

# SOLUTIONS TO *CALCULUS* VOLUME 1 BY TOM APOSTOL.

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SOLUTIONS TO VOLUME 1 *One-Variable Calculus, with an Introduction to Linear Algebra*

## I 2.5 Exercises - Introduction to set theory, Notations for designating sets, Subsets, Unions, intersections, complements.

### Exercise 10. Distributive laws

Let  $X = A \cap (B \cup C), Y = (A \cap B) \cup (A \cap C)$

Suppose  $x \in X$

$x \in A$  and  $x \in (B \cup C) \implies x \in A$  and  $x$  is in at least  $B$  or in  $C$

then  $x$  is in at least either  $(A \cap B)$  or  $(A \cap C)$

$x \in Y, X \subseteq Y$

Suppose  $y \in Y$

$y$  is at least in either  $(A \cap B)$  or  $A \cap C$

then  $y \in A$  and either in  $B$  or  $C$

$y \in X, Y \subseteq X$

$X = Y$

Let  $X = A \cup (B \cap C), Y = (A \cup B) \cap (A \cup C)$

Suppose  $x \in X$

then  $x$  is at least either in  $A$  or in  $(B \cap C)$

if  $x \in A, x \in Y$

if  $x \in (B \cap C), x \in Y \qquad x \in Y, X \subseteq Y$

Suppose  $y \in Y$

then  $y$  is at least in  $A$  or in  $B$  and  $y$  is at least in  $A$  or in  $C$

if  $y \in A$ , then  $y \in X$

if  $y \in A \cap B$  or  $y \in A \cup C, y \in X$  (various carvings out of  $A$ , simply )

if  $y \in (B \cap C), y \in X \qquad y \in X, Y \subseteq X$

$X = Y$

**Exercise 11.** If  $x \in A \cup A$ , then  $x$  is at least in  $A$  or in  $A$ . Then  $x \in A$ . So  $A \cup A \subseteq A$ . Of course  $A \subseteq A \cup A$ .

If  $x \in A \cap A$ , then  $x$  is in  $A$  and in  $A$ . Then  $x \in A$ . So  $A \cap A \subseteq A$ . Of course  $A \subseteq A \cap A$ .

**Exercise 12.** Let  $x \in A. y \in A \cup B$  if  $y$  is at least in  $A$  or in  $B$ .  $x$  is in  $A$  so  $x \in A \cup B. \implies A \subseteq A \cup B$ .

Suppose  $\exists b \in B$  and  $b \notin A. b \in A \cup B$  but  $b \notin A$ . so  $A \subseteq A \cup B$ .

**Exercise 13.** Let  $x \in A \cup \emptyset$ , then  $x$  is at least in  $A$  or in  $\emptyset$ . If  $x \in \emptyset$ , then  $x$  is a null element (not an element at all). Then actual elements must be in  $A. \implies A \cup \emptyset \subseteq A$ .

Let  $x \in A$ . Then  $x \in A \cup \emptyset. A \subseteq A \cup \emptyset. \implies A = A \cup \emptyset$ .

**Exercise 14.** From distributivity,  $A \cup (A \cap B) = (A \cup A) \cap (A \cup B) = A \cap (A \cup B)$ .

If  $x \in A \cap (A \cup B), x \in A$  and  $x \in A \cup B$ , i.e.  $x \in A$  and  $x$  is at least in  $A$  or in  $B$ .

$\implies x$  is in  $A$  and is in  $B$  or is not in  $B$ . Then  $x \in A. \implies A \cap (A \cup B) \subseteq A$ . Of course,  $A \subseteq A \cap (A \cup B)$ .

$\implies A \cap (A \cup B) = A \cup (A \cap B) = A$ .

**Exercise 15.**  $\forall a \in A, a \in C$  and  $\forall b \in B, b \in C$ . Consider  $x \in A \cup B$ .  $x$  is at least in  $A$  or in  $B$ . In either case,  $x \in C. \implies A \cup B \subseteq C$ .

### Exercise 16.

if  $C \subseteq A$  and  $C \subseteq B$ , then  $C \subseteq A \cap B$

$\forall c \in C, c \in A$  and  $c \in B$

$x \in A \cap B, x \in A$  and  $x \in B$ . Then  $\forall c \in C, c \in A \cap B. C \subseteq A \cap B$

**Exercise 17.**

(1)

if  $A \subset B$  and  $B \subset C$  then

$$\forall a \in A, a \in B. \forall b \in B, b \in C.$$

then since  $a \in B, a \in C, \exists c \in C$  such that  $c \notin B$ .

$$\forall a \in A, a \in B \text{ so } a \neq c \forall a. \implies A \subset C$$

(2) If  $A \subseteq B, B \subseteq C, A \subseteq C$  since,  $\forall a \in A, a \in B, \forall b \in B, b \in C$ . Then since  $a \in B, a \in C. A \subseteq C$

(3)  $A \subset B$  and  $B \subseteq C. B \subset C$  or  $B = C. A \subset B$  only. Then  $A \subset C$ .

(4) Yes, since  $\forall a \in A, a \in B$ .

(5) No, since  $x \neq A$  (sets as elements are different from elements)

**Exercise 18.**  $A - (B \cap C) = (A - B) \cup (A - C)$

Suppose  $x \in A - (B \cap C)$

then  $x \in A$  and  $x \notin B \cap C \implies x \notin B \cap C$

then  $x$  is not in even at least one  $B$  or  $C$

$$\implies x \in (A - B) \cup (A - C)$$

Suppose  $x \in (A - B) \cup (A - C)$

then  $x$  is at least in  $(A - B)$  or in  $(A - C) \implies x$  is at least in  $A$  and not in  $B$  or in  $A$  and not in  $C$

then consider when one of the cases is true and when both cases are true  $\implies x \in A - (B \cap C)$

**Exercise 19.**

Suppose  $x \in B - \bigcup_{A \in \mathcal{F}} A$

then  $x \in B, x \notin \bigcup_{A \in \mathcal{F}} A$

$$x \notin \bigcup_{A \in \mathcal{F}} A \implies x \notin A, \forall A \in \mathcal{F}$$

since  $\forall A \in \mathcal{F}, x \in B, x \notin A$ , then  $x \in \bigcap_{A \in \mathcal{F}} (B - A)$

Suppose  $x \in \bigcap_{A \in \mathcal{F}} (B - A)$

then  $x \in B - A_1$  and  $x \in B - A_2$  and ...

then  $\forall A \in \mathcal{F}, x \in B, x \notin A$

then  $x \notin$  even at least one  $A \in \mathcal{F}$

$$\implies x \in B - \bigcup_{A \in \mathcal{F}} A$$

Suppose  $x \in B - \bigcap_{A \in \mathcal{F}} A$

then  $x \notin \bigcap_{A \in \mathcal{F}} A$

then at most  $x \in A$  for  $\forall A \in \mathcal{F}$  but one

then  $x$  is at least in one  $B - A$

$$\implies x \in \bigcup_{A \in \mathcal{F}} (B - A)$$

Suppose  $x \in \bigcup_{A \in \mathcal{F}} (B - A)$

then  $x$  is at least in one  $B - A$

then for  $A \in \mathcal{F}, x \in B$  and  $x \notin A$

Consider  $\forall A \in \mathcal{F}$

$$\implies \text{then } x \in B - \bigcap_{A \in \mathcal{F}} A$$

**Exercise 20.**

(1) (ii) is correct.

Suppose  $x \in (A - B) - C$

then  $x \in A - B, x \notin C$

then  $x \in A$  and  $x \notin B$  and  $x \notin C$

$x \notin B$  and  $x \notin C \implies x \notin$  even at least  $B$  or  $C$

$$x \in A - (B \cup C)$$

Suppose  $x \in A - (B \cup C)$

then  $x \in A, x \notin (B \cup C)$

then  $x \in A$  and  $x \notin B$  and  $x \notin C$

$$\implies x \in (A - B) - C$$

To show that (i) is sometimes wrong,

Suppose  $y \in A - (B - C)$   
 $y \in A$  and  $y \notin B - C$   
 $y \notin B - C$   
then  $y \notin B$  or  $y \in C$  or  $y \notin C$   
(where does this lead to?)

Consider directly.

Suppose  $x \in (A - B) \cup C$   
then  $x$  is at least in  $A - B$  or in  $C$   
then  $x$  is at least in  $A$  and  $\notin B$  or in  $C$   
Suppose  $x = c \in C$  and  $c \notin A$

(2)

If  $C \subseteq A$ ,  
 $A - (B - C) = (A - B) \cup C$

**I 3.3 Exercises - The field axioms.** The goal seems to be to abstract these so-called real numbers into just  $x$ 's and  $y$ 's that are purely built upon these axioms.

**Exercise 1.** Thm. I.5.  $a(b - c) = ab - ac$ .

Let  $y = ab - ac$ ;  $x = a(b - c)$

Want:  $x = y$

$ac + y = ab$  (by Thm. I.2, possibility of subtraction)

Note that by Thm. I.3,  $a(b - c) = a(b + (-c)) = ab + a(-c)$  (by distributivity axiom)

$ac + x = ac + ab + a(-c) = a(c + (-c)) + ab = a(0 + b) = ab$

But there exists exactly one  $y$  or  $x$  by Thm. I.2.  $x = y$ .

Thm. I.6.  $0 \cdot a = a \cdot 0 = 0$ .

$0(a) = a(0)$  (by commutativity axiom)

Given  $b \in \mathcal{R}$  and  $0 \in \mathcal{R}$ ,  $\exists$  exactly one  $-b$  s.t.  $b - a = 0$

$0(a) = (b + (-b))a = ab - ab = 0$  (by Thm. I.5. and Thm. I.2)

Thm. I.7.

$ab = ac$

By Axiom 4,  $\exists y \in \mathcal{R}$  s.t.  $ay = 1$

since products are uniquely determined,  $yab = yac \implies (ya)b = (ya)c \implies 1(b) = 1(c)$   
 $\implies b = c$

Thm. I.8. Possibility of Division.

Given  $a, b, a \neq 0$ , choose  $y$  such that  $ay = 1$ .

Let  $x = yb$ .

$ax = ayb = 1(b) = b$

Therefore, there exists at least one  $x$  such that  $ax = b$ . But by Thm. I.7, there exists only one  $x$  (since if  $az = b$ , and so  $x = z$ ).

Thm. I.9. If  $a \neq 0$ , then  $b/a = b(a^{-1})$ .

Let  $x = \frac{b}{a}$  for  $ax = b$

$y = a^{-1}$  for  $ay = 1$

Want:  $x = by$

Now  $b(1) = b$ , so  $ax = b = b(ay) = a(by)$

$\implies x = by$  (by Thm. I.7)

Thm. I.10. If  $a \neq 0$ , then  $(a^{-1})^{-1} = a$ .

Now  $ab = 1$  for  $b = a^{-1}$ . But since  $b \in \mathcal{R}$  and  $b \neq 0$  (otherwise  $1 = 0$ , contradiction), then using Thm. I.8 on  $b$ ,  $ab = b(a) = 1$ ;  $a = b^{-1}$ .

Thm.I.11. If  $ab = 0$ ,  $a = 0$  or  $b = 0$ .

$ab = 0 = a(0) \implies b = 0$  or  $ab = ba = b(0) \implies a = 0$ . (we used Thm. I.7, cancellation law for multiplication)

Thm. I.12. Want:  $x = y$  if  $x = (-a)b$  and  $y = -(ab)$ .

$$ab + y = 0$$

$$ab + x = ab + (-a)b = b(a + (-a)) = b(a - a) = b(0) = 0$$

0 is unique, so  $ab + y = ab + x$  implies  $x = y$  (by Thm. I.1)

Thm. I.13. Want:  $x + y = z$ , if  $a = bx$ ,  $c = dy$ ,  $(ad + bc) = (bd)z$ .

$$(bd)(x + y) = bdx + bdy = ad + bc = (bd)z$$

So using  $b, d \neq 0$ , which is given, and Thm. I.7, then  $x + y = z$ .

Thm. I.14. Want:  $xy = z$  for  $bx = a$ ,  $dy = c$ ,  $ac = (bd)z$ .

$$(bd)(xy) = (bx)(dy) = ac = (bd)z$$

$b, d \neq 0$ , so by Thm. I.7,  $xy = z$ .

Thm.I.15. Want:  $x = yz$ , if  $bx = a$ ,  $dy = c$ ,  $(bc)z = ad$

$$(bc)z = b(dy)z = d(byz) = da$$

$$d \neq 0 \text{ so by Thm. I.7, by } z = a, byz = abx$$

$$b \neq 0 \text{ so by Thm. I.7, } yz = x$$

**Exercise 2.** Consider  $0 + z = 0$ . By Thm. I.2, there exists exactly one  $z$ ,  $z = -0$ . By Axiom 4,  $z = 0$ .  $0 = -0$ .

**Exercise 3.** Consider  $1(z)z(1) = 1$ . Then  $z = 1^{-1}$ . But by Axiom 4, there exists distinct 1 such that  $z(1) = 1$ , so  $z = 1$ .

**Exercise 4.** Suppose there exists  $x$  such that  $0x = 1$ , but  $0x = 0$  and 0 and 1 are distinct, so *zero has no reciprocal*.

**Exercise 5.**  $a + (-a) = 0$ ,  $0 + 0 = 0$ . Then

$$a + (-a) + b + (-b) = (a + b) + (-a) + (-b) = 0$$

$$-(a + b) = -a + (-b) = -a - b$$

**Exercise 6.**  $a + (-a) = 0$ ,  $b + (-b) = 0$ , so

$$a + (-a) + b + (-b) = a + (-b) + (-a) + b = (a - b) + (-a) + b = 0 + 0 = 0$$

$$-(a - b) = -a + b.$$

**Exercise 7.**

$$(a - b) + (b - c) = a + (-b) + b + (-c) = a + (b + (-b)) + (-c) = a - c$$

**Exercise 8.**

$$(ab)x = 1 \quad (ab)^{-1} = x$$

$$a(bx) = 1 \quad a^{-1} = bx$$

$$b(ax) = 1 \quad b^{-1} = ax$$

$$a^{-1}b^{-1} = (abx)x = 1(x) = (ab)^{-1}$$

**Exercise 9.** Want:  $x = y = z$ , if

$$z = \frac{a}{-b}$$

$$a = zt \quad b + t = 0$$

$$y = \frac{(-a)}{b}$$

$$by = u \quad a + u = 0$$

$$x = -\left(\frac{a}{b}\right)$$

$$\left(\frac{a}{b}\right) + x = v + x = 0 \quad vb = a$$

$$a + (-a) = vb + by = b(v + y) = 0$$

$$\text{if } b \neq 0, v + y = 0, \text{ but } v + x = 0$$

$$\text{by Thm. I.1, } x = y$$

$$b + t = 0, \text{ then } z(b + t) = zb + zt = zb + a = z(0) = 0$$

$$a + zb = 0 \implies -a = zb = by$$

$$\text{since } b \neq 0, z = y \text{ so } x = y = z$$

**Exercise 10.** Since  $b, d \neq 0$ , Let

$$z = \frac{ad - bc}{bd} \quad (bd)z = ad - bc \text{ by previous exercise or Thm. I.8, the possibility of division}$$

$$x = \frac{a}{b} \quad bx = a$$

$$t = \frac{-c}{d} \quad dt = -c \text{ (By Thm. I.3, we know that } b - a = b + (-a) \text{)}$$

$$dbx + bdt = (bd)(x + y) = ad - bc = (bd)z$$

$$b, d \neq 0, \text{ so } x + y = z$$

### I 3.5 Exercises - The order axioms.

**Theorem 1 (I.18).** If  $a < b$  and  $c > 0$  then  $ac < bc$

**Theorem 2 (I.19).** If  $a < b$  and  $c > 0$ , then  $ac < bc$

**Theorem 3 (I.20).** If  $a \neq 0$ , then  $a^2 > 0$

**Theorem 4 (I.21).**  $1 > 0$

**Theorem 5 (I.22).** If  $a < b$  and  $c < 0$ , then  $ac > bc$ .

**Theorem 6 (I.23).** If  $a < b$  and  $-a > -b$ . In particular, if  $a < 0$ , then  $-a > 0$ .

**Theorem 7 (I.24).** If  $ab > 0$ , then both  $a$  and  $b$  are positive or both are negative.

**Theorem 8 (I.25).** If  $a < c$  and  $b < d$ , then  $a + b < c + d$ .

**Exercise 1.**

(1) By Thm. I.19,  $-c > 0$

$$a(-c) < b(-c) \rightarrow -ac < -bc$$

$$-bc - (-ac) = ac - bc > 0. \text{ Then } ac > bc \text{ (by definition of } > \text{)}$$

(2)

$$a < b \rightarrow a + 0 < b + 0 \rightarrow a + b + (-b) < b + a + (-a) \rightarrow (a + b) - b < (a + b) + (-a)$$

$$\text{By Thm. I.18 } (a + b) + -(a + b) + (-b) < (a + b) - (a + b) + (-a)$$

$$-b < -a$$

(3)

If  $a = 0$  or  $b = 0$ ,  $ab = 0$ , but  $0 \not> 0$   
 If  $a > 0$ , then if  $b > 0$ ,  $ab > 0(b) = 0$ . If  $b < 0$ ,  $ab < 0(b) = 0$ . So if  $a > 0$ , then  $b > 0$ .  
 If  $a < 0$ , then if  $b > 0$ ,  $ab < 0(b) = 0$ . If  $b < 0$ ,  $ab > 0(b) = 0$ . So if  $a < 0$ , then  $b < 0$ .

(4)

$$a < c \text{ so } a + b < c + b = b + c$$

$$b < d \text{ so } b + c < d + c$$

By Transitive Law,  $a + b < d + c$

**Exercise 2.** If  $x = 0$ ,  $x^2 = 0$ .  $0 + 1 = 1 \neq 0$ . So  $x \neq 0$ .

$$\text{If } x \neq 0, x^2 > 0, \text{ and by Thm. I.21, } 1 > 0$$

$$x^2 + 1 > 0 + 0 = 0 \rightarrow x^2 + 1 \neq 0$$

$$\implies \nexists x \in \mathbb{R} \text{ such that } x^2 + 1 = 0$$

**Exercise 3.**

$$a < 0, b < 0, a + b < 0 + 0 = 0 \text{ ( By Thm. I.25)}$$

**Exercise 4.** Consider  $ax = 1$ .

$$ax = 1 > 0. \text{ By Thm. I.24, } a, x \text{ are both positive or } a, x \text{ are both negative}$$

**Exercise 5.** Define  $x, y$  such that  $ax = 1, by = 1$ . We want  $x > y$  when  $b > a$ .

$$xb - ax = xb - 1 > 0 \implies bx > 1 = by$$

$$b > 0 \text{ so } x > y$$

**Exercise 6.**

If  $a = b$  and  $b = c$ , then  $a = c$

If  $a = b$  and  $b < c$ , then  $a < c$

If  $a < b$  and  $b = c$ , then  $a < c$

If  $a < b$  and  $b < c$ , then  $a < c$  (by transitivity of the inequality)

$\implies a \leq c$

**Exercise 7.** If  $a \leq b$  and  $b \leq c$ , then  $a \leq c$ . If  $a = c$ , then by previous proof,  $a = b$ .

**Exercise 8.** If  $a \leq b$  and  $b \leq c$ , then  $a \leq c$ . If  $a = c$ , then by previous proof,  $a = b$ .

**Exercise 8.** If  $a$  or  $b$  is zero,  $a^2$  or  $b^2 = 0$ . By Thm. I.20,  $b^2 \geq 0$  or  $a^2 \geq 0$ , respectively.

Otherwise, if neither are zero, by transitivity,  $a^2 + b^2 > 0$ .

**Exercise 9.** Suppose  $a \geq x$ . Then  $a - x \geq 0$ .

If  $a \in \mathbb{R}$  so  $\exists y \in \mathbb{R}$ , such that  $a - y = 0$ .

Consider  $y + 1 \in \mathbb{R}$  (by closure under addition).

$$a - (y + 1) = a - y - 1 = 0 - 1 < 0 \text{ Contradiction that } a \geq y + 1$$

**Exercise 10.**

If  $x = 0$ , done.

If  $x > 0$ ,  $x$  is a positive real number. Let  $h = \frac{x}{2}$ .

$\implies \frac{x}{2} > x$  Contradiction.

### I 3.12 Exercises - Integers and rational numbers, Geometric interpretation of real numbers as points on a line, Upper bound of a set, maximum element, least upper bound (supremum), The least-upper-bound axiom (completeness axiom), The Archimedean property of the real-number system, Fundamental properties of the supremum and infimum.

We use Thm I.30, the Archimedean property of real numbers, alot.

**Theorem 9 (I.30).** If  $x > 0$  and if  $y$  is an arbitrary real number, there exists a positive integer  $n$  such that  $nx > y$ .

We will use the least upper-bound axiom (completeness axiom) alot for continuity and differentiation theorems later. Apostol states it as an axiom; in real analysis, the existence of a sup for nonempty, bounded sets can be shown with an algorithm to zoom into a sup with monotonically increasing and monotonically decreasing sequence of “guesses” and showing its difference is a Cauchy sequence.

**Axiom 1** (Least upper-bound axiom). Every nonempty set  $S$  of real numbers which is bounded above has a supremum; that is, there's a real number  $B$  s.t.  $B = \sup S$ .

**Exercise 1.**  $0 < y - x$ .

$$\implies n(y - x) > h > 0, n \in \mathbb{Z}^+, h \text{ arbitrary}$$

$$y - x > h/n \implies y > x + h/n > x$$

so let  $z = x + h/n$  Done.

**Exercise 2.**  $x \in \mathbb{R}$  so  $\exists n \in \mathbb{Z}^+$  such that  $n > x$  (Thm. I.29).

Set of negative integers is unbounded below because

If  $\forall m \in \mathbb{Z}^-, -x > -m$ , then  $-x$  is an upper bound on  $\mathbb{Z}^+$ . Contradiction of Thm. I.29.  $\implies \exists m \in \mathbb{Z}$  such that  $m < x < n$

**Exercise 3.** Use Archimedean property.

$x > 0$  so for 1,  $\exists n \in \mathbb{Z}^+$  such that  $nx > 1, x > \frac{1}{n}$ .

**Exercise 4.**  $x$  is an arbitrary real number. By Thm. I.29 and well-ordering principle, there exists a smallest  $n + 1$  positive integer such that  $x < n + 1$  (consider the set of all  $m + 1 > x$  and so by well-ordering principle, there must be a smallest element of this specific set of positive integers).

If  $x = n$  for some positive integer  $n$ , done.

Otherwise, note that if  $x < n$ , then  $n + 1$  couldn't have been the smallest element such that  $m > x$ .  $x > n$ .

**Exercise 5.** If  $x = n$ , done. Otherwise, consider all  $m > x$ . By well-ordering principle, there exists a smallest element  $n$  such that  $n > x$ .

If  $x + 1 < n$ , then  $x < n - 1$ , contradicting the fact that  $n$  is the smallest element such that  $x < n$ . Thus  $x + 1 > n$ .

**Exercise 6.**  $y - x > 0$ .

$$n(y - x) > h, h \text{ arbitrary}, n \in \mathbb{Z}^+$$

$$y > x + h/n = z > x$$

Since  $h$  was arbitrary, there are infinitely many numbers in between  $x, y$ .

**Exercise 7.**  $x = \frac{a}{b} \in \mathbb{Q}, y \notin \mathbb{Q}$ .

$$x \pm y = \frac{a \pm by}{b}$$

If  $a \pm by$  was an integer, say  $m$ , then  $y = \pm \left( \frac{a - mb}{b} \right)$  which is rational. Contradiction.

$$xy = \frac{a}{b} \frac{y}{1} = \frac{ay}{b}$$

If  $ay$  was an integer,  $ay = n, y = \frac{n}{a}$ , but  $y$  is irrational.  $\implies xy$  is irrational.

$$\frac{x}{y}$$

$y$  is not an integer

**Exercise 8.** Proof by counterexamples. We want that the sum or product of 2 irrational numbers is not always irrational. If  $y$  is irrational,  $y + 1$  is irrational, otherwise, if  $y + 1 \in \mathbb{Q}, y \in \mathbb{Q}$  by closure under addition.

$$\implies y + 1 - y = 1$$

Likewise,  $y \frac{1}{y} = 1$ .

**Exercise 9.**

$y - x > 0 \implies n(y - x) > k, n \in \mathbb{Z}^+, k$  arbitrary. Choose  $k$  to be irrational. Then  $k/n$  irrational.

$$y > \frac{k}{n} + x > x. \text{ Let } z = x + \frac{k}{n}, z \text{ irrational.}$$

**Exercise 10.**

(1) Suppose  $n = 2m_1$  and  $n + 1 = 2m_2$ .

$$2m_1 + 1 = 2m_2 \quad 2(m_1 - m_2) = 1 \quad m_1 - m_2 = \frac{1}{2}. \text{ But } m_1 - m_2 \text{ can only be an integer.}$$

(2) By the well-ordering principle, if  $x \in \mathbb{Z}^+$  is neither even and odd, consider the set of all  $x$ . There must exist a smallest element  $x_0$  of this set. But since  $x_0 \in \mathbb{Z}^+$ , then there must exist a  $n < x$  such that  $n + 1 = x_0$ .  $n$  is even or odd since it doesn't belong in the above set. So  $x_0$  must be odd or even. Contradiction.

(3)

$$(2m_1)(2m_2) = 2(2m_1m_2) \text{ even}$$

$$2m_1 + 2m_2 = 2(m_1 + m_2) \text{ even}$$

$$(2m_1 + 1) + (2m_2 + 1) = 2(m_1 + m_2 + 1) \implies \text{sum of two odd numbers is even}$$

$$(n_1 + 1)(n_2 + 1) = n_1n_2 + n_1 + n_2 + 2 + 1 = 2(2m_1m_2)$$

$$2(2m_1m_2) - (n_1 + n_2) - 1 \text{ odd, the product of two odd numbers } n_1, n_2 \text{ is odd}$$

(4) If  $n^2$  even,  $n$  is even, since for  $n = 2m, (2m)^2 = 4m^2 = 2(2m^2)$  is even.

$$a^2 = 2b^2. \quad 2(b^2) \text{ even. } a^2 \text{ even, so } a \text{ even.}$$

$$\text{If } a \text{ even } a = 2n. a^2 = 4n^2$$

$$\text{If } b \text{ odd, } b^2 \text{ odd. } b \text{ has no factors of } 2 \quad b^2 \neq 4n^2$$

Thus  $b$  is even.

(5) For  $\frac{p}{q}$ , If  $p$  or  $q$  or both are odd, then we're done.

Else, when  $p, q$  are both even,  $p = 2^l m, q = 2^n p, m, p$  odd.

$$\frac{p}{q} = \frac{2^l m}{2^n p} = \frac{2^{l-n} m}{p} \text{ and at least } m \text{ or } p \text{ odd}$$

**Exercise 11.**  $\frac{a}{b}$  can be put into a form such that  $a$  or  $b$  at least is odd by the previous exercise.

However,  $a^2 = 2b^2$ , so  $a$  even,  $b$  even, by the previous exercise, part (d) or 4th part. Thus  $\frac{a}{b}$  cannot be rational.

**Exercise 12.** The set of rational numbers satisfies the Archimedean property but not the least-upper-bound property.

Since  $\frac{p}{q} \in \mathbb{Q} \subseteq \mathbb{R}, n \frac{p_1}{q_1} > \frac{p_2}{q_2}$  since if  $q_1, q_2 > 0$ ,

$$\frac{np_1q_2}{q_1q_2} > \frac{q_1p_2}{q_1q_2} \quad np_1q_2 > q_1p_2$$

$n$  exists since  $(p_1q_2), (q_1p_2) \in \mathbb{R}$ .

The set of rational numbers does not satisfy the least-upper-bound property.

Consider a nonempty set of rational numbers  $S$  bounded above so that  $\forall x = \frac{r}{s} \in S, x < b$ .

Suppose  $x < b_1, x < b_2 \forall x \in S$ .

$$\frac{r}{s} < b_2 < nb_1 \text{ but likewise } \frac{r}{s} < b_1 < mb_2, n, m \in \mathbb{Z}^+$$

So it's possible that  $b_1 > b_2$ , but also  $b_2 > b_1$ .

**I 4.4 Exercises - An example of a proof by mathematical induction, The principle of mathematical induction, The well-ordering principle.** Consider these 2 proofs.

$$N + N + \cdots + N = N^2$$

$$(N-1) + (N-2) + \cdots + (N-(N-1)) + (N-N) = N^2 - \sum_{j=1}^N j = \sum_{j=1}^{N-1} j$$

$$N^2 + N = 2 \sum_{j=1}^N j \implies \sum_{j=1}^N j = \frac{N(N+1)}{2}$$

An interesting property is that

$$S = \sum_{j=m}^n j = \sum_{j=m}^n (n+m-j)$$

So that

$$\begin{aligned} \sum_{j=1}^N j &= \sum_{j=m}^N j + \sum_{j=1}^m j = \sum_{j=m}^N j + \frac{m(m+1)}{2} = \frac{N(N+1)}{2} \\ \sum_{j=m}^N j &= \frac{N(N+1) - m(m+1)}{2} = \frac{(N-m)(N+m+1)}{2} \end{aligned}$$

Another way to show this is the following.

$$\begin{array}{ccccccc} S = & 1+ & 2+ & \cdots + & (N-2)+ & (N-1)+ & N \\ \text{but } S = & N+ & N-1+ & \cdots + & 3+ & 2+ & 1 \\ \hline 2S = & (N+1)N & & & & & \\ S = & \frac{N(N+1)}{2} \end{array}$$

Telescoping series will let you get  $\sum_{j=1}^N j^2$  and other powers of  $j$ .

$$\sum_{j=1}^N (2j-1) = 2 \frac{N(N+1)}{2} - N = N^2$$

$$\sum_{j=1}^N (j^2 - (j-1)^2) = \sum_{j=1}^N (j^2 - (j^2 - 2j + 1)) = \sum_{j=1}^N (2j-1) = 2 \left( \frac{N(N+1)}{2} \right) - N = N^2$$

$$\sum_{j=1}^N (j^3 - (j-1)^3) = N^3 = \sum_{j=1}^N (j^3 - (j^3 - 3j^2 + 3j - 1)) = \sum_{j=1}^N (3j^2 - 3j + 1)$$

$$\implies 3 \sum_{j=1}^N j^2 = -3 \frac{N(N+1)}{2} + N = N^3 \implies \frac{2N^3 + 2N - 3N^2 - 3N}{2} = \frac{N(N+1)(2N+1)}{6} = \sum_{j=1}^N j^2$$

$$\sum_{j=1}^N j^4 - (j-1)^4 = N^4 = \sum_{j=1}^N j^4 - (j^4 - 4j^3 + 6j^2 - 4j + 1) = \sum_{j=1}^N 4j^3 - 6j^2 + 4j - 1 =$$

$$= 4 \sum_{j=1}^N j^3 - 6 \frac{N(N+1)(2N+1)}{6} + 4 \frac{N(N+1)}{2} - N = N^4$$

$$\implies \sum_{j=1}^N j^3 = \frac{1}{4} (N^4 + N(N+1)(2N+1) - 2N(N+1) + N) = \frac{1}{4} (N^4 + (2N)N(N+1) - N(N+1) + N)$$

$$= \frac{1}{4} (N^4 + 2N^3 + 2N^2 - N^2 - N + N) = \frac{1}{4} N^2 (N^2 + 2N + 1) = \frac{1}{4} \frac{(N(N+1))^2}{2}$$



**Exercise 1.** Induction proof.

$$\frac{1(1+1)}{2} + \sum_{j=1}^{N+1} j = \sum_{j=1}^n j + n + 1 = \frac{n(n+1)}{2} + n + 1 = \frac{n(n+1) + 2(n+1)}{2} = \frac{(n+2)(n+1)}{2}$$

**Exercise 6.**

(1)

$$A(k+1) = A(k) + k + 1 = \frac{1}{8}(2k+1)^2 + k + 1 = \frac{1}{8}(4k^2 + 4k + 1) + \frac{8k+8}{8} = \frac{(2k+3)^2}{8}$$

(2) The  $n = 1$  case isn't true.

(3)

$$1 + 2 + \dots + n = \frac{(n+1)n}{2} = \frac{n^2 + n}{2} < \frac{n^2 + n + \frac{1}{4}}{2}$$

$$\text{and } \left(\frac{2n+1}{2}\right)^2 \frac{1}{2} = \frac{(n+1/2)^2}{2} = \frac{n^2 + n + 1/4}{2}$$

**Exercise 7.**

$$(1+x)^2 > 1 + 2x + 2x^2$$

$$1 + 2x + x^2 > 1 + 2x + 2x^2$$

$$0 > x^2 \implies \text{Impossible}$$

$$(1+x)^3 = 1 + 3x + 3x^2 + x^3 > 1 + 3x + 3x^2$$

$$\implies x^3 > 0$$

By well-ordering principle, we could argue that  $n = 3$  must be the smallest number such that  $(1+x)^n > 1 + 2x + 2x^2$ . Or we could find, explicitly

$$(1+x)^n = \sum_{j=0}^n \binom{n}{j} x^j = 1 + nx + \frac{n(n-1)}{2} x^2 + \sum_{j=3}^n \binom{n}{j} x^j$$

and

$$\frac{n(n-1)}{2} > n$$

$$n^2 - n > 2n$$

$$n^2 > 3n$$

$$n > 3$$

**Exercise 8.**

$$a_2 \leq ca_1, a_3 \leq ca_2 \leq c^2 a_1$$

$$a_{n+1} \leq ca_n \leq ca_1 c^{n-1} = a_1 c^n$$

**Exercise 9.**

$$n = 1, \sqrt{1} = 1$$

$$\sqrt{1^2 + 1^2} = \sqrt{2} \sqrt{(\sqrt{2})^2 + 1^2} = \sqrt{3}$$

$$\sqrt{(\sqrt{n})^2 + 1^2} = \sqrt{n+1}$$

**Exercise 10.**

$$1 = qb + r$$

$$q = 0, b = 1, r = 1$$

$$2 = qb + r, q = 0, r = 2, b = 1, 2 \text{ or } r = 0, q = 2; q = 1, r = 0$$

$$\text{Assume } n = qb + r; 0 \leq r < b; b \in \mathbb{Z}^+, b \text{ fixed}$$

$$n+1 = qb + r + 1 = qb + 1 + r = qb + 1 + b - 1 = (q+1)b + 0$$

**Exercise 11.** For  $n > 1$ ,  $n = 2, 3$  are prime.  $n = 4 = 2(2)$ , a product of primes.

Assume the  $k-1$ th case. Consider  $\frac{k}{j}, 1 \leq j \leq k$ .

If  $\frac{k}{j} \in \mathbb{Z}^+$ , only for  $j = 1, j = k$ , then  $k$  prime.

If  $\frac{k}{j} \in \mathbb{Z}^+$ , for some  $1 < j < k, \frac{k}{j} = c \in \mathbb{Z}^+, c, j < k$ .

Thus  $k = cj$ .  $c, j$  are products of primes or are primes, by induction hypothesis. Thus  $k$  is a product of primes.

**Exercise 12.**  $n = 2$ .  $G_1, G_2$  are blonde.  $G_1$  has blue eyes. Consider  $G_2$ .  $G_2$  may not have blue eyes. Then  $G_1, G_2$  are not all blue-eyed.

## I 4.7 Exercises - Proof of the well-ordering principle, The summation notation.

### Exercise 1.

- (1)  $\frac{n(n+1)}{2} = \sum_{k=1}^4 k = 10$
- (2)  $\sum_{n=2}^5 2^{n-2} = \sum_{n=0}^3 2^n = 1 + 14 = 15$
- (3)  $2 \sum_{r=0}^3 2^{2r} = 2 \sum_{r=0}^3 4^r = 170$
- (4)  $\sum_{j=1}^4 j^j = 1 + 4 + 27 + 4^4 = 288$
- (5)  $\sum_{j=0}^5 (2j+1) = 2 \frac{5(6)}{2} + 6(1) = 36$
- (6)  $\sum \frac{1}{k(k+1)} = \sum_{k=1}^n \left( \frac{1}{k} - \frac{1}{k+1} \right) = 1 - \frac{1}{n+1} = \frac{n}{n+1}$

### Exercise 2.

- (1) Want:  $\sum_{k=1}^n (a_k + b_k) = \sum_{k=1}^n a_k + \sum_{k=1}^n b_k$  (additive property)

$$a_1 + b_1 = a_1 + b_1$$

$$(a_1 + b_1) + (a_2 + b_2) = (a_1 + a_2) + (b_1 + b_2)$$

$$(a_1 + b_1) + (a_2 + b_2) + (a_3 + b_3) = (a_1 + a_2 + a_3) + (b_1 + b_2 + b_3)$$

$$\sum_{k=1}^{n+1} (a_k + b_k) = \sum_{k=1}^n (a_k + b_k) + a_{n+1} + b_{n+1} = \sum_{k=1}^n a_k + a_{n+1} + \sum_{k=1}^n b_k + b_{n+1} = \sum_{k=1}^{n+1} a_k + \sum_{k=1}^{n+1} b_k$$

- (2) Want:  $\sum_{k=1}^n (ca_k) = c \sum_{k=1}^n a_k$  (homogeneous property).

$$ca_1 = (c)a_1$$

$$ca_1 + ca_2 = c(a_1 + a_2)$$

$$ca_1 + ca_2 + ca_3 = c(a_1 + a_2 + a_3)$$

$$\sum_{k=1}^{n+1} (ca_k) = c \sum_{k=1}^n a_k + ca_{n+1} = c \sum_{k=1}^{n+1} a_k$$

- (3) Want:  $\sum_{k=1}^n (a_k - a_{k-1}) = a_n - a_0$  (telescoping property)

$$\begin{aligned} \sum_{k=1}^n (a_k - a_{k-1}) &= \sum_{k=1}^n (a_k + (-a_{k-1})) = \sum_{k=1}^n a_k + \sum_{k=1}^n (-1)a_{k-1} = \\ &= a_n + \sum_{k=1}^{n-1} a_k + (-1) \sum_{k=1}^{n-1} a_k - a_0 = a_n - a_0 \end{aligned}$$

**Exercise 3.**  $\sum_{k=1}^n 1 = \sum_{k=1}^n (k - (k-1)) = n$

### Exercise 4.

$$k^2 - (k-1)^2 = k^2 - (k^2 - 2k + 1) = 2k - 1$$

$$\sum_{k=1}^n (2k-1) = \sum_{k=1}^n k^2 - (k-1)^2 = n^2 - 0 = n^2$$

**Exercise 5.**  $\sum_{k=1}^n k = \frac{n^2+n}{2} = \frac{n(n+1)}{2}$

### Exercise 6.

$$\begin{aligned} \sum_{k=1}^n k^3 - (k-1)^3 &= n^3 = \sum_{k=1}^n 3k^2 - 3k + 1 = 3 \left( \sum_{k=1}^n k^2 - \frac{n(n+1)}{2} + \frac{n}{3} \right) \\ \implies \sum_{k=1}^n k^2 &= \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} \end{aligned}$$

**Exercise 7.**

$$\begin{aligned}
k^4 - (k-1)^4 &= -(-4k^3 + 6k^2 + -4k + 1) = 4k^3 - 6k^2 + 4k - 1 \\
\sum_{k=1}^n k^4 - (k-1)^4 &= n^4 = 4 \sum_{k=1}^n k^3 - 6 \left( \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} \right) + 4 \left( \frac{n^2}{2} + \frac{n}{2} \right) - n \\
&\implies \frac{n^4 + 2n^3 + n^2}{4} = \sum_{k=1}^n k^3
\end{aligned}$$

**Exercise 8.**

(1)

$$\begin{aligned}
-\sum_{k=1}^n (x^{k+1} - x^k) &= -(x^{n+1} - x) = \\
&= (1-x) \sum_{k=1}^n x^k = x(1-x^n) \\
\implies \sum_{k=0}^n x^k &= \frac{x(1-x^n)}{1-x} + \frac{1-x}{1-x} = \frac{1-x^{n+1}}{1-x}
\end{aligned}$$

(2) If  $x = 1$ , the sum equals  $(n+1)$ .**Exercise 9.**

$$n = 1(-1)(3) + 5 = 2 = 2n$$

$$n = 2(-1)(3) + 5 + (-1)7 + 9 = 4 = 2n$$

$$n \sum_{k=1}^{2n} (-1)^k (2k+1) = 2n$$

$$\begin{aligned}
n+1 \sum_{k=1}^{2(n+1)} (-1)^k (2k+1) &= \sum_{k=1}^{2n} (-1)^k (2k+1) + (-1)^{2n+1} (4n+3) + (-1)^{2n+2} (4n+5) = \\
&= 2n + 2 = 2(n+1)
\end{aligned}$$

**Exercise 10.**

$$(1) a_m + a_{m+1} + \cdots + a_{m+n}$$

(2)

$$n = 1 \frac{1}{2} = \frac{1}{1} - \frac{1}{2} = \frac{1}{2}$$

$$\begin{aligned}
n+1 \sum_{k=n+2}^{2(n+1)} \frac{1}{k} &= \sum_{m=1}^{2n} \frac{(-1)^{m+1}}{m} - \frac{1}{n+1} + \frac{1}{2n+1} + \frac{1}{2n+2} = \sum_{m=1}^{2n} \frac{(-1)^{m+1}}{m} + \frac{1}{2(n+1)} + \frac{(-1)^{2n+1+1}}{(2n+1)} \\
&= \sum_{m=1}^{2(n+1)} \frac{(-1)^{m+1}}{m}
\end{aligned}$$

**Exercise 13.**

$$n = 12(\sqrt{2}-1) < 1 < 2 \text{ since } \frac{1}{2} > \sqrt{2}-1$$

$$n \text{ case } (\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n}) = n+1 - n = 1 < \frac{\sqrt{n+1} + \sqrt{n}}{2\sqrt{n}} = \frac{1}{2} \left( \sqrt{1 + \frac{1}{n}} + 1 \right)$$

$$n+1 \text{ case } (\sqrt{n+2} - \sqrt{n+1})(\sqrt{n+2} + \sqrt{n+1}) = n+2 - (n+1) = 1$$

$$\frac{\sqrt{n+2} + \sqrt{n+1}}{2\sqrt{n+1}} = \frac{1 + \sqrt{1 + \frac{1}{n+1}}}{2} > 1$$

So then, using the telescoping property,

$$\sum_{n=1}^{n-1} 2(\sqrt{n+1} - \sqrt{n}) = 2(\sqrt{m} - 1) < \sum_{n=1}^m \frac{1}{\sqrt{n}} < \sum_{n=1}^m 2(\sqrt{n} - \sqrt{n-1}) = 2(\sqrt{m} - 1) < 2\sqrt{m} - 1$$

**I 4.9 Exercises - Absolute values and the triangle inequality. Exercise 1.**

(1)  $|x| = 0$  iff  $x = 0$

If  $x = 0$ ,  $x = 0$ ,  $-x = -0 = 0$ . If  $|x| = 0$ ,  $x = 0$ ,  $-x = 0$ .

(2)

$$|-x| = \begin{cases} -x & \text{if } -x \geq 0 \\ x & \text{if } -x \leq 0 \end{cases} = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x \leq 0 \end{cases}$$

(3)  $|x - y| = |y - x|$  by previous exercise and  $(-1)(x - y) = y - x$  (by distributivity)

(4)  $|x|^2 = \begin{cases} (x)^2 & \text{if } x \geq 0 \\ (-x)^2 & \text{if } x \leq 0 \end{cases} = x^2$

(5)  $\sqrt{x^2} = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x \leq 0 \end{cases} = |x|$

(6) We want to show that  $|xy| = |x||y|$

$$|xy| = \begin{cases} xy & \text{if } xy \geq 0 \\ -xy & \text{if } xy \leq 0 \end{cases} = \begin{cases} xy & \text{if } x, y \geq 0 \text{ or } x, y \leq 0 \\ -xy & \text{if } x, -y \geq 0 \text{ or } -x, y \leq 0 \end{cases}$$

$$|x||y| = \begin{cases} x|y| & \text{if } x \geq 0 \\ -x|y| & \text{if } x \leq 0 \end{cases} = \begin{cases} xy & \text{if } x, y \geq 0 \\ -xy & \text{if } x, -y \geq 0 \\ -xy & \text{if } -x, y \geq 0 \\ xy & \text{if } -x, -y \geq 0 \end{cases}$$

(7) By previous exercise, since

$$\left| \frac{x}{y} \right| = |xy^{-1}| = |x||y^{-1}|$$

$$\left| \frac{1}{y} \right| = \begin{cases} \frac{1}{y} & \text{if } \frac{1}{y} \geq 0 \\ \frac{-1}{y} & \text{if } \frac{1}{y} \leq 0 \end{cases} \quad \frac{1}{|y|} = \begin{cases} \frac{1}{y} & \text{if } \frac{1}{y} \geq 0 \\ \frac{-1}{y} & \text{if } \frac{1}{y} \leq 0 \end{cases}$$

(8) We know that  $|a - b| \leq |a - c| + |b - c|$ .

$$\text{Let } c = 0 \implies |x - y| \leq |x| + |y|$$

(9)  $x = a - b, b - c = -y$ .

$$|x| \leq |x - y| + |-y| \quad |x| - |y| \leq |x - y|$$

(10)

$$||x| - |y|| = \begin{cases} |x| - |y| & \text{if } |x| - |y| \geq 0 \\ |y| - |x| & \text{if } |x| - |y| \leq 0 \end{cases}$$

$$|x| \leq |x - y| + |-y| \implies |x| - |y| \leq |x - y|$$

$$|y| \leq |y - x| + |-x| \implies |y| - |x| \leq |y - x| = |x - y|$$

**Exercise 4.**

$\implies$

$$\text{If } \forall k = 1 \dots n; a_k x + b_k = 0$$

$$\left( \sum_{k=1}^n a_k (-x a_k) \right)^2 = \left( x \sum_{k=1}^n a_k^2 \right)^2 = \left( \sum_{k=1}^n a_k^2 \right) \left( \sum_{k=1}^n (-x a_k)^2 \right) = \left( \sum_{k=1}^n a_k^2 \right) \left( \sum_{k=1}^n b_k^2 \right)$$

$$\Leftarrow \text{Proving } a_k x + b_k = 0 \text{ means } x = -\frac{b_k}{a_k}, a_k \neq 0$$

$$(a_1 b_1 + a_2 b_2 + \dots + a_n b_n)^2 = \sum_{j=1}^n a_j^2 b_j^2 + \sum_{j \neq q}^n a_j a_k b_j b_k = \sum_{j=1}^n a_j^2 b_j^2 + \sum_{j \neq k}^n a_j^2 b_k^2$$

$$\implies a_j^2 b_k^2 - a_j a_k b_j b_k = a_j b_k (a_j b_k - a_k b_j) = 0$$

$$\text{if } a_j, b_k \neq 0, a_j b_k - a_k b_j = 0 \implies a_k \left( \frac{b_j}{-a_j} \right) + b_k = 0$$

**Exercise 8.** The trick of this exercise is the following algebraic trick (“multiplication by conjugate”) and using telescoping property of products:

$$(1 - x^{2^j})(1 + x^{2^j}) = 1 - x^{2^j+2^j} = 1 - x^{2^{j+1}}$$

$$\prod_{j=1}^1 1 + x^{2^{j-1}} = \prod_{j=1}^1 \frac{1 - x^{2^j}}{1 - x^{2^{j-1}}} = \frac{1 - x^{2^n}}{1 - x}$$

if  $x = 1, 2^n$

**Exercise 10.**

$$x > 1$$

$$x^2 > x \quad x^{n+1} = x^n x > x^2 > x$$

$$x^3 > x^2 > x$$

$$0 < x < 1$$

$$x^2 < x \quad x^{n+1} = x^n x < x^2 < x \implies x^{n+1} < x$$

$$X^3 < x^2 < x$$

**Exercise 11.** Let  $S = \{n \in \mathbb{Z}^+ | 2^n < n!\}$ .

By well-ordering principle,  $\exists$  smallest  $n_0 \in S$ . Now

$2^4 = 16$ ,  $4! = 24$ . So  $S$  starts at  $n = 4$ .

**Exercise 12.**

(1)

$$\left(1 + \frac{1}{n}\right)^n = \sum_{j=0}^n \binom{n}{j} \left(\frac{1}{n}\right)^j = \sum_{k=0}^n \frac{n!}{(n-k)!k!} \left(\frac{1}{n}\right)^k$$

$$\prod_{r=0}^{k-1} \left(1 - \frac{r}{n}\right) = \prod_{r=0}^{k-1} \left(\frac{n-r}{n}\right) = \left(\frac{1}{n^k}\right) \frac{n!}{(n-k)!}$$

$$\sum_{k=1}^n \frac{1}{k!} \prod_{r=0}^{k-1} \left(1 - \frac{r}{n}\right) = \left(\frac{1}{n^k}\right) \frac{n!}{(n-k)!}$$

(2)

$$\left(1 + \frac{1}{n}\right)^n = 1 + \sum_{k=1}^n \left(\frac{1}{k!} \prod_{r=0}^{k-1} \left(1 - \frac{r}{n}\right)\right) < 1 + \sum_{k=1}^n \frac{1}{k!} < 1 + \sum_{k=1}^n \frac{1}{2^k} = 1 + \frac{\frac{1}{2} - \left(\frac{1}{2}\right)^{n+1}}{\frac{1}{2}} = 1 + \left(1 - \left(\frac{1}{2}\right)^n\right)$$

$< 3$

The first inequality obtained from the fact that if  $0 < x < 1$ ,  $x^n < x < 1$ . The second inequality came from the previous exercise, that  $\frac{1}{k!} < \frac{1}{2^k}$ .

$$\left(1 + \frac{1}{n}\right)^n = \sum_{k=0}^n \binom{n}{k} \left(\frac{1}{n}\right)^k = 1 + \frac{1}{n} + \sum_{k=1}^{n-1} \binom{n}{k} \left(\frac{1}{n}\right)^k = 1 + \frac{1}{n} + \sum_{k=2}^{n-1} \binom{n}{k} \left(\frac{1}{n}\right)^k + \frac{n}{1} \left(\frac{1}{n}\right) > > 2$$

**Exercise 13.**

(1)

$$S = \sum_{k=0}^{p-1} \left(\frac{b}{a}\right)^k = \frac{1 - \left(\frac{b}{a}\right)^p}{1 - \frac{b}{a}}$$

$$\sum_{k=0}^{p-1} b^k a^{p-1-k} = a^{p-1} \frac{1 - \left(\frac{b}{a}\right)^p}{1 - \frac{b}{a}} = \frac{b^p - a^p}{b - a}$$

(2)

(3) Given

$$n^p < \frac{(n+1)^{p+1} - n^{p+1}}{p+1} < (n+1)^p$$

We want

$$\sum_{k=1}^{n-1} k^p < \frac{n^{p+1}}{p+1} < \sum_{k=1}^n k^p$$

$$\begin{aligned}
n = 21^p &< \frac{2^{p+1}}{p+1} < 1^p + 2^p \\
p &= 1 \\
1 &< 2^2/2 = 2.2 < 1 + 2 = 3 \\
p &= 2 \\
1 &< 8/3 < 1 + 4 = 5
\end{aligned}$$

#### I 4.10 Miscellaneous exercises involving induction. Exercise 13.

- (1)
- (2)
- (3) Let  $n = 2$ .

$$\sum_{k=1}^{2-1} k^p = 1^p = 1, \frac{n^{p+1}}{p+1} = \frac{2^{p+1}}{p+1} \sum_{k=1}^2 k^p = 1 + 2^p$$

What makes this exercise hard is that **we have to use induction on  $p$  itself**. Let  $p = 1$ .

$$1 < \frac{2^{1+1}}{1+2} = 2 < 1 + 2^1 = 3$$

Now assume  $p$ th case. Test the  $p+1$  case.

$$\frac{2^{p+2}}{p+2} = \frac{2(p+1)}{p+2} \left( \frac{2^{p+1}}{p+1} \right) > 1$$

since  $p+2 < 2p+2 = 2(p+1)$  for  $p \in \mathbb{Z}^+$

For the right-hand inequality, we will use the fact just proven, that  $2^p - (p) > 0$  and  $p$ th case rewritten in this manner

$$(1 + 2^p) > \frac{2^{p+1}}{p+1} \implies (1 + 2^p)(p+1) > 2^{p+1}$$

So

$$\begin{aligned}
(p+2)(1 + 2^{p+1}) &= (p+2) + ((p+1) + 1)2^p(2) = (p+2) + 2(p+1)2^p + 2^p(2) > \\
&> (p+2) + 2(2^{p+1} - (p+1)) + 2^p(2) = -p + 2^{p+2} + 2^{p+1} > 2^{p+2}
\end{aligned}$$

So the  $n = 2$  case is true for all  $p \in \mathbb{Z}^+$ .

Assume  $n$ th case is true. We now prove the  $n+1$  case.

$$\begin{aligned}
\sum_{k=1}^n k^p &= \sum_{k=1}^{n-1} k^p + n^p < \frac{n^{p+1}}{p+1} + n^p < \frac{n^{p+1}}{p+1} + \frac{(n+1)^{p+1} - n^{p+1}}{p+1} = \frac{(n+1)^{p+1}}{p+1} \\
\sum_{k=1}^{n+1} k^p &= \sum_{k=1}^n k^p + (n+1)^p > \frac{n^{p+1}}{p+1} + \frac{(n+1)^{p+1} - n^{p+1}}{p+1} = \frac{(n+1)^{p+1}}{p+1}
\end{aligned}$$

We had used the inequality proven in part b,  $n^p < \frac{(n+1)^{p+1} - n^{p+1}}{p+1} < (n+1)^p$ .

**Exercise 14.** Use induction to prove a general form of Bernoulli's inequality.

$$\begin{aligned}
1 + a_1 &= 1 + a_1 \\
(1 + a_1)(1 + a_2) &= 1 + a_2 + a_1 + a_1a_2 \geq 1 + a_1 + a_2
\end{aligned}$$

Test the  $n+1$  case

$$\begin{aligned}
(1 + a_1)(1 + a_2) \dots (1 + a_{n+1}) &\geq (1 + a_1 + a_2 + \dots + a_n)(1 + a_{n+1}) = \\
&= 1 + a_1 + a_2 + \dots + a_n + a_{n+1} + a_{n+1}(a_1 + a_2 + \dots + a_n) \geq \\
&\geq 1 + a_1 + a_2 + \dots + a_n + a_{n+1}
\end{aligned}$$

Note that the last step depended upon the given fact that all the numbers were of the same sign.

For  $a_1 = a_2 = \dots = a_n = x$ , then we have  $(1+x)^n \geq 1 + nx$ .

$$(1+x)^n = \sum_{j=0}^n \binom{n}{j} x^j = 1 + nx$$

Since  $x$  and  $n$  are arbitrary, we can compare terms of  $x^j$ 's. Then  $x = 0$ .

**Exercise 15.**  $\frac{2!}{2^2} = \frac{1}{2}$ ,  $\frac{3!}{3^3} = \frac{2}{9} < 1$ .

So we've shown the  $n = 2, n = 3$  cases. Assume the  $n$ th case, that  $\frac{n!}{n^n} \leq \left(\frac{1}{2}\right)^k$ , where  $k$  is the greatest integer  $\leq \frac{n}{2}$ .

$$\frac{(n+1)!}{(n+1)^{n+1}} \geq \frac{(n+1)n^n \left(\frac{1}{2}\right)^k}{(n+1)^{n+1}} = \left(\frac{n}{n+1}\right)^n \left(\frac{1}{2}\right)^k = \left(1 - \frac{1}{n+1}\right)^n \left(\frac{1}{2}\right)^k < \frac{1}{2} \left(\frac{1}{2}\right)^k = \left(\frac{1}{2}\right)^{k+1}$$

where in the second to the last step, we had made this important observation:

$$k \leq \frac{n}{2} \implies k + \frac{1}{2} \leq \frac{n+1}{2} \implies \frac{1}{n+1} \leq \frac{1}{2k+1} < \frac{1}{2}$$

**Exercise 16.**

$$\begin{aligned} a_1 &= 1 < \frac{1+\sqrt{5}}{2} \\ a_2 &= 2 < \left(\frac{1+\sqrt{5}}{2}\right)^2 = \frac{1+2\sqrt{5}+5}{4} = \frac{6+2\sqrt{5}}{4} \\ a_{n+1} &= a_n + a_{n-1} < \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1+\sqrt{5}}{2}\right)^{n-1} = \left(\frac{1+\sqrt{5}}{2}\right)^n \left(1 + \frac{2}{1+\sqrt{5}}\right) = \\ &= \left(\frac{1+\sqrt{5}}{2}\right)^n \left(\frac{2(1-\sqrt{5})}{1-5} + \frac{4}{4}\right) = \left(\frac{1+\sqrt{5}}{2}\right)^{n+1} \end{aligned}$$

**Exercise 17.** Use Cauchy-Schwarz, which says

$$\left(\sum a_k b_k\right)^2 \leq \left(\sum a_k^2\right) \left(\sum b_k^2\right)$$

Let  $a_k = x_k^p$  and  $b_k = 1$ . Then Cauchy-Schwarz says

$$\left(\sum x_k^p\right)^2 \leq \left(\sum x_k^{2p}\right) n \implies \sum x_k^{2p} \geq \frac{(\sum x_k^p)^2}{n}$$

We define  $M_p$  as follows:

$$M_p = \left(\frac{\sum_{k=1}^n x_k^p}{n}\right)^{1/p}$$

So then

$$\begin{aligned} nM_p^p &= \sum_{k=1}^n x_k^p \\ M_{2p} &= \left(\frac{\sum_{k=1}^n x_k^{2p}}{n}\right)^{1/2p} \\ nM_{2p}^{2p} &= \sum_{k=1}^n x_k^{2p} \\ \sum_{k=1}^n x_k^{2p} &= nM_{2p}^{2p} \geq \frac{(nM_p^p)^2}{n} = nM_p^{2p} \\ M_{2p}^{2p} &\geq M_p^{2p} \implies M_{2p} \geq M_p \end{aligned}$$

**Exercise 18.**

$$\begin{aligned} \left(\frac{a^4 + b^4 + c^4}{3}\right)^{1/4} &\geq \left(\frac{a^2 + b^2 + c^2}{3}\right)^{1/2} = \frac{2^{3/2}}{3^{1/2}} \text{ since} \\ a^4 + b^4 + c^4 &\geq \frac{64}{3} \end{aligned}$$

**Exercise 19.**  $a_k = 1, \sum_{k=1}^n 1 = n$

Now consider the case of when not all  $a_k = 1$ .

$$a_1 = 1$$

$a_1 a_2 = 1$  and suppose, without loss of generality  $a_1 > 1$ . Then  $1 > a_2$ .

$$(a_1 - 1)(a_2 - 1) < 0$$

$$a_1 a_2 - a_1 - a_2 + 1 < 0 \implies a_1 + a_2 > 2$$

(consider  $n + 1$  case ) If  $a_1 a_2 \dots a_{n+1} = 1$ , then suppose  $a_1 > 1$ ,  $a_{n+1} < 1$  without loss of generality

$$b_1 = a_1 a_{n+1}$$

$$b_1 a_2 \dots a_n = 1 \implies b_1 + a_2 + \dots + a_n \geq n \text{ (by the induction hypothesis)}$$

$$(a_1 - 1)(a_{n+1} - 1) = a_1 a_{n+1} - a_1 - a_{n+1} + 1 < 0, b_1 < a_1 + a_{n+1} - 1$$

$$\implies a_1 + a_{n+1} - 1 + a_2 + \dots + a_n > b_1 + a_2 + \dots + a_n \geq n$$

$$\implies a_1 + a_2 + \dots + a_{n+1} \geq n + 1$$

### 1.7 Exercises - The concept of area as a set function. We will use the following axioms:

Assume a class  $\mathcal{M}$  of measurable sets (i.e. sets that can be assigned an area), set function  $a, a : \mathcal{M} \rightarrow \mathbb{R}$ .

•

**Axiom 2** (Nonnegative property).

$$(1) \quad \forall S \in \mathcal{M}, a(S) \geq 0$$

•

**Axiom 3** (Additive property). If  $S, T \in \mathcal{M}$ , then  $S \cup T, S \cap T \in \mathcal{M}$  and

$$(2) \quad a(S \cup T) = a(S) + a(T) - a(S \cap T)$$

•

**Axiom 4** (Difference property). If  $S, T \in \mathcal{M}, S \subseteq T$  then  $T - S \in \mathcal{M}$  and

$$(3) \quad a(T - S) = a(T) - a(S)$$

•

**Axiom 5** (Invariance under congruence). If  $S \in \mathcal{M}, T = S$ , then  $T \in \mathcal{M}, a(T) = a(S)$

•

**Axiom 6** (Choice of scale).  $\forall$  rectangle  $R \in \mathcal{M}$ , if  $R$  has edge lengths  $h, k$  then  $a(R) = hk$

•

**Axiom 7** (Exhaustion property). Let  $Q$  such that

$$(4) \quad S \subseteq Q \subseteq T$$

If  $\exists$  only one  $c$  such that  $a(S) \leq c \leq a(T), \forall S, T$  such that they satisfy Eqn. (4) then  $Q$  measurable and  $a(Q) = c$

#### Exercise 1.

- (1) We need to say that we consider a line segment or a point to be a special case of a rectangle allowing  $h$  or  $k$  (or both) to be zero.

Let  $T_l = \{ \text{line segment containing } x_0 \}, Q = \{x_0\}$ .

For  $Q$ , only  $\emptyset \subset Q$

By Axiom 3, let  $T = S$ .

$$a(T - S) = a(\emptyset) = a(T) - a(T) = 0$$

$$\emptyset \subset Q \subseteq T_l \implies a(\emptyset) \leq a(Q) \leq a(T_l) \implies 0 \leq a(Q) \leq 0$$

$$\implies a(Q) = 0$$

- (2)

$$a \left( \bigcup_{j=1}^N Q_j \right) = \sum_{j=1}^N a(Q_j)$$

if  $Q_j$ 's disjoint. Let  $Q_j = \{x_j\}$ .

Since  $a(Q_j) = 0$ . By previous part,  $a \left( \bigcup_{j=1}^N Q_j \right) = 0$



**Exercise 2.** Let  $A, B$  be rectangles. By Axiom 5,  $A, B$  are measurable. By Axiom 2,  $A \cap B$  measurable.

$$a(A \cap B) = \sqrt{a^2 + b^2}d + ab - \left(\frac{1}{2}ab + \sqrt{a^2 + b^2}d\right) = \frac{1}{2}ab$$

**Exercise 3.** Prove that every trapezoid and every parallelogram is measurable and derive the usual formulas for their areas.

A trapezoid is simply a rectangle with a right triangle attached to each end of it.  $T_r = R + T_1 + T_2$ .  $T_1, T_2$  are right triangles and so by the previous problem,  $T_1, T_2$  are measurable. Then  $T_r$  is measurable by the Additive property axiom (note that the triangles and the rectangle don't overlap).

We can compute the area of a trapezoid:

$$\begin{aligned} T_r = R + T_1 + T_2 &\implies a(T_r) = a(R) + a(T_1) + a(T_2) \\ a(T_r) &= hb_1 + \frac{1}{2}h(b_2 - b_1)/2 + \frac{1}{2}h(b_2 - b_1)/2 = \frac{1}{2}h(b_1 + b_2) \end{aligned}$$

$P = R$  (a parallelogram consists of a right triangle rotated by  $\pi$  and attached to the other side of the same right triangle; the two triangles do not overlap). Since two right triangles are measurable, the parallelogram,  $P$  is measurable.

Using the Additive Axiom,  $a(P) = 2a(T) = 2\frac{1}{2}bh = bh$

**Exercise 4.** A point  $(x, y)$  in the plane is called a lattice point if both coordinates  $x$  and  $y$  are integers. Let  $P$  be a polygon whose vertices are lattice points. The area of  $P$  is  $I + \frac{1}{2}B - 1$ , where  $I$  denotes the number of lattice points inside the polygon and  $B$  denotes the number on the boundary.

- (1) Consider one side of the rectangle lying on a coordinate axis with one end on the origin. If the rectangle side has length  $l$ , then  $l + 1$  lattice points lie on this side (you have to count one more point at the 0 point. Then consider the same number of lattice points on the opposite side. We have  $2(l + 1)$  lattice points so far, for the *boundary*.

The other pair of sides will contribute  $2(h - 1)$  lattice points, the  $-1$  to avoid double counting. Thus  $2(l + h) = B$ .

$I = (h - 1)(l - 1)$  by simply considering multiplication of  $(h - 1)$  rows and  $(l - 1)$  columns of lattice points inside the rectangle.

$$I + \frac{1}{2}B - 1 = hl - h - l + 1 + (l + h) - 1 = hl = a(R)$$

(2)

(3)

**Exercise 5.** Prove that a triangle whose vertices are lattice points cannot be equilateral.

My way: I will take, for granted, that we know an equilateral triangle has angles of  $\pi/3$  for all its angles.

Even if we place two of the vertices on lattice points, so that its length is  $2L$ , and put the midpoint and an intersecting perpendicular bisector on a coordinate axis (a picture would help), but the ratio of the perpendicular bisector to the third vertex to half the length of the triangle is  $\cot \pi/3 = \frac{1}{\sqrt{3}}$ . Even if we go down by an integer number  $L$ ,  $L$  steps down, we go "out" to the third vertex by an irrational number  $\sqrt{3}L$ . Thus, the third vertex cannot lie on a lattice point.

**Exercise 6.** Let  $A = \{1, 2, 3, 4, 5\}$  and let  $\mathcal{M}$  denote the class of all subsets of  $A$ . (There are 32 altogether counting  $A$  itself and the empty set  $\emptyset$ ). (My Note: the set of all subsets, in this case,  $\mathcal{M}$ , is called a *power set* and is denoted  $2^A$ . This is because the way to get the total number of elements of this power set,  $|2^A|$ , or the size, think of assigning to each element a "yes," if it's in some subset, or "no", if it's not. This is a great way of accounting for all possible subsets and we correctly get all possible subsets.) For each set  $S$  in  $\mathcal{M}$ , let  $n(S)$  denote the number of distinct elements in  $S$ . If  $S = \{1, 2, 3, 4\}$  and  $T = \{3, 4, 5\}$ ,

$$\begin{aligned} n(S \cup T) &= 5 \\ n(S \cap T) &= 2 \\ n(S - T) &= n(\{1, 2\}) = 2 \\ n(T - S) &= n(\{5\}) = 1 \end{aligned}$$

$n$  satisfies nonnegative property because by definition, there's no such thing as a negative number of elements. If  $S, T$  are subsets of  $A$ , so are  $S \cup T, S \cap T$  since every element in  $S \cup T, S \cap T$  is in  $S$ . Thus  $n$  could be assigned to it, so that it's *measurable*. Since  $n$  counts only distinct elements, then  $n(S \cup T) = n(S) + a(T) - a(S \cap T)$ , where  $-a(S \cap T)$  ensures there is no double counting of distinct elements. Thus, the Additive Property Axiom is satisfied.

For  $S \subseteq T$ , then  $\forall x \in T - S, x \in T, x \notin S$  Now  $S \subseteq T$ , so  $\forall x \in S, x \in T$ . Thus  $T - S$  is complementary to  $S$  "with respect to"  $T$ .  $n(S) + n(T - S) = n(T)$ , since  $n$  counts up distinct elements.

**1.11 Exercises - Intervals and ordinate sets, Partitions and step functions, Sum and product of step function. Exercise 4.**

(1)

$$[x + n] = y \leq x + n, y \in \mathbb{Z}; y - n \leq x$$

$$[x] + n = z + n \leq x + n$$

If  $y - n < z$ , then  $y < z + n \leq x + n$ . then  $y$  wouldn't be the greatest integer less than  $x + n$

$$\implies y = z + n$$

(2)

$$= y_2 \leq x - [x] = -y_2 \geq -x - y_2 - 1 \leq x$$

$$-x \geq y_1 = [-x] = -y_2 - 1 = -[x] - 1; \text{ ( and } y_1 = -y_2 - 1 \text{ since } -y_2 > -x )$$

$$\text{If } x \text{ is an integer } -[x] = [-x]$$

(3) Let  $x = q_1 + r_1, y = q_2 + r_2; 0 \leq r_1, r_2 < 1$ .

$$= [q_1 + q_2 + r_1 + r_2] = \begin{cases} q_1 + q_2 \\ q_1 + q_2 + 1 \text{ if } r_1 + r_2 \geq 1 \end{cases}$$

$$[x] + [y] = q_1 + q_2 \quad [x] + [y] + 1 = q_1 + q_2 + 1$$

(4)

$$\text{If } x \text{ is an integer, } [2x] = 2x = [x] + [x + \frac{1}{2}] = [x] + [x] = 2x$$

$$[x] + [x + \frac{1}{2}] = q + \begin{cases} q \\ 2q + 1 \text{ if } r > \frac{1}{2} \end{cases} \quad \text{if } r < \frac{1}{2}$$

$$[2x] = [2(q + r)] = [2q + 2r] = \begin{cases} 2q & \text{if } r < \frac{1}{2} \\ 2q + 1 & \text{if } r > \frac{1}{2} \end{cases}$$

(5)

$$[x] + [x + \frac{1}{3}] + [x + \frac{2}{3}] = q + \begin{cases} q \\ q + 1 \text{ if } r > \frac{2}{3} \end{cases} \quad \text{if } r < \frac{2}{3} + \begin{cases} q \\ q + 1 \text{ if } r > \frac{1}{3} \end{cases} = \begin{cases} 3q & \text{if } r < \frac{1}{3} \\ 3q + 1 & \text{if } \frac{1}{3} < r < \frac{2}{3} \\ 3q + 2 & \text{if } r > \frac{2}{3} \end{cases}$$

$$[3x] = [3(q + r)] = [3q + 3r] = \begin{cases} 3q & \text{if } r < \frac{1}{3} \\ 3q + 1 & \text{if } \frac{1}{3} < r < \frac{2}{3} \\ 3q + 2 & \text{if } r > \frac{2}{3} \end{cases}$$

**Exercise 5.** Direct proof.

$$[nx] = [n(q + r)] = \begin{cases} nq & \text{if } r < \frac{1}{n} \\ nq + 1 & \text{if } \frac{1}{n} < r < \frac{2}{n} \\ nq + n - 1 & \text{if } r > \frac{n-1}{n} \end{cases}$$

**Exercise 6.**

$$a(R) = hk = I_R + \frac{1}{2}B_R - 1$$

$$\sum_{n=a}^b [f(n)] = [f(a)] + [f(a+1)] + \cdots + [f(b)]$$

$[f(n)] = g \leq f(n), g \in \mathbb{Z}$ , so that if  $f(n)$  is an integer,  $g = f(n)$ , and if  $f(n)$  is not an integer,  $g$  is the largest integer such that  $g < f(n)$ , so that all lattice points included and less than  $g$  are included.

**Exercise 7.**

(1) Consider a right triangle with lattice points as vertices. Consider  $b + 1$  lattice points as the base with  $b$  length.

Start from the vertex and move across the base by increments of 1.

The main insight is that the slope of the hypotenuse of the right triangle is  $\frac{a}{b}$  so as we move 1 along the base, the hypotenuse (or the  $y$ -value, if you will) goes up by  $\frac{a}{b}$ . Now

$$(5) \quad \left\lceil \frac{na}{b} \right\rceil = \text{number of interior points at } x = n \text{ and below the hypotenuse line of the right triangle of sides } a, b,$$

including points on the hypotenuse

$$\sum_{n=1}^{b-1} \left[ \frac{na}{b} \right] + \frac{1}{2}((a+1)+b) - 1 = \frac{ab}{2}$$

Now  $\frac{(a-1)(b-1)}{2} = \frac{ab}{2} - \frac{a}{2} - \frac{b}{2} + \frac{1}{2}$

$$\implies \sum_{n=1}^{b-1} \left[ \frac{na}{b} \right] = \frac{(a-1)(b-1)}{2}$$

(2)  $a, b \in \mathbb{Z}^+$

$$\sum_{n=1}^{b-1} \left[ \frac{na}{b} \right] = \sum_{n=1}^{b-1} \left[ \frac{a(b-n)}{b} \right] \quad (\text{reverses order of summation})$$

$$\sum_{n=1}^{b-1} \left[ a - \frac{an}{b} \right] = \begin{cases} -\sum_{n=1}^{b-1} \left[ \frac{an}{b} - a \right] & \text{if } \frac{an}{b} - a \text{ is an integer (but } a(\frac{n}{b} - 1) \text{ can't be!)} \\ -\sum_{n=1}^{b-1} ([\frac{an}{b} - a] - 1) & \text{otherwise} \end{cases}$$

$$= -\sum_{n=1}^{b-1} ([\frac{an}{b} - a] - 1) = -\sum_{n=1}^{b-1} ([\frac{an}{b}] - a) - (b-1) =$$

$$= -\sum_{n=1}^{b-1} [\frac{an}{b}] + a(b-1) - (b-1)$$

$$\sum_{n=1}^{b-1} \left[ \frac{na}{b} \right] = \frac{(a-1)(b-1)}{2}$$

**Exercise 8.** Recall that for the step function  $f = f(x)$ , there's a partition  $P = \{x_0, x_1, \dots, x_n\}$  of  $[a, b]$  such that  $f(x) = c_k$  if  $x \in I_k$ .

Given that  $\chi_s(x) = \begin{cases} 1 & \forall x \in S \\ 0 & \forall x \notin S \end{cases}$ .

If  $x \in [a, b]$ , then  $x$  must only lie in one open subinterval  $I_j$ , since real numbers obey transitivity.

$$\sum_{k=1}^n c_k \chi_{I_k}(x) = c_j \text{ for } x \in I_j \implies \sum_{k=1}^n c_k \chi_{I_k}(x) = f(x) \forall x \in [a, b]$$

### 1.15 Exercises - The definition of the integral for step functions, Properties of the integral of a step function, Other notations for integrals. Exercise 1.

- (1)  $\int_1^3 [x] dx = (-1) + 1 + (2) = 2$
- (2)  $\int_{-1}^3 [x + \frac{1}{2}] dx = \int_{-1/2}^{7/2} [x] dx = (-1)\frac{1}{2} + (1)(1) + (2)(1) + \frac{1}{2}3 = 4$
- (3)  $\int_{-1}^3 ([x] + [x + \frac{1}{2}]) dx = 6$
- (4)  $\int_{-1}^3 2[x] dx = 4$
- (5)  $\int_{-1}^3 [2x] dx = \frac{1}{2} \int_{-2}^6 [x] dx = \frac{1}{2}((-2)1 + (-1) + (1) + 2 + 3 + 4 + 5) = 6$
- (6)  $\int_{-1}^3 [-x] dx = -\int_1^{-3} [x] dx = \int_{-3}^1 [x] dx = -3 + -2 + -1 = -6$

**Exercise 2.**

$$s = \begin{cases} 5/2 & \text{if } 0 < x < 2 \\ -1 & \text{if } 2 < x < 5 \end{cases}$$

**Exercise 3.**  $[x] = y \leq x$  so  $-y \geq -x$ .

$-y - 1 \leq -x$ , otherwise if  $-y - 1 \geq -x$ ,  $y + 1 \leq x$  and so  $y$  wouldn't be the largest integer  $\leq x$ .

$$\implies [x] + [-x] = y - y - 1 = -1$$

Or use Exercise 4(c), pp. 64.

$$\int_a^b ([x] + [-x]) dx = \int_a^b [x - x] dx = \int_a^b (-1) dx = a - b$$

**Exercise 4.**

- (1)  $n \in \mathbb{Z}^+$ ,  $\int_0^n [t] dt = \sum_{t=0}^{n-1} t = \frac{(n-1)(n-1+1)}{2} = \frac{(n-1)n}{2}$
- (2)

**Exercise 5.**

$$(1) \int_0^2 [t^2] dt = \int_1^2 [t^2] dt = 1(\sqrt{2} - 1) + 2(\sqrt{3} - \sqrt{2}) + 3(2 - \sqrt{3}) = 5 - \sqrt{2} - \sqrt{3}$$

$$(2) \int_{-3}^3 [t^2] dt = \int_0^3 [t^2] dt + \int_{-3}^0 [t^2] dt = \int_0^3 [t^2] dt + - \int_3^0 [t^2] dt = 2 \int_0^3 [t^2] dt$$

$$\int_2^3 [t^2] dt = 4(\sqrt{5} - 2) + 5(\sqrt{6} - \sqrt{5}) + 6(\sqrt{7} - \sqrt{6}) + 7(\sqrt{8} - \sqrt{7}) + 8(3 - \sqrt{8})$$

$$16 - \sqrt{5} - \sqrt{6} - \sqrt{7} - \sqrt{8}$$

$$\int_0^2 [t^2] dt + \int_2^3 [t^2] dt = 21 - 3\sqrt{2} - \sqrt{3} - \sqrt{5} - \sqrt{6} - \sqrt{7} - \sqrt{8}$$

$$\implies \int_{-3}^3 [t^2] dt = 42 - 2(3\sqrt{2} + \sqrt{3} + \sqrt{5} + \sqrt{6} + \sqrt{7})$$

**Exercise 6.**

$$(1) \int_0^n [t]^2 dt = \int_1^n [t]^2 dt = \sum_{j=1}^{n-1} j^2 = \frac{(n-1)n(2n-1)}{6}$$

$$(2) \int_0^x [t]^2 dt = \sum_{j=1}^{[x-1]} j^2 + q^2 r \text{ where } x = q + r, q \in \mathbb{Z}^+, 0 \leq r < 1.$$

$$\int_0^x [t]^2 dt = \frac{q(q-1)(2q-1)}{6} + q^2 r = 2(x-1) = 2(q+r-1)$$

$$\implies q(q-1)(2q-1) + 6q^2 r = 12q + 12r - 12$$

$$\implies x = 1, x = 5/2$$

**Exercise 7.**

(1)

$$\int_0^9 [\sqrt{t}] dt = \int_1^9 [\sqrt{t}] dt = 3(1) + 5(2) = 13$$

$$\int_0^1 6[\sqrt{t}] dt = 3(1) + 5(2) + 7(3) = 34 = \frac{(4)(3)(17)}{6}$$

$$\text{Assume } \int_0^{n^2} [\sqrt{t}] dt = \frac{n(n-1)(4n+1)}{6}$$

$$\begin{aligned} \int_0^{(n+1)^2} [\sqrt{t}] dt &= \int_0^{n^2} [\sqrt{t}] dt + \int_{n^2}^{(n+1)^2} [\sqrt{t}] dt = \frac{n(n-1)(4n+1)}{6} + n((n+1)^2 - n^2) = \\ &= \frac{(n^2 - n)(4n+1) + 6n(2n+1)}{6} = \frac{4n^3 + n^2 - 4n^2 - n + 12n^2 + 6n}{6} = \frac{4n^3 + 9n^2 + 5n}{6} \end{aligned}$$

indeed ,

$$\frac{(n+1)(n)(4(n+1)+1)}{6} = \frac{(n^2+n)(4n+5)}{6} = \frac{4n^3 + 5n^2 + 4n^2 + 5n}{6}$$

$$\textbf{Exercise 8.} \int_{a+c}^{b+c} f(x) dx = \int_{a+c-c}^{b+c-c} f(x - (-c)) dx = \int_a^b f(x+c) dx$$

$$\textbf{Exercise 9.} \int_{ka}^{kb} f(x) dx = \frac{1}{k} \int_{(ka)/k}^{(kb)/k} f\left(\frac{x}{1/k}\right) dx = k \int_a^b f(kx) dx$$

$$\textbf{Exercise 10.} \text{ Given } s(x) = (-1)^n n \text{ if } n \leq x < n+1; n = 0, 1, 2, \dots, p-1; s(p) = 0, p \in \mathbb{Z}^+. f(p) = \int_0^p s(x) dx.$$

So for  $f(3) = \int_0^3 s(x) dx$ , we need to consider  $n = 0, 1, 2$ .

$$s(0 \leq x < 1) = 0$$

$$s(1 \leq x < 2) = (-1)(1)$$

$$s(2 \leq x < 3) = 2;$$

$$s(3 \leq x < 4) = -3$$

So then

$$f(3) = (-1)(1) + 2(1) = 1$$

$$f(4) = 1 + (-3)(1) = -2$$

$$f(f(3)) = f(1) = 0$$

We obtain this formula

$$\begin{aligned}
 f(p) &= \begin{cases} \frac{p}{2}(-1)^{p+1} & p \text{ even} \\ \frac{p-1}{2}(-1)^{p+1} & p \text{ odd} \end{cases} \text{ since} \\
 f(p+1) &= f(p) + \int_p^{p+1} s(x)dx = \begin{cases} \frac{p-1}{2}(-1)^{p+1} & p \text{ even} \\ (-1)^p p & p \text{ odd} \end{cases} + (-1)^p p \\
 &= \begin{cases} \frac{-p}{2} & p \text{ even} \\ \frac{p-1}{2} & p \text{ odd} \end{cases} + \begin{cases} p & p \text{ even} \\ -p & p \text{ odd} \end{cases} = \begin{cases} \frac{p}{2} & p \text{ even} \\ \frac{-p-1}{2} & p \text{ odd} \end{cases} = \\
 &= \begin{cases} -\frac{(p+1)}{2} & \text{if } p+1 \text{ even} \\ \frac{p}{2} & \text{if } p+1 \text{ odd} \end{cases}
 \end{aligned}$$

Thus,  $p = 14, p = 15$ .

**Exercise 11.**

(1)

$$\begin{aligned}
 \int_a^b s(x)dx &= \sum_{k=1}^n s_k^3(x_k - x_{k-1}) \\
 \int_a^b s + \int_b^c s &= \sum_{k=1}^{n_1} s_k^2(x_k - x_{k-1}) + \sum_{k=n_1}^{n_2} s_k^3(x_k - x_{k-1}) = \sum_{k=1}^{n_2} s_k^3(x_k - x_{k-1}) = \int_a^c s(x)dx
 \end{aligned}$$

$$(2) \int_a^b (s+t) = \sum_{k=1}^{n_3} (s+t)_k^3(x_k - x_{k-1}) \neq \int_a^b s + \int_a^b t$$

$$(3) \int_a^b cs = \sum_{k=1}^n (cs)^3(x_k - x_{k-1}) \neq c \int_a^b s$$

(4) Consider these facts that are true, that  $x_{k-1} < x < x_k$ ,  $s(x) = s_k$ ;  $x_0 = a + c$ ,  $x_n = b + c$ ,  
 $x_{k-1} - c < x - c < x_l - c \implies y_{k-1} < y < y_k$  so then  $s(y + c) = s_k$ .

$$\begin{aligned}
 \sum_{k=1}^n s_k^3(x_k - x_{k-1}) &= \sum_{k=1}^k s_k^3(x_k - c - (x_{k-1} - c)) = \\
 &= \sum_{k=1}^n s_k^3(y_k - y_{k-1}) = \int_a^b s(y + c)dy
 \end{aligned}$$

$$(5) s < t, \int_a^b s = \sum_{k=1}^n s_k^3(x_k - x_{k-1}).$$

$$\text{if } 0 < s, s^3 < s^2t < st^2 < t^3$$

$$\text{if } s < 0t, s^3 < \text{ and } t^3 > 0$$

$$\text{if } s < t < 0, s^3 < s^2t, s(st) < t(ts) = t^2s \quad \begin{matrix} ts > t^2 \\ t^2s < t^3 \end{matrix}$$

$$s^3 < s^2t < t^2s < t^3$$

$$\text{Then } \int_a^b s < \int_a^b t.$$

**Exercise 12.**

$$(1) \int_a^b s + \int_b^c s = \sum_{k=1}^{n_1} s_k(x_k^2 - x_{k-1}^2) + \sum_{k=n_1}^{n_2} s_k(x_k^2 - x_{k-1}^2) = \sum_{k=1}^{n_3} s_k(x_k^2 - x_{k-1}^2) = \int_a^c s$$

$$(2) \int_a^b (s+t) = \sum_{k=1}^{n_3} (s+t)_k(x_k^2 - x_{k-1}^2) = \sum_{k=1}^{n_3} (s_k + t_k)(x_k^2 - x_{k-1}^2) = \sum_{k=1}^{n_3} s_k(x_k^2 - x_{k-1}^2) + \sum_{k=1}^{n_3} t_k(x_k^2 - x_{k-1}^2)$$

since  $P_3 = \{x_k\}$  is a finer partition than the partition for  $s, P_1, t, P_2$ , then consider

$$s_k(y_j^2 - y_{j-1}^2) = s_k((x_{k+1}^2 - x_k^2) + (x_k^2 - x_{k-1}^2)), \text{ so}$$

$$\begin{aligned}
 \sum_{k=1}^{n_3} s_k(x_k^2 - x_{k-1}^2) + \sum_{k=1}^{n_3} t_k(x_k^2 - x_{k-1}^2) &= \sum_{j=1}^{n_1} s_j(x_j^2 - x_{j-1}^2) + \sum_{j=1}^{n_2} t_j(x_j^2 - x_{j-1}^2) = \\
 &= \int_a^b s + \int_a^b t
 \end{aligned}$$

$$(3) \int_a^b cs = \sum_{k=1}^n cs_k(x_k^2 - x_{k-1}^2) = c \sum_{k=1}^n s_k(x_k^2 - x_{k-1}^2) = c \int_a^b s$$

$$(4) \int_{a+c}^{b+c} s(x)dx = \sum_{k=1}^n s_k(x_k^2 - x_{k-1}^2) \text{ where}$$

$$s(x) = s_k \text{ if } x_{k-1} < x < x_k$$

$$x(y+c) = s_k \text{ if } x_{k-1} < y+c < x_k \implies x_{k-1}-c < y < x_k-c \implies y_{k-1} < y < y_k$$

where  $P' = \{y_k\}$  is a partition on  $[a, b]$

$$\begin{aligned} \int_a^b s(y+c)dy &= \sum_{k=1}^n s_k(y_k^2 - y_{k-1}^2) = \\ &= \sum_{k=1}^n s_k((x_k-c)^2 - (x_{k-1}-c)^2) = \sum_{k=1}^n s_k(x_k^2 - 2x_kc + c^2 - (x_{k-1}^2 - 2x_{k-1}c + c^2)) = \\ &= \sum_{k=1}^n s_k(x_k^2 - x_{k-1}^2 - 2c(x_k - x_{k-1})) \neq \sum_{k=1}^n s_k(x_k^2 - x_{k-1}^2) \end{aligned}$$

$$(5) \text{ Since } x_k^2 - x_{k-1}^2 > 0, \int_a^b sdx = \sum_{k=1}^n s_k(x_k^2 - x_{k-1}^2) < \sum_{k=1}^n t_k(x_k^2 - x_{k-1}^2) = \int_a^b tdx$$

Note that we had shown previously that the integral doesn't change under finer partition.

**Exercise 13.**

$$\begin{aligned} \int_a^b s(x)dx \sum_{k=1}^n s_k(x_k - x_{k-1}); \int_a^b t(x)dx = \sum_{k=1}^{n_2} t_k(y_k - y_{k-1}) \\ P = \{x_0, x_1, \dots, x_n\}, Q = \{y_0, y_1, \dots, y_n\} \end{aligned}$$

Note that  $x_0 = y_0 = a; x_n = y_{n_2} = b$ .

Consider  $P \cup Q = R$ .  $R$  consists of  $n_3$  elements, (since  $n_3 \leq n + n_2$  some elements of  $P$  and  $Q$  may be the same.  $R$  is another partition on  $[a, b]$  (by partition definition) since  $x_k, y_k \in \mathbb{R}$  and since real numbers obey transitivity,  $\{x_k, y_k\}$  can be arranged such that  $a < z_1 < z_2 < \dots < z_{n_3-2} < b$  where  $z_k = x_k$  or  $y_k$ .

$$(s+t)(x) = s(x) + t(x) = s_j + t_k \text{ if } x_{j-1} < x < x_j; y_{j-1} < x < y_j$$

$$\text{If } x_{j-1} \leq y_{j-1}, \text{ let } z_{l-1} = y_{j-1}, x_{j-1} \text{ and}$$

$$\text{If } x_j \leq y_j, \text{ let } z_l = x_j, y_j$$

$$\text{Let } s_j = s_l; t_k = t_l$$

$$(s+t)(x) = s(x) + t(x) = s_l + t_l, \text{ if } z_{l-1} < x < z_l$$

$$\int_a^b (s(x) + t(x))dx = \int_a^b ((s+t)(x))dx = \sum_{l=1}^{n_3} (s_l + t_l)(z_l - z_{l-1}) = \sum_{l=1}^{n_3} s_l(z_l - z_{l-1}) + \sum_{l=1}^{n_3} t_l(z_l - z_{l-1})$$

In general, it was shown (Apostol I, pp. 66) that any finer partition doesn't change the integral  $R$  is a finer partition. So

$$\sum_{l=1}^{n_3} s_l(z_l - z_{l-1}) + \sum_{l=1}^{n_3} t_l(z_l - z_{l-1}) = \sum_{k=1}^n s_k(x_k - x_{k-1}) + \sum_{k=1}^{n_2} t_k(y_k - y_{k-1}) = \int_a^b s(x)dx + \int_a^b t(x)dx$$

**Exercise 14.** Prove Theorem 1.4 (the linearity property).

$$\begin{aligned} c_1 \int_a^b s(x)dx + c_2 \int_a^b t(x)dx &= c_1 \sum_{k=1}^n s_k(x_k - x_{k-1}) + c_2 \sum_{k=1}^{n_2} t_k(x_k - x_{k-1}) = \\ &= \sum_{l=1}^{n_3} c_1 s_l(z_l - z_{l-1}) + \sum_{l=1}^{n_3} c_2 t_l(z_l - z_{l-1}) = \sum_{l=1}^{n_3} (c_1 s_l + c_2 t_l)(z_l - z_{l-1}) = \\ &= \int_a^b (c_1 s + c_2 t)(x)dx \end{aligned}$$

We relied on the fact that we could define a finer partition from two partitions of the same interval.

**Exercise 15.** Prove Theorem 1.5 (the comparison theorem).

$$\begin{aligned}
s(x) &< t(x) \forall x \in [a, b]; \quad s(x)(z_l - z_{l-1}) < t(x)(z_l - z_{l-1}) \quad (z_l - z_{l-1} > 0) \\
\int_a^b s(x)dx &= \sum_{k=1}^n s_k(x_k - x_{k-1}) = \sum_{l=1}^{n_3} s_l(z_l - z_{l-1}) < \sum_{l=1}^{n_3} t_l(z_l - z_{l-1}) = \sum_{k=1}^{n_2} t_k(y_l - y_{k-1}) = \\
&= \int_a^b t(x)dx \\
&\implies \int_a^b s(x)dx < \int_a^b t(x)dx
\end{aligned}$$

**Exercise 16.** Prove Theorem 1.6 (additivity with respect to the interval).

Use the hint:  $P_1$  is a partition of  $[a, c]$ ,  $P_2$  is a partition of  $[c, b]$ , then the points of  $P_1$  along with those of  $P_2$  form a partition of  $[a, b]$ .

$$\int_a^c s(x)dx + \int_c^b s(x)dx = \sum_{k=1}^{n_1} s_k(x_k - x_{k-1}) + \sum_{k=1}^{n_2} s_k(x_k - x_{k-1}) = \sum_{k=1}^{n_3} s_k(x_k - x_{k-1}) = \int_a^b s(x)dx$$

**Exercise 17.** Prove Theorem 1.7 (invariance under translation).

$$\begin{aligned}
P' &= \{y_0, y_1, \dots, y_n\}; \quad y_k = x_k + c; \\
&\implies x_{k-1} + c < y < x_k + c \\
&\quad x_{k-1} < y - c < x_k \\
y_k - y_{k-1} &= x_k + c - (x_{k-1} + c) = x_k - x_{k-1} \\
s(y - c) &= s_k \text{ if } x_{k-1} < y - c < x_k, \quad k = 1, 2, \dots, n \\
\int_a^b s(x)dx &= \sum_{k=1}^n s_k(x_k - x_{k-1}) = \sum_{k=1}^n s_k(y_k - y_{k-1}) = \int_{y_0}^{y_n} s(y - c)dy = \int_{a+c}^{b+c} s(x - c)dx
\end{aligned}$$

**1.26 Exercises - The integral of more general functions, Upper and lower integrals, The area of an ordinate set expressed as an integral, Informal remarks on the theory and technique of integration, Monotonic and piecewise monotonic functions. Definitions and examples, Integrability of bounded monotonic functions, Calculation of the integral of a bounded monotonic function, Calculation of the integral  $\int_0^b x^p dx$  when  $p$  is a positive integer, The basic properties of the integral, Integration of polynomials.**

**Exercise 16.**  $\int_0^2 |(x-1)(3x-1)|dx =$

$$\begin{aligned}
\int_1^2 (x-1)(3x-1)dx &= \int_1^2 (3x^2 - 4x + 1)dx = (x^3 - 2x^2 + x)|_1^2 = 2 \\
\int_{1/3}^1 (1-x)(3x-1)dx &= -(x^3 - 2x^2 + x)|_{1/3}^1 = \frac{4}{27} \\
\int_0^{1/3} (x-1)(3x-1)dx &= \frac{4}{27}
\end{aligned}$$

So the final answer for the integral is  $62/27$ .

**Exercise 17.**  $\int_0^3 (2x-5)^3 dx = 8 \int_0^3 (x - \frac{5}{2})^3 dx = 8 \int_{-5/2}^{3-5/2} x^3 dx = 8 \frac{1}{4} x^4 \Big|_{-5/2}^{1/2} = \frac{39}{2}$

**Exercise 18.**  $\int_{-3}^3 (x^2-3)^3 dx = \int_0^3 (x^2-3)^3 + \int_{-3}^0 (x^2-3)^3 = \int_0^3 (x^2-3)^2 + - \int_0^3 (x^2-3)^3 = 0$

**2.4 Exercises - Introduction, The area of a region between two graphs expressed as an integral, Worked examples.**

**Exercise 15.**  $f = x^2, g = cx^3, c > 0$

For  $0 < x < \frac{1}{c}, cx < 1$  (since  $c > 0$ ). So  $cx^3 < x^2$  (since  $x^2 > 0$ ).

$$\begin{aligned}
\int f - g &= \int x^2 - cx^3 = \left( \frac{1}{3}x^3 - \frac{c}{4}x^4 \right) \Big|_0^{1/c} = \frac{1}{12c^3} \\
\int f - g &= \frac{2}{3} = \frac{1}{12c^3}; \quad \boxed{c = \frac{1}{2\sqrt{2}}}
\end{aligned}$$

**Exercise 16.**  $f = x(1 - x), g = ax$ .

$$\int f - g = \int_0^{1-a} x - x^2 - ax = \left( (1-a)\frac{1}{2}x^2 - \frac{1}{3}x^3 \right) \Big|_0^{1-a} = (1-a)^3 \frac{1}{6} = 9/2 \implies \boxed{a = -2}$$

**Exercise 17.**  $\pi = 2 \int_{-1}^1 \sqrt{1 - x^2} dx$

(1)

$$\int_{-3}^3 \sqrt{9 - x^2} dx = 3 \int_{-3}^3 \sqrt{1 - \left(\frac{x}{3}\right)^2} = 3(3) \int_{-1}^1 \sqrt{1 - x^2} = \boxed{\frac{9\pi}{2}}$$

Now

$$\int_{ka}^{kb} f\left(\frac{x}{k}\right) dx = k \int_a^b f dx$$

(2)

$$\int_0^2 \sqrt{1 - \frac{1}{4}x^2} dx = 2 \int_0^1 \sqrt{1 - x^2} dx = \frac{2\pi}{4} = \boxed{\frac{\pi}{2}}$$

(3)  $\int_{-2}^2 (x - 3)\sqrt{4 - x^2} dx$

$$\begin{aligned} \int_{-2}^2 x\sqrt{4 - x^2} dx &= (-1) \int_2^{-2} -x\sqrt{4 - x^2} \implies 2 \int_{-2}^2 x\sqrt{4 - x^2} = 0 \\ -3 \int_{-2}^2 \sqrt{4 - x^2} dx &= (-6)(2) \int_{-1}^1 \sqrt{1 - x^2} = \boxed{-6\pi} \end{aligned}$$

**Exercise 18.** Consider a circle of radius 1 and a twelve-sided dodecagon inscribed in it. Divide the dodecagon by isosceles triangle pie slices. The interior angle that is the vertex angle of these triangles is  $360/12 = 30$  degrees.

Then the length of the bottom side of each triangle is given by the law of cosines:

$$c^2 = 1 + 1 - 2(1)(1) \cos 30^\circ = 2 \left( 1 - \frac{\sqrt{3}}{2} \right) \implies c = \sqrt{2} \sqrt{1 - \frac{\sqrt{3}}{2}}$$

The height is given also by the law of cosines

$$h = 1 \cos 15^\circ = \sqrt{\frac{1 + \cos 30^\circ}{2}} = \sqrt{\frac{1 + \frac{\sqrt{3}}{2}}{2}}$$

The area of the dodecagon is given by adding up twelve of those isosceles triangles

$$(12) \frac{1}{2} \left( \sqrt{1 + \frac{\sqrt{3}}{2}} \left( \frac{1}{\sqrt{2}} \right) \right) \left( \sqrt{2} \sqrt{1 - \frac{\sqrt{3}}{2}} \right) = 3$$

So  $3 < \pi$ .

Now consider a dodecagon that's circumscribing the circle of radius 1.

$$(12) \frac{1}{2} \left( 2 \sqrt{\frac{1 - \frac{\sqrt{3}}{2}}{1 + \frac{\sqrt{3}}{2}}} \right) (1) = \boxed{12 \left( 2 - \frac{\sqrt{3}}{2} \right)} > \pi$$

**Exercise 19.**

(1)  $(x, y) \in E$  if  $x = ax_1, y = by_1$  such that  $x_1^2 + y_1^2 \leq 1$

$$\implies \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

(2)

$$\begin{aligned} y &= b \sqrt{1 - \left(\frac{x}{a}\right)^2} \\ 2 \int_{-a}^a b \sqrt{1 - \left(\frac{x}{a}\right)^2} &= 2ba \int_{-1}^1 \sqrt{1 - x^2} = ba \frac{\pi}{2} (2) = \pi ba \end{aligned}$$



**Exercise 20.** Let  $f$  be nonnegative and integrable on  $[a, b]$  and let  $S$  be its ordinate set.

Suppose  $x$  and  $y$  coordinates of  $S$  were expanded in different ways  $x = k_1 x_1, y = k_2 y_1$ .

If  $f(x_1) = y_1, g(x) = k_2 f\left(\frac{x}{k_1}\right) = k_2 y_1 = y$ .

integrating  $g$  on  $[k_1 a, k_1 b]$ ,

$$\int_{k_1 a}^{k_1 b} g(x) dx = \int_{k_1 a}^{k_1 b} k_2 f\left(\frac{x}{k_1}\right) dx = k_2 k_1 \int_a^b f(x) dx = k_2 k_1 A$$

## 2.8 Exercises - The trigonometric functions, Integration formulas for the sine and cosine, A geometric description of the sine and cosine functions. Exercise 1.

(1)  $\sin \pi = \sin 0 = 0$ . sine is periodic by  $2\pi$ , so by induction,  $\sin n\pi = 0$ .

$$\sin 2(n+1)\pi = \sin 2n\pi + 2\pi = \sin 2n\pi = 0$$

$$\sin (2(n+1) + 1)\pi = \sin (2n+3)\pi = \sin ((2n+1)\pi + 2\pi) = \sin (2n+1)\pi = 0$$

(2)  $\cos \pi/2 = \cos -\pi/2 = 0$

$$\text{by induction, } \cos \pi/2 + 2\pi j = \cos \pi/2(1 + 4j)$$

$$\cos -\pi/2 + 2\pi j = \cos (4j - 1)\pi/2, j \in \mathbb{Z}^+$$

### Exercise 2.

(1)  $\sin \pi/2 = 1, \sin \pi/2(1 + 4j) = 1, j \in \mathbb{Z}^+$ .

(2)  $\cos x = 1, \cos 0 = 1, \cos 2\pi j = 1$

### Exercise 3.

$$\sin x + \pi = -\sin x + \pi/2 + \pi/2 = \cos x + \pi/2 = -\sin x$$

$$\cos x + \pi = \cos x + \pi/2 + \pi/2 = -\sin x + \pi/2 = -\cos x$$

### Exercise 4.

$$\begin{aligned} \sin 3x &= \sin 2x \cos x + \sin x \cos 2x = 2 \sin x \cos^2 x + \sin x (\cos^2 x - \sin^2 x) = 3 \cos^2 x \sin x - \sin^3 x = \\ &= 3(1 - \sin^2 x) \sin x - \sin^3 x = 3 \sin x - 4 \sin^3 x \end{aligned}$$

$$\cos 3x = \cos 2x \cos x - \sin 2x \sin x = (\cos^2 x - \sin^2 x) \cos x - (2 \sin x \cos x) \sin x = \cos x - 4 \sin^2 x \cos x$$

$$\cos 3x = -3 \cos x + 4 \cos^3 x$$

### Exercise 5.

(1) This is the most direct solution. Using results from Exercise 4 (and it really helps to choose the cosine relationship, not the sine relationship),

$$\cos 3x = 4 \cos^3 x - 3 \cos x$$

$$x = \pi/6$$

$$\cos 3\pi/6 = 0 = 4 \cos^3 \pi/6 - 3 \cos \pi/6 = \cos \pi/6 (4 \cos^2 \pi/6 - 3) = 0$$

$$\implies \cos \pi/6 = \sqrt{3}/2, \sin \pi/6 = 1/2 \text{ (by Pythagorean theorem)}$$

(2)  $\sin 2\pi/6 = 2 \cos \pi/6 \sin \pi/6 = \sqrt{3}/2, \cos \pi/3 = 1/2$  (by Pythagorean theorem)

(3)  $\cos 2\pi/4 = 0 = 2 \cos \pi/4 - 1, \cos \pi/4 = 1/\sqrt{2} = \sin \pi/4$

Note that the most general way to solve a cubic is to use this formula. For  $x^3 + bx^2 + cx + d = 0$ ,

$$R = \frac{9bc - 27d - 2b^3}{54} \quad S = (R + \sqrt{Q^3 + R^2})^{1/3}$$

$$Q = \frac{3c - b^2}{9} \quad T = (R - \sqrt{Q^3 + R^2})^{1/3}$$

$$x_1 = S + T - b/3$$

$$x_2 = -1/2(S + T) - b/3 + 1/2\sqrt{-3}(S - T)$$

$$x_3 = -1/2(S + T) - b/3 - 1/2\sqrt{-3}(S - T)$$

### Exercise 6.

$$\tan x - y = \frac{\sin x - y}{\cos x - y} = \frac{\sin x \cos y - \sin y \cos x}{\cos x \cos y + \sin x \sin y} \left( \frac{\frac{1}{\cos x \cos y}}{\frac{1}{\cos x \cos y}} \right) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

if  $\tan x \tan y \neq -1$

Similarly,

$$\begin{aligned}\tan x + y &= \frac{\sin x + y}{\cos x + y} = \frac{\sin x \cos y + \sin y \cos x}{\cos x \cos y - \sin x \sin y} = \frac{\tan x + \tan y}{1 - \tan x \tan y}, \tan x \tan y \neq 1 \\ \cot x + y &= \frac{\cos x + y}{\sin x + y} = \frac{\cos x \cos y - \sin x \sin y}{\sin x \cos y + \sin y \cos x} = \frac{\cot x \cot y - 1}{\cot y + \cot x}\end{aligned}$$

**Exercise 7.**  $3 \sin x + \pi/3 = A \sin x + B \cos x = 3(\sin x \frac{1}{2} + \frac{\sqrt{3}}{2} \cos x) = \frac{3}{2} \sin x + \frac{3\sqrt{3}}{2} \cos x$

**Exercise 8.**

$$\begin{aligned}C \sin x + \alpha &= C(\sin x \cos \alpha + \cos x \sin \alpha) = C \cos \alpha \sin x + C \sin \alpha \cos x \\ A &= C \cos \alpha, B = C \sin \alpha\end{aligned}$$

**Exercise 9.** If  $A = 0$ ,  $B \cos x = B \sin \pi/2 + x = C \sin x + \alpha$  so  $C = B$ ,  $\alpha = \pi/2$  if  $A = 0$ .

