its inverse is well-defined.
For $\widetilde{s} = s \circ t : [c,d] \to M$,
piecewise smooth t is reparametrized, i.e.

$$t : [c,d] \to [a,b] \qquad (37)$$

Now,

$$\frac{d}{d\tau}\widetilde{s}(\tau) = \frac{d}{d\tau}\widetilde{s}(t(\tau)) = \dot{s}(t)\frac{dt}{d\tau}(\tau) \equiv \dot{s}\frac{dt}{d\tau}$$
$$v^k(t) = v^k(t(\tau)) = v^k(\tau)$$
$$\frac{dv^k}{d\tau}(t(\tau)) = \frac{dv^k}{dt}\frac{dt}{d\tau} = \frac{dt}{d\tau}(-v^i(\tau)\dot{s}^j(t)\Gamma^k_{\ ij}) = -v^i(\tau)\frac{d\widetilde{s}^j}{d\tau}\Gamma^k_{\ ij}$$

Consider

$$h_s(v_0) = v(b)$$

If $t(c) = a$,
$t(d) = b$

$$h_{\widetilde{s}}(v_0) = \widetilde{v}(d) = v(t(d)) = v(b) = h_s(v_0)$$

If $t(c) = b$,
$t(d) = a$

$$h_{\widetilde{s}}(h_s(v_0)) = h_{\widetilde{s}}(v(b)) = h_{\widetilde{s}}(v(t(c))) = h_{\widetilde{s}}(\widetilde{v}(c)) =$$
$$= \widetilde{v}(d) = v(t(d)) = v(a) = v_0$$

Thus,

$$\boxed{h_{\widetilde{s}} = h_s^{-1}}$$

□

I am working through Conlon (2008) [16] , Clarke and Santoro (2012) [17], and Schreiber and Waldorf (2007)[18], concurrently, for holonomy.

9. Path Groupoid of a smooth manifold; generalization of paths

cf. Schreiber and Waldorf (2007)[18].

Definition 36 (path)**.** ***path** is a smooth map $\gamma : [0,1] \to M$, between 2 pts. $x, y \in M$, which has a sitting instant; i.e. number $0 < \epsilon < \frac{1}{2}$ s.t.*

$$\gamma(t) = \begin{cases} x & \text{for } 0 \leq t < \epsilon \\ y & \text{for } 1 - \epsilon < t \leq 1 \end{cases} \qquad (38)$$

Denote the set of such paths by PM,

$$PM \equiv \{ \gamma \in \Gamma(M) | \text{ smooth } \gamma : [0,1] \to M \text{ s.t. } \exists 0 < \epsilon < \frac{1}{2} \text{ s.t. } \begin{cases} x & \text{for } 0 \leq t < \epsilon \\ y & \text{for } 1 - \epsilon < t \leq 1 \end{cases} \} \qquad (39)$$

cf. Def. 2.1. of Schreiber and Waldorf (2007)[18]
Define *composition*:

Given paths γ_1, γ_2; $\gamma_1(0) = x$, $\gamma_2(0) = y$,
$\gamma_1(1) = y$ $\gamma_2(1) = z$
define composition to be path

$$\gamma_2 \circ \gamma_1$$
$$(\gamma_2 \circ \gamma_1)(t) := \begin{cases} \gamma_1(2t) & \text{for } 0 \leq t \leq \frac{1}{2} \\ \gamma_2(2t-1) & \text{for } \frac{1}{2} \leq t \leq 1 \end{cases} \qquad (40)$$

$\gamma_2 \circ \gamma_1$ smooth since γ_1, γ_2 both constant near gluing pt., due to sitting instants ϵ_1, ϵ_2, respectively.
Define *inverse*:

$$\gamma^{-1} : [0,1] \to M \qquad (41)$$
$$\gamma^{-1}(t) := \gamma(1-t)$$

(so that $\gamma^1(t) = \begin{cases} y & \text{for } 1-\epsilon < 1-t \leq 1 \text{ or } 0 \leq t < \epsilon \\ x & \text{for } 0 \leq 1-t < \epsilon \text{ or } 1-\epsilon < t \leq 1 \end{cases}$)

Definition 37 (thin homotopy equivalent). *2 paths γ_1, γ_2 s.t. $\begin{gathered}\gamma_1(0)=\gamma_2(0)=x,\\ \gamma_1(1)=\gamma_2(1)=y\end{gathered}$ γ_1,γ_2 are thin homotopy equivalent, if $\exists$ smooth $h:[0,1]\times[0,1]\to M$ s.t.*

(1) $\exists\, 0<\epsilon<\frac{1}{2}$ *with*

(a) $h(s,t)=x$ *for* $0\leq t<\epsilon$
$h(s,t)=y$ *for* $1-\epsilon<t\leq 1$

(b)

(c) $h(s,t)=\gamma_1(t)$ *for* $0\leq s<\epsilon$
$h(s,t)=\gamma_2(t)$ *for* $1-\epsilon<s\leq 1$