If  $A \neq 0$ ,

$$\begin{aligned}A \sin x + B \cos x &= A(\sin x + \frac{B}{A} \cos x) = A(\sin x + \tan \alpha \cos x) \\ &= \frac{A}{\cos \alpha} (\cos \alpha \sin x + \sin \alpha \cos x) = \frac{A}{\cos \alpha} (\sin x + \alpha)\end{aligned}$$

where  $-\pi/2 < \alpha < \pi/4$ ,  $B/A = \tan \alpha$ ,  $C = \frac{A}{\cos \alpha}$

**Exercise 10.**  $C \sin x + \alpha = C \sin x \cos \alpha + C \cos x \sin \alpha$ .

$$C \cos \alpha = -2, C \sin \alpha = -2, C = -2\sqrt{2}, \alpha = \pi/4$$

**Exercise 11.** If  $A = 0$ ,  $C = B$ ,  $\alpha = 0$ . If  $B = 0$ ,  $A = -C$ ,  $\alpha = \pi/2$ . Otherwise,

$$A \sin x + B \cos x = B(\cos x + \frac{A}{B} \sin x) = \frac{B}{\cos \beta} (\cos x \cos \beta + \sin \beta \sin x) = C \cos x + \alpha$$

where  $\frac{A}{B} = \tan \beta$ ,  $\alpha = -\beta$ ,  $C = \frac{B}{\cos \beta}$ .

**Exercise 12.**

$$\sin x = \cos x = \sqrt{1 - \cos^2 x} \implies \cos x = 1/\sqrt{2} \implies x = \frac{\pi}{4}$$

Try  $5\pi/4$ .  $\sin 5\pi/4 = \cos 3\pi/4 = -\sin \pi/4 = -1/\sqrt{2}$ .

$\cos 5\pi/4 = -\sin 3\pi/4 = -\cos \pi/4 = -1/\sqrt{2}$ . So  $\sin 5\pi/4 = \cos 5\pi/4$ .  $x = 5\pi/4$  must be the other root.

So  $\theta = \pi/4 + \pi n$  (by periodicity of sine and cosine).

**Exercise 13.**

$$\begin{aligned}\sin x - \cos x &= 1 = \sqrt{1 - \cos^2 x} = 1 + \cos x \\ \implies 1 - \cos^2 x &= 1 + 2 \cos x + \cos^2 x \implies 0 = 2 \cos x (1 + \cos x) \\ \cos x &= -1, x = \pi/2 + 2\pi n\end{aligned}$$

**Exercise 14.**

$$\begin{aligned}\cos x - y + \cos x + y &= \cos x \cos y + \sin x \sin y + \cos x \cos y - \sin x \sin y = 2 \cos x \cos y \\ \cos x - y - \cos x + y &= \sin x \cos y - \sin y \cos x + \sin x \cos y + \sin y \cos x = 2 \sin x \cos y \\ \sin x - y + \sin x + y &= \sin x \cos y - \sin y \cos x + \sin x \cos y + \sin y \cos x = 2 \sin x \cos y\end{aligned}$$

**Exercise 15.**

$$\begin{aligned}\frac{\sin x + h - \sin x}{h} &= \frac{\sin(x + h/2) \cos h/2 + \cos(x + h/2) \sin h/2 - \sin(x + h) \cos h/2 - \cos(x + h/2) \sin h/2}{h} \\ &= \frac{\sin h/2}{h/2} \cos(x + h/2) \\ \frac{\cos x + h - \cos x}{h} &= \frac{\cos(x + h/2) \cos h/2 - \sin(x + h/2) \sin h/2 - (\cos(x + h/2) \cos h/2 + \sin(x + h/2) \sin h/2)}{h} \\ &= -\frac{\sin h/2}{h/2} \sin(x + h/2)\end{aligned}$$

**Exercise 16.**

(1)

$$\sin 2x = 2 \sin x \cos x$$

$$\text{if } \sin 2x = 2 \sin x \text{ and } x \neq 0, x \neq \pi n, \cos x = 1 \text{ but } x \neq \pi n \implies \boxed{x = 2\pi n}$$

(2)  $\cos x + y = \cos x \cos y - \sin x \sin y = \cos x + \cos y.$

$$\cos x \cos y - \cos x - \cos y = \sin y \sqrt{1 - \cos^2 x}$$

Letting  $A = \cos x, B = \cos y,$

$$A^2 B^2 + A^2 + B^2 - 2A^2 B - 2AB^2 + 2AB = 1 - A^2 - B^2 + A^2 B^2$$

$$A^2 + B^2 - A^2 B - AB^2 + AB = 1/2$$

$$B^2(1 - A) + B(A - A^2) + A^2 - 1/2 = 0$$

$$B = \frac{A(1 - A) \pm \sqrt{A^2(1 - A)^2 - 4(1 - A)(A^2 - 1/2)}}{1 - A} = A \pm \frac{1}{\sqrt{1 - A}}(A^2(1 - A) - 4(A^2 - 1/2))^{1/2} =$$

$$= A \pm \frac{1}{\sqrt{1 - A}}(-3A^2 - A^3 + 2)^{1/2}$$

Note that  $-1 \leq B \leq 1$ , but for  $|A| \leq 1$ .

Solve for the roots of  $-3A^2 - A^3 + 2$ ,  $A_0 = -1, -1 + \sqrt{3}, -1 - \sqrt{3}$ . So suppose  $\cos x = 9/10$ . Then there is no real number for  $y$  such that  $\cos y$  would be real and satisfy the above equation.

(3)  $\sin x + y = \sin x \cos y + \sin y \cos x = \sin x + \sin y$

$$\implies \sin y(1 - \cos x) + \sin y + -\cos x \sin y = 0, \implies y = 2\pi n$$

Checking our result, we find that  $\sin(2\pi n + y) = \sin 2\pi n + \sin y(1)$

(4)

$$\int_0^y \sin x dx = -\cos x|_0^y = -(\cos y - 1) = 1 - \cos y = \sin y$$

$$\implies 1 - \cos y = \sqrt{1 - \cos^2 y}$$

$$1 - 2\cos y + \cos^2 y = 1 - \cos^2 y \implies \cos y(\cos y - 1) = 0; y = \frac{2(j+1)\pi}{2}, 2\pi n$$

**Exercise 17.**  $\int_a^b \sin x dx = -\cos x|_a^b = -\cos b + \cos a$

(1)  $-\frac{\sqrt{3}}{2} + 1$

(2)  $-\frac{\sqrt{2}}{2} + 1$

(3)  $\frac{1}{2}$

(4)  $1$

(5)  $2$

(6)  $0$  We were integrating over one period, over one positive semicircle and over one negative semicircle.

(7)  $0$  We had integrated over two equal parts, though it only shaded in up to  $x = 1$ .

(8)  $-\frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2}$

**Exercise 18.**  $\int_0^\pi (x + \sin x) dx = (\frac{1}{2}x^2 - \cos x)|_0^\pi = \frac{\pi^2}{2} - (-1 - 1) = \frac{\pi^2}{2} + 2$

**Exercise 19.**  $\int_0^{\pi/2} (x^2 + \cos x) dx = (\frac{1}{3}x^3 + \sin x)|_0^{\pi/2} = \frac{1}{3}(\pi/2)^3 + 1$

**Exercise 20.**  $\int_0^{\pi/2} (\sin x - \cos x) dx = (-\cos x - \sin x)|_0^{\pi/2} = -1 - (-1) = 0$

**Exercise 21.**  $\int_0^{\pi/2} |\sin x - \cos x| dx = (\text{by symmetry}) 2 \int_0^{\pi/4} (\cos x - \sin x) dx = 2(\sin x + \cos x)|_0^{\pi/4} = 2(\sqrt{2} - 1)$

**Exercise 22.**  $\int_0^\pi (\frac{1}{2} + \cos t) dt = (\frac{1}{2}t + \sin t)|_0^\pi = \frac{\pi}{2}$

**Exercise 23.**

$$\int_0^{2\pi/3} (\frac{1}{2} + \cos t) dt + \int_{2\pi/3}^\pi -(\frac{1}{2} + \cos t) dt = (\frac{t}{2} + \sin t) \Big|_0^{2\pi/3} + (\frac{t}{2} + \sin t) \Big|_{2\pi/3}^\pi$$

$$= 2(\frac{\pi}{3} + \frac{\sqrt{3}}{2}) - \frac{\pi}{2} = \frac{\pi}{6} + \sqrt{3}$$

**Exercise 24.** If  $-\pi < x \leq -\frac{2\pi}{3}$ ,

$$\int_{-\pi}^x -(\frac{1}{2} + \cos t) dt = \int_x^{-\pi} (\frac{1}{2} + \cos t) dt = \left(\frac{t}{2} + \sin t\right) \Big|_x^{-\pi} = -\frac{\pi}{2} - \frac{x}{2} - \sin x$$

If  $-2\pi/3 \leq x \leq 2\pi/3$ ,

$$\begin{aligned} \int_{-\pi}^{-2\pi/3} -\left(\frac{1}{2} + \cos t\right)dt + \int_{-2\pi/3}^x \left(\frac{1}{2} + \cos t\right)dt &= \frac{-\pi/6}{+} \sqrt{3}/2 + (t/2 + \sin t)|_{-2\pi/3}^x \\ &= x/2 + \sin x - \pi/3 - \sqrt{3}/2 + \sqrt{3}/2 - \pi/6 = \frac{x}{2} + \sin x - \pi/3 \end{aligned}$$

If  $2\pi/3 \leq x \leq \pi$ ,

$$\sqrt{3}/2 + \int_{2\pi/3}^x -(1/2 + \cos t)dt = \sqrt{3}/2 + (t/2 + \sin t)|_{2\pi/3}^x = \pi/3 + \sqrt{3} - x/2 - \sin x$$

**Exercise 25.**  $\int_x^{x^2} (t^2 + \sin t)dt = (\frac{1}{3}t^3 + -\cos t) = \frac{x^6-x^3}{3} + \cos x - \cos x^2$

**Exercise 26.**  $\int_0^{\pi/2} \sin 2x dx = \left(\frac{-\cos(2x)}{2}\right)\Big|_0^{\pi/2} = (-1/2)(-1 - 1) = 1$

**Exercise 27.**  $\int_0^{\pi/3} \cos x/2 dx = 2 \sin x/2\Big|_0^{\pi/3} = 2\frac{1}{2} = 1$

**Exercise 28.**

$$\begin{aligned} \int_0^x \cos(a+bt)dt &= \int_0^x (\cos a \cos bt - \sin a \sin bt)dt = \left(\frac{\cos a}{b} \sin bt - \sin a(-\cos bt/b)\right)\Big|_0^x = \\ &= \frac{\cos a}{b} \sin bx + \frac{\sin a}{b}(\cos bx - 1) = \frac{1}{b} \sin a + bx - \sin a/b \\ \int_0^x \sin(a+bt)dt &= \int_0^x (\sin a \cos bt + \sin bt \cos a)dt = \left(\frac{\sin a}{b} \sin bt - \frac{\cos a}{b} \cos bt\right)\Big|_0^x = \\ &= \frac{1}{b}(\cos bx + a + \cos a) \end{aligned}$$

**Exercise 29.**

(1)

$$\begin{aligned} \int_0^x \sin^3 t dt &= \int_0^x \frac{3 \sin t - \sin 3t}{4} dt = \left(-\frac{3}{4} \cos t + \cos 3t/12\right)\Big|_0^x = -3/4(\cos x - 1) + \frac{\cos 3x - 1}{12} = \\ &= \frac{1}{3} - \frac{3}{4} \cos x + \frac{1}{12}(\cos 2x \cos x - \sin 2x \sin x) = 2/3 - 1/3 \cos x(2 + \sin^2 x) \end{aligned}$$

(2)

$$\begin{aligned} \int_0^x \cos^3 t dt &= \int_0^x \frac{1}{4}(\cos 3t + 3 \cos t)dt = \left(\frac{1}{4} \frac{\sin 3t}{3} + \frac{3}{4} \sin t\right)\Big|_0^x = \\ &= \frac{1}{12}(\sin 2x \cos x + \sin x \cos 2x) + \frac{3}{4} \sin x = \frac{1}{12}(2 \sin x \cos x + \sin x(2 \cos^2 x - 1)) = \\ &= \frac{\sin x \cos^2 x + 2 \sin x}{3} \end{aligned}$$

**Exercise 30.** Now using the definition of a periodic function,

$$f(x) = f(x+p); f(x+(n+1)p) = f(x+np+p) = f(x+np) = f(x)$$

and knowing that we could write any real number in the following form,

$$a = np + r; 0 \leq r < p, r \in \mathbb{R}; n \in \mathbb{Z}$$

then

$$\begin{aligned} \int_a^{a+p} f(x)dx &= \int_r^{r+p} f(x+np)dx = \int_r^{r+p} f(x)dx = \int_r^p f + \int_p^{r+p} f(x)dx = \\ &= \int_r^p f + \int_0^r f(x-p)dx = \int_r^p f + \int_0^r f = \int_0^p f \end{aligned}$$

**Exercise 31.**

(1)

$$\begin{aligned} \int_0^{2\pi} \sin nx dx &= \frac{1}{n} \int_0^{2\pi n} \sin x dx = \frac{1}{n}(-\cos x)\Big|_0^{2\pi n} = -\frac{1}{n}(1 - 1) = 0 \\ \int_0^{2\pi} \cos nx dx &= \frac{1}{n} \int_0^{2\pi n} \cos x dx = \frac{1}{n} \sin x\Big|_0^{2\pi n} = 0 \end{aligned}$$

(2)

$$\begin{aligned}\int_0^{2\pi} \sin nx \cos mx dx &= \int_0^{2\pi} \frac{1}{2} (\sin(n+m)x + \sin(n-m)x) dx = 0 + 0 = 0 \\ \int_0^{2\pi} \sin nx \sin mx dx &= \int_0^{2\pi} \frac{1}{2} (\cos(n-m)x - \cos(n+m)x) dx = 0 + 0 = 0 \\ \int_0^{2\pi} \cos nx \cos mx dx &= \int_0^{2\pi} \frac{1}{2} (\cos(n-m)x + \cos(n+m)x) dx = 0 + 0 = 0\end{aligned}$$

While

$$\begin{aligned}\int_0^{2\pi} \sin^2 nx dx &= \int_0^{2\pi} \frac{1 - \cos 2nx}{2} dx = \pi \\ \int_0^{2\pi} \cos^2 nx dx &= \int_0^{2\pi} \frac{1 + \cos 2nx}{2} dx = \pi\end{aligned}$$

**Exercise 32.** Given that  $x \neq 2\pi n$ ;  $\sin x/2 \neq 0$ ,

$$\begin{aligned}\sum_{k=1}^n 2 \sin x/2 \cos kx &= 2 \sin x/2 \sum_{k=1}^n \cos kx = \sum_{k=1}^n \sin(2k+1)\frac{x}{2} - \sin(2k-1)\frac{x}{2} = \sin(2n+1)\frac{x}{2} - \sin x/2 \\ &= \sin nx \cos x/2 + \sin x/2 \cos nx - \sin x/2 = \\ &= 2 \sin nx/2 \cos nx/2 \cos x/2 + \sin x/2 (1 - 2 \sin^2 nx/2) - \sin x/2 = \\ &= 2(\sin nx/2)(\cos(n+1)x/2)\end{aligned}$$

**Exercise 33.** Recall that

$$\begin{aligned}\cos(2k+1)x/2 - \cos(2k-1)x/2 &= \cos kx + x/2 - \cos kx - x/2 = \\ &= \cos kx \cos x/2 - \sin kx \sin x/2 - (\cos kx \cos x/2 + \sin kx \sin x/2) = \\ &= -2 \sin kx \sin x/2\end{aligned}$$

$$\begin{aligned}-2 \sin x/2 \sum_{k=1}^n \sin kx &= \sum_{k=1}^n (\cos(2k+1)x/2 - \cos(2k-1)x/2) = \cos(2n+1)x/2 - \cos x/2 = \\ &= \cos nx + x/2 - \cos x/2\end{aligned}$$

Now

$$\begin{aligned}\sin nx/2 \sin nx/2 + x/2 &= \sin nx/2 (\sin nx/2 \cos x/2 + \sin x/2 \cos nx/2) = \\ &= \sin^2 nx/2 \cos x/2 + \sin x/2 \cos nx/2 \sin nx/2 = \\ &= \left( \frac{1 - \cos nx}{2} \right) \cos x/2 + \frac{\sin nx}{2} \sin x/2 = \\ &= \frac{1}{2} (\cos x/2 - \cos x/2 \cos nx + \sin nx \sin x/2) = \frac{1}{2} (\cos x/2 - \cos(nx_x/2))\end{aligned}$$

Then

$$\begin{aligned}-2 \sin x/2 \sum_{k=1}^n \sin kx &= -2 \sin nx/2 \sin \frac{1}{2}(n+1)x \\ \sum_{k=1}^n \sin kx &= \frac{\sin nx/2 \sin \frac{1}{2}(n+1)x}{\sin x/2}\end{aligned}$$

**Exercise 34.** Using triangle OAP, not the right triangle, if  $0 < x < \pi/2$

$$\begin{aligned}\frac{1}{2} \cos x \sin x &< \frac{1}{2} \sin x < \frac{x}{2} \\ \implies \sin x &< x\end{aligned}$$

Now if  $0 > x > -\pi/2$ ,  $\sin x < 0$ ,

$$|\sin x| = -\sin x = \sin -x = \sin |x| < |x|$$

## 2.11 Exercises - Polar coordinates, The integral for area in polar coordinates.

Exercise 1.

$$\begin{aligned} (x-1)^2 + y^2 &= 1 \\ \implies x^2 - 2x + 1 + y^2 &= 1 \end{aligned} \quad \implies r^2 = 2r \cos \theta \text{ or } \boxed{r = 2 \cos \theta}$$

Exercise 2.

$$\begin{aligned} x^2 + y^2 - x &= \sqrt{x^2 + y^2} \\ \implies r^2 &= r(\cos \theta + 1) \text{ or } r = 1 + \cos \theta \end{aligned}$$

Exercise 3.

$$\begin{aligned} (x^2 + y^2)^2 &= x^2 - y^2; \quad y^2 \leq x^2 & r^2 &= \cos 2\theta \\ r^4 &= r^2(\cos^2 \theta - \sin^2 \theta) & \implies & r = \sqrt{\cos 2\theta} \text{ for } \cos 2\theta > 0 \end{aligned}$$

Exercise 4.

$$(x^2 + y^2)^2 = r^4 = |x^2 - y^2| = |r^2 \cos 2\theta| \quad \implies \begin{aligned} r^2 &= |\cos 2\theta| \\ r &= \sqrt{|\cos 2\theta|} \end{aligned}$$

Recall that we defined the measure of an angle to be 2 times the area of the sector divided by the radius squared (we chose this because it doesn't change with a change in circle size). Then if  $r^2(\theta)$  is integrable, then  $A = \int_{\theta_a}^{\theta_b} \frac{1}{2} \theta r^2 d\theta$ .

See sketches for the following exercises. Otherwise, we compute the area of the sector.

Exercise 5. Spiral of Archimedes:  $f(\theta) = \theta$ ;  $0 \leq \theta \leq 2\pi$

$$A = \int_0^{2\pi} \frac{1}{2} \theta^2 d\theta = \frac{1}{6} (2\pi)^3 = \boxed{\frac{4\pi^3}{3}}$$

Exercise 6. Circle tangent to  $y$ -axis:  $f(\theta) = 2 \cos \theta$ ;  $-\pi/2 \leq \theta \leq \pi/2$

$$A = \frac{1}{2} \int_{-\pi/2}^{\pi/2} 4 \cos^2 \theta d\theta = 2 \int_{-\pi/2}^{\pi/2} \frac{(1 + \cos 2\theta)}{2} d\theta = \boxed{\pi}$$

Exercise 7. Two circles tangent to  $y$ -axis:  $f(\theta) = 2|\cos \theta|$ ;  $0 \leq \theta \leq 2\pi$

$$\frac{1}{2} \int_0^{2\pi} 4 \cos^2 \theta d\theta = 2 \int_0^{2\pi} \frac{1 + \cos 2\theta}{2} d\theta = \boxed{2\pi}$$

Exercise 8. Circle tangent to  $x$ -axis:  $f(\theta) = 4 \sin \theta$ ;  $0 \leq \theta \leq \pi$

$$\frac{1}{2} \int_0^{\pi} 16 \sin^2 \theta d\theta = 8 \int_0^{\pi} \left( \frac{1 - \cos 2\theta}{2} \right) d\theta = \boxed{4\pi}$$

Exercise 9. Two circles tangent to  $x$ -axis:  $f(\theta) = 4|\sin \theta|$ ;  $0 \leq \theta \leq 2\pi$

$$\frac{1}{2} \int_0^{2\pi} 16 \sin^2 \theta d\theta = 8 \int_0^{2\pi} \frac{1 - \cos 2\theta}{2} d\theta = \boxed{8\pi}$$

Exercise 10. Rosepetal:  $f(\theta) = \sin 2\theta$ ;  $0 \leq \theta \leq \pi/2$

$$\frac{1}{2} \int_0^{\pi/2} \sin^2 2\theta d\theta = \frac{1}{2} \int_0^{\pi/2} d\theta \left( \frac{1 - \cos 4\theta}{2} \right) = \boxed{\frac{\pi}{8}}$$

Exercise 11. Four-leaved rose  $f(\theta) = |\sin 2\theta|$ ;  $0 \leq \theta \leq 2\pi$

$$\frac{1}{2} \int_0^{2\pi} \sin^2 2\theta d\theta = \frac{1}{2} \int_0^{2\pi} d\theta \left( \frac{1 - \cos 4\theta}{2} \right) = \boxed{\frac{\pi}{2}}$$

**Exercise 13.** Four-leaf clover:  $f(\theta) = \sqrt{|\cos 2\theta|}$ ;  $0 \leq \theta \leq 2\pi$

$$\begin{aligned} & \frac{1}{2} \int_0^{2\pi} |\cos 2\theta| d\theta = \\ &= \frac{1}{2} \left( \int_0^{\pi/4} c(2\theta) - \int_{\pi/4}^{3\pi/4} c(2\theta) + \int_{3\pi/4}^{5\pi/4} c(2\theta) - \int_{5\pi/4}^{7\pi/4} c(2\theta) + \int_{7\pi/4}^{2\pi} c(2\theta) \right) = \\ &= \frac{1}{2} \left( \left. \frac{s(2\theta)}{2} \right|_0^{\pi/4} - \left. \frac{s(2\theta)}{2} \right|_{\pi/4}^{3\pi/4} + \left. \frac{s(2\theta)}{2} \right|_{3\pi/4}^{5\pi/4} - \left. \frac{s(2\theta)}{2} \right|_{5\pi/4}^{7\pi/4} + \left. \frac{s(2\theta)}{2} \right|_{7\pi/4}^{2\pi} \right) = \\ &= \frac{1}{2} \left( \frac{1}{2} - \left( \frac{-1-1}{2} \right) + \left( \frac{1-(-1)}{2} \right) - \left( \frac{-1-1}{2} \right) + \left( \frac{-(-1)}{2} \right) \right) = \boxed{2} \end{aligned}$$

**Exercise 14.** Cardioid:  $f(\theta) = 1 + \cos \theta$ ;  $0 \leq \theta \leq 2\pi$

$$\frac{1}{2} \int_0^{2\pi} 1 + 2 \cos \theta + \cos^2 \theta = \frac{1}{2} (2\pi + \frac{1}{2}(2\pi)) = \boxed{\frac{3\pi}{2}}$$

**Exercise 15.** Limacon:  $f(\theta) = 2 + \cos \theta$ ;  $0 \leq \theta \leq 2\pi$

$$\frac{1}{2} \int_0^{2\pi} 4 + 4 \cos \theta + \cos^2 \theta = \frac{1}{2} (4(2\pi) + \frac{1}{2}(2\pi)) = \boxed{\frac{9\pi}{2}}$$

**2.17 Exercises - Average value of a function. Exercise 1.**  $\frac{1}{b-a} \int x^2 dx = \frac{1}{3}(b^2 + ab + a^2)$

**Exercise 2.**  $\frac{1}{1-0} \int x^2 + x^3 = \frac{7}{12}$

**Exercise 3.**  $\frac{1}{4-0} \int x^{1/2} = \frac{4}{3}$

**Exercise 4.**  $\frac{1}{8-1} \int x^{1/3} = \frac{45}{28}$

**Exercise 5.**  $\frac{1}{\pi/2-0} \int_0^{\pi/2} \sin x = \frac{2}{\pi}$

**Exercise 6.**  $\frac{1}{\pi/2-(-\pi/2)} \int \cos x = 2/\pi$

**Exercise 7.**  $\frac{1}{\pi/2-0} \int \sin 2x = -1/\pi(-1-1) = 2/\pi$

**Exercise 8.**  $\frac{1}{\pi/4-0} \int \sin x \cos x = \frac{1}{\pi}$

**Exercise 9.**  $\frac{1}{\pi/2-0} \int \sin^2 x = \frac{1}{\pi}(x - \sin 2x/2)|_0^{\pi} = \frac{1}{2}$

**Exercise 10.**  $\frac{1}{\pi-0} \int \cos^2 x = \frac{1}{2}$

**Exercise 11.**

$$(1) \frac{1}{a-0} \int x^2 = a^2/3 = c^2 \implies c = a/\sqrt{3}$$

$$(2) \frac{1}{a-0} \int x^n = \frac{1}{a} \frac{1}{n+1} x^{n+1} \Big|_0^a = \frac{a^n}{n+1} = c^n \implies c = \frac{a}{(n+1)^{1/n}}$$

**Exercise 12.**

$$\begin{aligned} A &= \int w f / \int w \quad \int w x^2 = k \int x \\ \int x^3 &= \frac{1}{4} x^4 = k \frac{1}{2} x^2; k = \frac{1}{2}, w = x \\ \int x^4 &= \frac{1}{5} x^5 = k \frac{1}{3} x^3; k = \frac{3}{5}, w = x^2 \\ \int x^5 &= \frac{1}{6} x^6 = k \frac{1}{4} x^4; k = \frac{2}{3}, w = x^3 \end{aligned}$$

**Exercise 13.**

$$A(f+g) = \frac{1}{b-a} \int f+g = \frac{1}{b-a} \int f + \frac{1}{b-a} \int g = A(f) + A(g)$$

$$A(cf) = \frac{1}{b-a} \int cf = c \left( \frac{1}{b-a} \right) \int f$$

$$A(f) = \frac{1}{b-a} \int f \leq \frac{1}{b-a} \int g = A(g)$$

**Exercise 14.**

$$\begin{aligned}
A(c_1f + c_2g) &= \frac{\int w(c_1f + c_2g)}{\int w} = \frac{c_1 \int wf}{\int w} + \frac{c_2 \int wg}{\int w} \\
&= c_1A(f) + c_2A(g) \\
f \leq g \quad w > 0 \text{ (nonnegative)} &\implies wf \leq wg
\end{aligned}$$

**Exercise 15.**

$$\begin{aligned}
A_a^b(f) &= \frac{1}{b-a} \int_a^b f = \frac{1}{b-a} \left( \int_a^c f + \int_c^b f \right) = \left( \frac{c-a}{b-a} \right) \left( \frac{\int_a^c f}{c-a} \right) + \frac{b-a-(c-a)}{b-a} \frac{\int_a^b f}{b-c} \\
&\quad a < c < b \\
&\quad 0 < \frac{c-a}{b-a} < 1 \quad \text{Let } t = \frac{c-a}{b-a} \\
&\implies A_a^b(f) = tA_a^c(f) + (1-t)A_c^b(f) \\
A_a^b(f) &= \frac{\int_a^b wf}{\int_a^b w} = \frac{\int_a^c w}{\int_a^b w} \frac{\int_a^c wf}{\int_a^c w} + \left( \frac{\int_a^b w - \int_a^c w}{\int_a^b w} \right) \frac{\int_c^b wf}{\int_c^b w} \\
0 < \frac{\int_a^c w}{\int_a^b w} < 1 &\text{ since } w \text{ is a nonnegative function. Let } t = \frac{\int_a^c w}{\int_a^b w} \\
&\implies A_a^b(f) = tA_a^c(f) + (1-t)A_c^b(f)
\end{aligned}$$

**Exercise 16.** Recall that  $x_{cm} = \frac{\int x\rho}{\int \rho}$  or  $r_{cm} = \frac{\int r dm}{M}$ .

$$\begin{aligned}
x_{cm} &= \frac{\int_0^L x}{\int_0^L 1} = \frac{L}{2} \\
I_{cm} &= \int r^2 dm = \int x^2(1) = L^3/3 \\
r^2 &= \frac{I_{cm}}{\int_0^L 1} = L^2/3 \implies r = \frac{L}{\sqrt{3}}
\end{aligned}$$

**Exercise 17.**

$$\begin{aligned}
x_{cm} &= \frac{\int_0^{L/2} x + \int_{L/2}^L 2x dx}{\frac{L}{2} + 2(L - L/2)} = \frac{yL^2}{12} \\
I_{cm} &= \int_0^{L/2} x^2 + \int_{L/2}^L 2x^2 = 5L^3/8 \quad r^2 = \frac{5L^3/8}{3L/2} = \frac{5L^2}{12} \implies r = \frac{\sqrt{5}L}{2\sqrt{3}}
\end{aligned}$$

**Exercise 18.**  $\rho(x) = x$  for  $0 \leq x \leq L$

$$\begin{aligned}
x_{cm} &= \frac{\int x dx}{\int x dx} = \frac{\frac{1}{3}x^3|_0^L}{\frac{1}{2}x^2|_0^L} = \frac{2}{3}L \\
I_{cm} &= \int x^2 dx = L^4/4 \\
r^2 &= \frac{L^4/4}{L^2/2} = L^2/2 \quad r = \frac{L}{\sqrt{2}}
\end{aligned}$$

**Exercise 19.**

$$\begin{aligned}
x_{cm} &= \frac{\int x dx + \int x \frac{L}{2} dx}{\int x dx + \int L/2} = \frac{\frac{1}{3}x^3|_0^{L/2} + \frac{L}{2}(x^2/2)|_{L/2}^L}{\frac{1}{2}x^2|_0^{L/2} + \frac{L}{2}(L - L/2)} = 11L/18 \\
I_{cm} &= \int x^2 dx + \int x^2 L/2 dx = L^4 31/192 \\
r^2 &= I_{cm}/(L^2 3/8) = L^2 31/72 \quad r = \frac{\sqrt{31}L}{6\sqrt{2}}
\end{aligned}$$



**Exercise 20.**  $\rho(x) = x^2$  for  $0 \leq x \leq L$

$$x_{cm} = \frac{\int x x^2 dx}{\int x^2} = 3L/4$$

$$I_{cm} = \int x^2 x^2 dx = L^5/5$$

$$r^2 = \frac{I_{cm}}{\frac{1}{3}L^3} = \frac{3}{5}L^2 \quad r = \sqrt{\frac{3}{5}}L$$

**Exercise 21.**

$$x_{cm} = \frac{\int_0^{L/2} x x^2 dx + \int_{L/2}^L x \frac{L^2}{4} dx}{\int_0^{L/2} x^2 dx + \int_{L/2}^L \frac{L^2}{4} dx} = 21L/32$$

$$I_{cm} = \int_0^{L/2} x^2 x^2 dx + \int_{L/2}^L x^2 \frac{L^2}{4} dx = 19L^5/240$$

$$r^2 = \frac{I_{cm}}{L^3/6} = 19L^2/40 \implies r = \sqrt{19}L/2\sqrt{10}$$

**Exercise 22.** Be flexible about how you can choose a convenient origin to evaluate the center-of-mass from

Let  $\rho = cx^n$

$$c \int_0^L x^n dx = \frac{1}{n+1} L^{n+1} c = M$$

$$\implies c = \frac{(n+1)M}{L^{n+1}}$$

$$c \int_0^L x x^n dx = c \frac{1}{n+2} L^{n+2} = \frac{n+1}{n+2} ML = \frac{3ML}{4}$$

$$x_{cm} = \frac{\int x \rho}{M} = \frac{3L}{4} \implies \int x \rho = \frac{n+1}{n+2} = \frac{3}{4} \implies n = 2$$

$$\boxed{\rho = \frac{3M}{L^3} x^2}$$

**Exercise 23.**

(1)

$$\frac{1}{\pi/2 - 0} \int 3 \sin 2t = \frac{6}{\pi}$$

(2)

$$\frac{1}{\pi/2 - 0} \int 9 \sin^2 2t = 9/2 \implies v_{rms} = 3\sqrt{2}/2$$

**Exercise 24.**  $T = 2\pi$  (just look at the functions themselves)

$$\frac{1}{2\pi} \int_0^{2\pi} 160 \sin t \sin(t - \pi/6) = 80\sqrt{3}$$

**2.19 Exercises - The integral as a function of the upper limit. Indefinite integrals.** **Exercise 1.**  $\int_0^x (1 + t + t^2) dt = x + \frac{1}{2}x^2 + \frac{1}{3}x^3$

**Exercise 2.**  $2y + 2y^2 + 8y^3/3$

**Exercise 3.**  $2x + 2x^2 + 8x^3/3 - (-1 + 1/2 + -1/3) = 2(x + x^2 + 4x^3/3) + 5/6$

**Exercise 4.**  $\int_1^{1-x} (1 - 2t + 3t^2) dt = (t - t^2 + t^3) \Big|_1^{1-x} = -2x + 2x^2 - x^3$

**Exercise 5.**  $\int_{-2}^x t^4 + t^2 = \frac{1}{5}t^5 + \frac{1}{3}t^3 \Big|_{-2}^x = \frac{x^5}{5} + \frac{x^3}{3} + \frac{40}{3}$

**Exercise 6.**  $\int_x^{x^2} t^4 + 2t^2 + 1 = \left( \frac{t^5}{5} + \frac{2}{3}t^3 + t \right) \Big|_x^{x^2} = \frac{1}{5}(x^{10} - x^5) + \frac{2}{3}(x^6 - x^3) + x^2 - x$

**Exercise 7.**  $\left( \frac{2}{3}t^{3/2} + t \right) \Big|_1^x = \frac{2}{3}(x^{3/2} - 1) + (x - 1)$

**Exercise 8.**  $\left( \frac{2}{3}t^{3/2} + \frac{4}{5}t^{5/4} \right) \Big|_x^{x^2} = \frac{2}{3}(x^3 - x^{3/2}) + \frac{4}{5}(x^{5/2} - x^{5/4})$

**Exercise 9.**  $\sin t \Big|_{i\pi}^x = \sin x$

**Exercise 10.**  $\left( \frac{t}{2} + \sin t \right) \Big|_0^{x^2} = \frac{x^2}{2} + \sin x^2$

**Exercise 11.**  $\left(\frac{1}{2}t + \cos t\right)\Big|_x^{x^2} = \frac{x^2-x}{2} + \cos x^2 - \cos x$

**Exercise 12.**  $\left(\frac{1}{3}u^3 + -\frac{1}{3}\cos 3u\right)\Big|_0^x = \frac{x^3}{3} + -\frac{1}{3}(\cos 3x - 1)$

**Exercise 13.**  $\left(\frac{1}{3}v^3 + \frac{\cos 3v}{-3}\right)\Big|_x^{x^2} = \frac{x^6-x^3}{3} + \frac{-1}{3}(\cos 3x^2 - \cos 3x)$

**Exercise 14.**  $\int \frac{1-\cos 2x}{2} + x = \left(\frac{1}{2}x - \frac{\sin 2x}{4} + \frac{1}{2}x^2\right)\Big|_0^y = \boxed{\frac{y}{2} - \frac{\sin 2y}{4} + \frac{y^2}{2}}$

**Exercise 15.**  $\left(\frac{-\cos 2w}{2} + 2\sin \frac{w}{2}\right)\Big|_0^x = \boxed{-\frac{(\cos 2x - 1)}{2} + 2\sin \frac{x}{2}}$

**Exercise 16.**  $\int_{-\pi}^x \left(\frac{1}{2} + \cos t\right)^2 dt = \int_{-\pi}^x \frac{1}{4} + \cos t + \cos^2 t = \frac{1}{4}(x + \pi) + \sin x + \frac{1}{2}\left(t + \frac{\sin 2t}{2}\right)\Big|_{-\pi}^x = \frac{3}{4}(x + \pi) + \sin x + \frac{1}{4}\sin 2x$

**Exercise 17.**  $\int_0^x (t^3 - t)dt = \frac{1}{3}\int_{\sqrt{2}}^x (t - t^3)dt$

Note that  $t^3 - t < 0$  for  $0 < t \leq 1$  and  $t^3 - t > 0$  for  $t > 1$ .  $t - t^3 < 0$  for  $t > \sqrt{2}$ .

$$\begin{aligned}\frac{1}{4}x^4 - \frac{1}{2}x^2 &= \frac{1}{3}\left(\frac{1}{2}t^2 - \frac{1}{4}t^4\right)\Big|_{\sqrt{2}}^x = \frac{1}{6}x^2 - \frac{1}{12}x^4 \\ \implies \frac{1}{3}x^4 - \frac{2}{3}x^2 &= 0 \implies x = 0, \boxed{x = \sqrt{2}}\end{aligned}$$

$\int_0^1 (t^3 - t)dt + \int_1^{\sqrt{2}} (t^3 - t)dt$  “cancel” each other out.

**Exercise 18.**  $f(x) = x - [x] - \frac{1}{2}$  if  $x$  is not an integer;  $f(x) = 0$  if  $x \in \mathbb{Z}$ .

For any real number,  $x = q + r$ ,  $0 \leq r < 1$ ,  $q \in \mathbb{Z}$ . So then

$$\begin{aligned}x - [x] &= r \\ f(x) &= r - \frac{1}{2}\end{aligned}$$

(1) To show the periodicity, consider

$$\begin{aligned}f(x+1) &= x+1 - [x+1] - \frac{1}{2} = r - \frac{1}{2} = f(x) \text{ since } x+1 = q+1+r, [x+1] = q+1 \\ x+1 - [x+1] &= r - \frac{1}{2}\end{aligned}$$

(2)  $P(x) = \int_0^x f(t)dt = \int_0^x (t - \frac{1}{2}) = \frac{1}{2}x^2 - \frac{1}{2}x$  because given  $0 < x \leq 1$ , then  $q = 0$  for  $x$ , so we can use  $r = t$ .  
To show periodicity,

$$\begin{aligned}P(x+1) &= \int_0^{x+1} f(t)dt = \int_0^1 f(t)dt + \int_1^{x+1} f(t)dt = 0 + \int_0^x f(t+1)dt = \int_0^x f(t)dt = P(x) \\ \text{since } \int_0^1 f(t)dt &= \frac{1}{2}(x^2 - x)\Big|_0^1 = 0\end{aligned}$$

(3) Since  $P$  itself is periodic by 1, then we can consider  $0 \leq x < 1$  only. Now  $x - [x] = r$  and  $P(x) = \frac{1}{2}(r^2 - r)$ . So  
 $P(x) = \frac{1}{2}((x - [x])^2 - (x - [x]))$ .

(4)

$$\begin{aligned}\int_0^1 (P(t) + c)dt &= 0 \implies \int_0^1 P(t)dt = -c \\ 0 \leq t \leq 1 \text{ so } P(t) &= \frac{1}{2}(t^2 - t) \\ \implies \int_0^1 P(t)dt &= \frac{1}{2}\left(\frac{1}{3}t^3 - \frac{1}{2}t^2\right)\Big|_0^1 = \frac{1}{2}\frac{-1}{6} \implies \boxed{c = \frac{1}{12}}\end{aligned}$$

$$(5) \quad Q(x) = \int_0^x (P(t) + c) dt$$

$$\begin{aligned} Q(x+1) &= \int_0^{x+1} (P(t) + c) dt = \int_0^1 (P(t) + c) dt + \int_1^{x+1} (P(t) + c) dt = \\ &= 0 + \int_0^x (P(t+1) + c) dt = \int_0^x (P(t) + c) dt = Q(x) \\ &\text{so without loss of generality, consider } 0 \leq x < 1 \\ \implies Q(x) \int_0^x \frac{1}{2}(t^2 - t) + \frac{1}{12} &= \boxed{\frac{1}{6}x^3 - \frac{1}{4}x^2 + \frac{x}{12}} \end{aligned}$$

**Exercise 19.**  $g(2n) = \int_0^{2\pi} f(t) dt$

Consider

$$\begin{aligned} \int_{-1}^1 f(t) dt &= \int_0^1 f(t) dt + \int_{-1}^0 f(t) dt = \int_0^1 f(t) dt + \frac{1}{-1} \int_1^0 f(-1t) dt = \\ &= \int_0^1 f + \int_1^0 f(t) dt = 0 \end{aligned}$$

Consider that  $\int_1^3 f(t) dt = \int_{-1}^1 f(t+2) dt = \int_{-1}^1 f(t) dt = 0$ . Then, by induction,

$$\int_1^{2n+1} f = \int_1^{2n-1} f + \int_{2n-1}^{2n+1} f(t) dt = 0 + \int_{-1}^1 f(t+2n) dt = \int_{-1}^1 f(t) dt = 0$$

(1)

$$\begin{aligned} g(2n) &= \int_0^1 f + \int_1^{2n-1} f + \int_{2n-1}^{2n} f = \int_0^1 f + \int_{-1}^0 f(t) dt = \int_0^1 f + - \int_1^0 f(-t) dt \\ &= \int_0^1 f + \int_1^0 f = 0 \end{aligned}$$

(2)

$$\begin{aligned} g(-x) &= \int_0^{-x} f = - \int_0^x f(-t) dt = \int_0^x f(t) dt = g(x) \\ g(x+2) &= \int_0^{x+2} f(t) dt = \int_0^2 f + \int_2^{x+2} f = \int_0^x f(t+2) dt = \int_0^x f(t) dt = g(x) \end{aligned}$$

**Exercise 20.**

(1)  $g$  is odd since

$$g(-x) = \int_0^{-x} f(t) dt = - \int_0^x f(-t) dt = - \int_0^x f(t) dt = -g(x)$$

Now

$$\begin{aligned} g(x+2) &= \int_0^{x+2} f = \int_0^2 f + \int_2^{x+2} f = g(2) + \int_0^x f(t+2) dt = g(2) + \int_0^x f(t) dt = g(2) + g(x) \\ \implies g(x+2) - g(x) &= g(2) \end{aligned}$$

(2)

$$\begin{aligned} g(2) &= \int_0^2 f = \int_1^2 f + \int_0^1 f = \int_1^2 f + A = \int_{-1}^0 f(t+2) dt + A = \int_{-1}^0 f(t) dt + A = \\ &- \int_1^0 f(-t) dt + A = 2A \end{aligned}$$

$$g(5) - g(3) = g(2)$$

$$\begin{aligned} g(3) &= g(2) + \int_2^3 f(t) dt = 2A + \int_0^1 f(t+2) dt = 2A + A = 3A \\ \implies g(5) &= 3A + 2A = 5A \end{aligned}$$

(3) The key observation is to see that  $g$  must repeat itself by a change of 2 in the argument. To make  $g(1) = g(3) = g(5)$ , they're different, **unless**  $A = 0$ !

**Exercise 21.** From the given, we can derive

$$g(x) = f(x+5), f(x) = \int_0^x g(t)dt$$

$$\implies f(5) = \int_0^5 g(t)dt = g(0) = 7$$

- (1) The **key insight** I uncovered was, when stuck, one of the things you can do, is to **think geometrically** and **draw a picture**.

$$g(-x) = f(-x+5) = g(x) = -f(x-5)$$

$$\implies -g(x) = f(x-5)$$

(2)

$$\int_0^5 f(t)dt = \int_{-5}^0 f(t+5)dt = \int_{-5}^0 g(t)dt = -\int_0^{-5} g(t)dt = \int_0^5 g(-t)dt = \int_0^5 g(t)dt = f(5) = 7$$

(3)

$$\int_0^x f(t)dt = \int_{-5}^{x-5} f(t+5)dt = \int_{-5}^{x-5} g(t)dt = \int_0^{x-5} g + \int_{-5}^0 g = f(x-5) + -\int_0^{-5} g(t)dt =$$

$$f(x-5) + \int_0^5 g(-t)dt = f(x-5) + f(5) = -g(x) + g(0)$$

where we've used  $f(x-5) = -g(x)$  in the second and third to the last step.

**3.6 Exercises - Informal description of continuity, The definition of the limit of a function, The definition of continuity of a function, The basic limit theorems. More examples of continuous functions, Proofs of the basic limit theorems.**  
Polynomials are continuous.

**Exercise 1.**  $\lim_{x \rightarrow 2} \frac{1}{x^2} = \frac{1}{\lim_{x \rightarrow 2} x^2} = \frac{1}{4}$

**Exercise 2.**  $\frac{\lim_{x \rightarrow 0} (25x^3 + 2)}{\lim_{x \rightarrow 0} (75x^7 - 2)} = -1$

**Exercise 3.**  $\lim_{x \rightarrow 2} \frac{(x-2)(x+2)}{(x-2)} = 4$

**Exercise 4.**  $\lim_{x \rightarrow 1} \frac{(2x-1)(x-1)}{x-1} = 1$

**Exercise 5.**  $\lim_{h \rightarrow 0} \frac{t^2 + 2th + h^2 - t^2}{h} = 2t$

**Exercise 6.**  $\lim_{x \rightarrow 0} \frac{(x-a)(x+a)}{(x+a)^2} = -1$

**Exercise 7.**  $\lim_{a \rightarrow 0} \frac{(x-a)(x+a)}{(x+a)^2} = 1$

**Exercise 8.**  $\lim_{x \rightarrow a} \frac{(x-a)(x+a)}{(x+a)^2} = 0$

**Exercise 9.**  $\lim_{t \rightarrow 0} \tan t = \frac{\lim_{t \rightarrow 0} \sin t}{\lim_{t \rightarrow 0} \cos t} = \frac{0}{1} = 0$

**Exercise 10.**  $\lim_{t \rightarrow 0} (\sin 2t + t^2 \cos 5t) = \lim_{t \rightarrow 0} \sin 2t + \lim_{t \rightarrow 0} t^2 \lim_{t \rightarrow 0} \cos 5t = 0 + 0 = 0$

**Exercise 11.**  $\lim_{x \rightarrow 0^+} \frac{|x|}{x} = 1$

**Exercise 12.**  $\lim_{x \rightarrow 0^-} \frac{|x|}{x} = -1$

**Exercise 13.**  $\lim_{x \rightarrow 0^+} \frac{\sqrt{x^2}}{x} = +1$

**Exercise 14.**  $\lim_{x \rightarrow 0^-} \frac{\sqrt{x^2}}{x} = -1$

**Exercise 15.**  $\lim_{x \rightarrow 0} \frac{2 \sin x \cos x}{x} = 2$

**Exercise 16.**  $\lim_{x \rightarrow 0} \frac{2 \sin x \cos x}{\cos 2x \sin x} = 2$

**Exercise 17.**  $\lim_{x \rightarrow 0} \frac{\sin x \cos 4x + \sin 4x \cos x}{\sin x} = 1 + \lim_{x \rightarrow 0} \frac{2 \sin 2x \cos 2x}{\sin x} = 1 + 2 \left( \lim_{x \rightarrow 0} \frac{2 \sin x \cos x \cos 2x}{\sin x} \right) = 5$

**Exercise 18.**  $\lim_{x \rightarrow 0} \frac{5 \sin 5x}{5x} - \lim_{x \rightarrow 0} \frac{3 \sin 3x}{3x} = 5 - 3 = 2$

**Exercise 19.**

$$\lim_{x \rightarrow 0} \frac{\sin \left( \frac{x+a}{2} + \frac{x-a}{2} \right) - \sin \left( \frac{x+a}{2} - \left( \frac{x-a}{2} \right) \right)}{x-a} =$$

$$= \lim_{x \rightarrow 0} \left( \frac{\sin \frac{x+a}{2} \cos \frac{x-a}{2} + \sin \frac{x-a}{2} \cos \frac{x+a}{2} - \left( \sin \frac{x+a}{2} \cos \frac{x-a}{2} - \sin \frac{x-a}{2} \cos \frac{x+a}{2} \right)}{x-a} \right) =$$

$$= \lim_{x \rightarrow a} \frac{2 \sin \frac{x-a}{2} \cos \frac{x+a}{2}}{x-a} = \cos a$$

**Exercise 20.**  $\lim_{x \rightarrow 0} \frac{2 \sin^2 x/2}{4(x/2)^2} = \frac{1}{2} \left( \lim_{x \rightarrow 0} \frac{\sin x/2}{x/2} \right) = \frac{1}{2}$

**Exercise 21.**  $\lim_{x \rightarrow 0} \frac{1 - \sqrt{1 - x^2}}{x^2} \left( \frac{1 + \sqrt{1 - x^2}}{1 + \sqrt{1 - x^2}} \right) = \lim_{x \rightarrow 0} \frac{1 - (1 - x^2)}{x^2(1 + \sqrt{1 - x^2})} = \boxed{\frac{1}{2}}$

**Exercise 22.**  $b, c$  are given.

$$\sin c = ac + b, a = \frac{\sin c - b}{c}, c \neq 0.$$

if  $c = 0$ , then  $b = 0, a \in \mathbb{R}$ .

**Exercise 23.**  $b, c$  are given.

$$2 \cos c = ac^2 + b, a = \frac{2 \cos c - b}{c^2}, c \neq 0.$$

If  $c = 0$ , then  $b = 2, a \in \mathbb{R}$ .

**Exercise 24.**

tangent is continuous for  $x \notin (2n + 1)\pi/2$

cotangent is continuous for  $x \notin 2n\pi$

**Exercise 25.**  $\lim_{x \rightarrow 0} f(x) = \infty$ . No  $f(0)$  cannot be defined.

**Exercise 26.**

(1)  $|\sin x - 0| = |\sin x| < |x|$ . Choose  $\delta = \epsilon$  for a given  $\epsilon$ .

Then  $\forall \epsilon > 0, \exists \delta > 0$  such that  $|\sin x - 0| < \epsilon$  when  $|x| < \delta$ .

(2)

$$|\cos x - 1| = |-2 \sin^2 x/2| = 2|\sin x/2|^2 < 2\left(\frac{x}{2}\right)^2 = \frac{|x|^2}{2} < 2\epsilon/2 = \epsilon$$

If we had chosen  $\delta_0 = \sqrt{2\epsilon}$  for a given  $\epsilon$ .  $|x - 0| < \delta = \sqrt{2\epsilon}$ .

(3)

$$|\sin x(\cos h - 1) + \cos x \sin h| \leq |\sin x| |\cos h - 1| + |\cos h| |\sin h| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

$$|\cos x + h - \cos x| = |\cos x \cos h - \sin x \sin h - \cos x| = |\cos x(\cos h - 1) - \sin x \sin h| \leq$$

$$\leq |\cos x| |\cos h - 1| + |\sin x| |\sin h| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

since  $\forall \epsilon > 0 \exists \delta_1, \delta_2 > 0$  such that  $|\cos h - 1| < \epsilon_0; |\sin h| < \epsilon$  whenever  $|h| < \min(\delta_1, \delta_2)$

Choose  $\delta_3$  such that if  $|h| < \delta_3; |\cos h - 1| < \frac{\epsilon}{2}; |\sin h| < \frac{\epsilon}{2}$

**Exercise 27.**  $f(x) - A = \sin \frac{1}{x} - A$ .

Let  $x = \frac{1}{n\pi}$ .

$$|f(x) - A| = |\sin n\pi - A| > ||\sin n\pi| - |A|| > |1 - |A||$$

Consider  $|x - 0| = |x| = \frac{1}{n\pi} \leq \delta(n)$ . Consider  $\epsilon_0 = \frac{|1 - |A||}{2}$ . Then suppose a  $\delta(n) \geq |x - 0|$  but  $|f(x) - A| > \epsilon_0$ . Thus, contradiction.

**Exercise 28.** Consider  $x \leq \frac{1}{n}, n \in \mathbb{Z}^+, n > M(n)$  ( $n$  is a given constant)

$$f(x) = \left\lceil \frac{1}{x} \right\rceil = [n] = n, \text{ for } m > M(n), x = \frac{1}{m} f(x) > M(n)$$

so  $\forall \epsilon > 0$ , we cannot find  $\delta = \frac{1}{n}$  such that  $|f(x) - A| < \epsilon$  for  $x < \delta$ .

So  $f(x) \rightarrow \infty$  as  $x \rightarrow 0^+$ .

Consider  $\frac{1}{n} \geq x > 0, n \in \mathbb{Z}^-; -n > M(n)$ .

$$f(x) = \left\lceil \frac{1}{x} \right\rceil = [n] = n < -M(n)$$

Since integers are unbounded, we can consider  $n < A$ , so that

$$|f(x) - A| > ||f| - |A|| = -n - |A| > M(n) - |A|. \text{ Choose } n \text{ such that } M(n) - |A| > 0$$

**Exercise 29.**

$$|f - A| = |(-1)^{[1/x]} - A| \geq ||(-1)^{[1/x]}| - |A|| = |1 - |A||$$

Choose  $\epsilon < |1 - |A||$ . Then  $\forall \delta > 0$  (such that  $|x| < \delta$ ),  $|f - A| > \epsilon$ . Thus there's no value for  $f(0)$  we could choose to make this function continuous at 0.

**Exercise 30.** Since

$$|f(x)| = |x| |(-1)^{[1/x]}| = |x|$$

So  $\forall \epsilon$ , let  $\delta = \epsilon$ .

**Exercise 31.**  $f$  continuous at  $x_0$ .

Choose some  $\epsilon_0, 0 < \epsilon_0 < \min(b - x_0, x_0 - a)$ . Then  $\exists \delta_0 = \delta(x_0, \epsilon_0)$ .

Consider  $\epsilon_1 = \frac{\epsilon_0}{2}$  and  $\delta_1 = \delta(x_0, \epsilon_1)$

Consider  $x_1 \in (x_0 - \delta_1, x_0 + \delta_1)$ , so that  $|f(x_1) - f(x_0)| < \epsilon_1$ .

Proceed to construct a  $\delta$  for  $x_1$ , some  $\delta(x_1; \epsilon_0)$

$$|x - x_1| = |x - x_0 + x_0 - x_1| < |x - x_0| + |x_0 - x_1|$$

Without loss of generality, we can specify  $x_1$  such that  $|x_0 - x_1| < \frac{\delta_1}{2}$ . Also, “pick” only the  $x$ ’s such that

$$\begin{aligned} |x - x_0| &< \frac{\delta_1}{2} < \delta_1 \\ \implies |x - x_1| &< \frac{\delta_1}{2} + \frac{\delta_1}{2} = \delta_1 \end{aligned}$$

Thus, “for these  $x$ ’s”

$$|f(x) - f(x_1)| = |f(x) - f(x_0) + f(x_0) - f(x_1)| < |f(x) - f(x_0)| + |f(x_1) - f(x_0)| < \epsilon_1 + \epsilon_1 = \epsilon_0$$

So  $\forall \epsilon_0, \exists \delta_1$  for  $x_1$ .  $f$  is continuous at  $x_1 \in (a, b)$ . Thus, there must be infinitely many points that are continuous in  $(a, b)$ , and at the very least, some or all are “clustered” around some neighborhood about the one point given to make  $f$  continuous.

**Exercise 32.** Given  $\epsilon = \frac{1}{n}$ ,  $|f(x)| = |x \sin \frac{1}{x}| = |x| |\sin 1/x| < |x|(1)$ .

Let  $\delta = \delta(n) = \frac{1}{n}$ , so that  $|x| < \frac{1}{n}$ .

$$\implies |f(x)| < \frac{1}{n}$$

**Exercise 33.**

(1) Consider  $x_0 \in [a, b]$ .

Choose some  $\epsilon_0, 0 < \epsilon_0 < \min(b - x_0, x_0 - a) \neq 0$ , ( $x_0$  could be  $a$  or  $b$ )

Consider, without loss of generality, only “ $x$ ’s” such that  $x \in [a, b]$ .

$$|f(x) - f(x_0)| \leq |x - x_0|$$

$$\text{Let } \delta_0 = \delta(\epsilon_0, x_0) = \epsilon_0 \implies |f(x) - f(x_0)| < \epsilon_0.$$

Since we didn’t specify  $x_0, \forall x_0 \in [a, b]$ ,  $f$  is continuous at  $x_0$ .

(2)

$$\begin{aligned} \left| \int_a^b f(x) dx - (b-a)f(a) \right| &= \left| \int_a^b (f(x) - f(a)) dx \right| \leq \int_a^b |f(x) - f(a)| dx \leq \\ &\leq \int_a^b |x - a| dx = \left( \frac{1}{2}x^2 - ax \right) \Big|_a^b = \frac{1}{2}(b-a)(b+a) - a(b-a) = \frac{(b-a)^2}{2} \end{aligned}$$

(3)

$$\begin{aligned} \left| \int_a^b f(x) dx - (b-a)f(c) \right| &= \left| \int_a^b (f(x) - f(c)) dx \right| \leq \int_a^b |f(x) - f(c)| dx \leq \int_a^b |x - c| dx = \\ &= \int_a^c (c - x) dx + \int_c^b (x - c) dx = c(c-a) - \frac{1}{2}(c-a)(c+a) + \frac{1}{2}(b-c)(b+c) - c(b-c) = \\ &= \frac{1}{2}((c-a)^2 + (b-c)^2) \end{aligned}$$

Draw a figure for clear, geometric reasoning.

Consider a square of length  $(b-a)$  and a  $45-45$  right triangle inside. From the figure, it’s obvious that right triangles of  $c-a$  length and  $(b-c)$  length lie within the  $(b-a)$  right triangle.

Compare the trapezoid of  $c-a, b-a$  bases with the  $b-a$  right triangle.

$$\frac{1}{2}(b-c)(b-a+c-a) = \frac{1}{2}(b-c)(b-c+2(c-a)) > \frac{1}{2}(b-c)^2$$

Indeed, the trapezoid and  $c - a$  right triangle equals the  $b - a$  trapezoid since

$$\begin{aligned}\frac{1}{2}(b-c)(b-a+c-a) + \frac{1}{2}(c-a)^2 &= \frac{1}{2}(b^2 - c^2 - 2ab + 2ac + c^2 - 2ca + a^2) = \frac{1}{2}(b-a)^2 \\ \implies \frac{1}{2}(b-a)^2 &> \frac{1}{2}(b-c)^2 + \frac{1}{2}(c-a)^2 \\ \text{so then } \left| \int_a^b f(x)dx - (b-a)f(c) \right| &\leq \frac{(b-a)^2}{2}\end{aligned}$$

### 3.8 Exercises - Composite functions and continuity.

**Exercise 1.**  $f(x) = x^2 - 2x$ ;  $g(x) = x + 1$

$$f(g(x)) = (x+1)^2 - 2(x+1) = x^2 - 1$$

$h : \mathbb{R} \rightarrow \mathbb{R}$

**Exercise 2.**  $f(x) = x + 1$ ;  $g(x) = x^2 - 2x$

$$h = x^2 - 2x + 1$$

$h : \mathbb{R} \rightarrow \mathbb{R}$

**Exercise 3.**  $f(x) = \sqrt{x}$  if  $x \geq 0$ ,  $g(x) = x^2$

$$h(x) = |x|$$

$h : \mathbb{R} \rightarrow \mathbb{R}^+$

**Exercise 4.**  $f(x) = \sqrt{x}$  if  $x \geq 0$ ,  $g(x) = -x^2$

$h$  undefined.  $D(h) = \emptyset$

**Exercise 5.**  $f(x) = x^2$ ;  $g(x) = \sqrt{x}$  if  $x \geq 0$

$$h(x) = x \text{ if } x \geq 0$$

$h : \mathbb{R}^+ \rightarrow \mathbb{R}^+$

**Exercise 6.**  $f(x) = -x^2$ ;  $g(x) = \sqrt{x}$  if  $x \geq 0$

$$h = -x \text{ if } x \geq 0$$

$h : \mathbb{R}^+ \rightarrow \mathbb{R}^-$

**Exercise 7.**  $f(x) = \sin x$ ;  $g(x) = \sqrt{x}$  if  $x \geq 0$

$$h(x) = \sin \sqrt{x} \text{ if } x \geq 0$$

$h : \mathbb{R}^+ \rightarrow [-1, 1]$

**Exercise 8.**  $f(x) = \sqrt{x}$  if  $x \geq 0$ ,  $g(x) = \sin x$

$$h(x) = \sqrt{\sin x} \text{ if } 2\pi j \leq x \leq 2\pi j + \pi; j \in \mathbb{Z}^+$$

$h : \{x | 2\pi j \leq x \leq 2\pi j + \pi; j \in \mathbb{Z}^+\} \rightarrow [0, 1]$

**Exercise 9.**  $f(x) = \sqrt{x}$  if  $x \geq 0$ ,  $g(x) = x + \sqrt{x}$  if  $x > 0$

$$h(x) = \sqrt{x + \sqrt{x}}$$

$h : \{x > 0\} \rightarrow \{x > 0\}$

**Exercise 10.**  $f(x) = \sqrt{x + \sqrt{x}}$  if  $x > 0$ ;  $g(x) = x + \sqrt{x}$  if  $x > 0$

$$h(x) = \sqrt{x + \sqrt{x + \sqrt{x + \sqrt{x}}}}$$

$h : \{x > 0\} \rightarrow \{x > 0\}$

**Exercise 11.**  $\lim_{x \rightarrow -2} \frac{x^3 + 8}{x^2 - 4} = \lim_{x \rightarrow -2} \frac{(x+2)(x^2 - 2x + 4)}{(x+2)(x-1)} = \boxed{-3}$

(used  $\lim fg = AB$  and  $\lim \frac{f}{g} = \frac{A}{B}$ )

**Exercise 12.**  $\lim_{x \rightarrow 4} \sqrt{1 + \sqrt{x}} = \sqrt{3}$ . Used continuity of composite function theorem.

**Exercise 13.**  $\lim_{t \rightarrow 0} \frac{\sin \tan t}{\sin t} = \lim_{t \rightarrow 0} \frac{\sin(\tan t)}{\tan t} \left( \frac{1}{\cos t} \right) = \lim_{t \rightarrow 0} \frac{\sin \tan t}{\tan t} \lim_{t \rightarrow 0} \frac{1}{\cos t} = 1$ .

(we used  $\lim fg = AB$  in the last step;  $\lim_{t \rightarrow 0} \frac{\sin(\tan t)}{\tan t}$  is from continuity of composite functions! this had been a composite function  $f \circ g$  of  $f = \sin x$  and  $g = \tan x$ ).

**Exercise 14.**  $\lim_{x \rightarrow \pi/2} \frac{\sin \cos x}{\cos x} = 1$  (by continuity of composite functions; this was a composite function  $f \circ g$  of  $f = \sin x$  and  $g = \cos x$ ).

**Exercise 15.**  $\lim_{t \rightarrow \pi} \frac{\sin(t-\pi)}{t-\pi} = 1$  (by continuity of composite functions; this was a composite function  $f \circ g$  of  $f = \sin x$  and  $g = t - \pi$ ).

**Exercise 16.**  $\lim_{x \rightarrow 1} \frac{\sin(x^2-1)}{x-1} = \lim_{x \rightarrow 1} \frac{\sin(x^2-1)}{x^2-1}(x+1) = 2$  (we used continuity of composite functions and product of limits).

**Exercise 17.** Since  $0 < x \sin \frac{1}{x} < x$ ; by squeeze principle,  $\lim_{x \rightarrow 0} x \sin \frac{1}{x} = 0$

**Exercise 18.**  $\lim_{x \rightarrow 0} \frac{1-\cos 2x}{x^2} = \lim_{x \rightarrow 0} \frac{2\sin^2 x}{x^2} = 2$

**Exercise 19.**  $\lim_{x \rightarrow 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{x} \left( \frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}} \right) = \boxed{1}$

**Exercise 20.**  $\lim_{x \rightarrow 0} \frac{1-\sqrt{1-4x^2}}{x^2} \left( \frac{1+\sqrt{1-4x^2}}{1+\sqrt{1-4x^2}} \right) = \boxed{2}$

**Exercise 21.**  $f(x) = \frac{x+|x|}{2} \quad g(x) = \begin{cases} x & \text{for } x < 0 \\ x^2 & \text{for } x \geq 0 \end{cases}$

$$h(x) = \begin{cases} x^2 & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$h(x)$  continuous everywhere.

**Exercise 22.**  $f(x) = \begin{cases} 1 & \text{if } |x| \leq 1 \\ 0 & \text{if } |x| > 1 \end{cases} \quad g(x) = \begin{cases} 2-x^2 & \text{if } |x| \leq 2 \\ 2 & \text{if } |x| > 2 \end{cases}$

$$h(x) = \begin{cases} 0 & \text{if } |x| \leq 1, \sqrt{3} \leq |x| \\ 1 & \text{if } 1 \leq |x| \leq \sqrt{3} \end{cases}$$

$h$  cont. for  $|x| < 1$ ,  $\sqrt{3} < |x|$ ,  $1 < |x| < \sqrt{3}$

**Exercise 23.**  $h(x) = g(f(x))$

$$h(x) = \begin{cases} 0 & \text{for } x < 0 \\ x^2 & \text{for } x \geq 0 \end{cases}$$

$h(x)$  continuous everywhere.

### 3.11 Exercises - Bolzano's theorem for continuous functions, The intermediate-value theorem for continuous functions.

These theorems form the foundation for continuity and will be valuable for differentiation later.

**Theorem 10** (Bolzano's Theorem).

Let  $f$  be cont. at  $\forall x \in [a, b]$ .

Assume  $f(a)$ ,  $f(b)$  have opposite signs.

Then  $\exists$  at least one  $c \in (a, b)$  s.t.  $f(c) = 0$ .

*Proof.* Let  $f(a) < 0$ ,  $f(b) > 0$ .

Want: Find one value  $c \in (a, b)$  s.t.  $f(c) = 0$

Strategy: find the largest  $c$ .

Let  $S = \{ \text{all } x \in [a, b] \text{ s.t. } f(x) \leq 0 \}$ .

$S$  is nonempty since  $f(a) < 0$ .  $S$  is bounded since all  $S \subseteq [a, b]$ .

$\implies S$  has a supremum.

Let  $c = \sup S$ .

If  $f(c) > 0$ ,  $\exists(c - \delta, c + \delta)$  s.t.  $f > 0$

$c - \delta$  is an upper bound on  $S$

but  $c$  is a least upper bound on  $S$ . Contradiction.

If  $f(c) < 0$ ,  $\exists(c - \delta, c + \delta)$  s.t.  $f < 0$

$c + \delta$  is an upper bound on  $S$

but  $c$  is an upper bound on  $S$ . Contradiction. □

**Theorem 11** (Sign-preserving Property of Continuous functions).

Let  $f$  be cont. at  $c$  and suppose that  $f(c) \neq 0$ .

then  $\exists(c - \delta, c + \delta)$  s.t.  $f$  be on  $(c - \delta, c + \delta)$  has the same sign as  $f(c)$ .

*Proof.* Suppose  $f(c) > 0$ .

$\forall \epsilon > 0$ ,  $\exists \delta > 0$  s.t.  $f(c) - \epsilon < f(x) < f(c) + \epsilon$  if  $c - \delta < x < c + \delta$  (by continuity).

Choose  $\delta$  for  $\epsilon = \frac{f(c)}{2}$ . Then

$$\frac{f(c)}{2} < f(x) < \frac{3f(c)}{2} \quad \forall x \in (c - \delta, c + \delta)$$

Then  $f$  has the same sign as  $f(c)$ . □



**Theorem 12** (Intermediate value theorem).

Let  $f$  be cont. at each pt. on  $[a, b]$ .

Choose any  $x_1, x_2 \in [a, b]$  s.t.  $x_1 < x_2$ . s.t.  $f(x_1) \neq f(x_2)$ .

Then  $f$  takes on every value between  $f(x_1)$  and  $f(x_2)$  somewhere in  $(x_1, x_2)$ .

*Proof.* Suppose  $f(x_1) < f(x_2)$

Let  $k$  be any value between  $f(x_1)$  and  $f(x_2)$

Let  $g = f - k$

$$g(x_1) = f(x_1) - k < 0$$

$$g(x_2) = f(x_2) - k > 0$$

By Bolzano,  $\exists c \in (x_1, x_2)$  s.t.  $g(c) = 0 \implies f(c) = k$

□

**Exercise 1.**  $f(0) = c_0$ .  $f(0) \geq 0$ .

Since  $\lim_{x \rightarrow \infty} \frac{c_k x^k}{c_{k-1} x^{k-1}} = \lim_{x \rightarrow \infty} \frac{c_k}{c_{k-1}} x = \infty \exists M > 0$  such that  $|c_n M^n| > |\sum_{k=0}^{n-1} c_k M^k|$ . So then

$$f(M) = c_n M^n + \sum_{k=0}^{n-1} c_k M^k \leq c_n$$

By Bolzano's theorem  $\exists b \in (0, M)$  such that  $f(b) = 0$ .

**Exercise 2.** Try a lot of values systematically. I also cheated by taking the derivatives and feeling out where the function changed direction.

- (1) If  $P(x) = 3x^4 - 2x^3 - 36x^2 + 36x - 8$ ,  $P(-4) = 168$ ,  $P(-3) = -143$ ,  $P(0) = -8$ ,  $P(\frac{1}{2}) = \frac{15}{16}$ ,  $P(1) = -7$ ,  $P(-3) = -35$ ,  $P(4) = 200$
- (2) If  $P(x) = 2x^4 - 14x^2 + 14x - 1$ ,  $P(-4) = 231$ ,  $P(-3) = -7$ ,  $P(0) = -1$ ,  $P(\frac{1}{2}) = \frac{1}{8}$ ,  $P(\frac{3}{2}) = -\frac{11}{8}$ ,  $P(2) = 2$
- (3) If  $P(x) = x^4 + 4x^3 + x^2 - 6x + 2$ ,  $P(-3) = 2$ ,  $P(-\frac{5}{2}) = -\frac{3}{16}$ ,  $P(-2) = 2$ ,  $P(\frac{1}{3}) = \frac{22}{81}$ ,  $P(\frac{1}{2}) = -\frac{3}{16}$ ,  $P(\frac{2}{3}) = -\frac{14}{81}$ ,  $P(1) = 2$ .

**Exercise 3.** Consider  $f(x) = x^{2j+1} - a$ .  $f(0) = -a > 0$ .

Since  $a$  is a constant, choose  $M < 0$  such that  $M^{2j+1} - a < 0$ .  $f(M) < 0$ .

By Bolzano's theorem, there is at least one  $b \in (M, 0)$  such that  $f(b) = b^{2j+1} - a = 0$ .

Since  $x^{2j+1} - a$  is monotonically increasing, there is exactly one  $b$ .

**Exercise 4.**  $\tan x$  is not continuous at  $x = \pi/2$ .

**Exercise 5.** Consider  $g(x) = f(x) - x$ . Then  $g(x)$  is continuous on  $[0, 1]$  since  $f$  is.

Since  $0 \leq f(x) \leq 1$  for each  $x \in [0, 1]$ , consider  $g(1) = f(1) - 1$ , so that  $-1 \leq g(1) \leq 0$ . Likewise  $0 \leq g(0) \leq 1$ .

If  $g(1) = 0$  or  $g(0) = 0$ , we're done ( $g(0) = f(0) - 0 = 0$ .  $f(0) = 0$ . Or  $g(1) = f(1) - 1 = 0$ ,  $f(1) = 1$ ).

Otherwise, if  $-1 \leq g(1) < 0$  and  $0 < g(0) \leq 1$ , then by Bolzano's theorem,  $\exists$  at least one  $c$  such that  $g(c) = 0$  ( $g(c) = f(c) - c = 0$ .  $f(c) = c$ ).

**Exercise 6.** Given  $f(a) \leq a$ ,  $f(b) \geq b$ ,

Consider  $g(x) = f(x) - x \leq 0$ . Then  $g(a) = f(a) - a \leq 0$ ,  $g(b) = f(b) - b \geq 0$ .

Since  $f$  is continuous on  $[a, b]$  (so is  $g$ ) and since  $g(a), g(b)$  are of opposite signs, by Bolzano's theorem,  $\exists$  at least one  $c$  such that  $g(c) = 0$ , so that  $f(c) = c$ .

### 3.15 Exercises - The process of inversion, Properties of functions preserved by inversion, Inverses of piecewise monotonic functions. Exercise 1. $D = \mathbb{R}$ , $g(y) = y - 1$

**Exercise 2.**  $D = \mathbb{R}$ ,  $g(y) = \frac{1}{2}(y - 5)$

**Exercise 3.**  $D = \mathbb{R}$ ,  $g(y) = 1 - y$

**Exercise 4.**  $D = \mathbb{R}$ ,  $g(y) = y^{1/3}$

**Exercise 5.**  $D = \mathbb{R}$ ,

$$g(y) = \begin{cases} y & \text{if } y < 1 \\ \sqrt{y} & \text{if } 1 \leq y \leq 16 \\ (\frac{y}{8})^2 & \text{if } y > 16 \end{cases}$$

**Exercise 6.**  $f(M_f) = f(f^{-1}(\frac{1}{n} \sum_{i=1}^n f(a_i))) = \frac{1}{n} \sum_{i=1}^n f(a_i)$

**Exercise 7.**  $f(a_1) \leq \frac{1}{n} \sum_{i=1}^n f(a_i) \leq f(a_n)$ . Since  $f$  is strictly monotonic.

$g$  preserves monotonicity.

$$\implies a_1 \leq M_f \leq a_n$$

**Exercise 8.**  $h(x) = af(x) + b$ ,  $a \neq 0$

$$M_h = H\left(\frac{1}{n} \sum_{i=1}^n h(a_i)\right) = H\left(\frac{1}{n} \sum_{i=1}^n (af(a_i) + b)\right) = H\left(a \frac{1}{n} \sum_{i=1}^n f(a_i) + b\right)$$

The inverse for  $h$  is  $g\left(\frac{h-b}{a}\right) = H(h) = h^{-1}$ . So then

$$M_h = g\left(\frac{1}{n} \sum_{i=1}^n f(a_i)\right) = M_f$$

The average is invariant under translation and expansion in ordinate values.

### 3.20 Exercises - The extreme-value theorem for continuous functions, The small-span theorem for continuous functions (uniform continuity), The integrability theorem for continuous functions.

Since for  $c \in [a, b]$ ,  $m = \min_{x \in [a, b]} f \leq f(c) \leq \max_{x \in [a, b]} f = M$

and  $\frac{\int_a^b f(x)g(x)dx}{\int_a^b g(x)dx} = f(c)$

**Exercise 1.**

$$g = x^9 > 0 \text{ for } x \in [0, 1]; f = \frac{1}{\sqrt{1+x}} m = \frac{1}{\sqrt{2}}, M = 1$$

$$\int_0^1 x^9 = \frac{1}{10} x^{10} \Big|_0^1 = \frac{1}{10}$$

$$\frac{1}{10\sqrt{2}} \leq \int_0^1 \frac{x^9}{\sqrt{1+x}} dx \leq \frac{1}{10}$$

**Exercise 2.**

$$\sqrt{1-x^2} = \frac{1-x^2}{\sqrt{1-x^2}}, f = \frac{1}{\sqrt{1-x^2}} g = (1-x^2) M = \frac{2}{\sqrt{3}}, m = 1$$

$$\int_0^{1/2} (1-x^2) dx = \left(x - \frac{1}{3}x^3\right) \Big|_0^{1/2} = \frac{11}{24}$$

$$\frac{11}{24} \leq \int_0^{1/2} \sqrt{1-x^2} dx \leq \frac{11}{24} \sqrt{\frac{4}{3}}$$

**Exercise 3.**

$$f = \frac{1}{1+x^6} g = 1-x^2+x^4 \int_0^a 1-x^2+x^4 = \left(x - \frac{1}{3}x^3 + \frac{1}{5}x^5\right) \Big|_0^a = a - \frac{a^3}{3} + \frac{a^5}{5}$$

$$m = \frac{1}{1+a^6} M = 1$$

$$\frac{1}{1+a^6} \left(a - \frac{a^3}{3} + \frac{a^5}{5}\right) \leq \int_0^a \frac{1}{1+x^2} dx \leq \left(a - \frac{a^3}{3} + \frac{a^5}{5}\right)$$

So if  $a = \frac{1}{10}$ ,  $(a - a^3/3 + a^5/5) = a - 0.333 \dots a^3 + 0.2a^5 = 0.099669$

**Exercise 4.** (b) is wrong, since it had chosen  $g = \sin t$ , but  $g$  needed to be nonnegative.

**Exercise 5.** At worst, we could have utilized the fundamental theorem of calculus.

$$\begin{aligned} \int \sin t^2 dt &= \int \left(\frac{1}{2t}\right) (2t \sin t^2) dt = \frac{1}{2c} (-\cos t^2) \Big|_{\sqrt{n\pi}}^{\sqrt{(n+1)\pi}} = \\ &= \frac{-1}{2c} ((-1)^{n+1} - (-1)^n) = \boxed{\frac{1}{c} (-1)^n} \end{aligned}$$

**Exercise 6.**  $\int_a^b (f)(1) = f(c) \int_a^b 1 = f(c)(b-a)$ . Then  $f(c) = \frac{\int_a^b f}{b-a} = 0$  for some  $c \in [a, b]$  by Mean-value theorem for integrals.

**Exercise 7.**  $f$  nonnegative. Consider  $f$  at a point of continuity  $c$ , and suppose  $f(c) > 0$ . Then  $\frac{1}{2}f(c) > 0$ .

$$|f(x) - f(c)| < \epsilon \implies f(c) - \epsilon < f(x) < f(c) + \epsilon$$

$$\text{Let } \epsilon = \frac{1}{2}f(c) \exists \delta > 0 \text{ for } \epsilon = \frac{1}{2}f(c)$$

$$\int_{c-\delta}^{c+\delta} f(x)dx > \frac{1}{2}f(c)(2\delta) = f(c)\delta > 0$$

But  $\int_a^b f(x)dx = 0$  and  $f$  is nonnegative.  $f(c) = 0$ .

**Exercise 8.**

$$m \int g \leq \int fg \leq M \int g \implies m \int g \leq 0 \leq M \int g \forall g$$

$$m \leq 0 \leq M \text{ for } \int g = 1 \text{ but also}$$

$$-m \leq 0 \leq -M \implies m \geq 0 \text{ } M \leq 0 \text{ for } \int g = -1$$

So because of this contradiction,  $m = M = 0$ . By intermediate value theorem,  $f = 0$ ,  $\forall x \in [a, b]$ .

#### 4.6 Exercises - Historical introduction, A problem involving velocity, The derivative of a function, Examples of derivatives, The algebra of derivatives.

**Exercise 1.**  $f' = 1 - 2x$ ,  $f'(0) = 1$ ,  $f'(1/2) = 0$ ,  $f'(1) = -1$ ,  $f'(10) = -19$

**Exercise 2.**  $f' = x^2 + x - 2$

- (1)  $f' = 0$ ,  $x = 1, -2$
- (2)  $f'(x) = -2$ ,  $x = 0, -1$
- (3)  $f' = 10$ ,  $x = -4, 3$

**Exercise 3.**  $f' = 2x + 3$

**Exercise 4.**  $f' = 4x^3 + \cos x$

**Exercise 5.**  $f' = 4x^3 \sin x + x^4 \cos x$

**Exercise 6.**  $f' = \frac{-1}{(x+1)^2}$

**Exercise 7.**  $f' = \frac{-1}{(x^2+1)^2}(2x) + 5x^4 \cos x + x^5(-\sin x)$

**Exercise 8.**  $f' = \frac{x-1-(x)}{(x-1)^2} = \frac{-1}{(x-1)^2}$

**Exercise 9.**  $f' = \frac{-1}{(2+\cos x)^2}(-\sin x) = \frac{\sin x}{(2+\cos x)^2}$

**Exercise 10.**

$$\frac{(2x+3)(x^4+x^2+1) - (4x^3+2x)(x^2+3x+2)}{(x^4+x^2+1)^2} = \frac{-2x^5-9x^4+12x^3-3x^2-2x+3}{(x^4+x^2+1)^2}$$

**Exercise 11.**

$$f' = \frac{(-\cos x)(2-\cos x) - (\sin x)(2-\sin x)}{(2-\cos x)^2} = \frac{-2\cos x - 2\sin x + 1}{(2-\cos x)^2}$$

**Exercise 12.**

$$f' = \frac{(\sin x + x \cos x)(1+x^2) - 2x(x \sin x)}{(1+x^2)^2} = \frac{\sin x + x \cos x + x^3 \cos x - x^2 \sin x}{(1+x^2)^2}$$

**Exercise 13.**

(1)

$$\frac{f(t+h) - f(t)}{h} = \frac{v_0 h - 32th - 16h^2}{h} = v_0 + 32t - 16h$$

$$f'(t) = v_0 - 32t$$

(2)  $t = \frac{v_0}{32}$

(3)  $-v_0$

(4)  $T = \frac{v_0}{16}$ ,  $v_0 = 16$  for 1sec.  $v_0 = 160$  for 10sec.  $\frac{v_0}{16}$  for  $T$ sec.

(5)  $f'' = -32$

(6)  $h = -20t^2$

**Exercise 14.**  $V = s^3$ ,  $\frac{dV}{ds} = 3s^2$

**Exercise 15.**

(1)  $\frac{dA}{dr} = 2\pi r = C$

$$(2) \frac{dV}{dr} = 4\pi r^2 = A$$

**Exercise 16.**  $f' = \frac{1}{2\sqrt{x}}$

**Exercise 17.**  $f' = \frac{-1}{(1+\sqrt{x})^2} \left( \frac{1}{2\sqrt{x}} \right)$

**Exercise 18.**  $f' = \frac{3}{2}x^{1/2}$

**Exercise 19.**  $\frac{3}{2}x^{-5/2}$

**Exercise 20.**  $f' = \frac{1}{2}x^{-1/2} + \frac{1}{3}x^{-2/3} + \frac{1}{4}x^{-3/4}x > 0$

**Exercise 21.**  $f' = -\frac{1}{2}x^{-3/2} - \frac{1}{3}x^{-4/3} - \frac{1}{4}x^{-5/4}$

**Exercise 22.**  $f' = \frac{\frac{1}{2}x^{-1/2}(1+x) - \sqrt{x}}{(1+x)^2} = \frac{1}{2\sqrt{x}(1+x)^2}$

**Exercise 23.**  $f' = \frac{(1+\sqrt{x}) - x\frac{1}{2}\frac{1}{\sqrt{x}}}{(1+\sqrt{x})^2} = \frac{1+\frac{1}{2}\sqrt{x}}{(1+\sqrt{x})^2}$

**Exercise 24.**

$$g = f_1 f_2$$

$$g' = f_1' f_2 + f_1 f_2' \frac{g'}{g} = \frac{f_1'}{f_1} + \frac{f_2'}{f_2}$$

$$g = f_1 f_2 \cdots f_n f_{n+1}$$

$$g' = (f_1 f_2 \cdots f_n)' f_{n+1} + (f_1 f_2 \cdots f_n) f_{n+1}';$$

$$\frac{g'}{g} = \frac{(f_1 f_2 \cdots f_n)'}{f_1 f_2 \cdots f_n} + \frac{f_{n+1}'}{f_{n+1}}$$

$$= \frac{f_1'}{f_1} + \frac{f_2'}{f_2} + \cdots + \frac{f_n'}{f_n} + \frac{f_{n+1}'}{f_{n+1}}$$

**Exercise 25.**

$$(\tan x)' = \left( \frac{\cos x}{\sin x} \right)' = \frac{\cos^2 x - (-\sin x) \sin x}{\cos^2 x} = \sec^2 x$$

$$(\cot x)' = \left( \frac{\cos x}{\sin x} \right)' = \frac{-\sin x \sin x - \cos x \cos x}{\sin^2 x} = -\csc^2 x$$

$$(\sec x)' = \frac{-1}{\cos^2 x} (-\sin x) = \tan x \sec x$$

$$(\csc x)' = \frac{-1}{\sin^2 x} \cos x = -\cot x \csc x$$

**Exercise 35.**

$$\begin{aligned} f' &= \frac{(2ax+b)(\sin x + \cos x) - (\cos x - \sin x)(ax^2 + bx + c)}{(\sin x + \cos x)^2} = \\ &= \frac{(2ax+b)(\sin x + \cos x) - (\cos x - \sin x)(ax^2 + bx + c)}{(\sin x + \cos x)^2} \end{aligned}$$

**Exercise 36.**

$$f' = a \sin x + (ax+b) \cos x + c \cos x + (cx+d)(-\sin x) = ax \cos x + (b+c) \cos x + (a-d) \sin x - cx \sin x$$

So then  $a=1, d=1, b=d, c=0$ .

**Exercise 37.**

$$\begin{aligned} g' &= (2ax+b) \sin x + (ax^2+bx+c) \cos x + (2dx+e) \cos x + (dx^2+ex+f)(-\sin x) = \\ &= ax^2 \cos x - dx^2 \sin x + (2a-e)x \sin x + (b+2d)x \cos x + (b-f) \sin x + (c+e) \cos x \end{aligned}$$

$$g = x^2 \sin x. \text{ So } d=-1, b=2, f=2, a=0, e=0, c=0.$$

**Exercise 38.**  $1+x+x^2+\cdots+x^n = \frac{x^{n+1}-1}{x-1}$

(1)

$$(1+x+x^2+\cdots+x^n)' = 1+2x+\cdots+nx^{n-1} = \frac{(n+1)x^n(x-1) - (1)(x^{n+1}-1)}{(x-1)^2}$$

$$= \frac{(n+1)(x^{n+1}-x^n) - x^{n+1}+1}{(x-1)^2} = \frac{nx^{n+1} - (n+1)x^n + 1}{(x-1)^2}$$

$$x(1+2x+\cdots+nx^{n-1}) = x+2x^2+\cdots+nx^n = \frac{nx^{n+2} - (n+1)x^{n+1} + x}{(x-1)^2}$$

(2)

$$\begin{aligned}
(x + 2x^2 + \dots + nx^n)' &= (1 + 2^2x^1 + \dots + n^2x^{n-1}) = \\
&= \frac{(n(n+2)x^{n+1} - (n+1)^2x^n + 1)(x-1)^2 - 2(x-1)(nx^{n+2} - (n+1)x^{n+1} + x)}{(x-1)^4} \\
x + 2^2x^2 + \dots + n^2x^n &= \frac{(n(n+2)x^{n+2} - (n+1)^2x^{n+1} + x)(x-1) - 2(nx^{n+3} - (n+1)x^{n+2} + x^2)}{(x-1)^3} = \\
&= \frac{n^2x^{n+3} + (-2n^2 - 2n + 1)x^{n+2} + (n+1)^2x^{n+1} - x^2 - x}{(x-1)^3}
\end{aligned}$$

**Exercise 39.**

$$\begin{aligned}
\frac{f(x+h) - f(x)}{h} &= \frac{(x+h)^n - x^n}{h} \\
(x+h)^n &= \sum_{j=0}^n \binom{n}{j} x^{n-j} h^j \\
\frac{(x+h)^n - x^n}{h} &= \frac{\sum_{j=1}^n \binom{n}{j} x^{n-j} h^j}{h} = \sum_{j=1}^n x^{n-j} h^{j-1} \binom{n}{j} \\
\lim_{h \rightarrow 0} \frac{(x+h)^n - x^n}{h} &= \binom{n}{1} x^{n-1} = \boxed{nx^{n-1}}
\end{aligned}$$

#### 4.9 Exercises - Geometric interpretation of the derivative as a slope, Other notations for derivatives.

**Exercise 6.**

(1)

$$\begin{aligned}
f &= x^2 + ax + b & \frac{f(x_2) - f(x_1)}{x_2 - x_1} &= \frac{x_2^2 - x_1^2 + a(x_2 - x_1)}{x_2 - x_1} \\
f(x_1) &= x_1^2 + ax_1 + b \\
f(x_2) &= x_2^2 + ax_2 + b & &= \boxed{x_2 + x_1 + a}
\end{aligned}$$

(2)

$$\begin{aligned}
f' &= 2x + a \\
m &= x_2 + x_1 + a = 2x + a \quad \boxed{x = \frac{x_2 + x_1}{2}}
\end{aligned}$$

**Exercise 7.** The line  $y = -x$  as slope  $-1$ .

$$\begin{aligned}
y &= x^3 - 6x^2 + 8x \quad y' = 3x^2 - 12x + 9 \\
3x^2 - 12x + 8 &= -1 \implies x = 3, 1
\end{aligned}$$

The line and the curve meet under the condition

$$-x = x^3 - 6x^2 + 8x \implies x = 3; f(3) = -3$$

At  $x = 0$ , the line and the curve also meet.

**Exercise 8.**  $f = x(1 - x^2)$ .  $f' = 1 - 3x^2$ .

$$f'(-1) = -2 \implies \boxed{y = -2x - 2}$$

For the other line,

$$\begin{aligned}
f'(a) &= 1 - 3a^2 \\
\implies y(-1) = 0 &= (1 - 3a^2)(-1) + b \implies b = 1 - 3a^2
\end{aligned}$$

Now  $f(a) = a(1 - a^2) = a - a^3$  at this point. The line and the curve must meet at this point.

$$\begin{aligned}
y(a) &= (1 - 3a^2)a + (1 - 3a^2) = \\
&= a - 3a^3 + 1 - 3a^2 = a - a^3 \\
\implies -2a^3 + 1 - 3a^2 &= 0 = a^3 - \frac{1}{2} + \frac{3}{2}a^2
\end{aligned}$$

The answer could probably be guessed at, but let's review some tricks for solving cubics.

First, do a translation in the  $x$  direction to center the origin on the point of inflection. Find the point of inflection by taking the second derivative.

$$f'' = 6a + 3 \implies a = -\frac{1}{2}$$

So

$$a = x - \frac{1}{2}$$

$$\implies (x - \frac{1}{2})^3 + \frac{3}{2}(x - \frac{1}{2})^2 - \frac{1}{2} = x^3 = \frac{3}{4}x - \frac{1}{4} = 0$$

Then recall this neat trigonometric fact:

$$\cos 3x = \cos 2x \cos x - \sin 2x \sin x = 4 \cos^3 x - 3 \cos x$$

$$\implies \cos^3 x = \frac{3}{4} \cos x - \frac{\cos 3x}{4} = 0$$

Particularly for this problem, we have  $\cos 3x = 1$ . So  $x = 0, 2\pi/3, 4\pi/3$ .  $\cos x = 1, -\frac{1}{2}$ . Plugging  $\cos x \rightarrow x$  back into what we have for  $a$ ,  $a = -1$ , which we already have in the previous part, and  $a = \frac{1}{2}$ . So

$$\boxed{f\left(\frac{1}{2}\right) = \frac{3}{8}}$$

$$\boxed{y(x) = \left(\frac{1}{4}x\right) + \frac{1}{4}}$$

**Exercise 9.**

$$f(x) = \begin{cases} x^2 & \text{if } x \leq c \\ ax + b & \text{if } x > c \end{cases}$$

$$f'(x) = \begin{cases} 2x & \text{if } x \leq c \\ a & \text{if } x > c \end{cases}$$

$$\boxed{a = 2c; b = -c^2}$$

**Exercise 10.**

$$f(x) = \begin{cases} \frac{1}{|x|} & \text{if } |x| > c \\ a + bx^2 & \text{if } |x| \leq c \end{cases}$$

Note that  $c \neq 0$  since  $|x| \leq c$ , for the second condition.

$$f'(x) = \begin{cases} -\frac{1}{x^2} & \text{if } x > c \\ \frac{1}{x^2} & \text{if } x < c \\ 2bx & \text{if } |x| \leq c \end{cases}$$

So  $\boxed{b = -\frac{1}{2c^3}, a = \frac{3}{2c}}.$

**Exercise 11.**

$$f' = \begin{cases} \cos x & \text{if } x \leq c \\ a & \text{if } x > c \end{cases}$$

**Exercise 12.**  $f(x) = \left( \frac{1-\sqrt{x}}{1+\sqrt{x}} \right) = \frac{1-A}{1+A}$

$$\begin{aligned} A &= \sqrt{x} \quad A' = a = \frac{1}{2}x^{-1/2} = \frac{1}{2A}; \quad A'' = -\frac{1}{4}x^{-3/2} = -\frac{1}{4A^3} \\ f' &= \frac{-A'(1+A) - A'(1-A)}{(1+A)^2} = \frac{-2A'}{(1+A)^2} = \frac{-1}{\sqrt{x}(1+\sqrt{x})^2} \\ f'' &= \frac{1}{(A(1+A^2))^2} (A'(1+A)^2 + A(2)(1+A)A') = \frac{3\sqrt{x}+1}{2x^{3/2}(1+\sqrt{x})^3} \\ f''' &= \frac{1}{2} \left( \frac{-\frac{1}{A^2}A'(A^2(1+A)^3) - (2AA'(1+A)^3 + 3A^2(1+A)^2A')(3+\frac{1}{A})}{(A^2(1+A)^3)^2} \right) \\ &= \frac{-3}{4} \left( \frac{\frac{1}{A} + 4 + 5A}{A^4(1+A^4)} \right) = -\frac{3}{4} \frac{(1+4\sqrt{x}+5x)}{\sqrt{x}(x+\sqrt{x})^4} \end{aligned}$$

**Exercise 13.**

$$\begin{aligned} P &= ax^3 + bx^2 + cx + d & P''(0) &= 2b = 10 \implies b = 5 \\ P' &= 3ax^2 + 2bx + c & P'(0) &= c = -1 & P(0) &= d = -2 \\ P'' &= 6ax + 2b \\ P(1) &= a + 5 + -1 + -2 = a + 2 = -2 \implies a = -4 \end{aligned}$$

**Exercise 14.**

$$\begin{aligned} (1) \quad fg &= 2, \quad \frac{f'}{g'} = 2 \frac{f'}{g} = 4 \frac{g'}{g} = 2f = \frac{1}{2}, \quad g = 4 \\ (2) \quad h' &= \frac{f'g - g'f}{g^2} = \frac{f'}{g} - \frac{g'}{g} \frac{f}{g} = 4 - 2\left(\frac{1}{8}\right) = \frac{15}{4} \\ (3) \quad k' &= f'g + fg' = 4g^2 + f2g = 64 + 4 = 68 \\ \lim_{x \rightarrow 0} \frac{g'(x)}{f'(x)} &= \frac{\lim_{x \rightarrow 0} g'(x)}{\lim_{x \rightarrow 0} f'(x)} = \boxed{\frac{1}{2}} \end{aligned}$$

**Exercise 15.**

- (1) True, by definition of  $f'(a)$ .  
 (2)

$$\lim_{h \rightarrow 0} \frac{f(a) - f(a-h)}{h} = -\lim_{h \rightarrow 0} \frac{f(a-h) - f(a)}{h} = \lim_{-h \rightarrow 0} \frac{f(a-h) - f(a)}{-h} = f'(a)$$

- True, by definition of  $f'(a)$ .  
 (3)

$$\lim_{t \rightarrow 0} \frac{f(a+2t) - f(a)}{t} = 2 \lim_{2t \rightarrow 0} \frac{f(a+2t) - f(a)}{2t} = 2f'(a)$$

- False.  
 (4)

$$\begin{aligned} \lim_{t \rightarrow 0} \frac{f(a+2t) - f(a) + f(a) - f(a+t)}{2t} &= \\ \lim_{2t \rightarrow 0} \frac{f(a+2t) - f(a)}{2t} + -\frac{1}{2} \lim_{t \rightarrow 0} \frac{f(a+t) - f(a)}{t} &= \\ f'(a) - \frac{1}{2}f'(a) &= \frac{1}{2}f'(a) \end{aligned}$$

False.

**Exercise 16.**

(1)

$$\begin{aligned}
D^*(f+g) &= \lim_{h \rightarrow 0} \frac{(f(x+h) + g(x+h))^2 - (f(x) + g(x))^2}{h} = \lim_{h \rightarrow 0} \frac{(F+G)^2 - (f+g)^2}{h} = \\
&= D^*f + D^*g + \lim_{h \rightarrow 0} \frac{2FG - 2fg}{h} \\
\lim_{h \rightarrow 0} \frac{2FG - 2fg}{h} &= \lim_{h \rightarrow 0} \frac{(2(FG) - 2fG)(F+f)}{(F+f)h} + \frac{(2fG - 2fg)(G+g)}{(g+G)h} = \\
&= \lim_{h \rightarrow 0} \frac{2g}{F+h} \lim_{h \rightarrow 0} \frac{F^2 - f^2}{h} + \lim_{h \rightarrow 0} \frac{2f}{G+g} \lim_{h \rightarrow 0} \frac{G^2 - g^2}{h} = \\
&= \frac{g}{f} D^*f + \frac{f}{g} D^*g \\
D^*(f-g) &= \lim_{h \rightarrow 0} \frac{(f(x+h) - g(x+h))^2 - (f(x) - g(x))^2}{h} = \\
&= \lim_{h \rightarrow 0} \frac{(F-G)^2 - (f-g)^2}{h} = \\
&= D^*f + D^*g + \lim_{h \rightarrow 0} \frac{2FG - 2fg}{h} \\
&= D^*f + D^*g - \frac{g}{f} D^*f + \frac{f}{g} D^*g \\
D^*(fg) &= \lim_{h \rightarrow 0} \frac{((fg)(x+h))^2 - ((fg)(x))^2}{h} = \\
&= \lim_{h \rightarrow 0} \frac{(f^2(x+h))(g^2(x+h)) - f^2(x)g^2(x+h) + (g^2(x+h) - g^2(x))f^2(x)}{h} = \\
&= g^2 D^*f + f^2 D^*g \\
D^*(f/g) &= \lim_{h \rightarrow 0} \frac{\frac{f^2(x+h)}{g^2(x+h)} - \frac{f^2(x)}{g^2(x)}}{h} = \lim_{h \rightarrow 0} \frac{\frac{f^2(x+h) - f^2(x)}{g^2(x+h)} + \frac{f^2(x)}{g^2(x+h)} - \frac{f^2(x)}{g^2(x)}}{h} = \\
&= \frac{D^*f}{g^2} + \frac{f^2}{g^4} (-D^*g) \text{ when } g(x) \neq 0
\end{aligned}$$

(2)

(3)

#### 4.12 Exercises - The chain rule for differentiating composite functions, Applications of the chain rule. Related rates and implicit differentiation. Exercise 1. $-2 \sin 2x - 2 \cos x$

Exercise 2.  $\frac{x}{\sqrt{1+x^2}}$

Exercise 3.  $-2x \cos x^2 + 2x(x^2 - 2) \sin x^2 + 2 \sin x^3 + 6x^3 \cos x^3$

Exercise 4.

$$\begin{aligned}
f' &= \cos(\cos^2 x)(-2 \cos x \sin x) \cos(\sin^2 x) + \sin(\cos^2 x) \sin(\sin^2 x)(2 \sin x \cos x) = \\
&= -\sin 2x(\cos(\cos 2x))
\end{aligned}$$

Exercise 5.

$$f' = n \sin^{n-1} x \cos x \cos nx + -n \sin nx \sin^n x$$

Exercise 6.

$$f' = \cos(\sin(\sin x))(\cos(\sin x))(\cos x)$$

Exercise 7.

$$f' = \frac{2 \sin x \cos x \sin x^2 - 2x \cos x^2 \sin^2 x}{\sin^2 x^2} = \frac{\sin 2x \sin x^2 - 2x \sin^2 x \cos x^2}{\sin^2 x^2}$$

Exercise 8.  $f' = \frac{1}{2} \sec^2 \frac{x}{2} + \frac{1}{2} \csc^2 \frac{x}{2}$

Exercise 9.  $f' = 2 \sec^2 x \tan x + -2 \csc^2 x \cot x$

Exercise 10.  $f' = \sqrt{1+x^2} + \frac{x^2}{\sqrt{1+x^2}} = \frac{1+2x^2}{\sqrt{1+x^2}}$

Exercise 11.  $f' = \frac{4}{(4-x^2)^{3/2}}$



**Exercise 12.**

$$f' = \frac{1}{3} \left( \frac{1+x^3}{1-x^3} \right)^{-2/3} \left( \frac{3x^2(2)}{(1-x^3)^2} \right) = \frac{2x^2}{(1-x^3)^2} \left( \frac{1+x^3}{1-x^3} \right)^{-2/3}$$

**Exercise 13. This exercise is important.** It shows a neat integration trick.

$$\begin{aligned} f(x) &= \frac{1}{\sqrt{1+x^2}(x+\sqrt{1+x^2})} = \frac{1}{\sqrt{1+x^2}(x+\sqrt{1+x^2})} \left( \frac{x-\sqrt{1+x^2}}{x-\sqrt{1+x^2}} \right) = \\ &= \frac{x-\sqrt{1+x^2}}{-\sqrt{1+x^2}} = 1 - \frac{x}{\sqrt{1+x^2}} \\ f' &= \frac{\sqrt{1+x^2} - \frac{x^2}{\sqrt{1+x^2}}}{1+x^2} = \frac{1}{(1+x^2)^{3/2}} \end{aligned}$$

**Exercise 14.**

$$\frac{1}{2}(x + \sqrt{x + \sqrt{x}})^{-1/2} \left( 1 + \frac{1}{2}(x + \sqrt{x})^{-1/2} \left( 1 + \frac{1}{2\sqrt{x}} \right) \right)$$

**Exercise 15.**

$$f' = (2+x^2)^{1/2}(3+x^3)^{1/3} + (1+x)x(2+x^2)^{-1/2}(3+x^2)^{1/3} + (1+x)(2+x^2)^{1/2}(3+x^3)^{-2/3}x^2$$

**Exercise 16.**

$$\begin{aligned} f' &= \frac{-1}{\left(1 + \frac{1}{x}\right)^2} \left( \frac{-1}{x^2} \right) = \frac{1}{(x+1)^2} g' = \frac{-1}{\left(1 + \frac{1}{f}\right)^2} \left( \frac{-1}{f^2} \right) f' = \frac{f'}{(f+1)^2} \\ g' &= \frac{(x+1)^{-2}}{\left(\frac{x}{x+1} + 1\right)^2} = \frac{1}{(2x+1)^2} \end{aligned}$$

**Exercise 17.**  $h' = f'g'$

$x$	$h$	$h'$	$k$	$k'$
0	$f(2) = 0$	$2(-5) = -10$	$g(1) = 0$	$1(5) = 5$
1	$f(0) = 1$	$5(1) = 5$	$g(3) = 1$	$-6(-2) = 12$
2	$f(3) = 2$	$4(1) = 4$	$g(0) = 2$	$-5(2) = -10$
3	$f(1) = 3$	$-2(-6) = 12$	$g(2) = 3$	$1(4) = 4$

**Exercise 18.**

$$\begin{aligned} g(x) &= xf(x^2) \\ g'(x) &= f(x^2) + x(2x)f'(x^2) = f(x^2) + 2x^2f'(x^2) \\ g''(x) &= 2xf'(x^2) + 4xf'(x^2) + 2x^2(2x)f''(x^2) = 6xf'(x^2) + 4x^3f''(x^2) \end{aligned}$$

$x$	$g(x)$	$g'(x)$	$g''(x)$
0	0	0	0
1	1	3	10
2	12	$6 + 8(3) = 30$	$12(3) + 32(0) = 36$

**Exercise 19.**

(1)

$$g' = \frac{df(x^2)}{dx^2} 2x = 2xf'$$

(2)

$$g' = 2 \sin x \cos x f' - 2 \cos x \sin x f' = (\sin 2x)(f'(\sin^2 x) - f'(\cos^2 x))$$

(3)

$$g' = \frac{df(f(x))}{d(f(x))} f'$$

(4)

$$g' = \frac{df(f(f(x)))}{d(f(f(x)))} \frac{d(f(f(x)))}{d(f(x))} \frac{df}{dx}$$

**Exercise 20.**  $V = s^3$ ,  $s = s(t)$   $\frac{dV}{dt} = 3s^2 \frac{ds}{dt}$ .

$$s = 5cm \ 75cm^3/sec$$

$$s = 10cm \ 300cm^3/sec$$

$$s = xcm \ 3x^2cm^3/sec$$

**Exercise 21.**

$$l = \sqrt{x^2 + h^2} \quad \frac{dl}{dt} = \frac{1}{l} x \frac{dx}{dt}$$

$$\frac{dx}{dt} = \frac{l}{x} \frac{dl}{dt} = \frac{10mi}{-\sqrt{10^2 - 8^2}} (-4mi/sec) = \frac{20}{3} \frac{mi}{sec} \left( \frac{3600sec}{1hr} \right)$$

**Exercise 22.**

$$l^2 = x^2 + s^2$$

$$2l \frac{dl}{dt} = 2x \frac{dx}{dt} \frac{dl}{dt} = \frac{x}{l} \frac{dx}{dt}$$

$$\frac{dl}{dt} \left( x = \frac{s}{2} \right) = 20\sqrt{5}$$

$$\frac{dl}{dt} (x = s) = 50\sqrt{2}$$

**Exercise 23.**

$$\frac{dl}{dt} = \frac{x}{l} \frac{dx}{dt} = \frac{3}{5} 12 = \boxed{\frac{36}{5} mi/hr}$$

**Exercise 24.** Given the preliminary information

$$\frac{r}{h} = \frac{2}{5} = \alpha, \quad V = \frac{1}{3} \pi r^2 h = \frac{1}{3} \pi \alpha^2 h^3$$

(1)

$$V = \frac{\pi r^2}{h^2} (h^2 y - h y^2 + \frac{1}{3} y^3)$$

$$\frac{dV}{dt} = \frac{\pi r^2}{h^2} (h^2 - 2h y + y^2) \frac{dy}{dt}$$

$$\frac{dy}{dt} = \frac{h^2}{\pi r^2} \left( \frac{1}{h^2 - 2h y + y^2} \right) \frac{dV}{dt} = \frac{10^2}{\pi 4^2} \left( \frac{1}{10^2 - 2(10)5 + 25} \right) 5 = \frac{5}{4\pi}$$

(2)

$$\frac{dV}{dt} = \pi \alpha^2 h^2 \frac{dh}{dt}, \quad \frac{dh}{dt} = \frac{1}{\pi \alpha^2 h^2} \frac{dV}{dt} = \frac{5}{4\pi}$$

**Exercise 25.**

$$\alpha = \frac{r}{h} = \frac{3}{2}$$

$$\frac{dV}{dt} = \pi \alpha^2 h^2 \frac{dh}{dt}$$

$$c - 1\pi \frac{9}{4} (2^2) 4 = 36\pi \implies \boxed{c = 36\pi + 1}$$

**Exercise 26.** The constraint equation, using Pythagorean theorem on the geometry of a bottom hemisphere, is

$$r^2 = R^2 - (R - h)^2 = 2Rh - h^2$$

So then

$$r \frac{dr}{dt} = (R - h) \frac{dh}{dt}$$

$$V = \int \pi r^2 dh \implies \frac{dV}{dh} = \pi r^2 = \pi(2Rh - h^2)$$

$$\implies \frac{dV}{dh} = \pi(2(10)(5) - 25) = 50\pi$$

$$\begin{aligned}
\frac{dV}{dt} &= \frac{dV}{dh} \frac{dh}{dt}, \implies \frac{dh}{dt} = \frac{dV}{dt} \left( \frac{1}{\pi(2Rh - h^2)} \right) \\
\left( \frac{r}{R - h} \right) \frac{dr}{dt} &= \frac{dV}{dt} \left( \frac{1}{\pi(2Rh - h^2)} \right) \\
\frac{dr}{dt} &= \frac{dV}{dt} \left( \frac{R - h}{r\pi(2Rh - h^2)} \right) = \\
\implies &= (5\sqrt{3}) \left( \frac{10 - 5}{\pi(2(10)5 - 25)^{3/2}} \right) = \frac{1}{15\pi}
\end{aligned}$$

**Exercise 27.** I suppose the area of the triangle is 0 at  $t = 0$ .

Now the point on vertex  $B$  moves up along the  $y$  axis according to  $y = 1 + 2t$ .  $y\left(\frac{7}{2}\right) = 8$ .

$$\begin{aligned}
A &= \frac{1}{2} \sqrt{(y-1) \frac{36}{7}} y \\
\frac{dA}{dt} &= \frac{1}{2} \left( \sqrt{\frac{36}{7}} \frac{1}{2\sqrt{y-1}} y + \sqrt{(y-1) \frac{36}{7}} \right) \frac{dy}{dt} = \\
&= \frac{1}{2} \left( \frac{6}{2(7)} 8 + 6 \right) (2) = \frac{66}{7}
\end{aligned}$$

**Exercise 28.** From the given information,  $h = 3r + 3$ . The volume formula is  $V = \frac{\pi R^2}{3} H$ . So then

$$\begin{aligned}
V &= \pi/3r^2(3r+3) = \pi r^3 + \pi r^2 \\
\frac{dV}{dr} &= \pi r(3r+2) \frac{dr}{dt}
\end{aligned}$$

With the given information, we get

$$\frac{dr}{dt} = \frac{1}{\pi(6)(20)}$$

Using this, we can plug this back in for the different case:

$$\frac{dV}{dt} = n = \pi(36)(110)/(120\pi) = \boxed{33}$$

**Exercise 29.**

$$(1) \frac{dy}{dt} = 2x \frac{dx}{dt}; \text{ when } x = \frac{1}{2}, y = \frac{1}{4}, \frac{dy}{dt} = \frac{dx}{dt}$$

$$(2) \boxed{t = \frac{\pi}{6}}$$

**Exercise 30.**

$$(1) 3x^2 + 3y^2y' = 0 \implies x^2 + y^2y' = 0$$

(2)

$$\begin{aligned}
2x + 2yy'^2 + y^2y'' &= 0 \implies y^2y'' = -2(x + yy'^2) \\
\implies y'' &= -2 \left( \frac{xy^4 + yx^4}{y^6} \right) = -2xy^{-5}
\end{aligned}$$

**Exercise 31.**

$$\frac{1}{2} \frac{1}{\sqrt{x}} + \frac{1}{2\sqrt{y}} y' = 0 \implies y' = \frac{-\sqrt{y}}{\sqrt{x}} < 0$$

**Exercise 32.**

$$\begin{aligned}
&\pm \sqrt{\frac{12 - 3x^2}{4}} \\
6x + 8yy' &= 0 \implies y' = \frac{-3x}{4y} \\
3 + 4(y'^2 + yy'') &= 0 \\
y'' &= \left( -\frac{3}{4} - y'^2 \right) \frac{1}{y} = \frac{-9}{4y^3}
\end{aligned}$$

**Exercise 33.**

$$\begin{aligned}
\sin xy + x \cos^2 xy (y + xy') + 4x &= 0 \\
y'x^2 \cos xy + xy \cos xy + \sin xy + 4x &= 0
\end{aligned}$$

**Exercise 34.**  $y = x^4$ .  $y^n = x^m$ .

$$y^n = x^m, y' n y^{n-1} = m x^{m-1}; y' = \frac{m x^{m-1}}{n y^{n-1}} = \frac{m}{n} \frac{x^{m-1}}{x^{m(1-1/n)}} =$$

$$y' = \frac{m}{n} x^{m/n-1}$$

#### 4.15 Exercises - Applications of differentiation to extreme values of functions, The mean-value theorem for derivatives.

Let's recap what was shown in the past two sections:

**Theorem 13** (Theorem 4.3).

Let  $f$  be defined on  $I$ .

Assume  $f$  has a rel. extrema at an int. pt.  $c \in I$ .

If  $\exists f'(c)$ ,  $f'(c) = 0$ ; the converse is not true.

*Proof.*  $Q(x) = \frac{f(x)-f(c)}{x-c}$  if  $x \neq c$ ,  $Q(c) = f'(c)$

$\exists f'(c)$ , so  $Q(x) \rightarrow Q(c)$  as  $x \rightarrow c$  so  $Q$  is continuous at  $c$ .

If  $Q(c) > 0$ ,  $\frac{f(x)-f(c)}{x-c} > 0$ . For  $x - c \geq 0$ ,  $f(x) \geq f(c)$ , thus contradicting the rel. max or rel. min. (no neighborhood about  $c$  exists for one!)

If  $Q(c) < 0$ ,  $\frac{f(x)-f(c)}{x-c} < 0$ . For  $x - c \geq 0$ ,  $f(x) \leq f(c)$ , thus contradicting the rel. max or rel. min. (no neighborhood about  $c$  exists for one!)

Converse is not true: e.g. saddle points. □

**Theorem 14** (Rolle's Theorem).

Let  $f$  be cont. on  $[a, b]$ ,  $\exists f'(x) \quad \forall x \in (a, b)$  and let

$$f(a) = f(b)$$

then  $\exists$  at least one  $c \in (a, b)$ , such that  $f'(c) = 0$ .

*Proof.* Suppose  $f'(x) \neq 0 \quad \forall x \in (a, b)$ .

By extreme value theorem,  $\exists$  abs. max (min)  $M, m$  somewhere on  $[a, b]$ .

$M, m$  on endpoints  $a, b$  (Thm 4.3).

$F(a) = f(b)$ , so  $m = M$ .  $f$  constant on  $[a, b]$ . Contradict  $f'(x) \neq 0$  □

**Theorem 15** (Mean-value theorem for Derivatives). Assume  $f$  is cont. everywhere on  $[a, b]$ ,  $\exists f'(x) \quad \forall x \in (a, b)$ .

$\exists$  at least one  $c \in (a, b)$  such that

$$(6) \quad f(b) - f(a) = f'(c)(b - a)$$

*Proof.*

$$h(x) = f(x)(b - a) - x(f(b) - f(a))$$

$$h(a) = f(a)b - f(a)a - af(b) + af(a)$$

$$h(b) = f(b)(b - a) - b(f(b) - f(a)) = bf(a) - af(b) = h(a)$$

$$\implies \exists c \in (a, b), \text{ such that } h'(c) = 0 = f'(c)(b - a) - (f(b) - f(a))$$
□

**Theorem 16** (Cauchy's Mean-Value Formula). Let  $f, g$  cont. on  $[a, b]$ ,  $\exists f', g' \quad \forall x \in (a, b)$

Then  $\exists c \in (a, b)$ .  $x$

$$(7) \quad f'(c)(g(b) - g(a)) = g'(c)(f(b) - f(a)) \quad (\text{note how it's symmetrical})$$

*Proof.*

$$h(x) = f(x)(g(b) - g(a)) - g(x)(f(b) - f(a))$$

$$h(a) = f(a)(g(b) - g(a)) - g(a)(f(b) - f(a)) = f(a)g(b) - g(a)f(b)$$

$$h(b) = f(b)(g(b) - g(a)) - g(b)(f(b) - f(a))$$

$$\implies h'(c) = f'(c)(g(b) - g(a)) - g'(c)(f(b) - f(a)) = 0 \quad (\text{by Rolle's Thm.})$$
□

**Exercise 1.** For any quadratic polynomial  $y = y(x) = Ax^2 + Bx + C$ ,

$$y(a) = Aa^2 + Ba + C$$

$$y(b) = Ab^2 + Bb + C$$

$$\frac{y(b) - y(a)}{b - a} = \frac{A(b - a)(b + a) + B(b - a)}{b - a} = A(b + a) + B$$

$$y' = 2Ax + B$$

$$y' \left( \frac{a + b}{2} \right) = A(a + b) + B$$

Thus the chord joining  $a$  and  $b$  has the same slope as the tangent line at the midpt.

**Exercise 2.** The contrapositive of a theorem is always true. So the contrapositive of Rolle's Theorem is

If  $\nexists$  at least one  $c \in (a, b)$  s.t.  $f'(c) = 0$ ,

then  $f(a) \neq f(b)$ .

$$g' = 3x^2 - 3 = 3(x^2 - 1) \implies g'(\pm 1) = 0$$

$$\text{Suppose } g(B) = 0, \quad B \in (-1, 1)$$

$$\text{then } \forall x \in (-1, 1), \quad x \neq B, \quad g(x) \neq g(B), \quad \text{so } g(x) \neq 0 \text{ for } x \neq B$$

$$\boxed{\text{so only at most one } B \in (-1, 1) \text{ s.t. } g(B) = 0}$$

**Exercise 3.**  $f(x) = \frac{3-x^2}{2}$  if  $x \leq 1$ ,  $f(x) = \frac{1}{x}$  if  $x \geq 1$ .

(1) See sketch.

(2)

$$f(x) = \begin{cases} \frac{3-x^2}{2} & \text{if } x \leq 1 \\ 1/x & \text{if } x \geq 1 \end{cases} \quad f(1) = 1 = f(1) = 1/1$$

$$f'(x) = \begin{cases} -x; & f'(1) = -1 & \text{for } x \leq 1 \\ -1/x^2; & f'(1) = -1 & \text{for } x > 1 \end{cases}$$

Then  $f(x)$  is cont. and diff. on  $[0, 2]$ .

For  $0 \leq a < b \leq 1$

$$\frac{\frac{3-b^2}{2} - \left(\frac{3-a^2}{2}\right)}{b-a} = \frac{-(a+b)}{2} = -c$$

Note that  $-1 \leq f' \leq 0$  for  $0 \leq x \leq 1$

For  $1 \leq a < b \leq 2$

$$\frac{\frac{1}{b} - \frac{1}{a}}{b-a} = \frac{-1}{ab} = \frac{-1}{c^2} \implies c = \sqrt{ab}$$

Note that  $-1 \leq f' \leq -1/4$

For  $0 \leq a \leq 1$ ,  $1 \leq b \leq 2$

$$\frac{\frac{1}{b} - \left(\frac{3-a^2}{2}\right)}{b-a} = \frac{2 - (3-a^2)b}{2b(b-a)} = -c \text{ or } \frac{-1}{c^2}$$

depending upon if  $0 \leq c \leq 1$  or  $1 \leq c \leq 2$ , respectively

For instance, for  $a = 0$ ,  $b = 2$ , then  $\frac{f(b)-f(a)}{b-a} = -1/2$ , so  $c = 1/2$  or  $c = \sqrt{2}$

**Exercise 4.**

$$f(1) = 1 - 1^{2/3} = 0 = f(-1) = 1 - ((-1)^2)^{3/2} = 0$$

$$f' = \frac{-2}{3}x^{-1/3} \neq 0 \quad \text{for } |x| \leq 1$$

This is possible since  $f$  is not differentiable at  $x = 0$ .

**Exercise 5.**  $x^2 = x \sin x + \cos x$ .  $g = xS + C - x^2$ .  $g' = S + xC - S - 2x = xC - 2x = x(C - 2)$ . Since  $|C| \leq 1$  then  $(C - 2)$  is negative for all  $x$ . Then for  $x \geq 0$ ,  $g' \leq 0$ . Since  $g(0) = 1$  and for  $x \rightarrow \pm\infty$ ,  $g \rightarrow \mp\infty$ , then we could conclude that  $g$  must become zero between 0 and  $\infty$  and  $-\infty$  and 0.

**Exercise 6.**

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

$$b = x + h$$

$$b - a = h$$

$$a = x$$

$$\implies f(x + h) - f(x) = hf'(x + \theta h)$$

$$x < x + \theta h < x + h$$

$$(1) f(x) = x^2, \quad f' = 2x.$$

$$(x + h)^2 - x^2 = 2xh + h^2 = h(2(x + \theta h))$$

$$\frac{2x + h}{2} - x = \theta h \implies \boxed{\theta = \frac{1}{2}} \quad \text{so then } \lim_{h \rightarrow 0} \theta = \frac{1}{2}$$

$$(2) f(x) = x^3, \quad f' = 3x^2.$$

$$(x + h)^3 - x^3 = 3x^2h + 3xh^2 + h^3 = h3(x + \theta h)^2 \implies \left( \sqrt{\frac{3x^2 + 3xh + h^2}{3}} - x \right) / h = \theta$$

$$\theta = \sqrt{\frac{3x^2 + 3xh + h^2}{3h^2}} - \frac{x}{h} = \frac{\sqrt{x^2 + xh + \frac{h^2}{3}} - x}{h} \left( \frac{\sqrt{x^2 + xh + \frac{h^2}{3}} + x}{\sqrt{x^2 + xh + \frac{h^2}{3}} + x} \right) =$$

$$= \frac{x + \frac{h}{3}}{x + \sqrt{x^2 + xh + \frac{h^2}{3}}}$$

$$\implies \lim_{h \rightarrow 0} \boxed{\theta = \frac{1}{2}}$$

**Notice the trick of multiplying by the conjugate** on top and bottom to get a way to evaluate the limit.

**Exercise 7.**  $f(x) = (x - a_1)(x - a_2) \dots (x - a_r)g(x)$ .

$$(1) a_1 < a_2 < \dots < a_r.$$

Since  $f(a_1) = f(a_2) = 0$ .  $f'(c) = 0$  for  $c_1 \in (a_1, a_2)$ .

Consider that  $f(a_2) = f(a_3) = 0$  as well as  $f'(c_2) = 0$  for  $c \in (a_2, a_3)$ .

Indeed, since  $f(a_j) = f(a_{j+1}) = 0$ ,  $f'(c) = 0$  for  $c \in (a_j, a_{j+1})$ .

Thus,  $\exists r - 1$  zero's.

$f^{(k)}$  has  $r - k$  zeros in  $[a, b]$ .

$$f^{(k)} = (x - a_1)(x - a_2) \dots (x - a_{r-k})g_k(x)$$

Since  $f(a_1) = f(a_2) = 0$ ,  $f^{(k+1)}(c_1) = 0$  for  $c_1 \in (a_1, a_2)$ .

$$f^{(k)}(a_j) = f^{(k)}(a_{j+1}) = 0, \quad f^{(k+1)}(c_j) = 0 \text{ for } c_j \in (a_j, a_{j+1})$$

$$\implies f^{(k)}(x) \text{ has at least } r - k \text{ zeros in } [a, b]$$

We had shown the above by induction.

- (2) We can conclude that there's at most  $r + k$  zeros for  $f$  (since  $f^{(k)}$  has exactly  $r$  zeros, the intervals containing the  $r$  zeros are definite).

**Exercise 8.** Using the mean value theorem

$$(1)$$

$$\frac{\sin x - \sin y}{x - y} = \cos c \implies \left| \frac{\sin x - \sin y}{x - y} \right| = |\cos c| \leq 1$$

$$\implies |\sin x - \sin y| \leq |x - y|$$

$$(2) x \geq y > 0.$$

$f(z) = z^n$  is monotonically increasing for  $n \in \mathbb{Z}$ .

By mean-value theorem,

$$\frac{x^n - y^n}{x - y} = nc^{n-1} \text{ for } y < c < x$$

$$\text{Since } 0 < y < c < x; \quad ny^{n-1} \leq \frac{x^n - y^n}{x - y} \leq nx^{n-1}.$$

**Exercise 9.** Let  $g(x) = \left(\frac{f(b)-f(a)}{b-a}\right)x + \left(\frac{bf(a)-af(b)}{b-a}\right)$ .

$$f - g = h \quad \begin{array}{l} h(a) = h(c) \\ h(c) = h(b) \end{array}$$

so  $\exists c_1 \in (a, c)$ ,  $c_2 \in (c, b)$  s. t.  $h'(c_1) = h'(c_2) = 0$  by Rolle's Thm.

Let  $h' = H$

since  $H(c_1) = H(c_2) = 0$  and  $H$  is cont. diff. on  $(c_1, c_2)$ . then

$$\exists c_3 \in (c_1, c_2) \text{ s.t. } H'(c_3) = h''(c_3) = 0$$

$$\text{Now } h'' = (f - g)'' = f'' \text{ so } f''(c_3) = 0$$

We've shown one exists; that's enough.

**Exercise 10.** Assume  $f$  has a derivative everywhere on an open interval  $I$ .

$$g(x) = \frac{f(x) - f(a)}{x - a} \text{ if } x \neq a; \quad g(a) = f'(a)$$

$$(1) \quad g = \left(\frac{1}{x-a}\right)f - \frac{1}{x-a}f(a). \quad f \text{ is cont. on } (a, b] \text{ since } \exists f' \quad \forall x \in (a, b).$$

$\frac{1}{x-a}$  is cont. on  $(a, b]$ . Then  $g$  is cont. on  $(a, b]$  (remember, you can add, subtract, multiply, and divide cont. functions to get cont. functions because the rules for taking limits allow so).

$$g \text{ is cont. at } a \text{ since } \lim_{x \rightarrow a} g = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} = f'(a).$$

By mean value theorem,

$$\left(\frac{f(x) - f(a)}{x - a}\right) = f'(c) = g(x) \quad \forall c \in (a, x) \quad \forall x \in (a, b]$$

Then  $\forall c \in [a, b]$ ,  $f'(c)$  ranges from  $f'(a)$  to  $g(b)$  since  $f'(c) = g(x)$  so whatever  $g(x)$  ranges from and to, so does  $f'(c)$ .

$$(2) \quad \text{Let } h(x) = \frac{f(x) - f(b)}{x - b} \text{ if } x \neq b; \quad h(b) = f'(b).$$

$h$  is cont. on  $[a, b)$  since  $\frac{1}{x-b}$  is cont.,  $f(x)$  is cont.

$$\lim_{x \rightarrow b} h = \lim_{x \rightarrow b} \frac{f(x) - f(b)}{x - b} = f'(b) \text{ so } h \text{ is cont. at } b.$$

$h$  is cont. on  $[a, b] \rightarrow h$  takes all values from  $h(a)$  to  $f'(b)$  on  $[a, b]$  (by intermediate value theorem).

By mean value theorem,

$$h(x) = \frac{f(b) - f(x)}{b - x} = f'(c_2) \quad \text{for } c_2 \in (x, b) \quad \forall x \in [a, b]$$

So then  $f'$  ranges from  $h(a)$  to  $f'(b)$  just like  $h$ .

$h(a) = g(b)$ . So then  $f'$  must range from  $f'(a)$  to  $f'(b)$

#### 4.19 Exercises - Applications of the mean-value theorem to geometric properties of functions, Second-derivative test for extrema, Curve sketching.

**Exercise 1.**  $f(x) = x^2 - 3x + 2$

$$(1) \quad f'(x) = 2x - 3 \quad x_0 = \frac{3}{2}.$$

$$(2) \quad f'(x) \geq 0 \text{ for } x \geq \frac{3}{2}$$

$$(3) \quad f'' = 2 > 0 \text{ for } \forall x \in \mathbb{R}$$

(4) See sketch.

**Exercise 2.**  $f(x) = x^3 - 4x$

$$(1) \quad f' = 3x^2 - 4 \quad x_c = \pm \frac{2}{\sqrt{3}}$$

$$(2) \quad f' \geq 0 \text{ when } |x| \geq \frac{2}{\sqrt{3}}$$

$$(3) \quad f'' = 6x \quad f'' \geq 0 \text{ when } x \geq 0$$

(4) See sketch.

**Exercise 3.**  $f(x) = (x - 1)^2(x + 2)$

$$(1) \quad f' = 3(x - 1)(x + 1) \quad f'(x) = 0 \text{ when } x = \pm 1$$

- (2)  $f' \geq 0$  when  $|x| \geq 1$   
 (3)  $f'' = 3(2x) = 6x$   $f'' \geq 0$  when  $x \geq 0$   
 (4) See sketch.

**Exercise 4.**  $f(x) = x^3 - 6x^2 + 9x + 5$

- (1)  $f' = 3x^2 - 12x + 9 = 3(x-3)(x-1)$   $f'(x) = 0$  when  $x = 3, 1$   
 (2)  
 $f'(x) > 0$  when  $x < 1, x > 3$   
 $f'(x) < 0$  when  $1 < x < 3$   
 (3)  $f'' = 6x - 12 = 6(x-2)$   $f'' \geq 0$  when  $x \geq 2$   
 (4) See sketch.

**Exercise 5.**  $f(x) = 2 + (x-1)^4$

- (1)  $f'(x) = 4(x-1)^3$ .  $f'(0) = 0$  when  $x = 1$   
 (2)  $f'(x) \geq 0$  when  $|x| \geq 1$   
 (3)  $f''(x) = 12(x-1)^2 > 0 \quad \forall x \neq 1$   
 (4) See sketch.

**Exercise 6.**  $f(x) = 1/x^2$

- (1)  $f' = -\frac{2}{x^3}$   $f'(x) = 0$  for no  $x$   
 (2)  $f' \geq 0$  when  $x \leq 0$   
 (3)  $f'' = \frac{6}{x^4} > 0 \quad \forall x \neq 0$   
 (4) See sketch.

**Exercise 7.**  $f(x) = x + 1/x^2$

- (1)  $f' = 1 - \frac{2}{x^3}$   $f'(x) = 0 = 1 - \frac{2}{x^3} \implies x_c = 2^{1/3}$   
 (2)  
 $f'(x) > 0$  when  $x < 0, 0 < x < 2^{1/3}$   
 $f'(x) < 0$  when  $x > 2^{1/3}$   
 (3)  $f'' = \frac{6}{x^4} > 0 \quad \forall x \neq 0$   
 (4) See sketch.

**Exercise 8.**  $f(x) = \frac{1}{(x-1)(x-3)}$

- (1)  $f' = \frac{-1}{(x-1)^2(x-3)^2}((x-3) + x-1) = \frac{(-2)(x-2)}{(x-1)^2(x-3)^2}$   
 $f'(x) = 0$  when  $x = 2$   
 (2)  $f' \geq 0$  when  $x \leq 2$   
 (3)

$$f'' = (-2) \left( \frac{(x-1)^2(x-3)^2 - (x-2)(2(x-1)(x-3)^2 + 2(x-3)(x-1)^2)}{(x-1)^4(x-3)^4} \right) =$$

$$= (6) \left( \frac{x^2 - 4x + \frac{13}{3}}{(x-1)^3(x-3)^3} \right)$$

$$x^2 - 4x + \frac{13}{3} > 0 \text{ since } 144 - 4(-3)(-13) = 144 + 12(-13) < 0 \text{ so}$$

$$f'' > 0 \text{ if } x > 3, x < 1$$

$$f'' < 0 \text{ if } 1 < x < 3$$

- (4) See sketch.

**Exercise 9.**  $f(x) = x/(1+x^2)$

- (1)  
 $f' = \frac{(1+x^2) - x(2x)}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2}$   
 $f'(x) = 0$  when  $x = \pm 1$   
 (2)  $f' \geq 0$  when  $|x| \leq 1$



(3)

$$f'' = \frac{-2x(1+x^2)^2 - 2(1+x^2)(2x)(1-x^2)}{(1+x^2)^4} = \frac{2x(x^2-3)}{(1+x^2)^3}$$

$$f'' > 0 \text{ when } x > \sqrt{3}$$

$$f'' < 0 \text{ when } 0 < x < \sqrt{3}$$

$$f'' > 0 \text{ when } -\sqrt{3} < x < 0$$

$$f'' < 0 \text{ when } x < -\sqrt{3}$$

(4) See sketch.

**Exercise 10.**  $f(x) = (x^2 - 4)/(x^2 - 9)$

(1)

$$f' = \frac{2x(x^2 - 9) - (x^2 - 4)(2x)}{(x^2 - 9)^2} = \frac{-10x}{(x^2 - 9)^2}$$

$$f'(0) = 0$$

$$(2) f' \geq 0 \text{ when } x \leq 0, \quad x \neq \pm 3$$

(3)

$$f'' = (-10) \left( \frac{(x^2 - 9)^2 - 2(x^2 - 9)(2x)x}{(x^2 - 9)^2} \right) = (30) \frac{(x^2 + 3)}{(x^2 - 9)^3}$$

$$f'' \geq 0 \text{ when } |x| \geq 3$$

(4) See the sketch.

**Exercise 11.**  $f(x) = \sin^2 x$

$$(1) f' = \sin 2x \text{ So then } f' = 0 \text{ when } x = \frac{\pi}{2}n$$

(2)

$$f' > 0 \text{ when } \begin{array}{l} 0 < x < \frac{\pi}{2} \\ \pi n < x < \frac{\pi}{2} + \pi n \end{array}$$

$$f' < 0 \text{ when } \frac{\pi}{2} + \pi n < x < \pi(n+1)$$

(3)

$$f'' = 2 \cos 2x \quad \begin{array}{l} f'' > 0 \text{ when } -\frac{\pi}{4} + \pi n < x < \frac{\pi}{4} + \pi n \\ f'' < 0 \text{ when } \frac{\pi}{4} + \pi n < x < \frac{3\pi}{4} + \pi n \end{array}$$

(4) See sketch.

**Exercise 12.**  $f(x) = x - \sin x$

$$(1) f' = 1 - \cos x \quad f' = 0 \text{ when } x = 2\pi n$$

$$(2) f' > 0 \text{ if } x \neq 2\pi n$$

(3)

$$f'' = \sin x \quad \begin{array}{l} f'' > 0 \text{ when } 2\pi n < x < 2\pi n + \pi \\ f'' < 0 \text{ when } 2\pi n + \pi < x < 2\pi(n+1) \end{array}$$

(4) See sketch.

**Exercise 13.**  $f(x) = x + \cos x$

$$(1) f' = 1 - \sin x \quad x = \frac{\pi}{2} + 2\pi n \quad f'(x) = 0$$

$$(2) f' > 0 \text{ if } x \neq \frac{\pi}{2} + 2\pi n$$

(3)

$$f'' = -\cos x \quad \begin{array}{l} f'' > 0 \text{ when } -\frac{\pi}{2} + 2\pi n < x < \frac{\pi}{2} + 2\pi n \\ f'' < 0 \text{ when } \frac{\pi}{2} + 2\pi n < x < \frac{3\pi}{2} + 2\pi n \end{array}$$

(4) See sketch.

**Exercise 14.**  $f(x) = \frac{1}{6}x^2 + \frac{1}{12} \cos 2x$

$$(1) f' = \frac{1}{3}x + \frac{-\sin 2x}{6} \quad f'(0) = 0$$

$$(2) f' \geq 0 \text{ when } x \geq 0$$

- (3)  $f'' = \frac{1}{3} - \frac{\cos 2x}{3} = \frac{1 - \cos 2x}{3}$   
 $x = \pi n$  for  $f'' = 0$ . Otherwise  $f'' > 0$  for  $x \neq \pi n$   
 (4) See sketch.

#### 4.21 Exercises - Worked examples of extremum problems.

##### Exercise 1.

$$\begin{aligned} A &= xy \\ P &= 2(x + y) = 2\left(x + \frac{A}{x}\right) \\ P' &= 2\left(1 - \frac{A}{x^2}\right) = 0 \\ x &= \sqrt{A} \\ P'' &= \frac{4A}{x^3} > 0 \text{ for } x > 0 \text{ so } x = \sqrt{A} \text{ minimizes } P \end{aligned}$$

##### Exercise 2.

$$\begin{aligned} A &= xy \quad L = 2x + y \\ A &= x(L - 2x) = Lx - 2x^2 \implies \frac{dA}{dx} = L - 4x = 0 \text{ when } x = \frac{L}{4} \quad y = \frac{L}{2} \\ A'' &= -4 \text{ so } x = \frac{L}{4} \text{ maximizes } A \end{aligned}$$

##### Exercise 3.

$$\begin{aligned} A &= xy \quad L = 2x + y = 2x + \frac{A}{x} \quad \frac{dL}{dx} = 2 + \frac{-A}{x^2} = 0 \text{ when } x = \frac{\sqrt{A}}{\sqrt{2}} \quad y = \sqrt{2}\sqrt{A} \\ L'' &= \frac{2A}{x^3} > 0 \text{ for } x = \sqrt{\frac{A}{2}} \text{ so } x \text{ minimizes } L \end{aligned}$$

**Exercise 4.**  $f = x^2 + y^2 = x^2 + (S - x)^2$   
 $f' = 2x + 2(S - x)(-1) = -2S + 4x \implies x = \frac{S}{2}$   
 $f'' = 4 > 0$  so  $x = \frac{S}{2}$  minimizes  $f$

**Exercise 5.**  $x^2 + y^2 = R > 0$

$f = x + y$

$$\begin{aligned} f' &= 1 + y' = 0 = 1 + \frac{-x}{y} = 0 \implies \boxed{y = x} \\ f'' &= y'' = \frac{-1 - y'^2}{y} \text{ for } y > 0, f'' < 0 \text{ so that } f \text{ is max. when } y = x \\ 2x + 2yy' &= 0 \\ \text{Note that } y' &= \frac{-x}{y} \quad 1 + y'^2 + yy'' = 0 \implies yy'' = -1 - y'^2 \\ x + yy' &= 0 \end{aligned}$$

##### Exercise 6.

$$\begin{aligned} l^2 &= (L - x)^2 + x^2 = L^2 - 2Lx + 2x^2 = A \\ \frac{dA}{dx} &= -2L + 4x = 0 \implies x = \frac{L}{2} \\ \frac{d^2A}{dx^2} &= 4 > 0 \implies A \text{ minimized} \\ l\left(x = \frac{L}{2}\right) &= \frac{L\sqrt{2}}{2} \end{aligned}$$

##### Exercise 7.

$$\begin{aligned} (x + \sqrt{L^2 - x^2})^2 &= A \\ A' &= 2(x + \sqrt{L^2 - x^2})\left(1 + \frac{-x}{\sqrt{L^2 - x^2}}\right) = 0 \text{ when } L^2 - x^2 = x^2 \text{ or } x = \frac{L}{\sqrt{2}} \\ \text{so then the side of the circumscribing and area-maximized square is } &\frac{L}{\sqrt{2}} + \sqrt{L^2 - \frac{L^2}{2}} = \frac{2L}{\sqrt{2}} \end{aligned}$$

**Exercise 8.**

$$A = (2x)(2\sqrt{R^2 - x^2}) = 4x\sqrt{R^2 - x^2}$$

$$A' = 4\left(\sqrt{R^2 - x^2} + \frac{-x^2}{\sqrt{R^2 - x^2}}\right) = 4\left(\frac{R^2 - 2x^2}{\sqrt{R^2 - x^2}}\right) \Rightarrow x = \frac{R}{\sqrt{2}}$$

since  $A' \geq 0$  when  $x \leq \frac{R}{\sqrt{2}}$ , so  $A$  is maximized at  $x = \frac{R}{\sqrt{2}}$

$2x = \frac{2R}{\sqrt{2}}$ ;  $2\sqrt{R^2 - x^2} = \frac{2R}{\sqrt{2}}$  so then the rectangle that has maximum size is a square.

**Exercise 9.** Prove that among all rectangles of a given area, the square has the smallest circumscribed circle.

$$A_0 = (2x)(2\sqrt{r^2 - x^2}) = 4x\sqrt{r^2 - x^2} \text{ (fix the area to be } A_0\text{)}$$

$$\left(\frac{A_0}{4x}\right)^2 = r^2 - x^2 \Rightarrow x^4 - x^2 r^2 + \frac{A_0^2}{16} = 0$$

$$\Rightarrow 0 = 2xr^2 + x^2 2r \frac{dr}{dx} - 4x^3$$

$$\frac{dr}{dx} = 0 \text{ (for extrema)} \Rightarrow x = \frac{r}{\sqrt{2}} \text{ and } \sqrt{r^2 - x^2} = \frac{r}{\sqrt{2}}$$

We could argue that we had found a minimum because at the “infinity” boundaries, the circumscribing circle would be infinitely large.

**Exercise 10.** Given a sphere of radius  $R$ , find the radius  $r$  and altitude  $h$  of the right circular cylinder with the largest lateral surface area  $2\pi rh$  that can be inscribed in the sphere.

$$R^2 = \left(\frac{h}{2}\right)^2 + r^2$$

$$A = 2\pi rh = 2\pi r\sqrt{4(R^2 - r^2)} = 4\pi r\sqrt{R^2 - r^2}$$

$$\frac{dA}{dr} = 4\pi \left(\sqrt{R^2 - r^2} + \frac{-r^2}{\sqrt{R^2 - r^2}}\right) = 4\pi \left(\frac{R^2 - 2r^2}{\sqrt{R^2 - r^2}}\right) \Rightarrow \boxed{r = \frac{R}{\sqrt{2}}}$$

$$\Rightarrow \boxed{h = \sqrt{2}R}$$

**Exercise 11.** Among all right circular cylinders of given lateral surface area, prove that the smallest circumscribed sphere has radius  $\sqrt{2}$  times that of the cylinder.

$$A_0 = 2\pi RH \quad (A_0 \text{ is the total lateral area of the cylinder})$$

$$r^2 = R^2 + \left(\frac{H}{2}\right)^2 = R^2 + \left(\frac{A_0}{4\pi R}\right)^2 = R^2 + \frac{A_0^2}{16\pi^2 R^2}$$

$$2r \frac{dr}{dR} = 2R + \frac{A_0^2}{16\pi^2} \left(\frac{-2}{R^3}\right) \Rightarrow \frac{dr}{dR} = 0 \Rightarrow R = \frac{\sqrt{A_0}}{2\sqrt{\pi}} \Rightarrow \frac{H}{2} = R$$

$$r^2 = R^2 + R^2 = 2R^2 \Rightarrow \boxed{r = \sqrt{2}R}$$

**Exercise 12.** Given a right circular cone with radius  $R$  and altitude  $H$ . Find the radius and altitude of the right circular cylinder of largest lateral surface area that can be inscribed in the cone.

$\frac{h}{R-r} = \frac{H}{R} = \alpha$  is the constraint (look, directly at the side, at the similar triangles formed)

$$A = 2\pi rh = 2\pi r\alpha(R - r) = 2\pi\alpha(rR - r^2)$$

$$\frac{dA}{dr} = 2\pi\alpha(R - 2r) = 0 \Rightarrow \boxed{r = \frac{R}{2}}; \quad \boxed{h = \frac{H}{2}}$$

$$A'' = 2\pi\alpha(R - 2r)$$

since  $\frac{dA}{dr} \geq 0$  when  $r \leq \frac{R}{2}$ ,  $r = \frac{R}{2}$  maximizes lateral surface area

**Exercise 13.** Find the dimensions of the right circular cylinder of maximum volume that can be inscribed in a right circular cone of radius  $R$  and altitude  $H$ .

Constraint:  $\frac{h}{R-r} = \frac{H}{R} = \alpha$

$$V = \pi r^2 h = \pi r^2 \alpha (R - r) = \pi r^2 \alpha (R - r) = \pi \alpha (Rr^2 - r^3)$$

$$\frac{dV}{dr} = \pi \alpha (2Rr - 3r^2) = r\pi \alpha (2R - 3r) \quad \boxed{r = \frac{2R}{3}}$$

since  $\frac{dV}{dr} \geq 0$  when  $r \leq \frac{2R}{3}$ ,  $r = \frac{2R}{3}$  maximizes volume

$$\boxed{h = \frac{1}{3}H}$$

**Exercise 14.** Given a sphere of radius  $R$ . Compute, in terms of  $R$ , the radius  $r$  and the altitude  $h$  of the right circular cone of maximum volume that can be inscribed in this sphere.

$$\begin{aligned} V &= \frac{\pi r^2}{3} (R + \sqrt{R^2 - r^2}) \\ \frac{dV}{dr} &= \frac{\pi}{3} \left( 2rR + 2r\sqrt{R^2 - r^2} + \frac{r^2(-r)}{\sqrt{R^2 - r^2}} \right) = \frac{\pi}{3} r \frac{(2R\sqrt{R^2 - r^2} + 2R^2 - 3r^2)}{\sqrt{R^2 - r^2}} = 0 \\ \implies &\boxed{r = \frac{2\sqrt{2}R}{3}; \quad h = \frac{4R}{3}} \end{aligned}$$

Considering the geometric or physical constraints, since  $\lim_{V \rightarrow \infty} V = \lim_{h \rightarrow \infty} V = 0$ , so then  $r = \frac{2\sqrt{2}R}{3}$  must maximize  $V$ .

**Exercise 15.** Find the rectangle of largest area that can be inscribed in a semicircle, the lower base being on the diameter.

$$\begin{aligned} A &= \sqrt{R^2 - x^2}x \\ A' &= \sqrt{R^2 - x^2} + \frac{-x^2}{\sqrt{R^2 - x^2}} = 0 \implies \boxed{x = \frac{R}{\sqrt{2}}; \quad h = \frac{R}{\sqrt{2}}} \end{aligned}$$

**Exercise 16.** Find the trapezoid of largest area that can be inscribed in a semicircle, the lower base being on the diameter.

$$\begin{aligned} A &= \frac{1}{2}h(2\sqrt{R^2 - h^2} + 2R) \\ \frac{dA}{dh} &= \sqrt{R^2 - h^2} + R + h \left( \frac{-h}{\sqrt{R^2 - h^2}} \right) \\ \frac{dA}{dh} &= 0 \implies \boxed{h = \frac{\sqrt{3}R}{2}} \\ \implies A &= \frac{5\sqrt{3}R^2}{8} \quad \sqrt{R^2 - h^2} = 2\sqrt{R^2 - \frac{3}{4}R^2} = 2\frac{R}{2} = \boxed{R} \end{aligned}$$

**Exercise 17.** An open box is made from a rectangular piece of material by removing equal squares at each corner and turning up the sides. Find the dimensions of the box of largest volume that can be made in this manner if the material has sides (a) 10 and 10; (b) 12 and 18

(1)

$$(x - 2r)(Y - 2r)r = (xy - 2rx - 2ry + 4r^2)r = xy - 2r^2x - 2r^2y + 4r^3 = V$$

$$\frac{dV}{dr} = xy - 4rx - 4ry + 12r^2 = 0$$

$$\implies r = \frac{4(x + y) \pm \sqrt{16(x + y)^2 - 4(12)xy}}{2(12)} = \frac{(x + y) \pm \sqrt{x^2 + y^2 - xy}}{6}$$

$$\frac{d^2V}{dr^2} = -4x - 4y + 24r = -4(x + y) + 24r$$

We can plug in our expression for  $r$  into the second derivative of  $V$ , the volume of the box, to find out that we want to pick the “negative” root from  $r$ , in order to maximize the box volume.

Then for  $x = 10$ ;  $y = 10$ , we have  $r = \frac{5}{3}$ , so that the box dimensions are  $\frac{5}{3} \times \frac{20}{3} \times \frac{20}{3}$ .

(2) 12 and 18

$$\implies 5 - \sqrt{7} \times 2 + 2\sqrt{7} \times 8 + 2\sqrt{7}$$

**Exercise 18.** If  $a$  and  $b$  are the legs of a right triangle whose hypotenuse is 1, find the largest value of  $2a + b$ .

$$\begin{aligned} L &= 2a + b = 2a + \sqrt{1 - a^2} & L' = 0 &\implies \left(\frac{a}{2}\right)^2 = 1 - a^2 \implies a = \frac{2}{\sqrt{5}} \\ L' &= 2 + \frac{-a}{\sqrt{1 - a^2}} \\ L'' &= (-1) \left( \frac{\sqrt{1 - a^2} - \frac{-a}{\sqrt{1 - a^2}} a}{1 - a^2} \right) = (-1) \left( \frac{1}{(1 - a^2)^{3/2}} \right) < 0 \text{ (so } a = \frac{2}{\sqrt{5}} \text{ maximizes } L) \end{aligned}$$

**Exercise 19.**  $2 + \frac{x^2}{600}$  gallons per hour.  $l_0 = 300$  mi  $x$  = constant speed.  $\frac{l_0}{x}$  = time spent.  $K$  = gas cost = 0.30.  $C$  = gas cost + driver labor cost =  $l_0 \left( \frac{2K}{x} + \frac{Kx}{600} + \frac{D}{x} \right)$

$$\begin{aligned} \frac{dC}{dx} &= l_0 \left( \frac{-2K}{x^2} + \frac{K}{600} - \frac{D}{x^2} \right) = 0 \xrightarrow{\frac{dC}{dx}=0} x = \sqrt{\frac{2K + D}{K}} 10\sqrt{6} \\ \frac{d^2C}{dx^2} &= l_0 \left( (-2K - D) \left( \frac{-2}{x^3} \right) \right) = l_0 \left( \frac{2(2K + D)}{x^3} \right) > 0 \\ \text{Thus, } C &\text{ is minimized if } x = \sqrt{\frac{2K + D}{K}} 10\sqrt{6} \\ \implies C_{min} &= (300) \left( \frac{\sqrt{2K + D} \sqrt{K}}{10\sqrt{6}} + \frac{\sqrt{K} \sqrt{2K + D} 10\sqrt{6}}{600} \right) = \boxed{3\sqrt{2}\sqrt{6 + 10D}} \end{aligned}$$

Remember that there is a **speed limit** of 60 mi/hr.

- (1)  $D = 0, \quad x = 20\sqrt{3} \quad C = 6\sqrt{3} \approx 10.39$
- (2)  $D = 1, \quad x = 40\sqrt{2} \quad C = 12\sqrt{2} \approx 16.97$
- (3)  $D = 2, \quad x = 60$  (because of the speed limit)  $C = 300 \left( \frac{2K}{60} + \frac{K(60)}{600} + \frac{D}{60} \right) = 5(2.4 + D) = 22.00$
- (4)  $D = 3, \quad x = 60 \quad C = 27.00$
- (5)  $D = 4, \quad x = 60 \quad C = 32.00$

**Exercise 20.**  $y = \frac{x}{x^2 + 1}$  Suppose the rectangle starts at  $x_0$  on the  $x$  axis. Then its  $y$  coordinate intersecting the curve, and thus the height of rectangle, must be  $y_0 = \frac{x_0}{x_0^2 + 1}$

$$\begin{aligned} \implies x_0 &= \frac{1}{2y_0} \pm \sqrt{\frac{1}{(2y_0)^2} - 1} \\ x_2 - x_1 &= \sqrt{\frac{1}{y_0^2} - 4} \end{aligned}$$

where  $x_2 - x_1$  is going to be the base of the rectangle. The volume of the cylinder,  $V$ , which is obtained from revolving the rectangle about the  $x$  axis, is going to be

$$\begin{aligned} V &= \pi y_0^2 (x_2 - x_1) = \pi y_0^2 \left( \sqrt{\frac{1}{y_0^2} - 4} \right) = \pi y_0 \sqrt{1 - 4y_0^2} \\ \frac{dV}{dy_0} &= \pi \left( \sqrt{1 - 4y_0^2} + \frac{y_0}{2\sqrt{1 - 4y_0^2}} (-8y_0) \right) = \pi \left( \frac{1 - 8y_0^2}{\sqrt{1 - 4y_0^2}} \right) \implies y_0 = \frac{1}{2\sqrt{2}} \end{aligned}$$

We could argue that  $V$  is maximized, since the “infinite” boundaries would yield a volume of 0 (imagine stretching and squeezing the rectangle inside the curve).

$$\text{Then } V_{max} = \pi \frac{1}{8} 2 = \boxed{\frac{\pi}{4}}$$

**Exercise 21.** Draw a **good diagram**. Note how the right triangle that you folded is now *reflected backwards*, so that this triangle’s right angle is on the left-hand side now.

The constraint is that the crease touches the left edge.

$$\begin{aligned} w_0 &= l \sin \alpha + l \sin \alpha \cos(2\alpha) = l \sin \alpha (1 + \cos(2\alpha)) = \\ &= 2l \sin \alpha \cos^2 \alpha \end{aligned}$$

Note that we will obtain a minimum crease because by considering the “physical infinite” boundary, we could make a big crease along the vertical half of the paper or the horizontal half of the paper.

So, isolating  $l$ , the length of the crease, and then taking the derivative,

$$l = \frac{w_0}{2 \sin(\alpha) \cos^2 \alpha} = \frac{w_0}{2} \csc(\alpha) \sec^2(\alpha)$$

$$\frac{dl}{d\alpha} = \frac{w_0}{2} (-\cot \alpha \csc \alpha \sec^2 \alpha + \csc \alpha 2 \sec \alpha \sec \alpha \tan \alpha) =$$

$$= \frac{w_0}{2} \left( \frac{-C}{S} \left( \frac{1}{S} \right) \left( \frac{1}{C^2} \right) + \frac{1}{S^2} \left( \frac{1}{C^2} \right) \left( \frac{S}{C} \right) \right) = \frac{w_0}{2} \left( \frac{-1}{S^2 C} + \frac{2}{C^3} \right)$$

$$\xrightarrow{\frac{dl}{d\alpha}=0} \boxed{\sin \alpha = \frac{1}{\sqrt{3}}} \text{ or } \tan \alpha = \frac{1}{\sqrt{2}}$$

where  $\alpha$  is the angle of the crease. The corresponding minimum length of the crease will be

$$l = \frac{w_0}{2} \frac{1}{\frac{1}{\sqrt{3}}^2} = \boxed{\frac{9\sqrt{3}}{2}}$$

### Exercise 22.

- (1) Consider the center of the circle  $O$ , the apex of the isosceles triangle that makes an angle  $2\alpha$ ,  $A$ , and one of its other vertices,  $B$ . Draw a line segment from  $O$  to  $B$  and simply consider the two triangles making up one half of the isosceles triangle. Find all the angles.

Angle  $AOB$  is  $\pi - 2\alpha$  by the geometry or i.e. inspection of the figure. The complement of that angle is  $2\alpha$ . Beforehand, we can get the length of the isosceles triangle leg from the law of cosines.

$$\cos(\pi - 2\alpha) = -\cos(2\alpha)$$

$$s^2 = R^2 + R^2 - 2R^2 \cos(\pi - 2\alpha) + 2R^2(1 + \cos(2\alpha)) = 2R^2(2\cos^2 \alpha) = 4R^2 \cos^2 \alpha$$

$$s = 2R \cos \alpha$$

**The constraint equation** is

$$(8) \quad P = 4R \cos \alpha + 2R \sin(2\alpha)$$

So then

$$P' = 4R(-\sin \alpha) + 4R \cos(2\alpha) = 0 \implies \cos 2\alpha = \sin \alpha$$

$$\sin \alpha = \frac{-\frac{1}{2} \pm \sqrt{\frac{1}{4} - 4(1)(-\frac{1}{2})}}{2(1)} = \frac{1}{2} > 0$$

$$\implies \boxed{P = 3\sqrt{3}R}$$

$P = 3\sqrt{3}$  is a max because

Look at the “boundary conditions” imposed on  $P$  by the physical-geometry.  $\alpha = 0$ , triangle is completely flattened,  $\alpha = \pi$ , triangle “completely disappears.”

- (2) I had originally thought to Reuse the constraint equation, Eqn. (8). *This is wrong!*

Think about the problem directly and for what it actually is; less wishful thinking.

Consider a fixed perimeter  $L$  and imagine  $L$  to be a string that can be stretched into an isosceles triangle. A “trivial” isosceles triangle is a collapsed triangle with two sides of length  $L/2$  only. Then the radius of the disk needs to be  $L/4$ .

Consider a general isosceles triangle with  $2\alpha$  as the vertex angle and isosceles sides of  $h$ . The perimeter for this triangle,  $P$ , is then

$$P = 2h + 2h \sin \alpha = 2h(1 + \sin \alpha)$$

$$\implies h = \frac{P}{2(1 + \sin \alpha)}$$

$$\frac{h}{2} = R \cos \alpha$$

We could try to extremize this equation.

$$\begin{aligned}\frac{dR}{d\alpha}(4\cos\alpha + 2\sin(2\alpha)) + R(-4\sin\alpha + 4\cos(2\alpha)) &= 0 \\ \frac{dR}{d\alpha} = 0 \implies \cos(2\alpha) = \sin\alpha \implies \sin\alpha = \frac{1}{\sqrt{3}} \quad \cos\alpha = \frac{\sqrt{2}}{\sqrt{3}} \\ R &= \frac{3P}{4\sqrt{2}(\sqrt{3}+1)}\end{aligned}$$

However, this is the *minimized*  $R$ , *minimized* radius for the smallest circle fitting a particular isosceles triangle of a fixed perimeter. We want smallest circle with a radius big enough to fit all the possible triangles. Thus  $R = \frac{L}{4}$

**Exercise 23.** The constraint equation on perimeter is

$$P = 2h + W + \pi\left(\frac{W}{2}\right) = 2h + W\left(1 + \frac{\pi}{2}\right)$$

Then intensity function, “normalized” is given by

$$I = Wh + \frac{\pi}{2}\left(\frac{W}{2}\right)^2\left(\frac{1}{2}\right) = \frac{Wp}{2} - \frac{W^2}{2} - \frac{3\pi}{16}W^2$$

So then

$$\frac{dI}{dW} = \frac{P}{2} - W - \frac{3\pi}{8}W = 0 \implies W = \boxed{\frac{P}{2 + \frac{3\pi}{4}}}$$

The height of the rectangle is

$$h = \frac{P - \frac{P}{2 + \frac{3\pi}{4}}\left(1 + \frac{\pi}{2}\right)}{2} = P\left(\frac{4 + \pi}{16 + 6\pi}\right)$$

**Exercise 24.** A log 12 feet long has the shape of a frustum of a right circular cone with diameters 4 feet and  $(4 + h)$  feet at its ends, where  $h \geq 0$ . Determine, as a function of  $h$ , the volume of the largest right circular cylinder that can be cut from the log, if its axis coincides with that of the log.

Remember to **label your diagram carefully**.

$$\begin{aligned}\frac{y}{x} &= \frac{\frac{h}{2} - y}{l_0 - x} = \frac{h/2}{l_0} \implies \frac{h/2 - \left(\frac{xh}{2l_0}\right)}{l_0 - x} \\ V &= \pi(H + y)^2(l_0 - x) = \pi\left(H + \frac{xh}{2l_0}\right)^2(l_0 - x)\end{aligned}$$

Note that  $V(x = 0) = \pi H^2 l_0 = \pi 4(12)$

$$\begin{aligned}\frac{dV}{dx} &= \pi\left(2\left(H + \frac{hx}{2l_0}\right)\frac{h}{2l_0}(l_0 - x) + \left(H + \frac{xh}{2l_0}\right)^2(-1)\right) = \xrightarrow{\frac{dV}{dx}=0} x = \frac{(h - H)2l_0}{3h} \\ &= \pi\left(H + \frac{hx}{2l_0}\right)\left(h - H - \frac{3xh}{2l_0}\right)\end{aligned}$$

$$V\left(x = \frac{(h - H)2l_0}{3h}\right) = \pi\left(l_0 - \frac{(h - H)2l_0}{3h}\right)\left(H + \frac{h}{2l_0}\frac{(h - H)2l_0}{3h}\right)^2 = \boxed{\pi l_0 \frac{(h + 2H)^3}{27h}}$$

where  $H = 2, l_0 = 12$

**Exercise 25.**

$$\begin{aligned}S &= \sum_{k=1}^n (x - a_k)^2 \implies \frac{dS}{dx} = \sum_{k=1}^n 2(x - a_k) = 0 \\ \implies nx &= \sum_{k=1}^n a_k \implies x = \frac{\sum_{k=1}^n a_k}{n}\end{aligned}$$

Since  $\lim_{x \rightarrow \pm\infty} S = +\infty$ ,  $x = \frac{\sum_{k=1}^n a_k}{n}$  minimizes  $S$ .

**Exercise 26. Hint: draw a picture.** Then observe that for  $f(x) \geq 24$ ,  $A$  must be greater than 0 (we’ll show that explicitly soon) and that  $f$  must have a *minimum* somewhere.

If  $A < 0$ , then consider  $f(x) = \frac{5x^7 + A}{x^5}$ . Consider  $x = \frac{-A^{1/7}}{6^{1/7}} > 0$ .

$$f(x) = \frac{-A^{1/7}}{6^{1/7}} = \frac{A}{6x^5} < 0$$

Thus,  $A > 0$ .

$$\frac{df}{dx} = 10x - 5Ax^{-6} = 5 \left( \frac{2x^7 - A}{x^6} \right) = 0$$

$$x = \left( \frac{A}{2} \right)^{1/7}$$

$$\frac{d^2f}{dx^2} = 10 + 30Ax^{-7} = 10 + 3 - A \left( \frac{2}{A} \right) = 70 > 0$$

Thus  $x = \left( \frac{A}{2} \right)^{1/7}$  minimizes  $f$  for  $A > 0$ .

$$f(x) = \left( \frac{A}{2} \right)^{1/7} = \frac{5 \left( \frac{A}{2} \right) + A}{\left( \frac{A}{2} \right)^{5/7}} = 24$$

$$\implies \boxed{A = 2 \left( \frac{24}{7} \right)^{7/2}}$$

**Exercise 27.** Consider  $f(x) = -\frac{x^3}{3} + t^2x$  over  $0 \leq x \leq 1$ .

$$f(0) = 0, f(1) = -\frac{1}{3} + t^2 \implies f(1) \geq 0 \text{ if } t^2 \geq \frac{1}{3}$$

$$f'(x) = -x^2 + t^2 = 0$$

$$\implies x^2 = t^2 \text{ but } x \geq 0, \text{ so } x = |t|$$

$$f(x^2 = t^2) = -\frac{1}{3}t^2(x) + t^2x = \frac{2}{3}t^2x > 0 \text{ for } 1 \geq x \geq 0$$

So the minimum isn't in the interior of  $[0, 1]$ . It's on the end points.

$$\boxed{m(t) = 0 \text{ for } |t| > \frac{1}{3} \quad m(t) = -\frac{1}{3} + t^2 \text{ for } |t| < \frac{1}{3}}$$

**Exercise 28.**

(1)

$$E(x, t) = \frac{|t - x|}{x}$$

$$M(t) = \max \frac{|t - x|}{x} \text{ as } x = a \rightarrow x = b$$

$$\frac{|t - x|}{x} = \begin{cases} \frac{t-x}{x} & \text{if } t \geq x \\ \frac{x-t}{x} & \text{if } t < x \end{cases}$$

$$\frac{d}{dt} \left( \frac{|t - x|}{x} \right) = \begin{cases} -\frac{t}{x^2} & \text{if } t \geq x \\ \frac{t}{x^2} & \text{if } t < x \end{cases}$$

Now  $t, x \geq a > 0$  (this is an important, given, fact). So  $x = t$  should be a relative minimum.

So the maximum occurs at either endpoints

$$\frac{t-a}{a} = E(a, t), \quad \frac{b-t}{b} = E(b, t)$$

By monotonicity on  $[a, t]$ ,  $(t, b]$ , and having shown the relative minimum of  $\frac{|t-x|}{x}$  at  $x = t$ , the maximum occurs at  $x = a$  or  $x = b$ , depending upon the relationship  $E(a, t) \geq E(b, t)$ .

(2)

$$M(t) = \begin{cases} \frac{t-a}{a} & \text{if } \frac{t-a}{a} > \frac{b-t}{b} \text{ i.e. } t \left( \frac{b+a}{ab} \right) > 2 \\ \frac{b-t}{b} & \text{if } \frac{b-t}{b} > \frac{t-a}{a} \end{cases} \quad \frac{dM}{dt} = \begin{cases} \frac{1}{a} & \text{if } t > 2 \frac{ab}{a+b} \\ -\frac{1}{b} & \text{if } t < \left( \frac{ab}{a+b} \right) 2 \end{cases}$$

Since  $\frac{dM}{dt} \geq 0$  when  $t \geq \frac{2ab}{a+b}$ ,  $M$  is minimized for  $\boxed{t = \frac{2ab}{a+b}}$



## 4.23 Exercises - Partial Derivatives.

**Exercise 8.**  $f(x, y) = \frac{x}{\sqrt{x^2 + y^2}}$ .

$$\begin{aligned} f_x &= \frac{1}{\sqrt{x^2 + y^2}} + \frac{-x^2}{(x^2 + y^2)^{3/2}} = \frac{y^2}{(x^2 + y^2)^{3/2}} & f_{xx} &= \frac{-3y^2x}{(x^2 + y^2)^{5/2}} \\ f_y &= \frac{-xy}{(x^2 + y^2)^{3/2}} & f_{yy} &= (-x) \left( \frac{x^2 - 2y^2}{(x^2 + y^2)^{5/2}} \right) \\ f_{xy} &= (-y) \left( \frac{(x^2 + y^2)^{3/2} - x \frac{3}{2}(x^2 + y^2)^{1/2}(2x)}{(x^2 + y^2)^3} \right) = \\ &= (-y) \left( \frac{-2x^2 + y^2}{(x^2 + y^2)^{5/2}} \right) & f_{yx} &= \frac{2y}{(x^2 + y^2)^{3/2}} + \frac{-3y^2y}{(x^2 + y^2)^{5/2}} = \frac{(-y)(-2x^2 + y^2)}{(x^2 + y^2)^{5/2}} \end{aligned}$$

**Exercise 9.**

(1)

$$\begin{aligned} z &= (x - 2y)^2 \\ z_x &= 2(x - 2y) = 2\sqrt{z} & x(2z) - 4zy &= (x - 2y)2\sqrt{z} = 2z \\ z_y &= 2(x - 2y)(-2) = -4\sqrt{z} \end{aligned}$$

(2)

$$\begin{aligned} z &= (x^4 + y^4)^{1/2} \\ z_x &= \frac{1}{z} 2x^3 & x(2z) - 4zy &= (x - 2y)2\sqrt{z} = 2z \\ z_y &= \frac{2y^3}{z} \end{aligned}$$

**Exercise 10.**

$$f = \frac{xy}{(x^2 + y^2)^2}, \quad f_x = \frac{y}{(x^2 + y^2)^2} + \frac{-4x^2y}{(x^2 + y^2)^3} = \frac{y^3 - 3x^2y}{(x^2 + y^2)^3}$$

So

$$\begin{aligned} f_{xx} &= \frac{-6xy(x^2 + y^2)^3 - 3(x^2 + y^2)^2(2x)(y^3 - 3x^2y)}{(x^2 + y^2)^6} = \\ &= \frac{12xy(x^2 - y^2)}{(x^2 + y^2)^4} \end{aligned}$$

By label symmetry,

$$f_{xx} + f_{yy} = \frac{12xy(x^2 - y^2)}{(x^2 + y^2)^4} + \frac{12yx(y^2 - x^2)}{(x^2 + y^2)^4} = 0$$

**5.5 Exercises - The derivative of an indefinite integral. The first fundamental theorem of calculus, The zero-derivative theorem, Primitive functions and the second fundamental theorem of calculus, Properties of a function deduced from properties of its derivative.**

Review the fundamental theorems of calculus, Thm. 5.1 and Thm. 5.3. Note the **differences** between the two.

**Theorem 17** (First fundamental theorem of calculus).

Let  $f$  be integrable on  $[a, x]$   $\forall x \in [a, b]$

Let  $c \in [a, b]$  and

$$(9) \quad A(x) = \int_c^x f(t)dt \quad \text{if } a \leq x \leq b$$

Then  $\exists A'(x) \quad \forall x \in (a, b)$  where  $f$  is continuous at  $x$  and

$$(10) \quad A'(x) = f(x)$$

**Theorem 18** (Second fundamental theorem of calculus).

Assume  $f$  continuous on open interval  $I$

Let  $P$  be any primitive of  $f$  on  $I$ , i.e.  $P' = f \quad \forall x \in I$

Then  $\forall c, x \in I$

$$(11) \quad P(x) = P(c) + \int_c^x f(t)dt$$

**Exercise 6.**  $\sqrt{2}\frac{2}{3}x^{3/2} + \sqrt{\frac{1}{2}}\frac{2}{3}x^{3/2} = \sqrt{2}x^{3/2}$

$$\int_a^b f = \sqrt{2}(b^{3/2} - a^{3/2})$$

**Exercise 7.**  $f = x^{3/2} - 3x^{1/2} + \frac{7}{2}x^{-1/2};$

$$P = \frac{2}{5}x^{5/2} - 2x^{3/2} + 7x^{1/2}$$

$$\int_a^b f = \frac{2}{5}(b^{5/2} - a^{5/2}) - 2(b^{3/2} - a^{3/2}) + 7(b^{1/2} - a^{1/2})$$

**Exercise 8.**  $P = \frac{3}{2}x^{4/3} - \frac{3}{2}x^{2/3}; \quad x > 0$

**Exercise 9.**  $P = -3 \cos x + \frac{x^6}{3}$

**Exercise 10.**  $P = \frac{3}{7}x^{7/3} - 5 \sin x$

**Exercise 11.**  $f'(x) = \frac{1}{x}$

$$f = \sum_{k=-\infty}^{\infty} a_k x^k$$

$$f' = \sum_{k=-\infty}^{\infty} k a_k x^{k-1} = \frac{1}{x}$$

Comparing terms, only  $k = 0$  would work, but the coefficient is unequivocally 0

**Exercise 12.**

$$\int_0^x |t| dt = \begin{cases} \int_0^x t dt & \text{if } x \geq 0 \\ \int_0^x -t dt & \text{if } x < 0 \end{cases} = \begin{cases} \frac{1}{2}x^2 & \text{if } x \geq 0 \\ -\frac{1}{2}x^2 & \text{if } x < 0 \end{cases} = \frac{1}{2}x|x|$$

**Exercise 13.**

$$\int_0^x (t + |t|)^2 dt = \begin{cases} \int_0^x (2t)^2 dt & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases} = \begin{cases} \frac{4}{3}x^3 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases} = \frac{2x^2}{3}(x + |x|)$$

**Exercise 14.** Using 1st. fund. thm. of calc.

$$\int_0^x f(t) dt = A(x) - A(0)$$

$$A'(x) = f(x)$$

$$\implies 2x + \sin 2x + 2x \cos 2x + -\sin 2x$$

$$f' = 2 + 4 \cos 2x + -4x \sin 2x - 2 \cos 2x$$

$$= 2 + 2 \cos 2x - 4x \sin 2x$$

$f\left(\frac{\pi}{4}\right) = \frac{\pi}{2}$ $f'\left(\frac{\pi}{4}\right) = 2 - \pi$
--

**Exercise 15.**  $\int_c^x f(t) dt = \cos x - \frac{1}{2} \quad f(x) = -\sin x \quad c = -\frac{\pi}{6}.$

**Exercise 16.** Suppose  $f(x) = \sin x - 1$  and  $c = 0$ .

$$\int_0^x t \sin t - t = \left( -t \cos t + \sin t - \frac{1}{2}t^2 \right) \Big|_0^x = \sin x - x \cos x - \frac{1}{2}x^2$$

So  $c = 0$ .

**Exercise 17.** For  $f(x) = -x^2 f(x) + 2x^{15} + 2x^{17}$  (found by taking the derivative of  $\int_0^x f = \int_x^1 t^2 f + \frac{x^{16}}{8} + \frac{x^{18}}{9} + C$ .) Suppose that  $f = 2x^{15}$ .

$$\implies \frac{x^{16}}{8} = -\frac{x^{18}}{9} + -\frac{1}{9} + \frac{x^{16}}{8} + \frac{x^{18}}{9} + C$$

$$\implies C = \frac{1}{9}$$

**Exercise 18.** By plugging in  $x = 0$  into the defined  $f(x)$ ,  $f(x) = 3 + \int_0^x \frac{1+\sin t}{2+t^2} dt$ , we get for  $p(x) = a + bx + cx^2$ ,

$$a = 3$$

Continuing on,

$$f' = \frac{1 + \sin x}{2 + x^2}; \quad f'' = \frac{(\cos x)(2 + x^2) - 2x(1 + \sin x)}{(2 + x^2)^2}$$

$$f'(0) = \frac{1}{2} = b \quad f''(0) = \frac{1}{2} + 2c; c = \frac{1}{4}$$

**Exercise 19.**

$$\begin{aligned}
f(x) &= \frac{1}{2} \int_0^x (x-t)^2 g(t) dt = \frac{1}{2} \int_0^x (x^2 - 2xt + t^2) g(t) dt = \\
&= \frac{1}{2} \left( x^2 \int_0^x g - 2x \int_0^x t g + \int_0^x t^2 g \right) \\
f' &= x \int_0^x g + \frac{x^2}{2} g(x) - \int_0^x t g - x(xg(x)) + \frac{1}{2} x^2 g(x) = \\
&= x \int_0^x g - \int_0^x t g \\
f'' &= \int_0^x g + xg - xg = \int_0^x g \quad \boxed{f''(1) = 2} \\
f''' &= g \quad \boxed{f'''(1) = 5}
\end{aligned}$$

**Exercise 20.**

$$\begin{aligned}
(1) \quad & \left( \int_0^x (1+t^2)^{-3} dt \right)' = (1+x^2)^{-3} \\
(2) \quad & \left( \int_0^{x^2} (1+t^2)^{-3} dt \right)' = (1+x^4)^{-3} (2x) = \frac{2x}{(1+x^4)^3} \\
(3) \quad & \left( \int_{x^3}^{x^2} (1+t^2)^{-3} dt \right)' = (1+x^4)^{-3} (2x) - (1+x^6)^{-3} (3x^2) = \frac{2x}{(1+x^4)^3} - \frac{3x^2}{(1+x^6)^3}
\end{aligned}$$

**Exercise 21.**

$$f'(x) = \left( \int_{x^3}^{x^2} \frac{t^6}{1+t^4} dt \right)' = \left( \frac{x^{12}}{1+x^8} \right) (2x) - \left( \frac{x^{18}}{1+x^{12}} \right) 3x^2$$

**Exercise 22.**

$$\begin{aligned}
(1) \quad & f(x) = 2x(1+x) + x^2 = 2x + 3x^2 \\
& f(2) = 16 \\
(2) \quad & \frac{d}{dx} \left( \int_{a(x)}^{b(x)} f(t) dt \right) = f(b)b' - f(a)a' \\
& 2x + 3x^2 = f(x^2)(2x) \\
& f(x^2) = \left( 1 + \frac{3x}{2} \right) \\
& f(2) = 1 + \frac{3\sqrt{2}}{2} \\
(3) \quad & \int_0^{f(x)} t^2 dt = x^2(1+x) \\
& 2x + 3x^2 = (f(x))^2 f'(x) \\
& \implies f^3(2) = 3(4)(3) = 9(4) = 36 \\
& f(2) = 36^{1/3} \\
(4) \quad & \frac{d}{dx} \left( \int_0^{x^2(1+x)} f(t) dt \right) = 1 = f(x^2(1+x))(2x(1+x) + x^2) \\
& x = 1 \quad (f(2))(5) = 1 \implies f(2) = \frac{1}{5}
\end{aligned}$$

**Exercise 23.**

$$\begin{aligned}
& a^3 - 2a \cos a + (2 - a^2) \sin a = \int_0^a f^2(t) dt \\
& 3x^2 - 2 \cos x + 2x \sin x + -2x \sin x + (2 - x^2) \cos x = 3x^2 - x^2 \cos x = f^2(x) \\
& f(x) = x\sqrt{3 - \cos x} \\
& \boxed{f(a) = a\sqrt{3 - \cos a}}
\end{aligned}$$

**Exercise 24.**  $f(t) = \frac{t^2}{2} + 2t \sin t$ 

$$\begin{aligned}
(1) \quad & f' = 2t + 2 \sin t + 2t \cos t \quad f'(\pi) = 2\pi - 2\pi = 0
\end{aligned}$$

(2)

$$f'' = 2 + 2 \cos t + 2 \cos t - 2t \sin t = 2 + 4 \cos t - 2t \sin t$$

$$f''\left(\frac{\pi}{2}\right) = 2 - \pi$$

$$(3) \quad f''\left(\frac{3\pi}{2}\right) = 0$$

$$(4) \quad f\left(\frac{5\pi}{2}\right) = \frac{25\pi^2}{8} + 5\pi$$

$$(5) \quad f(\pi) = \frac{\pi^2}{2}$$

**Exercise 25.**

(1)

$$\frac{df}{dt} = \frac{1 + 2 \sin \pi t \cos \pi t}{1 + t^2} = v(t)$$

$$a(t) = \frac{2\pi(\cos(2\pi t))(1 + t^2) - 2t(\sin(2\pi t))}{(1 + t^2)^2}$$

$$a(t=2) = a(t=1) = \frac{4\pi}{4} = \pi$$

$$(2) \quad v(t=1) = \frac{1}{2}$$

$$(3) \quad v(t) = \pi(t-1) + \frac{1}{2}; \quad t > 1$$

(4)

$$f(t) - f(1) = \int_1^t v(t) dt = \int_1^t \pi(t-1) + \frac{1}{2} = \left( \frac{\pi t^2}{2} - \pi t + \frac{1}{2}t \right) \Big|_1^t = \frac{\pi t^2}{2} - \pi t + \frac{t}{2} + \frac{\pi}{2} - \frac{1}{2}$$

**Exercise 26.**

(1)

$$f''(x) > 0 \forall x \quad f'(0) = 1; f'(1) = 0$$

$$\int_0^1 f''(t) dt = f'(1) - f'(0) = 0 - 1 < 0$$

Thus, it's impossible, since  $f''(x) > 0$ , so  $\int_0^1 f''(t) dt > 0$

(2)

$$\int_0^1 \left( 3 - \frac{\pi}{2} \sin \frac{\pi x}{2} \right) dx = \left( 3x + \cos \frac{\pi x}{2} \right) \Big|_0^1 = 3 - 1 = 2$$

$$f(x) = \frac{3x^2}{2} + \frac{2}{\pi} \sin \frac{\pi x}{2} + C$$

$$(3) \quad f''(0) > 0 \forall x \quad f'(0) = 1; f(x) \leq 100 \forall x > 0$$

$$\int_a^b f''(t) dt = f'(b) - f'(a); \quad \int_c^k f'(t) dt = f(k) - f(c)$$

$$\int_0^b f'' = f'(b) - f'(0) = f'(b) - 1 \geq 0 \text{ if } b \geq 0$$

$$\int_c^k (f'(b) - 1) db = f(k) - f(c) - (k - c) > 0 \text{ if } k > c > 0$$

$$f(k) - f(c) > k - c$$

$$f(x) \leq 100 \text{ is untrue for all } x > 0$$

$$f(k) - f(0) > k - 0$$

$$f(100) - f(0) > 100$$

$$(4) \quad f''(x) = e^x > 0 \quad f'(x) = e^x; \quad f'(0) = 1 \quad f(x) = e^x \quad \forall x < 0, e^x < 1$$

**Exercise 27.**  $f''(t) \geq 6.$   $b - a = \frac{1}{2}.$   $f'(0) = 0$ 

$$\int_a^b f'' = f'(b) - f'(a) \geq 6(b - a) = 3 \quad \text{since } b - a = \frac{1}{2}$$

$$\int_0^a f'' = f'(a) - f'(0) = f'(a) \geq 6(a - 0) = 6a$$

$$\text{If } a = \frac{1}{2}, f'(1/2) \geq 3$$

Then by intermediate value theorem, with  $f$  being continuous and  $f'(0) = 0$ ,  $f'(1/2) \geq 3$ ,  $f'$  must take on the value of 3 somewhere between 0 and 3. Thus there is an interval  $[a, b]$  of length  $1/2$  where  $f' \geq 3$ .

### 5.8 Exercises - The Leibniz notation for primitives, Integration by substitution.

**Exercise 1.**  $\int \sqrt{2x+1} dx = \frac{1}{3}(2x+1)^{3/2}$ .

**Exercise 2.**  $\int x\sqrt{1+3x} = \frac{2x}{9}(1+3x)^{3/2} - \frac{4}{135}(1+3x)^{5/2}$

**Exercise 3.**

$$\int x^2 \sqrt{x+1} = \frac{2x^2(x+1)^{3/2}}{3} - \frac{8x(x+1)^{5/2}}{15} + \frac{16(x+1)^{7/2}}{105}$$

since

$$\left( \frac{2x^2(x+1)^{3/2}}{3} \right)' = x^2(x+1)^{1/2} + \frac{4x(x+1)^{3/2}}{3}$$

$$\left( \frac{8x(x+1)^{5/2}}{15} \right)' = \frac{4}{3}x(x+1)^{3/2} + \frac{8(x+1)^{5/2}}{15}$$

$$\left( \frac{16(x+1)^{7/2}}{105} \right)' = \frac{8(x+1)^{5/2}}{15}$$

**Exercise 4.**

$$\int \frac{x dx}{\sqrt{2-3x}} = \frac{2x(2-3x)^{1/2}}{-3} + \frac{4(2-3x)^{3/2}}{-27}$$

$$\int_{-2/3}^{1/3} \frac{x dx}{\sqrt{2-3x}} = -2/9 - 4/27 - (8/9 - 32/27) = -2/27$$

**Exercise 5.**

$$\int \frac{(x+1)dx}{((x+1)^2+1)^3} = \frac{((x+1)^2+1)^{-2}}{-4}$$

**Exercise 6.**

$$\int \sin^3 x = \int \sin x(1 - \cos^2 x) = -\cos x + \frac{1}{3} \cos^3 x$$

**Exercise 7.**

$$\int x^{1/3}(1+x) = \frac{3}{4}x^{4/3} + \frac{3}{7}x^{7/3} = \frac{3}{4}(z-1)^{4/3} + \frac{3}{7}(z-1)^{7/3}$$

**Exercise 8.**  $\frac{\sin^{-2} x}{-2}$

**Exercise 9.**  $\frac{(4-\sin 2x)^{3/2}}{-3} \Big|_0^{\pi/4} = \frac{3^{3/2}-8}{-3}$

**Exercise 10.**  $(3 + \cos x)^{-1}$

**Exercise 11.**  $2 \cos^{-1/2} x$

**Exercise 12.**  $2 \cos \sqrt{x+1} \Big|_3^8 = 2(\cos 3 - \cos 2)$

**Exercise 13.**  $-\frac{\cos x^n}{n}$

**Exercise 14.**  $\frac{(1-x^6)^{1/2}}{-3}$

**Exercise 15.**

$$\int t(1+t)^{1/4} dt = \int (x-1)x^{1/4} dx = \frac{4}{9}x^{9/4} - \frac{4}{5}x^{5/4} = \frac{4}{9}(1+t)^{9/4} - \frac{4}{5}(1+t)^{5/4}$$

**Exercise 16.**  $\int (x^2+1)^{-3/2} dx = ?$

$$\left( \frac{x}{\sqrt{x^2+1}} \right)' = \frac{\sqrt{x^2+1} - x^2/\sqrt{x^2+1}}{x^2+1} = \frac{1}{(x^2+1)^{3/2}}$$

**Exercise 17.**

$$(8x^3+27)^{5/3} \left( \frac{3}{5} \right) \left( \frac{1}{24} \right) = \frac{1}{40}(8x^3+27)^{5/3}$$

**Exercise 18.**  $\frac{3}{2}(\sin x - \cos x)^{2/3}$

**Exercise 19.**

$$\begin{aligned}\int \frac{x dx}{\sqrt{1+x^2+(1+x^2)^{3/2}}} &= \int \frac{\frac{1}{2} du}{\sqrt{u+u^{3/2}}} = \\ &= \frac{1}{2} \int \frac{du}{\sqrt{u}\sqrt{1+u^{1/2}}} = 2(1+u^{1/2})^{1/2} = \\ &= 2(1+\sqrt{1+x^2})^{1/2} + C\end{aligned}$$

**Exercise 20.**

$$\int \frac{(x^2-2x+1)^{1/5} dx}{1-x} = \int \frac{-(x-1)^{2/5}}{x-1} dx = - \int (x-1)^{-3/5} dx = -5/2(x-1)^{2/5}$$

**Exercise 21.** Thm. 1.18. invariance under translation.  $\int_a^b f(x) dx = \int_{a+c}^{b+c} f(x-c) dx$ .

Thm. 1.19. expansion or contraction of the interval of integration.

$$\begin{aligned}\int_a^b f(x) dx &= \frac{1}{k} \int_{ka}^{kb} f\left(\frac{x}{k}\right) dx \\ \frac{y}{dy} = \frac{c}{dx} \quad \int_a^b f(x) dx &= \int_{a+c}^{b+c} f(y-c) dy \\ \frac{y}{dy} = \frac{kx}{kdx} \quad \int_a^b f(x) dx &= \frac{1}{k} \int_{ka}^{kb} f\left(\frac{y}{k}\right) dy\end{aligned}$$

**Exercise 22.**

$$\begin{aligned}F\left(\frac{x}{a}, 1\right) &= \int_0^{x/a} \frac{u^p}{(u^2+1^2)^q} du \quad \begin{matrix} u = \frac{t}{a} \\ du = \frac{dt}{a} \end{matrix} \\ F\left(\frac{x}{a}, 1\right) &= \frac{1}{a} \int_0^x \frac{(t/a)^p dt}{\left(\left(\frac{t}{a}\right)^2 + 1^2\right)^q} = \\ &= a^{-p-1+2q} \int_0^x \frac{t^p}{(t^2+1^2)^q} dt = a^{-p-1+2q} F(x, a)\end{aligned}$$

**Exercise 23.**

$$\begin{aligned}\int_x^1 \frac{dt}{1+t^2} &= F(1) - F(x) & u = \frac{1}{t} \\ \int_1^x \frac{dt}{1+t^2} &= F(x) - F(1) & du = \frac{-1}{t^2} dt, \frac{-1}{u^2} du = dt \\ \int_1^{1/x} \frac{dt}{1+t^2} &= F\left(\frac{1}{x}\right) - F(1) & \int_1^x \frac{dt}{1+t^2} = \int_1^{1/x} \frac{-du}{u^2(1+\frac{1}{u^2})} = \\ & &= - \int_1^{1/x} \frac{du}{u^2+1} = \int_{1/x}^1 \frac{dt}{t^2+1}\end{aligned}$$

**Exercise 24.**

$$\int_0^1 x^m (1-x)^n dx = - \int_1^0 (1-u)^m (u^n) du = \int_0^1 (1-x)^m x^n dx \quad \text{using } \begin{matrix} u = 1-x \\ x = 1-u \end{matrix}$$

**Exercise 25.**

$$\begin{aligned}\cos^m x \sin^m x &= \left(\frac{\sin 2x}{2}\right)^m = 2^{-m} \sin^m 2x \\ \int_0^{\pi/2} \cos^m x \sin^m x dx &= 2^{-m} \int_0^{\pi/2} \sin^m 2x dx = 2^{-m} \int_0^{\pi} \frac{1}{2} \sin^m x dx = 2^{-m-1} \int_0^{\pi} \sin^m x dx = \\ &= -2^{-m-1} \int_{\pi/2}^{-\pi/2} \sin^m \left(\frac{\pi}{2} - x\right) dx = 2^{-m-1} \int_{-\pi/2}^{\pi/2} \cos^m x dx = 2^{-m} \int_0^{\pi/2} \cos^m x dx\end{aligned}$$

**Exercise 26.**

(1)

$$\begin{aligned}
u = \pi - x \quad x = \pi - u \quad \int_0^\pi x f(\sin x) dx &= \int_\pi^0 (\pi - u) f(\sin(\pi - u)) (-du) = \int_0^\pi (\pi - u) f(\sin u) du = \\
&= \pi \int_0^\pi f(\sin x) dx - \int_0^\pi x f(\sin x) dx \\
\implies \int_0^\pi x f(\sin x) dx &= \frac{\pi}{2} \int_0^\pi f(\sin x) dx
\end{aligned}$$

(2)

$$\begin{aligned}
u = \cos x \quad du = -\sin x dx \quad \int_0^\pi \frac{x \sin x}{1 + \cos^2 x} dx &= \int_0^\pi \frac{x \sin x}{2 - \sin^2 x} = \frac{\pi}{2} \int_0^\pi \frac{\sin x}{2 - \sin^2 x} dx = \frac{\pi}{2} \int_0^\pi \frac{\sin x}{1 + \cos^2 x} dx = \\
&= -\frac{\pi}{2} \int_1^{-1} \frac{du}{1 + u^2} = \frac{\pi}{2} \int_{-1}^1 \frac{du}{1 + u^2} = \pi \int_{-1}^1 \frac{dx}{1 + x^2}
\end{aligned}$$

**Exercise 27.**

$$\begin{aligned}
x = \sin u \quad dx = \cos u \quad \int_0^1 (1 - x^2)^{n-\frac{1}{2}} dx &= \int_0^{\pi/2} (\cos^2 u)^{n-\frac{1}{2}} \cos u du = \int_0^{\pi/2} \cos^{2n} u du
\end{aligned}$$

**5.10 Exercises - Integration by Parts.****Exercise 1.**  $\int x \sin x = -x \cos x + \sin x$ **Exercise 2.**  $\int x^2 \sin x = -x^2 \cos x + 2x \sin x + 2 \cos x$ **Exercise 3.**  $\int x^3 \cos x = x^3 \sin x + 3x^2 \cos x - 6x \sin x + -6 \cos x$ **Exercise 4.**  $\int x^3 \sin x = -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x$ **Exercise 5.**  $\int \sin x \cos x = -\frac{1}{4} \cos 2x = -\frac{1}{4} (\cos^2 x - \sin^2 x)$ **Exercise 6.**  $\int x \sin x \cos x dx = \int \frac{x}{2} \sin 2x = -\frac{x \cos 2x}{4} + \frac{\sin 2x}{8}$ **Exercise 7.**  $\int \sin^2 x = \int \sin x \sin x = -\sin x \cos x + \int \cos^2 x$ 

$$\int \sin^2 x dx = \frac{-1}{4} \sin 2x + \frac{x}{2}$$

**Exercise 8.**

$$\begin{aligned}
\int \sin^n x dx &= -\cos x \sin^{n-1} x + \int (n-1) \sin^{n-2} x \cos^2 x \quad \begin{array}{l} u = \sin^{n-1} x \\ dv = \sin x dx \end{array} \\
&= -\cos x \sin^{n-1} x + \int (n-1) \sin^{n-2} x (1 - \sin^2 x) = \\
&= -\cos x \sin^{n-1} x + \int (n-1) \sin^{n-2} x - (n-1) \int \sin^n x \\
\int \sin^n x &= \frac{-1}{n} \sin^{n-1} x \cos x + \frac{(n-1)}{n} \int \sin^{n-2} x
\end{aligned}$$

**Exercise 9.**

(1)

$$\begin{aligned}
\int \sin^2 x &= \frac{-1}{2} \sin x \cos x + \frac{1}{2} \int 1 = \frac{-1}{2} \sin x \cos x + \frac{1}{2} x \\
\int_0^{\pi/2} \sin^2 x dx &= \frac{\pi}{4}
\end{aligned}$$

$$(2) \int_0^{\pi/2} \sin^4 x = \frac{-1}{4} \sin^3 x \cos x \Big|_0^{\pi/2} + \frac{3}{4} \int_0^{\pi/2} \sin^2 x = \frac{3\pi}{16}$$

$$(3) \int_0^{\pi/2} \sin^6 x = \frac{5}{6} \int_0^{\pi/2} \sin^4 x = \boxed{\frac{5\pi}{32}}$$

**Exercise 10.**

(1)

$$\begin{aligned}
\int \sin^3 x dx &= \frac{-1}{3} \sin^2 x \cos x + \frac{2}{3} \int \sin x = -\frac{1}{6} \sin 2x \cos x - \frac{2}{3} \cos x = \frac{-3}{4} \cos x + \frac{1}{12} \cos 3x \quad \text{since} \\
\frac{-3}{4} \cos x + \frac{1}{12} \cos 3x &= \frac{-3}{4} \cos x + \frac{1}{12} (\cos x \cos 2x - \sin 2x \sin x) = \\
&= -\frac{3}{4} \cos x + \frac{1}{12} (\cos x (1 - 2 \sin^2 x) - 2 \sin^2 x \cos x) = \boxed{\frac{-2}{3} \cos x - \frac{1}{3} \sin^2 x \cos x}
\end{aligned}$$

(2)

$$\int \sin^4 x dx = \frac{-1}{4} \sin^3 x \cos x + \frac{3}{4} \int \sin^2 x = \frac{-1}{4} \sin^3 x \cos x + \frac{3}{4} \left( \frac{x}{2} - \frac{\sin 2x}{4} \right) = \frac{-1}{4} \sin^3 x \cos x + \frac{3x}{8} - \frac{3 \sin 2x}{16}$$

$$\text{Now } \frac{1}{32} \sin 4x = \frac{1}{32} (2 \sin 2x \cos 2x) = \frac{1}{8} (\sin x \cos x (1 - 2 \sin^2 x)) = \frac{\sin 2x}{16} - \frac{1}{4} \sin^3 x \cos x$$

$$\implies \frac{-1}{4} \sin^3 x \cos x + \frac{3x}{8} - \frac{3 \sin 2x}{16} = \frac{3x}{8} - \frac{1}{4} \sin 2x + \frac{1}{32} \sin 4x$$

(3)

$$\begin{aligned} \int \sin^5 x dx &= \int \sin^4 x \sin x dx = -\cos x \sin^4 x + \int \cos^2 x 4 \sin^3 x = \\ &= -\cos x \sin^4 x + 4 \left( \int \sin^3 x - \int \sin^5 x \right) = -\cos x \sin^4 x + 4 \int \sin^3 x - 4 \int \sin^5 x \\ &\quad 5 \int \sin^5 x = -\cos x \sin^4 x + 4 \int \sin^3 x \\ 5 \int \sin^5 x &= -\cos x (1 - \cos^2 x)^2 + 4 \left( \frac{-3}{4} \cos x + \frac{1}{12} \cos 3x \right) \\ &= -\cos x (1 - 2 \cos^2 x + \cos^4 x) + -3 \cos x + \frac{1}{3} \cos 3x \\ &= -\cos x + 2 \cos^3 x - \cos^5 x - 3 \cos x + \frac{1}{3} (\cos x \cos 2x - \sin x \sin 2x) = \\ &= -4 \cos x + 2 \cos^3 x - \cos^5 x + \frac{1}{3} (4 \cos^3 x - 3 \cos x) = -5 \cos x + \frac{10 \cos^3 x}{3} - \cos^5 x \\ \int \sin^5 x dx &= -\cos x + \frac{2 \cos 3x}{3} - \frac{1}{5} \cos^5 x \end{aligned}$$

My solution to the last part of this exercise **conflicts** with what's stated in the book.

### Exercise 11.

(1)

$$\begin{aligned} \int x \sin^2 x dx &= \left( \int \sin^2 x \right) x - \int (\sin^2 t) = \frac{x^2}{2} - \frac{x \sin 2x}{4} - \left( \frac{x^2}{4} + \frac{\cos 2x}{8} \right) = \\ &= \frac{x^2}{8} - \frac{x \sin 2x}{4} - \frac{\cos 2x}{8} \\ &\quad \text{we had used } \int \sin^2 x = \frac{x}{2} - \frac{\sin 2x}{4} \end{aligned}$$

(2)

$$\begin{aligned} \int x \sin^3 x &= \frac{-3x}{4} \cos x + \frac{x}{12} \cos 3x - \int \frac{3}{4} \cos x + \frac{1}{12} \cos 3x = \\ &= \frac{-3x}{4} \cos x + \frac{x}{12} \cos 3x + \frac{3}{4} \sin x + \frac{-\sin 3x}{36} \\ \int \sin^3 x &= \frac{-3}{4} \cos x + \frac{1}{12} \cos 3x \end{aligned}$$

(3)

$$\begin{aligned} \int x^2 \sin^2 x dx &= x^2 \left( \frac{x}{2} - \frac{\sin 2x}{4} \right) - \int 2x \left( \frac{x}{2} - \frac{\sin 2x}{4} \right) = \frac{x^3}{2} - \frac{x^2 \sin 2x}{4} - \frac{1}{3} x^3 + \int \frac{x \sin 2x}{2} = \\ &= \frac{x^3}{6} - \frac{x^2 \sin 2x}{4} + \frac{1}{2} \left( \frac{-x \cos 2x}{2} + \frac{\sin 2x}{4} \right) = \\ &= \boxed{\frac{x^3}{6} - \frac{x^2 \sin 2x}{4} - \frac{x \cos 2x}{4} + \frac{\sin 2x}{8}} \end{aligned}$$



**Exercise 12.**

$$\begin{aligned}
\int \cos^n x dx &= \int \cos^{n-1} x \cos x dx = \cos^{n-1} x \sin x + \int (n-1) \cos^{n-2} x \sin^2 x dx = \\
&= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x dx - \int (n-1) \cos^n x dx \\
&\Rightarrow \int \cos^n x dx = \frac{\cos^{n-1} x \sin x}{n} + \left( \frac{n-1}{n} \right) \int \cos^{n-2} x dx
\end{aligned}$$

**Exercise 13.**

- (1)  $\int \cos^2 x = \frac{\sin 2x}{2} + \frac{1}{2}x$   
(2)  $\int \cos^3 x = \frac{\cos^2 x \sin x}{3} + \frac{2}{3} \sin x = \frac{3}{4} \sin x + \frac{1}{12} \sin 3x$  since

$$\begin{aligned}
\frac{1}{12} \sin 3x &= \frac{1}{12} (\sin 2x \cos x + \sin x \cos 2x) = \frac{1}{6} \sin x \cos^2 x + \frac{1}{12} \sin x (2 \cos^2 x - 1) = \\
&= \frac{1}{3} \sin x \cos^2 x - \frac{1}{12} \sin x
\end{aligned}$$

(3)

$$\begin{aligned}
\int \cos^4 x dx &= \frac{\cos^3 x \sin x}{4} + \frac{3}{4} \left( \frac{1}{2}x + \frac{1}{4} \sin 2x \right) = \frac{3}{8}x + \frac{3}{16} \sin 2x + \frac{\cos^3 x \sin x}{4} \\
\sin 4x &= 2 \sin 2x \cos 2x = 4 \sin x \cos x (2 \cos^2 x - 1) = 8 \sin x \cos^3 x - 2 \sin 2x \quad \text{then} \\
\int \cos^4 x dx &= \frac{3}{8}x + \frac{1}{4} \sin 2x + \frac{1}{32} \sin 4x
\end{aligned}$$

**Exercise 14.**

$$\begin{aligned}
\int \sqrt{1-x^2} dx &= x\sqrt{1-x^2} + \int \frac{x^2}{\sqrt{1-x^2}} dx \\
\int \frac{x^2}{\sqrt{1-x^2}} dx &= \int \frac{x^2 + 1}{\sqrt{1-x^2} + 1 - 1} = - \int \sqrt{1-x^2} + \frac{1}{\sqrt{1-x^2}} \\
&\quad x^2 = x^2 - 1 + 1 \\
&\Rightarrow \int \sqrt{1-x^2} dx = \frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2} \int \frac{1}{\sqrt{1-x^2}}
\end{aligned}$$

**Exercise 15.**

(1)

$$\begin{aligned}
\int (a^2 - x^2)^n dx &= x(a^2 - x^2)^n - \int n(a^2 - x^2)^{n-1} (-2x) x dx = x(a^2 - x^2)^n + 2n \int x^2 (a^2 - x^2)^{n-1} dx \\
\int x^2 (a^2 - x^2)^{n-1} dx &= \int ((x^2 - a^2) + a^2) (a^2 - x^2)^{n-1} dx = \int -(a^2 - x^2)^n + a^2 (a^2 - x^2)^{n-1} dx \\
&\Rightarrow \int (a^2 - x^2)^n dx = \frac{x(a^2 - x^2)^n}{2n+1} + \frac{2a^2 n}{2n+1} \int (a^2 - x^2)^{n-1} dx
\end{aligned}$$

(2)

$$\begin{aligned}
\int (a^2 - x^2) dx &= \frac{x(a^2 - x^2)}{3} + \frac{2a^2}{3}x = \frac{-x^3}{3} + a^2x \\
\int (a^2 - x^2)^{5/2} dx &= \frac{x(a^2 - x^2)^{5/2}}{6} + \frac{a^2 5}{6} \int (a^2 - x^2)^{3/2} dx \\
\int (a^2 - x^2)^{3/2} dx &= \frac{x(a^2 - x^2)^{3/2}}{4} + \frac{3a^2}{4} \int (a^2 - x^2)^{1/2} dx \\
\int (a^2 - x^2)^{1/2} dx &= a \int \sqrt{1 - \left(\frac{x}{a}\right)^2} dx = a^2 \int \cos^2 \theta d\theta = \sin \theta = \frac{x}{a} \\
&= a^2 \int \frac{1 + \cos 2\theta}{2} = a^2 \left( \frac{\theta}{2} + \frac{\sin 2\theta}{4} \right) = a^2 \left( \arcsin \frac{x}{a} + \frac{1}{2} \frac{x}{a} \sqrt{1 - \left(\frac{x}{a}\right)^2} \right) \quad \cos \theta d\theta = \frac{dx}{a} \\
&\int_0^a \sqrt{a^2 - x^2} = a^2 \left( \frac{\pi}{2} - 0 \right) = \frac{\pi a^2}{2} \\
&\int_0^a (a^2 - x^2)^{3/2} dx = \frac{3a^2}{4} \left( \frac{\pi a^2}{2} \right) = \frac{3\pi a^4}{8} \\
&\boxed{\int_0^a (a^2 - x^2)^{5/2} dx = \frac{5a^2}{6} \left( \frac{3\pi a^4}{8} \right) = \frac{5}{16} \pi a^6}
\end{aligned}$$

**Exercise 16.**  $I_n(x) = \int_0^x t^n (t^2 + a^2)^{-1/2} dt$

(1)

$$\begin{aligned}
I_n(x) &= (t^2 + a^2)^{1/2} t^{n-1} - \int (n-1) t^{n-2} (t^2 + a^2)^{1/2} = t^{n-1} (t^2 + a^2)^{1/2} - (n-1) \int \frac{t^{n-2} (t^2 + a^2)}{(t^2 + a^2)^{1/2}} \\
(n) I_n &= x^{n-1} (x^2 + a^2)^{1/2} - a^2 (n-1) \int \frac{t^{n-2}}{(t^2 + a^2)^{1/2}} = x^{n-1} \sqrt{x^2 + a^2} - (n-1) a^2 I_{n-2}
\end{aligned}$$

(2)  $n = 5; x = 2; a = \sqrt{5}$ .

$$\begin{aligned}
I_1(2) &= \int_0^2 x(x^2 + 5)^{-1/2} dx = (x^2 + 5)^{1/2} \Big|_0^2 = 3 - \sqrt{5} \\
5I_5(2) &= \int_0^2 t^5 (t^2 + 5)^{-1/2} dt = 2^{5-1} (4 + 5)^{1/2} - 5(5-1)I_3(2) = 48 - 20I_3(2) \\
3I_3(2) &= 2^2 \sqrt{4 + 5} - 5(3-1)I_1(2) = 12 - 10(3 - \sqrt{5}) \\
I_5(2) &= \frac{1}{5} \left( 48 - 20 \left( -6 + \frac{10\sqrt{5}}{3} \right) \right) = \boxed{\frac{168}{5} - \frac{40\sqrt{5}}{3}}
\end{aligned}$$

**Exercise 17.**

$$\begin{aligned}
\int t^3 (c + t^3)^{-1/2} dt &= \frac{t^2 (c + t^3)^{1/2}}{3} - \int \frac{2(c + t^3)^{1/2}}{3} \\
\int_{-1}^3 t^3 (4 + t^3)^{-1/2} dt &= \frac{t^2 (4 + t^3)^{1/2}}{3} \Big|_{-1}^3 - \frac{2}{3} \int_{-1}^3 (4 + t^3)^{1/2} = \boxed{2\sqrt{31} + \frac{2\sqrt{3}}{3} - \frac{2}{3} (11.35)}
\end{aligned}$$

**Exercise 18.**

$$\begin{aligned}
\int \frac{\sin^{n+1} x}{\cos^{m+1} x} dx &= \int \sin^n x \left( \frac{\sin x}{\cos^{n+1} x} \right) dx = \frac{\sin^n x}{m \cos^m x} - \int \frac{n \sin^{n-1} x}{m \cos^{m-1} x} \\
&\Rightarrow \int \frac{\sin^{n+1} x}{\cos^{m+1} x} dx = \frac{\sin^n x}{m \cos^m x} - \frac{n}{m} \int \frac{\sin^{n-1} x}{\cos^{m-1} x}
\end{aligned}$$

**Exercise 19.**

$$\begin{aligned}
 \int \frac{\cos^{m+1} x}{\sin^{n+1} x} dx &= \int \cos^m x \left( \frac{\cos x dx}{\sin^{n+1} x} \right) = \cos^m x \frac{1}{-n \sin^n x} - \int \frac{m \cos^{m-1} x}{-n \sin^n x} = \\
 &= \frac{\cos^m x}{-n \sin^n x} + \frac{m}{n} \int \frac{\cos^{m-1} x}{\sin^n x} \\
 \int \cot^2 x &= \int \frac{\cos^{1+1} x}{\sin^{1+1} x} = \frac{-1}{1} \frac{\cos^1 x}{\sin^1 x} - \frac{1}{1} \int dx = -\cot x - x \\
 \int \cot^4 x dx &= \int \frac{\cos^{3+1} x}{\sin^{3+1} x} = -\frac{1}{3} \frac{\cos^3 x}{\sin^3 x} - \int \frac{\cos^{3-1} x}{\sin^{3-1} x} = \frac{-1}{3} \cot^3 x - (-\cot x - x) = \boxed{\frac{-1}{3} \cot^3 x + \cot x + x}
 \end{aligned}$$

**Exercise 20.**

(1)

$$\int_0^2 t f''(t) dt = 2 \int_0^1 t f''(2t) dt \quad n = 2$$

(2)

$$\int_0^1 x f''(2x) dx = \frac{1}{2} \int_0^2 t f''(t) dt = \frac{1}{2} \left( t f'(t) \Big|_0^2 - \int_0^2 f'(t) dt \right) = \frac{1}{2} (2f'(2) - (f(2) - f(0))) = \boxed{4}$$

**Exercise 21.**

(1) Recall Theorem 5.5, the second mean-value theorem for integrals:

$$\begin{aligned}
 \int_a^b f(x)g(x)dx &= f(a) \int_a^c g(x)dx + f(b) \int_c^b g(x)dx \\
 \int_a^b \sin \phi(t) \left( \frac{\phi'(t)}{\phi'(t)} \right) dt &= \frac{1}{\phi'(a)} \int_a^c \phi'(t) \sin \phi(t) + \frac{1}{\phi'(b)} \int_a^b \phi'(t) \sin \phi(t) = \\
 &= \frac{1}{\phi'(a)} \cos \phi(t) \Big|_a^c + \frac{1}{\phi'(b)} (\cos \phi(b) - \cos \phi(a)) \leq \frac{4}{m} \quad \text{where } \frac{1}{m} \geq \frac{1}{\phi'(t)} \quad \forall t \in [a, b]
 \end{aligned}$$

(2)  $\phi(t) = t^2$ ;  $\phi'(t) = 2t > 2a$  if  $t > a$

$$\left| \int_a^x \sin t^2 dt \right| \leq \frac{4}{2a} = 2a$$

**5.11 Miscellaneous review exercises.**

**Exercise 1.**  $g(x) = x^n f(x)$ ;  $f(0) = 1$

$g'(x) = nx^{n-1}f(x) + x^n f'(x)$ ;  $g'(0) = 0$  especially if  $n \in \mathbb{Z}^+$  (just note that if negative integer values are included,  $g'(0)$  easily blows up)

$$g^j(x) = \sum_{k=0}^h \binom{j}{k} \frac{n!}{(n-k)!} x^{n-k} f^{j-k}(x)$$

If  $j < n$ , then  $g^j(0)$ , since each term contains some power of  $x$

If  $j \geq n$ ,

$$\begin{aligned}
 g^j(x) &= \sum_{k=0}^n \binom{j}{k} \frac{n!}{(n-k)!} x^{n-k} f^{(j-k)}(x) \\
 g^j(0) &= \frac{j!}{(j-n)!} f^{(j-n)}(0)
 \end{aligned}$$

If  $j = n$ ,  $g^n(0) = n!$

**Exercise 2.**

$$\begin{aligned}
 P(x) &= \sum_{j=0}^5 a_j x^j & P(0) &= 1 = a_0 \\
 P'(x) &= \sum_{j=1}^5 j a_j x^{j-1} & P'(0) &= 0 = a_1 & a_1 = a_2 = 0 \\
 P''(x) &= \sum_{j=2}^5 j(j-1) a_j x^{j-2} & P''(0) &= 0 = 2(1)a_2 \\
 P(x) &= a_5 x^5 + a_4 x^4 + a_3 x^3 + 1 & P(1) &= a_5 + a_4 + a_3 + 1 = 2 \\
 \implies P'(x) &= 5a_5 x^4 + 4a_4 x^3 + 3a_3 x^2 & P'(1) &= 5a_5 + 4a_4 + 3a_3 = 0 \\
 P''(x) &= 20a_5 x^3 + 12a_4 x^2 + 6a_3 x & P''(1) &= 20a_5 + 12a_4 + 6a_3 = 0
 \end{aligned}$$

Solve for the undetermined coefficients by Gauss-Jordan elimination process

$$\begin{aligned}
 \begin{bmatrix} 5 & 4 & 3 \\ 20 & 12 & 6 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a_5 \\ a_4 \\ a_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} & \begin{bmatrix} 5 & 4 & 3 & | & 0 \\ 20 & 12 & 6 & | & 0 \\ 1 & 1 & 1 & | & 1 \end{bmatrix} &= \begin{bmatrix} 1 & 0 & 0 & | & -15 \\ 0 & 1 & 1 & | & 10 \\ 1 & 0 & 0 & | & 6 \end{bmatrix} \\
 \implies a_5 = 6 & \quad a_4 = -15 & \quad a_3 = 10 & \quad \boxed{P(x) = 6x^5 - 15x^4 + 10x^3 + 1}
 \end{aligned}$$

**Exercise 3.** If  $f(x) = \cos x$  and  $g(x) = \sin x$ , Prove that  $f^{(n)} = \cos(x + \frac{n\pi}{2})$  and  $g^{(n)}(x) = \sin(x + \frac{n\pi}{2})$

$$\begin{aligned}
 f^{(n)}(x) &= \cos\left(x + \frac{n\pi}{2}\right) = \begin{cases} \sin x (-1)^{j+1} & \text{if } n = 2j + 1 \\ \cos x (-1)^j & \text{if } n = 2j \end{cases} \\
 g^{(n)}(x) &= \sin\left(x + \frac{n\pi}{2}\right) = \begin{cases} \cos x (-1)^j & \text{if } n = 2j + 1 \\ \sin x (-1)^j & \text{if } n = 2j \end{cases} \\
 f(x) &= \cos x & f^{(2j)}(x) &= \cos x (-1)^j \\
 f'(x) &= -\sin x & f^{(2(j+1))}(x) &= (\cos x (-1)^j)' = \cos x (-1)^{j+1} \\
 f''(x) &= -\cos x & f^{(2j+1)}(x) &= \sin x (-1)^{j+1} \\
 f'''(x) &= \sin x & f^{(2j+3)}(x) &= (\sin x (-1)^{j+1})' = \sin x (-1)^{j+2} \\
 f^{(4)}(x) &= \cos x & \\
 g(x) &= \sin x & g^{(2j)}(x) &= \sin x (-1)^j \\
 g'(x) &= \cos x & g^{(2(j+1))}(x) &= (\sin x (-1)^j)'' = \sin x (-1)^{j+1} \\
 g''(x) &= -\sin x & g^{(2j+1)}(x) &= \cos x (-1)^j \\
 g'''(x) &= -\cos x & g^{(2j+3)}(x) &= (\cos x (-1)^j)' = \cos x (-1)^{j+1} \\
 g^{(4)}(x) &= \sin x &
 \end{aligned}$$

**Exercise 4.**

$$\begin{aligned}
 h'(x) &= f'g + fg' \\
 h''(x) &= f''g + 2f'g' + fg'' \\
 h^{(n+1)} &= \sum_{k=0}^n \binom{n}{k} \left( f^{(k+1)}g^{(n-k)} + f^{(k)}g^{(n-k+1)} \right) = \\
 &= f^{(1)}g^{(n)} + fg^{(n+1)} + \sum_{k=1}^{n-1} \binom{n}{k} \left( f^{(k+1)}g^{(n-k)} + f^{(k)}g^{(n-k+1)} \right) + f^{(n+1)}g + f^{(n)}g^{(1)} \\
 &= f^{(1)}g^{(n)} + fg^{(n+1)} + \sum_{k=2}^n \frac{n!}{(n-k+1)!(k-1)!} \left( f^{(k)}g^{(n-k+1)} + \sum_{k=1}^{n-1} \frac{n!}{(n-k)!k!} f^{(k)}g^{(n-k+1)} \right) + \\
 &\quad + f^{(n+1)}g + f^{(n)}g^{(1)} \\
 \text{Now } \frac{n!}{(n-k+1)!(k-1)!} + \frac{n!}{(n-k)!k!} &= (k + (n-k+1)) \left( \frac{n!}{(n+1-k)!(k)!} \right) = \binom{n+1}{k} \text{ so then} \\
 h^{(n+1)} &= fg^{(n+1)} + \sum_{k=1}^n \binom{n+1}{k} f^{(k)}g^{(n+1-k)} + f^{(n+1)}g = \sum_{k=0}^{n+1} \binom{n+1}{k} f^{(k)}g^{(n+1-k)}
 \end{aligned}$$

By induction, this formula is true.