(2) *differential of h has at most rank 1 everywhere, i.e.*

$$\mathit{rank}(dh|_{(s,t)}) \leq 1 \qquad \forall (s,t)\in[0,1]\times[0,1] \tag{42}$$

cf. Def. 2.2. of Schreiber and Waldorf (2007)[18]

$\begin{aligned} h(s,t)&=\gamma_1(t) \text{ for } 0\leq s<\epsilon \\ h(s,t)&=\gamma_2(t) \text{ for } 1-\epsilon<s\leq 1\end{aligned}$ is the homotopy from γ_1 to γ_2, i.e. $\begin{aligned}h(0,t)&=\gamma_1(t)\\ h(1,t)&=\gamma_2(t)\end{aligned}$
and define an equivalence relation on PM.

Note that for $h:[0,1]\times[0,1]\to M$,

$$(Dh)|_{(s,t)} = \left[\frac{\partial h^i}{\partial s}, \frac{\partial h^i}{\partial t}\right]$$

$$\begin{gathered} P^1M \equiv \text{ set of thin homotopy classes of paths, i.e.} \\ P^1M = \{[\gamma] | \gamma_1 \in PM, \text{ if } \exists \text{ smooth } h:[0,1]\times[0,1]\to M \text{ s.t. } h \text{ thin homotopy of } \gamma_1 \text{ and } \gamma_2, \gamma_1\sim\gamma_2\}\end{gathered} \tag{43}$$

$\mathrm{pr}: PM \to P^1M$ is projection to classes.

Denote thin homotopy class of path γ, $\begin{aligned}\gamma(0)&=x,\\ \gamma(1)&=y\end{aligned}$ by $\overline{\gamma}$, or $[\gamma]$.

9.1. Reparametrization of thin homotopies.

Let $\beta:[0,1]\to[0,1]$, $\begin{aligned}\beta(0)&=0.\\ \beta(1)&=1\end{aligned}$

Then $\forall$ path γ, $\begin{aligned}\gamma(0)&=x,\\ \gamma(1)&=y\end{aligned}$ $\gamma\circ\beta$ is also a path $\begin{aligned}\gamma\circ\beta(0)&=x\\ \gamma\circ\beta(1)&=y\end{aligned}$ and

$$h(s,t) := \gamma(t\beta(1-s)+\beta(t)\beta(s) \tag{44}$$

defines a homotopy from γ to $\gamma\circ\beta$.

$$\gamma_1\circ\gamma_2 \in PM \xmapsto{\mathrm{pr}} [\gamma_1\circ\gamma_2] = [\gamma_1][\gamma_2] \in P^1M$$

Composition of thin homotopy classes of paths obeys following rules:

Lemma 3. $\forall$ *path* γ, $\begin{aligned}\gamma(0)&=x\\ \gamma(1)&=y\end{aligned}$

(1) $\overline{\gamma}\circ\overline{id_x} = \overline{\gamma} = \overline{id_y}\circ\overline{\gamma} \equiv [\gamma]1_x = [\gamma] = 1_y[\gamma]$

(2) *for paths* γ'; $\begin{aligned}\gamma'(0)&=y, & \gamma''(0)&=z\\ \gamma'(1)&=z & \gamma''(1)&=w\end{aligned}$

$$(\overline{\gamma}''\circ\overline{\gamma}')\circ\overline{\gamma} = \overline{\gamma}''\circ(\overline{\gamma}'\circ\overline{\gamma}) \equiv ([\gamma''][\gamma'])[\gamma] = [\gamma'']([\gamma'][\gamma]) \tag{45}$$

(3) $\overline{\gamma}\circ\overline{\gamma}^{-1} = \overline{id_y}$ *and* $\overline{\gamma^{-1}}\circ\overline{\gamma} = \overline{id_x} \equiv [\gamma][\gamma^{-1}] = 1_y$ *and* $[\gamma^{-1}][\gamma] = 1_x$

cf. Lemma 2.3. of Schreiber and Waldorf (2007)[18]

Definition 38 (path groupoid). *$\forall$ smooth manifold M, consider category whose set of objects is M, whose set of morphisms is P^1M, where class $[\gamma]$, $\begin{aligned}[\gamma](0)&=x\\ [\gamma](1)&=y\end{aligned}$ is a morphism from x to y and composition $[\gamma_1][\gamma_2]=[\gamma_1\circ\gamma_2]\in P^1M$ Lemma 3 are axioms of a category, 3rd. property says $\forall$ morphism is invertible. Hence, we've defined a groupoid, called* ***path groupoid*** *of M, $\mathcal{P}_1(M)$.*

So

$$\begin{aligned}\mathrm{Obj}(\mathcal{P}_1(M)) &= M \\ \mathrm{Mor}(\mathcal{P}_1(M)) &= P^1M\end{aligned}$$