**Exercise 5.**

(1)

$$\begin{aligned}
 f^2 + g^2 &= f(-g') + gf' \\
 Y = f^2 + g^2 \quad Y' &= 2ff' + 2gg' = 2(gf + 2g(-f)) = 0 \implies Y = C \\
 Y = C = f^2 + g^2 \quad f(0) &= 0; \quad g(0) = 1 \implies \boxed{C = 1}
 \end{aligned}$$

(2)

$$\begin{aligned}
 h &= (F - f)^2 + (G - g)^2 \\
 &= f'(x) = g(x), \quad g'(x) = -f(x); \quad f(0) = 0; \quad g(0) = 1 \\
 h' &= 2(F - f)(F' - f') + 2(G - g)(G' - g'); \quad h'(0) = 2(0) + 2(0) = 0 \\
 &= 2(F - f)(G - g) + 2(G - g)(-F + f) = 0 \quad \forall x \\
 h(x) = C &\implies h(x) = (F(x) - f(x))^2 + (G(x) - g(x))^2 \\
 h(0) &= 0 \quad \text{so } C = 0 \\
 &\implies F = f; \quad G = g
 \end{aligned}$$

**Exercise 6.**  $\frac{df}{du} 2x = 3x^2$   $f'(4) = \frac{3x}{2} = 3$  where we had used the substitution  
 $u = x^2$   $u = 4; x = 2$

**Exercise 7.**  $\frac{dg}{du} = u^{3/2}; \quad g(u) = \frac{2}{5}u^{5/2} \quad g(4) = \frac{2}{5}2^5 = \boxed{\frac{64}{5}}$

**Exercise 8.**

$$\begin{aligned}
 \int_0^x \frac{\sin t}{t+1} dt &= \frac{1}{0+1} \int_0^c \sin t dt + \frac{1}{x+1} \int_c^x \sin t dt = \\
 &= -\cos t \Big|_0^c + \frac{1}{x+1} (-\cos t \Big|_c^x) = -\cos c + 1 + \frac{-1}{x+1} (\cos x - \cos c) = \\
 &= \frac{-x \cos c + x - \cos c + 1 - \cos x + \cos c}{x+1} = \frac{x(1 - \cos c) + (1 - \cos c)}{x+1} > 0
 \end{aligned}$$

**Exercise 9.**  $y = x^2$  is the curve for  $C$ .  $y = \frac{1}{2}x^2$  is the curve for  $C_1$ .

$$\int_0^b (x^2 - \frac{1}{2}x^2) = \frac{1}{6}b^3 \quad P : (b, b^2)$$

Assume  $C_2$  is of the form  $kx^2$

$$\int_0^c (kx^2 - x^2) + \int_c^b (b^2 - x^2) = \frac{(k-1)}{3}c^3 + b^2(b-c) + -\frac{1}{3}(b^3 - c^3) \quad \begin{matrix} kc^2 = b^2 \\ k = \frac{b^2}{c^2} \end{matrix}$$

$$\Rightarrow \frac{(b^2 - c^2)c}{3} + b^3 - cb^2 - \frac{b^3}{3} + \frac{c^3}{3} = \frac{2b^3}{3} - \frac{2}{3}cb^2 = \frac{2}{3}b^2(b-c)$$

Now  $A(A) = A(B)$

$$\Rightarrow \frac{2}{3}b^2(b-c) = \frac{1}{6}b^3 \Rightarrow b = \frac{4}{3}c$$

$$\text{so then } \boxed{kx^2 = \frac{16}{9}x^2}$$

**Exercise 10.**

$$(1) |Q(h) - 0| = \left| \frac{f(h)}{h} \right| = \begin{cases} \frac{h^2}{|h|} & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

For now, consider  $0 < h < \delta$ ; let  $\delta(\epsilon; h=0) = \epsilon$

$$|Q(h) - 0| = \left| \frac{f(h)}{h} \right| = \begin{cases} h & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases} < \epsilon$$

(2)

$$\left| \frac{f(h) - f(0)}{h} - 0 \right| = \epsilon \Rightarrow f'(0) = 0$$

**Exercise 11.**  $\int (2 + 3x) \sin bxdx = -\frac{2}{5} \cos 5x + -\frac{3x}{5} \cos 5x + \frac{3}{25} \sin 5x$

**Exercise 12.**  $\frac{4}{3}(1 + x^2)^{3/2}$

**Exercise 13.**  $\left. \frac{(x^2-1)^{10}}{20} \right|_{-2}^1 = \frac{-3^{10}}{20}$

**Exercise 14.**

$$\frac{1}{3} \int_0^1 \frac{6x+7+2}{(6x+7)^3} dx = \frac{1}{3} \left( \frac{-(6x+7)^{-1}}{6} + \frac{-(6x+7)^{-2}}{6} \right) \Big|_0^1 = \frac{1}{3} \left( -\left(\frac{1}{13}\right) \left(\frac{1}{6}\right) + \frac{1}{42} + \frac{-1}{(6)13^2} + \frac{1}{(6)49} \right) =$$

$$= \boxed{\frac{37}{8281}}$$

**Exercise 15.**  $\int x^4(1+x^5)^5 dx = \frac{(1+x^5)^6}{30}$

**Exercise 16.**

$$\int_0^1 x^4(1-x)^{20} dx = \begin{matrix} u = 1-x \\ x = 1-u \end{matrix}$$

$$= -\int_1^0 (1-u)^4 u^{20} du = \int_0^1 (1-u)^4 u^{20} du = \int_0^1 (1+4(-u)+6u^2+4u^3+u^4) u^{20} du =$$

$$= \frac{1^{21}}{21} + \frac{-41^{22}}{22} + \frac{61^{23}}{23} + \frac{-41^{24}}{24} + \frac{1^{25}}{25} =$$

$$= \boxed{\frac{1}{265650}}$$

Make sure to check your arithmetic.

**Exercise 17.**  $\int_1^2 x^{-2} \sin \frac{1}{x} dx = \left( \cos \frac{1}{x} \right) \Big|_1^2 = \cos \frac{1}{2} - \cos 1$

**Exercise 18.**  $\int \sin(x-1)^{1/4} dx$

$$\begin{aligned}
 u &= (x-1)^{1/4} \\
 du &= \frac{1}{4}(x-1)^{-3/4} dx = \frac{1}{4} \frac{1}{u^3} dx \implies \int \sin(x-1)^{1/4} dx = \int (\sin u) 4u^3 du = 4 \int u^3 \sin u du \\
 \int u^3 \sin u du &= -u^3 \cos u + 3u^2 \sin u + 6u \cos u + 06 \sin u \\
 &= -(x-1)^{3/4} \cos(x-1)^{1/4} + 3(x-1)^{1/2} \sin(x-1)^{1/4} + 6(x-1)^{1/4} \cos(x-1)^{1/4} - 6 \sin(x-1)^{1/4} \\
 &\quad \sin(x-1)^{1/4} dx = \\
 &= \boxed{-4(x-1)^{3/4} \cos(x-1)^{1/4} + 24(x-1)^{1/4} \cos(x-1)^{1/4} + 12(x-1)^{1/4} \sin(x-1)^{1/4} - 24 \sin(x-1)^{1/4}}
 \end{aligned}$$

**Exercise 19.**  $\int x \sin x^2 \cos x^2 dx = (1/4) \sin^2 x^2 + C$

**Exercise 20.**  $\int \sqrt{1+3\cos^2 x} \sin 2x dx = \int \sqrt{1+3\cos^2 x} 2 \sin x \cos x dx = \int u^{1/2} \frac{du}{-3} = \frac{2u^{3/2}}{-9} = \boxed{\frac{-2}{9}(1+3\cos^2 x)^{3/2}}$ ,  
where we had used this substitution:

$$\begin{aligned}
 u &= 1+3\cos^2 x & du &= -6 \cos x \sin x dx \\
 \frac{du}{-3} &= 2 \cos x \sin x dx
 \end{aligned}$$

**Exercise 21.**  $\int_0^2 375x^5(x^2+1)^{-4} dx$

$$\begin{aligned}
 u &= x^2+1 \\
 du &= 2x dx \\
 (u-1)^2 &= x^4 \implies \int_0^2 \frac{375}{2} du (u-1)^2 u^{-4} = \frac{375}{2} \int_1^5 du \frac{(u^2-2u+1)}{u^4} = \frac{375}{2} \int_1^5 du \left( \frac{1}{u^2} - \frac{2}{u^3} + \frac{1}{u^4} \right) = \\
 &= \frac{375}{2} \left( \frac{-1}{u} + \frac{1}{u^2} + \frac{1}{-3u^3} \right) \Big|_1^5 = \boxed{64=2^6}
 \end{aligned}$$

**Exercise 22.**  $\int_0^1 (ax+b)(x^2+3x+2)^{-2} dx = \frac{3}{2}$

Since  $\left( \frac{-1}{x^2+3x+2} \right) \Big|_0^1 = \frac{-1}{6} + \frac{1}{2} = \frac{2}{3}$ ,

then if  $a = 9/2$ ,  $b = \frac{27}{2}$ , we'll obtain  $3/2$

**Exercise 23.**  $I_n = \int_0^1 (1-x^2)^n dx$

$$\begin{aligned}
 I_n &= \int_0^1 (1-x^2)^n dx = \boxed{x(1-x^2)^n \Big|_0^1 - \int_0^1 xn(1-x^2)^{n-1}(-2x) dx} = \\
 &= 2 \int_0^1 x^2 n (1-x^2)^{n-1} dx = 2n \int_0^1 ((x^2-1)+1)(1-x^2)^{n-1} dx = \implies I_n = \left( \frac{2n}{2n+1} \right) I_{n-1} \\
 &= 2n I_{n-1} - 2n I_n \\
 I_2 &= \int_0^1 (1-x^2)^2 dx = \int_0^1 dx (1-2x^2+x^4) = \left( x - \frac{2x^3}{3} + \frac{1}{5}x^5 \right) \Big|_0^1 = \boxed{\frac{8}{15}} \\
 I_3 &= \frac{6}{7} \frac{8}{15} \boxed{\frac{16}{35}} \\
 I_4 &= \frac{8}{9} I_3 = \boxed{\frac{128}{315}} \\
 I_5 &= \frac{10}{11} I_4 = \boxed{\frac{256}{693}}
 \end{aligned}$$

**Exercise 24.**  $F(m, n) = \int_0^x t^m (1+t)^n dt; \quad m > 0, \quad n > 0$

$$F(m, n) = \frac{t^{m+1}}{m+1} (1+t)^n \Big|_0^x - \int_0^x \frac{t^{m+1}}{m+1} n(1+t)^{n-1} dt = \frac{x^{m+1}(1+x)^n}{m+1} - \frac{n}{m+1} F(m+1, n-1)$$

$$(m+1)F(m, n) + nF(m+1, n-1) = x^{m+1}(1+x)^n$$

$$F(11, 1) = \int_0^x t^{11} (1+t)^1 dt = \int_0^x t^{11} + t^{12} = \left( \frac{t^{12}}{12} + \frac{t^{13}}{13} \right) \Big|_0^x = \frac{x^{12}}{12} + \frac{x^{13}}{13}$$

$$11F(10, 2) + 2 \left( \frac{x^{12}}{12} + \frac{x^{13}}{13} \right) = x^{11}(1+x)^2$$

$$F(10, 2) = \frac{x^{13}}{13} + \frac{x^{12}}{6} + \frac{x^{11}}{11}$$

**Exercise 25.**  $f(n) = \int_0^{\pi/4} \tan^n x dx$

(1)

Use this extremely **important** fact:  $\int_a^b fg = f(b) \int_c^b g + f(a) \int_a^c g$

$$f(n+1) = \int_0^{\pi/4} \tan^n x \tan x = \int_c^{\pi/4} \tan^n x < \int_0^{\pi/4} \tan^n x = f(n)$$

(2)

$$\begin{aligned} f(n+2) + f(n) &= \int_0^{\pi/4} \tan^n x \tan^2 x + \int_0^{\pi/4} \tan^n x = \int_0^{\pi/4} \tan^n x (\sec^2 x) = \\ &= \frac{\tan^{n+1} x}{n+1} \Big|_0^{\pi/4} = \frac{1}{n+1} \end{aligned}$$

(3)

$$f(n+2) + f(n) = \frac{1}{n+1} < f(n+1) + f(n) < 2f(n)$$

$$\frac{1}{n-1} = f(n-2) + f(n) > f(n-1) + f(n) > 2f(n)$$

$$\implies \frac{1}{n+1} < 2f(n) < \frac{1}{n-1}$$

**Exercise 26.**  $f(0), \quad f(\pi) = 2 \quad \int_0^\pi (f(x) + f''(x)) \sin x dx = 5$

$$\begin{aligned} \int_0^\pi f'' \sin x &= f' \sin x - \int f' \cos x = - \int f' \cos x = -f \cos x - \int f \sin x \\ \int_0^\pi (f + f'') \sin x dx &= \int f \sin x + -(f(\pi)(-1) - f(0)) - \int f \sin x = 2 + f(0) = 5 \implies f(0) = 3 \end{aligned}$$

**Exercise 27.**

$$\begin{aligned} \int_0^{\pi/2} \frac{\sin x \cos x}{x+1} dx &= \int_0^{\pi/2} \frac{\sin 2x}{2x+2} dx = \frac{1}{2} \int_0^\pi \frac{\sin x}{x+2} dx = \frac{1}{2} \left( \frac{-\cos x}{x+2} - \int \frac{\cos x}{(x+2)^2} \right) = \\ &= \frac{1}{2} \left( \frac{1}{\pi+2} + \frac{1}{2} - A \right) = \frac{4+\pi}{4(\pi+2)} - \frac{A}{2} \end{aligned}$$

**Exercise 28.**

$$\begin{aligned} \int \frac{dx}{x\sqrt{a+bx}} &= \frac{2\sqrt{a+bx}}{bx} + \frac{2}{b} \int \frac{\sqrt{a+bx}}{x^2} \\ \int \frac{\sqrt{a+bx}}{x} &= \frac{\frac{2}{3b}(a+bx)^{3/2}}{x} + \int \frac{\frac{2}{3b}(a+bx)^{3/2}}{x^2} = \frac{2}{3b} \sqrt{a+bx} \left( \frac{a}{x} + b \right) + \frac{2}{3b} \int \sqrt{a+bx} \left( \frac{a}{x^2} + \frac{b}{x} \right) \\ &\implies \int \frac{\sqrt{a+bx}}{x} = a \int \frac{dx}{x\sqrt{a+bx}} + 2\sqrt{a+bx} \end{aligned}$$



**Exercise 29.**

$$\begin{aligned}\int x^n \sqrt{ax+b} dx &= \frac{2x^n(ax+b)^{3/2}}{3a} - \int \frac{nx^{n-1}2(ax+b)^{3/2}}{3a} = \frac{2x^n(ax+b)^{3/2}}{3a} - \frac{2n}{3a} \int x^{n-1}(ax+b)\sqrt{ax+b} = \\ &= \frac{2x^n(ax+b)\sqrt{ax+b}}{3a} - \frac{2n}{3} \int x^n \sqrt{ax+b} - \frac{2nb}{3a} \int x^{n-1} \sqrt{ax+b} \\ \boxed{\int x^n \sqrt{ax+b} dx &= \frac{2}{(2n+3)a} \left( x^n(ax+b)^{3/2} - nb \int x^{n-1} \sqrt{ax+b} \right) + C} \quad n \neq \frac{-3}{2}\end{aligned}$$

**Exercise 30.**

$$\begin{aligned}\int \frac{x^m}{\sqrt{a+bx}} dx &= \frac{2x^m(a+bx)^{1/2}}{b} - \int \frac{mx^{m-1}2(a+bx)^{1/2}}{b} \\ &= \frac{2}{b} x^m(a+bx)^{1/2} - \frac{2m}{b} \int \frac{x^{m-1}(a+bx)}{\sqrt{a+bx}} = \frac{2}{b} x^m(a+bx)^{1/2} - 2m \int \frac{x^m}{\sqrt{a+bx}} - \frac{2ma}{b} \int \frac{x^{m-1}}{\sqrt{a+bx}} \\ \boxed{\int \frac{x^m}{\sqrt{a+bx}} dx &= \frac{1}{2m+1} \frac{2}{b} x^m(a+bx)^{1/2} - \frac{2ma}{b(2m+1)} \int \frac{x^{m-1}}{\sqrt{a+bx}}}\end{aligned}$$

**Exercise 31.**

$$\begin{aligned}\int \frac{dx}{x^n \sqrt{ax+b}} &= 2 \frac{\sqrt{ax+b}}{ax^n} + \int \frac{n2\sqrt{ax+b}}{ax^{n+1}} \sqrt{\frac{ax+b}{ax+b}} = \frac{2\sqrt{ax+b}}{ax^n} + \frac{2n}{a} \int \frac{ax+b}{x^{n+1}\sqrt{ax+b}} = \\ &= \frac{2\sqrt{ax+b}}{ax^n} + 2n \int \frac{1}{x^n \sqrt{ax+b}} + \frac{2nb}{a} \int \frac{1}{x^{n+1}\sqrt{ax+b}} \\ \Rightarrow (1-2n) \int \frac{dx}{x^n \sqrt{ax+b}} - \frac{2\sqrt{ax+b}}{ax^n} &= \int \frac{b \left( \frac{2n}{a} \right)}{x^{n+1}\sqrt{ax+b}} \\ \boxed{\int \frac{1}{x^n \sqrt{ax+b}} &= \frac{-\sqrt{ax+b}}{(n-1)bx^{n-1}} - \frac{(2n-3)a}{(2n-2)b} \int \frac{1}{x^{n-1}\sqrt{ax+b}}}\end{aligned}$$

**Exercise 32.** I derived the formulas for this and Exercise 33 by doing the **following trick**.

$$\begin{aligned}\boxed{(C^{m+1}S^{1-n})' &= (m+1)C^m(-S^{2-n}) + C^{m+2}(1-n)S^{-n} = -(m+1)C^mS^{2-n} + (1-n)C^mS^{-n}(1-S^2) = \\ &= -(m+1+1-n)C^mS^{2-n} + (1-n)C^mS^{-n} = -(m-n+2)C^mS^{2-n} + (1-n)C^mS^{-n}} \\ \Rightarrow \int C^mS^{-n} &= \frac{-(C^{m+1}S^{1-n})}{n-1} - \frac{(m-n+2)}{n-1} \int C^mS^{2-n}\end{aligned}$$

**Exercise 33.**

$$\begin{aligned}(C^{m-1}S^{1-n})' &= (m-1)C^{m-2}(-S^{2-n}) + (1-n)C^mS^{-n} \\ &= -(m-1)C^{m-2}(S^{-n})(1-C^2) + (1-n)C^mS^{-n} = \\ &= -(m-1)C^{m-2}S^{-n} + (m-1)C^mS^{-n} + (1-n)C^mS^{-n} \\ C^{m-1}S^{1-n} &= -(m-1) \int C^{m-2}S^{-n} + (m-n) \int C^mS^{-n} \\ \frac{m-1}{m-n} \int C^{m-2}S^{-n} + \frac{C^{m-1}S^{1-n}}{m-n} &= \int C^mS^{-n}\end{aligned}$$

**Exercise 34.**

$$(1) P'(x) - 3P(x) = 4 - 5x + 3x^2$$

$$\begin{aligned}P &= \sum_{j=0}^n a_j x^j \\ P' &= \sum_{j=1}^n a_j j x^{j-1} = \sum_{j=0}^{n-1} a_{j+1} (j+1) x^j \\ \Rightarrow \sum_{j=0}^{n-1} (a_{j+1}(j+1) - 3a_j) x^j &= 4 - 5x + 3x^2\end{aligned}$$

Generally, we can say

$$a_{j+1} = \frac{3}{j+1} a_j \quad \text{if } j \geq 3$$

We also have

$$\begin{aligned}a_1(1) - 3a_0 &= 4 \\a_2(2) - 3a_1 &= -5 \\a_3(3) - 3a_2 &= 3 \implies a_3 - a_2 = 1\end{aligned}$$

Then let  $a_2 = -1$  and  $a_3 = 0$ . So we have  $a_1 = 1$  and  $a_0 = -1$ .  $P(x) = -1 + x - x^2$  is one possible polynomial and we were only asked for one.

Suppose  $Q$  s.t.  $Q' - 3Q = 4 - 5x + 3x^2$  (another solution). Then

$$\begin{aligned}(P - Q)' - 3(P - Q) &= 0 \quad \forall x \\ \implies \frac{(P - Q)'}{P - Q} &= 3 \implies \ln(P - Q) = 3x \implies ke^{3x} = P - Q = k \sum_{j=0}^{\infty} \frac{(3x)^j}{j!} \\ Q &= -k \sum_{j=0}^{\infty} \frac{(3x)^j}{j!} + P\end{aligned}$$

Since we didn't specify what  $Q$  has to be, we find that, in general, any  $Q$  is  $P$  plus some "amount" of the homogeneous solution,  $ke^{3x}$ .

- (2) If  $Q(x)$  is a given polynomial, and suppose  $P$  is a polynomial solution to  $P'(x) - 3P(x) = Q(x)$ . Suppose  $R$  is another polynomial solution such that  $R'(x) - 3R(x) = Q(x)$ . Then just like above,  $P - R = ke^{3x}$ . If we wanted polynomial answers of finite terms, then  $k$  must be zero. Thus, there's at most only one polynomial solution  $P$ .

### Exercise 35. Bernoulli Polynomials.

$$(1) \quad P_1(x) = 1; \quad P'_n(x) = nP_{n-1}(x); \quad \int_0^1 P_n(x)dx = 0, \quad \text{if } n \geq 1$$

$$\begin{aligned}n = 1 \quad (1)(1) &= P'_1 \quad \int_0^1 (x + c) = \left(\frac{1}{2}x^2 + Cx\right)\Big|_0^1 = \frac{1}{2} + C = 0 \quad C = -1/2 \\ P_1 &= x - 1/2\end{aligned}$$

$$\begin{aligned}n = 2 \quad 2(x - 1/2) &= P'_2 \quad \int_0^1 (x^2 - x + C) = \left(\frac{1}{3}x^3 - \frac{1}{2}x^2 + Cx\right)\Big|_0^1 = \frac{-1}{6} + C = 0 \quad C = 1/6 \\ P_2 &= x^2 - x + 1/6\end{aligned}$$

$$\begin{aligned}n = 3 \quad 3(x^2 - x + \frac{1}{6}) &= P'_3 \quad \int_0^1 (x^3 - \frac{3x^2}{2} + \frac{x}{2} + C) = \frac{1}{4} - \frac{1^3}{2} + \frac{1}{4} + C = 0 \\ P_3 &= x^3 - \frac{3x^2}{2} + \frac{x}{2}\end{aligned}$$

$$\begin{aligned}n = 4 \quad 4(x^3 - \frac{3}{2}x^2 + \frac{x}{2}) &= P'_4 \quad \int_0^1 (x^4 - 2x^3 + x^2 + C) = \frac{1}{5} - \frac{1^4}{2} + \frac{1}{3}1^3 + C = 0 \quad C = \frac{-1}{30} \\ P_4 &= x^4 - 2x^3 + x^2 + \frac{-1}{30}\end{aligned}$$

$$\begin{aligned}n = 5 \quad 5(x^4 - 2x^3 + x^2 - \frac{1}{30}) &= P'_5 \quad \int_0^1 (x^5 - \frac{5x^4}{2} + \frac{5x^3}{3} - \frac{x}{6} + C) = \frac{1}{6} - \frac{(1)^5}{2} + \frac{5(1)^4}{12} + \frac{-(1)^2}{12} + C = 0; \quad C = 0 \\ P_5 &= x^5 - \frac{5x^4}{2} + \frac{5x^3}{3} - \frac{x}{6}\end{aligned}$$

- (2) The first, second, and up to fifth case has already been proven.

Assume the  $n$ th case, that  $P_n(t) = t^n + \sum_{j=0}^{n-1} a_j t^j$  (the general form of a polynomial of degree  $n$ ).

$$P'_{n+1} = (n+1)(t^n + \sum_{j=0}^{n-1} a_j t^j) \implies P_{n+1} = t^{n+1} + (n+1) \sum_{j=0}^{n-1} \frac{a_j t^{j+1}}{j+1} + C$$

(3) The first, second, and up to fifth case has already been proven.

Assume the  $n$ th case, that  $P_n(0) = P_n(1)$ .

$$\int P'_{n+1} = P_{n+1}(1) - P_{n+1}(0) \quad (\text{by the second fundamental theorem of calculus})$$

$$P'_{n+1} = (n+1)P_n$$

$$\int_0^1 (n+1)P_n(t) dt = 0 \quad (\text{by the given properties of Bernoulli polynomials})$$

$$\implies P_{n+1}(1) = P_{n+1}(0)$$

(4)  $P_n(x+1) - P_n(x) = nx^{n-1}$  is true for  $n = 1, 2$ , by quick inspection (and doing some algebra mentally).

$$P'_{n+1} = (n+1)P_n \implies \int P'_{n+1} = (n+1) \int P_n$$

$$P_{n+1}(x+1) - P_{n+1}(x) = P_{n+1}(a_1) + (n+1) \int_{a_1}^{x+1} P_n(t) dt - (P_{n+1}(a_2) + (n+1) \int_{a_2}^x P_n(t) dt)$$

$$a_1 = 1; a_2 = 0; \text{ so } P_{n+1}(1) - P_{n+1}(0) = 0 \quad (\text{from previous problems})$$

$$\begin{aligned} P_{n+1}(x+1) - P_{n+1}(x) &= (n+1) \left( \int_1^{x+1} P_n(t) dt - \int_0^x P_n(t) dt \right) = \\ \implies &= (n+1) \int_0^x P_n(t+1) - P_n(t) dt = (n+1) \int_0^x nt^{n-1} dt = (n+1)x^n \end{aligned}$$

(5)

$$\int_0^k P_n = \int_0^k \frac{P'_{n+1}}{n+1} = \frac{P_{n+1}(k) - P_{n+1}(0)}{n+1}$$

$$P_n(x+1) - P_n(x) = nx^{n-1}$$

$$\frac{P_n(x+1) - P_n(x)}{n} = x^{n-1}$$

$$\begin{aligned} \implies \sum_{x=1}^{k-1} \frac{P_{n+1}(x+1) - P_{n+1}(x)}{n+1} &= \sum_{x=1}^{k-1} x^n = \sum_{r=1}^{k-1} \frac{P_{n+1}(r+1) - P_{n+1}(r)}{n+1} = \\ &= \sum_{r=1}^{k-1} r^n = \frac{P_{n+1}(k) - P_{n+1}(0)}{n+1} \quad (\text{telescoping series and } P_{n+1}(1) = P_{n+1}(0)) \end{aligned}$$

(6) **This part was fairly tricky.** A horrible clue was that this part will rely directly on the last part (because of the way this question is asked), which gave us  $\sum_{j=1}^{x-1} j^n = \int_0^x P_n(t) dt = \frac{P_{n+1}(x) - P_{n+1}(0)}{n+1}$ .

Use induction. It can be easily verified, plugging in, that  $P_n(1-x) = (-1)^n P_n(x)$  is true for  $n = 0 \dots 5$ . Assume the  $n$ th case is true.

$$\begin{aligned} \int_0^x P_n(t) dt &= \int_1^{1-x} -P_n(1-u) du = - \int_1^{1-x} P_n(u) (-1)^n du \\ & \quad (\text{since } P_n(1-x) = (-1)^n P_n(x), \text{ assumed } n\text{th case is true}) \\ u = 1-t & \\ du = -dt & \\ &= (-1)^{n+1} \int_1^{1-x} P_n(t) dt = \\ &= (-1)^{n+1} \int_0^{1-x} P_n(t) dt \quad (\text{since } \int_0^1 P_n = 0) \\ \implies &= (-1)^{n+1} \left( \frac{P_{n+1}(1-x) - P_{n+1}(0)}{n+1} \right) = \frac{P_{n+1}(x) - P_{n+1}(0)}{n+1} \\ \implies &P_{n+1}(1-x) = (-1)^{n+1} P_{n+1}(x) \end{aligned}$$

In the second to last and last step, we had used  $(-1)^{n+1} P_{n+1}(0) = P_{n+1}(0)$ . For  $n+1$  even, this is definitely true. If  $n+1$  was odd,

Doing some algebra for the first five cases, we can show that  $P_{2j-1}(0) = 0$  for  $j = 2, 3$ . Assume the  $j$ th case is true. Since  $P_{2j-1}$  is a polynomial and  $P_{2j-1}(0) = 0$ , then the form of  $P_{2j-1}$  must be  $P_{2j-1} = \sum_{k=1}^{2j-1} a_k x^k$ . Using

$$P'_{n+1} = (n+1)P_n,$$

$$\begin{aligned} P_{2j}(x) - P_{2j}(0) &= 2j \int_0^x P_{2j-1} = 2j \sum_{k=1}^{2j-1} \int_0^x a_k t^k = 2j \sum_{k=1}^{2j-1} a_k \frac{1}{k+1} t^{k+1} \Big|_0^x = \\ &= 2j \sum_{k=1}^{2j-1} \frac{a_k}{k+1} x^{k+1} = 2j \sum_{k=2}^{2j} \frac{a_{k-1} x^k}{k} + P_{2j}(0) \\ P_{2j+1}(x) - P_{2j+1}(0) &= (2j+1) \int_0^x \left( \sum_{k=2}^{2j} (2j) \frac{a_{k-1} t^k}{k} + P_{2j}(0) \right) = (2j+1) \sum_{k=3}^{2j+1} \frac{2j a_{k-2} x^k}{k(k-1)} + (2j+1) P_{2j}(0) x \end{aligned}$$

If we take the integral from 0 to 1, then we find that  $P_{2j+1}(0) = 0$

(7) Using  $P_n(1-x) = (-1)^n P_n(x)$ , derived above,

$$\begin{aligned} P_{2j+1}(0) &= (-1)^{2j+1} P_{2j+1}(1) = (-1) P_{2j+1}(0) \\ \implies P_{2j+1}(0) &= 0 \\ P_{2j-1}(1 - \frac{1}{2}) &= (-1)^{2j-1} (P_{2j-1}(\frac{1}{2})) \\ \implies P_{2j-1}(\frac{1}{2}) &= 0 \end{aligned}$$

**Exercise 36.** There's a maximum at  $c$  for  $f$ , so  $f'(c) = 0$

$$\begin{aligned} \int_a^x f''(t) dt &= f'(x) - f'(a) \\ \int_0^c f''(t) dt &= f'(c) - f'(0) = -f'(0) & |f'(0)| &= \left| \int_c^0 f''(t) dt \right| \leq \int_0^c |f''| dt \leq mc \\ \int_c^a f''(t) dt &= f'(a) - f'(c) = f'(a) & |f'(a)| &\leq m(a-c) \end{aligned}$$

$$\boxed{|f'(0)| + |f'(a)| \leq ma}$$

**6.9 Exercises - Introduction, Motivation for the definition of the natural logarithm as an integral, The definition of the logarithm. Basic properties; The graph of the natural logarithm; Consequences of the functional equation  $L(ab) = L(a) + L(b)$ ; Logarithm referred to any positive base  $b \neq 1$ ; Differentiation and integration formulas involving logarithms; Logarithmic differentiation.**

**Exercise 1.**

(1)

$$\log x = c + (\ln |t|)|_e^x = c + \ln |x| - 1 \implies \ln \left( \frac{x}{|x|} \right) = c - 1$$

$$\boxed{c = 1}$$

(2)

$$\begin{aligned} f(x) &= \ln \frac{1+a}{1-a} + \ln \frac{1+b}{1-b} = \ln \left( \frac{(1+a)(1+b)}{(1-a)(1-b)} \right) = \ln \left( \frac{1+x}{1-x} \right) \\ \implies \frac{1+x}{1-x} &= \frac{(1+a)(1+b)}{(1-a)(1-b)} = \frac{1+a+b+ab}{1-b-a+ab} \end{aligned}$$

$$\boxed{x = \frac{a+b}{1+ab}}$$

**Exercise 2.**

$$(1) \log(1+x) = \log(1-x)$$

$$\implies \boxed{x = 0}$$

$$(2) \log(1+x) = 1 + \log(1-x)$$

$$\ln(1+x) = 1 + \ln(1-x) = \ln(e) + \ln(1-x) \ln e(1-x)$$

$$\implies 1+x = e - ex \implies \boxed{x = \frac{e-1}{1+e}}$$

(3)  $2 \log x = x \log 2$

$$\ln x^2 + \ln 2^{-x} = 0 = \ln 1 = \ln x^2 2^{-x} \implies \boxed{x = 2}$$

(4)  $\log(\sqrt{x} + \sqrt{x+1}) = 1$

$$\sqrt{x+1} = e - \sqrt{x} \implies x+1 = e^2 - 2\sqrt{x}e + x$$

$$2\sqrt{x}e = e^2 - 1 \implies \boxed{x = \left(\frac{e^2 - 1}{2e}\right)^2}$$

**Exercise 3.**

$$f = \frac{\ln x}{x}$$

$$f' = \frac{1 - \ln x}{x^2}$$

$$f'' = \frac{-2}{x^3} - \left(\frac{\frac{1}{x}x^2 - 2x \ln x}{x^4}\right) = \frac{2 \ln x - 3}{x^3}$$

$$f' < 0 \text{ when } x > e \quad f' > 0 \text{ when } 0 < x < e$$

$$\text{for } x^3 > 0, \quad 2 \ln x - 3 > 0 \implies f''(x) < 0 \text{ (concave), when } 0 < x < e^{3/2};$$

$$f''(x) > 0 \text{ (convex), when } x > e^{3/2}$$

**Exercise 4.**  $f(x) = \log(1 + x^2)$

$$f' = \frac{2x}{1 + x^2}$$

**Exercise 5.**  $f(x) = \log \sqrt{1 + x^2}$

$$f' = \frac{x}{1 + x^2}$$

**Exercise 6.**  $f(x) = \log \sqrt{4 - x^2}$

$$f' = \frac{1}{2} \frac{-2x}{4 - x^2} = \frac{-x}{4 - x^2}$$

**Exercise 7.**  $f(x) = \log(\log x)$

$$f' = \frac{1}{\ln x} \left(\frac{1}{x}\right) = \frac{1}{x \ln x}$$

**Exercise 8.**  $f(x) = \log x^2 \log x$

$$f' = (2 \log x + \log \log x)' = \frac{2}{x} + \frac{1}{x \log x}$$

**Exercise 9.**  $f(x) = \frac{1}{4} \log \frac{x^2 - 1}{x^2 + 1}$

$$f' = \frac{1}{4} (\log x^2 - 1 - \log x^2 + 1)' = \frac{1}{4} \left( \frac{2x}{x^2 - 1} - \frac{2x}{x^2 + 1} \right) = x \left( \frac{1}{x^4 - 1} \right)$$

**Exercise 10.**  $f(x) = (x + \sqrt{1 + x^2})^n$

$$\begin{aligned} \ln f &= n \ln(x + \sqrt{1 + x^2}) \\ \frac{f'}{f} &= n \left( \frac{1}{x + \sqrt{1 + x^2}} \right) \left( 1 + \frac{x}{\sqrt{1 + x^2}} \right) = \frac{n}{\sqrt{1 + x^2}} \\ f' &= (x + \sqrt{1 + x^2})^n \frac{n}{\sqrt{1 + x^2}} \end{aligned}$$

**Exercise 11.**  $f(x) = \sqrt{x+1} - \log(1 + \sqrt{x+1})$

$$f' = \frac{1}{2\sqrt{x+1}} - \frac{1}{1 + \sqrt{x+1}} \left( \frac{1}{2\sqrt{x+1}} \right) = \boxed{\frac{1}{2(1 + \sqrt{x+1})}}$$

**Exercise 12.**  $f(x) = x \log(x + \sqrt{1 + x^2}) - \sqrt{1 + x^2}$

$$f' = \log(x + \sqrt{1 + x^2}) + \frac{x}{x + \sqrt{1 + x^2}} \left( 1 + \frac{x}{\sqrt{1 + x^2}} \right) - \frac{x}{\sqrt{1 + x^2}}$$

**Note to self:** Notice how this had made some of the square root terms disappear.

**Exercise 13.**  $f(x) = \frac{1}{2\sqrt{ab}} \log \frac{\sqrt{a} + x\sqrt{b}}{\sqrt{a} - x\sqrt{b}}$

$$f = \frac{1}{2\sqrt{ab}}(\ln(\sqrt{a} + x\sqrt{b}) - \ln(\sqrt{a} - x\sqrt{b}))$$

$$f' = \frac{1}{2\sqrt{ab}}\left(\frac{1}{\sqrt{a} + x\sqrt{b}}\sqrt{b} - \frac{1}{\sqrt{a} - x\sqrt{b}}(-\sqrt{b})\right) = \boxed{\frac{-x\sqrt{b}}{\sqrt{a}(a - bx^2)}}$$

**Exercise 14.**  $f(x) = x(\sin(\log x) - \cos \log x)$

$$f' = \sin(\ln x) - \cos(\ln x) + (\cos(\ln x) + \sin(\ln x)) = 2 \sin(\ln x)$$

**Exercise 15.**  $f(x) = \log^{-1} x$

$$f' = \frac{-1}{x(\ln x)^2}$$

**Exercise 16.**  $\int \frac{dx}{2+3x} = \frac{1}{3} \ln(2+3x)$

**Exercise 17.**  $\int \log^2 x dx$

$$\begin{aligned} (x \ln^2 x)' &= \ln^2 x + 2 \ln x \\ (x \ln x - x)' &= \ln x + 1 - 1 = \ln x \end{aligned} \quad \implies x \ln^2 x - 2(x \ln x - x) = x \ln^2 x - 2x \ln x + 2x$$

**Exercise 18.**  $\int x \log x dx$

$$\begin{aligned} \left(\frac{x^2 \ln x}{2}\right)' &= x \ln x + \frac{x}{2} \\ \int x \ln x &= \frac{x^2 \ln x}{2} - \frac{x^2}{4} \end{aligned}$$

**Exercise 19.**  $\int x \log^2 x dx$

$$\begin{aligned} \left(\frac{x^2 \ln^2 x}{2}\right)' &= x \ln^2 x + x^2 \ln x \left(\frac{1}{x}\right) \\ \implies \int x \log^2 x &= \frac{x^2 \ln^2 x}{2} - \frac{x^2 \ln x}{2} - \frac{x^2}{4} \end{aligned}$$

**Exercise 20.**  $\int_0^{e^3-1} \frac{dt}{1+t}$

$$\int_0^{e^3-1} \frac{dt}{1+t} = \ln(1+t) \Big|_0^{e^3-1} = 3$$

**Exercise 21.**  $\int \cot x dx$

$$\int \frac{\cos x}{\sin x} dx = \ln |\sin x|$$

**Exercise 22.**  $\int x^n \log(ax) dx$  Solve the problem directly.

$$\begin{aligned} \int x^n \log ax &= \int x^n \log a + \int x^n \log x = \frac{x^{n+1}}{n+1} \log a + \int x^n \log x \\ \int x^n \ln x &= \frac{x^{n+1}}{n+1} \ln x - \int \frac{x^{n+1}}{n+1} \frac{1}{x} = \frac{x^{n+1}}{n+1} \log x - \frac{x^{n+1}}{(n+1)^2} \\ \implies \int x^n \log ax &= \frac{x^{n+1}}{n+1} \log a + \frac{x^{n+1}}{n+1} \log x - \frac{x^{n+1}}{(n+1)^2} \end{aligned}$$

**Exercise 23.**  $\int x^2 \log^2 x dx$

$$\int x^2 \log^2 x = \frac{1}{3} x^3 \ln^2 x - \int \frac{x^2}{3} 2 \ln x = \frac{x^3 \ln^2 x}{3} - \frac{2}{3} \left( \frac{x^3 \ln x}{3} - \frac{x^3}{9} \right) = \boxed{\frac{x^3 \ln^2 x}{3} - \frac{2x^3 \ln x}{9} + \frac{2x^3}{27}}$$

**Exercise 24.**  $\int \frac{dx}{x \log x}$

$$\int \frac{dx}{x \ln x} = \ln(\ln x) + C$$

**Exercise 25.**  $\int_1^{1-e^{-2}} \frac{\log(1-t)}{1-t} dt$

$$\int_0^{1-e^2} \frac{\ln(1-t)}{1-t} dt = -\frac{1}{2} (\ln(1-t))^2 \Big|_0^{1-e^{-2}} = \boxed{-2}$$

**Exercise 26.**  $\int \frac{\log |x|}{x\sqrt{1+\log |x|}} dx$

$$\begin{aligned} \int \frac{\log x}{x\sqrt{1+\log x}} &= \int (2(1+\log x)^{1/2})' \log x = 2(1+\log x)^{1/2} \log x - \int \frac{2(1+\log x)^{1/2}}{x} = \\ &= \boxed{2 \log x (1+\log x)^{1/2} - \frac{4}{3} (1+\log x)^{3/2}} \end{aligned}$$

**Exercise 27.** Derive

$$\int x^m \log^n x dx = \frac{x^{m+1}}{m+1} \ln^n x - \frac{n}{m+1} \int x^m \ln^{n-1} x$$

By inspection, we just needed integration by parts.

$$\int x^3 \ln^3 x = \frac{x^4}{4} \ln^3 x - \frac{3}{4} \int x^3 \ln^2 x = \frac{x^4}{4} \ln^3 x - \frac{3}{4} \left( \frac{x^4}{4} \ln^2 x - \frac{2}{4} \int x^3 \ln x \right) = \frac{x^4 \ln^3 x}{4} - \frac{3x^4 \ln^2 x}{16} + \frac{3x^4 \ln x}{32} - \frac{3x^4}{128}$$

**Exercise 28.** Given  $x > 0$ ,  $f(x) = x - 1 - \ln x$ ;  $g(x) = \ln x - 1 + \frac{1}{x}$

(1)

$$\begin{aligned} f' &= 1 - \frac{1}{x} & xg' &= f' \\ g' &= \frac{1}{x} - \frac{1}{x^2} = \frac{1}{x} f' \\ \text{so then if } f' &> 0, g' > 0; & f' < 0, & g' < 0 \\ \text{For } f' < 0 & 0 < x < 1 & f' > 0 & x > 1 & f'(1) = g'(1) = 0 \\ f(1) &= g(1) = 0 \\ x - 1 - \ln x &> 0 \text{ since } f(1) = 0 \text{ is a rel. min.} \\ 0 < \ln x - 1 + \frac{1}{x} &\text{ since } g(0) \text{ is a rel. min.} \end{aligned}$$

(2) See sketch.

**Exercise 29.**  $\lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 1$

(1)  $L(x) = \int_1^x \frac{1}{t} dt$ ;  $L'(x) = \frac{1}{x}$ ;  $L'(1) = 1$

(2) Use this theorem.

**Theorem 19** (Theorem I.31).

If 3 real numbers  $a, x$ , and  $y$  satisfy the inequalities

$$\begin{aligned} a &\leq x \leq a + \frac{y}{n} \\ \forall n \geq 1, \quad n \in \mathbb{Z}, \text{ then } x &= a \\ 1 - \frac{1}{x} < \ln x < x - 1 &\implies 1 - \frac{1}{x+1} < \ln x + 1 < x \\ 1 - \frac{x}{1+x} < \frac{\ln x + 1}{x} < 1 &\implies \boxed{\frac{\ln(1+x)}{x} = 1} \end{aligned}$$

**Exercise 30.** Using  $f(xy) = f(x) + f(y)$ ,

$$r = \frac{p}{q} \implies f(a^{p/q}) = f((a^{1/q})^p) = pf(a^{1/q}) = \frac{p}{q} f(a)$$

since

$$\begin{aligned} f(a) &= f(a); & f(a^2) &= f(aa) = 2f(a) & \text{If } f(a) &= f((a^{1/q})^q) = qf(a^{1/q}) \\ f(a^{p+1}) &= f(a^p) + f(a) = (p+1)f(a) & \text{then } f(a^{1/q}) &= \frac{1}{q} f(a) \end{aligned}$$

**Exercise 31.**  $\ln x = \int_1^{|x|} \frac{1}{t} dt$

(1)  $\ln x = \int_1^{|x|} \frac{1}{t} dt$  From this definition, then for  $n$  partitions,  $a_0 = 1, a_1 = 1 + \frac{b-1}{n}, \dots, a_n = b = x$

$$\frac{b-a}{n} = \frac{b-1}{n}; \quad \text{so if } a_k = 1 + k \left( \frac{b-1}{n} \right)$$

$$\sum_{k=1}^n \left( \frac{a_k - a_{k-1}}{a_k} \right) < \log x < \sum_{k=1}^n \left( \frac{a_k - a_{k-1}}{a_{k-1}} \right)$$

(2)  $\log x$  is greater than the step function integral consisting of rectangular strips within  $\frac{1}{x}$  and less than rectangular strips covering over  $\frac{1}{x}$

(3)  $a_k = 1 + k \implies \frac{a_k - a_{k-1}}{a_{k-1}} = \frac{1}{k}$

$$\begin{aligned} \sum_{k=1}^n \frac{1}{1+k} &< \ln(n+1) < \sum_{k=1}^n \frac{1}{k} \implies \sum_{k=2}^{n+1} \frac{1}{k} < \ln(n+1) < \sum_{k=1}^n \frac{1}{k} \\ &\implies \sum_{k=2}^n \frac{1}{k} < \ln(n) < \sum_{k=1}^{n-1} \frac{1}{k} \end{aligned}$$

**Exercise 32.**

(1)  $L(x) = \frac{1}{\ln b} \ln x = \log_b x$

$\log_a x = c \log_b x \implies \log_a a = 1$  (There must be a unique real number s.t.  $L(a) = 1$ )

$\log_a x = \frac{1}{\log_b a} \log_b x \implies \log_b x = \log_b a \log_a x$

(2) Changing labels for  $a, b$ :  $\log_b x = \frac{\log_a x}{\log_a b}$

**Exercise 33.**  $\log_e 10 = 2.302585$

$\log_{10} e = \frac{\log_e e}{\log_e 10} \implies \log_e 10 = \frac{1}{\log_{10} e} = \frac{1}{2.302585} \simeq 0.43429$

**Exercise 34.** Given  $\int_x^{xy} f(t)dt = B(y)$   $f(2) = 2$ , We Want  $A(x) = \int_1^x f(t)dt$

$$\begin{aligned} \int_x^{xy} f(t)dt &= B(xy) - B(x) \\ \frac{d}{dx} \int_x^{xy} f(t)dt &= \frac{d}{dx} (B(xy) - B(x)) = \frac{dB(xy)}{d(xy)} y - \frac{dB(x)}{dx} = f(xy)y - f(x) = 0 \\ &\implies f(xy) = \frac{f(x)}{y} \\ A(x) &= \int_1^x f(t)dt = \int_1^x f\left(2\right)\left(\frac{t}{2}\right)dt = \int_1^x \frac{f(2)}{(t/2)}dt = \boxed{4 \ln x} \end{aligned}$$

**Exercise 35.** Given  $\int_1^{xy} f(t)dt = y \int_1^x f(t)dt + x \int_1^y f(t)dt$ , and letting  $F$  be the antiderivative of  $f$ ,

$$\begin{aligned} F(xy) - F(1) &= y(F(x) - F(1)) + x(F(y) - F(1)) \\ \xrightarrow{d/dx} f(xy)(y) &= y(f(x)) + \int_0^y f(t)dt \xrightarrow{d/dx} f'(xy)y^2 = y(f'(x)) \\ \xrightarrow{x=1} f'(y) &= \frac{(f'(1))}{y} \xrightarrow{\int} f(y) = k \ln y + C \xrightarrow{y=1} f(1) = 0 + C = 3 \\ \int_1^{xy} (k \ln t + 3) &= y \int_1^x (k \ln t + 3) + x \int_1^y (k \ln t + 3) \\ k((xy) \ln xy - (xy) + 1) + 3(xy - 1) &= y(k((x) \ln x - x + 1) + 3(x - 1)) + x(k(y \ln y - y + 1) + 3(y - 1)) \\ &\implies k - 3 = ky - 3y - kxy + kx + 3xy - 3x \\ y(3 - k) + xy(k - 3) + k - 3 + (3 - k)x &= 0 \implies k = 3 \\ &\implies f(x) = 3 \ln x + 3 \end{aligned}$$

**Exercise 36.**

## 6.11 Exercises - Polynomial approximations to the logarithm.

**Exercise 1.**

**Theorem 20** (Theorem 6.5). If  $0 < x < 1$  and if  $m \geq 1$ ,

$$\ln \frac{1+x}{1-x} = 2\left(x + \frac{x^3}{3} + \cdots + \frac{x^{2m-1}}{2m-1}\right) + R_m(x)$$



where

$$\frac{x^{2m+1}}{2m+1} < R_m(x) \leq \frac{2-x}{1-x} \frac{x^{2m+1}}{2m+1}$$

$$R_m(x) = E_{2m}(x) - E_{2m}(-x)$$

where  $E_{2m}(x)$  is the error term for  $\log 1-x$

$$m = 5, \quad x = \frac{1}{3}$$

$$\ln\left(\frac{4/3}{2/3}\right) = \ln 2 \simeq 2 \left( \frac{1}{3} + \frac{(\frac{1}{3})^3}{3} + \frac{(\frac{1}{3})^5}{5} + \frac{(\frac{1}{3})^7}{7} + \frac{(\frac{1}{3})^9}{9} \right) \simeq 0.693146047$$

The error for  $m = 5, \quad x = \frac{1}{3}$  is

$$\frac{(\frac{1}{3})^{11}}{11} \leq R_5(x) \leq \frac{5}{2} \frac{(\frac{1}{3})^{11}}{11}$$

**Exercise 2.**  $\ln\left(\frac{1+x}{1-x}\right) = \ln \frac{3}{2} = \ln 3 - \ln 2$

$$x = \frac{1}{5}$$

$$m = 5$$

$$\ln\left(\frac{1+x}{1-x}\right) = 2 \left( \frac{1}{5} + \frac{(1/5)^3}{3} + \frac{(1/5)^7}{7} + \frac{(1/5)^9}{9} \right) = 0.405465104$$

$$\frac{(1/5)^{11}}{11} \simeq 0.000000002 < R_5(x) \leq \frac{9}{4} \frac{(1/5)^{11}}{11} \simeq 0.000000004$$

$$\Rightarrow \boxed{\begin{array}{c} \log 3 \simeq 1.098611 \\ 1.098611666 < \log 3 < 1.098612438 \end{array}}$$

**Exercise 3.**  $x = \frac{1}{9} \quad \ln\left(\frac{1+\frac{1}{9}}{1-\frac{1}{9}}\right) = \ln\left(\frac{10}{8}\right) = \ln \frac{5}{4} = \ln 5 - 2 \ln 2$

For  $m = 2$

$$2 \left( \frac{1}{9} + \frac{(\frac{1}{9})^5}{3} + \frac{(\frac{1}{9})^7}{5} \right) = 0.2231435$$

$$\frac{(\frac{1}{9})^7}{7} \simeq 0.000000003 < R_3(x) \leq \frac{17}{8} \frac{(\frac{1}{9})^7}{7} \simeq 0.000000063$$

$$1.609437 < \log 5 < 1.609438$$

**Exercise 4.**  $x = \frac{1}{6} \quad \ln\left(\frac{1+\frac{1}{6}}{1-\frac{1}{6}}\right) = \ln\left(\frac{7}{5}\right) = \ln 7 - \ln 5.$

For  $m = 3,$

$$\ln \frac{1+x}{1-x} \simeq 2 \left( \frac{1}{6} + \frac{(\frac{1}{6})^3}{3} + \frac{(\frac{1}{6})^5}{5} \right) = 0.336471193$$

$$\frac{(\frac{1}{6})^{2(3)+1}}{2(3)+1} = 0.000000051$$

The error bounds are

$$\frac{(\frac{1}{6})^{2(3)+1}}{2(3)+1} \left( \frac{2 - \frac{1}{6}}{1 - \frac{1}{6}} \right) = 0.000001123$$

$$1.945908703 < \ln 7 < 1.945910316$$

**Exercise 5.**

$$0.6931460 < \ln 2 < 0.6931476$$

$$\ln 3 = 1.098614$$

$$\ln 4 = 1.386293$$

$$\ln 5 = 1.609436 \quad \ln 6 = \ln 2 + \ln 3 = 1.791700$$

$$\ln 7 = 1.945909$$

$$\ln 8 = 3 \ln 2 = 2.0794404$$

$$\ln 9 = 2 \ln 3 = 2.197228$$

$$\ln 10 = \ln 5 + \ln 2 = 1.302577$$

**6.17 Exercises - The exponential function, Exponentials expressed as powers of  $e$ , The definition of  $e^x$  for arbitrary real  $x$ , The definition of  $a^x$  for  $a > 0$  and  $x$  real, Differentiation and integration formulas involving exponentials.**

**Exercise 1.**  $f' = 3e^{3x-1}$  **Exercise 2.**  $8xe^{4x^2}$  **Exercise 3.**  $-2xe^{-x^2}$  **Exercise 4.**  $\frac{1}{2\sqrt{x}}e^{\sqrt{x}}$  **Exercise 5.**  $\frac{-1}{x^2}e^{-1/x}$  **Exercise 6.**

$\ln 22^x$  **Exercise 7.**  $(2x \ln 2)2^{x^2}$  **Exercise 8.**  $\cos x e^{\sin x}$  **Exercise 9.**  $-2 \cos x \sin x e^{\cos^2 x}$  **Exercise 10.**  $\frac{1}{x}e^{\log x}$  **Exercise 11.**

$e^x e^{e^x}$  **Exercise 12.**  $e^{e^x}(e^x e^{e^x})$  **Exercise 13.**  $\int x e^x dx = x e^x - e^x$  **Exercise 14.**  $\int x e^{-x} dx = -x e^{-x} + -e^{-x}$  **Exercise 15.**

$\int x^2 e^x = x^2 e^x - 2x e^x + 2e^x$  **Exercise 16.**  $\int x^2 e^{-2x} dx = \frac{x^2 e^{-2x}}{-2} + \frac{x e^{-2x}}{-2} + \frac{-e^{-2x}}{4}$

**Exercise 17.**  $\int e^{\sqrt{x}} = e^{\sqrt{x}}(2\sqrt{x}) - \int \frac{1}{\sqrt{x}} e^{\sqrt{x}} = e^{\sqrt{x}}(2\sqrt{x}) - 2e^{\sqrt{x}} = \boxed{2(\sqrt{x}e^{\sqrt{x}} - e^{\sqrt{x}})}$

**Exercise 18.**  $\int x^3 e^{-x^2}.$

$$x^2 e^{-x^2} = -2x^3 e^{-x^2} + 2x e^{-x^2}$$

$$e^{-x^2} = -2x e^{-x^2}$$

$$\frac{1}{-2}(x^2 e^{-x^2} + e^{-x^2})' = \frac{1}{-2}(2x e^{-x^2} + -2x^3 e^{-x^2} + -2x e^{-x^2}) = \boxed{\frac{x^2 e^{-x^2} + e^{-x^2}}{-2}}$$

**Exercise 19.**  $e^x = b + \int_a^b e^t dt = b + e^x - e^a, \quad e^a = b.$

**Exercise 20.**

$$A = \int e^{ax} \cos bx dx$$

$$B = \int e^{ax} \sin bx dx$$

$$A = \frac{e^{ax}}{a} \cos bx - \int -\frac{b \sin bx e^{ax}}{a} = \frac{e^{ax}}{a} \cos bx + \frac{b}{a} B \implies aA + bB = e^{ax} \cos bx + C$$

$$B = \frac{e^{ax}}{a} \sin bx - \int \frac{e^{ax}}{a} b \cos bx \implies aB + bA = e^{ax} \sin bx + C$$

$$A = \frac{1}{a^2 + b^2} (a e^{ax} \cos bx + b e^{ax} \sin bx)$$

$$B = \frac{-b e^{ax} \cos bx + a e^{ax} \sin bx}{a^2 + b^2}$$

**Exercise 21.**

$$\ln f = x \ln x; \quad \frac{f'}{f} = \ln x + 1; \quad f' = x^x (\ln x + 1)$$

**Exercise 22.**

$$\ln \frac{f'}{f} = \frac{1}{1+x} + \frac{1}{1+e^{x^2}}(2xe^{x^2}), \quad f' = 1 + e^{x^2} + 2(1+x)xe^{x^2}$$

**Exercise 23.**  $f = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$

$$\ln f = \ln(e^x - e^{-x}) - \ln(e^x + e^{-x})$$

$$\frac{f'}{f} = \frac{1}{e^x - e^{-x}}(e^x + e^{-x}) - \frac{1}{e^x + e^{-x}}(e^x - e^{-x})$$

$$f' = 1 - \left( \frac{e^x - e^{-x}}{e^x + e^{-x}} \right)^2$$

**Exercise 24.**  $f' = (x^{a^a})' + (a^{x^a})' + (a^{a^x})'.$

$$f_1 = x^{a^a}$$

$$f_2 = a^{x^a}$$

$$f_3 = a^{a^x}$$

$$\ln f_1 = a^a \ln x$$

$$\ln f_2 = x^a \ln a$$

$$\ln f_3 = a^x \ln a$$

$$\frac{f'_1}{f_1} = \frac{a^a}{x}$$

$$\frac{f'_2}{f_2} = a x^{a-1} \ln a$$

$$\frac{f'_3}{f_3} = (\ln a)^2 a^x$$

$$f'_1 = x^{a^a-1} a^a$$

$$f'_2 = a^{x^a+1} x^{a-1} \ln a$$

$$f'_3 = (\ln a)^2 a^{x+a^x}$$

$$\implies f' = x^{a^a-1}a^a + a^{x^a+1}x^{a-1}\ln a + (\ln a)^2a^{x+a^x}$$

**Exercise 25.**

$$\begin{aligned} e^f &= \ln(\ln x); \\ f'e^f &= \frac{1}{\ln x} \left( \frac{1}{x} \right) \\ f' &= \frac{1}{\ln(\ln x)} \left( \frac{1}{\ln x} \right) \left( \frac{1}{x} \right) \end{aligned}$$

**Exercise 26.**  $e^f = e^x + \sqrt{1 + e^{2x}}$

$$\begin{aligned} (e^f)f' &= e^x + \frac{e^{2x}}{\sqrt{1 + e^{2x}}} \\ f' &= \frac{\sqrt{1 + e^{2x}}e^x + e^{2x}}{\sqrt{1 + e^{2x}}(e^x + \sqrt{1 + e^{2x}})} = \frac{e^x}{\sqrt{1 + e^{2x}}} \end{aligned}$$

**Exercise 27.**  $\ln f = x^x \ln x$

$$\begin{aligned} \frac{f'}{f} &= (x^x)' \ln x + \frac{x^x}{x} = (x^x(\ln x + 1)) \ln x + x^{x-1} \\ f' &= \boxed{x^{x+x^x}(\ln x + 1) \ln x + x^{x^x+x-1}} \end{aligned}$$

**Exercise 28.**  $\ln f = x \ln(\ln x)$

$$\begin{aligned} \frac{f'}{f} &= \ln(\ln x) + \frac{1}{\ln x} \\ f' &= (\ln x)^x (\ln(\ln x) + \frac{1}{\ln x}) \end{aligned}$$

**Exercise 29.**  $\ln f = (\ln x) \ln x$

$$\begin{aligned} \frac{f'}{f} &= \frac{2 \ln x}{x} \\ f' &= \boxed{2x^{\log x - 1} \ln x} \end{aligned}$$

**Exercise 30.**  $\ln f = x \ln(\ln x) - \ln x \ln x = x \ln \ln x - (\ln x)^2$

$$\begin{aligned} \frac{f'}{f} &= \ln(\ln x) + \frac{1}{\ln x} - 2 \frac{\ln x}{x} \\ f' &= \frac{(\ln x)^x}{x^{\ln x}} \left( \ln \ln x + \frac{1}{\ln x} - \frac{2 \ln x}{x} \right) \end{aligned}$$

**Exercise 31.**

$$\begin{aligned} \ln f_1 &= \cos x \ln \sin x & \ln f_2 &= \sin x \ln \cos x \\ \frac{f'_1}{f_1} &= -\sin x \ln \sin x + \frac{\cos^2 x}{\sin x} & \frac{f'_2}{f_2} &= \cos x \ln \cos x + \frac{-\sin^2 x}{\cos x} \\ f'_1 &= -\sin x \cos x (\ln \sin x)^2 + \frac{\cos^3 x \ln \sin x}{\sin x} & f'_2 &= \sin x \cos x (\ln \cos x)^2 - \frac{\sin^3 x \ln \cos x}{\cos x} \\ \implies f &= \sin x \cos x (-(\ln \sin x)^2 + (\ln \cos x)^2) + \frac{\cos^3 x \ln \sin x}{\sin x} + \frac{-\sin^3 x \ln \cos x}{\cos x} \end{aligned}$$

**Exercise 32.**  $\ln f = \frac{1}{x} \ln x$

$$\frac{f'}{f} = \frac{-1}{x^2} \ln x + \frac{1}{x^2} \implies f' = -x^{1/x-2} \ln x + x^{1/x-2}$$

**Exercise 33.**  $\ln f = 2 \ln x + \frac{1}{3} \ln(3-x) - \ln(1-x) + \frac{-2}{3} \ln(3+x)$

$$\begin{aligned} \frac{f'}{f} &= \frac{2}{x} + \frac{-1}{3(3-x)} - \frac{-1}{1-x} - \frac{2}{3} \frac{1}{3+x} \\ f' &= 2 \frac{x(3-x)^{1/3}}{(1-x)(3+x)^{2/3}} + \frac{1}{3} \left( \frac{x^2(3-x)^{-2/3}}{(1-x)(3+x)^{2/3}} \right) + \frac{x^2(3-x)^{1/3}}{(1-x)^2(3+x)^{2/3}} - \frac{2}{3} \frac{x^2(3-x)^{1/3}}{(1-x)(3+x)^{5/3}} \\ &= \frac{x(18-12x+\frac{4}{3}x^2+\frac{2}{3}x^3)}{(1-x)^2(3+x)^{5/3}(3-x)^{2/3}} \end{aligned}$$

**Exercise 34.**  $\ln f = \sum_{i=1}^n b_i \ln(x - a_i)$

$$\frac{f'}{f} = \sum_{i=1}^n \frac{b_i}{x - a_i} \implies \boxed{f' = \sum_{j=1}^n \frac{b_j}{x - a_j} \prod_{i=1}^n (x - a_i)^{b_i}}$$

**Exercise 35.**

(1) Show that  $f' = rx^{r-1}$  for  $f = x^r$  holds for arbitrary real  $r$ .

$$x^r = e^{r \ln x}$$

$$(e^{r \ln x})' = e^{r \ln x} \frac{r}{x} = rx^{r-1}$$

(2) For  $x \leq 0$ , by inspection of  $x^r = e^{r \log x}$ , then if  $x^r > 0$ , then the equality would remain valid. So then  $x^r = |x|^r = |x|^r$  and so

$$\ln |f(x)| = r \ln |x|$$

$$\frac{f'(x)}{f(x)} = r \frac{1}{x} \implies f'(x) = rx^{r-1}$$

**Exercise 36.** Use the definition  $a^x = e^{x \log a}$

(1)  $\log a^x = x \log a$

Taking the exponential is a well-defined inverse function to log so taking the log of both sides of the definition, we get  $\log a^x = x \log a$

(2)  $(ab)^x = a^x b^x$

$$(ab)^x = e^{x \log ab} = e^{x(\log a + \log b)} = a^x b^x$$

(3)  $a^x a^y = a^{x+y}$

$$a^{x+y} = e^{(x+y) \log a} = e^{x \log a} e^{y \log a} = a^x a^y$$

(4)  $(a^x)^y = (a^y)^x = a^{xy}$

$$(a^x)^y = e^{xy \log a} = (a^y)^x = a^{xy}$$

(5) If  $x = \log_a y$ ,

Using the definition

$$\log_b x = \frac{\log x}{\log b} \quad \text{if } b > 0, b \neq 1, \quad x > 0$$

so then

$$\log_a y = \frac{\log y}{\log a} = x \implies \log y = x \log a \implies e^{x \log a} = e^{\log y} = y = a^x$$

If  $y = a^x$ ,

$$\log y = x \log a \implies x = \frac{\log y}{\log a} = \log_a y$$

**Exercise 37.** Let  $f(x) = \frac{1}{2}(a^x + a^{-x})$  if  $a > 0$ .

$$f(x+y) = \frac{1}{2}(a^{x+y} + a^{-(x+y)})$$

$$f(x+y) + f(x-y) = \frac{1}{2}(a^{x+y} + a^{-(x+y)} + a^{x-y} + a^{-(x-y)})$$

$$f(x)f(y) = \frac{1}{4}(a^x + a^{-x})(a^y + a^{-y}) = \frac{1}{4}(a^{x+y} + a^{-x-y} + a^{-(x-y)} + a^{(x-y)})$$

**Exercise 38.**

$$f(x) = e^{cx}; f'(x) = ce^{cx}; \boxed{f'(0) = c}$$

$$\lim_{x \rightarrow 0} \frac{e^{cx} - 1}{x} = c \left( \lim_{x \rightarrow 0} \frac{e^{cx} - 1}{cx} \right) = c \left( \lim_{cx \rightarrow 0} \frac{e^{cx} - 1}{cx} \right) = c \frac{df(cx)}{d(cx)}(0) = \frac{d}{dx}(e^{cx})(0)$$

$$\lim_{x \rightarrow 0} \frac{e^{cx} - 1}{x} = f'(0) = c$$

**Exercise 39.**

$$g(x) = f(x)e^{-cx}$$

$$g'(x) = f'e^{-cx} + -cg = cg - cg = 0$$

$$\boxed{f = Ke^{kx}}$$

**Exercise 40.** Let  $f$  be a function defined everywhere on the real axis. Suppose also that  $f$  satisfies the functional equation

$$f(x+y) = f(x)f(y) \text{ for all } x \text{ and } y$$

(1)

$$f(0) = f(0)f(0) = f^2(0) \quad \begin{array}{l} \text{If } f(0) = 0, \text{ then we're done} \\ \text{If } f(0) \neq 0 \text{ then } f(0) = 1 \text{ (by dividing both sides by } f(0) \text{ )} \end{array}$$

(2) Take the derivative with respect to  $x$  on both sides of the functional equation.

$$\frac{df(x+y)}{d(x+y)} \frac{d(x+y)}{dx} = \frac{df(x)}{dx} f(y) \implies \frac{d(f(x+y))}{d(x+y)} = f'(x) \frac{f(x+y)}{f(x)}$$

$$\boxed{\text{Let } y = -x + y}$$

$$\frac{df(y)}{dy} = f'(x) \frac{f(y)}{f(x)} \implies f'(x)f(y) = f'(y)f(x)$$

$$(3) \quad \frac{f'(x)}{f(x)} = \frac{f'(y)}{f(y)} \quad \forall x, y$$

The only way they could do that for any arbitrary  $x$ , for any arbitrary  $y$  they one could choose on *either* side, is for them to both equal a constant

$$\implies \frac{f'(y)}{f(y)} = c$$

(4) Referring to Exercise 39 of the same section,  $f = e^{cx}$  since  $f'(0) = 1$

**Exercise 41.**

(1)

$$\begin{array}{ll} f = e^x - 1 - x & f(0) = e^0 - 1 - 0 = 0 \\ f' = e^x - 1 \geq 0 \text{ if } x \geq 0 & \left\{ \begin{array}{ll} e^x > 1 + x & \text{for } x > 0 \\ e^{-x} > 1 - x & \text{for } x < 0 \end{array} \right. \end{array}$$

(2)

$$\begin{array}{l} \int_0^x e^t = e^x - 1 > x + \frac{1}{2}x^2 \implies e^x > 1 + x + \frac{1}{2}x^2 \\ -e^{-x} > -1 + x - \frac{x^2}{2} \implies e^{-x} < 1 - x + \frac{x^2}{2} \end{array}$$

(3)

$$\begin{array}{l} \int_0^x e^t = e^x - 1 > x + \frac{1}{2}x^2 + \frac{1}{3*2}x^3 \implies e^x > 1 + x + \frac{1}{2}x^2 + \frac{1}{3*2}x^3 \\ -e^{-x} > -1 + x - \frac{x^2}{2} + \frac{x^3}{3*2} \implies e^{-x} < 1 - x + \frac{x^2}{2} - \frac{x^3}{3*2} \end{array}$$

(4) Suppose the  $n$ th case is true.

$$\begin{array}{l} e^x > \sum_{j=0}^n \frac{x^j}{j!} \\ e^x > 1 + \sum_{j=0}^n \frac{x^{j+1}}{(j+1)!} = 1 + \sum_{j=1}^{n+1} \frac{x^j}{j!} \\ -e^{-x} + 1 \left\{ \begin{array}{l} > \sum_{j=0}^{2m+1} \frac{x^{j+1}}{(j+1)!} \\ < \sum_{j=0}^{2m} \frac{x^{j+1}}{(j+1)!} \end{array} \right. = e^{-x} \left\{ \begin{array}{l} < \sum_{j=0}^{2m+2} \frac{x^j}{j!} \\ > \sum_{j=0}^{2m+1} \frac{x^j}{j!} \end{array} \right. \end{array}$$

**Exercise 42.** Using the result from Exercise 41,

$$\left(1 + \frac{x}{n}\right)^n = \sum_{j=0}^n \binom{n}{j} 1^{n-j} \left(\frac{x}{n}\right)^j = \sum_{j=0}^n \frac{n!}{(n-j)!j!} \frac{x^j}{n^j} = \sum_{j=0}^n \frac{n(n-1)\dots(n-j+1)}{j!} \frac{x^j}{n^j} < \sum_{j=0}^n \frac{x^j}{j!} < e^x$$

If you make this clever observation, the second inequality is easy to derive.

$$\begin{aligned} x > 0 \quad \frac{x}{n} > 0 \\ e^{-\frac{x}{n}} > 1 - \frac{x}{n} &\implies \left(e^{-x/n}\right)^n > \left(1 - \frac{x}{n}\right)^n \\ e^{-x} > (1 - x/n)^n &\implies e^x < (1 - x/n)^{-n} \end{aligned}$$

**Exercise 43.**  $f(x, y) = x^y = e^{y \ln x}$

$$\partial_x f = x^y y / x$$

$$\partial_y f = x^y \ln x$$

## 6.19 Exercises - The hyperbolic functions.

**Exercise 7.**

$$2 \sinh x \cosh x = 2 \frac{e^x - e^{-x}}{2} \frac{e^x + e^{-x}}{2} = \frac{1}{2} (e^{2x} - e^{-2x}) = \sinh 2x$$

**Exercise 8.**

$$\cosh^2 x + \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 + \left(\frac{e^x - e^{-x}}{2}\right)^2 = \frac{1}{4} (e^{2x} + 2 + e^{-2x} + e^{2x} - 2 + e^{-2x}) = \cosh 2x$$

**Exercise 9.**

$$\cosh x + \sinh x = \frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2} = e^x$$

**Exercise 10.**

$$\cosh x - \sinh x = \frac{e^x + e^{-x}}{2} - \left(\frac{e^x - e^{-x}}{2}\right) = e^{-x}$$

**Exercise 11.** Use induction.

$$\begin{aligned} (\cosh x + \sinh x)^2 &= \cosh^2 x + 2 \sinh x \cosh x + \sinh^2 x = \cosh 2x + \sinh 2x \\ (\cosh x + \sinh x)^{n+1} &= (\cosh x + \sinh x)(\cosh nx + \sinh nx) = \\ &= \cosh nx \cosh x + \cosh nx \sinh x + \sinh nx \cosh x + \sinh x \sinh nx = \\ &= \frac{e^{nx} + e^{-nx}}{2} \frac{e^x + e^{-x}}{2} + \frac{e^{nx} + e^{-nx}}{2} \frac{e^x - e^{-x}}{2} + \\ &= \frac{e^{nx} - e^{-nx}}{2} \frac{e^x + e^{-x}}{2} + \frac{e^{nx} - e^{-nx}}{2} \frac{e^x - e^{-x}}{2} \\ &= \cosh (n+1)x + \sinh (n+1)x \end{aligned}$$

**Exercise 12.**

$$\cosh 2x = \cosh^2 x + \sinh^2 x = 1 + 2 \sinh^2 x$$

## 6.22 Exercises - Derivatives of inverse functions, Inverses of the trigonometric functions.

**Exercise 1.**

$$(\cos x)' = -\sin x = -\sqrt{1 - \cos^2 x} D \arccos x = \frac{1}{-\sqrt{1 - x^2}} \quad -1 < x < 1$$

**Exercise 2.**

$$\begin{aligned} (\tan x)' &= \sec^2 x = \frac{\sin^2 x + \cos^2 x}{\cos^2 x} = \tan^2 x + 1 \\ D \arctan x &= \frac{1}{1 + x^2} \end{aligned}$$

**Exercise 3.**

$$(\cot x)' = -\csc^2 x = -\frac{(\sin^2 x + \cos^2 x)}{\sin^2 x} = -(1 + \cot^2 x) \implies \operatorname{arccot} x = -\frac{1}{1 + x^2}$$

**Exercise 4.**

$$(\sec y)' = \tan y \sec y = \sqrt{\sec^2 y - 1} \sec y; |\sec y| > 1 \forall y \in \mathbb{R}$$

If we choose to restrict  $y$  such that  $0 \leq y \leq \pi$ , then  $(\sec y)' > 0$ . Then we must make  $\sec y \rightarrow |\sec y|$ .

$$D \operatorname{arcsec} x = \frac{1}{|x| \sqrt{x^2 - 1}}$$

**Exercise 5.**

$$(\csc y)' = -\cot x \csc x = -\csc y(\sqrt{\csc^2 y - 1})$$

$$\text{Let } y \text{ such that } \frac{-\pi}{2} < y < \frac{\pi}{2} \quad (\csc y) < 0$$

$$\operatorname{Darccsc} x = \frac{1}{-|x|\sqrt{x^2 - 1}}$$

**Exercise 6.**

$$(\operatorname{xarccot} x)' = \operatorname{arccot} x - \frac{x}{1 + x^2}$$

$$\left( \frac{1}{2} \ln(1 + x^2) \right)' = \frac{1x}{(1 + x^2)}$$

$$\int \operatorname{arccot} x = \operatorname{xarccot} x + \frac{1}{2} \ln(1 + x^2) + C$$

**Exercise 7.**

$$(\operatorname{xarcsec} x)' = \operatorname{arcsec} x + \frac{x}{|x|\sqrt{x^2 - 1}}$$

$$\left( \frac{x}{|x|} \ln |x + \sqrt{x^2 - 1}| \right)' = \begin{cases} \frac{1 + \frac{x}{\sqrt{x^2 - 1}}}{|x + \sqrt{x^2 - 1}|} & x > 1 \\ -\frac{1 + \frac{x}{\sqrt{x^2 - 1}}}{|x + \sqrt{x^2 - 1}|} & x < -1 \end{cases} = \frac{x}{|x|\sqrt{x^2 - 1}}$$

$$\Rightarrow \int \operatorname{arcsec} x dx = \operatorname{xarcsec} x - \frac{x}{|x|} \log |x + \sqrt{x^2 - 1}| + C$$

Take a note of this exercise. **When dealing with**  $(\mp x^2 \pm 1)^{\frac{2j+1}{2}}$ ;  $j \in \mathbb{Z}$ ; **try**  $x \pm \sqrt{x^2 \pm 1}$  **combinations**. It'll work out.

**Exercise 8.**

$$(\operatorname{xarccsc} x)' = \operatorname{arccsc} x + \frac{x}{-|x|\sqrt{x^2 - 1}}$$

$$\left( \frac{x}{|x|} \ln |x + \sqrt{x^2 - 1}| \right)' = \begin{cases} \frac{1}{x + \sqrt{x^2 - 1}} \left( 1 + \frac{x}{\sqrt{x^2 - 1}} \right) = \frac{1}{\sqrt{x^2 - 1}} & x > 1 \\ \frac{-1}{\sqrt{x^2 - 1}} & x < -1 \end{cases}$$

$$\Rightarrow \int \operatorname{arccsc} x = \operatorname{xarccsc} x + \frac{x}{|x|} \ln |x + \sqrt{x^2 - 1}|$$

**Exercise 9.**

$$(x(\arcsin x)^2)' = (\arcsin x)^2 + \frac{2x \arcsin x}{\sqrt{1 - x^2}}$$

$$\left( \sqrt{1 - x^2} \arcsin x \right)' = \frac{-x}{\sqrt{1 - x^2}} \arcsin x + 1$$

$$\boxed{\int (\arcsin x)^2 = x(\arcsin x)^2 + 2\sqrt{1 - x^2} \arcsin x - 2x}$$

**Exercise 10.**

$$\left( \frac{-\arcsin x}{x} \right)' = \frac{1}{x^2} \arcsin x - \frac{1}{x\sqrt{1 - x^2}}$$

I would note how  $x$  is in the denominator of the second term. Again, reiterating,

$$(\sqrt{1 - x^2})' = \frac{-x}{\sqrt{1 - x^2}}$$

$$(y \pm \sqrt{\pm 1 \mp x^2})(y \mp \sqrt{\pm 1 \mp x^2}) = y^2 - (\pm 1 \mp x^2)$$

Multiply by its “conjugate.” As we see, choose  $y$  appropriately to get the desired denominator (that’s achieved after differentiation). Here, pick  $y = 1$ .

$$(\ln(1 + \sqrt{1 - x^2}))' = \frac{1}{1 + \sqrt{1 - x^2}} \left( \frac{-x}{\sqrt{1 - x^2}} \right) = \frac{-x(1 - \sqrt{1 - x^2})}{\sqrt{1 - x^2}(x^2)} = \frac{-(1 - \sqrt{1 - x^2})}{x\sqrt{1 - x^2}}$$

$$\Rightarrow \int \frac{\arcsin x}{x^2} = \ln \left| \frac{1 - \sqrt{1 - x^2}}{x} \right| - \frac{\arcsin x}{x} + C$$

**Exercise 11.**

(1)

$$D\left(\operatorname{arccot} x - \arctan \frac{1}{x}\right) = \frac{-1}{x^2 + 1} - \frac{1}{1 + \left(\frac{1}{x}\right)^2} \left(\frac{-1}{x^2}\right) = 0$$

(2)  $\operatorname{arccot} x - \arctan \frac{1}{x} = C$

Now  $\operatorname{arccot} x = \frac{\pi}{2} - \arctan x$ .

$$\frac{\pi}{2} - \arctan x - \arctan \frac{1}{x} = C \implies \frac{\pi}{2} - C = \arctan x + \arctan \frac{1}{x}$$

$$x \rightarrow \infty \implies \frac{\pi}{2} - C = \frac{\pi}{2} + 0 \implies C = 0$$

but  $x \rightarrow -\infty$

$$\frac{\pi}{2} - C = -\frac{\pi}{2} + 0 \implies C = \pi$$

There are problems with the choice of branches for  $\operatorname{arccot} x$ ,  $\arctan \frac{1}{x}$ , even though the derivatives work in all cases.

**Exercise 12.**

$$f' = \frac{1}{\sqrt{1 - \left(\frac{x}{2}\right)^2}} \left(\frac{1}{2}\right)$$

**Exercise 13.**

$$f' = \frac{-1}{\sqrt{1 - \left(\frac{1-x}{\sqrt{2}}\right)^2}} \left(\frac{-1}{\sqrt{2}}\right) = \frac{1}{\sqrt{1 + 2x - x^2}}$$

**Exercise 14.**  $f = \arccos \frac{1}{x}$ .

$$f' = \frac{-1}{\sqrt{1 - \left(\frac{1}{x}\right)^2}} \left(\frac{-1}{x^2}\right) = \frac{1}{\sqrt{x^2 - 1}|x|}$$

**Exercise 15.**

$$f(x) = \arcsin(\sin x) = \frac{1}{\sqrt{1 - \sin^2 x}} \cos x = \frac{\cos x}{|\cos x|}$$

**Exercise 16.**

$$\frac{1}{2 \sqrt{x}} - \frac{1}{x+1} \frac{1}{2\sqrt{x}} = \frac{\sqrt{x}}{2(x+1)}$$

**Exercise 17.**

$$\frac{1}{1+x^2} + \frac{x^2}{1+x^6}$$

**Exercise 18.**

$$\frac{1}{\sqrt{1 - \left(\sqrt{\frac{1-x^2}{1+x^2}}\right)^2}} \left(\frac{-2x(2)}{(1+x^2)^2}\right) = \frac{\sqrt{1+x^2}}{\sqrt{1+x^2 - (1-x^2)}} \frac{-4x}{(1+x^2)^2} = \frac{-4}{(1+x^2)^{3/2}\sqrt{2}}$$

**Exercise 19.**  $f = \arctan \tan^2 x$

$$\frac{1}{1 + \tan^4 x} (2 \tan x \sec^2 x) = \frac{2 \tan x \sec^2 x}{1 + \tan^4 x}$$

**Exercise 20.**

$$f' = \frac{1}{1 + (x + \sqrt{1+x^2})^2} \left(1 + \frac{x}{\sqrt{1+x^2}}\right) = \frac{\sqrt{1+x^2} + x}{1 + (x + \sqrt{1+x^2})^2}$$

**Exercise 21.**

$$f' = \frac{1}{\sqrt{1 - (\sin^2 x + \cos^2 x - 2 \sin x \cos x)}} (\cos x + \sin x) = \frac{\cos x + \sin x}{\sqrt{2 \sin x \cos x}}$$

**Exercise 22.**

$$f' = (\arccos \sqrt{1-x^2})' = \frac{1}{-\sqrt{1 - (1-x^2)}} = -\frac{1}{|x|}$$

**Exercise 23.**

$$f' = \frac{1}{1 + \left(\frac{1+x}{1-x}\right)^2} \left(\frac{2}{(1-x)^2}\right) = \frac{2}{(1-x)^2 + (1+x)^2} = \frac{2}{(1-2x+x^2+1+2x+x^2)} = \frac{1}{1+x^2}$$



**Exercise 24.**  $f = (\arccos(x^2))^{-2}$

$$f' = -2(\arccos x^2)^{-3} \left( \frac{-1}{\sqrt{1-x^2}} \right) (2x) = \frac{4x(\arccos x^2)^{-3}}{\sqrt{1-x^4}}$$

**Exercise 25.**

$$f' = \left( \frac{1}{\arccos \frac{1}{\sqrt{x}}} \right) \left( \frac{-1}{\sqrt{1-\frac{1}{x}}} \right) \left( \frac{-1}{2x^{3/2}} \right) = \frac{1}{2 \arccos \frac{1}{\sqrt{x}} \sqrt{x^3-x^2}}$$

**Exercise 26.**  $\frac{dy}{dx} = \frac{x+y}{x-y}$ .

$$\begin{aligned} \left( \arctan \frac{y}{x} \right)' &= \frac{1}{1 + \left( \frac{y}{x} \right)^2} \left( \frac{y'x - y}{x^2} \right) = \left( \frac{1}{2} \ln(x^2 + y^2) \right)' = \frac{1}{2} \frac{1}{(x^2 + y^2)} (2x + 2yy') \\ &\Rightarrow y' = \frac{x+y}{x-y} \end{aligned}$$

**Exercise 27.**

$$\begin{aligned} \ln y &= \ln(\arcsin x) - \frac{1}{2} \ln(1-x^2) \\ \frac{y'}{y} &= \frac{1}{\arcsin x} \left( \frac{1}{\sqrt{1-x^2}} \right) - \frac{1}{2} \frac{1}{1-x^2} (-2x) = \frac{1}{\arcsin x \sqrt{1-x^2}} + \frac{x}{1-x^2} \\ y &= \frac{\arcsin x}{\sqrt{1-x^2}} \\ y' &= \left( \frac{1}{1-x^2} \right) + \frac{(\arcsin x)x}{(1-x^2)^{3/2}} = \frac{\sqrt{1-x^2} + x(\arcsin x)}{(1-x^2)^{3/2}} \\ \ln y' &= \ln(\sqrt{1-x^2} + x \arcsin x) - \frac{3}{2} \ln(1-x^2) \\ \frac{y''}{y'} &= \frac{1}{\sqrt{1-x^2} + x \arcsin x} \left( \frac{-x}{\sqrt{1-x^2}} + \arcsin x + \frac{x}{\sqrt{1-x^2}} \right) - \frac{3}{2} \frac{(-2x)}{1-x^2} \\ y'' &= y' \frac{\arcsin x}{\sqrt{1-x^2} + x \arcsin x} + \frac{y' 3x}{1-x^2} = \boxed{\frac{\arcsin x}{(1-x^2)^{3/2}} + \frac{(\sqrt{1-x^2} + x \arcsin x)(3x)}{(1-x^2)^{3/2}}} \end{aligned}$$

**Exercise 28.**

$$\begin{aligned} f' &= \frac{1}{1+x^2} - 1 + x^2 = \frac{1-1-x^2+x^4}{1+x^2} = \frac{x^4}{1+x^2} \geq 0 \forall x \\ \text{since } f(0) &= \arctan 0 - 0 + 0 = 0, \quad \arctan x > x - \frac{x^3}{3}, \quad \forall x > 0 \end{aligned}$$

**Exercise 29.**

$$\int \frac{dx}{\sqrt{a^2-x^2}}, a \neq 0 \Rightarrow \boxed{\arcsin \frac{x}{a}}$$

**Exercise 30.**

$$\int \frac{dx}{\sqrt{2-(x+1)^2}} = \int \frac{dx}{\sqrt{2}\sqrt{1-\left(\frac{x+1}{\sqrt{2}}\right)^2}} = \arcsin \frac{x+1}{\sqrt{2}}$$

**Exercise 31.**

$$\int \frac{dx}{a^2 \left( 1 + \left( \frac{x}{a} \right)^2 \right)} = \frac{1}{a} \arctan \frac{x}{a}$$

**Exercise 32.**

$$\frac{dx}{a \left( 1 + \left( \frac{\sqrt{ba}}{\sqrt{a}} \right)^2 \right)} = \frac{1}{\sqrt{ba}} \arctan \frac{\sqrt{b}x}{\sqrt{a}}$$

**Exercise 33.**

$$\int \frac{dx}{\left(x - \frac{1}{2}\right)^2 + \frac{7}{4}} = \frac{4}{7} \int \frac{dx}{\left(\frac{2}{\sqrt{7}} \left(x - \frac{1}{2}\right)\right)^2 + 1} = \boxed{\frac{2}{\sqrt{7}} \arctan \frac{2\left(x - \frac{1}{2}\right)}{\sqrt{7}}}$$

**Exercise 34.**

$$\begin{aligned}\left(\frac{x^2 \arctan x}{2}\right)' &= x \arctan x + \frac{x^2}{2} \frac{1}{1+x^2} = x \arctan x + \frac{1}{2} \left(1 - \frac{1}{1+x^2}\right) \\ \left(\frac{1}{2}(x - \arctan x)\right)' &= \frac{1}{2} \left(1 - \frac{1}{1+x^2}\right) \\ \int x \arctan x &= x \arctan x + -\frac{1}{2}(x - \arctan x)\end{aligned}$$

**Exercise 35.**

$$\begin{aligned}\left(\frac{x^3}{3} \arccos x\right)' &= x^2 \arccos x + \frac{x^3}{3} \frac{-1}{\sqrt{1-x^2}} \\ (x^2 \sqrt{1-x^2})' &= 2x \sqrt{1-x^2} + -\frac{x^3}{\sqrt{1-x^2}} \\ ((1-x^2)^{3/2})' &= \frac{3}{2}(-2x)(1-x^2)^{1/2} = -3x(1-x^2)^{1/2}\end{aligned}$$

$$\boxed{\int x^2 \arccos x = \frac{x^3}{3} \arccos x - \frac{1}{3} x^2 \sqrt{1-x^2} - \frac{9}{2} (1-x^2)^{3/2}}$$

**Exercise 36.**

$$\begin{aligned}\left(\frac{x^2(\arctan x)^2}{2}\right)' &= x(\arctan x)^2 + x^2 \left(\frac{1}{1+x^2}\right) \arctan x = x(\arctan x)^2 + \left(1 - \left(\frac{1}{1+x^2}\right)\right) \arctan x \\ \left(\frac{(\arctan x)^2}{2}\right)' &= \frac{\arctan x}{1+x^2} \\ (x \arctan x)' &= \arctan x + \frac{x}{1+x^2} \\ \int x(\arctan x)^2 dx &= \frac{x^2(\arctan x)^2}{2} - \left(x \arctan x - \frac{\ln(1+x^2)}{2}\right) + \frac{(\arctan x)^2}{2}\end{aligned}$$

**Exercise 37.**

$$\begin{aligned}(\arctan \sqrt{x})' &= \left(\frac{1}{1+x}\right) \left(\frac{1}{2\sqrt{x}}\right) \\ (x \arctan \sqrt{x})' &= \arctan \sqrt{x} + \frac{\sqrt{x}}{2(1+x)} = \arctan \sqrt{x} + \frac{1}{2} \left(\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{x}(1+x)}\right) \\ (x \arctan \sqrt{x} + \arctan \sqrt{x} - x^{1/2})' &= \arctan \sqrt{x} + 0\end{aligned}$$

$$\boxed{\int \arctan \sqrt{x} = x \arctan \sqrt{x} + \arctan \sqrt{x} - x^{1/2}}$$

**Exercise 38.** From the previous exercise,

$$\int \frac{\arctan \sqrt{x}}{\sqrt{x}(1+x)} dx = (\arctan \sqrt{x})^2$$

**Exercise 39.** Let  $x = \sin u$

$$\int \sqrt{1-x^2} dx = \int \cos^2 u du = \frac{u}{2} + \frac{\sin 2u}{4} = \frac{\arcsin x}{2} + \frac{x\sqrt{1-x^2}}{4}$$

**Exercise 40.**

$$\begin{aligned}\int \frac{x e^{\arctan x}}{(1+x^2)^{3/2}} \\ \left(\frac{e^{\arctan x}}{\sqrt{1+x^2}}\right)' &= \frac{-x e^{\arctan x}}{(1+x^2)^{3/2}} + \frac{e^{\arctan x}}{(1+x^2)^{3/2}} \\ \left(\frac{x e^{\arctan x}}{\sqrt{1+x^2}}\right)' &= \frac{e^{\arctan x}}{\sqrt{1+x^2}} + -\frac{x^2 e^{\arctan x}}{(1+x^2)^{3/2}} + \frac{x e^{\arctan x}}{(1+x^2)^{3/2}} = \frac{e^{\arctan x}}{(1+x^2)^{3/2}} + \frac{x e^{\arctan x}}{(1+x^2)^{3/2}} \\ \frac{1}{2} \left(\frac{x e^{\arctan x}}{\sqrt{1+x^2}} - \frac{e^{\arctan x}}{\sqrt{1+x^2}}\right)' &= \frac{x e^{\arctan x}}{(1+x^2)^{3/2}}\end{aligned}$$

**Exercise 41.** From the previous exercise,

$$\frac{1}{2} \left( \frac{x e^{\arctan x}}{\sqrt{1+x^2}} + \frac{e^{\arctan x}}{\sqrt{1+x^2}} + C \right)$$

**Exercise 42.**

$$\begin{aligned} \text{Since } - \left( \frac{x(1+x^2)^{-1}}{2} \right)' &= \frac{x^2}{(1+x^2)^2} - \frac{1}{2(1+x^2)} \\ \int \frac{x^2}{(1+x^2)^2} dx &= \frac{-x}{2(1+x^2)} + \frac{1}{2} \arctan x \end{aligned}$$

**Exercise 43.**  $\arctan e^x$ .

**Exercise 44.**

$$\begin{aligned} \int \frac{\operatorname{arccote}^x}{e^x} dx \\ (\operatorname{arccote}^x)' &= \frac{-e^x}{1+e^{3x}} \\ -(e^{-x} \operatorname{arccote}^x)' &= e^{-x} \operatorname{arccote}^x + \frac{e^{-x}(-1)e^x}{1+e^{2x}} = e^{-x} \operatorname{arccote}^x + - \left( 1 - \frac{e^{2x}}{1+e^{2x}} \right) \\ (\ln(1+e^{2x}))' &= \frac{2e^{2x}}{1+e^{2x}} \\ \boxed{(-e^{-x} \operatorname{arccote}^x + x - \frac{1}{2} \ln(1+e^{2x}))'} &= e^{-x} \operatorname{arccote}^x \end{aligned}$$

**Exercise 45.**

$$\int \sqrt{\frac{a+x}{a-x}} dx = \int \frac{a+x}{\sqrt{a^2-x^2}} dx = \int \left( \frac{1a}{\sqrt{1-\left(\frac{x}{a}\right)^2}} + \frac{x}{\sqrt{a^2-x^2}} \right) dx = a \arcsin \frac{x}{a} + -\sqrt{a^2-x^2}$$

**Exercise 46.**

$$\begin{aligned} \int \sqrt{x-a} \sqrt{b-x} dx &= \int \sqrt{bx-ab-x^2+ax} dx = \\ &= \int \sqrt{-\left(x-\left(\frac{a+b}{2}\right)\right)\left(x-\left(\frac{a+b}{2}\right)\right) + \frac{a^2+b^2}{4} - \frac{2ab}{4}} = \\ &= \int \sqrt{\left(\frac{a-b}{2}\right)^2 - \left(x-\left(\frac{a+b}{2}\right)\right)^2} = \left(\frac{a-b}{2}\right) \int \sqrt{1-\left(\frac{x-\left(\frac{a+b}{2}\right)}{\left(\frac{a-b}{2}\right)}\right)^2} dx = \\ &= \left(\frac{a-b}{2}\right)^2 \int \sqrt{1-u^2} = \\ &= \boxed{\left(\frac{a-b}{2}\right)^2 \frac{\arcsin\left(\frac{2x-(a+b)}{a-b}\right)}{2} + \frac{2x-(a+b)}{2(a-b)^2} \sqrt{(a-b)^2 - (2x-(a+b))^2}} \end{aligned}$$

Since, recall,

$$\left( \frac{\arcsin x}{2} + \frac{1}{2} x \sqrt{1-x^2} \right)' = \frac{1}{2} \frac{1}{\sqrt{1-x^2}} + \frac{\sqrt{1-x^2}}{2} + \frac{1}{4} \frac{x(-2x)}{\sqrt{1-x^2}} = \sqrt{1-x^2}$$

**Exercise 47. Wow!**

$$\boxed{\int \frac{dx}{\sqrt{(x-a)(b-x)}}}$$

$$x-a = (b-a) \sin^2 u$$

$$dx = (b-a)(2) \sin u \cos u du$$

$$b-x = (a-b) \sin^2 u + b-a = \boxed{(b-a)(\cos^2 u)}$$

$$\int \frac{dx}{\sqrt{(x-a)(b-x)}} = \int \frac{(b-a)(2) \sin u \cos u du}{\sqrt{b-a} \cos u \sqrt{b-a} \sin u} = 2u = \boxed{2 \arcsin \sqrt{\frac{x-a}{b-a}}}$$

## 6.25 Exercises - Integration by partial fractions, Integrals which can be transformed into integrals of rational functions.