$\forall$ smooth $f:M\to N$, denote functor f_*

$$f_*:\mathcal{P}_1(M)\to\mathcal{P}_1(N) \tag{46}$$

with

$$\begin{aligned} f_*(x) &= f(x) \\ (f_*)([\gamma]) &:= [f\circ\gamma]\end{aligned}$$

If $\gamma\sim\gamma'$, for $f\circ\gamma$, $f\circ\gamma'$,

$f\circ h(s,t)$ with $\begin{aligned} f\circ h(0,t) &= f\circ\gamma(t)\ , \\ f\circ h(1,t) &= f\circ\gamma'(t)\end{aligned}$
so $f\circ h$ is a thin homotopy between $f\circ\gamma$, $f\circ\gamma'$ and so $[f\circ\gamma]$ *well-defined.*

Part 8. Complex Manifolds

EY : 20170123 I don't see many good books on Complex Manifolds for physicists other than Nakahara's. I will supplement this section on Complex Manifolds with external links to the notes of other courses that I found useful to myself.

Complex Manifolds - Lecture Notes Koppensteiner (2010) [13]

Lectures on Riemannian Geometry, Part II: Complex Manifolds by Stefan Vandoren Vandoren (2008) [14]

Part 9. Jets, Jet bundles, h-principle, h-Prinzipien

cf. Eliashberg and Misahchev (2002) [19]
cf. Ch. 1 Jets and Holonomy, Sec. 1.1 Maps and sections of Eliashberg and Misahchev (2002) [19].
Visualize $f : \mathbb{R}^n \to \mathbb{R}^q$ as graph $\Gamma_f \subset \mathbb{R}^n \times \mathbb{R}^q$.

Consider this graph as image of $\mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^q$, i.e.

$$x \mapsto (x, f(x))$$

$\mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^q$ is called section (by mathematicians),
$x \mapsto (x, f(x))$
is called *field* or $\mathbb{R}^q$-valued field (by physicists).
cf. Ch. 1 Jets and Holonomy, Sec. 1.2 Coordinate definition of jets of Eliashberg and Misahchev (2002) [19].

Definition 39 (r-jet)**.** *Given (smooth) $f : \mathbb{R}^n \to \mathbb{R}^q$, given $x \in \mathbb{R}^n$.*
r-jet of f at x - sequence of derivatives of f, up to order r, $\equiv$

$$J^r_f(x) = (f(x), f'(x) \dots f^{(r)}(x)) \tag{47}$$

$f^{(q)}$ consists of all partial derivatives $D^\alpha f$, $\alpha = (\alpha_1 \dots \alpha_n)$, $|\alpha| = \alpha_1 + \dots + \alpha_n = s$, ordered lexicographically.
e.g. $q = 1$,
$f : \mathbb{R}^n \to \mathbb{R}$.
1-jet of f at $x = J^1_f(x) = (f(x), f^{(1)}(x))$.

$$f^{(1)}(x) = \{ D^\alpha f | \alpha = (\alpha_1 \dots \alpha_n), |\alpha| = \alpha_1 + \dots + \alpha_n = 1 \} = \left(\frac{\partial f}{\partial x^1}, \frac{\partial f}{\partial x^2}, \dots \frac{\partial f}{\partial x^n} \right)$$

Let $d_r = d(n, r) =$ number of all partial derivatives D^α of order r of function $\mathbb{R}^n \to \mathbb{R}$.
Consider r-jet $J^r_f(x)$ of map $f : \mathbb{R}^n \to \mathbb{R}^q$ as pt. of space $\mathbb{R}^q \times \mathbb{R}^{qd_1} \times \mathbb{R}^{qd_2} \times \dots \times \mathbb{R}^{qd_r} = \mathbb{R}^{qN_r}$, where $N_r = N(n, r) = 1 + d_1 + d_2 + \dots + d_r$, i.e.

$$J^r_f(x) = (f(x), f^{(1)}(x), \dots f^{(r)}(x)) \in \mathbb{R}^q \times \mathbb{R}^{qd_1} \times \dots \times \mathbb{R}^{qd_r} = \mathbb{R}^{qN_r}$$

Exercise 1.
Given order r, consider n-tuple of (positive) integers $(r_1, r_2 \dots r_n)$ s.t. $r_1 + r_2 + \dots + r_n = r$, and $r_k \geq 0$.
Imagine $r_k =$ occupancy number, num ber of balls in kth cell. $(r_1 \dots r_n)$ describes a positive ocnfiguration of occupancy numbers, with indistinguishable balls; 2 distributions are distinguishable only if corresponding n-tuples $(r_1 \dots r_n)$ not identical.
Represent balls by stars, and indicate n cells by n spaces between $n + 1$ bars.
With $n+1$ bars, r stars, 2 bars are fixed. $n - 1$ bars and r stars to arrange linearly, so a total of $n - 1 + r$ objects to arrange. r stars indistinguishable amongst themselves, so choose r out of $n - 1 + r$ to be stars.

$$\Longrightarrow d_r = d(n, r) = \binom{n - 1 + r}{r} \tag{48}$$

Use *induction* (cf. Ch. 4 Binomial Coefficients).

$$N_0 = N(n, 0) = \binom{n - 1 + 0}{0} = 1$$
$$N_1 = N(n, 1) = 1 + \binom{n - 1 + 1}{1} = 1 + n = \frac{(n + 1)!}{n!1!}$$

Induction step:

$$N_{r-1} = N(n, r - 1) = \sum_{k=1}^{r-1} d_k + 1 = \binom{n + r - 1}{r - 1}$$

and so

$$N_r = N(n, r) = \sum_{k=1}^{r} d_k + 1 = \sum_{k=1}^{r} \binom{n - 1 + k}{k} + 1 = \sum_{k=1}^{r-1} \binom{n - 1 + k}{k} + \binom{n - 1 + r}{r} + 1 =$$
$$= \binom{n + r - 1}{r - 1} + \binom{n - 1 + r}{r} = \frac{(n + r - 1)!}{(r - 1)!n!} + \frac{(n - 1 + r)!}{r!(n - 1)!} = \frac{(n + r)!}{n!r!} = \binom{n + r}{r}$$

$$\begin{tikzcd} \mathbb{R}^{qN_r} & \\ \mathbb{R}^n \arrow[u, "J^r_f"] \arrow[r, "f"] & \mathbb{R}^q \end{tikzcd} \qquad \begin{tikzcd} J^r_f(x) & \\ x \arrow[u, mapsto, "J^r_f"] \arrow[r, mapsto, "f"] & f(x) \end{tikzcd}$$

Definition 40 (space of r-jets)**.** *space of r-jects of maps $\mathbb{R}^n \to \mathbb{R}^q$ or space of r-jets of sections $\mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^q \equiv$*

$$J^r(\mathbb{R}^n, \mathbb{R}^q) = \mathbb{R}^n \times \mathbb{R}^{qN_r} \equiv \mathbb{R}^n \times \mathbb{R}^q \times \mathbb{R}^{qd_1} \times \mathbb{R}^{qd_2} \times \dots \times \mathbb{R}^{qd_r} \tag{49}$$

e.g. $J^1(\mathbb{R}^n, \mathbb{R}^q) = \mathbb{R}^n \times \mathbb{R}^q \times M_{q \times n}$, where $M_{q \times n} = \mathbb{R}^{qn}$ is the space of $(q \times n)$-matrices.

Part 10. Morse Theory

10. Morse Theory introduction from a physicist

I needed some physical motivation to understand Morse theory, and so I looked at Hori, et. al. [15].
cf. pp. 43, Sec. 3.4 Morse Theory, from Ch. 3. Differential and Algebraic Topology of Hori, et. al. [15].
Consider smooth $f : M \to \mathbb{R}$, with non-degenerate critical points.
If no critical values of f between a and b ($a < b$), then subspace on which f takes values less than a is deformation retract of subspace where f less than b, i.e.

$$\{ x \in M | f(x) < b \} \times [0, 1] \xrightarrow{F} \{ x \in M | f(x) < b \}$$

$\forall\, x \in M$ s.t. $f(x) < b$,

$$\begin{aligned} & F(x, 0) = x \\ & F(x, 1) \in \{ x \in M | f(x) < a \} \end{aligned} \qquad \text{and } F(a', 1) = a' \qquad \forall\, a' \in M \text{ s.t. } f(a') < a$$

To show this, consider $-\nabla f / |\nabla f|^2$
Morse lemma: $\forall$ critical pt. p s.t. $\exists$ choice of coordinates s.t.

$$f = -(x_1^2 + x_2^2 + \dots + x_\mu^2) + x_{\mu+1}^2 + \dots + x_n^2 \tag{50}$$

where $f(p) = 0$ and p is at origin of these coordinates.

- difference between
  $$f^{-1}(\{ x \leq -\epsilon \}),\ f^{-1}(\{ x \leq +\epsilon \})$$
  can be determined by local analysis and only depends on μ, $\mu \equiv$ "Morse index" $=$ number of negative eigenvalues of Hessian of f at critical pt.
  Answer:

$$f^{-1}(\{ x \leq +\epsilon \}) \text{ can be obtained from } f^{-1}(\{ x \leq -\epsilon \}) \text{ by "attaching } \mu\text{-cell" along boundary } f^{-1}(0)$$

- "attaching μ-cell to X mean, take μ-ball $B_\mu = \{|x| \leq 1\}$ in μ-dim. space, identity pts. on boundary $S^{\mu-1}$ with pts. in the space X, through cont. $f : S^{\mu-1} \to X$, i.e. take
$$X \coprod B_\mu$$
with $x \sim f(x) \quad \forall\, x \in \partial B_\mu = S^{\mu-1}$.
- find homology of M,
  f defines chain complex C^*_f, kth graded piece C^{α_k}, α_k is number of critical pts. with index k.
$$\begin{gathered} \partial : C^k_p \to C^{k-1}_p \\ \partial x_a = \sum_b \Delta_{a,b} x_b \end{gathered} \tag{51}$$
where $\Delta_{a,b} :=$ signed number of lines of gradient flow from x_a to x_b, b labels pts. of index $k-1$.