**Exercise 1.**  $\int \frac{2x+3}{(x-2)(x+5)} = \int \left( \frac{1}{x-2} \right) + \left( \frac{1}{x+5} \right) = \ln(x-2) + \ln(x+5)$

**Exercise 2.**  $\int \frac{x dx}{(x+1)(x+2)(x+3)}$

$$\begin{aligned} \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{x+3} &= A(x^2+5x+6) + B(x^2+4x+3) + C(x^2+3x+2) \\ \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 5 & 4 & 3 \\ 6 & 3 & 2 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} &= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 1 & | & 0 \\ 5 & 4 & 3 & | & 1 \\ 6 & 3 & 2 & | & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & | & -1/2 \\ & 1 & 0 & | & 2 \\ & 0 & 1 & | & -3/2 \end{bmatrix} \\ A &= -1/2, \quad B = 2, \quad C = -3/2 \\ \Rightarrow \boxed{\frac{-1}{2} \ln(x+1) + 2 \ln(x+2) + \frac{-3}{2} \ln(x+3)} \end{aligned}$$

**Exercise 3.**  $\int \frac{x}{(x-2)(x-1)} = \int \frac{2}{x-2} + \frac{-1}{x-1} = 2 \ln x - 2 - \ln(x-1)$

**Exercise 4.**  $\int \frac{x^4+2x-6}{x^3+x^2-2x} dx$

$$\begin{aligned} \frac{x^4+2x-6}{x^3+x^2-2x} &= x-1 + \frac{3(x^2-2)}{x^3+x^2-2x} \quad (\text{do long division}) \\ \int x-1 + \frac{3(x^2-2)}{x(x+2)(x-1)} &= \frac{1}{2}x^2 - x + 3 \int \frac{x^2-2}{x(x+2)(x-1)} \\ \int \frac{x^2-2}{x(x+2)(x-1)} &= \int \frac{1}{x} + \frac{1/3}{x+2} + \frac{-1/3}{x-1} = \ln x + \frac{1}{3} \ln x + 2 - \frac{1}{3} \ln x - 1 \\ \Rightarrow \boxed{\frac{1}{2}x^2 - x + 3 \ln x + \ln x + 2 - \ln x - 1} \end{aligned}$$

**Exercise 5.**  $\int \frac{8x^3+7}{(x+1)(2x+1)^3} dx$

$$\begin{aligned} \frac{8x^3+7}{(x+1)(2x+1)^3} &= \frac{A}{(2x+1)^3} + \frac{B}{(2x+1)^2} + \frac{C}{(2x+1)} + \frac{D}{(x+1)} \\ 8x^3+7 &= A(x+1) + B(2x^2+3x+1) + (4x^3+8x^2+5x+1)C + D(8x^3+12x^2+6x+1) \\ \Rightarrow \begin{bmatrix} 0 & 0 & 4 & 8 \\ 0 & 2 & 8 & 12 \\ 1 & 3 & 5 & 6 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} &= \begin{bmatrix} 8 \\ 0 \\ 0 \\ 7 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & | & 12 \\ & 1 & 0 & 0 & | & -6 \\ & & 1 & 0 & | & 0 \\ & & 0 & 1 & | & 1 \end{bmatrix} \\ A &= 12, \quad B = -6, \quad C = 0, \quad D = 1 \\ \int \frac{12}{(2x+1)^3} + \frac{-6}{(2x+1)^2} + \frac{1}{x+1} &= \boxed{\frac{-6(2x+1)^{-2}}{2} + \frac{6(2x+1)^{-1}}{2} + \ln(x+1)} \end{aligned}$$

**Exercise 6.**  $\int \frac{4x^2+x+1}{(x-1)(x^2+x+1)}$

$$\begin{aligned} \frac{4x^2+x+1}{(x-1)(x^2+x+1)} &= \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1} \Rightarrow A(x^2+x+1) + (Bx+C)(x-1) = 4x^2+x+1 \\ \Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} &= \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix} \Rightarrow A = 2, \quad B = 2, \quad C = 1 \\ \Rightarrow \int \frac{2}{x-1} + \frac{2x+1}{x^2+x+1} &= \boxed{2 \ln|x-1| + \ln|x^2+x+1|} \end{aligned}$$

**Exercise 7.**  $\int \frac{x^4 dx}{x^4+5x^2+4}$

Doing the long division,  $\frac{x^4}{x^4+5x^2+4} = 1 - \left( \frac{5x^2+4}{(x^2+1)(x^2+4)} \right)$

$$\frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+4} = \frac{5x^2+4}{(x^2+1)(x^2+4)}$$

It could be seen that  $A+C=0$ ,  $4A+C=0$  so  $A=C=0$

$$\begin{aligned} B+D &= 5 & B &= \frac{-1}{3} \\ 4B+D &= 4 & D &= \frac{16}{3} \end{aligned}$$

$$\Rightarrow \int 1 - \frac{5x^2+4}{(x^2+1)(x^2+4)} = x - \int \frac{-1/3}{x^2+1} + \frac{16/3}{x^2+4} = \boxed{x + \frac{1}{3} \arctan x + 4/3 \arctan x/2 + C}$$

**Exercise 8.**  $\int \frac{x+2}{x(x+1)} dx = \int \frac{1}{x+1} + \frac{2}{x(x+1)} = \ln|x+1| + 2 \int \frac{1}{x} - \frac{1}{x+1} = \boxed{-\ln|x+1| + 2 \ln x}$

**Exercise 9.**  $\int \frac{dx}{x(x^2+1)^2} = \int \frac{A}{x} + \frac{Bx+C}{(x^2+1)} + \frac{Dx+E}{(x^2+1)^2}$

$$\frac{A(x^4+2x^2+1) + x(Bx+C)(x^2+1) + Dx^2 + Ex}{x(x^2+1)^2} = \frac{A(x^4+2x^2+1) + Bx^4 + Cx^3 + Bx^2 + Cx + Dx^2 + Ex}{x(x^2+1)^2}$$

$$\Rightarrow A=1; \quad B=-1; \quad D=-1; \quad C=0; \quad E=0$$

$$\int \frac{1}{x} + \frac{-x}{x^2+1} + \frac{-x}{(x^2+1)^2} = \boxed{\ln x + -\frac{\ln|x^2+1|}{2} + \frac{(x^2+1)^{-1}}{2}}$$

**Exercise 10.**  $\int \frac{dx}{(x+1)(x+2)^2(x+3)^3}$

**Exercise 11.**  $\int \frac{x}{(x+1)^2} dx$

$$\frac{x}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} \Rightarrow x = A(x+1) + B$$

$$\frac{1}{x+1} + \frac{-1}{(x+1)^2} \Rightarrow A=1; \quad B=-1$$

$$\int \left( \frac{1}{x+1} + \frac{-1}{(x+1)^2} \right) dx = \ln|x+1| + \frac{1}{x+1} + C$$

**Exercise 12.**  $\int \frac{dx}{x(x^2-1)} = \int \frac{dx}{x(x-1)(x+1)} = \int \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}$

$$A(x^2-1) + Bx(x+1) + Cx(x-1) = Ax^2 - A + Bx^2 + Bx + Cx^2 - Cx \Rightarrow A=-1, B=\frac{1}{2}=C$$

$$\int \frac{-1}{x} + \frac{1/2}{x-1} + \frac{1/2}{x+1} = -\ln|x| + \frac{1}{2} \ln|x-1| + \frac{1}{2} \ln|x+1|$$

**Exercise 13.**  $\int \frac{x^2 dx}{x^2+x-6} = \int \frac{x^2 dx}{(x+3)(x-2)}$

The easiest way to approach this problem is to notice that this is an improper fraction and to do long division:

$$\frac{x^2}{x^2+x-6} = 1 + \frac{6-x}{x^2+x-6}$$

$$\frac{A}{x+3} + \frac{B}{x-2} \Rightarrow \begin{aligned} 6 &= A(x-2) + B(x+3) \\ 6 &= Ax - 2A + Bx + 3B \\ 6 &= (A+B)x - 2A + 3B \end{aligned}$$

$$\begin{aligned} A &= \frac{-6}{5}; \quad B = \frac{6}{5} \\ A &= \frac{3B}{2} \Rightarrow B = -2/5 \end{aligned}$$

$$\Rightarrow \int 1 + \frac{-6/5}{x+3} + \frac{6/5}{x-2} + \frac{-3/5}{x+3} + \frac{-2/5}{x-2} = \boxed{\frac{-9}{5} \ln|x+3| + \frac{4}{5} \ln|x-2| + x + C}$$

**Exercise 14.**  $\int \frac{x+2}{(x-2)^2} = \int \frac{x-2+4}{(x-2)^2} = \ln|x-2| + \int \frac{4}{(x-2)^2} = \boxed{\ln|x-2| + -4(x-2)^{-1}}$

**Exercise 15.**  $\int \frac{dx}{(x-2)^2(x^2-4x+5)}$

Consider the denominator with its  $x^2-4x+5$ . Usually, we would try a partial fraction form such as  $\frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{Cx+D}{x^2-4x+5}$ , but the algebra will get messy. Instead, it helps to be clever here.

$$\begin{aligned} \frac{1}{(x-2)^2(x^2-4x+4+1)} &= \frac{1}{(x-2)^2((x-2)^2+1)} = \frac{1}{(x-2)^2} - \frac{1}{(x-2)^2+1} \\ \Rightarrow \int \frac{dx}{(x-2)^2(x^2-4x+5)} &= \int \frac{1}{(x-2)^2} - \frac{1}{(x-2)^2+1} = \boxed{-(x-2)^{-1} - \arctan(x-2) + C} \end{aligned}$$

**Exercise 16.**  $\int \frac{(x-3)dx}{x^3+3x^2+2x} = \int \frac{(x-3)dx}{x(x+2)(x+1)}$

$$\begin{aligned} \int \frac{(x-3)dx}{x(x+2)(x+1)} &= \int \frac{1}{(x+2)(x+1)} + -3 \int \frac{1}{x(x+2)(x+1)} \\ \frac{1}{(x+2)(x+1)} &= \frac{-1}{x+2} + \frac{1}{x+1} \\ \frac{1}{x(x+2)(x+1)} &= \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x+1} \end{aligned}$$

Now to solve for  $A, B, C$  in the last expression, it is useful to use Gaussian elimination for this system of three linear equations:

$$\begin{aligned} \begin{bmatrix} 1 & 1 & 1 \\ 3 & 1 & 2 \\ 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\ \begin{bmatrix} 1 & 1 & 1 & | & 0 \\ 3 & 1 & 2 & | & 0 \\ 2 & 0 & 0 & | & 1 \end{bmatrix} &= \begin{bmatrix} 0 & 1 & 0 & | & 1/2 \\ 0 & 0 & 1 & | & -1 \\ 1 & 0 & 0 & | & 1/2 \end{bmatrix} \\ \Rightarrow \frac{1}{x(x+2)(x+1)} &= \frac{1/2}{x} + \frac{1/2}{x+2} + \frac{-1}{x+1} \\ \Rightarrow -\ln|x+2| + \ln|x+1| + -3/2 \ln|x+1| &= \boxed{-5/2 \ln|x+2| + 4 \ln|x+1| - 3/2 \ln|x|} \end{aligned}$$

**Exercise 17.** Use partial fraction method to integrate  $\int \frac{1}{(x^2-1)^2}$ . Then **build the sum**.

$$\begin{aligned} \frac{A}{(x-1)^2} + \frac{B}{(x+1)^2} + \frac{C}{(x-1)} + \frac{D}{(x+1)} \\ (x^2-1)(x-1) &= x^3 - x^2 - x + 1 \\ \text{Now } (x^2-1)(x+1) &= x^3 + x^2 - x - 1 \\ \Rightarrow (x^2-1)(x+1) - (x^2-1)(x-1) &= 2x^2 - 2 \\ x^2 + 2x + 1 & \\ x^2 - 2x + 1 & \\ \Rightarrow (\text{summing the above two expressions we obtain}) & 2x^2 + 2 \\ \frac{-1}{x-1} + \frac{1/4}{x+1} + \frac{1/4}{(x-1)^2} + \frac{1/4}{(x+1)^2} & \\ \int \frac{dx}{(x^2-1)^2} = \frac{1}{4} \ln \left| \frac{x+1}{x-1} \right| + -\frac{1}{2} \left( \frac{x}{x^2-1} \right) & \end{aligned}$$

**Exercise 18.** Use the method of partial fractions, where we find that

$$\begin{aligned}\int \frac{(x+1)}{x^3-1} dx &= \int \frac{x+1}{(x-1)(x^2+x+1)} dx = \int \frac{-\frac{2}{3}x - \frac{1}{3}}{x^2+x+1} + \frac{\frac{2}{3}}{x-1} = \\ &= \boxed{-\frac{1}{3} \ln |x^2+x+1| + \frac{2}{3} \ln |x-1|}\end{aligned}$$

where we had used the following partial fraction decomposition for the given integrand

$$\begin{aligned}\frac{Ax+B}{x^2+x+1} + \frac{C}{x-1} &= \frac{x+1}{x^3-1} \\ Ax^2+Bx-Ax-B+Cx^2+Cx+C &= x+1 \\ 2Ax+B-A+2Cx+C &= 1 \quad (\text{where we used the trick to take the derivative of the above equation}) \\ \implies A &= -C \quad B-A-A=1 \\ -B+C &= 1; \quad A = \frac{2}{-3}, C = \frac{2}{3} \quad B = -\frac{1}{2}\end{aligned}$$

**Exercise 19.**  $\int \frac{x^4+1}{x(x^2+1)^2}$

Again, it helps to be clever here.

$$\begin{aligned}\int \frac{x^4+1}{x(x^2+1)^2} &= \int \frac{x^4+2x^2+1-2x^2}{x(x^2+1)^2} = \int \frac{(x^2+1)^2}{x(x^2+1)^2} + \frac{-2x}{(x^2+1)^2} = \\ &= \ln x + (x^2+1)^{-1} + C\end{aligned}$$

**Exercise 20.**  $\int \frac{dx}{x^3(x-2)}$

Working out the algebra for the partial fractions method, we obtain

$$\frac{1}{x^3(x-2)} = \left( \frac{-1/2}{x^3} \right) + \frac{-1/4}{x^2} + \frac{-1/8}{x} + \frac{1/8}{x-2}$$

So then

$$\boxed{\int \frac{dx}{x^3(x-2)} = \frac{1}{4x^2} + \frac{1}{4x} + \frac{-1}{8} \ln x + \frac{1}{8} \ln |x-2| + C}$$

**Exercise 21.**

$$\begin{aligned}\int \frac{1-x^3}{x(x^2+1)} &= -\int \frac{x^3-1}{x(x^2+1)} = -\int \frac{x^2}{x^2+1} + \int \frac{1}{x(x^2+1)} = \\ &= -\int \left( 1 - \frac{1}{x^2+1} \right) + \int \frac{1}{x} + \frac{-x}{x^2+1} = \\ &= \boxed{-x + \arctan x + \ln x - \ln |x^2+1| + C}\end{aligned}$$

**Exercise 22.**

$$\begin{aligned}\int \frac{dx}{x^4-1} &= \int \left( \frac{1}{x^2+1} \right) \left( \frac{1}{x^2-1} \right) = \int \frac{1/2}{x^2-1} - \frac{1/2}{x^2+1} = \\ &= \int \frac{1}{2} \left( \frac{1/2}{x-1} - \frac{1/2}{x+1} \right) - \frac{1/2}{x^2+1} = \\ &= \boxed{\frac{1}{4} \ln(x-1) - \frac{1}{4} \ln(x+1) - \frac{1}{2} \arctan x + C}\end{aligned}$$

**Exercise 23.**  $\boxed{\int \frac{dx}{x^4+1}}$

I had to rely on complex numbers.

Notice that with complex numbers, *you can split up polynomial power sums*

$$\begin{aligned}
x^4 + 1 &= (x^2 + i)(x^2 - i) = (x + ie(i\frac{\pi}{4}))(x - ie(i\frac{\pi}{4}))(x + e(i\frac{\pi}{4}))(x - e(i\frac{\pi}{4})) = \\
&= (x + e(i\frac{3\pi}{4}))(x - e(i\frac{3\pi}{4}))(x + e(i\frac{\pi}{4}))(x - e(i\frac{\pi}{4})) \\
&\frac{A}{(x + e(i\frac{3\pi}{4}))} + \frac{B}{(x - e(i\frac{3\pi}{4}))} + \frac{C}{(x + e(i\frac{\pi}{4}))} + \frac{D}{(x - e(i\frac{\pi}{4}))} = \frac{1}{x^4 + 1} \\
A(x^2 - i)(x - e(i\frac{3\pi}{4})) + B(x^2 - i)(x + e(i\frac{3\pi}{4})) + C(x^2 + i)(x - e(i\frac{\pi}{4})) + D(x^2 + i)(x + e(i\frac{\pi}{4})) &= 1 \\
x^3 : A + B + C + D &= 0 \\
x^2 : -Ae(-\frac{3\pi}{4}) + Be(i\frac{3\pi}{4}) - Ce(i\frac{\pi}{4}) + De(i\frac{\pi}{4}) &= 0 \\
\text{do the algebra} \rightarrow x^1 : -iA - iB + iC + iD &= 0 \\
x^0 : -e(i\frac{\pi}{4})A + Be(i\frac{\pi}{4}) + C(-e(i\frac{3\pi}{4})) + D(e(i\frac{3\pi}{4})) &= 1 \\
\Rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ -e(i\frac{3\pi}{4}) & e(i\frac{3\pi}{4}) & -e(i\frac{\pi}{4}) & e(i\frac{\pi}{4}) \\ -i & -i & i & i \\ -e(i\frac{\pi}{4}) & e(i\frac{\pi}{4}) & -e(i\frac{3\pi}{4}) & e(i\frac{3\pi}{4}) \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}
\end{aligned}$$

To do the complex algebra for the desired Gaussian elimination procedure, I treated the complex numbers as vectors and added them and rotated them when multiplied.

$$\begin{aligned}
\left[ \begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ \swarrow & \nwarrow & \swarrow & \nwarrow & 0 \\ \downarrow & \downarrow & \uparrow & \uparrow & 0 \\ \swarrow & \nwarrow & \swarrow & \nwarrow & 1 \end{array} \right] &= \left[ \begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ & 2 \nwarrow & & 2 \swarrow & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 2 \swarrow & 2 \nwarrow & 1 \end{array} \right] = \left[ \begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ & 2 \nwarrow & & 2 \swarrow & 0 \\ 0 & 0 & 1 & 1 & 0 \\ & & 0 & 4 \nwarrow & 1 \end{array} \right] = \left[ \begin{array}{cccc|c} 1 & & & & 0 \\ & \nwarrow & & & 0 \\ 0 & 0 & 1 & 0 & 1 \\ & & 0 & 1 & 1 \end{array} \right] = \left[ \begin{array}{cccc|c} 1 & & & & 0 \\ & 1 & & & 0 \\ & & 1 & & 0 \\ & & & 1 & 1 \end{array} \right]
\end{aligned}$$

$$\begin{aligned}
A &= \frac{1}{4}e(i\frac{3\pi}{4}) \\
B &= -\frac{1}{4}e(i\frac{3\pi}{4}) \\
C &= \frac{1}{4}e(i\frac{\pi}{4}) \\
D &= -\frac{1}{4}e(i\frac{\pi}{4}) \\
\Rightarrow \int \frac{1/4e(i\frac{3\pi}{4})}{x + e(i\frac{3\pi}{4})} + \frac{-1/4e(i\frac{3\pi}{4})}{x - e(i\frac{3\pi}{4})} + \frac{1/4e(i\frac{\pi}{4})}{x + e(i\frac{\pi}{4})} + \frac{-1/4e(i\frac{\pi}{4})}{x - e(i\frac{\pi}{4})} & \\
\left( \frac{1}{4} \right) \left( e(i\frac{3\pi}{4}) \ln(x + e(i\frac{3\pi}{4})) - e(i\frac{3\pi}{4}) \ln(x - e(i\frac{3\pi}{4})) + e(i\frac{\pi}{4}) \ln(x + e(i\frac{\pi}{4})) - e(i\frac{\pi}{4}) \ln(x - e(i\frac{\pi}{4})) \right) &
\end{aligned}$$

After doing some complex algebra,

$$\Rightarrow \frac{1}{4\sqrt{2}} \left( \ln \left| \frac{x^2 + \sqrt{2}x + 1}{x^2 - \sqrt{2}x + 1} \right| - 2 \arctan \left( \frac{1}{\sqrt{2}x - 1} \right) - 2 \arctan \left( \frac{1}{\sqrt{2}x + 1} \right) \right)$$

The computation could be done to do the derivative on this, so to check our answer and reobtain the integrand.

**Is there a way to solve this without complex numbers?**



**Exercise 24.**  $\int \frac{x^2 dx}{(x^2+2x+2)^2}$

$$\begin{aligned}\int \frac{x^2 dx}{(x^2+2x+2)^2} &= \int \frac{x^2+2x+2-2x-2}{(x^2+2x+2)^2} = \left( \int \frac{1}{x^2+2x+2} \right) + (x^2+2x+2)^{-1} \\ &= \int \frac{1}{x^2+2x+2} = \int \frac{1}{(x+1)^2+1} = \arctan(x+1) \\ \boxed{\int \frac{x^2 dx}{(x^2+2x+2)^2} &= \arctan(x+1) + \frac{1}{x^2+2x+2} + C}\end{aligned}$$

**Exercise 25.**  $\int \frac{4x^5-1}{(x^5+x+1)^2} dx$

$$\begin{aligned}(-(x^5+x+1)^{-1})' &= (x^5+x+1)^{-2}(5x^4+1) \quad (\text{doesn't work}) \\ (-x(x^5+x+1)^{-1})' &= (x^5+x+1)^{-2}(5x^5+x) + -(x^5+x+1)^{-1} = (x^5+x+1)^{-2}(5x^5+x-x^5-x-1) \\ &\Rightarrow \int \frac{4x^5-1}{(x^5+x+1)^2} dx = \boxed{-x(x^5+x+1)^{-2}}\end{aligned}$$

**Exercise 26.**  $\int \frac{dx}{2\sin x - \cos x + 5}$  (good example of the use of half angle substitution)

$$\begin{aligned}\int \frac{dx}{2\sin x - \cos x + 5} &= \int \frac{dx}{4SC + -C^2 + S^2 + 5} \left( \frac{\frac{1}{C^2}}{\frac{1}{C^2}} \right) = \int \frac{\sec^2 \frac{x}{2} dx}{4T - 1 + T^2 + 5(1+T^2)} = \\ &= \int \frac{\sec^2 \frac{x}{2} dx}{6T^2 + 4T + 4} = \int \frac{\sec^2 \frac{x}{2} dx}{6(T + \frac{1}{3})^2 + \frac{10}{3}} = \int \frac{2du}{6(u + \frac{1}{3})^2 + \frac{10}{3}} \quad \left( \text{where } \begin{array}{l} u = \tan \frac{x}{2} \\ du = \frac{\sec^2 \frac{x}{2}}{2} dx \end{array} \right) \\ &= \frac{3}{5} \int \frac{du}{\frac{9(u + \frac{1}{3})^2}{5} + 1} = \boxed{\frac{1}{\sqrt{5}} \arctan \left( \frac{3(\tan \frac{x}{2} + \frac{1}{3})}{\sqrt{5}} \right)}\end{aligned}$$

**Exercise 27.**  $\int \frac{dx}{1+a \cos x}$  ( $0 < a < 1$ ) Again, using the half-angle substitution,

$$\boxed{\begin{array}{l} u = \tan \frac{x}{2} \\ du = \frac{\sec^2 \frac{x}{2}}{2} dx \end{array}},$$

$$\begin{aligned}\frac{1}{a} \int \frac{dx}{\frac{1}{a} + \cos x} &= \frac{1}{a} \int \frac{dx}{\frac{1}{a} + C^2 - S^2} = \frac{1}{a} \int \frac{\sec^2 \frac{x}{2} dx}{\frac{1}{a} \sec^2 \frac{x}{2} + 1 - T^2} = \\ &= \frac{1}{a} \int \frac{\sec^2 \frac{x}{2} dx}{\frac{1}{a} + 1 + T^2(\frac{1}{a} - 1)} = \frac{1}{a} \int \frac{2du}{\frac{1}{a} + 1 + u^2(\frac{1}{a} - 1)} = \frac{2}{1+a} \int \frac{du}{1 + \left(u \sqrt{\frac{1-a}{1+a}}\right)^2} = \\ &= \frac{2}{1+a} \frac{\arctan \sqrt{\frac{1-a}{1+a}} u}{\sqrt{\frac{1-a}{1+a}}} \Rightarrow \boxed{\frac{2}{\sqrt{1-a^2}} \arctan \left( \sqrt{\frac{1-a}{1+a}} \tan \frac{x}{2} \right)}\end{aligned}$$

**Exercise 28.**  $\int \frac{dx}{1+a \cos x}$  Half-angle substitution.

$$\begin{aligned} \int \frac{dx}{1+a \cos x} &= \int \frac{dx}{1+a(C^2 - S^2)} = \int \frac{\sec^2 \frac{x}{2} dx}{\sec^2 \frac{x}{2} + a(1 - T^2)} = \\ u = \tan \theta/2 = T \\ du = \sec^2 \theta/2 \left(\frac{1}{2}\right) d\theta &\Rightarrow \int \frac{2du}{1+T^2+a(1-T^2)} = 2 \int \frac{du}{(1-a)T^2+(1+a)} = \frac{2}{1-a} \int \frac{du}{u^2 - \frac{a+1}{a-1}} = \\ &= \frac{2}{1-a} \int \left( \frac{1}{u - \sqrt{\frac{a+1}{a-1}}} - \frac{1}{u + \sqrt{\frac{a+1}{a-1}}} \right) \frac{1}{2\sqrt{\frac{a+1}{a-1}}} = \\ &= \sqrt{\frac{a-1}{a+1}} \left( \frac{1}{1-a} \right) \left( \ln(u - \sqrt{\frac{a+1}{a-1}}) - \ln(u + \sqrt{\frac{a+1}{a-1}}) \right) = \\ &= \boxed{\frac{-1}{\sqrt{a^2-1}} \left( \ln \left( \frac{\tan \frac{x}{2} - \sqrt{\frac{a+1}{a-1}}}{\tan \frac{x}{2} + \sqrt{\frac{a+1}{a-1}}} \right) \right)} \end{aligned}$$

**Exercise 29.**  $\int \frac{\sin^2 x}{1+\sin^2 x} dx$

$$\begin{aligned} \int \frac{s^2}{1+s^2} dx &= \int \frac{s^2+1-1}{1+s^2} dx = x - \int \frac{dx}{1+\sin^2 x} \\ \int \frac{dx}{1+\sin^2 x} &= \int \frac{dx}{1+(\frac{1-\cos 2x}{2})} = \int \frac{2dx}{3-\cos 2x} = \frac{2}{3} \int \frac{dx}{1-\frac{\cos 2x}{3}} = \frac{2}{3} \int \frac{dx}{1-(\frac{c^2-s^2}{3})} = \frac{2}{3} \int \frac{\sec^2 x dx}{\sec^2 x - (\frac{1-T^2}{3})} = \\ u = \tan x &\Rightarrow \frac{2}{3} \int \frac{du}{1+u^2 - (\frac{1-u^2}{3})} = \frac{2}{3} \int \frac{du}{\frac{2}{3} + \frac{4}{3}u^2} = \int \frac{du}{1+(\sqrt{2}u)^2} = \\ du = \sec^2 x dx & \\ &= \frac{1}{\sqrt{2}} \arctan \sqrt{2} \tan x \\ &\Rightarrow \boxed{\int \frac{\sin^2 x}{1+\sin^2 x} dx = x - \frac{1}{\sqrt{2}} \arctan(\sqrt{2} \tan x)} \end{aligned}$$

It seems like for here, when *dealing with squares of trig. functions*, “step up” to double angle.

**Exercise 30.**  $\int \frac{dx}{a^2 \sin^2 x + b^2 \cos^2 x}$  ( $ab \neq 0$ ) Take note, we need not change the angle to half-angle or double-angle.

$$\begin{aligned} \frac{1}{a^2 s^2 + b^2 c^2} &= \frac{1}{a^2(1-c^2) + b^2 c^2} = \frac{1}{a^2 + (b^2 - a^2)c^2} = \frac{1}{a^2(1+(kc)^2)} = \frac{\sec^2}{a^2(\sec^2 + k^2)} = \frac{\sec^2}{a^2(1+T^2 + k^2)} = \\ u = \tan x &\Rightarrow \frac{du}{a^2(1+u^2 + k^2)} = \frac{1/a^2 du}{(1+k^2)(1+\frac{u^2}{1+k^2})} = \frac{\sqrt{1+k^2}}{a^2(1+k^2)} \arctan \left( \frac{u}{\sqrt{1+k^2}} \right) \\ du = \sec^2 x dx & \\ &\Rightarrow \int \frac{dx}{a^2 \sin^2 x + b^2 \cos^2 x} = \boxed{\frac{1}{ab} \arctan \left( \frac{a \tan x}{b} \right)} \end{aligned}$$

**Exercise 31.**  $\int \frac{dx}{(a \sin x + b \cos x)^2}$  ( $a \neq 0$ )

Note it's a good idea to *simplify*, cleverly, your constants as much as you can.

$$\int \frac{dx}{(a \sin x + b \cos x)^2} = \frac{1}{a^2} \int \frac{dx}{(\sin x + k \cos x)^2}$$

Thus, only one constant,  $k$ , is only worried about.

$$\frac{1}{(s+kc)^2} = \frac{1}{s^2 + 2ksc + k^2c^2} = \frac{1/c^2}{t^2 + 2kt + k^2} = \frac{\sec^2}{(t+k)^2} =$$

$$u = \tan x \quad \Rightarrow \quad \frac{du}{(u+k)^2}$$

$$du = \sec^2 x dx$$

$$\Rightarrow \boxed{\frac{1}{a^2} \int \frac{1}{(s + \frac{b}{a}c)^2} = \frac{-1}{(a^2 \tan x + ab)}}$$

Again, *note*, **we need not** always step up or step down a half angle in the substitution.

**Exercise 32.** Note that we have a rational expression consisting of single powers of  $\sin$  and  $\cos$ . Then use the  $\tan \frac{\theta}{2}$  substitution.

$$\int \frac{\sin x}{1 + \cos x + \sin x} = \int \frac{2CS}{1 + 2CS + C^2 - S^2} = \int \frac{CS}{C(S+C)} = \int \frac{T}{(T+1)}$$

$$\text{where } \begin{array}{ll} C = \cos x/2 & u = \tan \theta/2 \\ S = \sin x/2 & du = \frac{\sec^2 \theta/2 d\theta}{2} \end{array}$$

$$\int \frac{u}{(u+1)} \left( \frac{2du}{u^2+1} \right) = \int \frac{2udu}{(u^2+1)(u+1)}$$

$$\frac{A}{u+1} + \frac{Bu+C}{u^2+1}$$

$$Au^2 + A + Bu^2 + Cu + Bu + C = u$$

$$A = -B \quad C + B = 1 \quad A + C = 0 \Rightarrow C = \frac{1}{2}; B = \frac{1}{2}; A = -\frac{1}{2}$$

$$2 \int \left( \frac{-1/2}{u+1} + \frac{\frac{1}{2}(u+1)}{u^2+1} \right) du = \int \frac{-1}{u+1} + \frac{u+1}{u^2+1}$$

$$= -\ln|u+1| + \frac{1}{2} \ln|u^2+1| + \arctan u =$$

$$= -\ln|\tan x/2 + 1| + \frac{1}{2} \ln|\sec^2 x/2| + \frac{x}{2}$$

$$\Rightarrow -(\ln|2|) + \frac{1}{2} \ln|2| + \frac{\pi}{4} = \boxed{-\frac{1}{2} \ln|2| + \frac{\pi}{4}}$$

**Exercise 33.**  $\int \sqrt{3-x^2} dx$

$$\int (x)' \sqrt{3-x^2} dx = x\sqrt{3-x^2} - \int \frac{x(-x)}{\sqrt{3-x^2}} = x\sqrt{3-x^2} - \int \frac{-x^2+3-3}{\sqrt{3-x^2}} = x\sqrt{3-x^2} - \int \sqrt{3-x^2} + 3 \int \frac{1}{\sqrt{3-x^2}}$$

$$\Rightarrow 2 \int \sqrt{3-x^2} = x\sqrt{3-x^2} + \sqrt{3} \int \frac{1}{\sqrt{1-(\frac{x}{\sqrt{3}})^2}}$$

$$\Rightarrow \boxed{\int \sqrt{3-x^2} = \frac{x}{2} \sqrt{3-x^2} + \frac{3}{2} \arcsin \frac{x}{\sqrt{3}}}$$

**Exercise 34.**  $\int \frac{1}{\sqrt{3-x^2}} dx = -(3-x^2)^{1/2} + C.$

$$\left( \arccos \frac{x}{\sqrt{3}} \right)' = \frac{1}{\sqrt{3}} \frac{-1}{\sqrt{1-\frac{x^2}{3}}} = -\frac{1}{\sqrt{3-x^2}}$$

$$\left( x\sqrt{3-x^2} \right)' = \sqrt{3-x^2} + \frac{-x^2}{\sqrt{3-x^2}}$$

$$\boxed{\frac{x\sqrt{3-x^2}}{2} + -\frac{3}{2} \arccos \frac{x}{\sqrt{3}}}$$

**Exercise 35.**  $\int \frac{\sqrt{3-x^2}}{x} dx = \int \sqrt{\frac{3}{x^2} - 1} dx.$

$$\begin{aligned}\frac{\sqrt{3}}{x} &= \sec \theta \\ \sqrt{3} \cos \theta &= x \\ dx &= -\sin \theta \sqrt{3} \\ \int \sqrt{\sec^2 \theta - 1} (-\sin \theta) \sqrt{3} &= \int \tan \theta \sin \theta (-\sqrt{3}) = -\sqrt{3} \int (\sec \theta - \cos \theta) = \\ &= -\sqrt{3} \ln |\sec \theta + \tan \theta| + \sqrt{3} \sin \theta = \\ &= \boxed{-\sqrt{3} \ln \left| \frac{\sqrt{3}}{x} + \sqrt{\frac{3}{x^2} - 1} \right| + \sqrt{3} \sqrt{1 - \frac{x^2}{3}}}\end{aligned}$$

**Exercise 36.**  $\int \sqrt{1 + \frac{1}{x}} dx$

$$\begin{aligned}\left(x \sqrt{1 + \frac{1}{x}}\right)' &= \sqrt{1 + \frac{1}{x}} + \frac{x}{2\sqrt{1 + \frac{1}{x}}} \left(\frac{-1}{x}\right) = \sqrt{1 + \frac{1}{x}} + \frac{-1/2}{\sqrt{x^2 + x}} \\ \left(\ln \left(x + \frac{1}{2} + \sqrt{x^2 + x}\right)\right)' &= \frac{1}{x + \frac{1}{2} + \sqrt{x^2 + x}} \left(1 + \frac{x + \frac{1}{2}}{\sqrt{x^2 + x}}\right) = \frac{1}{\sqrt{x^2 + x}} \\ \Rightarrow \int \left(1 + \frac{1}{x}\right) dx &= x \sqrt{1 + \frac{1}{x}} + \frac{1}{2} \ln \left(x + \frac{1}{2} + \sqrt{x^2 + x}\right)\end{aligned}$$

**Exercise 37.**

$$\begin{aligned}(x \sqrt{x^2 + 6})' &= \sqrt{x^2 + 5} + \frac{x^2}{\sqrt{x^2 + 5}} \\ (\ln(x + \sqrt{x^2 + b}))' &= \left(\frac{1}{x + \sqrt{x^2 + b}}\right) \left(1 + \frac{x}{\sqrt{x^2 + b}}\right) = \frac{1}{\sqrt{x^2 + b}} \\ \int \sqrt{x^2 + 5} &= \frac{1}{2} \left(x \sqrt{x^2 + 5} + 5 \ln(x + \sqrt{x^2 + 5})\right)\end{aligned}$$

**Exercise 38.**

$$\begin{aligned}\left(\ln \left(x + \frac{1}{2} + \sqrt{x^2 + x + 1}\right)\right)' &= \frac{1}{x + \frac{1}{2} + \sqrt{x^2 + x + 1}} \left(1 + \frac{x + \frac{1}{2}}{\sqrt{x^2 + x + 1}}\right) = \frac{1}{\sqrt{x^2 + x + 1}} \\ \int \frac{x}{\sqrt{x^2 + x + 1}} &= \int \frac{x + \frac{1}{2} - \frac{1}{2}}{\sqrt{x^2 + x + 1}} = (x^2 + x + 1)^{1/2} - \frac{1}{2} \ln(x + \frac{1}{2} + \sqrt{x^2 + x + 1})\end{aligned}$$

The trick is to note how I formed a “conjugate-able” sum from  $x^2 + x + 1$ ’s derivative.

**Exercise 39.**

$$\begin{aligned}\int \frac{dx}{\sqrt{x^2 + x}} &= \int \frac{dx}{\sqrt{(x + \frac{1}{2})^2 - \frac{1}{4}}} = \int \frac{2dx}{\sqrt{(2(x + \frac{1}{2}))^2 - 1}} \\ &= \left(\ln \left(\sqrt{(2(x + 1/2))^2 - 1} + 2(x + 1/2)\right)\right)' = \\ &= \frac{1}{\sqrt{(2(x + 1/2))^2 - 1} + 2(x + 1/2)} \left(2 + \frac{2(x + 1/2)2}{\sqrt{(2(x + 1/2))^2 - 1}}\right) = \\ &= \frac{2}{\sqrt{(2(x + 1/2))^2 - 1}} \\ &= \boxed{\int \frac{dx}{\sqrt{(x + \frac{1}{2})^2 - \frac{1}{4}}} = \ln \left(2 \left(x + \frac{1}{2}\right) + \sqrt{(2(x + 1/2))^2 - 1}\right) + C}\end{aligned}$$

**Exercise 40.**

**6.26 Miscellaneous review exercises. Exercise 1.**

$$\begin{aligned}
 f(x) &= \int_1^x \frac{\log t}{t+1} \\
 f\left(\frac{1}{x}\right) &= \int_1^{\frac{1}{x}} \frac{\log t}{t+1} dt = \int_1^x \frac{-\ln(u)}{\frac{1}{u}+1} \left(\frac{-1}{u^2} du\right) = \\
 &= \int_1^x \frac{\ln(u)}{u+u^2} du \quad \begin{array}{l} u = \frac{1}{t} \\ du = -\frac{1}{t^2} dt \end{array} \\
 f(x) + f\left(\frac{1}{x}\right) &= \int_1^x \frac{t \ln t + \ln t}{t(t+1)} dt = \int_1^x \frac{\ln t}{t} dt = \frac{(\ln t)^2}{2} \Big|_1^x = \boxed{\frac{(\ln x)^2}{2}} \\
 &\quad \boxed{f(2) + f\left(\frac{1}{2}\right) = \frac{1}{2}(\ln 2)^2}
 \end{aligned}$$

**Exercise 2.** Take the derivative of both sides, using the (first) fundamental theorem of calculus.

$$2ff' = f(x) \frac{\sin x}{2 + \cos x}; \implies 2f' = \frac{\sin x}{2 + \cos x}$$

At this point, it could be very easy to evaluate the integral by guessing at the solution.

$$(-\ln 2 + \cos x)' = \frac{\sin x}{2 + \cos x} \implies \boxed{f = -\frac{\ln |2 + \cos x|}{2} + C}$$

Otherwise, remember that for rational expressions involving single powers of sin and cos, we can make a  $u = \tan \theta/2$  substitution.

$$\begin{aligned}
 u &= \tan \frac{x}{2} & C &= \cos x/2, S = \sin x/2 \\
 2du &= \sec^2 \frac{x}{2} dx \\
 \int \frac{\sin x}{2 + \cos x} dx &= \int \frac{2SC dx}{2 + C^2 - S^2} = \int \frac{2T}{2 \sec^2 x/2 + 1 - T^2} \left( \frac{2du}{\sec^2 x/2} \right) = \\
 &= 4 \int \frac{udu}{(1+u^2)(3+u^2)} = 2 \int \frac{T}{T^2+1} - \frac{T}{T^2+3} \\
 &= 2 \left( \frac{1}{2} \ln T^2 + 1 - \frac{1}{2} \ln T^2 + 3 \right) = \left( \ln \left( \frac{T^2+1}{T^2+3} \right) \right) = \ln \left( \frac{2}{4+2\cos x} \right) \\
 \text{where } \tan^2 \frac{x}{2} &= \frac{\sin^2 \frac{x}{2}}{\cos^2 \frac{x}{2}} = \frac{1 - \cos x}{1 + \cos x}
 \end{aligned}$$

**Exercise 3.**

$$\int \frac{e^x}{x} dx = e^x - \int \frac{e^x(x-1)}{x^2} x dx = e^x - \int \frac{e^x(x-1)}{x} dx \dots$$

No way.

**Exercise 4.**  $\int_0^{\pi/2} \ln(e^{\cos x}) dx = -\cos x \Big|_0^{\pi/2} = \boxed{1}.$

**Exercise 5.**

(1)

$$\begin{aligned}
 f &= \sqrt{4x+2}x(x+1)(x+2) \\
 \ln f &= \frac{1}{2} \ln \left( \frac{4(x+2)}{x(x+1)(x+2)} \right) = \frac{1}{2} (\ln(4x+2) - \ln x - \ln(x+1) - \ln(x+2)) \\
 \frac{f'}{f} &= \frac{1}{2} \left( \left( \frac{4}{4x+2} \right) - \frac{1}{x} - \frac{1}{x+1} - \frac{1}{x+2} \right) & f'(1) &= -\frac{7}{12} \\
 \implies f' &= \frac{1}{2} \sqrt{\frac{4x+2}{x(x+1)(x+2)}} \left( \left( \frac{4}{4x+2} \right) - \frac{1}{x} - \frac{1}{x+1} - \frac{1}{x+2} \right)
 \end{aligned}$$

(2)

$$\begin{aligned}\int_1^4 \pi \frac{4x+2}{x(x+2)(x+1)} dx &= 2\pi \int_1^4 \frac{(2x+1)}{x(x+2)(x+1)} dx = \\ &= 2\pi \left( \frac{1}{2} \ln x + \frac{3}{2} \ln |x+2| + \ln |x+1| \right) \Big|_1^4 = \boxed{\pi \ln \frac{25}{8}}\end{aligned}$$

since we can find the antiderivative through partial fractions:

$$\begin{aligned}\frac{A}{x} + \frac{B}{x+2} + \frac{C}{x+1} &= \frac{2x+1}{x(x+2)(x+1)} \\ A(x^2+3x+2) + B(x^2+x) + C(x^2+2x) &= 2x+1 \\ \begin{bmatrix} 1 & 1 & 1 \\ 3 & 1 & 2 \\ 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} &= \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \implies \begin{bmatrix} 0 & 1 & 0 & -3/2 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1/2 \end{bmatrix}\end{aligned}$$

**Exercise 6.**

(1)

$$\log x = \int_1^x \frac{1}{t} dt \quad F(x) = \int_1^x \frac{e^t}{t} dt; \quad \begin{matrix} \text{if } x > 0 \\ e^t > 1 \text{ for } t > 0 \end{matrix}$$

If  $0 < x < 1$

$$\log x = \int_1^x \frac{1}{t} dt = \int_x^1 \frac{-1}{t} dt > - \int_x^1 \frac{e^t}{t} dt = F(x)$$

$$\boxed{\log x \leq F(x) \text{ for } x \geq 1}$$

(2)

$$\begin{aligned}F(x+a) - F(1+a) &= \int_1^{x+a} \frac{e^t}{t} dt - \int_1^{a+1} \frac{e^t}{t} dt = \int_{1-a}^x \frac{e^{t+a}}{t+a} dt - \int_{1-a}^1 \frac{e^{t+a}}{t+a} dt \\ &= e^a \int_1^x \frac{e^t}{t+a} dt\end{aligned}$$

(3)

$$\begin{aligned}\int_1^x \frac{e^{at}}{t} dt &= \int_a^{ax} \frac{e^t}{t} dt = \int_1^{ax} \frac{e^t}{t} dt + \int_a^1 \frac{e^t}{t} dt = \\ &= \boxed{F(ax) - F(a)} \\ \int_1^x \frac{e^t}{t^2} dt &= -\frac{1}{t} e^t - \int -\frac{1}{t} e^t dt = \boxed{-\left(\frac{e^x}{x} - e\right) + F(x)} \\ \int_1^x e^{1/t} dt &= \int_1^{1/x} \frac{-e^u}{u^2} du = -\left(\frac{-e^u}{u} - \int -\frac{e^u}{u} du\right) = \\ &= \boxed{xe^{1/x} - e - F(1/x)} \quad \begin{matrix} u = \frac{1}{t} \\ du = -\frac{1}{t^2} dt \end{matrix} \text{ where we used the substitution}\end{aligned}$$

**Exercise 7.**

(1)

$$e^x = F(x) - F(0); F(x) = e^x + F(0) \implies F(0) = 1 + F(0)$$

$0 \neq 1$ . False.

(2)

$$\begin{aligned}\frac{d}{dx} \int_0^{x^2} f(t) dt &= f(x^2)(2x) = -(2x) \ln 2e^{x^2 \ln 2} \quad f(x) = -\ln 2e^{x \ln 2} \\ \int_0^{x^2} -\ln 2e^{t \ln 2} dt &= -e^{t \ln 2} \Big|_0^{x^2} = -e^{x^2 \ln 2} + 1\end{aligned}$$

(3)

$$\begin{aligned}
f(x) &= 2f(x)f'(x); \implies f(x) = \frac{x}{2} + C \\
\int_0^x \left( \frac{1}{2}t + c \right) dt &= \left( \frac{t^2}{4} + ct \right) \Big|_0^x = \frac{x^2}{4} + cx \\
f^2(x) - 1 &= \frac{x^2}{4} + Cx + C^2 - 1 \\
\implies C &= \pm 1, f(x) = \frac{x}{2} + \pm 1
\end{aligned}$$

**Exercise 8.**

(1)

$$\begin{aligned}
\frac{f(x+h) - f(x)}{h} &= \frac{f(x)f(h) - f(x)}{h} = \frac{f(x)(hg(h))}{h} = f(x)g(h) \\
g(h) &\rightarrow 1 \text{ as } h \rightarrow 0 \text{ so } \implies f'(x) = f(x)
\end{aligned}$$

(2) Since for  $f(x) = e^x$ , we defined  $e^x$  such that  $f' = f$ , if

$$\begin{aligned}
(e^x + g)' &= e^x + g' = e^x + g \\
\implies e^x + g &= Ce^x \implies g = (C-1)e^x \text{ but } f'(0) = 1 \text{ so } \boxed{g = e^x}
\end{aligned}$$

**Exercise 9.**

(1)

$$\begin{aligned}
g(2x) &= 2e^x g(x) \\
g(3x) &= e^x g(2x) + e^{2x} g(x) = e^x 2e^x g + e^{2x} g = 3e^{2x} g
\end{aligned}$$

(2)

$$\text{Assume } g(nx) = ne^{(n-1)x} g$$

$$g((n+1)x) = e^x g(nx) + e^{(n+1)x} g(x) = ne^{nx} g(x) + e^{nx} g = (n+1)e^{nx} g$$

(3) From  $g(x+y) = e^y g(x) + e^x g(y)$ ,

$$g(0) = g(0) + g(0) \implies g(0) = 0$$

$$\begin{aligned}
\frac{g(x+h) - g(x)}{h} &= \frac{e^h g(x) + e^x g(h) - g(x)}{h} = g(x) \left( \frac{e^h - 1}{h} \right) + \frac{e^x g(h)}{h} \\
f'(0) &= 2 = \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} = \lim_{h \rightarrow 0} h \rightarrow 0 \frac{g(h)}{h}
\end{aligned}$$

$$(4) \quad g'(x) = g(x) + 2e^x \quad \boxed{C=2}$$

**Exercise 10.**

$$\begin{aligned}
\forall x \in \mathbb{R}, f(x+a) &= bf(x); f(x+2a) = bf(x+a) = b^2 f(x) \\
f(x+(n+1)a) &= f(x+na+a) = bf(x+na) = b^{n+1} f(x) \\
f(x+na) &= b^n f(x)
\end{aligned}$$

$$\boxed{f(x) = b^{x/a} g(x) \text{ where } g \text{ is periodic in } a}$$

**Exercise 11.**

$$\begin{aligned}
(\ln(fg))' &= \frac{f'}{f} + \frac{g'}{g} \implies (fg)' = f'g + fg' \\
\left( \ln \left( \frac{f}{g} \right) \right)' &= \frac{f'}{f} - \frac{g'}{g} \implies \frac{f'g - g'f}{g^2} = \left( \frac{f}{g} \right)'
\end{aligned}$$

**Exercise 12.**  $A = \int_0^1 \frac{e^t}{t+1} dt$ 

(1)

$$\begin{aligned}
&\int_{a-1}^a \frac{e^{-t}}{t-a-1} dt \\
u = t-a &\implies \int_{-1}^0 \frac{e^{-t-a}}{t-1} dt = - \int_1^0 \frac{e^{t-a}}{-t-1} dt = -e^{-a} \int_0^1 \frac{e^t}{t+1} dt = \boxed{-e^{-a} A}
\end{aligned}$$

$$(2) \quad \int_0^1 \frac{te^{t^2}}{t^2+1} dt = \int_0^1 \frac{\frac{1}{2} du e^u}{u+1} = \boxed{\frac{1}{2} A}$$

$$(3) \int_0^1 \frac{e^t}{(t+1)^2} dt = \left. \frac{-e^t}{t+1} \right|_0^1 - \int_0^1 \frac{-e^t}{t+1} = \boxed{\frac{-e^1}{2} + 1 + A}$$

$$(4) \int_0^1 e^t \ln(1+t) dt = e^t \ln(1+t) - \int \frac{e^t}{1+t} = \boxed{e \ln 2 - A}$$

**Exercise 13.**

$$(1) p(x) = c_0 + c_1 x + c_2 x^2; \quad f(x) = e^x p(x) \quad p' = c_1 + 2c_2 x \quad p'' = 2c_2$$

$$f' = f + e^x p'$$

$$f^{(n)}(x) = \sum_{j=0}^n \binom{n}{j} e^x (p(x))^j = f + e^x (c_1 + 2c_2 x) + \left( \frac{n!}{(n-1)!} \right) + e^x (2c_2) \left( \frac{n!}{(n-2)!2!} \right)$$

$$f^{(n)}(0) = c_0 + c_1 n + n(n-1)c_2$$

(2) See generalization below.

(3)

$$p = \sum_{j=0}^m a_j x^j; \quad p(0) = a_0; \quad p^{(k)}(x) = \sum_{j=0}^m a_j \frac{j!}{(j-k)!} x^{j-k} = p^{(k)}(0) = a_k k!$$

$$f^{(n)}(x) = \sum_{j=0}^n \binom{n}{j} e^x p^{(j)}(x)$$

$$f^{(n)}(0) = \sum_{j=0}^n \binom{n}{j} p^{(j)}(0) = \sum_{j=0}^n \binom{n}{j} a_j j! = \boxed{\sum_{j=0}^n \frac{n!}{(n-j)!} a_j}$$

So for  $m = 3$ , then  $f^{(n)}(0) = a_0 + na_1 + n(n-1)a_2 + n(n-1)(n-2)a_3$

**Exercise 14.**  $f(x) = x \sin ax$ ;  $f^{(2)} = -a^2 x \sin ax + 2a \cos ax$

$$f^{(2n)}(x) = (-1)^n (a^{2n} x \sin ax - 2na^{2n-1} \cos ax)$$

$$f^{(2n+1)}(x) = (-1)^n (a^{2n+1} x \cos ax + a^{2n} \sin ax + 2na^{2n} \sin ax)$$

$$f^{(2n+2)}(x) = (-1)^n (-a^{2n+2} x \sin ax + a^{2n+1} \cos ax (2n+2))$$

**Exercise 15.**

$$\begin{aligned} \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{1}{k+m+1} &= \sum_{k=0}^n (-1)^k \binom{n}{k} \int_0^1 t^{k+m} dt = \int_0^1 \sum_{k=0}^n (-1)^k \binom{n}{k} t^{k+m} dt = \\ &= \int_0^1 t^m \sum_{k=0}^n \binom{n}{k} (-t)^k dt = \int_0^1 t^m (1-t)^n dt = \\ &= - \int_1^0 (1-u)^m u^n du = \int_0^1 (1-u)^m u^n du = \\ &= \int_0^1 \sum_{j=0}^m \binom{m}{j} (-u)^j u^n du = \boxed{\sum_{j=0}^m (-1)^j \binom{m}{j} \int_0^1 t^{j+n} dt} \\ &\quad u1-t \\ &\quad du = -dt \end{aligned}$$

**Exercise 16.**  $F(x) = \int_0^x f(t) dt$

(1)

$$F(x) = \int_0^x (t + |t|)^2 dt = \begin{cases} \int_0^x (2t)^2 dt = \frac{4}{3}x^3 & \text{if } t, x \geq 0 \\ \int_0^x 0 dt = 0 & \text{if } t, x < 0 \end{cases}$$

(2)

$$\begin{aligned} F(x) &= \int_0^x f(t) dt = \begin{cases} \int_0^x (1-t^2) dt & \text{if } |t| \leq 1 \\ \int_0^x (1-|t|) dt & \text{if } |t| > 1 \end{cases} = \\ &= \begin{cases} \left( t - \frac{1}{3}t^3 \right) \Big|_0^x = x - \frac{x^3}{3} & \text{if } |x| \leq 1 \\ \begin{cases} \frac{2}{3} + \int_1^x (1-t) dt = x - \frac{x^2}{2} + \frac{1}{6} & \text{if } x > 1 \\ -\frac{2}{3} + \int_{-1}^x (1+t) dt = x + \frac{x^2}{2} - \frac{1}{6} & \text{if } x < -1 \end{cases} & \text{if } |x| \geq 1 \end{cases} \end{aligned}$$



(3)  $f(t) = e^{-|t|}$

$$\begin{aligned} F(x) &= \int_0^x f(t)dt = \int_0^x e^{-|t|}dt = \\ &= \begin{cases} \int_0^x e^{-t}dt = e^{-t}|_0^x = 1 - e^{-x} & \text{if } x \geq 0 \\ \int_0^x e^t dt = e^t|_0^x = e^x - 1 & \text{if } x < 0 \end{cases} \end{aligned}$$

(4)  $f(t) = \max. \text{ of } 1 \text{ and } t^2$

$$\begin{aligned} F(x) &= \int_0^x f(t)dt = \begin{cases} \int_0^x 1dt = x & \text{if } |x| \leq 1 \\ 1 + \int_1^x t^2 dt & \text{if } x > 1 \\ -\int_0^x f & \text{if } x < -1 \end{cases} = \\ &= \begin{cases} x & \text{if } |x| \leq 1 \\ 1 + \frac{1}{3}t^3|_1^x = \frac{x^3}{3} + \frac{2}{3} & \text{if } x > 1 \\ -\int_x^{-1} t^2 + -\int_{-1}^0 1 = \frac{1}{3}t^3|_{-1}^x - 1 = \frac{x^3}{3} + \frac{2}{3} & \text{if } x < -1 \end{cases} \end{aligned}$$

**Exercise 17.**  $\int \pi f^2 = \pi \int_0^a f^2 = a^2 + a.$

$$\begin{aligned} \left( \frac{x^2 + x}{\pi} \right)' &= \sqrt{\frac{2x+1}{\pi}} \\ \text{for } \int_0^a \frac{2x+1}{\pi} &= (x^2 + x)|_0^a = a^2 + a \end{aligned}$$

**Exercise 18.**  $f(x) = e^{-2x}.$

(1)  $A(t) = \int_0^t e^{-2x} dx = \frac{e^{-2x}}{-2} \Big|_0^t = \frac{e^{-2t}-1}{-2}$

(2)  $V(t) = \pi \int_0^t e^{-4x} dx = \frac{\pi}{-4} (e^{-4t} - 1)$

(3)

$$y = e^{-2x} \implies \frac{\ln y}{-2} = x$$

$$\begin{aligned} W(t) &= \pi \int_{e^{-2t}}^1 \left( \frac{\ln y}{-2} \right)^2 dy = \frac{\pi}{4} (y(\ln y)^2 - (2(y \ln y - y))) \Big|_{e^{-2t}}^1 = \\ &= \frac{\pi}{4} (2 - (e^{-2t} 4t^2 - (2(e^{-2t}(-2t) - e^{-2t})))) = \\ &= \left( \frac{\pi}{2} - \pi t e^{-2t} - \frac{\pi}{2} e^{-2t} \right) \end{aligned}$$

where the antiderivative used was  $(y(\ln y)^2)' = (\ln y)^2 + 2 \ln y$

(4)

$$\frac{\frac{\pi}{4}(1 - e^{-4t})}{\left( \frac{1 - e^{-2t}}{2} \right)} = \frac{\pi \left( \frac{e^{-4t}-1}{t} \right)}{\frac{e^{-2t}-1}{t}} = \boxed{\pi}$$

where we used the limit  $\lim_{x \rightarrow 0} \frac{e^{cx} - 1}{x} = c$

**Exercise 19.**  $\sinh c = \frac{3}{4}$

(1)

$$\begin{aligned} e^c &= e^x + \sqrt{e^{2x} + 1} \\ \frac{e^c - e^{-c}}{2} &= \frac{e^x + \sqrt{e^{2x} + 1} - \frac{1}{e^x + \sqrt{e^{2x} + 1}}}{2} = \\ &= \frac{e^{2x} + 2e^x \sqrt{e^{2x} + 1} + e^{2x} + 1 - 1}{2(e^x + \sqrt{e^{2x} + 1})} = e^x = \frac{3}{4} \end{aligned}$$

$$\boxed{x = \ln 3 - 2 \ln 2}$$

(2)

$$\begin{aligned}
\frac{e^c - e^{-c}}{2} &= \frac{e^x - \sqrt{e^{2x} - 1} - \frac{1}{e^x - \sqrt{e^{2x} - 1}}}{2} = \\
&= \frac{e^{2x} - 2e^x \sqrt{e^{2x} - 1} + e^{2x} - 1 - 1}{2(e^x - \sqrt{e^{2x} - 1})} \\
&= \frac{e^x}{+ e^x - \sqrt{e^{2x} - 1}} - 1 = e^x - \frac{1}{e^c} = \frac{3}{4} \\
&\implies \boxed{x = \ln 5 - 2 \ln 2}
\end{aligned}$$

**Exercise 20.**(1) True.  $\ln(2^{\log 5}) = \ln 5^{\ln 2} = (\ln 2) \ln 5$ .(2)  $\log_3 5 = \frac{\log_2 5}{\log_2 3}$  This is a true fact.

$$\begin{aligned}
\frac{\log_3 5}{\log_2 3} &= \frac{\log_2 5}{(\log_2 3)^2} = \log_2 5 \\
&\implies 1 = \log_2 3 \quad \text{False}
\end{aligned}$$

(3) **Use induction**

$$n = 1 \quad 1^{-1/2} < 2\sqrt{1}$$

$$n = 2 \quad 1 + \frac{1}{\sqrt{2}} < 2\sqrt{2}$$

$$\begin{aligned}
n+1 \text{ case} \quad \sum_{k=1}^{n+1} k^{-1/2} &= \sum_{k=1}^n k^{-1/2} + \frac{1}{\sqrt{(n+1)}} < \\
&< 2\sqrt{n} \left( \frac{\sqrt{(n+1)}}{\sqrt{(n+1)}} \right) + \frac{1}{\sqrt{(n+1)}}
\end{aligned}$$

Now  $(n + \frac{1}{2})^2 = n^2 + n + \frac{1}{2} > n^2 + n$ , certainly. So then

$$\begin{aligned}
n + \frac{1}{2} &> \sqrt{n^2 + n} \implies n + 1 > \sqrt{n^2 + n} \\
&\implies \sum_{k=1}^{n+1} k^{-1/2} < 2\sqrt{n+1}
\end{aligned}$$

(4)

$$f = (\cosh x - \sinh x - 1) = \frac{e^x + e^{-x} - e^x + e^{-x}}{2} - 1 = e^{-x} - 1 < 0 \text{ for } x > 0$$

False.

**Exercise 21.** For  $0 < x < \frac{\pi}{2}$ ,

$$(\sin x)' = \cos x > 0 \text{ for } 0 < x < \frac{\pi}{2}$$

$$(x - \sin x)' = 1 - \cos x \geq 0 \text{ for } 0 < x < \frac{\pi}{2}$$

$$(x - \sin x)(x = 0) = 0 \implies \sin x < x$$

**Exercise 22.**

$$\frac{1}{t} < \frac{1}{t} \left( \frac{x+1}{t} \right) \quad \text{if } 0 < x < t < x+1$$

$$\int_x^{x+1} \frac{1}{t} dt = \ln(x+1) - \ln x; \int_x^{x+1} \frac{x+1}{t} dt = \left[ \frac{1}{x} \right] \text{ So } \ln \frac{x+1}{x} < \frac{1}{x}$$

**Exercise 23.**

$$(x - \sin x)' = 1 - \cos x \geq 0 \quad \forall x > 0$$

since  $(x - \sin x)(x = 0) = 0$ ;  $(x - \sin x)'(x = 0) = 0$ , then  $x - \sin x > 0$  in general for  $x > 0$ 

$$(\sin x - \left( x - \frac{x^3}{6} \right))' = \cos x - 1 + \frac{x^2}{2} > 0 \quad \forall x > 0$$

$$\implies x - \frac{x^3}{6} < \sin x < x$$

**Exercise 24.**  $(x^b + y^b)^{1/b} < (x^a + y^a)^{1/a}$  if  $x > 0$ ,  $y > 0$  and  $0 < a < b$

$(x^n + y^n)^{1/n} = x(1 + (\frac{y}{x})^n)^{1/n}$ . Without loss of generality, assume  $x < y$ .

Consider  $(1 + A^n)^{1/n}$ ,  $A$  constant.

$$\begin{aligned} ((1 + A^n)^{1/n})' &= \left( \exp \left( \frac{1}{n} \ln(1 + A^n) \right) \right)' = (1 + A^n)^{1/n} \left( \frac{-1}{n^2} \ln(1 + A^n) + \frac{1}{n} \left( \frac{1}{1 + A^n} \right) (\ln A) A^n \right) = \\ &= (1 + A^n)^{1/n} \left( \frac{-(1 + A^n) \ln(1 + A^n) + n(\ln A) A^n}{n^2(1 + A^n)} \right) = \\ &= (1 + A^n)^{1/n} \left( \frac{-\ln(1 + A^n) - A^n \ln(1 + A^n) + A^n \ln A^n}{n^2(1 + A^n)} \right) = \\ &= \frac{(1 + A^n)^{\frac{1-n}{n}}}{n^2} \left( \frac{-\ln(1 + A^n) + A^n \ln \left( \frac{A^n}{1 + A^n} \right)}{n^2(1 + A^n)} \right) < 0 \quad \text{since } \ln \left( \frac{A^n}{1 + A^n} \right) < 0 \\ &\implies (x^b + y^b)^{1/b} < (x^a + y^a)^{1/a} \quad \text{if } b > a \end{aligned}$$

**Exercise 25.**

(1)

$$\int_0^x e^{-t} t = (-te^{-t} - e^{-t}) \Big|_0^x = -xe^{-x} - e^{-x} + 1$$

(2)

$$\begin{aligned} \int_0^x t^2 e^{-t} dt &= -t^2 e^{-t} \Big|_0^x - \int -e^{-t} (2t) dt = -x^2 e^{-x} + 2 \int t e^{-t} dt = \\ &= -x^2 e^{-x} + -2xe^{-x} - 2e^{-x} + 2 \end{aligned}$$

(3)

$$\begin{aligned} \int_0^x t^3 e^{-t} dt &= -t^3 e^{-t} \Big|_0^x - 3 \int_0^x t^2 (-e^{-t}) dt = -x^3 e^{-x} + 3 \int_0^x t^2 e^{-t} dt = \\ &= -x^3 e^{-x} + 3(2)(e^{-x}) \left( e^x - 1 - x - \frac{x^2}{2!} \right) \end{aligned}$$

(4) Assume the induction hypothesis, that

$$\begin{aligned} \int_0^x t^n e^{-t} dt &= n! e^{-x} \left( e^x - \sum_{j=0}^n \frac{x^j}{j!} \right) \\ \int_0^x t^{n+1} e^{-t} dt &= -t^{n+1} e^{-t} \Big|_0^x - \int (n+1)t(-e^{-t}) = -x^{n+1} e^{-x} + (n+1)n! e^{-x} \left( e^x - \sum_{j=0}^n \frac{x^j}{j!} \right) \\ &= \boxed{(n+1)! e^{-x} \left( e^x - \sum_{j=0}^{n+1} \frac{x^j}{j!} \right)} \end{aligned}$$

**Exercise 26.** Consider the hint  $a_1 \sin x + b \cos x = A(a \sin x + b \cos x) + B(a \cos x - b \sin x)$ . Solve for  $A, B$  in terms of  $a_1, b_1, a, b$ . Matching up term by term the coefficients for sin and cos separately,

$$Aa^2 - abB = aa_1 \quad -Aab + Bb^2 = -a_1b$$

$$Ab^2 + Bab = b_1b \quad Aab + Ba^2 = ab_1$$

$$\boxed{A = \frac{aa_1 + bb_1}{a^2 + b^2}}$$

$$\boxed{B = \frac{ab_1 - a_1b}{a^2 + b^2}}$$

So if not both  $a, b = 0$ ,

$$\begin{aligned} \int \frac{a_1 \sin x + b_1 \cos x}{a \sin x + b \cos x} &= \int \frac{A(a \sin x + b \cos x) + B(a \cos x - b \sin x)}{a \sin x + b \cos x} = \\ &= \boxed{Ax + B \ln |a \sin x + b \cos x| + C} \end{aligned}$$

**Exercise 27.**

(1)

$$\begin{aligned} f'(x^2) &= \frac{1}{x} \\ \frac{df}{du} &= u^{-1/2} \\ f(x^2) &= 2x - 1 \end{aligned}$$

(2)

$$f'(\sin^2 x) = 1 - \sin^2 x \, f'(u) = 1 - u \, f = u - \frac{1}{2}u^2 + C \implies \boxed{f(x) = x - \frac{x^2}{2} + \frac{1}{2}}$$

(3)

$$f'(\sin X) = (1 - \sin^2 x) f'(u) = u - \frac{1}{3}u^3 + C \quad \boxed{f(x) = x - \frac{x^3}{3} + \frac{1}{3}}$$

(4)

$$f'(\ln x) = \begin{cases} 1 & \text{for } x \leq 1 \\ x & \text{for } x > 1 \end{cases} = \begin{cases} 1 & \text{for } 0 < x \leq 1 \\ e^{\ln x} & x > 1 \end{cases}$$

$$\boxed{f(y) = \begin{cases} y & \text{for } y < 0 \\ e^y - 1 & \text{for } y > 0 \end{cases}}$$

**Exercise 28.**

(1)

$$Li(x) = \int_2^x \frac{dt}{\ln t} \text{ if } x \geq 2$$

$$Li(x) = \frac{x}{\ln x} - 2 \frac{1}{\ln 2} - \int_2^x \frac{-1}{(\ln x)^2} dt = \frac{x}{\ln x} + \int_2^x \frac{dt}{(\ln x)^2} - \frac{2}{\ln 2}$$

(2)

$$Li(x) = \frac{x}{\ln x} - \frac{2}{\ln x} - \frac{2}{(\ln 2)^2} + \frac{x}{(\ln x)^2} - \int_a^x \frac{-2}{(\ln t)^3} dt \quad C_2 = \frac{-2}{\ln x} + - \sum_{j=2}^2 \frac{2(j-1)!}{(\ln 2)^j}$$

$$Li(x) = \frac{x}{\ln x} + \sum_{k=1}^{n-1} \frac{k!x}{\ln^{k+1} x} + n! \left( \frac{x}{\ln^{n+1} x} - \int_2^x \frac{-(n+1)dt}{\ln^{n+2} t} \right) \quad C_n = -2 \frac{1}{\ln 2} - \sum_{j=2}^n \frac{2(j-1)!}{(\ln 2)^j}$$

$$Li(x) = \frac{x}{\ln x} + \sum_{k=1}^n \frac{k!x}{\ln^{k+1} x} + (n+1)! \int_2^x \frac{dt}{\ln^{(n+1)+1} t} \quad C_{n+1} = -2 \frac{1}{\ln 2} - \sum_{j=2}^{n+1} \frac{2(j-1)!}{(\ln 2)^j}$$

(3)

$$u = \ln t$$

$$Li(x) = \int_2^x \frac{dt}{\ln t} \quad \begin{aligned} du &= \frac{1}{t} dt \\ e^u du &= dt \, e^u = t \end{aligned}$$

$$\boxed{Li(x) = \int_{\ln 2}^{\ln x} \frac{e^t dt}{t}}$$

(4)

$$c = 1 + \frac{1}{2} \ln 2$$

$$\int_{c-1}^{x-1} \frac{e^{2(u+1)} du}{u} = e^2 \int_{c-1}^{x-1} \frac{e^{2u}}{u} du =$$

$$t = u + 1 \implies = e^2 \frac{1}{2} \int_{\frac{2(c-1)}{3}}^{\frac{2(x-1)}{3}} \frac{e^t}{\frac{t}{2}} dt =$$

$$= \boxed{e^2 Li(e^{2(x-1)})}$$

(5)

$$f(x) = e^4 Li(e^{2(x-2)}) - e^2 Li(e^{2(x-1)})$$

$$= \int_c^x \frac{e^{2t}}{t-2} - \int_c^x \frac{e^{2t}}{(t-1)} \Rightarrow \boxed{f'(x) = \frac{e^{2x}}{x-2} + -\frac{e^{2x}}{x-1} = e^{2x} \left( \frac{1}{x^2 - 3x + 2} \right)}$$

**Exercise 29.**  $f(x) = \log |x|$  if  $x < 0$ .  $\forall x < 0 \exists$  uniquely  $\ln |x|$  since  $f' = \frac{1}{x} < 0 \forall x < 0$ .

$$-e^y = x(y) = g(y) \quad \boxed{D = \mathbb{R}}$$

Recall Theorem 3.10.

**Theorem 21.** Assume  $f$  is strictly increasing and continuous on an interval  $[a, b]$ . Let  $c = f(a)$ ,  $d = f(b)$  and let  $-g$  be the inverse of  $f$ .

That is  $\forall y \in [c, d]$ , Let  $g(y)$  be that  $x \in [a, b]$ , such that  $y = f(x)$ .

Then

- (1)  $-g$  is strictly increasing on  $[c, d]$
- (2)  $-g$  is continuous on  $[c, d]$

**Exercise 30.**  $f(x) = \int_0^x (1+t^3)^{-1/2} dt$  if  $x \geq 0$ .

(1)

$$f'(x) = \frac{1}{\sqrt{1+x^3}} > 0 \text{ for } x > 0$$

(2)

$$g'(x) = \frac{1}{f'(x)} = \sqrt{1+x^3} \quad g''(x) = \frac{3x^2}{2\sqrt{1+x^3}}$$

**7.4 Exercises - Introduction, The Taylor polynomials generated by a function, Calculus of Taylor polynomials.** Use the following theorems for the following exercises.

**Theorem 22** (Properties of Taylor polynomials, Apostol Vol. 1. Theorem 7.2.). (1) *Linearity*  $T_n(c_1 f + c_2 g) = c_1 T_n(f) + c_2 T_n(g)$

(2) *Differentiation*  $(T_n f)' = T_{n-1}(f')$

(3) *Integration.* If  $g(x) = \int_a^x f(t) dt$   
 $T_{n+1}g(x) = \int_a^x T_n f(t) dt$

**Theorem 23** ( Substitution Property, Apostol Vol. 1. Theorem 7.3. ). Let  $g(x) = f(cx)$ ,  $c$  is a constant.

$$(12) \quad T_n g(x; a) = T_n f(cx; ca)$$

This theorem is useful for finding new Taylor polynomials without having to find the  $j$ th derivatives of the desired function.

**Theorem 24.**  $P_n$  is a polynomial of degree  $n \geq 1$ .

Let  $f, g$  be 2 functions with derivatives of order  $n$  at 0.

$$(13) \quad f(x) = P_n(x) + x^n g(x)$$

where  $g(x) \mapsto 0$  as  $x \mapsto 0$ .

Then  $P_n = T_n(f, x=0)$ .

**Exercise 3.**

$$T_n f(x) = \sum_{j=0}^n \frac{f^{(j)}(a)}{j!} (x-a)^j$$

$$a^x = e^{x \ln a}$$

$$(a^x)' = (a^x) \ln a$$

$$(a^x)^{(n+1)} = (a^x (\ln a)^n)' = a^x (\ln a)^{n+1}$$

$$T_n(a^x) = \sum_{j=0}^n \frac{(\ln a)^j}{j!} x^j$$

**Exercise 4.**

$$\begin{aligned}\left(\frac{1}{1+x}\right)' &= \frac{-1}{(1+x)^2}; \left(\frac{1}{1+x}\right)'' = \frac{(-1)^2 2}{(1+x)^3} \\ \left(\frac{1}{1+x}\right)^{(n+1)} &= \left(\frac{(-1)^n n!}{(1+x)^{n+1}}\right)' = \frac{(-1)^{n+1} (n+1)!}{(1+x)^{n+2}} \\ T_n\left(\frac{1}{1+x}\right) &= \sum_{j=0}^n (-1)^j x^j\end{aligned}$$

**Exercise 5.** Use Theorem 7.4. Theorem 7.4 says for  $f(x) = P_n(x) + x^n g(x)$ ,  $P_n(x)$  is the Taylor polynomial.

$$\begin{aligned}\frac{1}{1-x^2} &= \left(\sum_{j=0}^n (x^2)^j\right) + \frac{(x^2)^{n+1}}{1-x^2} = \left(\sum_{j=0}^n x^{2j}\right) + \frac{x^{2n+2}}{1-x^2} \\ \frac{x}{1-x^2} &= \left(\sum_{j=0}^n x^{2j+1}\right) + \frac{(x^{2n+3})}{1-x^2} \\ T_{2n+1}\left(\frac{x}{1-x^2}\right) &= \sum_{j=0}^n x^{2j+1}\end{aligned}$$

**Exercise 6.**

$$(\ln(1+x))' = \frac{1}{1+x} \quad T_n\left(\frac{1}{1+x}\right) = \sum_{j=0}^n (-x)^j \quad T_n(\ln(1+x)) = \sum_{j=0}^n (-1)^j \frac{x^{j+1}}{j+1} = \sum_{j=1}^n (-1)^{j+1} \frac{x^j}{j}$$

**Exercise 7.**

$$\begin{aligned}\left(\log \sqrt{\frac{1+x}{1-x}}\right)' &= \sqrt{\frac{1-x}{1+x}} \sqrt{\frac{1-x}{1+x}} \frac{1}{(1-x)^2} = \frac{1}{(1+x)(1-x)} = \frac{1}{1-x^2} \\ \int \frac{1}{1-x^2} &= \log \sqrt{\frac{1+x}{1-x}} \text{ so } \int \sum_{j=0}^n x^{2j} = \sum_{j=0}^n \frac{x^{2j+1}}{2j+1} = T_{2n+1}\left(\ln \sqrt{\frac{1+x}{1-x}}\right)\end{aligned}$$

**Exercise 8.**

$$T_n\left(\frac{1}{2-x}\right) = T_n\left(\frac{1/2}{1-x/2}\right) = \frac{1}{2} T_n\left(\frac{1}{1-(\frac{1}{2})x}\right) = \frac{1}{2} \left(\sum_{j=0}^n \left(\frac{1}{2}\right)^j x^j\right) = \sum_{j=0}^n \frac{x^j}{2^{j+1}}$$

**Exercise 9.** We can show this in two ways.

We could write out the actual polynomial expansion.

$$(1+x)^\alpha = \sum_{j=0}^n \binom{\alpha}{j} 1^{\alpha-j} x^j = \sum_{j=0}^n \binom{\alpha}{j} x^j$$

or determine each of the coefficients of the Taylor polynomial.

$$\begin{aligned}((1+x)^\alpha)' &= \alpha(1+x)^{\alpha-1}; ((1+x)^\alpha)'' = \alpha(\alpha-1)(1+x)^{\alpha-2} \\ ((1+x)^\alpha)^{(n+1)} &= \left(\frac{\alpha!}{(\alpha-n)!} (1+x)^{\alpha-n}\right)' = \frac{\alpha!}{(\alpha-(n+1))!} (1+x)^{\alpha-(n+1)}\end{aligned}$$

**Exercise 10.** Use the substitution theorem, Apostol Vol.1. Thm. 7.3., to treat  $\cos 2x$ .

$$\begin{aligned}T_{2n}(\cos x) &= \sum_{j=0}^n \frac{(-1)^j x^{2j}}{(2j)!}; T_{2n}(\cos 2x) = \sum_{j=0}^n \frac{(-1)^j (2x)^{2j}}{(2j)!} \\ T_{2n}(\sin x^2) &= T_{2n}\left(\frac{1}{2}(1 - \cos 2x)\right) = \frac{1}{2} \left(1 - \sum_{j=0}^n \frac{(-1)^j (2x)^{2j}}{(2j)!}\right) = \sum_{j=1}^n \frac{(-1)^{j+1} 2^{2j-1} x^{2j}}{(2j)!}\end{aligned}$$

**7.8 Exercises - Taylor's formula with remainder, Estimates for the error in Taylor's formula, Other forms of the remainder in Taylor's formula.** We will use Theorem 7.7, which we learn in the preceding sections, extensively.

**Theorem 25.** If for  $j = 1, \dots, n+1$ ,  $m \leq f^{(j)}(t) \leq \forall t \in I$ ,  $I$  containing  $a$ ,

$$(14) \quad m \frac{(x-a)^{n+1}}{(n+1)!} \leq E_n(x) \leq M \frac{(x-a)^{n+1}}{(n+1)!} \quad \text{if } x > a$$

$$(15) \quad m \frac{(a-x)^{n+1}}{(n+1)!} \leq (-1)^{n+1} E_n(x) \leq M \frac{(a-x)^{n+1}}{(n+1)!} \quad \text{if } x < a$$

$$(16) \quad E_n(x) = \frac{1}{n!} \int_a^x (x-t)^n f^{(n+1)}(t) dt$$

**Exercise 1.** For  $a = 0$ ,  $|\sin^{(j)}(x)| \leq 1$  for  $\forall x \in \mathbb{R}$ .

$$E_n(x) \leq \frac{(x)^{2n+1}}{(2n+1)!} \text{ if } x > 0; (-1)^{2n+1} E_{2n}(x) \leq (+1) \frac{(-x)^{2n+1}}{(2n+1)!}$$

$$\implies \boxed{E_{2n}(x) \leq \frac{|x|^{2n+1}}{(2n+1)!}}$$

**Exercise 2.**

$$\cos x = \sum_{k=0}^n \frac{(-1)^k x^{2k}}{(2k)!} + E_{2n+1}(x) \quad |\cos^{(j)}(x)| \leq 1$$

$$E_{2n+1}(x) \leq \frac{x^{2n+2}}{(2n+2)!}; (-1)^{2n+2} E_{2n+1}(x) \leq (1) \frac{(-x)^{2n+2}}{(2n+2)!}$$

$$\implies |E_{2n+1}(x)| \leq \frac{|x|^{2n+2}}{(2n+2)!}$$

**Exercise 3.**

$$\arctan x = \sum_{k=0}^{n-1} \frac{(-1)^k x^{2k+1}}{2k+1} + E_{2n}(x)$$

$$\sum_{k=0}^{n-1} \frac{(-1)^k x^{2k+1}}{2k+1} \left( \frac{(2k)!}{(2k)!} \right) = \sum_{k=0}^{n-1} \frac{(-1)^k (2k)! x^{2k+1}}{(2k+1)!} \implies f^{(2k+1)}(0) = (-1)^k (2k)!$$

$$\frac{f^{(2n+1)}(0) x^{2n+1}}{(2n+1)!} = \frac{(-1)^n (2n)! x^{2n+1}}{(2n+1)!} = \frac{(-1)^n x^{2n+1}}{2n+1} \leq \frac{x^{2n+1}}{2n+1}$$

Note how  $j$ th derivative  $(\arctan x)^{(j)}$  changes sign with each differentiation for  $f^{(2j+1)}(0)$ . Then we can always pick a small enough closed interval with  $a = 0$  as a left or right end point to make the  $f^{(2j+1)}(0)$  value the biggest for  $f^{(2j+1)}(t)$ .

**Exercise 4.**

(1)

$$x^2 = \sin x = x - \frac{x^3}{6} \implies \frac{x^3}{6} + x^2 - x = \frac{x}{6} \left( x - (-3 + \sqrt{15}) \right) \left( x - (-3 - \sqrt{15}) \right)$$

$$\boxed{x = \sqrt{15} - 3}$$

(2)

$$E_4(r; 0) = \frac{1}{4!} \int_0^r (r-t)^4 \cos t dt > 0 \sin r - r^2 = 0 + E_4(r) \leq \frac{r^5}{5!} < \frac{r}{5!} = \frac{3}{5(2)(5)(4)(3)(2)1} = \frac{\frac{3}{4}}{(5)(4)(2)5} < \frac{1}{200}$$

**Exercise 5.**

$$\arctan r - r^2 = r - \frac{r^3}{3} - r^2 + E_4(r; 0) = 0 + E_4(r; 0)$$

$$E_4(r, 0) = \frac{1}{4!} \int_0^r (x-t)^4 f^{(5)}(t) dt \leq \frac{M(r^5)}{5!} \frac{r^5}{5!} = 0.065536 < \frac{7}{100}$$

$$E_4(r, 0) < E_j(r, 0); j > 4$$

the 5th degree term is  $\frac{f^{(5)}(0)}{5!} r^5 = \frac{24}{5!} r^5 > 0$

so  $r^2 - \arctan r = -E_4(r, 0) < 0$

**Exercise 6.** Apply long division on the fraction in the integrand.

$$\begin{aligned}\int_0^1 \frac{1+x^{30}}{1+x^{60}} dx &= \int_0^1 1 + \frac{x^{30}-x^{60}}{1+x^{60}} dx = 1 + \int_0^1 x^{30} \left( \frac{1-x^{30}}{1+x^{60}} \right) dx = \\ &= 1 + c \left. \frac{1}{31} x^{31} \right|_0^1 = 1 + \frac{c}{31}\end{aligned}$$

**Exercise 7.**  $\int_0^{1/2} \frac{1}{1+x^4} dx$ .

$$\begin{aligned}\frac{1}{1+x^4} &= \sum_{j=0}^{\infty} (-x^4)^j = \sum_{j=0}^{n-1} (-x^4)^j + E_n = 1 - x^4 + x^8 \dots \\ \frac{16}{17} \frac{x^{4n+1}}{(n+1)!} &\leq E_n(x; 0) \leq \frac{1}{(n+1)!} (x^4)^{n+1} \\ \Rightarrow \int_0^{1/2} E_n &= \frac{\left(\frac{1}{2}\right)^{4n+5}}{(n+1)!} \left( \frac{1}{4n+5} \right) \\ \int_0^{1/2} \frac{1}{1+x^4} &\simeq \frac{1}{2} + \frac{-1}{5} \left( \frac{1}{2} \right)^5 \\ 0.493852 &< 0.49375 < 0.493858\end{aligned}$$

**Exercise 8.**

(1)

$$\begin{aligned}0 \leq x \leq \frac{1}{2} \quad \sin x &= x - \frac{x^3}{3!} + E_4(x) \\ |E_4(x)| &\leq \frac{M|x|^5}{5!} = \frac{\sin \frac{1}{3}|x|^5}{5!} \leq \frac{1}{5!} \left( \frac{1}{2} \right)^5\end{aligned}$$

(2)

$$\begin{aligned}\sin x^2 &= x^2 - \frac{x^6}{6} + E_4(x^2) \\ \int_0^{\sqrt{2}/2} \sin x^2 &= \left( \frac{1}{3} x^3 - \frac{x^7}{42} \right) \Big|_0^{\sqrt{2}/2} = \sqrt{2} \left( \frac{1}{12} - \frac{1}{42(16)} \right) \\ E_4(x^2) &\leq \frac{\sqrt{2}}{64(5!)} \\ \int_0^{\sqrt{2}/2} \sin x^2 &\leq \sqrt{2} \left( \frac{55}{672} + \frac{1}{64(5!)} \right) = 0.1159\end{aligned}$$

**Exercise 9.**

$$\begin{aligned}\sin x &= x - \frac{x^3}{6} + \frac{x^5}{5!} E_6(x; 0) \leq \frac{(1)x^7}{7!} \\ \frac{\sin x}{x} &= 1 - \frac{x^2}{6} + \frac{x^4}{5!} + \frac{E_6(x; 0)}{x} \leq 1 - \frac{x^2}{6} + \frac{x^4}{5!} + \frac{x^6}{7!} \\ \int_0^1 \frac{\sin x}{x} dx &= 1 - \frac{1}{3} \left( \frac{1}{6} \right) + \frac{1}{5(5!)} + \frac{1}{7(7!)} = 0.9461 + \frac{1}{7(7!)} = 0.9461 + 0.0000283\end{aligned}$$

**Exercise 10.**  $\alpha = \arctan \frac{1}{5}, \beta = 4\alpha - \frac{\pi}{4}$ .

(1)

$$\begin{aligned}\tan(A+B) &= \frac{(\tan A + \tan B)}{1 - \tan A \tan B}; A = B = \alpha; \tan 2\alpha = \frac{2 \tan \alpha}{1 - \tan^2 \alpha} = \frac{2/5}{24/25} = 5/12 \\ \tan 4\alpha &= \frac{2(\tan 2\alpha)}{1 - \tan^2(2\alpha)} = \frac{2(5/12)}{1 - (5/12)^2} = \frac{10/12}{119/144} = \frac{120}{119} A = 4\alpha, B = -\frac{\pi}{4} \\ \tan(4\alpha - \frac{\pi}{4}) &= \tan \beta = \frac{\tan 4\alpha + \tan(-\frac{\pi}{4})}{1 - \tan 4\alpha \tan(-\frac{\pi}{4})} = \frac{\frac{120}{119} + -\frac{119}{119}}{1 - \frac{120}{119}(-1)} = \frac{1}{239}\end{aligned}$$

$4 \arctan \frac{1}{5} = \frac{\pi}{4} + \arctan \frac{1}{239}$	This is incredible.
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(2)

$$\begin{aligned}
T_{11}(\arctan x) &= \sum_{k=0}^{6-1} \frac{(-1)^k x^{2k+1}}{2k+1} + E_{2(6)}(x) \\
&= x + \frac{-x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \frac{x^{11}}{11} \dots \\
&\implies 3.158328957 < 16 \arctan \frac{1}{5} < 3.158328958
\end{aligned}$$

(3)

$$\begin{aligned}
T_3(\arctan x); x &= \frac{1}{239} \\
-0.016736304 &< -4 \arctan \frac{1}{239} < -0.016736304
\end{aligned}$$

(4)

$$\begin{aligned}
&3.141592625 \\
3.158328972 - 0.016736300 &= 3.141592672
\end{aligned}$$

### 7.11 Exercises - Further remarks on the error in Taylor's formula. The $o$ -notation; Applications to indeterminate forms.

**Exercise 1.**

$$2^x = \exp x \ln 2 = \boxed{1 + (x \ln 2) + \frac{x^2 (\ln 2)^2}{2!}} + o(x^2)$$

**Exercise 2.**

$$\begin{aligned}
x(\cos x) &= ((x-1)+1)(\cos x) = (x-1)\cos 1 + (-\sin 1)(x-1)^2 - \frac{\cos 1(x-1)^3}{2} + \cos 1 + \\
&+ (-\sin 1)(x-1) - \frac{\cos 1(x-1)^2}{2} + \frac{\sin 1(x-1)^3}{3!} = \\
&= \cos 1 + (\cos 1 - \sin 1)(x-1) + \left(-\sin 1 - \frac{\cos 1}{2}\right)(x-1)^2 + \left(\frac{\sin 1 - 3\cos 1}{3!}\right)(x-1)^3 + o(x-1)^3
\end{aligned}$$

**Exercise 3.** Just treat the argument of  $\sin x - x^2$  just like  $u$  with  $u \rightarrow 0$ .

$$\begin{aligned}
\sin(x-x^2) &= (x-x^2) - \frac{(x-x^2)^3}{3!} + \frac{(x-x^2)^5}{5!} + o(x-x^2)^5 = \\
&= (x-x^2) - \frac{1}{6}(x^3-3x^4+3x^5-x^6) + \frac{1}{120}(x^5-5x^6+10x^7-10x^8+5x^9+x^{10}) = \\
&= (x-x^2) - \frac{1}{6}x^3 - \frac{1}{2}x^4 + \frac{61}{120}x^5 - \frac{25}{120}x^6
\end{aligned}$$

**Exercise 4.**

$$\begin{aligned}
\log x &= \log(1+(x-1)) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} \\
&\implies \boxed{a=0; b=1, c=\frac{-1}{2}}
\end{aligned}$$

**Exercise 5.**

$$\begin{aligned}
\cos x &= 1 - \frac{1}{2}x^2 + o(x^3) \text{ as } x \rightarrow 0 & 1 - \cos x &= \frac{1}{2}x^2 + o(x^3) \\
&& \frac{1 - \cos x}{x^2} &= \frac{1}{2} + \frac{o(x^3)}{x^2} \\
&\text{since } \frac{1 - \cos x}{x^2} = \frac{1}{2} + o(x), \frac{1 - \cos x}{x^2} \rightarrow \frac{1}{2} \text{ as } x \rightarrow 0 \\
\cos x &= 1 - \frac{1}{2}x^2 + \frac{x^4}{4!} + o(x^5) \implies \cos 2x = 1 - 2x^2 + \frac{2}{3}x^4 + o(x^5) \\
\frac{1 - \cos 2x - 2x^2}{x^4} &= \frac{-\frac{2}{3}x^4 - o(x^5)}{x^4} = \frac{-2}{3} - o(x) \rightarrow -\frac{2}{3} \text{ as } x \rightarrow 0
\end{aligned}$$

**Exercise 6.**

$$\lim_{x \rightarrow 0} \frac{\sin ax}{\sin bx} = \lim_{x \rightarrow 0} \frac{ax - \frac{(ax)^3}{3!} + o(x^4)}{bx + o(x^2)} = \frac{a}{b}$$

**Exercise 7.**

$$\lim_{x \rightarrow 0} \frac{\sin 2x}{\cos 2x \sin 3x} = \lim_{x \rightarrow 0} \frac{(2x) + \frac{(2x)^3}{3!} + o(x^4)}{\left(1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} + o(x^5)\right) \left((3x) - \frac{(3x)^3}{3!} + o(x^4)\right)} = \boxed{\frac{2}{3}}$$

**Exercise 8.**

$$\lim_{x \rightarrow 0} \frac{\sin x - x}{x^3} = \boxed{-\frac{1}{6}}$$

**Exercise 9.**

$$\lim_{x \rightarrow 0} \frac{\ln 1 + x}{e^{2x} - 1} = \lim_{x \rightarrow 0} \frac{x - o(x)}{2x + o(x)} = \frac{1}{2}$$

**Exercise 10.** Don't do the trig. identity.

$$\lim_{x \rightarrow 0} \frac{1 - \cos^2 x}{x \tan x} = \lim_{x \rightarrow 0} \frac{1 - \left(1 - \frac{x^2}{2!} + o(x^2)\right)^2}{x \left(x + \frac{x^3}{6!} + o(x^3)\right)} = \lim_{x \rightarrow 0} \frac{1 - (1 + x^2 + o(x^2))}{x^2 + o(x^2)} = 1$$

**Exercise 11.**

$$\lim_{x \rightarrow 0} \frac{x + \frac{x^3}{6} + o(x^4)}{x - \frac{x^3}{3}} = 1$$

**Exercise 12.**

$$\lim_{x \rightarrow 0} \frac{e^{x \ln a} - 1}{e^{x \ln b} - 1} = \lim_{x \rightarrow 0} \frac{x \ln a + o(x)}{x \ln b + o(x)} = \boxed{\ln a / b}$$

**Exercise 13.**

$$\lim_{x \rightarrow 1} \frac{(x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} + o(x-1)^4}{(x+2)(x+1)} = \boxed{\frac{1}{3}}$$

**Exercise 14.**  $\boxed{1}$ .

**Exercise 15.**

$$\lim_{x \rightarrow 0} \frac{x(e^x + 1) - 2(e^x - 1)}{x^3} = \lim_{x \rightarrow 0} \frac{x(2 + x + \frac{x^2}{2}) - 2(x + x^2/2 + x^3/6)}{x^3} = \lim_{x \rightarrow 0} \frac{x^3(\frac{1}{6})}{x^3} = \boxed{\frac{1}{6}}$$

**Exercise 16.**

$$\lim_{x \rightarrow 0} \frac{\ln(1+x) - x}{1 - \cos x} = \lim_{x \rightarrow 0} \frac{x - \frac{x^2}{2} + \frac{x^3}{3} + o(x^3) - x}{x^3/2} = \boxed{-1}$$

**Exercise 17.**

$$\lim_{x \rightarrow \pi/2} \frac{\cos x}{x - \frac{\pi}{2}} = \frac{0 + -1(x - \frac{\pi}{2})}{x - \frac{\pi}{2}} = \boxed{-1}$$

**Exercise 18.**  $\boxed{1/6}$

**Exercise 19.**

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\cosh x - \cos x}{x^2} &= \lim_{x \rightarrow 0} \frac{\frac{e^x + e^{-x}}{2} - \cos x}{x^2} = \\ &= \lim_{x \rightarrow 0} \frac{1 + x + \frac{x^2}{2} - 2\left(1 - \frac{x^2}{2}\right) + o(x^3)}{x^2} = \boxed{2} \end{aligned}$$

**Exercise 20.**

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{3 \tan 3x - 12 \tan x}{3 \sin 4x - 12 \sin x} &= \lim_{x \rightarrow 0} \frac{(4x) + \frac{(4x)^3}{3} - 4\left(x + \frac{x^3}{3}\right) + o(x^4)}{(4x) - \frac{(4x)^3}{3!} - 4\left(x - \frac{x^3}{3!}\right) + o(x^4)} = \\ &= \frac{4^3 - 4}{\frac{-4^3 + 4}{2}} = \boxed{-2} \end{aligned}$$

**Exercise 21.**

$$\begin{aligned}
\lim_{x \rightarrow 0} \frac{a^x - a^{\sin x}}{x^3} &= \lim_{x \rightarrow 0} \frac{e^{x \ln a} - e^{\sin x \ln a}}{x^3} = \\
&= \lim_{x \rightarrow 0} \frac{1 + x \ln a + \frac{(x \ln a)^2}{2!} + \frac{(x \ln a)^3}{6} - \left(1 + \sin x \ln a + \frac{\sin^2 x (\ln a)^2}{2} + \frac{\sin^3 x (\ln a)^3}{3!}\right) + o(x^4)}{x^3} = \\
&= \lim_{x \rightarrow 0} \frac{\ln a \left(x - \left(x - \frac{x^3}{3!}\right)\right) + \frac{(x^2 \ln a)^2}{2!} - \frac{(\ln a)^2}{2} (x^2) + \frac{(\ln a)^3}{6} (x^3 - x^3) + o(x^4)}{x^3} = \boxed{\frac{\ln a}{6}}
\end{aligned}$$

**Exercise 22.**

$$\begin{aligned}
\lim_{x \rightarrow 0} \frac{\cos \sin x - \cos x}{x^4} &= \lim_{x \rightarrow 0} \frac{1 - \frac{\sin^2 x}{2!} + \frac{\sin^4 x}{4!} + o(x^5) - \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!}\right)}{x^4} = \\
&= \lim_{x \rightarrow 0} \frac{\frac{1}{2} \left(x^2 - \left(x^2 - \frac{x^4}{3}\right)\right) + \left(\frac{x^4 - x^4}{4!}\right) + o(x^5)}{x^4} = \boxed{\frac{1}{6}}
\end{aligned}$$

**Exercise 23.**

$$\lim_{x \rightarrow 1} x^{\frac{1}{1-x}} = \lim_{x \rightarrow 1} 1 e^{\frac{1}{1-x} \ln x} = \exp \left( \lim_{x \rightarrow 1} \frac{\ln x}{1-x} \right) = \exp \left( \lim_{x \rightarrow 1} \frac{(x-1) + o(x-1)^2}{1-x} \right) = \boxed{e^{-1}}$$

**Exercise 24.**

$$\begin{aligned}
\lim_{x \rightarrow 0} (x + e^{2x})^{1/x} &= \exp \lim_{x \rightarrow 0} \frac{1}{x} \ln (x + e^{2x}) \\
\lim_{x \rightarrow 0} \frac{\ln (x + e^{2x})}{x} &= \lim_{x \rightarrow 0} \frac{\ln (1 + x + e^{2x} - 1)}{x} = \\
&= \lim_{x \rightarrow 0} \frac{x + e^{2x} - 1 + o(x^2)}{x} = \lim_{x \rightarrow 0} \frac{3x + o(x^2)}{x} = 3 \\
\Rightarrow \lim_{x \rightarrow 0} (x + e^{2x})^{1/x} &= e^3
\end{aligned}$$

**Exercise 25.**

$$\begin{aligned}
\lim_{x \rightarrow 0} \frac{(1+x)^{1/x} - e}{x} &= \lim_{x \rightarrow 0} \frac{e^{\frac{1}{x} \ln (1+x)} - e}{x} = \lim_{x \rightarrow 0} \frac{e^{\frac{x + \frac{x^2}{2} + o(x^2)}{x}} - e}{x} = \\
&= \lim_{x \rightarrow 0} \frac{e^{1 - \frac{x}{2} + o(x)} - e}{x} = \lim_{x \rightarrow 0} \frac{e(1 + \frac{-x}{2} + o(x)) - e}{x} = \frac{-e}{2}
\end{aligned}$$

**Exercise 26.**

$$\lim_{x \rightarrow 0} \left( \frac{(1+x)^{1/x}}{e} \right)^{1/x} = \lim_{x \rightarrow 0} \left( \exp \left( \frac{1}{x} \ln (1+x) \right) - 1 \right)^{1/x} = \lim_{x \rightarrow 0} e^{\frac{x - \frac{x^2}{2} + o(x^3) - x}{x^2}} = e^{-1/2}$$

**Exercise 27.**

$$\begin{aligned}
(\arcsin x)' &= \frac{1}{\sqrt{1-x^2}} \\
(\arcsin x)'' &= \frac{x}{(1-x^2)^{3/2}} \\
(\arcsin x)''' &= \frac{1}{(1-x^2)^{3/2}} + \frac{3x^2}{(1-x^2)^{5/2}} \\
\exp \left( \lim_{x \rightarrow 0} \frac{1}{x^2} \left( \ln \left( \frac{\arcsin x}{x} \right) \right) \right) &= \exp \lim_{x \rightarrow 0} \frac{1}{x^2} \ln \left( 1 + \left( \frac{\arcsin x}{x} - 1 \right) \right) = \\
&= \exp \left( \lim_{x \rightarrow 0} \frac{1}{x^2} \ln \left( 1 + \frac{x + x^3/6 + o(x^4) - x}{x} \right) \right) = \exp \left( \lim_{x \rightarrow 0} \frac{1}{x^2} \ln \left( 1 + \frac{x^2}{6} + o(x^3) \right) \right) = \\
&= \exp \left( \lim_{x \rightarrow 0} \frac{1}{x^2} \left( \frac{x^2 + o(x^3)}{6} \right) \right) = \boxed{e^{1/6}}
\end{aligned}$$

$$\textbf{Exercise 28.} \lim_{x \rightarrow 0} \left( \frac{1}{x} - \frac{1}{e^x - 1} \right) = \lim_{x \rightarrow 0} \left( \frac{e^x - 1 - x}{x(e^x - 1)} \right) = \lim_{x \rightarrow 0} \frac{\frac{x^2}{2} + o(x^3)}{x^2 + o(x^3)} = \boxed{\frac{1}{2}}$$

**Exercise 29.**

$$\begin{aligned}\lim_{x \rightarrow 1} \left( \frac{1}{\log x} - \frac{1}{x-1} \right) &= \lim_{x \rightarrow 1} \left( \frac{(x-1) - \log x}{(x-1) \log x} \right) = \\ &= \lim_{x \rightarrow 1} \frac{(x-1) - ((x-1) - \frac{(x-1)^2}{2} + o((x-1)^3))}{(x-1)((x-1) + o((x-1)^2))} = \boxed{\frac{1}{2}}\end{aligned}$$

**Exercise 30.**

$$\lim_{x \rightarrow 0} \frac{e^{ax} - e^x - x}{x^2} = \lim_{x \rightarrow 0} \frac{1 + ax + \frac{(ax)^2}{2} + o(x^3) - 1 - x - \frac{x^2}{2} - x}{x^2}$$

if  $a = 2$ , the limit is  $\boxed{2}$

**Exercise 31.**

(1) Prove  $\int_0^x f(t)dt = o\left(\int_0^x g(t)dt\right)$  as  $x \rightarrow 0$ , given  $f(x) = o(g(x))$ .

Consider  $\lim_{x \rightarrow 0} \frac{\int_0^x f(t)dt}{\int_0^x g(t)dt}$ .

Since  $f, g$  have derivatives in some interval containing 0,  $f, g$  continuous and differentiable for  $|t| \leq x$ .

$$\lim_{x \rightarrow 0} \frac{\int_0^x f(t)dt}{\int_0^x g(t)dt} = \lim_{x \rightarrow 0} \frac{\left(\frac{A(x)-A(0)}{x}\right)}{\left(\frac{B(x)-B(0)}{x}\right)} = \frac{f(0)}{g(0)} = \frac{\lim_{x \rightarrow 0} f(x)}{\lim_{x \rightarrow 0} g(x)} = 0$$

We can do the second to last step since  $f, g$  have derivatives at 0 and thus are continuous about 0.

(2) Consider  $\lim_{x \rightarrow 0} \frac{x}{e^x} = 0$ . However,  $\lim_{x \rightarrow 0} \frac{1}{e^x} = 1$ .

**Exercise 32.**

(1) Use long division to find that

$$\frac{1}{1+g(x)} = 1 - g(x) + g^2(x) + \frac{-g^3(x)}{1+g(x)} = 1 - g(x) + g^2(x) + o(g^2(x))$$

(2)

**Exercise 33.** Given  $\lim_{x \rightarrow 0} \left(1 + x + \frac{f(x)}{x}\right)^{1/x} = e^3$ , **use the hint.**

$$\lim_{x \rightarrow 0} g(x) = A, \quad \text{then } G(x) = A + o(1) \text{ as } x \rightarrow 0$$

Then

$$\begin{aligned}g(x) &= e^3 + o(1) = \left(1 + x + \frac{f}{x}\right)^{1/x} \\ x(e^3 + o(1))^x &= x + x^2 + f(x) \implies \boxed{f(0) = 0} \\ x \exp x \ln(e^3 + o(1)) &= x + x^2 + f(x) \\ 1 \exp x \ln(e^3 + o(1)) + x \exp x \ln(e^3 + o(1)) \left(\ln e^3 + o(1) + \frac{x}{e^3 + o(1)}(o'(1))\right) &= 1 + 2x + f'(0) \\ 1 + 3(0) + 0 &= 1 + 0 + f'(0) \implies f'(0) = 0\end{aligned}$$

We need to assume that in general  $o(1) = x + kx^2 + o(x^2)$ .

$$\begin{aligned}2 \exp x \ln(e^3 + o(1)) \left(\ln(e^3 + o(1)) + \frac{x}{e^3 + o(1)}(o'(1))\right) &+ \\ + x \exp x \ln(e^3 + o(1)) \left(\ln(e^3 + o(1)) + \frac{x}{e^3 + o(1)}o'(1)\right) &+ \\ x \exp x \ln(e^3 + o(1)) \left(\frac{2o'(1)}{(e^3 + o(1))} + \frac{x o''(1)}{e^3 + o(1)} - \frac{x}{(e^3 + o(1))}(o'(1))^2\right) &= 2 + f''(x)\end{aligned}$$

$$\xrightarrow{x \rightarrow 0} 2(3) = 2 + f'(0) \implies f'(0) = 4$$

To evaluate  $\lim_{x \rightarrow 0} \left(1 + \frac{f(x)}{x}\right)^{1/x}$ , consider a Taylor series expansion of  $f$ .

$$\begin{aligned}\lim_{x \rightarrow 0} \left(1 + \frac{f(x)}{x}\right)^{1/x} &= \lim_{x \rightarrow 0} \left(1 + \frac{0 + 0 + 4\frac{x^2}{2} + o(x^3)}{x}\right)^{1/x} = \lim_{x \rightarrow 0} (1 + x(2 + o(x)))^{1/x} = \\ &= \lim_{y \rightarrow 0} \lim_{x \rightarrow 0} (1 + x(2 + o(y)))^{1/x} = \lim_{y \rightarrow 0} \exp 2 + o(y) = e^2\end{aligned}$$

### 7.13 Exercises - L'Hopital's rule for the indeterminate form 0/0. Exercise 1.

$$\lim_{x \rightarrow 2} \frac{3x^2 + 2x - 16}{x^2 - x - 2} = \lim_{x \rightarrow 2} \frac{(3x + 8)(x - 2)}{(x - 2)(x + 1)} = \boxed{\frac{14}{3}}$$

#### Exercise 2.

$$\lim_{x \rightarrow 3} \frac{x^2 - 4x + 3}{2x^2 - 13x + 21} = \lim_{x \rightarrow 3} \frac{(x - 3)(x - 1)}{(2x - 1)(x - 3)} = \boxed{-2}$$

#### Exercise 3.

$$\lim_{x \rightarrow 0} \frac{\sinh x - \sin x}{x^3} = \frac{0}{0} = \lim_{x \rightarrow 0} \left(\frac{\cosh x - \cos x}{3x^2}\right) = \frac{0}{0} = \lim_{x \rightarrow 0} \left(\frac{\sinh x + \sin x}{6x}\right) = \lim_{x \rightarrow 0} \left(\frac{\cosh x + \cos x}{6}\right) = \boxed{\frac{1}{3}}$$

#### Exercise 4.

$$\lim_{x \rightarrow 0} \frac{(2 - x)e^x - x - 2}{x^3} = \lim_{x \rightarrow 0} \frac{-e^x + (2 - x) - 1}{3x^2} = \lim_{x \rightarrow 0} \frac{(2 - x)(1 + x + x^2/2 + x^3/6 + o(x^3))}{x^3} = \boxed{-\frac{1}{6}}$$

#### Exercise 5.

$$\lim_{x \rightarrow 0} \frac{\log(\cos ax)}{\log(\cos bx)} = \lim_{x \rightarrow 0} \frac{-\cos bx \sin ax}{-\cos ax \sin bx} = \lim_{x \rightarrow 0} \frac{\cos ax a^2}{\sin bx b^2} = \frac{a^2}{b^2}$$

#### Exercise 6. When it doubt, **Taylor expand**.

$$\begin{aligned}\lim_{x \rightarrow 0^+} \frac{x - \sin x}{(x \sin x)^{3/2}} &= \lim_{x \rightarrow 0^+} \frac{1 - \cos x}{\frac{3}{2}(x \sin x)^{1/2}(\sin x + x \cos x)} = \frac{2}{3} \lim_{x \rightarrow 0^+} \frac{1 - (1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^4))}{e^{\frac{1}{2} \ln(x \sin x)}(x - \frac{x^3}{6} + o(x^3) + x - \frac{x^3}{2} + o(x^3))} = \\ &= \frac{2}{3} \lim_{x \rightarrow 0^+} \left( \frac{\frac{x^2}{2} - \frac{x^4}{24} + o(x^4)}{\sqrt{x(x + \frac{x^3}{6} + o(x^3))}(2x - \frac{2}{3}x^3 + o(x^3))} \right) = \\ &= \frac{2}{3} \lim_{x \rightarrow 0^+} \left( \frac{\frac{1}{2} - \frac{x^2}{24} + o(x^2)}{\sqrt{1 + \frac{x^2}{6} + o(x^2)}(2 + \frac{-2}{3}x^2 + o(x^2))} \right) = \frac{2}{3} \lim_{x \rightarrow 0^+} \left( \frac{1/2}{2} \right) = \boxed{\frac{1}{6}}\end{aligned}$$

Notice in the third step how in general we deal with powers,  $(x \sin x)^{1/2}$ , is to convert it into exponential form,  $e^{\frac{1}{2} \ln(x \sin x)}$ , but it wasn't necessary.