  Gradient flow line is path $x(t)$ s.t. $\dot{x} = \nabla(f)$, with $\begin{aligned} x(-\infty) &= x_a \\ x(+\infty) &= x_b \end{aligned}$
- To define this number ($\Delta_{a,b}$?), construct moduli space of such lines of flow (???) by intersecting outward and inward flowing path spaces from each critical point, and then show this moduli space is oriented, 0-dim. manifold (pts. with signs)
- $\partial^2 = 0$ proof
  ∂, boundary of space of paths connecting critical points, whose index differs by $2 =$ union over compositions of paths between critical pts. whose index differs by 1.
  $\implies$ coefficients of ∂^2 are sums of signs of pts. in 0-dim. space, which is boundary of 1-dim. space.
  These signs must therefore add to 0, so $\partial^2 = 0$.

Hori, et. al. [15] is good for physics, but there isn't much thorough, step-by-step explanations of the math. I will look at Hirsch (1997) [6] and Shastri (2011) [5] at the same time.

10.1. Introduction, definitions of Morse Functions, for Morse Theory.

cf. Ch. 6, Morse Theory of Hirsch (1997) [6], Section 1. Morse Functions, pp. 143-

Recall for TM, $T_xM \xrightarrow{\varphi} \mathbb{R}^n$.
Cotangent bundle T^*M defined likewise:
$$T^*_xM \xrightarrow{\varphi} \text{ dual vector space } (\mathbb{R}^n)^* = L(\mathbb{R}^n, \mathbb{R})$$
i.e.
$$T^*M = \bigcup_{x \in M} (M^*_x) \qquad M^*_x = L(M_x, \mathbb{R})$$
If chart (φ, U) on M, natural chart on T^*M is
$$\begin{gathered} T^*U \to \varphi(U) \times (\mathbb{R}^n)^* \\ \lambda \in M^*_x \mapsto (\varphi(x), \lambda\varphi^{-1}_x) \end{gathered}$$

Projection map
$$\begin{gathered} p : T^* \to M \\ M^*_x \mapsto x \end{gathered}$$
Let C^{r+1} map, $1 \leq r \leq \omega$, $f : M \to \mathbb{R}$, $\forall\, x \in M$, linear map $T_xf : M_x \to \mathbb{R}$ belongs to M^*_x
$$T_xf = Df_x \in M^*_x$$
Then
$$\begin{gathered} Df : M \to T^*M \\ x \mapsto Df_x = Df(x) \end{gathered}$$
is C^r section of T^*M.

Definition 41. ***critical point*** *x of f is zero of Df, i.e.*
$$Df(x) = 0 \tag{52}$$
of vector space M^*_x.

Thus, set of critical pts. of f is counter-image of submanifold $Z^* \subset T^*M$ of zeros.
Note $Z^* \approx M$, codim. of Z^* is $n = \dim M$.

Definition 42. ***Morse function*** *f if $\forall$ critical pts. of f are nondegenerate.*

Note set of critical pts. closed discrete subset of M.
Let open $U \subset \mathbb{R}^n$, let C^2 map $g : U \to \mathbb{R}$,
critical pt. $p \in U$ nondegenerate iff

- linear $D(Dg)(p) : \mathbb{R}^n \to (\mathbb{R}^n)^*$ bijective
- identify $L(\mathbb{R}^n, (\mathbb{R}^n)^*)$ with space of bilinear maps $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $\implies$ equivalent to condition that symmetric bilinear $D^2g(p) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ non-degenerate
- *$n \times n$ Hessian matrix*
$$\left[\frac{\partial^2 g}{\partial x^i \partial x^j}(p) \right]$$
  has rank n

Hessian of g at critical pt. p is quadratic form H_pf associated to bilinear form $D^2g(p)$
$$\implies H_pf(y) = D^2g(p)(y,y) = \sum_{i,j} \frac{\partial^2 g}{\partial x^i \partial x^j}(p) y^i y^j$$
Let open $V \subset \mathbb{R}^n$, suppose C^2 diffeomorphism $h : V \to U$.
Let $q = h^{-1}(p)$, so q is critical pt. of $gh : V \to \mathbb{R}$.
$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "H_q(gh)"] \arrow[d, "Dh(q)"'] & \mathbb{R} \\ \mathbb{R}^n \arrow[ur, "H_pg"'] & \end{tikzcd}$$
(quadratic) form (H_pf) invariant under diffeomorphisms.
Let C^2 $f : M \to \mathbb{R}$.
$\forall$ critical pt. x of f, define
Hessian quadratic form
$$\begin{gathered} H_xf : M_x \to \mathbb{R} \\ H_xf : M_x \xrightarrow{D\varphi_x} \mathbb{R}^n \xrightarrow{H_{\varphi(x)}(f\varphi^{-1})} \mathbb{R} \end{gathered}$$
where φ is any chart at x.
Thus, critical pt. of a C^2 real-valued function nondegenerate iff associated Hessian quadratic form is nondegenerate.
Let Q nondegenerate quadratic form on vector space E.
Q negative definite on subspace $F \subset E$ if $Q(x) < 0$ whenever $x \in F$ nonzero.
Index of $Q \equiv \mathrm{Ind}Q$, is largest possible dim. of subspace on which Q is negative definite.
cf. 1.1. Morse's Lemma of Ch. 6, pp. 145, Morse Theory of Hirsch (1997) [6]