#### Exercise 7. Do L'Hopital's first.

$$\begin{aligned}\lim_{x \rightarrow a^+} \frac{\sqrt{x} - \sqrt{a} + \sqrt{x - a}}{\sqrt{x^2 - a^2}} &= \lim_{x \rightarrow a^+} \frac{\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{x-a}}}{\frac{x}{\sqrt{x^2 - a^2}}} = \frac{1}{2} \lim_{x \rightarrow a^+} \frac{\sqrt{x^2 - a^2}}{x^{3/2}} + \frac{\sqrt{x+a}}{x} = \\ &= \frac{1}{2} \lim_{x \rightarrow a^+} \frac{\sqrt{x^2 - a^2} + x^{1/2}\sqrt{x+a}}{x^{3/2}} = \frac{1}{2} \frac{\sqrt{2a}}{a^{3/2}} = \frac{\sqrt{2}}{2\sqrt{a}}\end{aligned}$$

#### Exercise 8. Do L'Hopital's at the second step.

$$\begin{aligned}\lim_{x \rightarrow 1^+} \frac{\exp(x \ln x) - x}{1 - x + \ln x} &= \lim_{x \rightarrow 1^+} \frac{\exp x \ln x (\ln x + 1) - 1}{-1 + \frac{1}{x}} = \lim_{x \rightarrow 1^+} \frac{\exp(x \ln x)(\ln x + 1)^2 + \frac{1}{x} \exp(x \ln x)}{-1/x^2} = \\ &= \lim_{x \rightarrow 1^+} x^2 \exp(x \ln x)(\ln x + 1)^2 + x \exp(x \ln x) = 1 + 1 = \boxed{2}\end{aligned}$$

**Exercise 9.** Keep doing L'Hopital's.

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\arcsin 2x - 2 \arcsin x}{x^3} &= \lim_{x \rightarrow 0} \frac{\frac{2}{\sqrt{1-(2x)^3}} - 2 \frac{1}{\sqrt{1-x^2}}}{3x^2} = \\ &= \frac{2}{3} \lim_{x \rightarrow 0} \frac{-\frac{1}{2}(1-(2x)^2)^{-3/2}(-8x) - (-\frac{1}{2})(1-x^2)^{-3/2}(-2x)}{6x} = \\ &= \frac{1}{9} \lim_{x \rightarrow 0} \frac{4(1-(2x)^2)^{-3/2} + 4x(-\frac{3}{2})(1-(2x)^2)^{-5/2}(-8x) + -(1-x^2)^{-3/2} - x(-\frac{3}{2})(1-x^2)^{-5/2}(-2x)}{1} = \\ &= \frac{1}{9} \frac{4-1}{1} = \boxed{\frac{1}{3}} \end{aligned}$$

**Exercise 10.**

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{x \cot x - 1}{x^2} &= \lim_{x \rightarrow 0} \frac{x \cos x - \sin x}{x^2 \sin x} = \lim_{x \rightarrow 0} \frac{\cos x - x \sin x - \cos x}{2x \sin x + x^2 \cos x} = \lim_{x \rightarrow 0} \frac{-\sin x}{2 \sin x + x \cos x} = \\ &= - \lim_{x \rightarrow 0} \frac{\cos x}{2 \cos x + \cos x + -x \sin x} = - \lim_{x \rightarrow 0} \frac{\cos x}{3 \cos x - x \sin x} = \boxed{\frac{1}{3}} \end{aligned}$$

**Exercise 11.**

$$\lim_{x \rightarrow 1} \frac{\sum_{k=1}^n x^k - n}{x - 1} = \lim_{x \rightarrow 1} \frac{\sum_{k=1}^n kx^{k-1}}{1} = \sum_{k=1}^n k = \frac{n(n+1)}{2}$$

**Exercise 12.**

$$\begin{aligned} \lim_{x \rightarrow 0+} \frac{1}{x\sqrt{x}} \left( a \arctan \frac{\sqrt{x}}{a} - b \arctan \frac{\sqrt{x}}{b} \right) &= \lim_{x \rightarrow 0+} \frac{\left( a \frac{1}{1+\frac{x}{a^2}} \left( \frac{1}{2a\sqrt{x}} \right) - b \frac{1}{1+\frac{x}{b^2}} \left( \frac{1}{2b\sqrt{x}} \right) \right)}{\frac{3}{2}x^{1/2}} = \\ &= \frac{1}{3} \lim_{x \rightarrow 0+} \frac{a^2}{a^2+x} \left( \frac{1}{x} \right) - \frac{b^2}{b^2+x} \left( \frac{1}{x} \right) = \frac{1}{3} \lim_{x \rightarrow 0+} \frac{(b^2+x)a^2 - b^2(a^2+x)}{(a^2+x)(x+b^2)x} = \\ &= \frac{1}{3} \lim_{x \rightarrow 0+} \frac{(a^2-b^2)}{(a^2-x)(b^2+x)} = \boxed{\frac{1}{3} \frac{a^2-b^2}{a^2b^2}} \end{aligned}$$

**Exercise 13.**

$$\begin{aligned} \frac{(\sin 4x)(\sin 3x)}{x \sin 2x} &= \frac{(2 \sin 2x \cos 2x)(\sin 3x)}{x \sin 2x} = \frac{2(-2 \sin 2x \sin 3x + \cos 2x 3 \cos 3x)}{1} = \boxed{6} \text{ as } x \rightarrow 0 \text{ otherwise} \\ \frac{2 \cos 2x \sin 3x}{x} &\rightarrow \frac{4}{\pi} \text{ as } x \rightarrow \frac{\pi}{2} \end{aligned}$$

We used L'Hopital's at the second to last step for  $x \rightarrow 0$ .

**Exercise 14.**

$$\begin{aligned} \lim_{x \rightarrow 0} (x^{-3} \sin 3x + ax^{-2} + b) &= 0 \\ \lim_{x \rightarrow 0} \frac{\sin 3x + ax + bx^3}{x^3} &= \frac{3 \cos 3x + a + 3bx^2}{3x^2} = \frac{-9 \sin 3x + 6bx}{6x} = \frac{-27 \cos 3x + 6b}{6} = \frac{-27 + 6b}{6} = 0 \end{aligned}$$

So  $\boxed{b = \frac{9}{2}, a = -3}.$

**Exercise 15.**

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{1}{bx - \sin x} \int_0^x \frac{t^2 dt}{\sqrt{a+t}} &= \frac{\frac{x^3}{3}}{b - \cos x} = \frac{2x}{\frac{1}{2\sqrt{a+x}}(1 - \cos x) + \sqrt{a+x}(\sin x)} = \frac{2x\sqrt{a+x}}{\frac{1}{2}(1 - \cos x) + (a+x) \sin x} = \\ &= 2 \lim_{x \rightarrow 0} \frac{\sqrt{a+x} + \frac{x}{2\sqrt{a+x}}}{\frac{\sin x}{2} + \sin x + (a+x) \cos x} = 2 \frac{\sqrt{a}}{a} = 1 \\ &\implies \boxed{a = 4} \end{aligned}$$

Note that we had dropped the limit notation in some earlier steps and applied L'Hopital's a number of times, and we also rearranged the denominator and numerator cleverly at each step.

**Exercise 16.**

(1)

$$\begin{aligned}
&\text{angle } ABC \text{ is } \frac{x}{2}, \text{ length } BC \text{ is } \tan \frac{x}{2} \\
&2 \tan \frac{x}{2} \cos \frac{x}{2} = 2 \sin \frac{x}{2} \text{ is the base length of } ABC \\
&\tan \frac{x}{2} \sin \frac{x}{2} \text{ is the height of triangle } ABC \\
\Rightarrow T(x) &= \tan \frac{x}{2} \sin^2 \frac{x}{2} = \frac{1 - \cos^2 \frac{x}{2}}{\cos \frac{x}{2}} \sin \frac{x}{2} = \tan \frac{x}{2} - \frac{1}{2} \sin x
\end{aligned}$$

(2)

$$S(x) = \left(\frac{x}{2\pi}\right)(\pi(1)) - \frac{1}{2} \cos \frac{x}{2} (2 \sin \frac{x}{2}) = \frac{x}{2} - \frac{\sin 2}{2}$$

(3) Use L'Hopital's theorem.

$$\begin{aligned}
\frac{T(x)}{S(x)} &= \frac{\tan \frac{x}{2} - \frac{1}{2} \sin x}{\frac{x - \sin x}{2}} \xrightarrow{\frac{d}{dx}} \frac{\frac{1}{2} \sec^2 \frac{x}{2} - \frac{\cos x}{2}}{\frac{1 - \cos x}{2}} \\
&\xrightarrow{\frac{d}{dx}} \frac{\sec^2 \frac{x}{2} \tan \frac{x}{2} + \sin x}{\sin x} \xrightarrow{\frac{d}{dx}} \frac{\sec^2 \frac{x}{2} \tan^2 \frac{x}{2} + \frac{1}{2} \sec^4 \frac{x}{2} + \cos x}{\cos x} \rightarrow \\
&\xrightarrow{x \rightarrow 0} \frac{3}{2}
\end{aligned}$$

**Exercise 17.** Use L'Hopital's rule.

$$\begin{aligned}
I(t) &= \frac{E}{R}(1 - e^{-\frac{Rt}{L}}) \\
\lim_{R \rightarrow 0} I(t) &= \lim_{R \rightarrow 0} \frac{E(-1)(e^{-Rt/L}) \left(\frac{-t}{L}\right)}{1} = \boxed{\frac{Et}{L}}
\end{aligned}$$

**Exercise 18.**

$$\begin{aligned}
c - k &\rightarrow 0 \text{ since } c \rightarrow k \\
k - c &= u \\
k - u &= c \\
f(t) &= \frac{A(\sin kt - \sin ct)}{c^2 - k^2} = \frac{A(\sin(kt) - \sin(k - u)t)}{-u(2k - u)} = \\
&= \frac{A(\cos(k - u)t)(t)}{-(2k - u) + u} \rightarrow \frac{-At \cos kt}{2k}
\end{aligned}$$

**7.17 Exercises - The symbols  $+\infty$  and  $-\infty$ . Extension of L'Hopital's rule; Infinite limits; The behavior of  $\log x$  and  $e^x$  for large  $x$ .****Exercise 15.** Use L'Hopital's at the second to last step.

$$\lim_{x \rightarrow 1^-} (\ln x)(\ln(1 - x)) = \lim_{x \rightarrow 1^-} \frac{\ln(1 - x)}{\frac{1}{\ln x}} = \lim_{x \rightarrow 1^-} \frac{\left(\frac{1}{1-x}\right)(-1)}{\frac{-1}{(\ln x)^2} \left(\frac{1}{x}\right)} = \lim_{x \rightarrow 1^-} \frac{x}{1-x} (\ln x)^2 = \lim_{x \rightarrow 1^-} \frac{2 \ln x \left(\frac{1}{x}\right)}{(-1)} = \boxed{0}$$

**Exercise 16.** Persist in using L'Hopital's and trying all possibilities systematically.

$$\begin{aligned}
\lim_{x \rightarrow 0^+} x^{x^x - 1} &= \lim_{x \rightarrow 0^+} e^{(x^x - 1) \ln x} = \lim_{x \rightarrow 0^+} e^{(e^{x \ln x} - 1) \ln x} = \exp \lim_{x \rightarrow 0^+} \frac{\ln x}{(e^{x \ln x} - 1)^{-1}} = \\
&= \exp \lim_{x \rightarrow 0^+} \frac{1/x}{(-1)(e^{x \ln x} - 1)^{-2} (e^{x \ln x} (\ln x + 1))} = \exp - \lim_{x \rightarrow 0^+} \left( \frac{e^{2x \ln x} - 2e^{x \ln x} + 1}{e^{x \ln x} (x \ln x + x)} \right) = \\
&= \exp - \lim_{x \rightarrow 0^+} \frac{e^{x \ln x} (\ln x + 1) + e^{-x \ln x} (-\ln x - 1)}{(\ln x + 1 + 1)} = \exp - \lim_{x \rightarrow 0^+} \frac{e^{x \ln x} - e^{-x \ln x}}{1 + \frac{1}{\ln x + 1}} = \boxed{1}
\end{aligned}$$

**Exercise 17.**

$$\begin{aligned}
\lim_{x \rightarrow 0^+} (x^{x^x} - 1) &= \lim_{x \rightarrow 0^+} (e^{x^x \ln x} - 1) = \lim_{x \rightarrow 0^+} (e^{e^{x \ln x} \ln x} - 1) = (e^{\lim_{x \rightarrow 0^+} e^{x \ln x} \ln x} - 1) = \\
&= e^{\lim_{x \rightarrow 0^+} x \ln x \lim_{x \rightarrow 0^+} \ln x} - 1 = 0 - 1 = -1
\end{aligned}$$

We used

$$\lim_{x \rightarrow 0^+} x^\alpha \log x = 0 \forall \alpha > 0$$

since

$$t = \frac{1}{x}, x^\alpha \log x = \frac{-\log t}{t^\alpha} \rightarrow 0 \text{ as } t \rightarrow \infty.$$

**Exercise 18.**

$$\begin{aligned} \lim_{x \rightarrow 0^-} e^{\sin x \ln(1-2^x)} &= \exp \left( \lim_{x \rightarrow 0^-} \frac{\ln(1-e^{x \ln 2})}{1/\sin x} \right) = \exp \left( \lim_{x \rightarrow 0^-} \frac{\frac{1}{1-e^{x \ln 2}} (-\ln 2 e^{x \ln 2})}{\frac{-1}{\sin^2 x} \cos x} \right) = \\ &= \exp \left( \lim_{x \rightarrow 0^-} \frac{(\sin^2 x) \ln 2 e^{x \ln 2}}{(1-e^{x \ln 2}) \cos x} \right) = \exp \left( (\ln 2) \lim_{x \rightarrow 0^-} \frac{2 \sin x \cos x}{-\ln 2 e^{x \ln 2}} \right) = \boxed{1} \end{aligned}$$

**Exercise 19.**

$$\lim_{x \rightarrow 0^+} e^{\frac{1}{\ln x} \ln x} = \boxed{e}$$

**Exercise 20.** At the end, L'Hopital's could be used to verify that indeed  $\sin x \ln \sin x \rightarrow 0$  as  $x \rightarrow 0$ .

$$\lim_{x \rightarrow 0^+} e^{\sin x \ln \cot x} = e^{\lim_{x \rightarrow 0^+} \sin x (\ln \cos x - \ln \sin x)} = e^{\lim_{x \rightarrow 0^+} -\sin x \ln \sin x} = \boxed{1}$$

**Exercise 21.** Rewrite tan into sin and cos and use L'Hopital's.

$$\begin{aligned} \lim_{x \rightarrow \frac{\pi}{4}} (\tan x)^{\tan 2x} &= \lim_{x \rightarrow \frac{\pi}{4}} e^{\tan 2x \ln \tan x} = \exp \lim_{x \rightarrow \frac{\pi}{4}} \frac{1}{\cos 2x} (\ln \sin x - \ln \cos x) = \\ &= \exp \lim_{x \rightarrow \frac{\pi}{4}} \frac{\frac{1}{\sin x} \cos x - \frac{1}{\cos x} (-\sin x)}{-2 \sin 2x \cos 2x} = \exp \lim_{x \rightarrow \frac{\pi}{4}} \frac{1}{-\sin^2 2x} = \boxed{e^{-1}} \end{aligned}$$

**Exercise 22.**

**Exercise 23.** Use L'Hopital's theorem, taking derivatives of top and bottom.

$$\lim_{x \rightarrow 0^+} \exp \frac{e}{1 + \ln x} \ln x = \exp e \lim_{x \rightarrow 0} \frac{\ln x}{1 + \ln x} = \exp e \lim_{x \rightarrow 0} \frac{1/x}{1/x} = e^e$$

**Exercise 24.** Rewrite tan into sin and cos and take out sin since we could do the limit before doing L'Hopital's.

$$\lim_{x \rightarrow 1} (2-x)^{\tan(\pi x/2)} = \lim_{x \rightarrow 1} e^{\tan \frac{\pi x}{2} \ln(2-x)} = e^{\lim_{x \rightarrow 1} \frac{\sin \pi x/2 \ln(2-x)}{\cos \pi x/2}} = \exp \lim_{x \rightarrow 1} \frac{\frac{(-1)}{2-x}}{\frac{\pi}{2} \sin \pi x/2} = \boxed{\exp \frac{-2}{\pi}}$$

**Exercise 26.**

$$\begin{aligned} \lim_{x \rightarrow +\infty} \exp \left( x \ln \left( \frac{x+c}{x-c} \right) \right) &= \exp \lim_{x \rightarrow +\infty} \frac{\ln \left( \frac{1+c/x}{1-c/x} \right)}{1/x} = \exp \lim_{x \rightarrow +\infty} \frac{\frac{1}{\left( \frac{1+c/x}{1-c/x} \right)} \left( \frac{(x-c)-(x+c)}{(x-c)^2} \right)}{\frac{-1}{x^2}} = \\ &= \exp(2c) = 4 \implies \boxed{c = \ln 2} \end{aligned}$$

**Exercise 27.**

$$(1+x)^c = \exp(c \ln(1+x)) = \exp(c(x - o(x))) = 1 + c(x - o(x)) + o(x - o(x)) = 1 + cx + o(x)$$

$$x^2 \left( 1 + \frac{1}{x^2} \right)^{1/2} - x^2 = x^2 \left( 1 + \frac{1}{x^2} \right)^{1/2} - x^2$$

Let  $x^2 = \frac{1}{t}$ . So  $t \rightarrow 0$  as  $x \rightarrow +\infty$ .

$$\implies \frac{(1+t)^{1/2} - 1}{t} = \frac{1 + \frac{1}{2}t - 1 + o(t)}{t} = \boxed{\frac{1}{2}}$$

**Exercise 28.**

$$(x^5 + 7x^4 + 2)^c - x = x^5 \left( 1 + \frac{7}{x} + \frac{2}{x^5} \right)^c - x$$

Let  $\frac{1}{t} = x$  and guess that  $c = \frac{1}{5}$

$$\begin{aligned} \left( \frac{1}{t^5} + \frac{7}{t^4} + 2 \right)^c - \frac{1}{t} &= (1 + (7t + 2t^5))^{1/5} / t - \frac{1}{t} = \\ &= \frac{1 + \frac{1}{5}(7t + 2t^5) + o(t) - 1}{t} = \boxed{\frac{7}{5}} \end{aligned}$$



**Exercise 29.**

$$\begin{aligned}
 g(x) &= xe^{x^2} & f(x) &= \int_1^x g(t) \left(t + \frac{1}{t}\right) dt \\
 g'(x) &= e^{x^2} + 2x^2 e^{x^2} & f'(x) &= g(x) \left(x + \frac{1}{x}\right) - g(1)2; \\
 g''(x) &= 2xe^{x^2} + 4xe^{x^2} + 4x^3 e^{x^2} = 6xe^{x^2} + 4x^3 e^{x^2} & f''(x) &= g'(x)(x + 1/x) + g(x)(1 - 1/x^2) \\
 \frac{f''(x)}{g''(x)} &= \frac{(e^{x^2} + 2x^2 e^{x^2})(x + 1/x) + xe^{x^2}(1 - \frac{1}{x^2})}{6xe^{x^2} + 4x^3 e^{x^2}} = \frac{2xe^{x^2} + 2x^3 e^{x^2} + 2xe^{x^2}}{(6xe^{x^2} + 4x^3 e^{x^2})} = \\
 &= \frac{4x + 2x^3}{6x + 4x^3} = \boxed{\frac{1}{2}} \text{ as } x \rightarrow \infty
 \end{aligned}$$

**Exercise 30.**

$$\begin{aligned}
 g(x) &= x^c e^{2x} & f(x) &= \int_0^x e^{2t}(3t^2 + 1)^{1/2} dt \\
 g'(x) &= cx^{c-1} e^{2x} + 2x^c e^{2x} & f'(x) &= e^{2x}(3x^2 + 1)^{1/2} - 1
 \end{aligned}$$

Guessing that  $c = 1$

$$\frac{f'(x)}{g'(x)} = \frac{e^{2x}(3x^2 + 1)^{1/2} - 1}{2x^c e^{2x} + cx^{c-1} e^{2x}} = \frac{\sqrt{3}x(1 + \frac{1}{3x^2})^{1/2} - e^{-2x}}{2x + 1} = \boxed{\frac{\sqrt{3}}{2}}$$

So  $\boxed{c = 1}$ .

**Exercise 31.**

**Exercise 32.**

(1)

$$P\left(1 + \frac{r}{m}\right), P\left(1 + \frac{r}{m}\right)^2, \dots, P\left(1 + \frac{r}{m}\right)^m$$

For each year, there are the just previously shown  $m$  compoundings, so for  $n$  years,

$$P\left(1 + \frac{r}{m}\right)^{mn}$$

(2)

$$2 = e^{rt}$$

$$\frac{\ln 2}{r} = t = \boxed{11.55 \text{ years}}$$

(3)

$$\begin{aligned}
 2P_0 &= P_0 \left(1 + \frac{r}{m}\right)^{mt} \\
 \ln 2 &= mt \ln(1 + r/m)
 \end{aligned}$$

$$t = \frac{\ln 2}{m \ln(1 + r/m)} = \frac{\ln 2}{4 \ln(1 + 0.06/4)} = \boxed{11.64 \text{ years}}$$

**7.17 Exercises - The symbols  $+\infty$  and  $-\infty$ . Extension of L'Hopital's rule; Infinite limits; The behavior of  $\log x$  and  $e^x$  for large  $x$ . Exercise 1.**

$$\begin{aligned}
 \lim_{x \rightarrow 0} \frac{e^{-1/x^2}}{x^{1000}} &= \lim_{u \rightarrow \infty} \frac{e^{-u}}{u^{500}} = \lim_{u \rightarrow \infty} \frac{u^{500}}{e^u} = 0 & u &= \frac{1}{x^2} \\
 & & x^2 &= \frac{1}{u} \\
 & & x^{1000} &= \frac{1}{u^{500}}
 \end{aligned}$$

Where we had used Theorem 7.11, which are two very useful limits for log and exp.

**Theorem 26.** If  $a, b > 0$ ,

$$(17) \quad \lim_{x \rightarrow +\infty} \frac{(\log x)^b}{x^a} = 0$$

$$(18) \quad \lim_{x \rightarrow +\infty} \frac{x^b}{e^{ax}} = 0$$

*Proof.* Trick - use the definition of the logarithm as an integral.

If  $c > 0, t \geq 1, t^c \geq 1 \implies t^{c-1} \geq t^{-1}$ .

$$0 < \ln x = \int_1^x \frac{1}{t} dt \leq \int_1^x t^{c-1} dt = \frac{1}{c}(x^c - 1) < \frac{x^c}{c}$$

$$\implies 0 < \frac{(\ln x)^b}{x^a} < \frac{x^{cb-a}}{c^b}$$

$$\text{Choose } c = \frac{a}{2b}, \frac{x^{cb-a}}{c^b} = \frac{x^{-a/2}}{c^b} \rightarrow 0 \text{ as } x \rightarrow \infty$$

$$\text{then } \frac{(\ln x)^b}{x^a} \rightarrow 0 \text{ as } x \rightarrow \infty$$

For exp, Let  $t = e^x$ .  $\ln t = x$ .  $\frac{x^b}{e^{ax}} = \frac{(\ln t)^b}{t^a} \rightarrow 0$  as  $t \rightarrow \infty$  as  $x \rightarrow \infty$ . □

**Exercise 2.**

$$\lim_{x \rightarrow 0} \frac{\sin \frac{1}{x}}{\arctan \frac{1}{x}} = \lim_{x \rightarrow 0} \frac{\frac{1}{x} - o\left(\frac{1}{x}\right)}{\frac{1}{x} - o\left(\frac{1}{x}\right)} = \boxed{1}$$

**Exercise 3.** Use L'Hopital's.

$$\lim_{x \rightarrow \frac{\pi}{2}} \frac{\tan 3x}{\tan x} = \lim_{x \rightarrow \frac{\pi}{2}} \frac{\cos x}{\cos 3x} = \lim_{x \rightarrow \frac{\pi}{2}} \frac{-\sin x}{-3 \sin 3x} = \boxed{-\frac{1}{3}}$$

**Exercise 4.** Use L'Hopital's.

$$\lim_{x \rightarrow \infty} \frac{\ln(a + be^x)}{\sqrt{a + bx^2}} = \lim_{x \rightarrow \infty} \frac{\frac{1}{a+be^x}(be^x)}{\frac{bx}{\sqrt{a+bx^2}}} = \lim_{x \rightarrow \infty} \frac{\frac{ae^{-x}+b}{1}}{\frac{1}{\sqrt{\frac{a}{x^2}+b}}} = \frac{1}{\sqrt{b}}$$

**Exercise 5.** Make the substitution  $x = \frac{1}{t}$ .

$$\lim_{x \rightarrow \infty} x^4 \left( \cos \frac{1}{x} - 1 + \frac{1}{2x^2} \right) = \lim_{t \rightarrow 0} \frac{1}{t^4} \left( \cos t - 1 + \frac{t^2}{2} \right) = \lim_{t \rightarrow 0} \frac{t^4/4! + o(t^4)}{t^4} = \frac{1}{4!} = \frac{1}{120}$$

**Exercise 6.**

$$\lim_{x \rightarrow \pi} \frac{\ln |\sin x|}{\ln |\sin 2x|} = \lim_{x \rightarrow \pi} \frac{\frac{1}{\sin x} \cos x}{\frac{2}{\sin 2x} \cos 2x} = -\frac{1}{2} \lim_{x \rightarrow \pi} \frac{\sin 2x}{\sin x} = -\frac{1}{2} \lim_{x \rightarrow \pi} \frac{2 \cos 2x}{\cos x} = \boxed{1}$$

**Exercise 7.**

$$\begin{aligned} \lim_{x \rightarrow \frac{1}{2}^-} \frac{\ln(1-2x)}{\tan \pi x} &= \lim_{x \rightarrow \frac{1}{2}^-} \frac{\left(\frac{1}{1-2x}\right)(-2)}{(\sec^2 \pi x)\pi} = \lim_{x \rightarrow \frac{1}{2}^-} \frac{-2(\cos^2 \pi x)}{(1-2x)\pi} = \\ &= -\frac{2}{\pi} \lim_{x \rightarrow \frac{1}{2}^-} \frac{2 \cos \pi x \pi - \sin \pi x}{-2} = 1 \end{aligned}$$

**Exercise 8.**

$$\lim_{x \rightarrow \infty} \frac{\cosh x + 1}{e^x} = \lim_{x \rightarrow \infty} \frac{e^{x+1} + e^{-x-1}}{2e^x} = \frac{1}{2} \lim_{x \rightarrow \infty} e^1 + \frac{1}{e^{2x+1}} = \boxed{\frac{e}{2}}$$

**Exercise 9.**

$$\lim_{x \rightarrow \infty} \frac{a^x}{x^b} = \lim_{x \rightarrow \infty} \frac{e^{x \ln a}}{x^b} \rightarrow \infty; a > 1$$

$$\text{since } \lim_{x \rightarrow \infty} \frac{x^b}{e^{ax}} \text{ (in this case, } \ln a > 0 \text{)}$$

**Exercise 10.**

$$\lim_{x \rightarrow \frac{\pi}{2}} \frac{\tan x - 5}{\sec x + 4} = \lim_{x \rightarrow \frac{\pi}{2}} \frac{\sec^2 x}{\tan x \sec x} = \lim_{x \rightarrow \frac{\pi}{2}} \frac{\sec x}{\tan x} = \lim_{x \rightarrow \frac{\pi}{2}} \frac{1}{\cos x} \frac{\cos x}{\sin x} = \boxed{1}$$

## 8.5 Exercises - Introduction, Terminology and notation, A first-order differential equation for the exponential function, First-order linear differential equations.

The ordinary differential equation theorems we will use are

$$(19) \quad y' + P(x)y = 0$$

$$A(x) = \int_a^x P(t)dt$$

$$(20) \quad y = be^{-A(x)}$$

Consider  $y' + P(x)y = Q(x)$ ;  $A(x) = \int_a^x P(t)dt$ .

Let  $h(x) = g(x)e^{A(x)}$ ;  $g$  a solution.

$$h'(x) = (g' + Pg)e^A = Qe^A$$

$$\xrightarrow{\text{2nd. fund. thm. of calc.}} h(x) = h(a) + \int_a^x Q(t)e^{A(t)}dt$$

since  $h(a) = g(a)$

$$(21) \quad y = g(x) = e^{-A(x)} \left( \int_a^x Q(t)e^{A(t)}dt + b \right)$$

We had done some of these problems previously, using an integration constant  $C$ , but following Apostol's notation for  $y(a) = b$  for initial conditions is far more advantageous and superior as we seem clearly the dependence upon the initial conditions - so some of the solutions for the exercises will show corrections to the derived formula using Apostol's notation for  $y(a) = b$  initial conditions.

**Exercise 1.**  $A(x) = \int_0^x (-3)dt = -3x$

$$y = e^{3x} \left( \int_0^x e^{2t} e^{-3t} dt + 0 \right) = e^{3x} (-e^{-t}) \Big|_0^x = \boxed{-e^{2x} + e^{3x}}$$

**Exercise 2.**  $y' - \frac{2}{x}y = x^4$ .  $A(x) = \int_1^x \left( -\frac{2}{t} \right) dt = -2 \ln x$ .

$$y = e^{-A(x)} \left( \int_a^x Q(t)e^{A(t)}dt + b \right) = e^{2 \ln x} \left( \int_1^x t^4 e^{-2 \ln t} dt + 1 \right) = x^2 \left( \int_1^x t^2 dt + 1 \right) =$$

$$= -x^2 + \frac{x^2}{3}(x^3 - 1) = \frac{x^5}{3} + \frac{2x^2}{3}$$

**Exercise 3.**  $y' + y \tan x = \sin 2x$  on  $(-\frac{\pi}{2}, \frac{\pi}{2})$ , with  $y = 2$  when  $x = 0$ .

$$A(x) = \int_0^x P(t)dt = \int_0^x \tan t dt = -\ln |\cos t| \Big|_0^x = -\ln \cos x$$

$$y = e^{-A(x)} \left( \int_a^x Q(t)e^{A(t)}dt + b \right) = \cos x \left( \int_0^x \sin 2t e^{-\ln \cos t} dt + 2 \right) =$$

$$= \cos x \left( \int_0^x 2 \sin t dt + 2 \right) = -2 \cos^2 x + 4 \cos x$$

**Exercise 4.**  $y' + xy = x^3$ .  $y = 0, x = 0$ .

$$A(x) = \frac{1}{2}x^2$$

$$y = e^{-\frac{1}{2}x^2} \left( \int_0^x t^3 e^{\frac{t^2}{2}} dt + 0 \right) = e^{-\frac{x^2}{2}} \left( t^2 e^{\frac{t^2}{2}} - 2e^{\frac{t^2}{2}} \right) \Big|_0^x =$$

$$= x^2 - 2 + 2e^{-\frac{x^2}{2}}$$

**Exercise 5.**  $y' + y = e^{2t}$ .  $y = 1, t = 0$ .

$$y(x) = e^{-x} \left( \int_0^x e^{2t} e^t dt + 1 \right) = e^{-x} \left( \frac{e^{3t}}{3} \Big|_0^x + 1 \right)$$

$$A = x$$

$$= \boxed{\frac{e^{2x}}{3} + \frac{2}{3}e^{-x}}$$

**Exercise 6.**  $y' \sin x + y \cos x = 1; (0, \pi). \implies y' + \cot xy = \csc x$

$$A(x) = \int_a^x \cot t dt = \ln \left( \frac{\sin x}{\sin a} \right)$$

$$y(x) = e^{-\ln \left( \frac{\sin x}{\sin a} \right)} \left( \int (\csc t) e^{\ln \left( \frac{\sin t}{\sin a} \right)} dt + b \right) = \left( \frac{\sin a}{\sin x} \right) \left( \frac{x-a}{\sin a} + b \right)$$

indeed  $y = \frac{x-a}{\sin x} + \frac{b \sin a}{\sin x}$

$$x \rightarrow 0 \text{ for } y = \frac{x-a}{\sin x} + \frac{b \sin a}{\sin x} = \frac{x+b \sin a-a}{\sin x}$$

$$\boxed{b \sin a = a} \text{ for } x \rightarrow 0$$

$$\boxed{a - b \sin a = \pi} \text{ for } x \rightarrow \pi$$

**Exercise 7.**

$$x(x+1)y' + y = x(x+1)^2 e^{-x^2} \implies y' + \frac{1}{x(x+1)}y = (x+1)e^{-x^2}$$

$$A(x) = \int_a^x \left( \frac{1}{t} - \frac{1}{t+1} \right) dt = \ln \frac{x}{a} - \ln \frac{x+1}{a+1}$$

$$e^{A(x)} = \frac{(a+1)x}{a(x+1)}$$

$$y = \frac{a(x+1)}{(a+1)x} \left( \int_a^x (t+1)e^{-t^2} \left( \frac{(a+1)t}{a(t+1)} \right) dt + b \right) = \frac{(x+1)}{x} \left( \int_a^x t e^{-t^2} dt + b \frac{a(x+1)}{(a+1)x} \right) =$$

$$= \frac{x+1}{2x} (e^{-a^2} - e^{-x^2}) + \frac{a(x+1)b}{(a+1)x}$$

It's easy to see that the last equation above goes to 0 as  $x \rightarrow -1$ .

$$y = (e^{-a^2} - e^{-x^2})(1/2)(1 + \frac{1}{x}) + \frac{ab}{a+1}(1 + \frac{1}{x}) =$$

$$\lim_{x \rightarrow 0} y = \frac{1}{x} \left( \frac{e^{-a^2} - e^{-x^2}}{2} + \frac{ab}{a+1} \right) \boxed{a=0}$$

**Exercise 8.**  $y' + y \cot x = 2 \cos x$  on  $(0, \pi)$ .

$$A(x) = \int_a^x \cot t dt = \ln \left( \frac{\sin x}{\sin a} \right) e^{A(x)} = \frac{\sin x}{\sin a}$$

$$y = \frac{\sin a}{\sin x} \left( \int 2 \cos t \frac{\sin t}{\sin a} + b \right) = \frac{\sin a}{\sin x} \left( -\frac{\cos 2t}{2 \sin a} \Big|_a^x + b \right) =$$

$$= \frac{\cos(2a) + \cos 2x}{2 \sin x} + \frac{\sin a}{\sin x}$$

$$\boxed{y = \sin x} \quad \boxed{y = \frac{\cos(2a) - \cos(2x)}{2 \sin x} + \frac{\sin a}{\sin x}}$$

**Exercise 9.**  $(x-2)(x-3)y' + 2y = (x-1)(x-2)$ .  $y' + \frac{2}{(x-2)(x-3)}y = \frac{x-1}{x-3}$ .

$$A(x) = \int_a^x \frac{2dt}{(t-2)(t-3)} = 2 \int_a^x \left( \frac{1}{t-3} - \frac{1}{t-2} \right) dt = 2 (\ln |t-3| - \ln |t-2|) \Big|_a^x = 2 \ln \left| \frac{x-3}{a-3} \right| \left| \frac{a-2}{x-2} \right|$$

$$\text{If } (-\infty, 2), (3, +\infty), \frac{x-3 \leq 0}{x-2 \leq 0} \frac{3-x}{2-x} = \frac{x-3}{x-2}$$

$$\text{If } (2, 3), x-3 < 0, \text{ but } x-2 > 0$$

$$\text{If } (-\infty, 2), (3, \infty) y = b \left( \frac{x-2}{x-3} \right)^2 \left( \left| \frac{a-3}{a-2} \right| \right)^2 + \left( \frac{x-2}{x-3} \right)^2 \left( x + \frac{1}{x-2} - a - \frac{1}{a-2} \right)$$

$$\begin{aligned}
\text{If } (2, 3)y &= b \left( \frac{x-2}{x-3} \right)^2 \left( \left| \frac{a-3}{a-2} \right| \right)^2 + \left( \frac{x-2}{3-x} \right)^2 \left| \frac{a-3}{a-2} \right|^2 \int \left( \frac{t-1}{t-3} \right) \left| \frac{a-2}{a-3} \right|^2 \left( \frac{3-t}{t-2} \right)^2 = \\
&= b \left( \frac{x-2}{3-x} \right)^2 \left| \frac{a-3}{a-2} \right|^2 + \frac{x-2}{3-x} \left( - \int \frac{(t-1)(3-t)}{(t-2)^2} \right) = \\
&= b \left( \frac{x-2}{x-3} \right)^2 \left| \frac{a-3}{a-2} \right|^2 + \left( \frac{x-2}{x-3} \right)^2 (x + (x-2)^{-1} - a - (a-2)^{-1})
\end{aligned}$$

**Exercise 10.**  $s(x) = \frac{\sin x}{x}; x \neq 0$   $s(0) = 1$ ,  $T(x) = \int_0^x s(t)dt$   $f(x) = xT(x)$

$$\begin{aligned}
f' &= T + x(s(x)) = T + \sin x \\
xf' - f &= x \sin x \\
y' - \frac{y}{x} &= \sin x \\
A(x) &= \int_a^x \frac{-1}{t} dt = -\ln \frac{x}{a}, e^{-A(x)} = \frac{x}{a} \\
y &= \frac{x}{a} \left( \int_a^x \sin t \frac{a}{t} + b \right) = xT + \frac{bx}{a} \\
y(0) &= 0 \neq 1 \text{ since } P(x) = \frac{1}{x} \text{ is not continuous at } x = 0
\end{aligned}$$

**Exercise 11.**

$$\begin{aligned}
f(x) &= 1 + \frac{1}{x} \int_1^x f(t)dt \\
xf(x) &= x + \int_1^x f(t)dt \implies f(x) + xf' = 1 + f(x) \implies f' = \frac{1}{x} \\
&\implies \boxed{f(x) = \ln|x| - C} \\
\ln|x| - C &= 1 + \frac{1}{x} \int_1^x (\ln|t| - c) dt = \\
&= 1 + \frac{1}{x} (t \ln|t| - t - ct)|_1^x = \ln|x| - C + \frac{1+C}{x} \implies \boxed{C = -1} \quad \boxed{f(x) = \ln|x| + 1}
\end{aligned}$$

**Exercise 12.** Rewrite the second property we want  $f$  to have:

$$\int_1^x f(t)dt = \frac{1-f(x)}{x} \implies f(x) = -\frac{1}{x^2} - \left( \frac{f'x - f}{x^2} \right)$$

So then

$$\begin{aligned}
f' + \left( x - \frac{1}{x} \right) f &= -\frac{1}{x} \\
A(x) &= \int P(t)dt = \int \left( t - \frac{1}{t} \right) dt = \left( \frac{1}{2}x^2 - \ln x \right) - \left( \frac{1}{2}a^2 - \ln a \right) \\
e^{A(x)} &= e^{\frac{x^2-a^2}{2}} \frac{a}{x}; e^{-A(x)} = \frac{x}{a} e^{\frac{a^2-x^2}{2}} \\
\boxed{f(x) &= \frac{x}{a} e^{\frac{a^2-x^2}{2}} \left( \int_a^x \frac{-1}{t^2} a e^{\frac{t^2-a^2}{2}} dt + b \right)}
\end{aligned}$$

**Exercise 13.**

$$\begin{aligned}
v &= y^k \\
v' + kPv &= kQ = ky^{k-1}y' + kPy^k = kQ \text{ where } k = 1 - n \\
v' &= ky^{k-1}y' \\
&\implies y'y^{-n} + Py^{1-n} = Q \implies \boxed{y' + Py = Qy^n}
\end{aligned}$$

**Exercise 14.**  $y' - 4y = 2e^x y^{1/2}$

$$n = \frac{1}{2} k = 1 - \frac{1}{2} = \frac{1}{2}; v = y^{1/2}; v' + \frac{1}{2}(-4)v = \frac{1}{2}(2e^x) = v' - 2v = e^x$$

$$A(x) = \int_a^x P(t)dt = -2(x - a) = -2x$$

$$v = e^{2x} \left( \int e^t e^{-2t} dt + b \right) = e^{2x} (-e^{-x} + b) = \boxed{be^{2x} - e^x}$$

$$\implies y = (1 + \sqrt{2})^2 e^{4x} - 2(1 + \sqrt{2})e^{3x} + e^{2x}$$

$$y(x) = b^2 - 2b + 1 = 2$$

$$b = 1 + \sqrt{2}$$

**Exercise 15.**  $y' - y = -y^2(x^2 + x + 1), n = 2, k = 1 - n = -1, v = y^k, v = y^{-1}.$

$$v' + kPv = kQ, v' + (-1)(-1)v = (-1)(-(x^2 + x + 1)) = v' + v = x^2 + x + 1$$

$$A(x) = \int P(t)dt = \int_0^x 1dt = x$$

$$v = e^{-x} \left( \int (t^2 + t + 1)e^t dt + b \right) = e^{-x} (t^2 e^t - t e^t + 2e^t) \Big|_0^x + b e^{-x} = (x^2 - x + 2) - (2e^{-x}) + b e^{-x}$$

$$y = \frac{1}{x^2 - x + 2 - 2e^{-x}}$$

**Exercise 16.**

$$v' + -\frac{1}{x}v = 2x^2$$

$$v = e^{\ln x} \left( \int 2x^2 e^{-\ln x} dx + C \right) = x(x^2 + C) = x^3 + Cx$$

Then since  $v = y^k, k = 1 - n,$

$$y = (x^3 + Cx)^2 = x^2(x^2 + C)^2; x \neq 0$$

$$\boxed{y = x^2(x^2 - 1)^2}$$

Check:

$$\begin{aligned} y' &= 2x(x^2 - 1)^2 + x^3(4)(x^2 - 1) \\ 2x^2 + x^4(4)(x^2 - 1) - 2x^2(x^2 - 1)^2 &= 4x^3 \end{aligned}$$

**Exercise 17.**  $xy' + y = y^2 x^2 \log x$  on  $(0, +\infty)$  with  $y = \frac{1}{2}$  when  $x = \frac{1}{2}, x \neq 0.$

$$y' + \frac{y}{x} = y^2 x \log x$$

$$k = 1 - n = 1 - 2 = -1, v = y^k = y^{-1}$$

$$v' + kPv = kQ; \quad v' + -\left(\frac{1}{x}\right)v = -1x \log x$$

$$\begin{aligned} v &= x \left( \int \frac{-x \log x}{x} dt + C \right) = x \left( \int -\log t dt + C \right) = \boxed{y = \frac{1}{Cx + x^2 - x^2 \log x}} \quad C = -\frac{1}{2}(1 - \log \frac{1}{2}) + 2 \\ &= x(-(x \log x - x) + C) = -x^2 \log x + x^2 + Cx \end{aligned}$$

Check:

$$\begin{aligned} y' &= (C + 2x - 2x \log x - x)(-y^2) \\ y^2(-C - 2x + 2x \log x + x + C + x - x \log x) \end{aligned}$$

$$y = \frac{1}{2}; x = 1, y(x = 1) = \frac{1}{2}, b = 2$$

$$\boxed{y = \frac{1}{x(x \ln x - x + 3)}}$$

**Exercise 18.**  $2xyy' + (1+x)y^2 = e^x$  on  $(0, +\infty)$ ,  $y = \sqrt{e}$  when  $x = 1$ ;  $y = -\sqrt{e}$  when  $x = 1$ .

If  $x \neq 0, y \neq 0$ ,

$$\begin{aligned} y' + \frac{(1+x)}{2x}y &= \frac{e^2}{2x}y^{-1} \\ k = 1 - n = 2; \quad v &= y^k = y^2 \\ v' + 2Pv &= 2Q; \quad v' + 2\left(\frac{1+x}{2x}\right)v = \frac{2e^x}{2x} \implies v' + \frac{1+x}{x}v = \frac{e^x}{x} = v' + P_v = Q_v \\ A_v &= \int_a^x P_v = \int_a^x \frac{1+t}{t}dt = \ln\left(\frac{x}{a}\right) + (x-a); \quad e^{\int_a^x P_v dt} = e^{\ln x/a + x-a} = \frac{x}{a}e^{x-a} \\ v &= \frac{ae^{-x+a}}{x} \left( \int_a^x \frac{e^t}{t} \frac{t}{a} e^{t-a} dt + b_v \right) = \frac{a}{x} e^{-x+a} \left( \frac{1}{2a} e^{2t-a} \Big|_a^x + b_v \right) \\ &= \frac{a}{x} e^{-x+a} \left( \frac{e^{2x-a}}{2a} - \frac{e^a}{2a} + b_v \right) \\ y &= \pm \sqrt{\frac{a}{x} e^{-x+a} \left( \frac{e^{2x-a}}{2a} - \frac{e^a}{2a} + b_v \right)} \end{aligned}$$

Now  $y^k = v$

$$y^2 = v = e = \frac{1}{1} e^{-1+1} \left( \frac{e^{2-1}}{2(1)} - \frac{e^1}{2} + b_v \right) = b_v = e$$

$$(1) \quad b_v = e \implies y = \sqrt{\frac{1}{x} e^{-x+1} \left( \frac{e^{2x-1}}{2} - \frac{e}{2} + e \right)}$$

$$(2) \quad b_v = e \implies y = -\sqrt{\frac{1}{x} e^{-x+1} \left( \frac{e^{2x-1}}{2} - \frac{e}{2} + e \right)}$$

$$(3) \quad \text{If we could take } a = 0, \text{ then } \lim_{x \rightarrow 0} y = \pm \sqrt{\frac{e^x}{2x} - \frac{e^{-x+2a}}{2x} + \frac{b_v a e^{-x+a}}{x}} = \pm \sqrt{\frac{1+x-e^{2(0)}(1-x)+2b_v 0e^{-x+a}}{2x}} = \pm 1$$

If we consider  $\lim_{x \rightarrow 0} y^2$ , and let  $a$  go to 0, then

$$v = \frac{1}{x} (e^x - e^{-x}) \implies \frac{\sinh x}{x}$$

**Exercise 19.** The Riccati equation is  $y' + P(x)y + Q(x)y^2 = R(x)$ .

If  $u$  is a known solution,  $y = u + \frac{1}{v}$  is also a solution if  $v$  satisfies a first-order ODE.

$$\begin{aligned} (u + 1/v)' &= u' + \frac{-1}{v^2} v' \\ y' + Py + Qy^2 &= R \implies u' + \frac{-v'}{v^2} + P(u + \frac{1}{v}) + Q(u^2 + \frac{2u}{v} + \frac{1}{v^2}) = R \\ &\implies \boxed{v' - Pv = Q(2uv + 1)} \end{aligned}$$

**Exercise 20.**  $y' + y + y^2 = 2, y = 1, -2$ .

(1) If  $-2 \leq b < 1$ ,

$$\begin{aligned} y' + y + y^2 &= 2 \quad P = 1, Q = 1, R = 2 \\ v' + (-P - 2Qu)v &= Qy = u + \frac{1}{v} \\ u &= 1 \\ v' + (-1 - 2(1)(1))v &= v' - 3v = Q = 1 \\ v &= e^{3x} \left( \int 1e^{-3t} dt + b \right) = e^{3x} \left( \frac{e^{-3t}}{-3} \Big|_a^x + b \right) = be^{3x} - \frac{1}{3} (1 - e^{3x-3a}) \\ y &= 1 + \frac{3}{3be^{3x} - (1 - e^{3x-3a})} \\ b &= 1 + \frac{1}{b} \\ y(0) &= 1 + \frac{3}{3b - (1 - e^{-3a})} \implies b^2 - b - 1 = 0 \\ b &= \frac{1 \pm \sqrt{5}}{2} \end{aligned}$$

$$y = 1 + \frac{3}{3be^{3x} - (1 - e^{3x})} \quad b = \frac{1 - \sqrt{5}}{2}$$

(2)

$$u = -2$$

$$v' + (-1 - 2(1)(-2))v = v' + 3v = 1$$

$$v = e^{-3x} \left( \int_a^x e^{3t} dt + b \right) = e^{-3x} \left( \frac{e^{3x} - e^{3a}}{3} \right) + be^{-3x} = \frac{1 - e^{3a-3x}}{3} + be^{-3x}$$

$$y = -2 + \frac{3}{1 - e^{3a-3x} + 3be^{-3x}}; y(0) = -2 + \frac{3}{1 - e^{3a} + 3b} \xrightarrow{a=0} y(0) = -2 + \frac{3}{3b} \Rightarrow b = -1 \pm \sqrt{2}$$

$$b \geq 1 \text{ or } b < -2, y = -2 + \frac{3}{1 - e^{-3x} + 3be^{-3x}} \quad b = -1 \pm \sqrt{2}$$

## 8.7 Exercises - Some physical problems leading to first-order linear differential equations.

### Exercise 3.

(1)  $y' = -\alpha y(t)$ .  $y(T) = y_0 e^{-\alpha T} = \frac{y_0}{n}$ .  $n = e^{\alpha T}$  so the relationship between  $T$  and  $n$  doesn't depend upon

$$y_0. \quad \boxed{\frac{1}{k} \ln e = T}.$$

(2)  $f(a) = y_0 e^{-ka}$ ;  $f(b) = y_0 e^{-kb}$ .

$$\frac{f(a)}{f(b)} = e^{-ka+kb} \Rightarrow \ln \frac{f(a)}{f(b)} = -k(a-b); \quad \frac{1}{a-b} \ln \frac{f(a)}{f(b)} = -k$$

$$f(t) = y_0 \exp \left( \frac{\ln \frac{f(a)}{f(b)} t}{a-b} \right); \quad f(a) = y_0 \left( \frac{f(b)}{y_0} \right)^{a/b}; \quad \Rightarrow \frac{f(a)}{(f(b))^{a/b}} = y_0^{1-\frac{a}{b}}$$

$$\begin{aligned} f(t) &= \left( \frac{f(a)}{(f(b))^{a/b}} \right)^{\frac{b}{b-a}} \left( \frac{f(a)}{f(b)} \right)^{\frac{t}{a-b}} = \frac{(f(a))^{\frac{b}{b-a}}}{f(b)^{\frac{a}{b-a}}} \frac{f(a)^{-\frac{t}{b-a}}}{f(b)^{\frac{t}{a-b}}} = \\ &= \frac{f(a)^{\frac{b-t}{b-a}}}{(f(b))^{\frac{a-t}{b-a}}} = \boxed{f(a)^{\frac{b-t}{b-a}} f(b)^{\frac{t-a}{b-a}}} \\ w(t) &= \frac{b-t}{b-a}; \quad 1-w(t) = \frac{b-a-(b-t)}{b-a} = \frac{t-a}{b-a} \end{aligned}$$

**Exercise 4.**  $F = mv' = w_0 - \frac{3}{4}v$

$$\begin{aligned} w_0 &= 192 \\ \frac{w_0}{g} &= 6 = m \quad v' = \frac{w_0}{m} - \frac{1}{8}v \Rightarrow v' + \frac{1}{8}v = \frac{w_0}{m} = 32 \end{aligned}$$

$$v(t) = e^{-\frac{t}{8}} \left( \int_0^t \frac{w_0}{m} e^{\frac{t}{8}} dt + b \right) = e^{-\frac{t}{8}} \left( \frac{8w_0}{m} (e^{t/8} - 1) + 0 \right) = (256(1 - e^{-t/8}))$$

$$v(10) = 256(1 - e^{-5/4}) = 256(1 - 37/128) = 182$$

$$F = mv' = w_0 - 12v \quad v' + \frac{12}{m}v = \frac{w_0}{m} = v' + 2v = 32$$

$$v(t) = e^{-2t} \left( \int_{t_0}^t e^{2x} 32 dx + b \right) = e^{-2t} (16(e^{2t} - e^{2t_0}) + b) = (16(1 - e^{2(t_0-t)}) + be^{-2t})$$

$$v(10) = be^{-2(10)} = 182 \Rightarrow b = 182e^{20}$$

$$v(t) = 16 + 166e^{20-2t}$$

So then

$$v(t) = \begin{cases} 256(1 - e^{-t/8}) & \text{if } t < 10 \\ 16 + 166e^{20-2t} & \text{if } t > 10 \end{cases}$$

### Exercise 7.



(1)

$$\begin{aligned}
y'(t) &= (y - M)k = ky - kM & y' + -ky &= -kM \\
y &= e^{kt} \left( \int -kM e^{-kt} dt + b \right) = e^{kt} (M(e^{-kt} - 1) + b) & M &= 60^\circ \\
y(0) &= b = 200^\circ \\
y_f &= e^{kT} (M(e^{-kT} - 1) + 200) = M + e^{kT} (200 - M) \\
\frac{1}{T} \ln \left( \frac{y_f - M}{b - M} \right) &= k = \frac{1}{T} (\ln(60) - \ln(140)) = \frac{1}{T} \ln \left( \frac{3}{7} \right) = \frac{1}{30} (\ln 3 - \ln 7) \\
\boxed{y(t) &= 60 + 140e^{\frac{\ln 3 - \ln 7}{30} t}}
\end{aligned}$$

(2)

$$\begin{aligned}
y(t) &= 60 + 140e^{-kt}; \quad k = \frac{(\ln 7 - \ln 3)}{30} \\
\ln \left( \frac{Y - 60}{140} \right) &= -kt \implies t_f = \frac{(\ln 140 - \ln(T - 60))}{k} \text{ for } 60 < T \leq 200
\end{aligned}$$

$$(3) \quad t_f = \frac{\ln(140) - \ln(30)}{k} = \frac{30}{\ln 7/3} \ln \frac{14}{3} = 54 \text{ minutes}$$

$$(4) \quad M = M(t) = M_0 - \alpha t \quad \alpha = \frac{1}{10}$$

$$\begin{aligned}
y &= e^{kt} \left( \int -kM e^{-kt} dt + b \right) = e^{kt} \left( \int_0^t -k(M_0 - \alpha u) e^{-ku} du + b \right) = \\
&= -ke^{kt} \int_0^t (M_0 e^{-ku} - \alpha u e^{ku}) du + be^{kt} = \\
&= -ke \left( \frac{M_0 e^{-ku}}{-k} \Big|_0^t - \alpha \left( \frac{u e^{-ku}}{-k} + \frac{-e^{-ku}}{k^2} \right) \Big|_0^t \right) + be^{kt} = \\
&= -ke^{kt} \left( \frac{M_0}{k} (1 - e^{-kt}) - \alpha \left( \frac{te^{-kt}}{-k} - \frac{e^{-kt}}{k^2} + \frac{1}{k^2} \right) \right) + be^{kt} = \\
y(t) &= -M_0 e^{kt} + M_0 - \alpha t - \alpha/k + \alpha e^{kt}/k + be^{kt} = (-M_0 + \alpha/k + b)e^{kt} + (M_0 - \alpha t - \alpha/k) =
\end{aligned}$$

$$\boxed{y(t) = (140 + \frac{3}{(\ln 3 - \ln 7)})e^{-kt} + (60 - \frac{t}{10} - (\frac{3}{\ln 3/7}))}$$

**Exercise 8.**  $y'(t) = -k(y - M_0); \quad y' + ky = kM_0.$

$$\begin{aligned}
y &= e^{-kt} \left( \int_{t_i}^{t_f} kM_0 e^{ku} du + b \right) = e^{-kt_f} (M_0(e^{kt_f} - e^{kt_i}) + b) = M_0(1 - e^{-k(t_f - t_i)}) + be^{-kt_f} \\
y(t_f) - M_0 &= -M_0 e^{-k(t_f - t_i)} + be^{-kt_f} \\
\implies 65 - M_0 &= -M_0 e^{-k(5)} + 75e^{-k(5)} = (75 - M_0)e^{-5k} \implies \ln \left( \frac{65 - M_0}{75 - M_0} \right) = -5k; \quad k = \frac{1}{5} \ln \left( \frac{75 - M_0}{65 - M_0} \right) \\
60 - M_0 &= -M_0 e^{-k(5)} + 65e^{-k(5)} = (65 - M_0)e^{-5k} \implies \ln \left( \frac{60 - M_0}{65 - M_0} \right) = -5k \\
&\implies \frac{65 - M_0}{75 - M_0} = \frac{60 - M_0}{65 - M_0} \\
&\boxed{M_0 = 55}
\end{aligned}$$

**Exercise 9.**

Let  $y(t)$  = absolute amount of salt.

Water is leaving according to  $w(t) = w_0 + (3 - 2)t = w_0 + t$ .

Salt leaving =  $\left( \frac{2 \text{ gal}}{\text{min.}} \right) \left( \frac{y(t) \text{ salt}}{w(t)} \right)$

So then

$$y' = \frac{-2y}{w_0 + t} \implies \ln y = -2(\ln(w_0 + t) - \ln w_0) = -2 \ln \left( \frac{w_0 + t}{w_0} \right)$$

is the equation of motion given by the problem.

$$y(t) = C e^{\ln\left(\frac{w_0+t}{w_0}\right)^{-2}} = C \left(\frac{w_0+t}{w_0}\right)^{-2}$$

$$\boxed{y(t) = 50 \left(\frac{100}{100+t}\right)^2} \quad y(t = 60 \text{ min.}) = 50 \left(\frac{100}{160}\right)^2 = 50 \frac{25}{64} = \frac{625}{32} \simeq 19.53$$

### Exercise 10.

Let  $y$  be the dissolved salt (total amount of) at  $t$  time. The (total) amount of water at any given time in the tank is  $w = w_0 + t$ . There is dissolved salt in mixture that is leaving the tank at any minute. There is also salt from undissolved salt in the tank that is “coming into” the dissolved salt, adding to the amount of dissolved salt in the mixture. Thus

$$y'(t) = (-2) \left(\frac{y}{w}\right) + \alpha \left(\frac{y}{w} - 3\right); \quad \alpha = \frac{-1 \text{ gal}}{3 \text{ min}}$$

We obtained  $\alpha$  easily by considering only the dissolving part and how it dissolves 1 pound of salt per minute if the salt concentration,  $\frac{y}{w}$  was zero, i.e. water is fresh.

$$\begin{aligned} y'(t) &= \frac{-7}{3} \frac{y}{w} + 1; & y' &= \frac{7}{3} \frac{y}{w_0+t} = 1 \\ P &= \frac{7/3}{w_0+t} \\ \int P &= \int \frac{7/3}{w_0+t} = \frac{7}{3} \ln(w_0+t) \Big|_0^t = \frac{7}{3} \ln\left(\frac{w_0+t}{w_0}\right) \\ y &= e^{-\ln\left(\frac{w_0+t}{w_0}\right)^{-7/3}} \left( \int_0^t (1) e^{\ln\left(\frac{w_0+u}{w_0}\right)^{7/3}} du + b \right) = \\ &= \left(\frac{w_0}{w_0+t}\right)^{7/3} \left( \int_0^t \left(\frac{w_0+u}{w_0}\right)^{7/3} du + b \right) = \left(\frac{w_0}{w_0+t}\right)^{7/3} \left( \frac{3w_0}{10} \left( \left(\frac{w_0+t}{w_0}\right)^{10/3} - 1 \right) + b \right) = \\ y &= \left(\frac{100}{100+60}\right)^{7/3} \left( \frac{3(100)}{10} \left( \left(\frac{100+60}{100}\right)^{10/3} - 1 \right) + 50 \right) \simeq 54.78 \text{ lbs.} \end{aligned}$$

### Exercise 11. $LI'(t) + RI(t) = V(t) \quad V(t) = E \sin \omega t$ .

$$\begin{aligned} I(t) &= I(0)e^{-Rt/L} + e^{-Rt/L} \int_0^t \frac{V(x)}{L} e^{Rx/L} dx \\ I(t) &= I(0)e^{-Rt/L} + e^{-Rt/L} \frac{E}{L} \int_0^t \sin \omega x e^{Rx/L} dx \end{aligned}$$

Using  $\int e^{ax} \sin bx dx = ae^{ax} \sin bx - be^{ax} \cos bx$

$$I(t) = I(0)e^{-\frac{Rt}{L}} + \frac{Ee^{-Rt/L}}{L} \left( \frac{\frac{R}{L}e^{\frac{Rt}{L}} \sin \omega t - \omega e^{\frac{Rt}{L}} \cos \omega t}{\left(\frac{R}{L}\right)^2 + \omega^2} + \frac{\omega}{\left(\frac{R}{L}\right)^2 + \omega^2} \right)$$

$I(0) = 0$  So

$$\begin{aligned} I(t) &= \frac{E}{L} \frac{\frac{R}{L} \sin \omega t - \omega \cos \omega t}{\left(\frac{R}{L}\right)^2 + \omega^2} + \frac{E\omega L}{R^2 + (\omega L)^2} e^{-Rt/L} = \frac{E(R \sin \omega t - \omega L \cos \omega t)}{R^2 + (\omega L)^2} + \frac{E\omega L}{R^2 + (\omega L)^2} e^{-Rt/L} = \\ &\sin \alpha = \frac{\omega L}{\sqrt{R^2 + (\omega L)^2}} \\ \Rightarrow I(t) &= \frac{E \sin(\omega t - \alpha)}{\sqrt{R^2 + (\omega L)^2}} + \frac{E\omega L}{(R^2 + \omega^2 L^2)} e^{-Rt/L} \\ L &= 0 \quad \sin \alpha = 0 \end{aligned}$$

**Exercise 12.**

$$E(t) = \begin{cases} E & \text{if } 0 < a \leq t < b \\ 0 & \text{otherwise} \end{cases}$$

$$I(t) = e^{-Rt/L} \int_0^t 0 = 0 \text{ for } t < a$$

$$I(t) = e^{-Rt/L} \int_a^t \frac{E}{L} e^{Rx/L} dx = \frac{E}{L} e^{-Rt/L} \frac{L}{R} (e^{Rt/L} - e^{Ra/L}) = \frac{E}{R} (1 - e^{\frac{R(a-t)}{L}})$$

$$I(b) = \frac{E}{R} (1 - e^{R(a-b)/L})$$

$$\text{for } t > b, \quad I(t) = K e^{-Rt/L}$$

$$\Rightarrow I(t) = \frac{E e^{-Rt/L}}{R} (e^{\frac{Rb}{L}} - e^{\frac{Ra}{L}}) \quad \text{for } I(b) = I(b)$$

**Exercise 13.** From Eqn. 8.22,  $\frac{dx}{dt} = kx(M - x)$

$$\frac{dx}{dt} = kx(M - x) = kMx - kx^2; \quad \Rightarrow \frac{dx}{kMx - kx^2} = dt = \frac{1/kdx}{x(M - x)} = dt$$

$$\Rightarrow kdt = \left( \frac{1}{x} + \frac{1}{M - x} \right) \frac{1}{M} dx = \frac{\ln x + \ln(M - x)}{M}$$

$$Mk(t - t_i) = \ln \frac{x}{M - x}; \quad e^{Mk(t-t_i)}(M - x) = x$$

$$x(t) = \frac{M e^{Mk(t-t_i)}}{1 + e^{Mk(t-t_i)}} = \boxed{\frac{M}{1 + e^{-Mk(t-t_i)}}}$$

**Exercise 14.** Note that we are given *three equally spaced times*.

$$M = x_2 + x_2 e^{-\alpha(t_2-t_0)}; \quad \frac{M - x_2}{x_2} = e^{-\alpha(t_2-t_0)}$$

$$\frac{M - x_2}{x_2} \left( \frac{x_1}{M - x_1} \right) = e^{-\alpha t_2 + \alpha t_0 + \alpha t_1 - \alpha t_0} = e^{-\alpha(t_2-t_1)}$$

$$\frac{M - x_3}{x_3} \left( \frac{x_2}{M - x_2} \right) = e^{-\alpha(t_3-t_2)} = \frac{M - x_2}{x_2} \left( \frac{x_1}{M - x_1} \right)$$

$$(M - x_3)(M - x_1)x_2^2 = x_1x_3(M - x_2)^2 = x_1x_3(M^2 - 2Mx_2 + x_2^2) = x_2^2(M^2 - M(x_1 + x_3) + x_1x_3)$$

$$(x_2^2 - x_1x_3)M^2 = M(x_2^2(x_1 + x_3) - 2x_2x_1x_3) = (-x_1(x_3 - x_2) + x_3(x_2 - x_1))x_2$$

$$\Rightarrow \boxed{M = x_2 \frac{(x_3(x_2 - x_1) - x_1(x_3 - x_2))}{x_2^2 - x_1x_3}}$$

**Exercise 15.**

$$\frac{dx}{dt} = k(t)Mx - k(t)x^2 \quad \frac{dx}{Mx - x^2} = k(t)dt \Rightarrow M \int_{t_i}^t k(u)du = \ln \left( \frac{x}{M - x} \right)$$

$$\frac{x}{M - x} = e^{M \int_{t_i}^t k(u)du}$$

$$x = \frac{M e^{M \int_{t_i}^t k(u)du}}{1 + e^{M \int_{t_i}^t k(u)du}} = \boxed{\frac{M}{1 + e^{-M \int_{t_i}^t k(u)du}}}$$

**Exercise 16.**

$$(1) \quad M = 23 \frac{92(23-3.9) - 3.9(92-23)}{23^2 - 3.9(92)} = 201$$

$$(2)$$

$$M = 122 \left( \frac{150(122 - 92) - 92(150 - 122)}{(122)^2 - 92(150)} \right) = 122 \left( \frac{150(30) - 92(28)}{(122)^2 - 92(150)} \right) = 216$$

$$(3) \text{ Reject.}$$

**8.14 Exercises - Linear equations of second order with constant coefficients, Existence of solutions of the equation  $y'' + by = 0$ , Reduction of the general equation to the special case  $y'' + by = 0$ , Uniqueness theorem for the equation  $y'' + by = 0$ , Complete solution of the equation  $y'' + by = 0$ , Complete solution of the equation  $y'' + ay' + by = 0$ .**

**Exercise 1.**  $y'' - 4y = 0$   $y = c_1 e^{2x} + c_2 e^{-2x}$ .

**Exercise 2.**  $y'' + 4y = 0$   $y = c_1 \cos(2x) + c_2 \sin(2x)$ .

Use Theorem 8.7.

**Theorem 27.** Let  $d = a^2 - 4b$  be the discriminant of  $y'' + ay' + by = 0$ .

Then  $\forall$  solutions on  $(-\infty, \infty)$  has the form

$$(22) \quad y = e^{-ax/2}(c_1 u_1(x) + c_2 u_2(x))$$

where

(1) If  $a = 0$ , then  $u_1(x) = 1$  and  $u_2(x) = x$

(2) If  $d > 0$ , then  $u_1(x) = e^{kx}$  and  $u_2(x) = e^{-kx}$ , where  $k = \frac{\sqrt{d}}{2}$

(3) If  $d < 0$ , then  $u_1(x) = \cos kx$  and  $u_2(x) = \sin kx$ ; where  $k = \frac{1}{2}\sqrt{-d}$

**Exercise 3.**  $y'' - 4y' = 0$ ;  $a = -4$ .

$$y = e^{2x}(c_1 e^{2x} + c_2 e^{-2x}) = \boxed{c_1 e^{4x} + c_2}$$

**Exercise 4.**  $y'' + 4y' = 0$

$$y = e^{-2x}(c_1 e^{2x} + c_2 e^{-2x}) = \boxed{c_1 + c_2 e^{-4x}}$$

**Exercise 5.**  $y'' - 2y' + 3y = 0$   $d = 4 - 4(3) = -8 \implies \boxed{y = e^{-x}(c_1 \sin \sqrt{2}x + c_2 \cos \sqrt{2}x)}$

**Exercise 8.**  $y'' - 2y' + 5y = 0$   $d = -16$   $y = e^x(c_1 \cos 2x + c_2 \sin 2x)$

**Exercise 9.**  $y'' + 2y' + y = 0$   $d = 4 - 4(1)(1) = 0$   $y = e^{-x}(1 + x)$

**Exercise 10.**  $y'' - 2y' + y = 0$   $d = 4 - 4(1)(1) = 0$   $y = e^x(1 + x)$

**Exercise 11.**  $y'' + \frac{3}{2}y' = 0$   $y = 1$ ,  $y' = 1$ ;  $x = 0$   $d = \frac{9}{4} > 0$

$$y = e^{-\frac{3}{4}x}(c_1 e^{\frac{3x}{4}} + c_2 e^{-\frac{3x}{4}}) = c_1 + c_2 e^{-\frac{3x}{2}}$$

$$c_2 = \frac{-2}{3} \implies y = \frac{5}{3} + \frac{-2}{3} e^{-\frac{3x}{2}}$$

**Exercise 12.**  $y'' + 25y = 0$ ;  $y = -1$ ,  $y' = 0$ ,  $x = 3$ .

$$y = c_1 \sin 5x + c_2 \cos 5x$$

$$-1 = c_1 \sin 15 + c_2 \cos 15$$

$$y' = 5c_1 \cos 5x - 5c_2 \sin 5x$$

$$0 = 5c_1 \cos 15 - 5c_2 \sin 15 \implies c_1 \cos 15 = c_2 \sin 15$$

$$-1c_1(\sin 15 + \frac{\cos^2 15}{\sin 15}) = c_1 \left( \frac{1}{\sin 15} \right)$$

$$\begin{aligned} c_1 &= -\sin 15 \\ c_2 &= -\cos 15 \end{aligned}$$

$$\boxed{y = -\sin 15 \sin 5x - \cos 15 \cos 5x}$$

**Exercise 13.**  $y'' - 4y' - y = 0$ ;  $y = 2$ ;  $y' = -1$  when  $x = 1$

$$d = 10 - 4(1)(-1) = 20$$

$$y = c_1 e^{(2+\sqrt{5})x} + c_2 e^{(2-\sqrt{5})x}$$

$$y(x=1) = c_1 e^{2+\sqrt{5}} + c_2 e^{2-\sqrt{5}} = 2$$

$$y'(x=1) = (2+\sqrt{5})c_1 e^{2+\sqrt{5}} + (2-\sqrt{5})c_2 e^{2-\sqrt{5}} = -1 \implies (5+2\sqrt{5})c_1 e^{2+\sqrt{5}} + (5-2\sqrt{5})c_2 e^{2-\sqrt{5}} = 0$$

$$c_1 = \frac{2\sqrt{5}-5}{5+2\sqrt{5}} c_2 e^{-2\sqrt{5}}$$

$$\left( \frac{2\sqrt{5}-5}{5+2\sqrt{5}} \right) c_2 e^{2-\sqrt{5}} + c_2 e^{2-\sqrt{5}} = 2 = \frac{4\sqrt{5}c_2 e^{2-\sqrt{5}}}{5+2\sqrt{5}} \implies \frac{5+2\sqrt{5}}{2\sqrt{5}} e^{\sqrt{5}-2} = c_2 \quad c_1 = \frac{2\sqrt{5}-5}{2\sqrt{5}} e^{-2-\sqrt{5}}$$

$$\boxed{y = \frac{2\sqrt{5}-5}{2\sqrt{5}} e^{-2-\sqrt{5}} e^{(2+\sqrt{5})x} + \frac{5+2\sqrt{5}}{2\sqrt{5}} e^{\sqrt{5}-2} e^{(2-\sqrt{5})x}}$$

**Exercise 14.**  $y'' + 4y' + 5y = 0$ , with  $y = 2$  and  $y' = y''$  when  $x = 0$

$$16 - 4(1)(5) = -4$$

$$y = e^{-2x}(c_1 \sin 2x + c_2 \cos 2x) \quad y(x=0) = c_2 = 2$$

$$y' = -2e^{-2x}(c_1 \sin 2x + 2 \cos 2x) + 2e^{-2x}(c_1 \cos 2x - 2 \sin 2x)$$

$$y'' = 4e^{-2x}(c_1 \sin 2x + 2 \cos 2x) - 8e^{-2x}(c_1 \cos 2x - 2 \sin 2x) + 4e^{-2x}(-c_1 \sin 2x - 2 \cos 2x)$$

$$y'(0) = -2(2) + 2(c_1) = -4 + 2c_1$$

$$y''(0) = 4(2) + (c_1) + 4(-2) = -c_1 = -4 + 2c_1 \quad c_1 = \frac{4}{3}$$

$$y = e^{-2x}\left(\frac{4}{3} \sin 2x + 2 \cos 2x\right)$$

**Exercise 15.**  $y'' - 4y' + 29y = 0$

$$d = 16 - 4(1)(29) = -100 \implies u = e^{2x}(c_1 \sin 5x + c_2 \cos 5x).$$

$$v : \quad y'' + 4y' + 13y = 0$$

$$d = 10 - 4(13) = -36 \implies v = e^{-2x}(b_1 \sin 3x + b_2 \cos 3x)$$

$$v(0) = b_2$$

$$u(0) = 1(0 + c_2) = c_2 = 0$$

$$u = e^{2x}c_1 \sin 5x$$

$$u' = 2e^{2x}c_1 \sin 5x + e^{2x}c_1 5 \cos 5x$$

$$u'\left(\frac{\pi}{2}\right) = 1 = 2e^\pi c_1(1) \quad c_1 = \frac{1}{2e^\pi}$$

$$u'(0) = \frac{1}{2e^\pi} 5$$

$$u'(0) = v'(0) \implies b_1 = \frac{5}{6e^\pi}$$

$$v = e^{-2x}b_1 \sin 3x$$

$$v'(0) = 3b_1$$

$$u = \frac{1}{2e^\pi} e^{2x} \sin 5x$$

$$v = e^{-2x} \frac{5}{6e^\pi} \sin 3x$$

**Exercise 16.**

$$y'' - 3y' - 4y = 0 \quad u \quad 9 - 4(1)(-4) = 25 \quad u = e^{\frac{3x}{2}}(c_1 e^{\frac{5x}{2}} + c_2 e^{-\frac{5x}{2}})$$

$$y'' + 4y' - 5y = 0 \quad v \quad 16 - 4(1)(-5) = 36 \quad v = e^{-2x}(b_1 e^{3x} + b_2 e^{-3x})$$

$$u(0) = c_1 + c_2 = 0 \quad v(0) = b_1 + b_2 = 0 \implies u = 2e^{\frac{3x}{2}}c_1 \left(\sinh\left(\frac{5x}{2}\right)\right)$$

$$v = 2b_1 e^{-2x}(\sinh(3x))$$

$$u' = c_1 \frac{3}{2} e^{\frac{3x}{2}} (e^{\frac{5x}{2}} - e^{-\frac{5x}{2}}) + 2e^{\frac{3x}{2}} c_1 \frac{5}{2} \cosh\left(\frac{5x}{2}\right)$$

$$v' = -4b_1 e^{-2x} \sinh(3x) + 6b_1 e^{-2x} \cosh(3x)$$

$$u'(0) = 5c_1$$

$$v'(0) = 6b_1$$

$$c_1 = \frac{6b_1}{5}$$

**Exercise 17.**

$$y'' + ky = 0$$

Assume  $k > 0$

$$d = -4(k)$$

$$y = c_1 \sin \sqrt{k}x + c_2 \cos \sqrt{k}x$$

$$y(0) = c_2 = 0$$

$$\frac{\sqrt{d}}{2} = \frac{\sqrt{-4k}}{2} = \sqrt{-k}$$

$$y(1) = c_1 \sin \sqrt{k}1 = 0 \implies \sqrt{k} = n\pi$$

$$k < 0; \quad -k = \kappa > 0$$

$$y = c_1 e^{\sqrt{\kappa}x} + c_2 e^{-\sqrt{\kappa}x}; \quad y(0) = c_1 + c_2 = 0 \quad y = c_1 \sinh \sqrt{\kappa}x$$

$$y = c_1 \sinh \sqrt{\kappa}1 = 0 \quad c_1 = 0$$

so if  $k < 0$ , there are no nontrivial solutions satisfying  $f_k(0) = f_k(1) = 0$

**Exercise 18.**  $y'' + k^2 y = 0$        $d = -4k^2 < 0$        $\frac{\sqrt{d}}{2} = \frac{2k}{2} = k > 0$

$$\begin{aligned}
 y &= c_1 \sin kx + c_2 \cos kx & y(a) &= b = c_1 \sin ka + c_2 \cos ka \\
 y' &= kc_1 \cos kx - c_2 k \sin kx & y'(a) &= m = kc_1 \cos ka - c_2 k \sin ka \\
 kb \cos ka &= kc_1 \cos ka \sin ka + c_2 k \cos^2 ka \\
 m \sin ka &= kc_1 \cos ka \sin ka - c_2 k \sin^2 ka \\
 kb \cos ka - m \sin ka &= c_2 k \implies c_2 = \frac{kb \cos ka - m \sin ka}{k} \\
 c_1 \sin ka &= b - c_2 \cos ka = \frac{kb}{k} - \left( \frac{kb \cos ka - m \sin ka}{k} \right) \cos ka = \\
 &= \frac{kb(1 - \cos^2 ka) + m \sin ka \cos ka}{k} = \frac{kb \sin^2 ka + m \sin ka \cos ka}{k} \\
 c_1 &= \frac{kb \sin ka + m \cos ka}{k} \\
 y &= \left( \frac{kb \sin ka + m \cos ka}{k} \right) \sin kx + \left( \frac{kb \cos ka - m \sin ka}{k} \right) \cos kx \\
 k = 0 &\implies y = mx - ma + b
 \end{aligned}$$

**Exercise 19.**

(1)  $y = k_1 \sin x + k_2 \cos x$

$$\begin{aligned}
 b_2 &= k_1 \sin(a_2) + k_2 \cos(a_2) = k_1 s_2 + k_2 c_2 & b_2 c_1 &= k_1 s_2 c_1 + k_2 c_2 c_1 \\
 b_1 &= k_1 \sin(a_1) + k_2 \cos(a_1) = k_1 s_1 + k_2 c_1 & - (b_1 c_2 &= k_1 s_1 c_2 + k_2 c_1 c_2) \implies k_1 = \frac{b_2 c_1 - b_1 c_2}{s_2 c_1 - s_1 c_2} \\
 & \implies b_2 c_1 - b_1 c_2 = k_1 (s_2 c_1 - s_1 c_2) \\
 s_1 b_2 &= k_1 s_1 s_2 + k_2 s_1 c_2 & & \\
 - (s_2 b_1 &= k_1 s_1 s_2 + k_2 c_1 s_2) \implies k_2 = \frac{s_1 b_2 - s_2 b_1}{s_1 c_2 - c_1 s_2} \\
 y &= \left( \frac{b_2 \cos a_1 - b_1 \cos a_2}{\sin a_2 \cos a_1 - \sin a_1 \cos a_2} \right) \sin x + \left( \frac{b_2 \sin a_1 - b_1 \sin a_2}{\sin a_1 \cos a_2 - \cos a_1 \sin a_2} \right) \cos x \\
 y &= \boxed{\frac{b_2 \cos a_1 - b_1 \cos a_2}{\sin(a_2 - a_1)} \sin x + \frac{b_2 \sin a_1 - b_1 \sin a_2}{\sin(a_1 - a_2)} \cos x} \text{ if } a_2 - a_1 \neq \pi n \\
 \text{otherwise, if } a_2 - a_1 &= \pi n; & \begin{matrix} b_2 c_1 - b_1 c_2 = 0 \\ b_2 s_1 - b_1 s_2 = 0 \end{matrix} & b_2 c_1 = b_1 (-1)^n c_1; \quad \text{if } c \cos(a_1) \neq 0, \quad b_2 = b_1 (-1)^n
 \end{aligned}$$

(2) It's true if  $a_1 = a_2 = \frac{\pi}{4}$ ;       $b_1 = b_2$ .

(3)  $y'' + k^2 y = 0$

$$\begin{aligned}
 y &= A \sin kx + B \cos kx & \begin{bmatrix} S_1 & C_1 \\ S_2 & C_2 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} &= \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \\
 y(a_1) &= A \sin ka_1 + B \cos ka_1 = b_1 = AS_1 + BC_1 & \begin{bmatrix} A \\ B \end{bmatrix} &= \begin{bmatrix} C_2 & -C_1 \\ -S_2 & S_1 \end{bmatrix} \left( \frac{1}{S_1 C_2 - C_1 S_2} \right) \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \\
 y(a_2) &= A \sin ka_2 + B \cos ka_2 = b_2 = AS_2 + BC_2 & & \\
 S_1 C_2 - C_1 S_2 &= \sin ka_1 \cos ka_2 - \cos ka_1 \sin ka_2 = \sin(k(a_1 - a_2)) \\
 y &= \frac{b_1 \cos(ka_2) - b_2 \cos(ka_1)}{\sin(k(a_1 - a_2))} \sin ka_1 + \frac{-b_1 \sin(ka_2) + b_2 \sin(ka_1)}{\sin(k(a_1 - a_2))} \sin ka_2 \\
 \text{if } k(a_1 - a_2) &\neq \pi n \quad k = 0; & y &= \left( \frac{b_2 - b_1}{a_2 - a_1} \right) (x - a_1) + b_1
 \end{aligned}$$

**Exercise 20.**

(1)

$$u_1(x) = e^x; \quad u_2(x) = e^{-x} \quad u'_2 = -e^{-x} \quad u''_2 = e^{-x} \implies y'' - y = 0$$

(2)

$$\begin{aligned}
u_1 &= e^{2x} & u_2 &= xe^{2x} \\
u_1' &= 2e^{2x} & u_2' &= e^{2x} + 2xe^{2x} \\
u_1'' &= 4e^{2x} & u_2'' &= 2e^{2x} + 2e^{2x} + 4xe^{2x} = 4e^{2x} + 4xe^{2x} = 4e^{2x}(1+x) \\
u_2'' - 4u_2' + 4u_2 &= 0 \implies \boxed{y'' - 4y' + 4y = 0}
\end{aligned}$$

(3)

$$\begin{aligned}
u_1(x) &= e^{-x/2} \cos x; & u_2(x) &= e^{-x/2} \sin x \\
u_1' &= -\frac{1}{2}e^{-x/2} \cos x - e^{-x/2} \sin x & u_2' &= \frac{-1}{2}e^{-x/2} \sin x + e^{-x/2} \cos x \\
u_1'' &= \frac{1}{4}e^{-x/2} \cos x + e^{-x/2} \sin x - e^{-x/2} \cos x & u_2'' &= \frac{1}{4}e^{-x/2} \sin x - e^{-x/2} \cos x - e^{-x/2} \sin x = \\
&= \frac{-3}{4}e^{-x/2} \cos x + e^{-x/2} \sin x & &= \frac{-3}{4}e^{-x/2} \sin x - e^{-x/2} \cos x \\
& & u_2'' + u_2' + \frac{5}{4}u_2 &= 0 \\
& & \boxed{y'' + y' + \frac{5}{4}y = 0}
\end{aligned}$$

$$(4) \quad u_1(x) = \sin(2x+1); \quad u_2(x) = \sin(2x+2)$$

$$\begin{aligned}
u_1' &= 2 \cos(2x+1) & u_2' &= 2 \cos(2x+2) \\
u_1'' &= -4 \sin(2x+1) & u_2'' &= -4 \sin(2x+2) \\
& \boxed{y'' + 4y = 0}
\end{aligned}$$

(5)

$$\begin{aligned}
u_1 &= \cosh x \\
u_1' &= \sinh x & \boxed{y'' - y = 0} \\
u_1'' &= \cosh x
\end{aligned}$$

**Exercise 21.**  $w = u_1 u_2' - u_2 u_1'$ .

$$(1) \quad w = 0 \quad \forall x \in \text{open interval } I,$$

$$\left(\frac{u_2}{u_1}\right)' = \frac{u_2' u_1 - u_1' u_2}{u_1^2} = 0 \implies \frac{u_2}{u_1} = c$$

If  $\frac{u_2}{u_1}$  is not constant, then  $w(0) \neq 0$  for at least one  $c$  in  $I$  (otherwise, it'd be constant).

$$(2) \quad w' = u_1 u_2'' - u_2 u_1''$$

**Exercise 22.**

$$(1) \quad w' + aw = u_1 u_2'' - u_2 u_1'' + a(u_1 u_2' - u_2 u_1') = u_1(-bu_2) + -u_2(bu_1) = 0$$

$$w(x) = w(0)e^{-ax} \quad \text{if } w(0) \neq 0, \text{ then } w(x) \neq 0 \quad \forall x.$$

$$(2) \quad u_1 \neq 0 \text{ If } w(0) = 0, \quad w(x) = 0 \quad \forall x, \text{ so } \frac{u_2}{u_1} \text{ constant. If } \frac{u_2}{u_1} \text{ constant, } w(0) = 0 \text{ since from the previous part.}$$

**Exercise 23.** Recall the properties of the Wronskian.

$$(1) \quad \text{If } W(x) = v_1(x)v_2' - v_2v_1' = v_1v_2' - v_2v_1' = 0 \quad \forall x \in I, \\ \text{then } \frac{v_2}{v_1} \text{ constant on } I$$

$$(2) \quad W' = v_1 v_2'' - v_2 v_1''$$

$$(3) \quad W' + aW = 0 \text{ if } v_1, v_2 \text{ are solutions to } y'' + ay' + by = 0$$

$$W(x) = W(0)e^{-ax}$$

$$\text{So if } W(0) \neq 0, \quad W(x) \neq 0 \quad \forall x$$

Consider adding together the solutions and the solution's derivatives into some function  $f$ . By the linearity of the differential equation, we know that  $f$  is also a solution since it is a linear superposition of solutions.

$$\begin{aligned} y(x) &= Av_1(x) + Bv_2(x) & y'(x) &= Av'_1(x) + Bv'_2(x) \\ f(0) &= Av_1(0) + Bv_2(0) & f'(0) &= Av'_1(0) + Bv'_2(0) \end{aligned}$$

$$\begin{bmatrix} v_1(0) & v_2(0) \\ v'_1(0) & v'_2(0) \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} f(0) \\ f'(0) \end{bmatrix} \Rightarrow \frac{1}{W(0)} \begin{bmatrix} v'_2(0) & -v_2(0) \\ -v'_1(0) & v_1(0) \end{bmatrix} \begin{bmatrix} f(0) \\ f'(0) \end{bmatrix} = \begin{bmatrix} A \\ B \end{bmatrix}$$

since  $W(0) \neq 0$ , this division is allowed above

$$\text{so } y(x) = \left( \frac{v'_2(0)f(0) - v_2(0)f'(0)}{W(0)} \right) v_1(x) + \left( \frac{v_1(0)f'(0) - v'_1(0)f(0)}{W(0)} \right) v_2(x)$$

$f(0), f'(0)$  are initial conditions for  $y$ .  $f(0), f'(0)$  are arbitrary.

But since  $W(0) \neq 0$ ,  $W(0) = v_1(0)v'_2(0) - v_2(0)v'_1(0)$ , we can do things like

$$f(0) = \frac{v'_2(0)f(0) - v_2(0)f'(0)}{v_1(0)v'_2(0) - v_2(0)v'_1(0)} v_1(0) + \frac{v_2(0)f'(0) - v'_1(0)f(0)}{W(0)} v_2(0)$$

### 8.17 Exercises - Nonhomogeneous linear equations of second order with constant coefficients, Special methods for determining a particular solution of the nonhomogeneous equation $y'' + ay' + by = R$ .

**Exercise 1.**  $y'' - y = x$  homogeneous solution  $c_1 e^x + c_2 e^{-x}$ .  $y_p = -x$

$$y = c_1 e^x + c_2 e^{-x} - x$$

**Exercise 2.**  $y'' - y' = x^2$  For the homogeneous solution

$$y'' - y' = 0 \quad d = (-1)^2 - 4(1)(0) = 1 \quad y_h = e^{\frac{x}{2}} (c_1 e^{\frac{x}{2}} + c_2 e^{-\frac{x}{2}}) = c_2 e^x + c_1$$

$$y_p = Ax^3 + Bx^2 + Cx + D$$

$$y'_p = 3Ax^2 + 2Bx + C$$

$$y''_p = 6Ax + 2B$$

$$y = c_1 + c_2 e^x + \frac{-1}{3} x^3 + x^2$$

**Exercise 3.**  $y'' + y' = x^2 + 2x$

$$e^{-\frac{x}{2}} (c_1 e^{-\frac{x}{2}} + c_2 e^{\frac{x}{2}}) = c_1 e^{-x} + c_2$$

$$P = Ax^3 + Bx^2 + Cx + D; \quad P' = 3Ax^2 + 2Bx + C \quad P'' = 6Ax + 2B$$

$$3Ax^2 + 2Bx + C + 6Ax + 2B = x^2 + 2x \quad A = \frac{1}{3} \quad B = 0 \quad C = 0$$

$$y = c_1 e^{-x} + c_2 + \frac{1}{3} x^3$$

**Exercise 4.**  $y'' - 2y' + 3y = x^3$   $u = e^x (c_1 \sin \sqrt{2}x + c_2 \cos \sqrt{2}x)$

$$3(Ax^3 + Bx^2 + Cx + D)$$

$$2(3Ax^2 + 2Bx + C) \quad A = \frac{1}{3} \quad B = \frac{2}{3} \quad C = \frac{8}{9} \quad D = \frac{16}{27}$$

$$(6Ax + 2B)$$

$$y = C_1 e^x \sin \sqrt{2}x + C_2 e^x \cos \sqrt{2}x + \frac{1}{3} x^3 + \frac{2}{3} x^2 + \frac{8}{9} x + \frac{16}{27}$$

**Exercise 5.**  $y'' - 5y' + 4y = x^2 - 2x + 1$

$$y_h = e^{\frac{5x}{2}} \left( e^{\frac{3x}{2}} + e^{-\frac{3x}{2}} \right) = c_1 e^{4x} + c_2 e^x$$

$$d = \sqrt{25 - 4(4)} = 3$$

$$4(Ax^2 + Bx + C)$$

$$4Ax^2 + 4Bx + 4C$$

$$-5(2Ax + B)$$

$$-10Ax - 5B$$

$$2A$$

$$2A$$

$$A = \frac{1}{4} \quad B = \frac{1}{8}$$

$$\frac{1}{2} - \frac{5}{8} + 4C = 1$$

$$C = \frac{9}{32}$$

$$y = c_1 e^{4x} + c_2 e^x + \frac{1}{4} x^2 + \frac{1}{8} x + \frac{9}{32}$$



**Exercise 6.**

$$\begin{aligned}
 y'' + y' - 6y &= 2x^3 + 5x^2 - 7x + 2 \\
 y_h &= e^{\frac{-x}{2}} \left( e^{\frac{-5x}{2}} + e^{\frac{5x}{2}} \right) = e^{-3x} + e^{2x} \\
 d &= \sqrt{1 - 4(-6)} = 5 \\
 y_p &= Ax^3 + Bx^2 + Cx + D \quad -6Ax^3 - 6Bx^2 - 6Cx - 6D \\
 y'_p &= 3Ax^2 + 2Bx + C \quad \implies \quad 3Ax^2 + 2Bx + C \\
 y''_p &= 6Ax + 2B \quad 6Ax + 2B \\
 A &= \frac{-1}{3} \quad B = -1 \quad C = \frac{1}{2} \quad D = \frac{-7}{12} \\
 \boxed{y} &= C_1 e^{-3x} + C_2 e^{2x} - \frac{1}{3}x^3 - x^2 + \frac{1}{2}x - \frac{7}{12}
 \end{aligned}$$

**Exercise 7.**

$$\begin{aligned}
 y'' - 4y &= e^{2x} & v_1 &= e^{2x} & v'_2 &= -2e^{-2x} \\
 y_h &= c_1 e^{2x} + c_2 e^{-2x} & v'_1 &= 2e^{2x} & v_2 &= e^{-2x}
 \end{aligned}$$

Use Theorem 8.9.

**Theorem 28.** Let  $v_1, v_2$  be solutions to  $L(y) = 0$  where  $L(y) = y'' + ay' + by$

Let  $W = v_1 v'_2 - v_2 v'_1$ . Then  $L(y) = R$  where

$$y_p = t_1 v_1 + t_2 v_2$$

$$\begin{aligned}
 (23) \quad t_1 &= - \int v_2 \frac{R}{W(x)} dx; \quad t_2(x) = \int v_1 \frac{R}{W} dx \\
 t_1 &= - \int e^{-2x} \frac{e^{2x}}{-4} = \frac{1}{4}x & w &= e^{2x}(-2)e^{-2x} - e^{-2x}2e^{2x} = -4 \\
 t_2 &= \int e^{2x} \frac{e^{2x}}{-4} = \frac{e^{4x}}{-16} \\
 y_p &= \frac{x}{4}e^{2x} + \frac{e^{2x}}{-16} \\
 y &= c_1 e^{2x} + c_2 e^{-2x} + \frac{x}{4}e^{2x} + \frac{e^{2x}}{-16}
 \end{aligned}$$

**Exercise 8.**

$$\begin{aligned}
 y'' + 4y &= e^{-2x} & w &= \sin 2x - \sin 2x(2) - 2 \cos 2x \cos 2x = 2 \\
 y_h &= c_1 \sin 2x + c_2 \cos 2x \\
 t_1 &= - \int \frac{\cos 2x e^{-2x} dx}{2} = \frac{-1}{2} \int e^{-2x} \cos 2x dx & (e^{-2x} \cos 2x)' &= -2e^{-2x} \cos 2x + -2e^{-2x} \sin 2x \\
 t_2 &= \int \frac{\sin 2x e^{-2x}}{2} = \frac{1}{2} \int e^{-2x} \sin 2x dx & (e^{-2x} \sin 2x)' &= -2e^{-2x} \sin 2x + 2e^{-2x} \cos 2x \\
 \left( \frac{e^{-2x} \sin 2x - e^{-2x} \cos 2x}{4} \right)' &= e^{-2x} \cos 2x & t_1 &= \frac{e^{-2x} \cos 2x - e^{-2x} \sin 2x}{8} \\
 \left( \frac{e^{-2x} \sin 2x + e^{-2x} \cos 2x}{-4} \right)' &= e^{-2x} \sin 2x & t_2 &= \frac{e^{-2x} \sin 2x + e^{-2x} \cos 2x}{-8} \\
 y_p &= \frac{e^{-2x} \sin 2x \cos 2x - e^{-2x} \sin^2 2x}{8} + \frac{e^{-2x} \sin 2x \cos 2x + e^{-2x} \cos^2 2x}{-8} = \boxed{\frac{e^{-2x}}{8}} \\
 \boxed{y} &= c_1 \sin 2x + c_2 \cos 2x + \frac{e^{-2x}}{8}
 \end{aligned}$$

**Exercise 9.**  $y'' + y' - 2y = e^x$   $d^2 = 1 - (4)(1)(-2) = 9$

$$y_h = e^{-\frac{1}{2}x} \left( e^{\frac{3x}{2}} + e^{-\frac{3x}{2}} \right) = e^x + e^{-2x}$$

$$(xe^x)' = e^x + xe^x$$

$$+(xe^x)'' = +(2e^x + xe^x)$$

$$\implies 3e^x + 2xe^x$$

$y = c_1 e^x + c_2 e^{-2x} + \frac{1}{3} x e^x$

**Exercise 10.**  $y'' + y' - 2y = e^{2x}$ .  $y_h = e^{-\frac{x}{2}} \left( c_1 e^{\frac{3x}{2}} + c_2 e^{-\frac{3x}{2}} \right) = c_1 e^x + c_2 e^{-2x}$ .

$$W(x) = v_1 v_2' - v_2 v_1' = e^x(-2)e^{-2x} - e^{-2x}e^x = -3e^{-x}$$

$$t_1 = - \int \frac{v_2 R}{W} \quad t_2 = \int \frac{v_1 R}{W}$$

$$t_1 = - \int \frac{e^{-2x} e^{2x}}{-3e^{-x}} = \frac{1}{3} e^x \quad t_2 = \int \frac{e^x e^{2x}}{-3e^{-x}} = -\frac{1}{12} e^{4x}$$

$$y_1 = t_1 v_1 + t_2 v_2 = \frac{1}{3} e^x e^x + -\frac{1}{12} e^{4x} e^{-2x} = \frac{1}{4} e^{2x}$$

$$y = c_1 e^x + c_2 e^{-2x} + \frac{1}{4} e^{2x}$$

**Exercise 11.**  $y'' + y' - 2y = e^x + e^{2x}$ .

Consider solutions to Exercise 9,10.

$$L(y_a) = e^x; L(y_b) = e^{2x}; L(y_a + y_b) = e^x + e^{2x}$$

$$\implies \boxed{y = c_1 e^x + c_2 e^{-2x} + \frac{1}{3} x e^x + \frac{1}{4} e^{2x}}$$

**Exercise 12.**  $y'' - 2y' + y = x + 2xe^x$ .  $d = 4 - 4(1) = 0$ . Recall the definition to be learned for this section of exercises:

**Theorem 29.** Let  $d = a^2 - 4b$  be the discriminant of  $y'' + ay' + by = 0$ . Then every solution of this equation on  $(-\infty, \infty)$  has the form

(24)  $y = e^{-ax/2} (c_1 u_1(x) + c_2 u_2(x))$

- (1) If  $d = 0$  then  $u_1 = 1, u_2 = x$
- (2) If  $d > 0$ ,  $u_1 = e^{kx}; u_2 = e^{-kx}, k = \frac{\sqrt{d}}{2}$
- (3) If  $d < 0$ ,  $u_1 = \cos kx, u_2 = \sin kx, k = \frac{\sqrt{-d}}{2}$

$$y_h = e^x (c_1 + c_2 x) = c_1 e^x + c_2 x e^x$$

$$t_1 = \int \frac{-x e^x (2x e^x)}{e^{2x}} = \frac{2x^3}{-3} \quad t_2 = \int \frac{e^x (2x e^x)}{e^{2x}} = x^2$$

$$W(x) = e^x (e^x + x e^x) - (x e^x) (e^x) = e^{2x}$$

$$y_p = \frac{2x^3}{-3} e^x + x^3 e^x = \boxed{\frac{x^3 e^x}{3}}$$

$y = \frac{x^3 e^x}{3} + c_1 e^x + c_2 x e^x$

**Exercise 13.**  $y'' + 2y' + y = \frac{e^{-x}}{x^2}$

$$y_h = e^{-x} (c_1 + c_2 x) = c_1 e^{-x} + c_2 x e^{-x}$$

$$t_1 = \int \frac{-x e^{-x} \left( \frac{e^{-x}}{x^2} \right)}{e^{-2x}} = -\ln x \quad t_2 = \int \frac{e^{-x} \left( \frac{e^{-x}}{x^2} \right)}{e^{-2x}} = \frac{-1}{x}$$

$$W = e^{-x} (e^{-x} + -x e^{-x}) - (-e^{-x}) (x e^{-x}) = e^{-2x}$$

$$y_p = -\ln x e^{-x} + x e^{-x} \left( \frac{-1}{x} \right) = -\ln x e^{-x} - e^{-x}$$

$y = c_1 e^{-x} + c_2 x e^{-x} + (-\ln x - 1) e^{-x}$

**Exercise 14.**  $y'' + y = \cot^2 x$ .  $y_h = c_1 \sin x + c_2 \cos x$ .

$$t_1 = \int \frac{-\cos x}{-1} \cot^2 x dx = \int \frac{\cos x \cos^2 x}{\sin^2 x} = \int \frac{\cos x (1 - \sin^2 x)}{\sin^2 x} = -\frac{1}{\sin x} + -\sin x$$

$$t_2 = \int \frac{\sin x}{-1} \cot^2 x = -\int \frac{\cos^2 x}{\sin x} = -\int \frac{1 - \sin^2 x}{\sin x} = \ln |\csc x + \cot x| - \cos x$$

$$y_p = -1 - \sin^2 x + \cos x \ln |\csc x + \cot x| - \cos^2 x = \boxed{-2 + \cos x \ln |\csc x + \cot x|}$$

$$\boxed{y = c_1 \sin x + c_2 \cos x - 2 + \cos x \ln |\csc x + \cot x|}$$

**Exercise 15.**  $y'' - y = \frac{2}{1+e^x}$

$$y_h = c_1 e^x + c_2 e^{-x} \implies W(x) = -e^x e^{-x} - e^x e^{-x} = -2$$

$$\begin{aligned} t_1 &= -\int \frac{e^{-x} \frac{2}{1+e^x}}{-2} = \int \frac{e^{-x}}{1+e^x} = \\ &= \int \frac{1}{e^x} - \frac{1}{1+e^x} = -e^{-x} - \int \frac{1}{1+e^x} = -e^{-x} + \int \frac{-e^{-x}}{e^{-x}+1} = -e^{-x} + \ln(1+e^{-x}) \\ t_2 &= \int \frac{e^x \frac{2}{1+e^x}}{-2} = -\ln|1+e^x| \end{aligned}$$

$$\boxed{y = -1 + e^x \ln(1+e^{-x}) + -e^{-x} \ln(1+e^x) + c_1 e^x + c_2 e^{-x}}$$

**Exercise 16.**  $y'' + y' - 2y = \frac{e^x}{1+e^x}$

$$\text{Discriminant: } \frac{-1 \pm \sqrt{1^2 - 4(-2)}}{2} = -2, 1 \implies y_h = c_1 e^x + c_2 e^{-2x} \implies W = e^x(-2)e^{-2x} - e^{-2x}e^x = -3e^{-x}$$

$$\begin{aligned} t_1 &= -\int \frac{e^{-2t} \left( \frac{e^t}{1+e^t} \right)}{-3e^{-t}} = \frac{1}{3} \int \frac{1}{1+e^t} = \frac{-1}{3} \ln(1+e^{-x}) \\ t_2 &= \int \frac{e^t \left( \frac{e^t}{1+e^t} \right)}{-3e^{-t}} = \frac{1}{-3} \int \frac{e^{3t}}{1+e^t} \xrightarrow{u=e^t} \frac{-1}{3} \int \frac{u^2 du}{1+u} = \\ &= \frac{-1}{3} \int u + \frac{-u}{u+1} = \frac{-1}{3} \int u + -1 + \frac{1}{u+1} = \frac{-1}{3} \left( \frac{1}{2} u^2 - u + \ln u + 1 \right) \\ &= \frac{-1}{6} e^{2x} + \frac{e^x}{3} + \frac{-1}{3} \ln(e^x + 1) \end{aligned}$$

$$\boxed{y_1 = \frac{-1}{3} e^x \ln(1+e^{-x}) + \frac{-1}{6} + \frac{e^{-x}}{3} - \frac{e^{-2x}}{3} \ln(e^x + 1) + c_1 e^x + c_2 e^{-2x}}$$

**Exercise 17.**  $y' + 6y' + 9y = f(x)$ ; where  $f(x) = 1$  for  $1 \leq x \leq 2$ .  $f(x) = 0$  for all other  $x$ .

$$d = 36 - 4(1)(9) = 0$$

$$y_h = e^{-3x}(c_1 + c_2 x) = c_1 e^{-3x} + c_2 x e^{-3x}$$

$$W(x) = e^{-3x}(e^{-3x} - 3xe^{-3x}) - (xe^{-3x})(-3e^{-3x}) = e^{-6x}$$

$$t_1(x) = \begin{cases} a < 1 < x & \int_1^x -te^{3t} dt = \left( \frac{-te^{3t}}{3} + \frac{e^{3t}}{9} \right) \Big|_1^x = \frac{-3xe^{3x} + e^{3x}}{9} + \frac{2}{9} e^3 \\ a < 1 < 2 < x & \int_1^2 -te^{3t} dt = \frac{-6e^6 + e^6}{9} + \frac{2e^3}{9} = \frac{-5e^6}{9} + \frac{2e^3}{9} \\ 1 < a < x < 2 & \int_a^x -te^{3t} dt = \left( \frac{-te^{3t}}{3} + e^{3t} \right) \Big|_a^x = \frac{-xe^{3x}}{3} + \frac{e^{3x}}{9} + \frac{ae^{3a}}{3} - \frac{e^{3a}}{9} \end{cases}$$

$$t_2(x) = \int \frac{e^{-3t} f(t)}{e^{-6t}} = \int e^{3t} f(t) = \int_a^x e^{3t} = \frac{1}{3} (e^{3x} - e^{3a})$$

$$y_1 = e^{-3x} \left( \frac{-3xe^{3x} + e^{3x}}{9} + C \right) + \frac{x}{3}$$

$$\boxed{y = c_1 e^{-3x} + c_2 x e^{-3x} + \frac{1}{9} \text{ when } 1 \leq x \leq 2; \text{ otherwise } y = y_h}$$

**Exercise 18.** Start from  $y'' - k^2 y = R(x)$ . Suppose  $L(y_p) = y_p'' - k^2 y_p = R(x)$ .

$$y_h = c_1 \sinh(kx) + c_2 \cosh(kx); \quad L(y_h) = 0$$

So consider  $L(y_p + y_h) = R(x)$ ;  $y_p + y_h = y_1$  is a nother particular solution.

The key to this problem is to apply the integration directly on the ODE itself, not to go the other way around by differentiating the supposed particular solution.

$$\begin{aligned} \frac{\int_0^x dt \sinh(k(x-t))}{\int_0^x dt \sinh(k(x-t))} &\rightarrow \int_0^x dt \frac{d^2 y}{dt^2}(t) \sinh(k(x-t)) - k^2 \int_0^x dt y(t) \sinh(k(x-t)) = \int_0^x dt R(t) \sinh(k(x-t)) dt \\ &\int_0^x y'' \sinh(\kappa) = -y'(0) \sinh(kx) - \int y' \cosh(\kappa)(-k) = \\ &= -y'(0) \sinh(kx) + k(y(x) - y(0) \cosh(kx) + k \int y(t) \sinh(\kappa)) = \\ &= -y'(0) \sinh(kx) + ky(x) - ky(0) \cosh(kx) + k^2 \int y(t) \sinh(\kappa) \\ &\Rightarrow y(x) - \frac{y'(0) \sinh(kx)}{k} - y(0) \cosh(kx) = \frac{1}{k} \int_0^x dt R(t) \sinh(k(x-t)) dt \end{aligned}$$

Now note that  $L(y_h) = 0$ , so applying  $\int_0^x dt \sinh(k(x-t))$  results in 0 still.