Lemma 4 (Morse's Lemma)**.** *Let $p \in M$ be nondegenerate critical pt. of index k of C^{r+2} map $f : M \to \mathbb{R}$, $1 \leq r \leq \omega$. Then $\exists\, C^r$ chart (φ, U) at p s.t.*
$$f\varphi^{-1}(u_1 \dots u_n) = f(p) - \sum_{i=1}^k u_i^2 + \sum_{i=k+1}^n u_i^2 \tag{53}$$

Let ${}^TQ \equiv Q^T$ denote tranpose of matrix Q.

Lemma 5. *Let $A = diag\{a_1, \dots, a_n\}$ diagonal $n \times n$ matrix, with diagonal entries ± 1. Then $\exists$ neighborhood N of A in vector space of symmetric $n \times n$ matrices, C^∞ map*

$$P : N \to GL(n, \mathbb{R}) \tag{54}$$

s.t. $P(A) = I$, and if $P(B) = Q$, then $Q^T BQ = A$

Proof. Let $B = [b_{ij}]$ be symmetri matrix near A s.t. $b11 \neq 0$ and b_{11} has same sign as a_1.
Consider $x = Ty$ where

$$x_1 = \left[y_1 - \frac{b_{12}}{b_{11}} y_2 - \dots - \frac{b_{1n}}{b_{11}} y_n \right] / \sqrt{|b_n|}$$
$$x_k = y_k \text{ for } k = 2, \dots n$$

□

11. Lagrange multipliers

From *wikipedia:Lagrange multiplier*, https://en.wikipedia.org/wiki/Lagrange_multiplier, find local minima (maxima), pt. $a \in N$, s.t. $\exists$ neighborhood U s.t. $f(x) \geq f(a)$ $(f(x) \leq f(a))$ $\forall x \in U$.
For $f : U \to \mathbb{R}$, open $U \subset \mathbb{R}^n$, find $x \in U$ s.t. $D_x f \equiv Df(x) = 0$, check if Hessian $H_x f < 0$.
Maxima may not exit since U open.
References:
Relative Extrema and Lagrange Multipliers
Other interesting links:
The Lagrange Multiplier Rule on Manifolds and Optimal Control of nonlinear systems

Part 11. Classical Mechanics applications

cf. Arnold, Kozlov, Neishtadt (2006) [20].
If known forces $\mathbf{F}_1 \dots \mathbf{F}_n$ acts on points, then

$$\sum_{i=1}^n \langle m_i \ddot{\mathbf{r}}_i - \mathbf{F}_i, \xi_i \rangle = 0 \tag{55}$$

cf. Eq. (1.26) of Arnold, Kozlov, Neishtadt (2006) [20], where $\xi_1, \dots \xi_n$ are arbitrary tangent vectors to M, $\xi_i, \dots \xi_n \in TM$.
$\sum_{i=1}^n \langle m_i \ddot{\mathbf{r}}_i - \mathbf{F}_i, \xi_i \rangle$ called "general equation of dynamics" or d'Alembert-Lagrange principle.

Part 12. Classical Mechanics

12. Classical Mechanics

12.1. Structure of Galilean Space-Time.
cf. Sec. 3.1 - *Structure of Galilean Space-Time* of Prástaro (1996) [12].
Mechanics assumes a particular simple formulation if formulated with respect to some spacetime manifold.
In Galilean spacetime, it's possible to naturally recognize absolute objects, and others that depend on frames.
cf. Def. 3.1 of Prástaro (1996) [12]

Definition 43 (Galilean spacetime structure)**.** (1) *Galilean spacetime structure* $:= (\mathcal{G}, g)$ *where $\mathcal{G}$ is (fiber bundle space-time)*

$$\mathcal{G} \equiv \{\tau : M \to T\} \tag{56}$$

where
*M = 4-dim. affine manifold (**space-time**); corresponding structure is $(M, \mathbf{M}, \alpha)$,*

(2) *T = 1-dim. affine space (**time**), corresponding affine structure is $(T, \mathbf{T}, \beta)$*

(3) *τ = surjective affine mapping, of constant rank 1, s.t. $\forall p \in M$ associates its time $\tau(p) \in T$*
Put $\mathbf{S} = ker(\underline{\tau}) \equiv ker(D\tau) \in M$,
where $\underline{\tau} \equiv D\tau$, D is symbol of derivative.
Define

$$g : M \to vS_2^0(M) \equiv M \times S_2^0(\mathbf{S})$$
$$g(p) = (p, \underline{g}) \equiv (p, Dg), \forall p \in M$$

where $\underline{g} \equiv Dg$ is a Euclidean structure on $\mathbf{S}$. g is called vertical metric field.