With  $y_p(x) = \frac{1}{k} \int_0^x dt R(t) \sinh(k(x-t)) dt + \frac{y'(0) \sinh(kx)}{k} + y(0) \cosh(kx)$ , we can add a homogeneous solution of  $\frac{y'(0) \sinh(kx)}{k} + y(0) \cosh(kx)$  to  $y_p(x)$  to obtain

$$y_1(x) = \frac{1}{k} \int_0^x dt R(t) \sinh(k(x-t)) dt$$

Now for  $y'' - 9y = e^{3x}$ ,

$$\begin{aligned} y_1(x) &= \frac{1}{3} \int_0^x dt (e^{3t}) \sinh(3(x-t)) dt = \frac{1}{6} \int_0^x dt e^{3t} \left( \frac{e^{3x-3t} - e^{-3x+3t}}{2} \right) = \frac{1}{6} \int_0^x dt (e^{3x} - e^{-3x+6t}) \\ &= \frac{1}{6} \left( e^{3x} x - e^{-3x} \frac{1}{6} e^{6t} \Big|_0^x \right) = \frac{1}{6} x e^{3x} - e^{-3x} \frac{e^{6x} - 1}{36} = \frac{1}{6} \left( x e^{3x} - \frac{e^{3x}}{6} + \frac{e^{-3x}}{6} \right) \\ y_1' &= \frac{1}{6} (e^{3x} + 3x e^{3x} - \frac{3}{2} e^{3x} - \frac{3}{2} e^{-3x}) \\ y_1'' &= \frac{1}{6} \left( 3e^{3x} + 3e^{3x} + 9x e^{3x} - \frac{9}{2} e^{3x} + \frac{9}{2} e^{-3x} \right) = \frac{1}{4} e^{3x} + \frac{3}{4} e^{-3x} + \frac{3}{2} x e^{3x} \\ y_1'' - 9y_1 &= \frac{1}{2} e^{3x} + \frac{1}{2} e^{-3x} \end{aligned}$$

Thus, we need to add homogeneous parts to our particular solution to make it work. So if

$$y_p = \frac{x e^{3x}}{6} - \frac{e^{3x}}{y} + \frac{e^{-3x}}{6}$$

then it could be checked easily with some computation, that this satisfies the ODE.

**Exercise 19.** Start from  $y'' + k^2 y = R(x)$

Again, note that if  $L(y_p) = y_p'' + k^2 y_p = R(x)$ ,  $L(y_p + y_h) = R(x) + 0 = R(x)$ , so  $y_1 = y_p + y_h$  is also a particular solution.

$$\begin{aligned} \frac{\int_0^x dt \sin k(x-t)}{\int_0^x dt \sin k(x-t)} &\rightarrow \int_0^x dt \sin k(x-t) \frac{d^2 y}{dt^2} + k^2 \int_0^x dt \sin k(x-t) y = \int_0^x R(t) \sin k(x-t) \\ \int_0^x dt \sin k(x-t) \frac{d^2 y}{dt^2} &= -y'(0) \sin(kx) + k \int_0^x y'(t) \cos(k(x-t)) dt = \\ &= -y'(0) \sin(kx) + k(y(x) - y(0) \cos(kx) - k \int_0^x y(t) \sin(k(x-t)) dt) = \\ &= ky(x) - ky(0) \cos(kx) - y'(0) \sin(kx) - k^2 \int_0^x y(t) \sin(k(x-t)) dt \end{aligned}$$

$$\Rightarrow y(x) = \frac{1}{k} \int_0^x dt R(t) \sin k(x-t) + y(0) \cos(kx) + \frac{y'(0) \sin(kx)}{k}$$

We can add  $y_h$  with  $c_1 = -y(0)$ ,  $c_2 = \frac{-y'(0)}{k}$

$$y_1 = \frac{1}{k} \int_0^x dt R(t) \sin k(x-t)$$

Now for  $y'' + 9y = \sin 3x$ , then  $k = 3$ ,

$$\begin{aligned} y_1 &= \frac{1}{3} \int_0^x \sin 3t \sin 3(x-t) dt \\ &= \frac{1}{3} \int_0^x \sin 3t (\sin 3x \cos 3t - \cos 3x \sin 3t) dt \\ \int_0^x sc &= \int_0^x \frac{s(6t)}{2} dt = \frac{-1}{12} (c(6x) - 1) \\ \int_0^x s^2 &= \int_0^x \frac{1 - \cos(6t)}{2} = \frac{x}{2} - \frac{\sin(6t)}{12} \Big|_0^x = \frac{x}{2} - \frac{\sin(6x)}{12} \\ y_1 &= \frac{1}{3} \left( \sin 3x \frac{-1}{12} (\cos(6x) - 1) - \cos 3x \left( \frac{x}{2} - \frac{\sin(6x)}{12} \right) \right) = \boxed{\frac{\sin 3x}{18} - \frac{x \cos 3x}{6}} \end{aligned}$$

It could be shown with some computation that this particular solution satisfies the ODE without having to add or subtract parts of a homogeneous solution.

**Exercise 20.**  $y'' + y = \sin x$

$$y_h = c_1 \sin x + c_2 \cos x \quad \Rightarrow W(x) = -s^2 - c^2 = -1$$

$$\begin{aligned} t_1 &= - \int \frac{cs}{-1} = \frac{-\cos 2x}{4}; \quad t_2 = \int \frac{ss}{-1} = - \int \frac{1 - \cos 2x}{2} = - \left( \frac{x}{2} - \frac{\sin 2x}{4} \right) \\ y_p &= - \frac{\sin x \cos 2x}{4} + \left( \frac{\sin 2x - 2x}{4} \right) \cos x = \frac{\sin x \cos^2 x + \sin^3 x - 2x \cos x}{4} \\ &\quad \boxed{y = c_1 \sin x + c_2 \cos x + \frac{\sin x \cos^2 x + \sin^3 x - 2x \cos x}{4}} \end{aligned}$$

**Exercise 21.**  $y'' + y = \cos x$

$$y_h = c_1 \sin x + c_2 \cos x = c_1 S + c_2 C \quad W(x) = -1$$

$$\begin{aligned} t_1 &= \int \frac{-CC}{-1} = \int \frac{1 + \cos 2x}{2} = \frac{x + \frac{\sin 2x}{2}}{2}; \quad t_2 = \int \frac{SC}{-1} = \frac{\cos 2x}{4} \\ y_p &= \frac{x \sin x}{2} + \frac{\sin 2x \sin x}{4} + \frac{\cos x \cos 2x}{4} \\ &\Rightarrow \boxed{y = \frac{x \sin x}{2} + \frac{\sin 2x \sin x}{4} + \frac{\cos x \cos 2x}{4} + c_1 \sin x + c_2 \cos x} \end{aligned}$$

**Exercise 22.**  $y'' + 4y = 3x \cos x$

$$y_h = c_1 \sin 2x + c_2 \cos 2x \quad W(x) = -\sin^2 2x(2) + -\cos^2 2x(2) = -2$$

$$\begin{aligned} t_1 &= \int \frac{-\cos(2x)(3x \cos x)}{-2} = \frac{3}{2} \int x \cos x \cos(2x) = \frac{3}{2} \int xc(1 - 2s^2) = \frac{3}{2} \int xc - 3 \int xcs^2 = \\ &= \frac{3}{2}(xs + c) - 3 \int x \left( \frac{c^3}{3} \right)' = \frac{3}{2}(xs + c) - (xs^3 - \int s^3) = \frac{3}{2}(xs + c) - xs^3 + \int s(1 - c^2) = \\ &= \frac{3}{2}(xs + c) - xs^3 + -c + \frac{1}{3}c^3 = \frac{3}{2}xs + \frac{c}{2} - xs^3 + \frac{1}{3}c^3 \end{aligned}$$

$$\begin{aligned}
t_2 &= \int \frac{\sin(2x)(3x \cos x)}{-2} = -3 \int x s c^2 = \int x(c^3)' = x c^3 - \int c^3 = x c^3 - \int c(1-s^2) = \\
&= x c^3 - s + \frac{1}{3} s^3 \\
y_p &= \left( \frac{3}{2} x s + \frac{c}{2} - x s^3 + \frac{1}{3} c^3 \right) (2s c) + (x c^3 - s + \frac{s^3}{3})(1-2s^2) = (\text{lots of algebra}) = \\
&= \boxed{x c^2 + \frac{2}{3} s} \\
&\implies \boxed{y = c_1 \sin 2x + c_2 \cos 2x + x \sin x - \frac{2}{3} \cos x}
\end{aligned}$$

Remember, *persistence is key* to work through the algebra, quickly.

**Exercise 23.**  $y'' + 4y = 3x \sin x$ . From the work above, we could guess at the solution.

$$\begin{aligned}
(xs)' &= s + xc \\
(xs)'' &= 2c - xs \\
(xs)'' + 4(xs) &= 2c - xs + 4xs = 2c + 3xs \quad (c)'' + 4c = 3c \implies \left(\frac{-2}{3}c\right)'' + 4\left(\frac{-2}{3}c\right) = -2c \\
&\implies \boxed{y_h = xs - \frac{2}{3}c} \\
&\boxed{y = x \sin x - \frac{2}{3} \cos x + c_1 \sin 2x + c_2 \cos 2x}
\end{aligned}$$

**Exercise 24.**  $y'' - 3y' = 2e^{2x} \sin x$  Guessing and stitching together the solution seems easier to me.

$$\begin{aligned}
(e^{2x}c)' &= 2e^{2x}c - e^{2x}s \\
(e^{2x}c)'' &= 4e^{2x}c - 4e^{2x}s - e^{2x}c = 3e^{2x}c - 4e^{2x}s \\
(e^{2x}c)'' - 3(e^{2x}c)' &= 3e^{2x}c - 4e^{2x}s - 6e^{2x}c + 3e^{2x}s = \\
&= -3e^{2x}c - e^{2x}s \\
(e^{2x}s)' &= 2e^{2x}s + e^{2x}c \\
(e^{2x}s)'' &= 4e^{2x}s + 4e^{2x}c - se^{2x} = \\
&= 3e^{2x}s + 4e^{2x}c \\
(e^{2x}s)'' - 3(e^{2x}s)' &= 3e^{2x}s + 4e^{2x}c - 6e^{2x}s - 3e^{2x}c = \\
&= -3e^{2x}s + e^{2x}c \\
(3e^{2x}s)'' - 3(3e^{2x}s)' + (e^{2x}c)'' - 3(e^{2x}c)' &= -10e^{2x}s \\
\implies y_p &= \frac{e^{2x}(3 \sin x + \cos x)}{5} \implies \boxed{y = c_1 \sin \sqrt{3}x + c_2 \cos \sqrt{3}x + \frac{e^{2x}(3 \sin x + \cos x)}{-5}}
\end{aligned}$$

**Exercise 25.**  $y'' + y = e^{2x} \cos 3x$ .

$$\begin{aligned}
(e^{2x}c(3x))'' &= 4e^{2x}c(3x) + 12e^{2x}s(3x) - 9e^{2x}c(3x) = -5e^{2x}c(3x) - 12e^{2x}s(3x) \\
(e^{2x}s(3x))'' &= 4e^{2x}s(3x) + 12e^{2x}c(3x) - 9e^{2x}s(3x) = -5e^{2x}s(3x) + 12e^{2x}c(3x) \\
L(e^{2x}c(3x) - 3e^{2x}s(3x)) &= -40e^{2x} \cos(3x) \\
\boxed{y = c_1 \sin x + c_2 \cos x + \frac{e^{2x} \cos(3x) - 3e^{2x} \sin(3x)}{-40}}
\end{aligned}$$

### 8.19 Exercises - Examples of physical problems leading to linear second-order equations with constant coefficients.

In exercises 1-5, a particle is assumed to be moving in simple harmonic motion, according to the equation  $y = C \sin(kx + \alpha)$ . The velocity of the particle is defined to be the derivative  $y'$ . The frequency of the motion is the reciprocal of the period. (Period =  $2\pi/k$ , frequency =  $k/2\pi$ )

**Exercise 1.** Find the amplitude  $C$  if the frequency is  $1/\pi$  and if the initial values of  $y$  and  $y'$  (when  $x = 0$ ) are 2 and 4, respectively.

$$\begin{aligned}
\text{frequency} &= \frac{k}{2\pi} = \frac{1}{\pi} \implies k = 2 \\
y(x=0) &= C \sin \alpha \\
y'(x=0) &= C \cos \alpha \implies \frac{y(x=0)}{y'(x=0)} = \frac{1}{k} \tan \alpha = \frac{1}{2} \\
\alpha &= \frac{\pi}{4} \text{ and } \boxed{C = 2\sqrt{2}}
\end{aligned}$$

**Exercise 2.** Find the velocity when  $y$  is zero, given that the amplitude is 7 and the frequency is 10.

$$\begin{aligned} y &= C \sin(kx + \alpha) & C &= 7 & \frac{k}{2\pi} &= 10 \implies k = 20\pi \\ y' &= Ck \sin(kx + \alpha) \\ y(x = \frac{-\alpha}{k}) &= 0 \implies y'(x = \frac{-\alpha}{k}) = \boxed{140\pi} \end{aligned}$$

**Exercise 3.**

$$\begin{aligned} y &= A \cos(mx + \beta) \\ y &= A \cos(mx + \beta) = A \cos \beta \cos(mx) - A \sin \beta \sin(mx) \implies \boxed{k = m} \text{ (since } x \text{ is arbitrary)} \\ y &= C \sin kx + \alpha = C \cos \alpha \sin kx + C \sin \alpha \cos kx \\ -A \sin \beta &= C \cos \alpha \\ A \cos \beta &= C \sin \alpha \implies \tan \alpha = \pm \cot \beta = \mp \tan\left(\frac{\pi}{2} - \beta\right) = \tan\left(\beta - \frac{\pi}{2}\right) \\ &\implies \boxed{\alpha = \beta - \frac{\pi}{2}} \text{ and } \boxed{|C| = |A|} \end{aligned}$$

**Exercise 4.**  $\frac{2\pi}{T} = 4\pi$

$$y = C \cos(kx + \alpha) = C \cos(kx) = \boxed{3 \cos(4\pi x)}$$

**Exercise 5.**  $y = C \cos(x + \alpha) \quad y_0 = C \cos(x_0 + \alpha)$

$$y' = -C \sin(x + \alpha) = \pm v_0$$

$$v_0^2 + y_0^2 = C^2 \sin^2(x + \alpha) + C^2 \cos^2(x_0 + \alpha) = C^2 \implies \boxed{C = \sqrt{v_0^2 + y_0^2}}$$

**Exercise 6.**

$$\begin{aligned} y &= C \cos(kx + \alpha) & y' &= -kC \sin(kx + \alpha) \\ y(0) &= C \cos(\alpha) = 1 & y'(0) &= -kC \sin(\alpha) = 2 \\ y''(0) &= -k^2 C \cos(\alpha) = -12 & \frac{y''(0)}{y(0)} &= -k^2 = \frac{-12}{1} \implies k = 2\sqrt{3} \\ \frac{y'(0)}{y(0)} &= \frac{-kC \sin(\alpha)}{C \cos(\alpha)} = \frac{2}{1} = 2 = -k \tan(\alpha) \implies \boxed{\alpha = \frac{-\pi}{6}} \end{aligned}$$

**Exercise 7.**  $k = \frac{2\pi}{3} \quad y = -C \sin(kx)$

$$\boxed{y = -C \sin \frac{2\pi x}{3}}; \quad C > 0$$

**Exercise 8.** Let's first solve the homogenous equation.

$$\begin{aligned} y'' + y &= 0 \\ y_h &= C_1 \sin x + C_2 \cos x & t_1 &= \int_0^x \frac{-\cos t(1)}{-1} = \sin x \\ W(x) &= -S^2 - C^2 = 1 & t_2 &= \int_0^x \frac{(\sin t)(1)}{-1} = (\cos x - 1) \\ &\text{for } 0 \leq x \leq 2\pi \text{ otherwise, for } x > 2\pi, t_1 = 0, t_2 = 0 \\ y_1 &= \sin^2 x + \cos^2 x + (1 - \cos x) & y(0) &= 0 = c_2 \\ y'(x) &= C_1 \cos x + \sin x & y'(0) &= c_1 = 1 \\ &y = \sin x + (1 - \cos x) \\ &\implies \boxed{I(t) = \sin t + (1 - \cos t)} \quad 0 \leq t \leq 2\pi \end{aligned}$$

**Exercise 9.**

(1) Consider large  $t$ . Then  $I(t) = F(t) + A \sin(\omega t + \alpha) \rightarrow A \sin(\omega t + \alpha)$

$$\begin{aligned}
 I &= AS(\omega t + \alpha) = A(S(\omega t)C(\alpha) + C(\omega t)S(\alpha)) \\
 I' &= \omega AC(\omega t + \alpha) = \omega A(C(\omega t)C(\alpha) - S(\omega t)S(\alpha)) \\
 I'' &= -\omega^2 AS(\omega t + \alpha) = -\omega^2 A(S(\omega t)C(\alpha) + C(\omega t)S(\alpha)) \\
 I'' + RI' + I &= I'' + I' + I = \\
 &= A((- \omega^2 C(\alpha) + -\omega S(\alpha) + C(\alpha))S(\omega t) + (-\omega^2 S(\alpha) + \omega C(\alpha) + S(\alpha))C(\omega t)) = S(\omega t) \\
 &\implies \tan(\alpha) = \frac{-\omega}{1 - \omega^2}
 \end{aligned}$$

With  $\tan \alpha = \frac{-\omega}{1 - \omega^2}$  and the trig identities  $t^2 + 1 = \sec^2$ ,  $\frac{1}{C^2} = \sec^2$ , and  $S^2 + C^2 = 1$ , we can get

$$C = \frac{1 - \omega^2}{\sqrt{1 - \omega^2 + \omega^4}}$$

$$S = \frac{-\omega}{\sqrt{1 - \omega^2 + \omega^4}}$$

Note that the sign of  $S$  is fixed by  $\tan$ .

$$\begin{aligned}
 A(-\omega^2 C(\alpha) + -\omega S(\alpha) + C(\alpha))S(\omega t) &= S(\omega t) \implies A = \frac{1}{(1 - \omega^2)C(\alpha) - \omega S(\alpha)} = \frac{D}{(1 - \omega^2)^2 - \omega(-\omega)} \\
 &\implies \boxed{A = \frac{1}{\sqrt{\omega^4 - \omega^2 + 1}}}
 \end{aligned}$$

We could immediately see that  $\omega = \frac{1}{\sqrt{2}}$ ,  $f = \frac{1}{2\pi\sqrt{2}}$  will maximize  $A$ .

(2) We could have, from the beginning, considered the problem with any  $R$ , in general.

$$\begin{aligned}
 A((- \omega^2 C(\alpha) + -R\omega S(\alpha) + C(\alpha))S(\omega t) + (-\omega^2 S(\alpha) + \omega RC(\alpha) + S(\alpha))C(\omega t)) &= S(\omega t) \\
 \tan \alpha = \frac{-\omega R}{1 - \omega^2} \quad S(\alpha) &= \frac{-\omega R}{\sqrt{(\omega R)^2 + (1 - \omega^2)^2}} \\
 C(\alpha) &= \frac{1 - \omega^2}{\sqrt{(\omega R)^2 + (1 - \omega^2)^2}} \\
 \implies A &= \frac{1}{-\omega^2 C(\alpha) - \omega RS(\alpha) + C(\alpha)} = \frac{1}{\sqrt{(\omega R)^2 + (1 - \omega^2)^2}} \\
 (\omega R)^2 + (1 - \omega^2)^2 &= \omega^4 + \omega^2(-2 + R^2) + 1 \xrightarrow{\frac{d}{d\omega}} 4\omega^3 + 2\omega(-2 + R^2) = 2\omega(2\omega^2 + (-2 + R^2)) = 0 \\
 \omega &= 0 \\
 \implies 2\omega^2 &= 2 - R^2 \xrightarrow{\text{to have resonances}} \boxed{R < \frac{1}{\sqrt{2}}}
 \end{aligned}$$

**Exercise 10.** A spaceship is returning to earth. Assume that the only external force acting on it is the action of gravity, and that it falls along a straight line toward the center of the earth. The rocket fuel is consumed at a constant rate of  $k$  pounds per second and the exhaust material has a constant speed of  $c$  feet per second relative to the rocket.

Let  $M(t) = M$  be the mass of the *rocket + fuel* combination at time  $t$ . With  $+y$  direction being towards earth, then the equation of motion is  $F_g = +M(t)g$ , where  $g = 9.8m/s^2$ .

$M(t)v(t) = Mv_R$  is the momentum of the rocket.

$M(t+h) = M(t) - \Delta m = M - \Delta m$  is the change in mass of the rocket due to spent fuel.

$v_e$  = velocity of the exhaust in the lab frame =  $c + v_R(t)$

$$\begin{aligned}
 \Delta p &= \Delta m(c + v_R) + (M - \Delta m)v_R(t+h) - Mv_R = M(v_R(t+h) - v_R) + -\Delta m(v_R(t+h) - v_R) + \Delta mc \\
 \frac{\Delta p}{\Delta t} &= M \left( \frac{v_R(t+h) - v_R}{\Delta t} \right) + -\Delta m \left( \frac{v_R(t+h) - v_R}{\Delta t} \right) + \left( \frac{\Delta m}{\Delta t} \right) c = M(t)g
 \end{aligned}$$



$$Mv'_R + \frac{kc}{g} = M(t)g$$

$$\text{Now } M(t) = M_0 - \frac{kt}{g} \implies v'_R = g - \frac{kc/g}{M} = g - \frac{kc/g}{M_0 - \frac{kt}{g}}$$

$$v_R = gt - \frac{kc}{g-k} \ln \left( M_0 - \frac{kt}{g} \right) = gt + c \ln \left( M_0 - \frac{kt}{g} \right)$$

$$y_R = \frac{gt^2}{2} + c \frac{g}{k} \left( \left( \frac{k}{g}t - M_0 \right) \ln \left( M_0 - \frac{kt}{g} \right) - \frac{kt}{g} \right) + \frac{M_0cg}{k} \ln M_0$$

**Exercise 11.**

$$Mv'_R = \frac{-kc}{g}$$

$$v'_R = \frac{-kc}{g} \left( \frac{1}{M_0 - \frac{kt}{g}} \right) \implies y_R = c \frac{g}{k} \left( \left( \frac{kt}{g} - M_0 \right) \ln \left( M_0 - \frac{kt}{g} \right) - \frac{kt}{g} \right) + \frac{M_0cg \ln M_0}{k}$$

$$v_R = c \ln \left( M_0 - \frac{kt}{g} \right)$$

$$M_0g = w \implies y_R = c \frac{g}{k} \left( \frac{kt-w}{g} \ln \left( \frac{w-kt}{g} \right) - \frac{kt}{g} \right) + \frac{wc}{k} \ln \frac{w}{g}$$

We could've also solved this problem with an initial velocity of  $v_0$  and gravity. Then

$$v_R(t) = gt + c \ln \left( 1 - \frac{kt}{M_0g} \right) + v_0$$

$$y(t) = v_0t + \frac{1}{2}gt^2 + c \left( \left( t - \frac{M_0g}{k} \right) \ln \left( 1 - \frac{kt}{M_0g} \right) - t \right)$$

**Exercise 12.**

$$Mv_R = (M - \Delta m)(v_R(t+h)) + 0$$

$$M(v_R(t+h) - v_R(t)) = (\Delta m)v_R(t+h)$$

$$Mv'_R = \frac{k}{g}v_R \implies \frac{v'_R}{v_R} = \frac{k}{g(M_0 - \frac{kt}{g})} = \frac{k}{M_0g(1 - \frac{kt}{M_0g})}$$

$$\ln v_R = (k/w) \ln \left( 1 - \frac{kt}{w} \right) \left( \frac{-w}{k} \right) = -\ln \left( 1 - \frac{kt}{w} \right)$$

$$v_R = \frac{v_0}{1 - \frac{kt}{w}} \implies x(t) = v_0 \left( \frac{-w}{k} \right) \ln \left( 1 - \frac{kt}{w} \right) = \frac{-v_0w}{k} \ln \left( 1 - \frac{kt}{w} \right)$$

## 8.22 Exercises - Remarks concerning nonlinear differential equations, Integral curves and direction fields.

**Exercise 1.**  $2x + 3y = C \implies y' = \frac{-2}{3}$

**Exercise 2.**  $y = Ce^{-2x} \implies y' = -2y$

**Exercise 3.**  $x^2 - y^2 = c \implies yy' = x \implies y' = \frac{x}{y}; \quad y \neq 0$

**Exercise 4.**  $xy = c \implies y' = \frac{-y}{x}; \quad x \neq 0$

**Exercise 5.**  $y^2 = cx \implies \frac{y^2}{x} = c \implies y' = \frac{y}{2x} \quad x \neq 0$

**Exercise 6.**  $x^2 + y^2 + 2Cy = 1$

$$\frac{x^2}{y} + y - \frac{1}{y} = -2C$$

$$\frac{2xy - y'x^2}{y^2} + y' + \frac{1}{y^2}y' = 0$$

$$y' = \frac{-2xy}{1 + y^2 - x^2}$$

**Exercise 7.**  $y = C(x-1)e^x$

$$\frac{y}{(x-1)e^x} = C$$

$$\frac{y'(x-1)e^x - (e^x + (x-1)e^x)y}{(x-1)^2 e^{2x}} = 0$$

$$y' = \frac{xy}{x-1}$$

**Exercise 8.**  $y^4(x+2) = C(x-2)$

$$\frac{y^4(x+2)}{x-2} = C$$

$$\frac{(4y^3y'(x+2) + y^4)(x-2) - y^4(x+2)}{(x-2)^2} = 0 \implies 4y^3y'(x+2) + y^4 = \frac{y^4(x+2)}{x-2}$$

$$y' = \frac{y}{(x-2)(x+2)}$$

**Exercise 9.**  $y = c \cos x \implies y' = -\tan xy$

**Exercise 10.**  $\arctan y + \arcsin x = C$

$$\frac{1}{1+y^2}y' + \frac{1}{\sqrt{1-x^2}} = 0 \implies y' = \frac{-(1+y^2)}{\sqrt{1-x^2}}$$

**Exercise 11.** All circles through the points  $(1, 0)$  and  $(-1, 0)$ .

Start with the circle equation:  $(x-A)^2 + (y-B)^2 = R^2$

$$(1, 0): (1-A)^2 + B^2 = R^2 \implies -(1-2A+A^2+B^2=R^2)$$

$$(-1, 0): (-1-A)^2 + B^2 = 1+2A+A^2+B^2 = R^2$$

$$\implies 4A = 0, A = 0 \quad 1+B^2 = R^2$$

$$x^2 + (y \pm \sqrt{R^2-1})^2 = R^2$$

$$x^2 + y^2 - 2By + (R^2-1) = R^2 \implies x^2 + y^2 - 2By = 1$$

$B$  depends upon  $R$ , the radius of the circles, so we could use  $B$  as the parameter for the family of circles.

$$x^2 + y^2 - 1 = 2By$$

$$\frac{x^2}{y} + y - \frac{1}{y} = 2B \implies \frac{2xy - y'x^2}{y^2} + y' + \frac{1}{y^2}y' = 0$$

$$y' = \frac{2xy}{y^2 - x^2 + 1}$$

**Exercise 12.**

$$(x+A)^2 + (y+B)^2 = r^2$$

$$(1+A)^2 + (1+B)^2 = 1+2A+A^2+1+2B+B^2 = r^2$$

$$-((-1+A)^2 + (-1+B)^2) = -(1-2A+A^2+1-2B+B^2 = r^2)$$

$$\implies 4A+4B=0 \implies A=-B$$

$$(x-B)^2 + (y+B)^2 = r^2$$

$$2(x-B) + 2(y+B)y' = 0 \implies y' = \frac{B-x}{y+B}$$

$$(y+B)y' = B-x$$

$$(1-B)^2 + (1+B)^2 = r^2 \implies \sqrt{2}\sqrt{(1+B^2)} = r \text{ or } \sqrt{\frac{r^2}{2}-1} = B$$

$$\implies (\text{so } B \text{ could be treated as a parameter for the family of curves})$$

## 8.24 Exercises - First-order separate equations.

**Exercise 1.**  $y' = x^3/y^2$

$$\frac{1}{3}y^3 = \frac{1}{4}x^4 + C \implies y^3 = \frac{3}{4}x^4 + C$$

**Exercise 2.**  $\tan x \cos y = -y' \tan y$

$$\ln |\cos x| = \frac{1}{\cos y}$$

**Exercise 3.**  $(x+1)y' + y^2 = 0$

$$\frac{1}{y} = \ln(x+1) + c$$

**Exercise 4.**  $y' = (y-1)(y-2)$

$$\left(\frac{1}{y-2} + \frac{-1}{y-1}\right)y' = 1 \implies \ln(y-2) - \ln(y-1) = x$$

$$\frac{y-2}{y-1} = e^x$$

**Exercise 5.**  $y\sqrt{1-x^2}y' = x$

$$\frac{1}{2}y^2 = -\sqrt{1-x^2} \implies y^2 = -2\sqrt{1-x^2}$$

**Exercise 6.**  $(x-1)y' = xy$

$$\ln y = \int 1 + \frac{1}{x-1} = x + \ln|x-1|$$

$$\boxed{y = e^x(x-1) + C}$$

**Exercise 7.**  $(1-x^2)^{1/2}y' + 1 + y^2 = 0$

$$\arctan y = \arccos x + C$$

**Exercise 8.**  $xy(1+x^2)y' - (1+y^2) = 0$

$$\frac{1}{2} \ln(1+y^2) = \int \left(\frac{1}{x} - \frac{x}{1+x^2}\right) + C = \ln x - \frac{1}{2 \ln|1+x^2|}$$

$$y^2 = k \left(\frac{x}{\sqrt{1+x^2}}\right)^2$$

**Exercise 9.**  $(x^2-4)y' = y$

$$\ln y = \frac{-1}{2} \operatorname{arctanh}\left(\frac{x}{2}\right)$$

since  $\int \frac{1}{x^2-4} dx = \frac{1}{4} \int \frac{dx}{\left(\frac{x}{2}\right)^2 - 1} =$

$$= \frac{1}{2} \int \frac{du}{u^2 - 1} \text{ (where } u = \frac{x}{2} \text{)}$$

$$(\tanh(u))' = \frac{\cosh^2 u - \sinh^2 u}{\cosh^2 u} = 1 - \tanh^2 u$$

$$\implies \boxed{y = k \exp\left(-\frac{1}{2} \operatorname{arctanh}\left(\frac{x}{2}\right)\right)}$$

**Exercise 10.**  $xyy' = 1 + x^2 + y^2 + x^2y^2$

$$\frac{1}{2} \ln(1+y^2) = \ln x + \frac{1}{2}x^2 + C \implies \boxed{y^2 = kx^2 e^{x^2} - 1}$$

**Exercise 11.**  $yy' = e^{x+2y} \sin x$

$$\frac{ye^{-2y}}{-2} - \frac{e^{-2y}}{4} = \frac{e^x \sin x - e^x \cos x}{2} + C$$

$$(2y+1)e^{-2y} = -2e^x(\sin x - \cos x) + C$$

**Exercise 12.**  $xdx + ydy = xy(xdy - ydx)$

$$\begin{aligned} y(1-x^2)dy &= x(-y^2-1)dx \\ \frac{ydy}{1+y^2} &= \frac{xdx}{x^2-1} \implies \frac{1}{2} \ln|1+y^2| = \frac{1}{2} \ln|x^2-1| + C \\ 1+y^2 &= (x^2-1)K \implies \boxed{y^2 = K(x^2-1) - 1} \end{aligned}$$

**Exercise 13.**  $f(x) = 2 + \int_1^x f(t)dt$

$$\begin{aligned} f'(x) &= f(x) \implies f(x) = Ce^x \\ \implies Ce^x &= 2 + Ce^x - Ce^1 \quad C = \frac{2}{e} \\ \boxed{f(x) &= \frac{2}{e}e^x} \end{aligned}$$

**Exercise 14.**  $f(x)f'(x) = 5x \quad f(0) = 1$

$$\begin{aligned} f(x)^2 &= 5x^2 + C \implies f(x) = \pm\sqrt{5x^2 + C} \\ \boxed{f(x) &= \sqrt{5x^2 + 1}} \end{aligned}$$

**Exercise 15.**  $f'(x) + 2xe^{f(x)} = 0 \quad f(0) = 0$

$$\begin{aligned} e^{-y}y' &= -2x \implies -e^{-y} = -x^2 + C \\ y &= \ln(x^2 + C)^{-1} \implies \boxed{y = -\ln(x^2 + 1)} \end{aligned}$$

**Exercise 16.**  $f^2(x) + (f'(x))^2 = 1$

$$\begin{aligned} f &= -1 \\ y'^2 &= 1 - y^2 \quad y' = \pm\sqrt{1-y^2} \\ \pm \arcsin(y) &= x + c \implies \boxed{f(x) = \pm \sin(x + c)} \end{aligned}$$

**Exercise 17.**

$$\begin{aligned} \int_a^x f(t)dt &= K(x-a) \implies \boxed{f(x) = k > 0} \\ f &> 0 \quad \forall x \in \mathcal{R} \end{aligned}$$

**Exercise 18.**

$$\begin{aligned} \int_a^x f(t)dt &= k(f(x) - f(a)) \xrightarrow{\frac{d}{dx}} f(x) = kf'(x) \\ \implies f(x) &= Ce^{\frac{1}{k}x}; \quad C > 0 \end{aligned}$$

**Exercise 19.**  $\int_a^x (f(t))dt = k(f(x) + f(a)) \implies f(x) = Ce^{\frac{x}{k}}$

$$\begin{aligned} kCe^{\frac{x}{k}} - kCe^{\frac{a}{k}} &= kCe^{\frac{x}{k}} + kCe^{a/k} \implies 2kCe^{a/k} = 0 \implies C = \\ \boxed{f &= 0} \end{aligned}$$

**Exercise 20.**  $\int_a^x f(t)dt = kf(x)f(a); \quad f(x) = kf'(x)f(a)$

$$\begin{aligned} \frac{1}{kf(a)} &= \frac{f'(x)}{f(x)} \implies \ln f(x) = \left(\frac{1}{kf(a)}\right)x + C; \implies f(x) = C \exp\left(\frac{x}{kf(a)}\right) \\ \int_a^x f(t)dt &= \left(kf(a)Ce^{\frac{t}{kf(a)}}\right)\Big|_a^x = kf(a)Ce^{\frac{x}{kf(a)}} - kf(a)Ce^{\frac{a}{kf(a)}} = kf(a)Ce^{\frac{x}{kf(a)}} - kf(a)Ce^{\frac{a}{kf(a)}} \\ f(a) \left(e^{\frac{x}{kf(a)}} - e^{\frac{a}{kf(a)}}\right) &= Ce^{\frac{x}{kf(a)}} e^{\frac{a}{kf(a)}} \\ \xrightarrow{x=a} 0 &= Ce^{\frac{2a}{kf(a)}} \implies C = 0 \\ \implies \boxed{f &= 0} \end{aligned}$$

## 8.26 Exercises - Homogeneous first-order equations.

**Exercise 1.**  $f(tx, ty) = f(x, y)$  homogeneity (or homogeneity of zeroth order).

$$y' = f(x, y) = \left(\frac{x}{v}\right)' = \frac{v - xv'}{v^2} = f\left(x, \frac{x}{v}\right) = f\left(1, \frac{1}{v}\right)$$

$$v - v^2 f\left(1, \frac{1}{v}\right) = xv' \implies \boxed{\ln x = \int \frac{dv}{v - v^2 f\left(1, \frac{1}{v}\right)}}$$

**Exercise 2.**  $y' = \frac{-x}{y} \implies \frac{1}{2}y^2 = -\frac{1}{2}x^2 + C \implies \boxed{y^2 = -x^2 + C}$

**Exercise 3.**  $y' = 1 + \frac{y}{x}$

$$\frac{y}{x} = v \implies y' = v + xv' = 1 + v \implies v = \ln x$$

$$\boxed{y = x(\ln x + C)}$$

**Exercise 4.**  $y' = \frac{x^2 + 2y^2}{xy}$

$$y' = \frac{x^2 + 2y^2}{xy} = \frac{x}{y} + \frac{2y}{x}$$

$$v = \frac{y}{x} \implies v + xv' = \frac{1}{v} + 2v \implies \frac{v'}{\frac{1}{v} + v} = \frac{1}{x}$$

$$\frac{1}{2} \ln |1 + v^2| = \ln x + C \implies \boxed{y^2 = (Cx^2 - 1)x^2}$$

**Exercise 5.**  $(2y^2 - x^2)y' + 3xy = 0$

$$\text{if } 2y^2 \neq x^2, y' = \frac{3xy}{x^2 - 2y^2}$$

$$\begin{aligned} y &= vx \\ y' &= v'x + v \implies y' = v'x + v = \frac{3vx^2}{x^2 - 2v^2x^2} = \frac{3v}{1 - 2v^2} \\ &\implies \frac{1 - 2v^2}{2v(1 + v^2)} v' = \frac{1}{x} \\ \frac{1}{2} \left( \frac{1}{v} + \frac{-3v}{1 + v^2} \right) v' &= \frac{1}{x} \implies \frac{1}{2} \ln v + \frac{-3}{2} \ln(1 + v^2) = \ln x + C \implies \frac{v}{(1 + v^2)^3} = Cx^2 = \frac{y/x}{\left(\frac{x^2 + y^2}{x^2}\right)^3} \end{aligned}$$

$$\boxed{yx^3 = C(x^2 + y^2)^3}$$

However,

$$y' = \frac{3xy}{x^2 - 2y^2}$$

$$\begin{aligned} v &= \frac{y}{x} \\ y' &= v'x + v \implies v'x + v = \frac{3x^2v}{x^2 - 2v^2x^2} = \frac{3v}{1 - 2v^2} \\ v'x &= \frac{3v}{1 - 2v^2} - \frac{(v - 2v^3)}{1 - 2v^2} = \frac{2(v + v^3)}{1 - 2v^2} \implies \left( \frac{1}{v} + \frac{-3v}{1 + v^2} \right) v' = \frac{2}{x} \implies \ln v + \frac{3}{2} \ln |1 + v^2| = 2 \ln x + C \\ \frac{v}{(1 + v^2)^{3/2}} &= Cx^2 \implies \frac{y^2/x^4}{(x^2 + y^2)^3} = Cx^4 \\ &\implies \boxed{y^2 = C(x^2 + y^2)^3} \end{aligned}$$

**Exercise 6.**  $xy' = y - \sqrt{x^2 + y^2}$

$$\begin{aligned} \implies y' &= \frac{y}{x} = \sqrt{1 + \left(\frac{y}{x}\right)^2} \\ v'x &= -\sqrt{1 + v^2} \\ v = \frac{y}{x} \quad v'x + v &= v - \sqrt{1 + v^2} \implies \frac{-v'}{\sqrt{1 + v^2}} = \frac{1}{x} \\ vx &= y \implies \ln(v + \sqrt{1 + v^2}) = \ln x + C \text{ since} \\ (\ln(v + \sqrt{1 + v^2}))' &= \frac{1}{v + \sqrt{1 + v^2}} \left(1 + \frac{v}{\sqrt{1 + v^2}}\right) = \frac{1}{\sqrt{1 + v^2}} \\ v + \sqrt{1 + v^2} &= Cx \implies 1 + v^2 = C^2x^2 - 2vCx + v^2 \\ \implies v &= \frac{Cx}{2} - \frac{1}{2Cx} \implies \boxed{y = \frac{Cx^2}{2} - \frac{1}{2C}} \end{aligned}$$

**Exercise 7.**  $x^2y' + xy + 2y^2 = 0$

$$\begin{aligned} x^2y' &= -2y^2 - xy \implies y' = \frac{-2y^2}{x^2} - \frac{y}{x} \quad v = \frac{y}{x} \implies \begin{aligned} v'x + v &= -2v^2 - v \\ v'x &= -2(v^2 + v) \end{aligned} \\ \frac{v'}{v(v+1)} &= \frac{-2}{x} = \int v' \left( \frac{1}{v} - \frac{1}{v+1} \right) = -2 \ln x + C \implies \frac{v}{v+1} = \frac{C}{x^2} \\ \boxed{y} &= \frac{-Cx}{C - x^2} \end{aligned}$$

**Exercise 8.**  $y^2 + (x^2 - xy + y^2)y' = 0$

$$\begin{aligned} \left(\frac{y}{x}\right)^2 + \left(1 - \frac{y}{x} + \left(\frac{y}{x}\right)^2\right)y' &= 0 \\ \text{Let } v = \frac{y}{x} \implies \frac{-v^2}{1 - v + v^2} &= v'x + v \\ v'x = \frac{-v^2}{1 - v + v^2} - v &= \frac{-v(1 + v^2)}{1 - v + v^2} \implies \frac{v^2 - v + 1}{v(1 + v^2)}v' = \frac{-1}{x} = \left(\frac{1}{v} + \frac{-1}{v^2 + 1}\right)v' = -\frac{1}{x} \\ \implies \ln v - \arctan v &= -\ln x + C \implies \ln(vx) = \arctan x + C \quad \boxed{\ln y = \arctan \frac{y}{x} + C} \end{aligned}$$

**Exercise 9.**  $y' = \frac{y(x^2 + xy + y^2)}{x(x^2 + 3xy + y^2)}$

$$\begin{aligned} y' &= \frac{y(x^2 + xy + y^2)}{x(x^2 + 3xy + y^2)} = \left(\frac{y}{x}\right) \left(\frac{1 + \frac{y}{x} + \frac{y^2}{x^2}}{1 + \frac{3y}{x} + \frac{y^2}{x^2}}\right) \xrightarrow{v=\frac{y}{x}} v'x + v = v \left(\frac{1 + v + v^2}{1 + 3v + v^2}\right) = v + \frac{-2v^2}{v^2 + 3v + 1} \\ v'(1 + \frac{3}{v} + \frac{1}{v^2}) &= \frac{-2}{x} \implies v + 3 \ln v + \frac{-1}{v} = -2 \ln x + C \\ \boxed{\frac{y}{x} + 3 \ln y - \frac{x}{y}} &= \ln x + C \end{aligned}$$

**Exercise 10.**  $y' = \frac{y}{x} + \sin \frac{y}{x}$

$$\begin{aligned} \frac{y}{x} &= x \implies v + v'x = v + \sin v \\ \frac{y'}{y} &= v + v'x \implies \frac{v'}{\sin v} = \frac{1}{x} \\ -\ln \csc v + \cot v &= \ln x + C \implies \boxed{\csc v + \cot v = \frac{K}{x}} \end{aligned}$$

**Exercise 11.**  $x(y + 4x)y' + y(x + 4y) = 0$

$$\begin{aligned} y' &= \frac{-y(x + 4y)}{x(y + 4x)} = \frac{-\frac{y}{x}(1 + \frac{4y}{x})}{\frac{y}{x} + 4} \xrightarrow{v=\frac{y}{x}} v + xv' = \frac{-v(1 + 4v)}{v + 4} \\ xv' &= \frac{-5v(1 + v)}{v + 4} \implies \frac{-5}{x} = \frac{v + 4}{v(1 + v)}v' = \left(\frac{4}{v} + \frac{-3}{1 + v}\right)v' \\ \int &\rightarrow 4 \ln v - 3 \ln(1 + v) = -5 \ln x + C \implies \boxed{(yx)^4 = (x + y)^3 C} \end{aligned}$$

## 8.28 Miscellaneous review exercises - Some geometrical and physical problems leading to first-order equations.

**Exercise 1.**

$$2x + 3y = C \quad y' = -\frac{2}{3} \quad g' = \frac{3}{2} \implies g - \frac{3}{2}x = C$$

**Exercise 2.**

$$xy = C \xrightarrow{d/dx} y + xy' = 0 \implies y' = -y/x \quad x \neq 0 \implies g' = x/g \implies \frac{1}{2}g^2 = \frac{1}{2}x^2 + C$$

**Exercise 3.**  $x^2 + y^2 + 2Cy = 1$

$$\begin{aligned} x + yy' + Cy' &= 0 \implies y'(y + C) = -x \\ y' &= \frac{-x}{y + C} = \frac{-x}{y + \frac{1-x^2-y^2}{2y}} = \frac{-2xy}{y^2 - x^2 + 1} \\ \xrightarrow{\text{orthogonal curves}} y' &= \frac{y^2 - x^2 + 1}{2xy} = \left(\frac{1}{2x}\right)y + \frac{1}{2}\left(\frac{1}{x} - x\right)y^{-1} \end{aligned}$$

Recognize that this is a *Ricatti equation* and we know how to solve them.

$$\begin{aligned} y' + \frac{-1}{2x}y &= y^{-1} \left(\frac{-x}{2} + \frac{1}{2x}\right) \quad n = -1 \\ k &= 1 - n = 1 - (-1) = 2 \\ v = y^k &= y^2 \quad v' + 2\left(\frac{-1}{2x}\right)v = \frac{2}{2}\left(\frac{1}{x} - x\right) \\ A(x) &= \int_a^x P(t)dt = \int_a^x \frac{-1}{t} = \ln \frac{a}{x} \quad \int_a^x Qe^A = \int_a^x \left(\frac{1}{t} - t\right) \frac{a}{t} = \frac{-a}{x} + 1 - a(x - a) \\ \boxed{y^2 = v} &= -1 + \frac{x}{a} - x(x - a) + \frac{bx}{a} \end{aligned}$$

**Exercise 4.**  $y^2 = Cx$ .

$$\begin{aligned} \frac{y^2}{x} &= C \xrightarrow{d/dx} \frac{2yy'x - y^2}{x^2} = 0 \\ y' &= \frac{y}{2x} \implies y' = \frac{1}{\left(\frac{-y}{2x}\right)} = \frac{-2x}{y} \\ \implies &\boxed{y^2 + 2x^2 = C} \end{aligned}$$

**Exercise 5.**  $x^2y = C$ .

$$\begin{aligned} 2xy + x^2y' &= 0 \quad y' = -\frac{2y}{x} \implies y' = \frac{x}{2y} \\ \frac{1}{2}y^2 &= \frac{x^2}{4} + C \\ \boxed{2y^2 - x^2} &= C \end{aligned}$$

**Exercise 6.**  $y = Ce^{-2x}$

$$\begin{aligned} e^{2x}y &= C \implies 2e^{2x}y + e^{2x}y' = 0 \\ y' &= -2y \xrightarrow{\text{invert}} y' = \frac{1}{2y} \\ \implies &\boxed{y^2 = x + C} \end{aligned}$$

**Exercise 7.**  $x^2 - y^2 = C$

$$\begin{aligned} 2x - 2yy' &= 0 \\ y' &= \frac{x}{y} \implies y' = \frac{-y}{x} \\ \implies \ln y &= -\ln x + C \implies \boxed{y = \frac{C}{x}} \end{aligned}$$

**Exercise 8.**  $y \sec x = C$

$$\begin{aligned} y' \sec x + y \tan x \sec x &= 0 \\ y' &= -y \tan x \xrightarrow{\text{(invert)}} y' = \frac{1}{y \tan x} \\ \boxed{\frac{1}{2}y^2} &= \ln |\sin x| + C \end{aligned}$$

**Exercise 9.** All circles through the points  $(1, 0)$  and  $(-1, 0)$  From Sec. 8.22, Ex.10, we had obtained  $y' = \frac{2xy}{y^2 - x^2 + 1}$

$$\implies y' = \frac{x^2 - y^2 - 1}{2xy} = \frac{x^2 - 1}{2yx} - \frac{y}{2x} \implies y' + \frac{1}{2x}y = \frac{x^2 - 1}{2x}y^{-1}$$

Recognize this is a *Ricatti equation*.

For  $y' + Py = Qy^n$ , in this case,  $n = -1$ , and so  $k = 1 - n = 1 - (-1) = 2$ .

Then  $v = y^k$  and  $v' + kPv = kQ$ . In this case,

$$v' + 2\left(\frac{1}{2x}\right)v = 2\left(\frac{x^2 - 1}{2x}\right) = v' + \frac{1}{x}v = \frac{x^2 - 1}{x} = x - 1/x.$$

$$\begin{aligned} A(x) &= \int_a^x P(t)dt = \int_a^x \frac{1}{t} = \ln \frac{x}{a} \\ \int \left(t - \frac{1}{t}\right) \exp\left(\ln \frac{t}{a}\right) dt &= \int \left(\frac{t^2}{a} - \frac{1}{a}\right) dt = \left(\frac{\frac{1}{3}t^3}{a} - \frac{t}{a}\right) \Big|_a^x \\ e^{-\ln \frac{x}{a}} &= \frac{a}{x} \\ \implies y^2 = v &= \frac{\frac{1}{3}x^3 - x - \frac{1}{3}a^3 + a}{x} + \frac{ba}{x} \end{aligned}$$

**Exercise 10.** All circles through the points  $(1, 1)$  and  $(-1, -1)$ .

$$\begin{aligned} (x + A)^2 + (y + B)^2 &= r^2 \\ (1 + A)^2 + (1 + B)^2 &= 1 + 2A + A^2 + 1 + 2B + B^2 = r^2 \\ -((-1 + A)^2 + (-1 + B)^2) &= -(1 - 2A + A^2 + 1 - 2B + B^2 = r^2) \\ \implies 4A + 4B &= 0 \implies A = -B \\ (x - B)^2 + (y + B)^2 &= r^2 \\ 2(x - B) + 2(y + B)y' &= 0 \implies y' = \frac{B - x}{y + B} \\ (y + B)y' &= B - x \\ (1 - B)^2 + (1 + B)^2 = r^2 &\implies \sqrt{2}\sqrt{(1 + B^2)} = r \text{ or } \sqrt{\frac{r^2}{2} - 1} = B \\ \implies (\text{so } B \text{ could be treated as a parameter for the family of curves}) \\ y' &= \frac{-1}{\left(\frac{B-x}{y+B}\right)} = \frac{y+B}{x-B} \\ \frac{y'}{y+B} &= \frac{1}{x-B} \implies \boxed{y = C(x-B) - B} \end{aligned}$$

**Exercise 11.** With  $(0, Y) = Q$  the point that moves up wards along the positive  $y$ -axis and

$P = (x, y)$  being the point  $P$  that pursues  $Q$ ,

$y' = \frac{Y-y}{X-x} = \frac{Y-y}{0-x}$  is the slope of the tangent line on a point on the trajectory of  $P$ .

The condition given, that the distance of  $P$  from the  $y$ -axis is  $k$  the distance of  $Q$  from the origin, is



$$kY = x.$$

$$y' = \frac{\left(\frac{1}{k}\right)x - y}{x} = f(x, y)$$

$f(x, y)$  is homogeneous of zero order  $\implies y = vx$  (try this substitution)

$$y' = v'x + v = \frac{1}{k} - v \implies \frac{v'}{\frac{1}{k} - 2v} = \frac{1}{x}$$

$$\implies -\frac{1}{2} \ln \left( \frac{1}{k} - 2v \right) = \ln x + C \implies y = \frac{x}{2k} - \frac{1}{2C^2x}$$

$$\xrightarrow{(1,0)} \boxed{y = \frac{x}{2k} = \frac{1}{2kx}} \quad k = \frac{1}{2} \quad y = x - \frac{1}{x}$$

**Exercise 12.**

$$y = \frac{x}{2k} - \frac{1}{2kx}$$

**Exercise 13.**  $y = f(x)$ .

$$n \int_0^x f(t) dt = xf(x) - \int_0^x f(t) dt; (n+1) \int_0^x f(t) dt = xy$$

$$ny = xy'$$

$$\implies (n_1)y = y + xy' \implies \frac{n}{x} = \frac{y'}{y}$$

$$n \ln x = \ln y$$

$$\boxed{y = Cx^n} \text{ of } y = Cx^{1/n}$$

**Exercise 14.**

$$n \int_0^x \pi f^2(t) dt = \int_0^x (\pi(y(x))^2 - \pi(f(t))^2) dt$$

$$(n+1) \int_0^x \pi f^2(t) dt = xy^2(x) = xy^2; \quad (n+1)f^2(x) = y^2 + 2xyy'$$

$$y' = \frac{ny}{2x} \implies \ln y = \frac{n}{2} \ln x + C$$

$$\boxed{y = Cx^{n/2}} \text{ or } y = Cx^{1/2n}$$

**Exercise 15.**

$$\pi \int_0^x f^2(t) dt = x^2 f(x) \quad \pi f^2(x) = 2xf + x^2 f' \implies f' = \frac{\pi f^2 - 2xf}{x^2}$$

The left hand side of the last expression shown is homogeneous. Thus do the  $y = vx$  substitution.

$$v'x + v = \frac{\pi v^2 x^2 - 2x^2 v}{x^2} = \pi v^2 - 2v$$

$$\frac{v'x}{\pi v^2 - 3v} = 1 \implies \frac{v'}{\pi(v^2 - \frac{3v}{\pi})} = \frac{1}{x} = \frac{1}{3} \left( \frac{1}{v - \frac{3}{\pi}} - \frac{1}{v} \right) v'$$

$$\ln \left( v - \frac{3}{\pi} \right) - \ln v = \ln \left( \frac{v - 3/\pi}{v} \right) = 3 \ln x + C$$

$$vx - \frac{3}{\pi} x C v x^4$$

$$y - \frac{3}{\pi} x = C y x^3 \implies \boxed{y = \frac{3x/\pi}{1 + \frac{x^3}{2}}}$$

**Exercise 16.**  $A = \int_0^a f; \quad B = \int_a^1 f$

$$A - B = \int_0^a f + \int_1^a f = 2f(a) + 3a + b$$

$$\xrightarrow{d/dx} 2f(a) = 2f'(a) + 3 \implies 1 = \frac{f'(a)}{f(a) - \frac{3}{2}}$$

So then

$$a + C = \ln\left(y - \frac{3}{2}\right); \quad f(a) = Ce^a + \frac{3}{2}$$

$$f(1) = 0 = Ce^1 + \frac{3}{2}; \quad \Rightarrow C = -\frac{3}{2e}$$

$$\boxed{f(x) = -\frac{3}{2}e^{x-1} + \frac{3}{2}}$$

To find  $b$ ,

$$2\left(-\frac{3}{2}e^{a-1} + \frac{3}{2}a\right) + \frac{3}{2}e^{-1} + \frac{3}{2} - \frac{3}{2} = 2f(a) + 3a + b =$$

$$= 2\left(-\frac{3}{2}e^{a-1} + \frac{3}{2}\right) + 3a + b$$

$$\Rightarrow \boxed{b = \frac{3}{2}e^{-1} - 3}$$

**Exercise 17.**

$$A(x) = \int_0^x \left( f(t) - \left( \left( \frac{y(x)-1}{x} \right) t + 1 \right) \right) dt = x^3 =$$

$$= \int_0^x f + - \left( \frac{y(x)-1}{x} \right) \frac{1}{2}x^2 - x = \int_0^x f + - \left( \frac{y(x)-1}{2} \right) x - x$$

$$\xrightarrow{d/dx} f(x) + -\frac{1}{2}(y'x + y) - \frac{1}{2} = 3x^2$$

$$y' = \frac{-2(3x^2 + \frac{1}{2} - y/2)}{x} = -6x - x^{-1} + \frac{y}{x}$$

As a leap of faith, try  $y = vx$  substitution to solve  $y' = -6x - x^{-1} + \frac{y}{x}$ .

$$v'x + v = -6x - x^{-1} + v \quad v' = -6 - x^{-2}; \quad v = -6x + x^{-1} + C$$

$$\boxed{y = -6x^2 + 5x + 1}$$

**Exercise 18.** Assuming no friction at the orifice and energy conservation.

$$mgh = \frac{1}{2}mv_f^2$$

(imagine how the top layer of water is now at the bottom of the tank (final “potential energy configurations”))

$V_f = \sqrt{2gh}$  (how fast water is rushing out)

$A_0$  = cross-sectional area of the orifice.

$$\frac{dV}{dt} = A \frac{dh}{dt} = -c\sqrt{2gh}A_0$$

$$2h^{1/2} \Big|_{h_i}^{h_f} = -c\sqrt{2g} \frac{A_0}{A} t$$

$$\Rightarrow T = \frac{\sqrt{2}A}{c\sqrt{g}A_0} \left( \sqrt{h_f} - \sqrt{h_i} \right) = 59.6 \text{ sec}$$

Note that we included the discharge coefficient  $C = 0.6$ .

**Exercise 19.**  $\frac{dV}{dt} = -c\sqrt{2gh}A_0 + \gamma_0 = A \frac{dh}{dt} = -\kappa h^{1/2} + \gamma_0$ .  $\kappa = c\sqrt{2g}A_0$ .

$$\begin{aligned}
\frac{Adh}{\gamma_0 - \kappa h^{1/2}} &= dt = (A/\gamma_0) \frac{dh}{1 - \frac{\kappa}{\gamma_0} h^{1/2}} \\
\left( \ln \left( 1 - ah^{1/2} \right) \right)' &= \frac{1}{1 - ah^{1/2}} \left( -\frac{a}{2} \frac{1}{h^{1/2}} \right) \quad (\text{where } a = \frac{\kappa}{\gamma_0}) \\
(h^{1/2})' &= \frac{1}{2h^{1/2}} \left( \frac{1 - ah^{1/2}}{1 - ah^{1/2}} \right) = \frac{(\frac{1}{2} - \frac{a}{2} h^{1/2})}{h^{1/2} (1 - ah^{1/2})} \\
\Rightarrow \int (A/\gamma_0) \frac{dh}{1 - \frac{\kappa}{\gamma_0} h^{1/2}} &= - (A/\gamma_0) \left( \frac{2\gamma_0^2}{\kappa^2} \ln \left( 1 - \frac{\kappa}{\gamma_0} h^{1/2} \right) + \frac{2\gamma_0 h^{1/2}}{\kappa} \right) \Big|_{h_i}^{h_f} = \\
&= T \\
\Rightarrow \exp \left( \frac{-\gamma_0}{A} t - \frac{2\gamma_0}{\kappa} (h_f^{1/2} - h_i^{1/2}) \right) \frac{\kappa^2}{2\gamma_0^2} &= \frac{1 - \frac{\kappa}{\gamma_0} h_f^{1/2}}{1 - \frac{\kappa}{\gamma_0} h_i^{1/2}} \\
\stackrel{t \rightarrow \infty}{\rightarrow} h_f &= \frac{\gamma_0^2}{\kappa^2} = \frac{(100 \text{ in}^3/\text{s})^2}{c^2(2)(32 \text{ ft}/\text{s}^2)(5/3 \text{ in}^2)^2(12 \text{ in}/1 \text{ ft})} = (25/24)^2
\end{aligned}$$

**Exercise 20.**

$$\begin{aligned}
V_0 &= \frac{1}{3} \pi R_0^2 H_0 \\
V(h) &= V_0 - \frac{1}{3} \pi h \left( h \frac{R_0}{H_0} \right)^2 = V_0 - \frac{1}{3} \pi \frac{R_0^2}{H_0^2} h^3 = V_0 - \alpha h^3 \\
mg(H_0 - h) &= \frac{1}{2} m v_f^2; \quad \sqrt{2g(H_0 - h)} = v_f \quad (\text{energy conservation}) \\
cA_0 v_f &= cA_0 \sqrt{2gH_0 \left( 1 - \frac{h}{H_0} \right)} = \beta \sqrt{1 - \frac{h}{H_0}} \\
\frac{dV}{dt} &= -\beta \sqrt{1 - h/H} = -3\alpha h^2 \frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{\beta}{3\alpha h^2} \sqrt{1 - \frac{h}{H}} \\
\int \frac{h^2/H_0^2}{\sqrt{1 - \frac{h}{H_0}}} &= \frac{\beta/H_0^2}{3\alpha} T = \\
&= H_0 \int \frac{u^2 du}{\sqrt{1 - u}} = H_0 \int \frac{(1 - y)^2 (-dy)}{\sqrt{y}} = -H_0 \int \frac{1 - 2y + y^2}{\sqrt{y}} = \\
&= -H_0 \left( 2y^{1/2} - \frac{2}{3} y^{3/2} + \frac{2}{5} y^{5/2} \right) \Big|_{h_i}^{h_f} = \\
&= -H_0 \left( 2 \left( 1 - \frac{h}{H_0} \right)^{1/2} - \frac{4}{3} \left( 1 - \frac{h}{H_0} \right)^{3/2} + \frac{2}{5} \left( 1 - \frac{h}{H_0} \right)^{5/2} \right) \Big|_{h_i}^{h_f} = \\
&= \frac{cA_0 \sqrt{2gH_0}/H_0^2}{3 \left( \frac{1}{3} \pi \frac{R_0^2}{H_0^2} \right)} T
\end{aligned}$$

For  $h_i = 0$ ,  $h_f = H$ ,

$$H_0(2(1) - 4/3(1) + 2/5) = H_0(16/15) = cA_0 \sqrt{2gH_0} T / (\pi R_0^2); T = \frac{\frac{16}{15} \sqrt{H_0} \pi R_0^2}{cA_0 \sqrt{2g}} = \frac{2}{9} \frac{\pi R_0^2 \sqrt{H_0}}{A_0}$$

**Exercise 21.**  $m^2x - m + (1 - x) = 0 \implies (m^2 - 1)x + 1 - m = 0, \implies m = 1$

**Exercise 22.** Given  $x + y^3 + 6xy^2y' = 0$

Notice that  $(2xy^3)' = 2(y^3 + 3xy^2y') = 2y^3 + 6xy^2y'$ . So then

$$x + y^3 + 6xy^2y' = (2xy^3)' + -y^3 + x = 0$$

Let  $u = 2xy^3$ . So then  $y^3 = \frac{u}{2x}$ . Thus, we have

$$u' - \frac{u}{2x} = -x$$

This equation is linear because it is a linear combination of  $y'$  and  $y$ . Now we can use the formula for solving first-order linear differential equations of the form  $u' + Pu = Q$ .

$$u = e^{-A} \left( \int_{x_0}^{x_f} Q e^A dx + u(x_0) \right) \quad \text{for } A = \int P dt$$

$$\Rightarrow A = \int \frac{-1}{2t} dt = \frac{-1}{2} \ln t \text{ and } u = x^{1/2} \left( \frac{-2}{3} (x^{3/2} - x_0^{3/2}) + u(x_0) \right) = \frac{-2}{3} x^2 + C x^{1/2}$$

$$\Rightarrow \boxed{y^3 = \frac{-x}{3} + \frac{C}{x^{1/2}}} \text{ for } x > 0 \text{ or } y^3 = \frac{-x}{3} \forall x$$

**Exercise 23.** Given  $(1 + y^2 e^{2x})y' + y = 0$ , let  $\boxed{y = u e^{mx}}$ ,  $u$  unknown.

$$y' = u' e^{mx} + m u e^{mx}$$

$$(1 + u^2 e^{2mx} e^{2x})(u' e^{mx} + m u e^{mx}) + u e^{mx} = 0 = u' e^{mx} + u'(u^2 e^{3mx+2x}) + m u e^{mx} + m u^3 e^{3mx} e^{2x} + u e^{mx}$$

$$u'(e^{mx} + u^2 e^{(3m+2)x}) + m u e^{mx} + m u^3 e^{3mx+2x} + u e^{mx} = 0$$

$$u' = \frac{-(m+1)u - m u^3 e^{(2m+2)x}}{1 + u^2 e^{2(m+1)x}}$$

$\boxed{\text{Let } m = -1}.$

$$\Rightarrow u' = \frac{u^3}{1 + u^2} \Rightarrow \frac{1 + u^2}{u^3} du = \left( \frac{1}{u^3} + \frac{1}{u} \right) du = dx \Rightarrow u^{-2}/-2 + \ln u - C = x$$

$$y = u e^{-x} \text{ or } u = y e^x$$

$$\Rightarrow \boxed{\ln y = x + \frac{e^{-2x}}{2y^2} + C}$$

**Exercise 24.** Given  $f$  s.t.  $2f'(x) = f\left(\frac{1}{x}\right)$  if  $x > 0$ ,  $f(1) = 2$  and  $x^2 y'' + a x y' + b y = 0$

(1)

$$2f''(x) = \frac{d}{dx} f\left(\frac{1}{x}\right) = (f')\left(\frac{-1}{x}\right)$$

$$\Rightarrow x^2 y'' = \frac{-1}{2} f' = \frac{-1}{4} f\left(\frac{1}{x}\right)$$

$$\boxed{a = 0; \quad b = \frac{1}{4}}$$

(2)

$$f(x) = C x^n$$

$$f' = n C x^{n-1} \quad \Rightarrow n(n-1) C x^n = \frac{-1}{4} C x^n$$

$$f'' = n(n-1) C x^{n-2}$$

$$n^2 - n + \frac{1}{4} \Rightarrow \boxed{n = \frac{1}{2}}$$

**Exercise 28.** Choose the units for time to be in days first - we can convert into years later.

If no one died from accidental death, then the population will grow by  $e$ . That means, with  $x = x(t)$  being the population at time  $t$

$$\frac{dx}{dt} = x \quad \Rightarrow x = C e^t$$

which makes sense because if  $C$  is the original population number, then after 1 year,  $x = C e$ .

With  $t$  in days, we have a decrease of  $\frac{1}{100}x$  in population each day due to death. Add up the changes from the decrease due to deaths and the increase due to growth for the DE:

$$\frac{dx}{dt} = \frac{1}{365}x - \frac{1}{100}x = \frac{100 - 365}{36500}x = \frac{-265}{36500}$$

$$\Rightarrow x = C e^{\frac{-265}{36500}t}$$

Change  $t$  units to years by multiplying the “time constant”  $\frac{-265}{36500}$  by 365 days.

$$x = 365 \exp(-2.65t)$$

To get the total fatalities, simply integrate the deaths during each year.

$$\int_0^t y = \frac{365}{100} \frac{365}{2.65} (-\exp(-2.65t) + 1)$$

**Exercise 29.** For constant gravity,  $\Delta K = -\Delta U$

$$\implies -(0 - mgh) = \frac{1}{2}mv_f^2$$

$$v_f = \sqrt{2gh} = (6.37 \times 10^8 \text{ cm}) \left( \frac{1 \text{ m}}{2.54 \text{ cm}} \right) \left( \frac{1 \text{ ft}}{12 \text{ in}} \right) = 6.93 \frac{\text{mi}}{\text{sec}} = 24940 \frac{\text{mi}}{\text{hr}}$$

The constant energy formula could also be obtained by considering

$$F = \frac{GM_em}{-r^2} = m \frac{d^2r}{dt^2} = -\partial_r U$$

**Exercise 30.** Let  $y = f(x)$  be the solution to  $y' = \frac{2y^2+x}{3y^2+5}$   $f(0) = 0$

(1)  $y'(0) = 0$  as easily seen. Now  $y'' = \frac{(4yy'+1)(3y^2+5)-(6yy')(2y^2+x)}{(3y^2+5)^2}$ , so then

$y''(0) = \frac{1}{5} > 0$ . It is a minimum.

(2)  $f'(x) \geq 2/3 \quad \forall x \geq 10/3$ .  $a = 2/3$  since  $f$  will be above this tangent line.

Suppose, in the “worst case,”  $f'(x) = 0$  for  $0 \leq x \leq 2/3$ . Then  $f(x) = 0$  for  $0 \leq x \leq 2/3$ . Then the tangent line must be at  $y = 0$  at  $x = 10/3$  to remain below the graph of  $f(x)$ .

$$\implies \frac{2}{3}x - 20/9 < f(x)$$

(3) Since  $f'(x) \geq \frac{2}{3}$  for each  $x \geq \frac{10}{3}$ , then  $f \rightarrow \infty$  for  $x \rightarrow \infty$  (otherwise,  $f$  would have to decrease somewhere, which would contradict the given fact about  $f$ ). Rewrite the DE for  $y'$  to be

$$y' = \frac{2y^2+x}{3y^2+5} \implies (3y^2+5)y' = 2y^2+x \implies (3+\frac{5}{y^2})y' = 2+\frac{x}{y^2}$$

Consider

$$y' = \frac{2y^2+x}{3y^2+5} = \frac{2+\frac{x}{y^2}}{3+\frac{5}{y^2}}$$

specifically,  $\frac{x}{y^2}$ . Now  $y$  must, at the very least, have some linear increase because we had already shown that  $y' \geq \frac{2}{3}$ .

So  $y^2$  would go to infinity faster than linear  $x$ . Thus  $\lim_{x \rightarrow \infty} y' = \frac{2}{3}$ . So then  $(3+\frac{5}{y^2})y' \xrightarrow{x \rightarrow \infty} (3+0)\frac{2}{3} = 2 = 2+\frac{x}{y^2}$ .

$$0 = \frac{x}{y^2}$$

$$(4) y' = \frac{2+\frac{x}{y^2}}{3+\frac{5}{y^2}} \xrightarrow{x \rightarrow \infty} \frac{2}{3}. \implies y = \frac{2}{3}x \text{ or } \frac{y}{x} = \frac{2}{3}.$$

**Exercise 31.** Given a function  $f$  which satisfies the differential equation  $xf''(x) + 3x(f'(x))^2 = 1 - e^{-x}$

(1)  $c \neq 0$  for an extremum.

$$cf''(c) + 3c(f'(c))^2 = cf''(c) = 1 - e^{-c} \implies f''(c) = \frac{1-e^{-c}}{c} > 0$$

(2) *Cleverly*, consider the *limit*.

$$xf''(x) + 3x(f'(x))^2 = 1 - e^{-x} \implies f''(x) + 3(f'(x))^2 = \left( \frac{1-e^{-x}}{x} \right) \xrightarrow{x \rightarrow 0} f''(0) + 0 = 1$$

So a critical point at  $x = 0$  would be a minimum.

(3) We'll have to “cheat” a little and use the idea of power series early on here.

$f'' + 3(f')^2 = \frac{1-e^{-x}}{x}$  suggests that we consider the Taylor series of  $e^{-x}$ .

$$\frac{1-e^{-x}}{x} = \frac{-\sum_{j=1}^{\infty} \frac{(-x)^j}{j!}}{x} = \sum_{j=0}^{\infty} \frac{(-1)^j x^j}{(j+1)!}$$

This further suggests that  $f$  itself has a power series representation because its first and second order derivatives are simply a combination of infinitely many terms containing powers of  $x$ .

Then suppose  $f = \sum_{j=0}^{\infty} a_j x^j$ .

$$\begin{aligned}
 f'' + 3(f')^2 &= \frac{1 - e^{-x}}{x} \implies f' = \sum_{j=0}^{\infty} (j+1)a_{j+1}x^j \\
 f'' &= \sum_{j=0}^{\infty} (j+2)(j+1)a_{j+2}x^j \\
 \implies \sum_{j=0}^{\infty} (j+2)(j+1)a_{j+2}x^j + 3 \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (j+1)(k+1)a_{j+1}a_{k+1}x^{j+k} &= \sum_{j=0}^{\infty} \frac{(-1)^j x^j}{(j+1)!} \\
 \text{If } f(0) &= 0 \quad a_0 = 0 \\
 \text{If } f'(0) &= 0 \quad a_1 = 0
 \end{aligned}$$

Then  $f = a_2 x^2 + \sum_{j=3}^{\infty} a_j x^j$ . Consider the  $x^0$  terms in the DE.  $(f')^2$  doesn't contribute, because  $f'$ 's leading order term is  $x^1$ . So then

$$2(1)a_2 + 0 = 1 \implies \boxed{a_2 = \frac{1}{2}} \quad \text{i.e. } f = \frac{1}{2}x^2 + \sum_{j=3}^{\infty} a_j x^j$$

$$\boxed{A = \frac{1}{2}} \text{ in order for } f(x) \leq Ax^2$$

## 9.6 Exercises - Historical introduction, Definitions and field properties, The complex numbers as an extension of the real numbers, The imaginary unit $i$ , Geometric interpretation. Modulus and argument.

**Exercise 6.** Let  $f$  be a polynomial with real coefficients.

(1) Since  $\overline{z_1 z_2}$

$$\overline{(z_1^{n+1})} = \overline{z_1^n z_1} = \overline{z_1^n} \overline{z_1} = \overline{z_1}^{n+1}$$

$$\overline{f(z)} = \overline{\sum a_j z^j} = \sum \overline{a_j z^j} = \sum \overline{a_j} \overline{z^j} = f(\overline{z})$$

(2) If  $f(z) = 0$ , then  $\overline{f(z)} = f(\overline{z}) = 0$  as well.

**Exercise 7.** The three ordering axioms are

Ax. 7 If  $x, y \in \mathbb{R}^+$ ,  $x + y, xy \in \mathbb{R}^+$

Ax. 8  $\forall x \neq 0$ ,  $x \in \mathbb{R}^+$  or  $-x \in \mathbb{R}^+$  but not both

Ax. 9  $0 \notin \mathbb{R}^+$

$x < y$  means  $y - x$  positive.

Suppose  $i$  positive:  $i(i) = -1$  but  $-1$  is not positive.

Suppose  $-i$  is positive.  $-i(-i) = -1$  but  $-1$  is not positive.

$i$  is neither positive nor negative so Ax. 8 is not satisfied.

**Exercise 8.** Ax. 8, Ax. 9 are satisfied.

$(a + ib)(c + id) = (ac - bd, ad + bc)$  so Ax. 7 might not be satisfied.

**Exercise 9.** Ax. 7, Ax. 8, Ax. 9 are trivially satisfied (all are positive).

**Exercise 10.**  $x > y$   $x > y$  is well defined. Ax. 8 is satisfied.

For Ax. 7,  $(\frac{3}{2}, 1), (1, \frac{1}{2})$  contradicts Ax. 7 since we required the product to be positive as well if the factors are positive. We found this particular counterexample by considering factors  $(a, b), (c, d)$ , so the product of the two is  $(ac - bd, ad + bc)$  and so we need  $ac - bd - ad - bc = a(c - d) - b(c + d) < 0$

**Exercise 11.** See sketch.

**Exercise 12.**

$$\begin{aligned}
 w &= \frac{az + b}{cz + d} \\
 w &= \frac{(az + b)(c\bar{z} + d)}{(cz + d)(c\bar{z} + d)} = \frac{ac|z|^2 + adz + bc\bar{z} + bd}{c^2|z|^2 + cd(z + \bar{z}) + d^2} \\
 w + -\bar{w} &= \frac{ac|z|^2 + adz + bc\bar{z} + bd - ac|z|^2 - ad\bar{z} - bc\bar{z} - bd}{|cz + d|^2} = \boxed{\frac{(ad - bc)(z - \bar{z})}{|cz + d|^2}} \\
 &\quad \text{If } ad - bc > 0
 \end{aligned}$$

$$w - \bar{w} = 2Imw = \frac{ad - bc}{|cz + d|^2} 2Imz; \quad \frac{ad - bc}{|az + d|^2} > 0$$

So  $Imw$  has the same sign as  $Imz$

**9.10 Exercises - Complex exponentials, Complex-valued functions, Examples of differentiation and integration formulas.**

**Exercise 7.**

(1)

$$\begin{aligned}
 \text{if } m \neq n, \int_0^{2\pi} e^{ix(n-m)} dx &= \left. \frac{e^{ix(n-m)}}{i(n-m)} \right|_0^{2\pi} = \frac{1-1}{i(n-m)} = 0 \\
 \text{if } m = n, \int_0^{2\pi} e^{ix(0)} dx &= 2\pi
 \end{aligned}$$

(2)

$$\begin{aligned}
 \int_0^{2\pi} e^{inx} e^{-imx} dx &= \int_0^{2\pi} (\cos nx + i \sin nx)(\cos mx - i \sin mx) = \\
 &= \int_0^{2\pi} \cos nx \cos mx + \sin nx \sin mx + i(\sin nx \cos mx - \sin mx \cos nx) \\
 \int_0^{2\pi} e^{-inx} e^{-imx} dx &= \int_0^{2\pi} (\cos nx - i \sin nx)(\cos mx - i \sin mx) = \\
 &= \int_0^{2\pi} \cos nx \cos mx - \sin nx \sin mx + i(-\sin nx \cos mx - \sin mx \cos nx)
 \end{aligned}$$

Summing the two equations above

$$\begin{aligned}
 0 &= \int_0^{2\pi} 2 \cos nx \cos mx + 2 \int_0^{2\pi} i - \sin mx \cos nx \\
 \Rightarrow \int_0^{2\pi} \cos nx \cos mx &= 0, \quad \int_0^{2\pi} \sin mx \cos nx = 0
 \end{aligned}$$

$$\begin{aligned}
 \int_0^{2\pi} e^{inx} e^{-imx} dx &= \int_0^{2\pi} (\cos nx + i \sin nx)(\cos mx - i \sin mx) = \\
 &= \int_0^{2\pi} \cos nx \cos mx + \sin nx \sin mx + i(\sin nx \cos mx - \sin mx \cos nx) \\
 \int_0^{2\pi} e^{inx} e^{imx} dx &= \int_0^{2\pi} (\cos nx + i \sin nx)(\cos mx + i \sin mx) = \\
 &= \int_0^{2\pi} \cos nx \cos mx - \sin nx \sin mx + i(\sin nx \cos mx + \sin mx \cos nx)
 \end{aligned}$$

Subtract the two equations above

$$\begin{aligned}
 \int_0^{2\pi} \sin nx \sin mx - i \int_0^{2\pi} \sin mx \cos nx &= 0 \\
 \Rightarrow \int_0^{2\pi} \sin nx \sin mx &= 0
 \end{aligned}$$

$$\begin{aligned}
\int_0^{2\pi} e^{inx} e^{-inx} &= 2\pi = \int_0^{2\pi} (\cos nx + i \sin nx)(\cos nx - i \sin nx) = \\
&= \int_0^{2\pi} \cos^2 nx + \sin^2 nx \\
\int_0^{2\pi} e^{inx} e^{inx} &= 0 = \int_0^{2\pi} \cos^2 nx - \sin^2 nx + i(2 \cos nx \sin nx) \\
\implies \int_0^{2\pi} \cos^2 nx - \sin^2 nx &= 0 \quad \int_0^{2\pi} \cos nx \sin nx = 0
\end{aligned}$$

Summing the two results above, we obtain

$$\begin{aligned}
2\pi &= \int_0^{2\pi} 2 \cos^2 nx \\
\implies \int_0^{2\pi} \cos^2 nx &= \pi
\end{aligned}$$

$$\text{Then also, } \int_0^{2\pi} \sin^2 nx = \pi$$

#### Exercise 8.

$$\begin{aligned}
z &= r e^{i\theta} = r e^{i(\theta+2\pi m)}, \quad m \in \mathbb{Z} \\
z^{1/n} &= r^{1/n} e^{i(\theta/n+2\pi m/n)} \quad m = 0, 1, \dots, n-1 \\
\implies z^{1/n} &= R e^{i\alpha} \epsilon^m = z_1 \epsilon^m
\end{aligned}$$

The roots are spaced equally by an angle  $2\pi/n$

$$\begin{aligned}
i &= e^{i\pi/2+i2\pi n} \implies i^{1/3} = e^{i\pi/6}, e^{i5\pi/6}, e^{i3\pi/2} \\
i^{1/4} &= e^{i\pi/8}, e^{5i\pi/8}, e^{9i\pi/8}, e^{13i\pi/8} \\
-i &= e^{-i\pi/2+i2\pi n} \implies (-i)^{1/4} = e^{-i\pi/8}, e^{3i\pi/8}, e^{7i\pi/8}, e^{11i\pi/8}
\end{aligned}$$

#### Exercise 9.

$$\begin{aligned}
e^{iu} e^{iv} &= e^{i(u+v)} = \cos u + v + i \sin u + v = \\
&= (\cos u + i \sin u)(\cos v + i \sin v) = \cos u \cos v - \sin u \sin v + i(\cos v \sin u + \cos u \sin v) \\
&\implies \sin u + v = \cos v \sin u + \cos u \sin v \\
&\implies \cos u + v = \cos u \cos v - \sin u \sin v \\
\sin^2 z + \cos^2 z &= \left( \frac{e^{iz} - e^{-iz}}{2i} \right)^2 + \left( \frac{e^{iz} + e^{-iz}}{2} \right)^2 = \\
&= \frac{-(e^{2iz} + e^{-2iz} - 2) + (e^{2iz} + 2 + e^{-2iz})}{4} = 1 \\
\cos iy &= \frac{e^{iyy} + e^{-iyy}}{2} = \cosh y \quad \sin iy = \frac{e^{iyy} - e^{-iyy}}{2i} = i \sinh y \\
e^{iz} &= e^{i(x+iy)} = e^{ix} e^{-y} = (\cos x + i \sin x) e^{-y} \\
e^{-iz} &= e^{-i(x+iy)} = e^{-ix} e^y = (\cos x - i \sin x) e^y
\end{aligned}$$

Thus it is clear, by mentally adding and subtracting the above results that

$$\begin{aligned}
\cos z &= \cos x \cosh y - i \sin x \sinh y \\
\implies \sin z &= i \cos x \sinh y + \sin x \cosh y
\end{aligned}$$

#### Exercise 10.

- (1)  $\text{Log}(-1) = i\pi \quad \log(i) = \ln 1 + i\frac{\pi}{2} = i\frac{\pi}{2}$
- (2)  $\text{Log}(z_1 z_2) = \text{Log}(|z_1||z_2|e^{i(\theta_1+\theta_2)}) = \ln|z_1||z_2| + i(\theta_1 + \theta_2 + 2n\pi) = \text{Log}z_1 + \text{Log}z_2 + i2\pi n$
- (3)  $\text{Log}(z_1/z_2) = \text{Log}(|z_1|/|z_2|e^{i(\theta_1-\theta_2)}) = \ln\frac{|z_1|}{|z_2|} + i(\theta_1 - \theta_2 + 2n\pi) = \text{Log}z_1 - \text{Log}z_2 + i2\pi n$
- (4)  $\exp(\text{Log}z) = \exp(\ln|z| + i\theta + i2\pi n) = z$

#### Exercise 11.



(1)

$$1^i = e^{i \operatorname{Log} 1} = e^{i(2\pi n)} = e^{-2\pi n} = 1 \text{ if } n = 0$$

$$i^i = e^{i \operatorname{Log} i} = e^{i(i\frac{\pi}{2} + i2\pi n)} = e^{-\frac{\pi}{2} - 2\pi n} = e^{-\pi/2} \text{ if } n = 0$$

$$(-1)^i = e^{i \operatorname{Log} -1} = e^{i(i\pi + i2\pi n)} = e^{-\pi - 2\pi n} = e^{-\pi}$$

$$(2) \quad z^a z^b = e^{a \operatorname{Log} z} e^{b \operatorname{Log} z} = e^{a \operatorname{Log} z + b \operatorname{Log} z} = e^{(a+b) \operatorname{Log} z} = z^{a+b}$$

(3)

$$(z_1 z_2)^w = e^{w \operatorname{Log} z_1 z_2} = e^{w(\operatorname{Log} z_1 + \operatorname{Log} z_2 + 2\pi m i)}$$

$$(z_1^w z_2^w) = e^{w \operatorname{Log} z_1} e^{w \operatorname{Log} z_2} = e^{w(\operatorname{Log} z_1 + \operatorname{Log} z_2)}$$

$m = 0$  is the condition required for equality.

**Exercise 12.**

$$\text{if } L(u) = P, L(v) = Q,$$

$$L(u + iv) = (u + iv)'' + a(u + iv)' + b(u + iv) = u'' + au' + bu + i(v'' + av' + bv) = L(u) + iL(v) = P + iQ = R$$

$$\text{if } L(f) = R$$

$$L(u + iv) = L(u) + iL(v) = P + iQ$$

then, equating real and imaginary parts,  $L(u) = P, L(v) = Q$

**Exercise 13.**

$$L(y) = -\omega^2 y + ai\omega y + by = Ae^{i\omega x} \implies (-\omega^2 + ai\omega + b)B = A$$

We cannot let  $(-\omega^2 + ai\omega b) = 0$  for a nontrivial solution. Thus  $b \neq \omega^2$  or  $a\omega \neq 0$ .

$$B = \frac{A}{-\omega^2 + ai\omega + b}$$

**Exercise 14.**

$$L(\hat{y}) = ce^{i\omega x}; \hat{y} = Be^{i\omega x} = \frac{c}{-\omega^2 + ai\omega + b} e^{i\omega x}$$

$$\implies \hat{y} = \frac{c}{\sqrt{(b - \omega^2)^2 + (a\omega)^2}} e^{i(\omega x - \alpha)} \text{ where } \tan \alpha = \frac{a\omega}{b - \omega^2}$$

$$\implies \Re \hat{y} = \frac{c}{\sqrt{(b - \omega^2)^2 + (a\omega)^2}} \cos(\omega x - \alpha)$$

**Exercise 15.**

$$\Im(\hat{y}) = \frac{c}{\sqrt{(b - \omega^2)^2 + (a\omega)^2}} \sin(\omega x + \alpha)$$

$$\implies A = \frac{c}{\sqrt{(b - \omega^2)^2 + (a\omega)^2}}; \quad -\tan \alpha = \frac{a\omega}{b - \omega^2}$$

**10.4 Exercises - Zeno's paradox, Sequences, Monotonic sequences of real numbers. Exercise 1.** Converges to 0.

$$f(n) = \frac{n}{n+1} - \frac{n+1}{n} = \frac{n^2 - (n^2 + 2n + 1)}{n(n+1)} = \frac{-2n - 1}{n^2 + n} = \frac{-\frac{2}{n} - \frac{1}{n^2}}{1 + \frac{1}{n}} \xrightarrow{n \rightarrow \infty} 0$$

**Exercise 2.** Converges to -1.

$$f(n) = \frac{n^3 - (n^3 + n + n^2 + 1)}{(n+1)n} = \frac{n^2 - n - 1}{n(n+1)} = \frac{-1 - \frac{1}{n} - \frac{1}{n^2}}{1 + \frac{1}{n}} \xrightarrow{n \rightarrow \infty} -1$$

**Exercise 3.** Diverges since

$$\left| \cos \frac{n\pi}{2} - L \right| \geq \left| \left| \cos \frac{n\pi}{2} \right| - |L| \right| \geq |1 - |L||$$

Choosing  $\epsilon_1 = \frac{|1 - |L||}{2}$ ,  $|\cos \frac{n\pi}{2} - L| > \epsilon_1$  for  $n = 4m$ .

**Exercise 4.**  $f(n) = \frac{1}{5} + \frac{3}{5n} - \frac{2}{5n^2} \rightarrow \lim_{n \rightarrow \infty} f(n) = \frac{1}{5}$

**Exercise 5.**  $f(x) = \frac{x}{2^x} = \frac{x}{\exp(x \ln 2)} \rightarrow 0$  since  $\lim_{x \rightarrow \infty} \frac{x^\alpha}{(e^x)^\beta} = 0$ .

**Exercise 6.**  $f(n) = 1 + (-1)^n = 0$  or  $1$ .

Thus, choosing  $\epsilon_1 = \frac{|1 - |L||}{2}$ ;

$$|f(n) - L| \geq ||f(n)| - |L|| = |1 - |L|| > \epsilon_1 \quad \text{for any } n = 2m$$

**Exercise 7.**  $f(n) = \frac{1 + (-1)^n}{n}$ .

Suppose  $\epsilon = \frac{3}{N}$ .

So for  $n > N$ ,  $\frac{1}{N} > \frac{1}{n}$ ,  $n \geq N = N(\epsilon) = 3/\epsilon$ .

$$|f(n)| = \left| \frac{1 + (-1)^n}{n} \right| \leq \frac{2}{n} < \frac{3}{n} < \frac{3}{N} = \epsilon$$

**Exercise 8.**  $f(n) = \frac{(-1)^n}{n} + \frac{1+(-1)^n}{2}$

$$\begin{aligned} |f(n) - L| &= \left| \frac{(-1)^n}{n} + \frac{1+(-1)^n}{2} - L \right| \geq \left| L - \frac{(-1)^n}{n} \right| - \left| \frac{1+(-1)^n}{2} \right| \geq \\ &\geq \left| |L| - \left| \frac{(-1)^n}{n} \right| \right| - \left| \frac{1+(-1)^n}{2} \right| = \left| |L| - \frac{1}{n} \right| = \left| \frac{1+(-1)^n}{2} \right| \geq \\ &\geq \left| |L| - \frac{1}{n} - \frac{1}{2} \right| \end{aligned}$$

Thus, consider

$$\left| |L| - \frac{1}{n} - \frac{1}{2} \right| > \left| \frac{1}{N} - |L| - 1 \right| = \epsilon_0 \text{ for } n > N$$

**Exercise 9.**  $f(x) = \exp\left(\frac{1}{x} \ln 2\right)$ ;  $\lim_{x \rightarrow \infty} f(x) = 0$ .

**Exercise 10.**

$$|f(n) - L| = |n^{(-1)^n} - L| \geq ||n^{(-1)^n}| - |L|| = ||n| - |L|| = |n| - |L| > N - |L|$$

Thus, for  $n > N$ ,  $N(\epsilon) = \epsilon + |L|$ , so then  $|f(n) - L| > \epsilon$ .

**Exercise 11.**  $f(n) = \frac{n^{2/3} \sin n!}{n+1}$ .

$$|f(n)| = \left| \frac{n^{2/3} \sin(n!)}{n+1} \right| = \left| \frac{\sin(n!)}{n^{1/3} + n^{-2/3}} \right| \leq \left| \frac{1}{n^{1/3}} \right|$$

Thus, for  $n > N$ ,  $N(\epsilon) = \frac{1}{\epsilon^3}$ ,  $|f(n)| < \epsilon$ .

**Exercise 12.** Converges, since

$$\begin{aligned} f(n) - \frac{1}{3} &= \frac{3^{n+1} + 3(-2)^n - 3^{n+1} - (-2)^{n+1}}{3(e^{n+1} + (-2)^{n+1})} = \left( \frac{(-2)^n(3+2)}{3(3^{n+1} + (-2)^{n+1})} \right) = \\ &= \left( \frac{5}{3} \frac{(-2)^n}{3^{n+1} + (-2)^{n+1}} \right) \\ \left| f(n) - \frac{1}{3} \right| &= \frac{5}{3} \left| \frac{(-2)^n}{3^{n+1} + (-2)^{n+1}} \right| \leq \left| \frac{-(-2)^{n+1}}{3^{n+1} + (-2)^{n+1}} \right| = \\ &= \left| \frac{-1}{\left(\frac{3}{-2}\right)^{n+1} + 1} \right| = \left| \frac{1}{1 + \left(\frac{3}{-2}\right)^{n+1}} \right| < \\ &< \frac{1}{\left(\frac{3}{2}\right)^{n+1}} < \frac{1}{\left(\frac{3}{2}\right)^n} \end{aligned}$$

For  $n > N$ , consider  $\epsilon = \left(\frac{2}{3}\right)^{-N}$ , i.e.  $N = \frac{-\ln \epsilon}{\ln 2/3} = N(\epsilon)$ . Thus

$$\boxed{L = \frac{1}{3};}$$

**Exercise 13.**

$$\begin{aligned} f(n) &= \sqrt{n+1} - \sqrt{n} \\ f(n) &= (\sqrt{n+1} - \sqrt{n}) \left( \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} \right) = \frac{n+1-n}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}} \\ |f(n)| &= \left| \frac{1}{\sqrt{n+1} + \sqrt{n}} \right| \leq \frac{1}{2\sqrt{n}}; \end{aligned}$$

So then  $\forall \epsilon$ , we have  $\epsilon = \frac{1}{2\sqrt{N}}$  and for  $n > N$ ,  $\frac{1}{2\sqrt{n}} < \frac{1}{2\sqrt{N}} = \epsilon$ .

Thus  $f(n)$  converges to 0.

**Exercise 14.**

$$f(n) = na^n = n \exp n \ln a = \frac{n}{\exp n \ln \frac{1}{a}} \rightarrow 0 \text{ since } \lim_{x \rightarrow \infty} \frac{x^a}{(e^x)^b} = 0$$

**Exercise 15.**  $f(n) = \frac{\log_a n}{n}$ ,  $a > 1$ .  $\lim_{n \rightarrow \infty} f(n) = 0$  since  $\lim_{x \rightarrow \infty} \frac{(\log x)^a}{x^b} = 0$  for  $a > 0, b > 0$

**Exercise 16.**  $\lim_{n \rightarrow \infty} f(n) = 0$

**Exercise 17.**  $\lim_{n \rightarrow \infty} f(n) = e^2$ .

**Exercise 18.**

$$\begin{aligned} \left| 1 + \frac{n}{n+1} \cos \frac{n\pi}{2} - L \right| &\geq \left| \left| 1 + \frac{n}{n+1} \cos \frac{n\pi}{2} \right| - |L| \right| = \left| \left| 1 + \frac{-n}{n+1} \right| - |L| \right| = \\ &= \left| \frac{1}{n+1} - |L| \right| > |L| \end{aligned}$$

Choose  $\epsilon_0 = \frac{|L|}{2}$ . For any  $N$ , for  $n > N$ ,  $|f(n) - L| > \epsilon_0$ .

**Exercise 19.**

$$\begin{aligned} 1 + \frac{i}{2} &= \frac{\sqrt{5}}{2} e^{i\alpha} & \left( \frac{\sqrt{5}}{2} e^{i\alpha} \right)^{-n}, & \left( \frac{2}{\sqrt{5}} \right)^n e^{-ni\alpha} \\ 1 + \left( \frac{1}{2} \right)^2 &= \frac{5}{4} & \lim_{n \rightarrow \infty} \left( \frac{2}{\sqrt{5}} \right)^n e^{-ni\alpha} &= 0 \\ \tan \alpha &= \frac{1}{2} \end{aligned}$$

**Exercise 20.**  $\lim_{n \rightarrow \infty} f(n) = \lim_{n \rightarrow \infty} e^{-\pi i n/2}$  diverges since

$$|e^{i\pi n/2} - L| \geq ||e^{-i\pi n/2}| - |L|| = |1 - |L|| > |L|$$

So for  $\epsilon_0 = \frac{|L|}{2}$ , for  $n > N$ ,  $|f(n) - |L|| > \epsilon_0$

**Exercise 21.**  $f(n) = \frac{1}{n} e^{-i\pi n/2}$   $|f(n)| = \frac{1}{n}$ .

Suppose  $N(\epsilon) = \frac{1}{\epsilon}$ ; then for  $n > N$ ,  $|f(n)| < \epsilon_0$

**Exercise 22.**  $|f(n) - L| \geq ||ne^{-\pi i n/2}| - |L|| = |n - |L||$

Consider  $\epsilon_0 = |1 - |L||$  for  $n > N > 1$ ,  $|f(n) - L| > \epsilon_0$ .

**Exercise 23.**  $a_n = \frac{1}{n}$ .

$$\begin{aligned} |a_n| &= \left| \frac{1}{n} \right| < \frac{1}{N}; & N(\epsilon) &= \frac{1}{\epsilon} \\ \epsilon &= 1, 0.1, 0.01, 0.001, 0.0001 \\ N &= 1, 10, 100, 1000, 10000 \end{aligned}$$

**Exercise 24.**  $|a_n - 1| = \left| \frac{n+1-n}{n+1} \right| = \frac{1}{n+1} < \frac{1}{n}$

$N(\epsilon) = \frac{1}{\epsilon} = 1, 10, 100, 1000, 10000$ .

**Exercise 25.**  $|a_n| = \frac{1}{n}$

$N(\epsilon) = 1, 10, 100, 1000, 10000$ .

**Exercise 26.**  $|a_n| = \left| \frac{1}{n!} \right| \leq \frac{1}{\exp n \ln n} < \frac{1}{\exp n}$

For  $n > N$ ,  $\frac{1}{\exp N} = \epsilon$ , so that  $N = \ln 1/\epsilon$ .

$$N(\epsilon) = 1, 2, 4, 6, 9$$

**Exercise 27.**  $a_n = \frac{2n}{n^2+1}$ ;  $|a_n| = \left| \frac{2}{n^2+1/n} \right| \leq \left| \frac{2}{n^2} \right|$ .

$$N(\epsilon) = \frac{\sqrt{2}}{\sqrt{\epsilon}} = 1, 4, 14, 44, 141$$

**Exercise 28.**  $|a_n| = \left| \frac{9}{10} \right|^n = \left( \frac{9}{10} \right)^n = e^{n \ln 9/10}$

$$\begin{aligned} N(\epsilon) &= \frac{\ln \epsilon}{\ln \left( \frac{9}{10} \right)} = \frac{-\ln 1/\epsilon}{\ln (9/10)} = \\ &= 1, 21, 43, 65, 87 \end{aligned}$$

**Exercise 30.** If  $\forall \epsilon > 0, \exists N \in \mathbb{Z}^+$  such that  $n > N, |a_n| < \epsilon$ .

$$|a_n|^2 < |a_n|\epsilon < \epsilon^2$$

$$|a_n^2| < \epsilon^2$$

So for  $\forall \epsilon_1 > 0, \epsilon_1 = \epsilon^2$  and  $\exists N = N(\epsilon) = N(\epsilon_1)$ , so that  $|a_n^2| < \epsilon_1$ .

**Exercise 31.**

$$|a_n + b_n - (A + B)| = ||a_n - A||b_n - B|| \leq |a_n - A| + |b_n - B| < \epsilon + \epsilon = 2\epsilon$$

$$\forall \epsilon > 0, \exists N_A, N_B \in \mathbb{Z}^+, |a_n - A| < \epsilon \text{ if } n > N_A; |b_n - B| < \epsilon \text{ if } n > N_B$$

$$\text{Consider } \max(N_A, N_B) = N_{A+B}$$

$$|a_n + b_n - (A + B)| < 2\epsilon$$

$$\forall \epsilon_1 > 0, \epsilon_1 = 2\epsilon, \text{ then } \exists N_{A+B} = N_{A+B}(\epsilon_1) \in \mathbb{Z}^+ \text{ such that}$$

$$|(a_n + b_n) - (A + B)| < \epsilon_1 \text{ if } n > N_{A+B}$$

$$|ca_n - cA| = c|a_n - A| < c\epsilon$$

$$\forall \epsilon_1 > 0, \epsilon_1 = c\epsilon; \text{ then } \exists N_{cA} = N(\epsilon) = N(\epsilon_1) \in \mathbb{Z}^+ \text{ such that}$$

$$|ca_n - cA| < \epsilon_1 \text{ for } n > N(\epsilon_1)$$

**Exercise 32.** Given  $\lim_{n \rightarrow \infty} a_n = A$ ,

$$\lim_{n \rightarrow \infty} (a_n - A)(a_n + A) = \lim_{n \rightarrow \infty} (a_n - A) \lim_{n \rightarrow \infty} (a_n + A) = 0(2A) = 0$$

$$2a_nb_n = (a_n + b_n)^2 - a_n^2 - b_n^2$$

$$2 \lim_{n \rightarrow \infty} a_nb_n = \lim_{n \rightarrow \infty} (a_n + b_n)^2 - \lim_{n \rightarrow \infty} a_n^2 - \lim_{n \rightarrow \infty} b_n^2 = \left( \lim_{n \rightarrow \infty} (a_n + b_n) \right)^2 - A^2 - B^2 = 2AB$$

$$\implies \lim_{n \rightarrow \infty} a_nb_n = AB$$

**Exercise 33.**  $\binom{\alpha}{n} = \frac{\alpha(\alpha-1)(\alpha-2)\dots(\alpha-n+1)}{n!}$

(1)

$$\binom{-\frac{1}{2}}{1!} = \frac{-1/2}{1}$$

$$\binom{-\frac{1}{2}}{2!} = \frac{(-\frac{1}{2})(-\frac{3}{2})}{2!} = \frac{5}{8}$$

$$\binom{-\frac{1}{2}}{3!} = \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{3!} = -\frac{5}{16}$$

$$\binom{-\frac{1}{2}}{4} = \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})(-\frac{7}{2})}{4!} = \frac{35}{128}$$

$$\binom{-\frac{1}{2}}{5} = \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})(-\frac{7}{2})(-\frac{9}{2})}{5!} = \frac{-63}{256}$$

$$(2) a_n = (-1)^n \binom{-\frac{1}{2}}{n}.$$

$$a_1 = \frac{1}{2} > 0. \quad a_2 = \frac{3}{8} > 0$$

$$a_{n+1} = (-1)^{n+1} \binom{-\frac{1}{2}}{n+1} = (-1)^{n+1} \binom{\alpha}{n} \left( \frac{-\frac{1}{2} - (n+1) + 1}{n+1} \right) = \frac{a_n(-1) \left( \frac{-1}{2} - n \right)}{n+1} =$$

$$= \frac{a_n(n+1/2)}{n+1} > 0$$

$$a_{n+1} = \left( \frac{n+1/2}{n+1} \right) a_n < a_n$$

**Exercise 34.**

(1)

$$t_n - s_n = \frac{1}{n} \sum_{k=1}^n f\left(\frac{k}{n}\right) - \frac{1}{n} \sum_{k=0}^{n-1} f\left(\frac{k}{n}\right) = \frac{1}{n} \left( f(1) + \sum_{k=1}^{n-1} f\left(\frac{k}{n}\right) - \left( f(0) + \sum_{k=1}^{n-1} f\left(\frac{k}{n}\right) \right) \right) =$$

$$= \frac{1}{n} (f(1) - f(0))$$

Since  $f\left(\frac{k}{n}\right) \leq f(t) \leq f\left(\frac{k+1}{n}\right)$  for  $\frac{k}{n} \leq t \leq \frac{k+1}{n}$ , by  $f$  being monotonically increasing,

$$\implies s_n \leq \int_0^1 f(x)dx \leq t_n \quad (\text{from definition of integral})$$

$$0 \leq \int_0^1 f(x)dx - s_n \leq t_n - s_n = \frac{1}{n}(f(1) - f(0))$$

(2) Use Theorem 1.9.

**Theorem 30.** Every function  $f$  which is bounded on  $[a, b]$  has a lower integral  $\underline{I}(f)$  and an upper integral  $\bar{I}(f)$  satisfying

$$\int_a^b s(x)dx \leq \underline{I}(f) \leq \bar{I}(f) \leq \int_a^b t(x)dx$$

for all step functions  $s$  and  $t$  with  $s \leq f \leq t$ . The function  $f$  is integrable on  $[a, b]$  iff its upper and lower integrals are equal,

$$\int_a^b f(x)dx = \underline{I}(f) = \bar{I}(f)$$

Since  $f(x)$  is integrable, then  $\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} t_n = \int_0^1 f(x)dx$

$$(3) \left(\frac{b-a}{n}\right) = \Delta, s_n = \frac{1}{\Delta} \sum_{k=0}^{n-1} f(a+k\Delta), t_n = \frac{1}{\Delta} \sum_{k=1}^n f(a+k\Delta)$$

So by increasing monotonicity of  $f$ ,  $s_n \leq \int_a^b f(x)dx \leq t_n$ .

$$t_n - s_n = \frac{1}{\Delta} \left( f(b) + \sum_{k=1}^{n-1} f(a+k\Delta) - \sum_{k=1}^{n-1} f(a+k\Delta) - f(a) \right) = \frac{f(b) - f(a)}{\Delta}$$

$$0 \leq \int_a^b f(x)dx \leq \frac{f(b) - f(a)}{\Delta}$$

**Exercise 35.**

$$(1) \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \left(\frac{k}{n}\right)^2 = \int_0^1 t^2 dt = \frac{1}{3}$$

$$(2) \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{n+k} = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \frac{1}{1+\frac{k}{n}} = \int_0^1 \frac{1}{1+x} dx = \ln 2$$

$$(3) \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \frac{1}{1+\left(\frac{k}{n}\right)^2} = \int_0^1 \frac{1}{1+x^2} dx = \arctan x \Big|_0^1 = \boxed{\frac{\pi}{4}}$$

$$(4) \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{\sqrt{n^2+k^2}} = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \frac{1}{\sqrt{1+\left(\frac{k}{n}\right)^2}} = \int_0^1 \frac{1}{\sqrt{1+x^2}} dx = \ln(x + \sqrt{1+x^2}) \Big|_0^1 = \ln(1 + \sqrt{2})$$

$$(5) \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{n} \sin \frac{k\pi}{n} = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \sin \frac{k\pi}{n} = \int_0^1 \sin \pi x dx = \left( -\frac{\cos \pi x}{\pi} \right) \Big|_0^1 = \frac{-(-1-1)}{\pi} = \boxed{\frac{2}{\pi}}$$

$$(6) \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{n} \sin^2 \frac{k\pi}{n} = \int_0^1 \sin^2 x dx = \boxed{\frac{1}{2}}$$

## 10.9 Exercises - Infinite series, The linearity property of convergent series, Telescoping series, The geometric series.