Thus, given $(M, \mathbf{M}, \alpha)$, $\forall (O, \{\mathbf{e}_i\}_{1 \leq i \leq d}$, $\{\mathbf{e}_i\}_{i=1\dots d}$, is basis of $\mathbf{M}$,

$$M \cong \mathbb{R}^4, \text{and } \exists \{x^\alpha : M = \mathbb{R}^4 \to \mathbb{R}\}_{\alpha=1\dots4}$$

12.2. Fundamental Theorems of (Classical) Dynamics.
cf. Sec. 3.4 - *Fundamental Theorems of Dynamics* of Prástaro (1996) [12].
cf. Thm. 3.20 of Prástaro (1996) [12]

Theorem 19 (Momentum Theorem)**.** *Variation of the free part of momentum of the observed motion of 1 body, in time interval $\Delta t \equiv [0, t]$ is equal to the corresponding impulse:*

$$I[0, t] \equiv \int_0^t F dt \equiv \left(\int_0^t F^j dt \right) \mathbf{e}_j \tag{57}$$

where $\{\mathbf{e}_j\}_{1 \leq j \leq 3}$ is a fixed basis of $\mathbf{S}$

$$\overline{p}_\psi(t) - \overline{p}_\psi(0) = I[0, t] \tag{58}$$

Proof.

$$\overline{p}_\psi = \mu \dot{m}_\psi = \overline{f}_\psi \Longrightarrow \dot{\overline{p}}_\psi = \dot{p}^j \mathbf{e}_j = F^j \mathbf{e}_j \Longrightarrow \int_{[0,t]} \dot{p}^j dt = \int_{[0,t]} F^j dt$$

□

13. Fluid Mechanics, Fluid Flow

13.1. Mass Conservation for Fluid Flow, Continuum media.
The mass of fluid in some volume $V_0 \subset N$ is $\int_{V^0} \rho \text{vol}^n$, where ρ is fluid density, $\rho \in C^\infty(N)$.
The total mass of fluid flowing out of volume V_0 is

$$\frac{d}{dt} \int_{V_0} \rho \text{vol}^n = \int_{V_0} \mathcal{L}_{\frac{\partial}{\partial t} + \mathbf{u}} (\rho \text{vol}^n) = \int_{V_0} \frac{\partial}{\partial t} \rho \text{vol}^n + \int_{V_0} \mathcal{L}_u \rho \text{vol}^n$$
$$\int_{V_0} \mathcal{L}_u \rho \text{vol}^n = \int_{V_0} di_{\mathbf{u}} \rho \text{vol}^n + i_{\mathbf{u}} d\rho \text{vol}^n = \int_{V_0} di_{\mathbf{u}} \rho \text{vol}^n + 0 = \int_{V_0} di_{\mathbf{u}} \rho \text{vol}^n = \int_{\partial V_0} i_{\mathbf{u}} \rho \text{vol}^n$$

Now

$$i_u \text{vol}^n = i_u \frac{\sqrt{g}}{n!} \epsilon_{i_1 \dots i_n} dx^{i_1} \wedge \dots \wedge dx^{i_n}$$

$$i_u dx^{i_1} \wedge \dots \wedge dx^{i_n} = u^{i_1} dx^{i_2} \wedge \dots \wedge dx^{i_n} - dx^{i_1} \wedge u^{i_2} dx^{i_3} \wedge \dots \wedge dx^{i_n} + \dots + (-1)^{n+1} dx^{i_1} \wedge \dots \wedge dx^{i_{n-1}} u^{i_n} = \epsilon^{i_1 \dots i_n}_{j_1 \dots j_n} u^{j_1} dx^{j_2} \wedge \dots \wedge dx^{j_n}$$

$$\Longrightarrow i_u \text{vol}^n = \frac{\sqrt{g}}{(n-1)!} \epsilon_{j_1 \dots j_n} u^{j_1} dx^{j_2} \wedge \dots \wedge dx^{j_n} \tag{59}$$

We can also rewrite Eq. 59 to be a "surface differential":

$$\begin{gathered} \int_{V_0} \mathbf{d} i_{\mathbf{u}} \rho \text{vol}^n = \int_{\partial V_0} i_{\mathbf{u}} \rho \text{vol}^n = \\ \int_{\partial V_0} \rho \frac{\sqrt{g}}{(n-1)!} u^{j_1} \epsilon_{j_1 j_2 \dots j_n} dx^{j_2} \wedge \dots \wedge dx^{j_n} \equiv \int_{\partial V_0} \rho \mathbf{u} \cdot d\mathbf{S} \equiv \int_{\partial V_0} \rho \langle \mathbf{u}, d\mathbf{S} \rangle \end{gathered} \tag{60}$$

If $\sqrt{g}=1$, $n=2$,

$$i_u \text{vol}^2 = (u^1 dx^2 - u^2 dx^1) = u\cdot n_1 dx^2 + u \cdot n_2 dx^1 = u \cdot ndS$$

with $n_1 = e_1$ and $n_2 = -e_2$.

Now

$$di_u \rho \text{vol}^n =$$

$$= \frac{\partial(\sqrt{g}\rho u^{j_1})}{\partial x^k} \frac{\epsilon_{j_1\dots j_n}}{(n-1)!} dx^k \wedge dx^{j_2} \wedge \dots \wedge dx^{j_n} = \frac{\partial(\sqrt{g}\rho u^k)}{\partial x^k}\frac{\epsilon_{j_1\dots j_n}}{n!} dx^{j_1}\wedge \dots \wedge dx^{j_n} = \frac{1}{\sqrt{g}}\frac{\partial(\sqrt{g}\rho u^k)}{\partial x^k}\text{vol}^n =$$

$$= \frac{\partial(\rho u^k)}{\partial x^k}\text{vol}^n + \rho u^k \frac{\partial \ln \sqrt{g}}{\partial x^k}\text{vol}^n = \text{div}(\rho u)\text{vol}^n + \rho u^k \frac{\partial \ln\sqrt{g}}{\partial x^k}\text{vol}^n$$

Now if $\sqrt{g}=1$, then

$$\frac{d}{dt}\int_{V_0} \rho \text{vol}^n = \int_{V_0} \frac{\partial \rho}{\partial t}\text{vol}^n + \int_{V_0} di_u \rho \text{vol}^n = \int_{V_0}\frac{\partial\rho}{\partial t}\text{vol}^n + \int_{V_0}\text{div}(\rho u)\text{vol}^n \Longrightarrow \frac{\partial \rho}{\partial t} + \text{div}(\rho u) = 0$$

which is the so-called mass continuity equation. $j=\rho u$ is the mass flux density.

Thus,

$$\text{(mass conservation)}$$

$$m = m(t) := \int_{V_0} \rho \text{vol}^n,\ V_0 \subset N$$

(61)
$$\dot{m} \equiv \frac{d}{dt}m(t) = \int_{V_0}\left(\frac{\partial\rho}{\partial t}\text{vol}^n + \mathbf{d}i_{\mathbf{u}}\rho\text{vol}^n\right) = \boxed{\int_{V_0}\frac{\partial \rho}{\partial t}\text{vol}^n + \int_{\partial V_0}\rho\mathbf{u}\cdot d\mathbf{S}}$$

$$\text{if } \sqrt{g}=1, \text{ and } \dot{m}=0, \text{ then}$$

$$\frac{\partial\rho}{\partial t} + \text{div}(\rho\mathbf{u}) = 0$$

TODO: 20190804 Frankel (2012) [**?**] in pp. 138 and onwards, for Sec. 4.3. Differentiation of Integrals posed the rightful question, "How does one compute the rate of change of an integral when the domain of integration is also changing?" Revisit the derivation from a Lie derivative and 1-parameter flow point of view.

14. Thermodynamics

Let Σ be a (topological) manifold.
Suppose U is a global coordinate on Σ:

$$\text{(First Law of Thermodynamics (energy conservation))}$$

(62)
$$\boxed{dU = Q - W \text{ or } Q = dU + W}$$

where $dU, Q, W \in \Omega^1(\Sigma)$ (i.e. dU,Q,W are 1-forms over manifold Σ).

Consider a path in Σ, γ, $\gamma:\mathbb{R}\to\Sigma$,
and using a chart $(U, S^1, \dots, S^n)$ (e.g. $n=1$, $S^1 = v$ for volume)

$$\gamma(t) = (U(t), S^1(t), \dots, S^n(t))$$

$$\dot{\gamma} \in \mathfrak{X}(\Sigma),\ \dot{\gamma} = \dot{U}\frac{\partial}{\partial U} + \dot{S}^i\frac{\partial}{\partial S^i}$$

Now

$$dU(\dot{\gamma}) = \dot{\gamma}(U) = \dot{U}\frac{\partial}{\partial U}U + 0 = \dot{U}$$

$$Q(\dot{\gamma}) = Q(t)dt\left(\dot{\gamma}\frac{\partial}{\partial t}\right) = Q(t)\dot{\gamma}$$

$$\Longrightarrow \dot{U} = Q(t)\dot{\gamma} - W(t)\dot{\gamma}$$

Recall that for enthalpy H, $H := U + pV$, $H = H(\sigma, p)$

TODO 20190804 Derive and check convection form of enthalpy against both Kittle and Kroemer plus thermodynamics and Sonntag, et. al.

Incomplete:

$$dH$$

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