**Exercise 1.**  $\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)} = \sum_{n=1}^{\infty} \frac{1/2}{2n-1} - \frac{1/2}{2n+1} = \frac{1}{2}$

**Exercise 2.**  $\sum_{n=1}^{\infty} \frac{2}{3^{n-1}} = 2 \sum_{n=0}^{\infty} \frac{1}{3^n} = 2 \frac{1}{1-1/3} = \boxed{3}$

**Exercise 3.**  $\sum_{n=2}^{\infty} \frac{1}{n^2-1} = \sum_{n=2}^{\infty} \frac{1/2}{n-1} - \frac{1/2}{n+1} = \sum_{n=2}^{\infty} \left( \frac{1/2}{n-1} - \frac{1/2}{n} \right) + - \left( \frac{1/2}{n+1} - \frac{1/2}{n} \right) = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$

**Exercise 4.**  $\sum_{n=1}^{\infty} \frac{2^n + 3^n}{6^n} = \sum_{n=1}^{\infty} \left( \frac{1}{3} \right)^n + \sum_{n=1}^{\infty} \left( \frac{1}{2} \right)^n = \frac{1/3}{1-1/3} + \frac{1/2}{1-1/2} = \frac{1}{2} + 1 = \frac{3}{2}$

**Exercise 5.**  $\sum_{n=1}^{\infty} \frac{\sqrt{n+1} - \sqrt{n}}{\sqrt{n^2+n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} = 1$

**Exercise 6.**

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{n}{(n+1)(n+2)(n+3)} &= \sum_{n=1}^{\infty} \frac{3/2}{(n+2)(n+3)} + \frac{-1/2}{(n+1)(n+2)} = \sum_{n=1}^{\infty} \frac{1}{(n+2)(n+3)} + \frac{1}{2} \left( \frac{-1}{6} \right) = \\ &= \sum_{n=1}^{\infty} \frac{1}{1+2} - \frac{1}{n+3} - \frac{1}{12} = \frac{1}{3} - \frac{1}{12} = \boxed{\frac{1}{4}} \end{aligned}$$

**Exercise 7.**  $\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} - \frac{1}{(n+1)^2} = 1$

**Exercise 8.**  $\sum_{n=1}^{\infty} \frac{2^n + n^2 + n}{2^{n+1}n(n+1)} = 1 = \sum_{n=1}^{\infty} \frac{1}{2n(n+1)} + \frac{1}{2^{n+1}} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1} + \frac{1/4}{1-1/2} = \frac{1}{2} + \frac{1}{2} = 1$

**Exercise 9.**  $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(2n+1)}{n(n+1)} = \sum_{n=1}^{\infty} (-1)^{n-1} \left( \frac{1}{n} + \frac{1}{n+1} \right).$

$$\begin{aligned} (-1)^{n-1} \left( \frac{1}{n} + \frac{1}{n+1} \right) + (-1)^n \left( \frac{1}{n+1} + \frac{1}{n+2} \right) &= (-1)^n \left( \frac{-1}{n} + \frac{1}{n+2} \right) \\ \sum_{j=1}^{\infty} \left( \frac{-1}{2j-1} + \frac{1}{2j+1} \right) &= \frac{1}{2} \sum_{j=1}^{\infty} \left( \frac{-1}{(j-1/2)} + \frac{1}{(j+1/2)} \right) = \frac{1}{2} \frac{-1}{1/2} = -1 \end{aligned}$$

**Exercise 10.**

$$\begin{aligned} \sum_{n=2}^{\infty} \frac{\log((1 + \frac{1}{n})^n (1+n))}{(\log n^n)(\log(n+1)^{n+1})} &= \sum_{n=2}^{\infty} \frac{\log((\frac{n+1}{n})^n (1+n))}{\log(n+1)^{n+1} \log n^n} = \\ &= \sum_{n=2}^{\infty} \frac{\log(n+1)^{n+1} - \log n^n}{\log(n+1)^{n+1} \log n^n} = \sum_{n=2}^{\infty} \frac{1}{\log n^n} - \frac{1}{\log(n+1)^{n+1}} = \\ &= \frac{1}{2 \log 2} = \log_2 \sqrt{e} \end{aligned}$$

since if  $\frac{1}{2 \log 2} = y$ , then  $y = \log_2 \sqrt{2}$ .

**Exercise 11.**  $\sum_{n=1}^{\infty} n x^n = x \frac{d}{dx} \sum_{n=1}^{\infty} x^n = x \left( \frac{x}{1-x} \right)' = \frac{x}{(1-x)^2}.$

**Exercise 12.**

$$\begin{aligned} \sum_{n=1}^{\infty} n^2 x^n &= x \frac{d}{dx} \sum_{n=1}^{\infty} n x^n = x \left( \frac{x}{(1-x)^2} \right)' = \\ &= x \frac{(1-x)^2 + 2(1-x)x}{(1-x)^4} = \frac{x(1-x^2)}{(1-x)^4} = \frac{x(1+x)}{(1-x)^3} \end{aligned}$$

**Exercise 13.**

$$\begin{aligned} \sum_{n=1}^{\infty} n^3 x^n &= x \frac{d}{dx} \sum_{n=1}^{\infty} n^2 x^n = x \left( \frac{x+x^2}{(1-x)^3} \right)' = \\ &= x \frac{(1+2x)(1-x)^3 + 3(1-x)^2(x)(x+1)}{(1-x)^6} = x \frac{(1+2x)(1-x) + 3x(x+1)}{(1-x)^4} = \frac{x(x^2+4x+1)}{(1-x)^4} \end{aligned}$$

**Exercise 14.**  $\sum_{n=1}^{\infty} n^4 x^4 = x \frac{d}{dx} \sum_{n=1}^{\infty} n^3 x^3 = x \left( \frac{x^3+4x^2+x}{(1-x)^4} \right)'.$

$$\begin{aligned} \ln \left( \frac{x^3+4x^2+x}{(1-x)^4} \right) &= \ln(x^3+4x^2+x) - 4 \ln(1-x) \\ (\ln f)' &= \frac{1}{f} f' = \frac{3x^2+8x+1}{x^3+4x^2+x} + 4 \frac{1}{1-x}; \\ f' &= \frac{(3x^2+8x+1)(1-x)}{(1-x)^5} + \frac{4(x^3+4x^2+x)}{(1-x)^5} = \\ &= \frac{3x^2+8x+1-3x^3-8x^2-x+4x^3+16x^2+4x}{(1-x)^5} = \frac{x^3+11x^2+11x+1}{(1-x)^5} \\ &\Rightarrow \frac{x^4+11x^3-11x^2+x}{(1-x)^5} \end{aligned}$$

**Exercise 15.**  $\sum_{n=1}^{\infty} \frac{x^n}{n} = x \sum_{n=1}^{\infty} \int_0^x t^{n-1} dt = \int_0^x dt \sum_{n=1}^{\infty} t^{n-1} = \int_0^x \frac{1}{1-t} dt = -\ln(1-x).$

**Exercise 16.**

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{x^{2n-1}}{2n-1} &= \sum_{j=1}^{\infty} \int_0^x t^{2j-2} dt = \int_0^x dt \sum_{j=1}^{\infty} (t^2)^{j-1} = \\ &= \int_0^x \frac{dt}{1-t^2} = \int_0^x dt \left( \frac{1/2}{1-t} + \frac{1/2}{1+t} \right) = \frac{1}{2} \ln \left( \frac{1+x}{1-x} \right) \end{aligned}$$

**Exercise 17.**  $\sum_{n=0}^{\infty} (n+1)x^n = \sum_{n=0}^{\infty} \frac{d}{dx} x^{n+1} = \frac{d}{dx} \left( \frac{x}{1-x} \right) = \frac{1}{(1-x)^2}$

**Exercise 18.**

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{2!} x^n &= \sum_{n=0}^{\infty} \left( \frac{x^{n+2}}{2} \right)'' = \frac{d^2}{dx^2} \frac{x^2}{2} \left( \frac{1}{1-x} \right) = \frac{d}{dx} \left( \frac{x}{1-x} + \frac{x^2}{2(1-x)^2} \right) \\ &= \frac{1}{1-x} + \frac{x}{(1-x)^2} + \frac{x}{(1-x)^2} + \frac{x^3}{(1-x)^3} = \frac{1-2x+x^2+2x-2x^2+x^2}{(1-x)^3} = \frac{1}{(1-x)^3}\end{aligned}$$

**Exercise 19.**

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(n+1)(n+2)(n+3)}{3!} x^n &= \sum_{n=0}^{\infty} \frac{d^3}{dx^3} \frac{x^{n+3}}{3!} = \frac{1}{3} \frac{d}{dx} \sum_{n=0}^{\infty} \frac{d^2}{dx^2} \frac{x^{(n+1)+2}}{2} = \\ &= \frac{1}{3} \frac{d}{dx} \sum_{n=1}^{\infty} \frac{d^2}{dx^2} \frac{x^{n+2}}{2} = \frac{1}{3} \frac{d}{dx} \left( \sum_{n=0}^{\infty} \frac{d^2}{dx^2} \frac{x^{n+2}}{2} - \frac{d^2}{dx^2} \left( \frac{x^2}{2} \right) \right) = \\ &= \frac{1}{3} \frac{d}{dx} \left( \frac{1}{(1-x)^3} - 1 \right) = \frac{1}{3} \frac{-3(-1)}{(1-x)^4} = \frac{1}{(1-x)^4}\end{aligned}$$

**Exercise 20.**  $\sum_{n=1}^{\infty} n^k x^n = \frac{P_k(x)}{(1-x)^{k+1}}$

$$\begin{aligned}\sum_{n=1}^{\infty} n^{k+1} x^n &= x \frac{d}{dx} \sum_{n=1}^{\infty} n^k x^n = x \frac{d}{dx} \left( \frac{P_n(x)}{(1-x)^{k+1}} \right) = x \left( \frac{P'_k(x)(1-x)^{k+1} + (k+1)(1-x)^k P_k(x)}{(1-x)^{2k+2}} \right) = \\ &= x \left( \frac{P'_k(x)(1-x) + (k+1)P_k(x)}{(1-x)^{k+2}} \right) = \frac{(k+1)xP_k(x) + x(1-x)P'_k(x)}{(1-x)^{k+2}}\end{aligned}$$

$((k+1)P_k(x) + (1-x)P'_k(x))x$  has  $x$  as its lowest degree term from  $xP'_k(x)$  and  $(k+1)x^{k+1} + -kx^{k+1} = x^{k+1}$  highest degree term is obtained from  $(k+1)P_k(x) + -xP'_k(x)$ .

**Exercise 21.**  $\sum_{n=0}^{\infty} \binom{n+k}{k} x^n = \frac{1}{(1-x)^{k+1}} = \frac{d^k}{dx^k} \sum_{n=0}^{\infty} \frac{x^{n+k}}{k!}$ .

$$\begin{aligned}\sum_{n=0}^{\infty} \binom{n+k+1}{k+1} x^n &= \sum_{n=0}^{\infty} \frac{d^{k+1}}{dx^{k+1}} \frac{x^{n+k+1}}{(k+1)!} = \frac{1}{(k+1)} \frac{d}{dx} \sum_{n=0}^{\infty} \frac{d^k}{dx^k} \frac{x^{(n+1)+k}}{k!} = \\ &= \frac{1}{k+1} \frac{d}{dx} \sum_{k=1}^{\infty} \frac{d^k}{dx^k} \frac{x^{n+k}}{k!} = \left( \frac{1}{k+1} \right) \frac{d}{dx} \sum_{n=0}^{\infty} \frac{d^k}{dx^k} \frac{x^{n+k}}{k!} - \frac{d^k}{dx^k} \frac{x^k}{k!} = \\ &= \left( \frac{1}{k+1} \right) \frac{d}{dx} \left( \frac{1}{(1-x)^{k+1}} - 1 \right) = \frac{1}{(1-x)^{k+2}}\end{aligned}$$

**Exercise 22.**

- (1)  $\sum_{n=2}^{\infty} \frac{n-1}{n!} = \sum_{n=2}^{\infty} \frac{1}{(n-1)!} - \sum_{n=2}^{\infty} \frac{1}{n!} = \sum_{n=1}^{\infty} \frac{1}{n!} - \sum_{n=2}^{\infty} \frac{1}{n!} = \boxed{1}$ .
- (2)  $\sum_{n=2}^{\infty} \frac{n}{n!} + \sum_{n=2}^{\infty} \frac{1}{n!} = \sum_{n=2}^{\infty} \frac{1}{(n-1)!} + \sum_{n=0}^{\infty} \frac{1}{n!} - 1 - 1 = \sum_{n=1}^{\infty} \frac{1}{n!} + \sum_{n=0}^{\infty} \frac{1}{n!} - 2 = \sum_{n=0}^{\infty} \frac{2}{n!} - 3 = \boxed{2e-3}$ .
- (3)

$$\begin{aligned}\sum_{n=2}^{\infty} \frac{(n-1)(n+1)}{n!} &= \sum_{n=2}^{\infty} \frac{n^2}{n!} + \sum_{n=2}^{\infty} \frac{-1}{n!} = \sum_{n=2}^{\infty} \frac{n}{(n-1)!} + \sum_{n=2}^{\infty} \frac{-1}{n!} = \\ &= \sum_{n=1}^{\infty} \frac{n+1}{n!} + \sum_{n=2}^{\infty} -\frac{1}{n!} = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} + 1 = \sum_{n=0}^{\infty} \frac{1}{n!} + 1 = \boxed{e+1}\end{aligned}$$

**Exercise 23.**

- (1)  $x \frac{d}{dx} \left( x \frac{d}{dx} \sum_{n=1}^{\infty} \frac{x^n}{n!} \right) = x \frac{d}{dx} \left( x \sum_{n=1}^{\infty} \frac{nx^{n-1}}{n!} \right) = x \frac{d}{dx} \sum_{n=1}^{\infty} \frac{nx^n}{n!} = \sum_{n=1}^{\infty} \frac{n^2 x^n}{n!} = x \frac{d}{dx} \left( x \frac{d}{dx} e^x \right) = x^2 e^x + x e^x$
- (2)  $x \frac{d}{dx} \left( \sum_{n=1}^{\infty} \frac{n^2 x^n}{n!} \right) = \sum_{n=1}^{\infty} \frac{n^3 x^n}{n!} = x \frac{d}{dx} ((x^2 + x)e^x) = x((2x+1)e^x + (x^2+x)e^x) = (x^3 + 3x^2 + x)e^x$

$$x = 1 \quad \boxed{k=5}$$

**Exercise 24.**

- (1)  $\sum_{n=2}^{\infty} (-1)^n = \sum_{n=2}^{\infty} (-1)^n (n - (n-1))$ . Identical.
- (2)  $\sum_{n=2}^{\infty} (1-1) = \sum_{n=2}^{\infty} (-1)^n$ . Not identical.
- (3) Not identical.  $\sum_{n=2}^{\infty} (-1)^n$  vs.  $(\sum_{n=2}^{\infty} (-1+1)) + 1$ .
- (4) Identical.  $\sum_{n=0}^{\infty} \left( \frac{1}{2} \right)^n = 1 + \sum_{n=1}^{\infty} \left( \left( \frac{1}{2} \right)^{n-1} - \left( \frac{1}{2} \right)^n \right) = \sum_{n=1}^{\infty} \left( \frac{1}{2} \right)^n (2-1) = \sum_{n=0}^{\infty} \left( \frac{1}{2} \right)^n$

**Exercise 25.**

(1)

$$1 + x^2 + x^4 + \cdots + x^{2n} + \cdots = \frac{1}{1-x^2} \quad \text{if } |x| < 1$$

$$\implies 1 + 0 + x^2 + 0 + x^4 + \cdots = \frac{1}{1-x^2} \quad \text{if } |x| < 1$$

(2) Thm. 10.2.  $\sum_{n=1}^{\infty} (\alpha a_n + \beta b_n) = \alpha \sum_{n=1}^{\infty} a_n + \beta \sum_{n=1}^{\infty} b_n$ . So then

$$\sum_{j=0}^{\infty} x^j - \sum_{j=0}^{\infty} \frac{x^j + (-x)^j}{2} = \sum_{j=0}^{\infty} \frac{x^j - (-x)^j}{2} = \sum_{j=0}^{\infty} x^{2j+1} = \frac{1}{1-x} - \frac{1}{1-x^2} = \frac{x}{1-x^2}$$

(3)

$$\sum_{j=0}^{\infty} (x^2)^j + \sum_{j=0}^{\infty} x^j = \sum_{j=0}^{\infty} (x^j - x^{2j}) = \frac{x}{1-x^2}$$

**10.14 Exercises - Tests for convergence, Comparison tests for series of nonnegative terms, The integral test.** We'll be using the integral test.

**Theorem 31** (Integral Test).

Let  $f$  be a positive decreasing function, defined for all real  $x \geq 1$ .

For  $\forall n \geq 1$ , let  $s_n = \sum_{k=1}^n f(k)$  and  $t_n = \int_1^n f(x)dx$ .

Then both sequences  $\{s_n\}$  and  $\{t_n\}$  converge or both diverge.

**Exercise 1.**

$$\begin{aligned} \frac{3}{4j-3} + \frac{-1}{4j-1} &= \frac{3(4j-1) + (-1)(4j-3)}{(4j-3)(4j-1)} = \frac{8j}{(4j-3)(4j-1)} \\ \sum_{j=1}^n \frac{j}{(4j-3)(4j-1)} &= \sum_{j=1}^n \left( \frac{3/8}{4j-3} + \frac{-1/8}{4j-1} \right) \\ \int_1^n \left( \frac{3/8}{4x-3} + \frac{-1/8}{4x-1} \right) dx &= \left( (3/8) \frac{\ln(4x-3)}{4} + (-1/8) \frac{\ln(4x-1)}{4} \right) \Big|_1^n = \\ &= \frac{1}{32} \ln \frac{(4x-3)^3}{4x-1} \Big|_1^n = \frac{1}{32} \ln \left( \frac{3(4n-3)^3}{4n-1} \right) \\ \lim_{n \rightarrow \infty} \frac{1}{32} \ln \left( \frac{3(4n-3)^3}{4n-1} \right) &= \lim_{n \rightarrow \infty} \int_1^n \frac{xdx}{(4x-3)(4x-1)} dx = \infty, \quad \text{so } \sum_{j=1}^n \frac{j}{(4j-3)(4j-1)} \text{ diverges as well} \end{aligned}$$

**Exercise 2.**

$$\begin{aligned} \sum_{j=1}^{\infty} \frac{\sqrt{2j-1} \log(4j+1)}{j(j+1)} &= \sum_{j=1}^{\infty} a_j \\ a_j &\leq \frac{\sqrt{4j+1} \log(4j+1)}{j(j+1/4)} = \frac{4 \log(4j+1)}{j(4j+1)^{1/2}} \left( \frac{(4j+1)}{(4j+1)} \right) = 4 \frac{\log(4j+1)(4+1/j)}{(4j+1)^{3/2}} \leq \\ &\leq \frac{16 \log(4j+1)}{(4j+1)^{3/2}} = b_j \end{aligned}$$

Now use the integral test on  $\sum b_j$  to determine the convergence of  $\sum b_j$ .

$$\begin{aligned} \int_1^n \frac{\log(ax+1)}{(ax+1)^{3/2}} dx &= \int \left( \frac{(-2)}{a(ax+1)^{1/2}} \right)' \log(ax+1) dx = \\ &= \frac{-2}{a(ax+1)^{1/2}} \log(ax+1) - \int \frac{-2}{a(ax+1)^{1/2}} \left( \frac{a}{ax+1} \right) dx = \\ &= \frac{-2}{a(ax+1)^{1/2}} \log(ax+1) + \frac{-4}{a(ax+1)^{1/2}} \\ \lim_{n \rightarrow \infty} \int_1^n \frac{\log(ax+1)}{(ax+1)^{3/2}} dx &= \lim_{n \rightarrow \infty} \left( \frac{-2 \log(an+1)}{a(an+1)^{1/2}} + \frac{2 \log(a+1)}{a(a+1)^{1/2}} + \frac{-4}{a} \left( \left( \frac{1}{an+1} \right)^{1/2} - \frac{1}{(a+1)^{1/2}} \right) \right) = \\ &= \frac{2 \log(a+1)}{a(a+1)^{1/2}} + \frac{4}{a(a+1)^{1/2}} \end{aligned}$$

Then by integral test,  $\sum b_j$  converges. Since  $\sum b_j$  converges, then  $\sum a_j$  converges by comparison test.



**Exercise 3.**  $\sum_{j=1}^{\infty} \frac{j+1}{2^j}$ .

$$\begin{aligned}\int_1^n \frac{x+1}{e^{x \ln 2}} dx &= \int_1^n (xe^{-x \ln 2} + e^{-x \ln 2}) dx = \left( \frac{xe^{-x \ln 2}}{-\ln 2} + \frac{e^{-x \ln 2}}{-\ln 2} + \frac{-e^{-x \ln 2}}{(\ln 2)^2} \right) \Big|_1^n \\ &= \frac{ne^{-n \ln 2}}{-\ln 2} + \frac{e^{-\ln 2}}{\ln 2} + - \left( \frac{1}{(\ln 2)^2} + \frac{1}{\ln 2} \right) e^{-\ln 2} + \left( \frac{1}{(\ln 2)^2} + \frac{1}{\ln 2} \right) e^{-\ln 2} \\ \lim_{n \rightarrow \infty} \int_1^n \frac{x+1}{e^{x \ln 2}} dx &= \boxed{\left( \frac{2}{\ln 2} + \frac{1}{(\ln 2)^2} \right) \left( \frac{1}{2} \right)}\end{aligned}$$

By integral test,  $\sum_{j=1}^{\infty} \frac{j+1}{2^j}$  converges.

**Exercise 4.**  $\sum_{j=1}^{\infty} \frac{j^2}{2^j}$ .

$$\begin{aligned}\int_1^n \frac{x^2}{2^x} dx &= \int_1^n \frac{x^2}{e^{x \ln 2}} = \left( \frac{x^2 e^{-x \ln 2}}{-\ln 2} + \frac{2x e^{-x \ln 2}}{-(-\ln 2)^2} + \frac{2e^{-x \ln 2}}{(-\ln 2)^3} \right) \Big|_1^n \\ \lim_{n \rightarrow \infty} \int_1^n \frac{x^2}{2^x} dx &= e^{-\ln 2} \left( \frac{1}{\ln 2} + \frac{2}{(\ln 2)^2} + \frac{2}{(\ln 2)^3} \right) = \frac{1}{2} \left( \frac{1}{\ln 2} + \frac{2}{(\ln 2)^2} + \frac{2}{(\ln 2)^3} \right)\end{aligned}$$

By integral test,  $\sum_{j=1}^{\infty} \frac{j^2}{2^j}$  converges.

**Exercise 5.**

$$\sum_{j=1}^{\infty} \frac{|\sin jx|}{j^2} = \sum_{j=1}^{\infty} a_j \leq \sum_{j=1}^{\infty} \frac{1}{j^2}$$

$\sum_{j=1}^{\infty} \frac{1}{j^2}$  converges since

$$\lim_{n \rightarrow \infty} \int_1^n \frac{1}{x^s} dx = \lim_{n \rightarrow \infty} \frac{x^{-s+1}}{-s+1} \Big|_1^n = \lim_{n \rightarrow \infty} \left( \frac{1}{1-s} \right) \left( \frac{1}{n^{s-1}} - 1 \right) = \frac{1}{s-1} \text{ if } s > 1$$

$\sum_{j=1}^{\infty} \frac{|\sin jx|}{j^2}$  converges by comparison test and integral test.

**Exercise 6.**

$$\sum_{j=1}^{\infty} \frac{2+(-1)^j}{2^j} = \sum_{j=1}^{\infty} \left( \frac{1}{2^{2j-1}} + \frac{3}{2^{2j}} \right) = \frac{2(1/4)}{1-1/4} + \frac{3(1/4)}{1-1/4} = \boxed{\frac{4}{3}}$$

**Exercise 7.**  $\sum_{j=1}^{\infty} \frac{j!}{(j+2)!}$ .

$$a_j = \frac{j!}{(j+2)!} = a_j = \frac{1}{(j+1)(j+2)} \leq \frac{1}{j^2} = b_j$$

Since  $\sum b_j$  converges,  $\sum a_j$  converges, by comparison test.

**Exercise 8.**  $\sum_{j=2}^{\infty} \frac{\log j}{j\sqrt{j+1}} = \sum_{j=2}^{\infty} a_j \leq \sum_{j=2}^{\infty} \frac{\log j}{j^{3/2}}$

$$\begin{aligned}\int_2^n \frac{\log x}{x^{3/2}} dx &= \int_2^n (-2x^{-1/2})' \log x = (-2x^{-1/2} \log x) \Big|_2^n - \int_2^n -\frac{2x^{-1/2}}{x} dx = \\ &= (-2) \left( \frac{\log n}{n^{1/2}} - \frac{\log 2}{2^{1/2}} \right) + -4x^{-1/2} \Big|_2^n \\ \lim_{n \rightarrow \infty} \int_2^n \frac{\log x}{x^{3/2}} dx &= 2^{1/2} \log 2 + \frac{4}{\sqrt{2}}\end{aligned}$$

So  $\sum a_j$  converges by comparison test.

**Exercise 9.**  $\sum_{j=1}^{\infty} \frac{1}{\sqrt{j(j+1)}} = \sum_{j=1}^{\infty} a_j$ . Let  $b_j = \frac{1}{j}$ .

$$\lim_{j \rightarrow \infty} \frac{a_j}{b_j} = \lim_{j \rightarrow \infty} \frac{j}{\sqrt{j(j+1)}} = \lim_{j \rightarrow \infty} \frac{1}{\sqrt{1+1/j}} = 1$$

By limit comparison test, since  $\sum b_j$  diverges,  $\sum a_j$  diverges.

**Exercise 10.**  $\sum_{j=1}^{\infty} \frac{1+\sqrt{j}}{(j+1)^3-1} = \sum_{j=1}^{\infty} a_j$

$$b_j = \frac{1}{j^{5/2}}$$

$$\lim_{j \rightarrow \infty} \frac{a_j}{b_j} = \lim_{j \rightarrow \infty} \left( \frac{1+\sqrt{j}}{(j+1)^3-1} \right) j^{5/2} = \lim_{j \rightarrow \infty} \frac{j^3+j^{5/2}}{(j+1)^3-1} = \lim_{j \rightarrow \infty} \frac{1+1/j^{1/2}}{(1+1/j)^3-1/j^3}$$

By limit comparison test, since  $\sum b_j$  converges,  $\sum a_j$  converges.

**Exercise 11.**  $\sum_{j=2}^{\infty} \frac{1}{(\log j)^s} = \sum a_j$

If  $s \leq 0$ ,  $\sum a_j$  diverges since  $\lim_{j \rightarrow \infty} a_j \neq 0$

If  $0 < s \leq 1$ ,  $\frac{1}{(\log j)^s} > \frac{1}{j^s}$ , and since  $\sum_{j=2}^{\infty} \frac{1}{j^s}$  diverges for  $0 < s < 1$ , so does  $\sum \frac{1}{(\log j)^s}$

$$\int (\log x)^{-s} = \int \left( \frac{1}{x} (\log x)^{-s} \right) x = \int \left( \frac{(\log x)^{-s+1}}{(1-s)} \right)' x = \frac{(\log x)^{-s+1}}{(1-s)} x - \int \frac{(\log x)^{1-s}}{1-s}$$

Thus, if  $s > 1$  has any decimal part, or is an integer, its integral will diverge, so that by integral test, the series diverges.

**Exercise 12.**  $\sum_{j=1}^{\infty} \frac{|a_j|}{10^j}$ ;  $|a_j| < 10$ .

$$\sum_{j=1}^{\infty} \frac{|a_j|}{10^j} < \sum_{j=1}^{\infty} \frac{10}{10^j} = \sum_{j=0}^{\infty} \frac{1}{10^j} = \frac{1}{1 - 1/10} = \boxed{\frac{10}{9}}$$

**Exercise 13.**  $\sum_{j=1}^{\infty} \frac{1}{1000j+1} < \sum_{j=1}^{\infty} \frac{1}{1000j} = \frac{1}{1000} \sum_{j=1}^{\infty} \frac{1}{j}$

The series diverges since  $\sum \frac{1}{j}$  diverges.

**Exercise 14.**

$$\sum_{j=1}^{\infty} \frac{j \cos^2(j\pi/3)}{2^j} \leq \sum_{j=1}^{\infty} \frac{j}{2^j} = \sum_{j=1}^{\infty} \frac{j}{e^{j \ln 2}}$$

$$\int_1^{\infty} \frac{x}{e^{kx}} = \int_1^{\infty} x e^{-kx} = \left( \frac{x e^{-kx}}{-k} - \frac{e^{-kx}}{(-k)^2} \right) \Big|_1^{\infty} = \frac{e^{-k}}{k} + \frac{e^{-k}}{(-k)^2}$$

**Exercise 15.**  $\sum_{j=3}^{\infty} \frac{1}{j \log j (\log(\log j))^s}$

$$\int \frac{1}{x \log x (\log(\log x))^s} = \int \left( \frac{(\ln(\ln x))^{-s+1}}{-s+1} \right)' = \begin{cases} \frac{(\ln(\ln x))^{-s+1}}{-s+1} & \text{if } s > 1, s < 1, s \neq 0 \\ \ln(\ln(\ln x)) & \text{if } s = 1 \\ \ln(\ln x) & \text{if } s = 0 \end{cases}$$

Converges, by integral test, for  $s > 1$

**Exercise 16.** Converges by integral test since

$$\int_1^{\infty} x e^{-x^2} = \left( \frac{e^{-x^2}}{-2} \right) \Big|_1^{\infty} = 0 + \boxed{\frac{e^{-1}}{2}}$$

**Exercise 17.** I drew a picture to help me see what's going on.

$$\frac{\sqrt{x}}{1+x^2} \leq \frac{\sqrt{x}}{1}$$

$$\int_0^{1/n} \sqrt{x} dx = \frac{2}{3} x^{3/2} \Big|_0^{1/n} = \frac{2}{3} \left( \frac{1}{n} \right)^{3/2}$$

$$\sum_{j=2}^{\infty} \frac{2}{3} \left( \frac{1}{j} \right)^{3/2} = \frac{2}{3} \sum_{j=2}^{\infty} \frac{1}{j^{3/2}}$$

So  $\sum_1 n \int_0^{1/n} \frac{\sqrt{x}}{1+x^2} dx$  converges by comparison test.

**Exercise 18.**

$$\begin{aligned}
 (e^{-\sqrt{x}})' &= e^{-\sqrt{x}} \frac{-1}{2\sqrt{x}} \\
 (\sqrt{x}e^{-\sqrt{x}})' &= \frac{-e^{-\sqrt{x}}}{2} + \frac{1}{2\sqrt{x}}e^{-\sqrt{x}}(\sqrt{x}e^{-\sqrt{x}} + e^{-\sqrt{x}})' = \frac{e^{-\sqrt{x}}}{2} \\
 \int_n^{n+1} e^{-\sqrt{x}} dx &= 2(\sqrt{x} + 1)e^{-\sqrt{x}} \Big|_n^{n+1} = 2 \left( \frac{\sqrt{x}}{e^{\sqrt{x}}} + \frac{1}{e^{\sqrt{x}}} \right) \Big|_n^{n+1} = \\
 &= 2 \left( \frac{\sqrt{n+1}}{e^{\sqrt{n+1}}} + \frac{1}{e^{\sqrt{n+1}}} - \frac{\sqrt{n}}{e^{\sqrt{n}}} - \frac{1}{e^{\sqrt{n}}} \right) \\
 &= 2 \sum_{j=1}^{\infty} \left( \frac{\sqrt{j+1}}{e^{\sqrt{j+1}}} - \frac{\sqrt{j}}{e^{\sqrt{j}}} + \frac{1}{e^{\sqrt{j+1}}} - \frac{1}{e^{\sqrt{j}}} \right) = \boxed{-\frac{1}{e}}
 \end{aligned}$$

**Note the use of telescoping sum in the last step.** The series converges.

**Exercise 19.**  $f$  is nonnegative and increasing.  $\xrightarrow{f \text{ increasing}} f(k) \leq f(x) \leq f(k+1), k \leq x \leq k+1$

$$\begin{aligned}
 \int_k^{k+1} f(k) dx &\leq \int_k^{k+1} f(x) dx \leq \int_k^{k+1} f(k+1) dx \implies f(k) \leq \int_k^{k+1} f(x) dx \leq f(k+1) \xrightarrow{\Sigma} \\
 \xrightarrow{\Sigma} \sum_{k=1}^{n-1} f(k) &\leq \sum_{k=1}^{n-1} \int_k^{k+1} f(x) dx = \int_1^n f(x) dx \leq \sum_{k=1}^{n-1} f(k+1) = \sum_{k=2}^n f(k)
 \end{aligned}$$

Note that for the summation on the integral, we simply relied on the properties of integrals to get the final integral.

$$\int_1^n f(x) dx = \int_1^n \log x dx = (x \ln x - x) \Big|_1^n = n \ln n - n + 1.$$

$$\begin{aligned}
 \sum_{k=1}^{n-1} \ln k &\leq \int_1^n \ln x \leq \sum_{k=2}^n \ln(k) \implies \sum_{k=1}^{n-1} \ln k \leq n \ln n - n + 1 \leq \sum_{k=2}^n \ln(k) \\
 \exp \left( \sum_{k=1}^{n-1} \ln k \right) &= (n-1)! \leq n^n e^{-n+1} \\
 \exp \left( \sum_{k=2}^n \ln k \right) &= (n)! \geq n^n e^{-n+1} \\
 \implies \frac{e^{1/n}}{e} &< \frac{(n!)^{1/n}}{n} < \frac{e^{1/n} n^{1/n}}{e}
 \end{aligned}$$

**10.16 Exercises - The root test and the ratio test for series of nonnegative terms. Exercise 1.**  $\sum_{j=1}^{\infty} \frac{(j!)^2}{(2j)!}$

$$\frac{((j+1)!)^2}{(2j+2)!} \frac{(2j)!}{(j!)^2} = \frac{(j+1)^2}{(2j+2)(2j+1)} = \frac{j^2+2j+1}{4j^2+6j+2} \xrightarrow{j \rightarrow \infty} \frac{1}{4}$$

Converges by ratio test.

**Exercise 2.**  $\sum_{j=1}^{\infty} \frac{(j!)^2}{2^{j^2}}$ .

$$\frac{((j+1)!)^2}{2^{(j+1)^2}} \frac{2^{j^2}}{(j!)^2} = \frac{(j+1)^2 2^{j^2}}{2^{j^2+2j+1}} = \frac{j^2+2j+1}{2e^{j \ln 2}} \xrightarrow{j \rightarrow \infty} 0$$

Converges by ratio test.

**Exercise 3.**  $\sum_{j=1}^{\infty} \frac{2^j j!}{j^j}$

$$\frac{2^{j+1}(j+1)!}{(j+1)^{j+1}} \frac{j^j}{2^j j!} = \frac{2(j+1)}{(j+1)} \left( \frac{1}{1+1/j} \right)^j \xrightarrow{j \rightarrow \infty} \frac{2}{e} < 1$$

Converges by ratio test.

**Exercise 4.**  $\sum_{j=1}^{\infty} \frac{3^j j!}{j^j}$

$$\frac{3^{j+1}(j+1)!}{(j+1)^{j+1}} \left( \frac{j^j}{3^j j!} \right) = 3 \left( \frac{1}{(1+1/j)^j} \right) \xrightarrow{j \rightarrow \infty} \frac{3}{e} > 1$$

Diverges by ratio test.

**Exercise 5.**  $\sum_{j=1}^{\infty} \frac{j!}{3^j}$ .

$$\frac{(j+1)!}{3^{j+1}} \frac{3^j}{j!} = \frac{j+1}{3}$$

Diverges by ratio test.

**Exercise 6.**  $\sum_{j=1}^{\infty} \frac{j!}{2^{2j}}$

$$\frac{(j+1)!}{2^{2(j+1)}} \frac{2^{2j}}{j!} = \frac{j+1}{4}$$

Diverges.

**Exercise 7.**  $\sum_{j=2}^{\infty} \frac{1}{(\log j)^{1/j}}$

Draw a picture to see what's going on.

$$\begin{aligned} \sum_{j=2}^{\infty} \frac{1}{(\log j)^{1/j}} &= \sum_{j=2}^{\infty} \exp\left(\frac{-1}{j} \ln \log j\right) \\ 0 &< \frac{\ln \log j}{j} \quad \text{for } j > 3 \\ \frac{\ln(\log j)}{j} &< \frac{\ln j}{j} \implies \lim_{j \rightarrow \infty} \frac{\ln(\log j)}{j} < \lim_{j \rightarrow \infty} \frac{\ln j}{j} = 0 \\ &\implies \lim_{j \rightarrow \infty} \frac{-1}{j} \ln(\log j) = 0 \quad \text{so then} \\ &\lim_{j \rightarrow \infty} \exp\left(\frac{-1}{j} \ln(\log j)\right) = 1 \end{aligned}$$

$\sum_{j=2}^{\infty} \frac{1}{(\log j)^{1/j}}$  diverges because the  $a_j$  term doesn't go to zero.

**Exercise 8.**  $\sum_{j=1}^{\infty} (j^{1/j} - 1)^j$

$$((j^{1/j} - 1)^j)^{1/j} = (e^{\frac{1}{j} \ln j} - 1) \xrightarrow{j \rightarrow \infty} 0$$

Converges by root test.

**Exercise 9.**  $\sum_{j=1}^{\infty} e^{-j^2}$

$$(e^{-j^2})^{1/j} = e^{-j} \xrightarrow{j \rightarrow \infty} 0$$

Converges by root test.

**Exercise 10.** I systematically tried ratio test and then root test. Both were inconclusive.

Consider comparison with  $\sum \frac{1}{j}$ .

$$\left(\frac{e^{j^2} - j}{je^{j^2}}\right) j = \frac{e^{j^2} - j}{e^{j^2}} = 1 - \frac{j}{e^{j^2}} \xrightarrow{j \rightarrow \infty} 1$$

By limit comparison test, since  $\sum \frac{1}{j}$  diverges, so does  $\sum \left(\frac{1}{j} - \frac{1}{e^{j^2}}\right)$ .

**Exercise 11.**  $\sum_{j=1}^{\infty} \frac{(1000)^j}{j!} = e^{1000}$

**Exercise 12.**  $\sum_{j=1}^{\infty} \frac{j^{j+1/j}}{(j+1/j)^j}$

$$a_j^{1/j} = \frac{j^{1/j^2}}{1 + \frac{1}{j^2}} \quad \lim_{j \rightarrow \infty} a_j^{1/j} = \lim_{j \rightarrow \infty} \frac{\exp\left(\frac{1}{j^2} \ln j\right)}{1 + \frac{1}{j^2}} = 1$$

Note that root test is inconclusive.

$$\begin{aligned} a_j &= \frac{\exp\left(\frac{1}{j} \ln j\right)}{\left(1 + \frac{1}{j}\right)^j} \geq \frac{\exp\left(\frac{1}{j} \ln j\right)}{\left(1 + \frac{1}{j^2}\right)^{j^2}} \\ \lim_{j \rightarrow \infty} a_j &\geq \lim_{j \rightarrow \infty} \frac{\exp\left(\frac{1}{j} \ln j\right)}{\left(1 + \frac{1}{j^2}\right)^{j^2}} = \frac{1}{e} > 0 \end{aligned}$$

Diverges since  $\lim_{j \rightarrow \infty} a_j > 0$ .

**Exercise 13.**  $\sum_{j=1}^{\infty} \frac{j^3(\sqrt{2}+(-1)^j)^j}{3^j}$ .

$$\left( \frac{j^3(\sqrt{2}+(-1)^j)^j}{3^j} \right)^{1/j} = \frac{j^{3/j}(\sqrt{2}+(-1)^j)}{3} = \frac{e^{\frac{3}{j} \ln j}(\sqrt{2}+(-1)^j)}{3} \xrightarrow{j \rightarrow \infty} \frac{(\sqrt{2}+(-1)^j)}{3} < 1$$

Converges by root test.

**Exercise 14.**  $\sum_{j=1}^{\infty} r^j |\sin jx|$ .

If  $0 < r < 1$ .

$$\sum_{j=1}^{\infty} r^j |\sin jx| < \sum_{j=1}^{\infty} r^j$$

so by comparison test,  $\sum_{j=1}^{\infty} r^j |\sin jx|$  converges for  $0 < r < 1$

If  $r \geq 1$ ,

$$\lim_{j \rightarrow \infty} r^j |\sin jx| \neq 0 \text{ so } \sum_{j=1}^{\infty} r^j |\sin jx| \text{ diverges, unless } jx = \pi j$$

**Exercise 15.**

(1)  $c_j = b_j - \frac{b_{j+1}a_{j+1}}{a_j} > 0 \quad \forall j \geq N$ . Then there must be a positive number  $r$  that's in between  $c_j$  and 0.

$$a_j b_j - a_{j+1} b_{j+1} \geq r a_j$$

$$r \sum_{j=N}^n a_j \leq \sum_{j=N}^n (a_j b_j - a_{j+1} b_{j+1}) = a_N b_N - a_{n+1} b_{n+1} \leq a_N b_N$$

$$\implies \sum_{j=N}^n a_j \leq \frac{a_N b_N}{r}$$

(2)  $c_n < 0$

$$b_j - \frac{b_{j+1}a_{j+1}}{a_j} < 0 \quad \sum \frac{1}{b_j} \text{ diverges, so}$$

$$a_j b_j < b_{j+1} a_{j+1} \implies \frac{b_j}{b_{j+1}} < \frac{a_{j+1}}{a_j} \quad \lim_{j \rightarrow \infty} \frac{b_j}{b_{j+1}} \geq 1 \text{ by ratio test}$$

$$\xrightarrow{j \rightarrow \infty} 1 \leq \frac{b_j}{b_{j+1}} < \frac{a_{j+1}}{a_j}$$

So by ratio test,  $\sum a_j$  diverges.

**Exercise 16.**  $b_{n+1} = n$ ;  $b_n = n - 1$ .

$$c_n = n - 1 - \frac{na_{n+1}}{a_n} \geq r \implies \frac{a_{n+1}}{a_n} \leq 1 - \frac{1}{n} - \frac{r}{n}.$$

Using Exercise 15,  $\sum a_n$  converges.

$\sum \frac{1}{b_n}$  diverges since  $\sum \frac{1}{b_n}$  is a harmonic series of  $s = 1$ .

$$n - 1 - \frac{na_{n+1}}{a_n} \leq 0 \implies 1 - \frac{1}{n} \leq \frac{a_{n+1}}{a_n}$$

**Exercise 17.** For some  $N \geq 1$ ,  $s > 1$ ,  $M > 0$ , and given that

$$\frac{a_{n+1}}{a_n} = 1 - \frac{A}{n} + \frac{f(n)}{n^s} = 1 - \left( \frac{A - \frac{f(n)}{n^{s-1}}}{n} \right)$$

Consider  $A - \frac{f(n)}{n^{s-1}}$ .

Since  $|f(n)| < M$ ,  $f(n)$  is finite, so consider  $s$  larger than 1 and  $n$  going to infinity so that  $\frac{f(n)}{n^{s-1}} \rightarrow 0$ .

Using Exercise 16, for  $\sum a_j$  to converge,  $A - \frac{f(n)}{n^{s-1}} = 1 + r$  where  $r > 0$ , for all  $n \geq N$ , where  $N$  is some positive number.

Let  $r = \frac{M}{N^{s-1}}$  so that

$$A = 1 + r + \frac{f(n)}{n^{s-1}} > 1$$

If  $A > 1$ , then  $\sum a_n$  converges.

If  $A = 1$ , then consider using Exercise 15 and  $b_n = n \log n$ .

$$\begin{aligned}
 c_n &= b_n - b_{n+1} \frac{a_{n+1}}{a_n} = (n-1) \log(n-1) - n \log n \left( \frac{a_{n+1}}{a_n} \right) \\
 &= (n-1) \log(n-1) - n \log n \left( 1 - \frac{1}{n} + \frac{f(n)}{n^s} \right) = (n-1) \log \left( \frac{n-1}{n} \right) - n \log n \left( \frac{f(n)}{n^s} \right) = \\
 &= -(n-1) \log \left( \frac{n}{n-1} \right) - n \log n \left( \frac{f(n)}{n^s} \right) \\
 &\quad \text{since } \frac{\log n}{n^{s-1}} \xrightarrow{n \rightarrow \infty} 0, \\
 &\quad -(n-1) \log \left( \frac{n}{n-1} \right) - n \log n \left( \frac{f(n)}{n^s} \right) < 0 \quad \text{for } n \text{ large enough} \\
 &\quad \text{since } c_n < 0 \text{ for } n \geq N \text{ for some } N > 0, \text{ then by Exercise 15, } \sum a_n \text{ is divergent.}
 \end{aligned}$$

Given that  $A < 1$ , then for  $A - \frac{f(n)}{n^{s-1}}$ , choose  $N > 0$  so that  $\frac{M}{n^{s-1}} < \epsilon < 1$  and that  $A - \frac{f(n)}{n^{s-1}} \leq A + \frac{M}{n^{s-1}} = A + \epsilon \leq 1$ . We can always choose  $\epsilon$  small enough because there's always a real number in between  $A$  and 1 (Axiom of Archimedes).

$$\begin{aligned}
 A - \frac{f(n)}{n^{s-1}} \leq 1 &\implies -\left(A - \frac{f(n)}{n^{s-1}}\right) \geq -1 \\
 \implies \text{using Exercise 16, } \frac{a_{n+1}}{a_n} &= 1 - \frac{A + \frac{f(n)}{n^{s-1}}}{n} \geq 1 - \frac{1}{n} \text{ for all } n \geq N
 \end{aligned}$$

**Exercise 18.**

$$\left( \frac{1 \cdot 3 \cdot 5 \dots (2n+1)}{2 \cdot 4 \cdot 6 \dots (2n+2)} \cdot \frac{2 \cdot 4 \cdot 6 \dots (2n)}{1 \cdot 3 \cdot 5 \dots (2n-1)} \right)^k = \left( \frac{2n+1}{2n+2} \right)^k \xrightarrow{n \rightarrow \infty} 1$$

Ratio test fails.

$$\begin{aligned}
 \frac{a_{n+1}}{a_n} &= \left( \frac{2n+1}{2n+2} \right)^k = \left( 1 + \frac{-1}{2n+2} \right)^k = \left( 1 + \frac{-1/2}{n+1} \right)^k = \\
 &= \sum_{j=0}^k \binom{k}{j} \left( \frac{-1/2}{n+1} \right)^j = 1 + k \left( \frac{-1/2}{n+1} \right) + \sum_{j=2}^k \binom{k}{j} \left( \frac{-1/2}{n+1} \right)^j
 \end{aligned}$$

$$\text{Note that for } k < \infty, \sum_{j=2}^k \binom{k}{j} \left( \frac{-1/2}{n+1} \right)^j < \infty$$

$$\text{Let } \left| \sum_{j=2}^k \binom{k}{j} \left( \frac{-1/2}{n+1} \right)^j \right| \leq M$$

$$k/2 = A > 1 \text{ or } k > 2 \text{ means } \sum a_j \text{ converges} \quad k/2 = A \leq 1 \text{ or } k \leq 2 \text{ means } \sum a_j \text{ diverges}$$

**10.20 Exercises - Alternating series, Conditional and absolute convergence, The convergence tests of Drichlet and Abel.** We will be using Leibniz's test alot, initially.

**Theorem 32 (Leibniz's Rule).** If  $a_j$  is a monotonically decreasing sequence with limit 0,

$$\sum_{j=1}^{\infty} (-1)^{j-1} a_j \text{ converges.}$$

$$\text{If } S = \sum_{j=1}^{\infty} a_j, \quad s_n = \sum_{j=1}^n (-1)^{j-1} a_j,$$

$$0 < (-1)^j (S - s_j) < a_{j+1}$$

**Exercise 1.**  $\sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{\sqrt{j}}$ .  $\lim_{j \rightarrow \infty} \frac{1}{\sqrt{j}} = 0$  Converges conditionally.

**Exercise 2.**  $\sum_{j=1}^{\infty} (-1)^j \frac{\sqrt{j}}{j+100}$   $\lim_{j \rightarrow \infty} \frac{\sqrt{j}}{j+100} = 0$ . Converges by Leibniz's test.

$\frac{1}{\sqrt{j+100}} \geq \frac{1}{101\sqrt{j}}$ , so by comparison test, the series diverges absolutely. So the alternating series converges conditionally by comparison test.

**Exercise 3.**  $\sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j^s}$  If  $s > 1$ , then the series absolutely converges.  $\lim_{j \rightarrow \infty} \frac{1}{j^s} = 0$  if  $s > 0$ . Converges conditionally for  $0 < s < 1$ . Otherwise, if  $s < 0$  the series diverges absolutely.

**Exercise 4.**  $\sum_{j=1}^{\infty} (-1)^j \left( \frac{1 \cdot 3 \cdot 5 \dots (2j-1)}{2 \cdot 4 \cdot 6 \dots (2j)} \right)^3$ .

$$\frac{a_{j+1}}{a_j} = \frac{1 \cdot 3 \cdot 5 \cdot (2j+1)}{2 \cdot 4 \cdot 6 \dots (2j+2)} \frac{2 \cdot 4 \cdot 6 \dots (2j)}{1 \cdot 3 \cdot 5 \dots (2j-1)} = \frac{2j+1}{2j+2}$$

Absolutely converges.

**Exercise 5.**  $\sum_{j=1}^{\infty} \frac{(-1)^{j(j-1)/2}}{2^j}$  converges since  $\lim_{j \rightarrow \infty} \frac{(-1)^{j(j-1)/2}}{2^j} = 0$ ;  $\sum_{j=1}^{\infty} \frac{1}{2^j} = \frac{1/2}{1-1/2} = \boxed{1}$ . Absolutely converges.

**Exercise 6.**  $\sum_{j=1}^{\infty} (-1)^j \left( \frac{2j+100}{3j+1} \right)^j$ .

$$\begin{aligned} \exp \left( j \ln \left( \frac{2j+100}{3j+1} \right) \right) &= \exp \left( j \ln \left( \frac{2}{3} + \frac{298}{9j+3} \right) \right) \leq \exp \left( j \left( \ln \left( \frac{2}{3} \right) + \frac{1}{\frac{3}{2}} \left( \frac{298}{9j+3} \right) \right) \right) = \\ &= \exp \left( j \ln \frac{2}{3} + \frac{146}{3+1/j} \right) \\ 0 &\leq \lim_{j \rightarrow \infty} \exp \left( j \ln \left( \frac{2j+100}{3j+1} \right) \right) \leq \lim_{j \rightarrow \infty} \\ &\quad \exp \left( j \ln \frac{2}{3} + \frac{146}{3+1/j} \right) = 0 \\ \Rightarrow \lim_{j \rightarrow \infty} \exp \left( j \ln \left( \frac{2j+100}{3j+1} \right) \right) &= 0 \end{aligned}$$

So the alternating series converges.

$$\begin{aligned} \left( \frac{2j+100}{3j+1} \right) &< \frac{2j+100}{3j} < \frac{2.5j}{3j} = \frac{5}{6} \quad (\text{for } j \geq 200) \\ \Rightarrow \left( \frac{2j+100}{3j+1} \right)^j &< \left( \frac{5}{6} \right)^j \quad \text{for } j \geq 200 \end{aligned}$$

So the series absolutely converges by comparison with a geometric series.

**Exercise 7.**  $\sum_{j=2}^{\infty} \frac{(-1)^j}{\sqrt{j} + (-1)^j}$ .

$$\lim_{j \rightarrow \infty} \frac{1}{\sqrt{j} + (-1)^j} \text{ doesn't exist since ???}$$

To show divergence, we usually think of either *taking the general term and finding the limit* (and if it goes to a nonzero constant, then it diverges), or we use ratio, root, comparison test on the general term. Since this is an alternating series, I've observed that the general term *is a sum of two adjacent terms, one even and one odd*.

$$\begin{aligned} \frac{(-1)^{2j}}{\sqrt{2j} + (-1)^{2j}} + \frac{(-1)^{2j+1}}{\sqrt{2j+1} + (-1)^{2j+1}} &= \frac{1}{\sqrt{2j} + 1} + \frac{-1}{\sqrt{2j+1} + 1} = \frac{\sqrt{2j+1} - 1 - (\sqrt{2j} + 1)}{(\sqrt{2j} + 1)(\sqrt{2j+1} + 1)} = \\ &= \frac{\sqrt{2j+1} - \sqrt{2j} - 2}{(\sqrt{2j} + 1)(\sqrt{2j+1} + 1)} = \frac{\sqrt{2j} \sqrt{1 + \frac{1}{2j}} - \sqrt{2j} - 2}{(\sqrt{2j} + 1)(\sqrt{2j} \sqrt{1 + \frac{1}{2j}} + 1)} = \\ \xrightarrow{\text{for } j \text{ large}} &\simeq \frac{\sqrt{2j} \left( 1 + \frac{1}{4j} \right) - \sqrt{2j} - 2}{(\sqrt{2j} + 1)(\sqrt{2j} \left( 1 + \frac{1}{4j} \right) + 1)} = \frac{-2}{j} \left( \frac{1 - \frac{1}{4\sqrt{2j}}}{2 - \frac{1}{2j} + \frac{1}{2\sqrt{2j}^{3/2}}} \right) \end{aligned}$$

Every term, since we considered any  $j$ , will contain  $-2$ . So we factor it out. Then

$$\frac{1}{j} \left( \frac{1 - \frac{1}{4\sqrt{2j}}}{2 - \frac{1}{2j} + \frac{1}{2\sqrt{2j}^{3/2}}} \right) > \frac{1}{j} \left( \frac{1 - \frac{1}{4\sqrt{2j}}}{4 - \frac{1}{\sqrt{2j}}} \right) = \frac{1}{4j}$$

By comparison test to  $\frac{1}{j}$  the series diverges.

**Exercise 8.** Using the theorem

**Theorem 33.**

Assume  $\sum |a_j|$  converges

Then  $\sum a_j$  converges and  $|\sum a_j| \leq \sum |a_j|$ .

So using the contrapositive,

If  $\sum a_j$  diverges,  
 $\sum |a_j|$  diverges.

$$\frac{1}{j^{1/j}} = \frac{1}{e^{\frac{1}{j} \ln j}} \quad \lim_{j \rightarrow \infty} \frac{1}{j^{1/j}} = \frac{1}{\exp\left(\lim_{j \rightarrow \infty} \frac{1}{j} \ln j\right)} = 1$$

Diverges absolutely.

**Exercise 9.**  $\sum_{j=1}^{\infty} (-1)^j \frac{j^2}{1+j^2}$  Diverges absolutely.

$$\begin{aligned} \frac{(2j)^2}{1+(2j)^2} - \frac{(2j-1)^2}{1+(2j-1)^2} &= \frac{4j^2}{1+4j^2} \left( \frac{4j^2-4j+2}{4j^2-4j+2} \right) - \frac{(4j^2-4j+1)(1+4j^2)}{(4j^2-4j+2)(1+4j^2)} \\ &= \frac{4j-1}{2(1+4j^2)(2j^2-2j+1)} \\ \frac{4j-1}{2(1+4j^2)(2j^2-2j+1)} (j^3) &= \frac{4-1/j}{2(4+1/j^2)(2-2/j+1/j^2)} = \frac{1}{4} \end{aligned}$$

By limit comparison test, with  $\sum \frac{1}{j^3}$ ,  $\sum_{j=1}^{\infty} \frac{j^2}{1+j^2}$  converges.

**Exercise 10.**  $\sum_{n=1}^{\infty} \frac{(-1)^n}{\log(e^n + e^{-n})}$

$$\lim_{n \rightarrow \infty} \frac{1}{\log(e^n + e^{-n})} = \lim_{n \rightarrow \infty} \frac{1}{n} = 0$$

The series converges.

$$\lim_{n \rightarrow \infty} \frac{n}{\log(e^n + e^{-n})} = \lim_{n \rightarrow \infty} \frac{n}{\log e^n + \log(1 + e^{-2n})} = \lim_{n \rightarrow \infty} \frac{1}{\left(\frac{n + \log(1 + e^{-2n})}{n}\right)} = 1$$

Since  $\sum \frac{1}{n}$  diverges,  $\sum \frac{1}{\log(e^n + e^{-n})}$  diverges.

**Exercise 11.**  $\sum_{j=1}^{\infty} \frac{(-1)^j}{j \log^2(j+1)}$

$\lim_{j \rightarrow \infty} \frac{1}{j \log^2(j+1)} = 0$  so by Leibniz's test, the alternating series.

$$\begin{aligned} \frac{1}{n \log^2(n+1)} &< \frac{1}{n \log^2(n)} \\ \int \frac{1}{n \log^2 n} &= \int \left( \frac{-1}{\log n} \right)' = \frac{-1}{\log n} \xrightarrow{n \rightarrow \infty} \frac{1}{\log 2} \end{aligned}$$

Converges by comparison test to  $\frac{1}{n \log^2 n}$ , which converges by integral test. So the series absolutely converges.

**Exercise 12.**  $\sum_{j=1}^{\infty} \frac{(-1)^j}{\log(1+1/j)}$  diverges absolutely.

$$\begin{aligned} \frac{(-1)}{\log\left(1 + \frac{1}{2j-1}\right)} + \frac{1}{\log\left(1 + \frac{1}{2j}\right)} &= \frac{-1}{\log\left(\frac{2j}{2j-1}\right)} + \frac{1}{\log\left(\frac{2j+1}{2j}\right)} = \frac{-\log\left(\frac{2j+1}{2j}\right) + \log\left(\frac{2j}{2j-1}\right)}{\log\left(\frac{2j}{2j-1}\right) \log\left(\frac{2j+1}{2j}\right)} = \\ &= \frac{\log\left(\frac{(2j)^2}{4j^2-1}\right)}{\log\left(\frac{2j}{2j-1}\right) \left(\log\left(1 + \frac{1}{2j}\right)\right)} = \frac{\log\left(1 + \frac{1}{4j^2-1}\right)}{\log\left(1 + \frac{1}{2j-1}\right) \log\left(1 + \frac{1}{2j}\right)} = \\ &= \frac{\frac{1}{4j^2-1} + o\left(\frac{1}{4j^2-1}\right)}{\left(\frac{1}{2j-1} + o\left(\frac{1}{2j-1}\right)\right) \left(\frac{1}{2j} + o\left(\frac{1}{2j}\right)\right)} \approx \\ &\approx \frac{4j^2-2j}{4j^2-1} = \frac{1-\frac{1}{2j}}{1-\frac{1}{4j^2}} \xrightarrow{j \rightarrow \infty} 1 \end{aligned}$$

So the alternating series diverges.

**Exercise 13.**  $\sum_{j=1}^{\infty} \frac{(-1)^j j^{37}}{(j+1)!}$  Use the ratio test.

$$\frac{a_{j+1}}{a_j} = \frac{(j+1)^{37} (j+1)!}{(j+2)! j^{37}} = \left(\frac{1}{j+2}\right) \left(1 + \frac{1}{j}\right)^{37} \rightarrow 0$$



Converges for  $\sum |a_j|$ . Then  $\sum a_j$  converges. The series absolutely converges.

**Exercise 14.**  $\sum_{n=1}^{\infty} (-1)^n \int_n^{n+1} \frac{e^{-x}}{x} dx$

$$\int_n^{n+1} \frac{e^{-x}}{x} dx \leq \int_n^{n+1} \frac{1}{e^{2x}} dx = \frac{e^{-2x}}{-2} \Big|_n^{n+1} = \left( \frac{-1}{2} \right) \left( \frac{1}{e^{2(n+1)}} - \frac{1}{e^{2n}} \right) = \frac{e^2 - 1}{2e^{2n+2}} < 1$$

Converges absolutely.

**Exercise 15.**  $\sum_{j=1}^{\infty} \sin(\log j)$

$\lim_{j \rightarrow \infty} \sin(\log j)$  doesn't exist. So the series is divergent.

**Exercise 16.**  $\sum_{j=1}^{\infty} \log \left( j \sin \frac{1}{j} \right)$  Note that

$$\begin{aligned} \log \left( j \sin \frac{1}{j} \right) &= \log \left( \frac{\sin 1/j}{1/j} \right) \\ \lim_{j \rightarrow \infty} \log \left( \frac{\sin 1/j}{1/j} \right) &= \log \left( \lim_{j \rightarrow \infty} \frac{\sin 1/j}{1/j} \right) = \log 1 = 0 \\ \sin \frac{1}{j} &= \sum_{k=0}^{\infty} \frac{\left( \frac{1}{j} \right)^{2k+1}}{(2k+1)!} (-1)^k \\ \log \left( j \sum_{k=0}^{\infty} \frac{\left( \frac{1}{j} \right)^{2k+1}}{(2k+1)!} (-1)^k \right) &= \log \left( 1 + \frac{-1}{6j^2} + \sum_{k=2}^{\infty} \frac{\left( \frac{1}{j} \right)^{2k}}{(2k+1)!} (-1)^k \right) \geq \\ &\geq \log \left( 1 - \frac{1}{6j^2} \right) \geq \frac{-1}{6j^2} \end{aligned}$$

The series absolutely converges.

**Exercise 17.**  $\sum_{j=1}^{\infty} (-1)^j \left( 1 - j \sin \frac{1}{j} \right)$

$$\begin{aligned} \left( 1 - x \sin \frac{1}{x} \right)' &= -\sin \frac{1}{x} - x \cos \frac{1}{x} \left( \frac{-1}{x^2} \right) = \\ &= -\sin \frac{1}{x} + \frac{1}{x} \cos \frac{1}{x} = \frac{-x \sin \frac{1}{x} + \cos \frac{1}{x}}{x} \\ \sin \frac{1}{x} &= \sum_{j=0}^{\infty} \frac{\left( \frac{1}{x} \right)^{2j+1}}{(2j+1)!} (-1)^j \\ \frac{-x \sin \frac{1}{x} + \cos \frac{1}{x}}{x} &= \\ &= \frac{-1 + \frac{\left( \frac{1}{x} \right)^3 (+1)}{3!} + \sum_{j=2}^{\infty} \frac{\left( \frac{1}{x} \right)^{2j+1} (-1)^j}{(2j+1)!} + 1 - \left( \frac{1}{x} \right)^2 / 2 + \sum_{j=2}^{\infty} \frac{\left( \frac{1}{x} \right)^{2j} (-1)^j}{(2j)!}}{x} < 0 \text{ for } x \text{ large enough} \end{aligned}$$

$\sum_{j=1}^{\infty} (-1)^j \left( 1 - j \sin \frac{1}{j} \right)$  converges since  $a_j = 1 - j \sin \frac{1}{j}$  is monotonically decreasing sequence with limit 0.

$$\begin{aligned} 1 - j \sin \frac{1}{j} &= 1 - j \sum_{k=0}^{\infty} \frac{\left( \frac{1}{j} \right)^{2k+1}}{(2k+1)!} (-1)^k = 1 - j \left( \frac{1}{j} + \sum_{k=1}^{\infty} \frac{\left( \frac{1}{j} \right)^{2k+1}}{(2k+1)!} (-1)^k \right) = \\ &= 1 - \left( 1 + \sum_{k=1}^{\infty} \frac{\left( \frac{1}{j} \right)^{2k}}{(2k+1)!} (-1)^k \right) = \sum_{k=1}^{\infty} \frac{\left( \frac{1}{j} \right)^{2k}}{(2k+1)!} \leq \frac{1}{6j^2} \end{aligned}$$

The series converges absolutely since the term itself is a series that is dominated by  $\frac{1}{6j^2}$ , so that by comparison test, the series must converge.

**Exercise 18.**  $\sum_{j=1}^{\infty} (-1)^j \left(1 - \cos \frac{1}{j}\right)$ .

$$\left(\cos \frac{1}{x}\right)' = \left(-\sin \frac{1}{x} \left(\frac{-1}{x^2}\right)\right) = \frac{-1}{x^2} \sin \frac{1}{x} < 0$$

$\sum_{j=1}^{\infty} (-1)^j (1 - \cos \frac{1}{j})$  converges since  $a_j = (1 - \cos \frac{1}{j})$  is monotonically decreasing to 0

$$(1 - \cos \frac{1}{j}) = 1 - \sum_{k=0}^{\infty} \frac{(1/j)^{2k} (-1)^k}{(2k)!} = \sum_{k=1}^{\infty} \frac{(1/j)^{2k} (-1)^{k+1}}{(2k)!} \leq \frac{1}{2j^2}$$

So the series converges absolutely, by comparison test with  $\sum \frac{1}{j^2}$  which converges.

**Exercise 19.**  $\sum_{j=1}^{\infty} (-1)^j \arctan \frac{1}{2j+1}$ .

$$\left(\arctan \left(\frac{1}{2j+1}\right)\right)' = \frac{1}{1 + \left(\frac{1}{2j+1}\right)^2} \left(\frac{-1}{(2j+1)^2}\right) (2) = \frac{-2}{(2j+1)^2 + 1} < 0$$

$\sum_{j=1}^{\infty} (-1)^j \arctan \frac{1}{2j+1}$  converges, since  $a_j = \arctan \frac{1}{2j+1}$  is monotonically decreasing to 0

$$\begin{aligned} \frac{1}{1+x^2} &= (\arctan x)' = \sum_{j=0}^{\infty} (-x^2)^j = \sum_{j=0}^{\infty} (-1)^j x^{2j} \\ \implies \arctan x &= \sum_{j=0}^{\infty} (-1)^j \frac{x^{2j+1}}{2j+1} \end{aligned}$$

$$\begin{aligned} \arctan \frac{1}{2j+1} &= \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{1}{2j+1}\right)^{2k+1}}{(2k+1)} = \frac{1}{2j+1} + (-1) \left(\frac{\frac{1}{(2j+1)^3}}{3}\right) + \sum_{k=2}^{\infty} \frac{(-1)^k \left(\frac{1}{2j+1}\right)^{2k+1}}{2k+1} > \\ &> \frac{1}{2j+1} + (-1) \frac{1}{3(2j+1)^3} = \frac{3(4j^2 + 4j + 1) + (-1)}{3(2j+1)^3} = \frac{12j^2 + 12j + 2}{3(2j+1)^3} > \frac{2j}{(2j+1)^2} > \frac{2}{9} \frac{1}{j} \text{ for } j > 2 \end{aligned}$$

So by comparison test to  $\sum \frac{1}{j}$ ,  $\sum \arctan \frac{1}{2j+1}$  diverges absolutely. The series is conditionally convergent.

**Exercise 20.**  $\sum_{j=1}^{\infty} (-1)^j \left(\frac{\pi}{2} - \arctan \log j\right)$

$$\left(\frac{\pi}{2} - \arctan \log n\right)' = \left(\frac{-1}{1 + (\log n)^2}\right) \left(\frac{1}{n}\right)$$

$$\begin{aligned} \frac{\pi}{2} - \arctan \log n &\geq \frac{\pi}{2} - \arctan (n-1) \rightarrow \frac{\pi}{2} - \arctan n \quad \text{just change indices} \\ \int_0^n \left(\frac{\pi}{2} - \arctan x\right) dx &= \frac{\pi}{2} x - \left(x \arctan x - \frac{1}{2} \ln(1+x^2)\right) \Big|_0^n \\ &= \frac{\pi}{2} n - \left(n \arctan n - \frac{1}{2} (\ln(1+n^2))\right) = n \left(\frac{\pi}{2} - \arctan n\right) + \frac{1}{2} \ln(1+n^2) \\ \lim_{n \rightarrow \infty} \left(n \left(\frac{\pi}{2} - \arctan n\right) + \frac{1}{2} \ln(1+n^2)\right) &\rightarrow \infty \end{aligned}$$

Then by the integral test,  $\sum \frac{\pi}{2} - \arctan \log n$  diverges absolutely. So the alternating series is conditionally convergent.

**Exercise 21.**  $\sum_{j=1}^{\infty} \log \left(1 + \frac{1}{|\sin j|}\right)$

$\lim_{j \rightarrow \infty} \log \left(1 + \frac{1}{|\sin j|}\right)$  doesn't exist and  $\log \left(1 + \frac{1}{|\sin j|}\right) > 0 \quad \forall j$  so the series diverges.

**Exercise 22.**  $\sum_{j=2}^{\infty} \sin\left(j\pi + \frac{1}{\log j}\right)$

$$\begin{aligned} & \sin\left(2j\pi + \frac{1}{\log 2j}\right) + \sin\left((2j+1)\pi + \frac{1}{\log(2j+1)}\right) = \\ &= \sin(2j\pi) \cos \frac{1}{\log 2j} + \sin\left(\frac{1}{\log 2j}\right) \cos(2\pi j) + \sin(2j+1)\pi \cos \frac{1}{\log(2j+1)} + \cos(2j+1)\pi \sin\left(\frac{1}{\log(2j+1)}\right) = \\ &= \sin \frac{1}{\log 2j} - \sin \frac{1}{\log(2j+1)} = \\ &= \sum_{k=0}^{\infty} \frac{\left(\frac{1}{\log 2j}\right)^{2k+1} (-1)^k}{(2k+1)!} - \sum_{k=0}^{\infty} \frac{\left(\frac{1}{\log(2j+1)}\right)^{2k+1} (-1)^k}{(2k+1)!} \end{aligned}$$

$$\begin{aligned} & \sin\left(\frac{1}{\log 2j}\right) - \sin\left(\frac{1}{\log(2j+1)}\right) = \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \left( \left(\frac{1}{\log 2j}\right)^{2k+1} - \left(\frac{1}{\log(2j+1)}\right)^{2k+1} \right) \end{aligned}$$

$$0 < \log 2j < \log(2j+1) \text{ so } \left(\frac{1}{\log 2j}\right)^{2k+1} - \left(\frac{1}{\log(2j+1)}\right)^{2k+1} > 0$$

and since for  $j > 1$ ,  $\left(\frac{1}{\log 2j}\right)$  and  $\left(\frac{1}{\log(2j+1)}\right)$  are  $< 1$  and so we are adding smaller and smaller amounts

$$\begin{aligned} & < \frac{1}{\log 2j} - \frac{\log(2j+1)}{\log 2j} = \\ &= \frac{\log(2j+1) - \log 2j}{\log 2j \log(2j+1)} \leq \frac{\log\left(1 + \frac{1}{2j}\right)}{(\log(2j))^2} \leq \frac{\frac{1}{2j}}{(\log(2j))^2} \\ & \int_1^n \frac{1}{2j(\log 2j)^2} = -\frac{1}{\log 2j} \Big|_1^n \xrightarrow{n \rightarrow \infty} \frac{1}{\log 2} \end{aligned}$$

So the series converges by using integral test, showing that  $\sum \frac{1}{2j(\log 2j)^2}$  converges, so by comparison test, the series converges.

**Exercise 33.**  $\sum_{n=1}^{\infty} n^n z^n$

$$\begin{aligned} \left| \sum_{n=1}^{\infty} n^n z^n \right| &= \left| \sum_{j=1}^{\infty} (jz)^j \right| = \sum (e^{\ln j} z)^j \\ &= \sum_{j=1}^n (e^{\ln j} z)^j = \\ &= \frac{e^{\ln j} z - (e^{\ln j} z)^{n+1}}{1 - e^{\ln j} z} \rightarrow \infty \end{aligned}$$

So  $z = 0$

**Exercise 34.**  $\sum_{j=1}^{\infty} \frac{(-1)^j z^{3j}}{j!} = \sum_{j=1}^{\infty} \frac{(-z^3)^j}{j!} = e^{-z^3} \cdot \boxed{\mathbb{C}}$ .

**Exercise 35.**  $\sum_{j=0}^{\infty} \frac{z^j}{3^j} = \sum_{j=0}^{\infty} \left(\frac{1}{3}\right)^j z^j$   
 $\sum z^j$  be convergent or  $\sum_{j=1}^n z^j$  bounded.

$\boxed{|z| < 3}$  and  $|z| = 3$  if  $z \neq 3$

**Exercise 36.**  $\sum_{j=1}^{\infty} \frac{z^j}{j^j} \boxed{\{z\} = \mathbb{C}}$  since

$$\left| \left(\frac{z}{j}\right) \right| < 1 \text{ for } j \geq N > |z|$$

**Exercise 37.**  $\sum_{j=1}^{\infty} \frac{(-1)^j}{z+j}$

By Leibniz's Rule, since  $\frac{1}{z+j} \xrightarrow{j \rightarrow \infty} 0$ , then the series converges. However,  $z$  cannot be equal to any negative integer since one term in the series will then blow up.

**Exercise 38.**  $\sum_{j=1}^{\infty} \frac{z^j}{\sqrt{j}} \log \left( \frac{2j+1}{j} \right)$ .

$$\frac{z^j}{\sqrt{j}} \log \left( 2 + \frac{1}{j} \right) = z^j \frac{\log \left( 2 + \frac{1}{j} \right)}{\sqrt{j}}$$

Since  $\lim_{j \rightarrow \infty} \frac{\log \left( 2 + \frac{1}{j} \right)}{\sqrt{j}} = 0$  so that  $\frac{\log \left( 2 + \frac{1}{j} \right)}{\sqrt{j}}$  is a monotonically convergent sequence.

Then by Dirichlet's test,  $\sum_{j=1}^n z^j$  must be bounded.  $|z| > 1$ , and  $|z| = 1$  if  $z \neq 1$ .

**Exercise 39.**  $\sum_{j=1}^{\infty} \left( 1 + \frac{1}{5j+1} \right)^{j^2} |z|^{17j} = \sum_{j=1}^{\infty} \left( 1 + \frac{1}{5j+1} \right)^{j^2} (|z|^{17})^j$

$$\begin{aligned} & \left( \left( 1 + \frac{1}{5j+1} \right)^j |z|^{17} \right)^j \\ \lim_{j \rightarrow \infty} \left( 1 + \frac{1}{5j+1} \right)^j & \leq \lim_{j \rightarrow \infty} \left( 1 + \frac{1/5}{j} \right)^j = e^{1/5} \\ e^{1/5} |z|^{17} < 1 & \implies |z| < e^{-1/85} \end{aligned}$$

**Exercise 40.**  $\sum_{j=0}^{\infty} \frac{(z-1)^j}{(j+2)!}$

$$\left| \sum_{j=0}^{\infty} \frac{(z-1)^j}{(j+2)!} \right| \leq \sum_{j=0}^{\infty} \frac{|(z-1)|^j}{(j+2)!} \leq \sum_{j=0}^{\infty} \frac{|(z-1)|^{j+2}}{(j+2)!} = \boxed{e^{|z-1|} - 1 - \frac{|z-1|}{1!}}$$

The series converges  $\forall z$ .

**Exercise 41.**

$$\sum_{j=1}^{\infty} \frac{(-1)^j (z-1)^j}{j} = \sum_{j=1}^{\infty} \frac{(1-z)^j}{j} = \log(1 - (1-z)) = \boxed{\log z}$$

So the series converges  $\forall z$  except for  $z = 0$ .

**Exercise 42.**  $\sum_{j=1}^{\infty} \frac{(2z+3)^j}{j \log(j+1)}$

$$\begin{aligned} \lim_{j \rightarrow \infty} \frac{1}{j \log(j+1)} &= 0 \text{ so } \frac{1}{j \log(j+1)} \text{ is a monotonically convergent sequence} \\ |2z+3| &< 1 \\ \sum (2z+3)^j & \text{ then } \sum (2z+3)^j \text{ converges} \\ |z + \frac{3}{2}| &< \frac{1}{2} \\ \left( \frac{1}{x \log(x+1)} \right)' &= \frac{-1}{(x \log(x+1))^2} \left( \log(x+1) + \frac{x}{x+1} \right) < 0; \quad \text{for } x > 0 \end{aligned}$$

By Dirichlet's test,  $\sum_{j=1}^{\infty} \frac{(2z+3)^j}{j \log(j+1)}$  converges for  $\boxed{\left| z + \frac{3}{2} \right| \leq \frac{1}{2}; \quad z \neq -1}$ .

**Exercise 43.**  $\sum_{j=1}^{\infty} \frac{(-1)^j}{(2j-1)} \left( \frac{1-z}{1+z} \right)^j = \sum_{j=1}^{\infty} \frac{(-1)^{j/2}}{j-\frac{1}{2}} \left( \frac{1-z}{1+z} \right)^j$

$\implies \sum_{j=1}^{\infty} \frac{\left( \frac{z-1}{z+1} \right)^j}{2j-1} = \lim_{j \rightarrow \infty} \frac{1}{2j-1} = 0$  and  $\frac{1}{2j-1}$  is monotonically decreasing.

So  $\frac{1}{2j-1}$  is a monotonically decreasing convergent sequence of real terms.

For  $\left| \frac{z-1}{z+1} \right| < 1$ ,

$$\begin{aligned} \left| \frac{z-1}{z+1} \right| < 1 & \implies |z-1| < |z+1| \\ (u-1)^2 + v^2 &< (u+1)^2 + v^2 \\ u^2 - 2u + 1 &< u^2 + 2u + 1 & \boxed{\Re(z) > 0; \quad z \neq 0} \\ -u &< u \end{aligned}$$

**Exercise 44.**

$$\sum_{j=1}^{\infty} \left( \frac{z}{2z+1} \right)^j = \sum_{j=1}^{\infty} \left( \frac{1}{2} + \frac{-1/2}{2z+1} \right)^j = \sum_{j=1}^{\infty} \left( \frac{1}{2} \right)^j \left( 1 - \frac{1}{2z+1} \right)^j$$

$\left( \frac{1}{2} \right)^j$  is a monotonically decreasing, convergent sequence

Now we want  $\left| 1 - \frac{1}{2z+1} \right| = \left| \frac{2z+1-1}{2z+1} \right| = \left| \frac{2z}{2z+1} \right| < 1$

$$\begin{aligned} |2z| &< |2z+1| && \text{if } \Re(z) = -\frac{1}{4} \\ |z| &< |z+1/2| \\ u^2 + v^2 &< (u+1/2)^2 + v^2 = u^2 + u + 1/4 + v^2 && \frac{2(-\frac{1}{4} + iv)}{2(-\frac{1}{4} + iv) + 1} = \frac{-\frac{1}{2} + 2iv}{\frac{1}{2} + 2iv} \\ \frac{-1}{4} < u &\implies \boxed{\Re(z) > -\frac{1}{4}} && \implies \boxed{z \neq -\frac{1}{4}} \end{aligned}$$

**Exercise 45.**  $\sum_{j=1}^{\infty} \frac{j}{j+1} \left( \frac{z}{2z+1} \right)^j$ .

$$\left( \frac{x}{x+1} \right)' = \frac{(x+1) - x}{(x+1)^2} = \frac{1}{(x+1)^2} > 0$$

$\frac{j}{j+1}$  is a monotonically increasing and convergent sequence

$$\begin{aligned} \frac{1}{2} + \frac{-1}{2(2z+1)} &= \left| \frac{z}{2z+1} \right| < 1 \\ \left| \frac{2z+1}{z} \right| > 1 &\implies \boxed{\left| 2 + \frac{1}{z} \right| > 1} \end{aligned}$$

**Exercise 46.**  $\sum_{j=1}^{\infty} \frac{1}{(1+|z|^2)^j} = \sum_{j=1}^{\infty} \left( \frac{1}{1+|z|^2} \right)^j$

$$\begin{aligned} \frac{1}{1+|z|^2} &< 1 \\ 1 &< 1 + |z|^2 \quad \forall z \text{ except } z = 0 \\ 0 &< |z|^2 \end{aligned}$$

**Exercise 47.**  $\sum_{j=1}^{\infty} (-1)^j \frac{2^j \sin^{2j} x}{j}$

Use Dirichlet's Test.  $\frac{1}{j}$  is a monotonically decreasing sequence converging to zero. Consider  $(-2)^j \sin^{2j} x$ . The condition for convergence is

$$\begin{aligned} |(-2 \sin^2 x)| &< 1 \quad \text{for } j \geq N \text{ for some } N \\ \implies x &\in \left( -\frac{\pi}{4} + n\pi, \frac{\pi}{4} + n\pi \right), \quad n \in \mathbb{Z} \end{aligned}$$

**Exercise 49.**  $\sum a_j$  converges.

$\sum a_j \frac{1}{a_j}$  diverges.

Then since  $\sum a_j$  is a convergent series (by Abel's test),  $\frac{1}{a_j}$  is a divergent sequence.

Then  $\sum \frac{1}{a_j}$  is divergent (since  $\lim_{j \rightarrow \infty} \frac{1}{a_j}$  doesn't exist).

**Exercise 50.**  $\sum |a_j|$  converges.

$\sum |a_j|$  converges, then  $\sum a_j$  converges.

$|a_j|^2 = a_j^2$ .  $|a_j|$  converges, then

$$\begin{aligned} \lim_{j \rightarrow \infty} |a_j| &= 0 \\ \lim_{j \rightarrow \infty} |a_j|^2 &= 0 \quad \lim_{j \rightarrow \infty} \frac{|a_j|^2}{a_j^2} = 1 \end{aligned}$$

By limit comparison test,  $\sum a_j^2$  converges.

Counterexample:  $\sum \left( \frac{1}{j} \right)^2$  converges, but  $\sum \frac{1}{j}$  diverges.

**Exercise 51.** Given  $\sum a_j$ ,  $a_j \geq 0$ .  $\sum a_j$  converges.

$$\begin{aligned} \lim_{j \rightarrow \infty} a_j &= 0 \\ \sum \sqrt{a_j} \frac{1}{(j)^p} & \quad \lim_{j \rightarrow \infty} \sqrt{a_j} = \left( \lim_{j \rightarrow \infty} a_j \right)^{1/2} = 0 \\ \int \sum_{j=0}^{n-1} x^j &= \sum_{j=0}^{n-1} \frac{x^{j+1}}{j+1} = \sum_{j=1}^n \frac{x^j}{j} = \int \frac{1-x^n}{1-x} \end{aligned}$$

A counterexample would be  $\frac{\sqrt{a_j}}{j^p} = \sqrt{\frac{a_j}{j}}$ .

**Exercise 52.**

(1)  $\sum a_j$  converges absolutely, then if  $\sum |a_j|$  converges,  $\sum a_j^2$  converges.

$$\begin{aligned} \frac{a_j^2}{1+a_j^2} &= 1 + \frac{-1}{1+a_j^2} \\ \frac{a_j^2}{1+a_j^2} &\leq a_j^2 \text{ since } \sum a_j^2 \text{ converges, } \sum \frac{a_j^2}{1+a_j^2} \text{ converges} \end{aligned}$$

(2)  $\sum a_j$  converges absolutely,  $\lim_{j \rightarrow \infty} |a_j| = 0$

$$\sum \left| \frac{a_j}{1+a_j} \right| = \sum |a_j| \left( \frac{1}{|1+a_j|} \right).$$

By Abel's test, since

$$\lim_{j \rightarrow \infty} \frac{1}{|1+a_j|} = \frac{1}{|1+\lim_{j \rightarrow \infty} a_j|} = 1 \quad \text{shows that } \frac{1}{|1+a_j|} \geq 0 \text{ is a monotonically convergent sequence}$$

By Abel's test,  $\sum \frac{a_j}{1+a_j}$  is convergent.

## 10.22 Miscellaneous review exercises - Rearrangements of series. Exercise 1.

(1)

$$\begin{aligned} a_j &= \sqrt{j+1} - \sqrt{j} = \sqrt{j} \sqrt{1 + \frac{1}{j}} - \sqrt{j} = \sqrt{j} \left( 1 + \frac{1}{2} \left( \frac{1}{j} \right) + o\left( \frac{1}{j} \right) + -1 \right) = \\ &= \sqrt{j} \left( \frac{1}{2} \left( \frac{1}{j} \right) + o\left( \frac{1}{j} \right) \right) \\ \lim_{j \rightarrow \infty} a_j &= 0 \end{aligned}$$

(2)

$$\begin{aligned} a_j &= (j+1)^c - j^c = j^c \left( \left( 1 + \frac{1}{j} \right)^c - 1 \right) = (j^c) \left( 1 + c \left( \frac{1}{j} \right) + o\left( \frac{1}{j} \right) - 1 \right) = \\ &= j^c \left( c \left( \frac{1}{j} \right) + o\left( \frac{1}{j} \right) \right) = \left( c j^{c-1} + j^c o\left( \frac{1}{j} \right) \right) \end{aligned}$$

if $c > 1$ , $a_j$ diverges if $c = 1$ , $\lim_{j \rightarrow \infty} a_j = 1$ if $c < 1$ , $\lim_{j \rightarrow \infty} a_j = 0$
---

**Exercise 2.**

(1)

$$(1+x^n)^{\frac{1}{n}} = \exp \left( \frac{1}{n} \ln(1+x^n) \right) = \exp \left( \frac{1}{n} \sum_{j=1}^{\infty} \frac{(x^n)^j (-1)^{j-1}}{j} \right) \xrightarrow{n \rightarrow \infty} \boxed{1}$$

$$(2) \lim_{n \rightarrow \infty} (a^n + b^n)^{1/n} = \lim_{n \rightarrow \infty} a \left( 1 + \left( \frac{b}{a} \right)^n \right)^{1/n} = a \text{ if } a > b.$$

**Exercise 3.**  $a_{n+1} = \frac{a_n + a_{n-1}}{2} = \frac{a_{n-1} + a_{n-2}}{2^2} + \frac{a_{n-2} + a_{n-3}}{2^2}$

## 10.24 Exercises - Improper integrals.

**Exercise 1.**  $\int_0^\infty \frac{x}{\sqrt{x^4+1}} dx$

$$\lim_{x \rightarrow \infty} \left( \frac{x}{\sqrt{x^4+1}} \right) \left( \frac{1}{1/x} \right) = \lim_{x \rightarrow \infty} \frac{x^2}{x^2 \sqrt{1+1/x^4}} = 1$$

Since  $\int_1^\infty \frac{1}{x}$  diverges, so does  $\int_0^\infty \frac{x}{\sqrt{x^4+1}} dx$

**Exercise 2.**

$$\begin{aligned} \int_{-\infty}^\infty e^{-x^2} dx &= \int_0^\infty e^{-x^2} dx + \int_0^{-\infty} e^{-x^2} dx = \int_0^\infty e^{-x^2} dx + - \int_\infty^0 e^{-x^2} dx = 2 \int_0^\infty e^{-x^2} dx \\ \int_0^\infty e^{-x^2} dx &\leq \int_0^\infty e^{-x} dx = -e^{-x} \Big|_0^\infty = \boxed{1} \end{aligned}$$

Converges by theorem.

**Exercise 3.**  $\int_0^\infty \frac{1}{\sqrt{x^3+1}} dx$

**Exercise 4.**  $\int_0^\infty \frac{1}{\sqrt{e^x}} dx$

**Exercise 5.**  $\int_{0+}^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}} dx$

**Exercise 6.**  $\int_{0+}^1 \frac{\log x}{\sqrt{x}} dx$

**Exercise 7.**  $\int_{0+}^{1-} \frac{\log x}{1-x} dx$

**Exercise 8.**  $\int_{-\infty}^\infty \frac{x}{\cosh x} dx$

**Exercise 9.**  $\int_{0+}^{1-} \frac{dx}{\sqrt{x} \log x}$

**Exercise 10.**  $\int_0^\infty \frac{dx}{x(\log x)^s}$

**11.7 Exercises - Pointwise convergence of sequences of functions, Uniform convergence of sequences of functions, Uniform convergence of sequences of functions, Uniform convergence and continuity, Uniform convergence and integration, A sufficient condition for uniform convergence, Power series. Circle of convergence. Exercise 1.**  $\sum_{j=0}^\infty \frac{z^j}{2^j} = \sum_{j=0}^\infty \left(\frac{z}{2}\right)^j$

Using the *comparison test*,

$$\left|\frac{z}{2}\right|^j \leq t^j; \quad \left|\frac{z}{2}\right| < 1 \implies |z| < 2$$

$$\text{Suppose } |z| = 2, z \neq 2 \quad \sum_{j=0}^\infty \left(\frac{z}{2}\right)^j = \sum_{j=0}^\infty (e^{2i\theta})^j$$

Now  $\sum_{j=0}^n (e^{2i\theta})^j \leq \frac{1}{\sin \theta} + 1$ , since

$$\sum_{j=1}^n (e^{2i\theta})^j = \frac{e^{2i\theta} - e^{2i\theta(n+1)}}{1 - e^{2i\theta}} = \frac{e^{in\theta} - e^{-i\theta n} e^{i\theta n + i\theta}}{-e^{-i\theta} + e^{i\theta}} = \frac{\sin n\theta e^{i\theta n + i\theta}}{\sin \theta} < \frac{1}{\sin \theta}$$

So  $\sum_{j=0}^\infty \left(\frac{z}{2}\right)^j$  converges for  $|z| = 2, z \neq 2$

**Exercise 2.**  $\sum_{j=0}^\infty \frac{z^j}{(j+1)2^j}$

Use *ratio test*.

$$\frac{a_{j+1}}{a_j} = \frac{z^{j+1}}{(j+2)2^{j+1}} \frac{(j+1)2^j}{z^j} = \frac{z(j+1)}{2(j+2)} = \frac{z(1+1/j)}{2(1+2/j)} \xrightarrow{j \rightarrow \infty} \frac{z}{2}$$

If  $|z| < 2$ ,  $\sum_{j=0}^\infty a_j$  converges, if  $|z| > 2$ ,  $\sum a_j$  diverges.

$$\text{If } |z| = 2, \sum \frac{z^j}{(j+1)2^j} = \sum (e^{2i\theta})^j \left(\frac{1}{j+1}\right)$$

Now  $\frac{1}{j+1}$  is a monotonically decreasing sequence of real terms.

$\sum (e^{2i\theta})^j$  is a bounded series.

By Dirichlet's test,  $\sum a_j$  converges if  $|z| = 2, z \neq 2$

**Exercise 3.**  $\sum_{j=0}^\infty \frac{(z+3)^j}{(j+1)2^j}$

Use ratio test:

$$\frac{a_{j+1}}{a_j} = \left( \frac{(z+3)^{j+1}}{(j+2)2^{j+1}} \right) \frac{(j+1)2^j}{(z+3)^j} = \frac{(z+3)}{2} \left( \frac{j+1}{j+2} \right) \xrightarrow{j \rightarrow \infty} \frac{z+3}{2}$$

Using Theorem 11.7,

**Theorem 34** (Existence of a circle of convergence).

$$\begin{aligned} & \text{Assume } \sum a_j z^j \begin{array}{l} \text{converges for at least } z_1 \neq 0 \\ \text{diverges for at least one } z_2 \neq 0 \end{array} \\ & \exists r > 0, \quad \text{such that } \sum a_j z^j \begin{array}{l} \text{absolutely converges for } |z| < r \\ \text{diverges for } |z| > r \end{array} \end{aligned}$$

We can plug in real numbers to satisfy the condition  $\frac{|z+3|}{2} < 1$  for convergence.  
 $\sum a_j$  converges for  $|z+3| < 2$ ; diverges for  $|z+3| > 2$ .

Consider  $|z+3| = 2$ ;  $z \neq 1$   $\sum a_j = \sum (e^{2i\theta})^j \left(\frac{1}{j+1}\right)$ . Since  $\frac{1}{j+1}$  is a monotonically decreasing sequence of real numbers and  $\sum (e^{2i\theta})^j$  is a bounded series, by Dirichlet's test,  $\sum a_j$  converges for  $|z+3| = 2$ ;  $z \neq -1$ .

**Exercise 4.**  $\sum_{j=1}^{\infty} \frac{(-1)^j 2^{2j} z^{2j}}{2j} = - \sum_{j=1}^{\infty} \frac{(-1)^{j-1} (2z)^{2j}}{(2j)}$ . Look at what the terms look like.

Consider using Leibniz's Rule, Theorem 10.14.

**Theorem 35** (Leibniz's rule). If  $a_j$  is a monotonic decreasing sequence and  $\lim_{j \rightarrow \infty} a_j = 0$ , then  $\sum_{j=1}^{\infty} (-1)^{j-1} a_j$  converges.

$$\begin{aligned} & \frac{(-1)^j 2^{2j} z^{2j}}{2j} = \frac{(-1)^j (2z)^{2j}}{2j} \\ \text{Consider } \left| \frac{(2z)^{2j}}{2j} \right| &= \frac{|2z|^{2j}}{2j} = \frac{(2|z|)^{2j}}{2j} \\ \text{Consider } 2|z| &= M^{1/2} < \infty \\ \implies \frac{(2|z|)^{2j}}{2j} &= \frac{M^j}{2j} = \frac{e^{j \ln M}}{2j} \end{aligned}$$

$$\text{converges for } 0 < 2|z| = M \leq 1 \implies |z| \leq \frac{1}{2}$$

(we use Theorem 11.6 at this point, because real numbers are included in complex numbers).

**Theorem 36.** Assume  $\sum a_j z^j$  converges for some  $z = z_1 \neq 0$ .

- (1)  $\sum a_j z^j$  converges absolutely  $\forall z$  with  $|z| < |z_1|$ .
- (2)  $\sum a_j z^j$  converges uniformly on every circular disk with center at 0 and  $R < |z_1|$

We had first used Leibniz's test to find  $az_1$  on the real line.

$$\frac{2^{4j} z^{4j} (4j(1 - \frac{1}{4z^2}) - 2)}{4j(4j - 2)} = \frac{2^{4j} z^{4j} ((1 - \frac{1}{4z^2}) - \frac{2}{4j})}{(4j - 2)} \xrightarrow{j \rightarrow \infty} \frac{2^{4j} z^{4j} (1 - \frac{1}{4z^2})}{(4j - 2)}$$

If  $|z| > \frac{1}{2}$ , the series diverges (by  $a_j$ th general term test).

**Exercise 5.**  $\sum_{j=1}^{\infty} (1 - (-2)^j) z^j$ .

**Try limit comparison test .**

**The first step** is to test *absolute convergence* first; it's easier.

$$\frac{|(1 - (-2)^j) z^j|}{|(-2z)^j|} = \left| \left( \frac{1}{-2} \right)^j - 1 \right| \xrightarrow{j \rightarrow \infty} 1$$

According to limit comparison test, for  $\sum (1 - (-2)^j) z^j$  to converge,  $\sum (-2z)^j$  must converge.  
 So if  $|z| < \frac{1}{2}$ , then  $\sum_{j=1}^{\infty} (1 - (-2)^j) z^j$  absolutely converges.

If  $|z| = \frac{1}{2}$ ,  $z \neq \frac{-1}{2}$ ,

$$\begin{aligned} \sum (1 - (-2)^j) z^j &= \sum z^j - \sum (-e^{2i\theta})^j = \\ &= 0 - \sum (e^{2i\theta + \pi i})^j < \frac{1}{\sin(\theta + \pi)} \end{aligned}$$

If  $z = \frac{-1}{2}$ ,

$$\sum \left( -\frac{1}{2} \right)^j - \sum 1^j \rightarrow \infty$$



**Exercise 6.**  $\sum_{j=1}^{\infty} \frac{j!z^j}{j^j}$

A **very big hint** is to use Exercise 19 on pp. 399, in the section for Exercises 10.14.

Since  $\sum_{j=1}^{n-1} f(j) \leq \int_1^n f(x)dx$

$$\sum_{j=1}^{n-1} \ln j \leq \int_1^n \ln x = n \ln n - n + 1 \leq \sum_{j=2}^n \ln j$$

$$(n-1)! \leq n^n e^{-n} n \leq n!$$

$$\frac{n!}{n^n} \geq n e^{-n}$$

$$\lim_{n \rightarrow \infty} \frac{n!}{n^n} \geq \lim_{n \rightarrow \infty} n e^{-n} = 0$$

$$\lim_{n \rightarrow \infty} \frac{n!}{n^n} \leq \lim_{n \rightarrow \infty} \frac{n^2}{e^n} = 0$$

$$\Rightarrow \boxed{\lim_{n \rightarrow \infty} \frac{n!}{n^n} = 0}$$

So then since  $\frac{n!}{n^n}$  is a monotonically decreasing convergent sequence of real terms; if  $\sum z$  is a bounded series, then by Dirichlet's test,  $\sum \frac{n!z^n}{n^n}$  is convergent.

$|z| < 1$ ;  $|z| = 1$  if  $z \neq 1$

Try the ratio test, because it's clear from the results of the ratio test where convergence and divergence begins and ends.

$$\frac{(n+1)!|z|^{n+1}}{(n+1)^{n+1}} \frac{n^n}{n!|z|^n} = \left(\frac{n}{n+1}\right)^n |z| =$$

$$= \left(\frac{1}{1+1/n}\right)^n |z| \xrightarrow{n \rightarrow \infty} \frac{1}{e} |z|$$

Converges for  $\boxed{|z| < e}$ .

Now try plugging in a complexified  $e$ :

$$\frac{e^j}{\left(\frac{j^j}{j!}\right)} = \frac{j!e^j e^{i2\theta j}}{(j)^j} \geq \frac{j^j e^{-j} j e^j e^{i2\theta j}}{j^j} e^{i2\theta j} \rightarrow \infty$$

So the series diverges for  $|z| = e$ .

**Exercise 7.**  $\sum_{j=0}^{\infty} \frac{(-1)^j (z+1)^j}{j^2+1}$

By the ratio test,

$$\frac{|z+1|^{j+1}}{j^2+2j+2} \frac{j^2+1}{|z+1|^j} = |z+1| \frac{1+1/j^2}{1+2/j+2/j^2} \xrightarrow{j \rightarrow \infty} |z+1|$$

The series absolutely converges for  $|z+1| < 1$ .

For  $z = 0$ ,  $\sum \frac{(-1)^j}{j^2+1}$  converges since  $\frac{1}{j^2+1} \xrightarrow{j \rightarrow \infty} 0$

For  $z = -2$ ,  $\sum \frac{(-1)^j (-1)^j}{j^2+1} = \sum \frac{1}{j^2+1}$  and  $\frac{1}{j^2+1} < \frac{1}{j^2}$ , but  $\sum \frac{1}{j^2}$  is convergent (by integral test). So the series converges for  $z = -2$ .

By Dirichlet's test, the series converges as well, if we treat  $a_j = (-1)^j (z+1)^j$  and  $b_j = \frac{1}{j^2+1}$  to be the monotonically decreasing sequence.

$\Rightarrow \boxed{|z+1| \leq 1}$  for convergence x

**Exercise 8.**  $\sum_{j=0}^{\infty} a^{j^2} z^j$ ,  $0 < a < 1$

Use the *root test*.

$$(a^{j^2} z^j)^{1/j} = a^j z \xrightarrow{j \rightarrow \infty} 0$$

So the series converges  $\forall z \in \mathbb{C}$

**Exercise 9.**  $\sum_{j=1}^{\infty} \frac{(j!)^2}{(2j)!} z^j$

Use the *ratio test*

$$\frac{a_{j+1}}{a_j} = \frac{((j+1)!)^2 z^{j+1}}{(2(j+1))!} \frac{(2j)!}{(j!)^2 z^j} = \frac{(j+1)^2 z}{(2j+2)(2j+1)} \xrightarrow{j \rightarrow \infty} \frac{1(z)}{4}$$

for  $|z| < 4$ ,  $\sum a_j$  absolutely converges

for  $|z| > 4$ ,  $\sum a_j$  diverges

Let's test the boundary for convergence.

$$\begin{aligned} \frac{(j!)^2}{(2j)!} 4^j &= \frac{(j!)^2}{(2j)!} e^{j \ln 4} = \frac{(j!)^2}{(2j-2)! (2j)(2j-1)} \frac{e^{j \ln 4}}{(2j-2)!} \geq \frac{(j!)^2}{(2j-2)!} \frac{e^{j \ln 4}}{(2j)^2} \geq \\ &\geq \frac{(j!)^2}{(2j-2)!} \frac{j^{2j}}{(j!)^2} = \frac{j^{2j}}{(2j-2)!} \rightarrow \infty \end{aligned}$$

where we had used

$$(n-1)! \leq n^n e^{-n} n \leq n! \implies n e^{-n} \leq \frac{n!}{n^n}$$

$$\frac{n}{e^n} \leq \frac{n!}{n^n} \implies \frac{n^n}{n!} \leq \frac{e^n}{n}$$

**Exercise 10.**  $\sum_{j=1}^{\infty} \frac{3^{\sqrt{j}} z^j}{j} = \sum_{j=1}^{\infty} \frac{e^{\sqrt{j} \ln 3} z^j}{j}$

$$\begin{aligned} \frac{e^{\sqrt{j+1} \ln 3} z^{j+1}}{j+1} \frac{j}{e^{\sqrt{j} \ln 3} z^j} &= \left( \frac{1}{j+1} \right) e^{(\sqrt{j+1} - \sqrt{j}) \ln 3} z \\ \sqrt{j+1} - \sqrt{j} &= \sqrt{1 + \frac{1}{j}} - 1 \simeq 1 + \frac{1}{2} \left( \frac{1}{j} \right) - 1 = \frac{1}{2j} \quad (\text{for large } j) \\ (\text{for large } j) \quad &\left( \frac{j}{j+1} \right) e^{\frac{1}{2j}} z \rightarrow 0 \\ &\implies \text{Converges} \quad \forall z \in \mathbb{C} \end{aligned}$$

**Exercise 11.**  $\sum_{j=1}^{\infty} \left( \frac{1 \cdot 3 \cdot 5 \dots (2j-1)}{2 \cdot 4 \cdot 6 \dots (2j)} \right)^3 z^j$

$$\frac{a_{j+1}}{a_j} = \left( \frac{1 \cdot 3 \cdot 5 \dots (2j+1)}{2 \cdot 4 \cdot 6 \dots (2j+2)} \right)^3 z^{j+1} \left( \frac{2 \cdot 4 \cdot 6 \dots (2j)}{1 \cdot 3 \cdot 5 \dots (2j-1)} \right)^3 \frac{1}{z^j} = \left( \frac{2j+1}{2j+2} \right)^3 z = \left( 1 + \frac{-1/2}{j+2} \right)^3 z$$

If  $|z| < 1$ , it converges by ratio test, if  $|z| = 1$ , then it converges by Gauss test

$$\begin{aligned} \frac{a_{j+1}}{a_j} &= \left( 1 + \frac{-1/2}{j+2} \right)^3 z = \sum_{k=0}^3 \binom{3}{k} \left( \frac{-1/2}{j+2} \right)^k |z| = \\ &= |z| \left( 1 + 3 \left( \frac{-1/2}{j+2} \right) + 3 \left( \frac{-1/2}{j+2} \right)^2 + \left( \frac{-1/2}{j+2} \right)^3 \right) \xrightarrow{j \rightarrow \infty} |z| \end{aligned}$$

diverges for  $|z| > 1$  (by ratio test)

**Exercise 12.**  $\sum_{j=1}^{\infty} \left( 1 + \frac{1}{j} \right)^{j^2} z^j$

$$\begin{aligned} \left( \left( 1 + \frac{1}{j} \right)^{j^2} z^j \right)^{1/j} &= \left( 1 + \frac{1}{j} \right)^j z \xrightarrow{j \rightarrow \infty} e^1 z \\ |z| < \frac{1}{e}, \quad &\sum \left( 1 + \frac{1}{j} \right)^{j^2} z^j \text{ converges by root test} \\ |z| > \frac{1}{e}, \quad &\sum \left( 1 + \frac{1}{j} \right)^{j^2} z^j \text{ diverges by root test} \end{aligned}$$

$$\boxed{\frac{1}{e} = r}$$

**Exercise 13.**  $\sum_{j=0}^{\infty} (\sin aj) z^j \quad a > 0$

$$|\sin(aj) z^j| \leq |z|^j$$

By comparison test,  $\sum_{j=0}^{\infty} (\sin aj) z^j$  converges, since  $\sum_{j=0}^{\infty} |z|^j$  converges absolutely, for  $|z| < 1$ .

Note that if  $a = \pi$ , the series is zero.

$\sum_{j=0}^{\infty} (\sin aj) \rightarrow \infty$  for  $a = 2\pi$  so r=1 indeed.

**Exercise 14.**  $\sum_{j=0}^{\infty} (\sinh aj) z^j = \sum_{j=0}^{\infty} \left( \frac{e^{aj} - e^{-aj}}{2} \right) z^j = \frac{1}{2} \left( \sum_{j=0}^{\infty} (e^a z)^j - \sum_{j=0}^{\infty} \left( \frac{z}{e} \right)^j \right); \quad a > 0$

If  $|z| < \frac{1}{e^a}$ , then  $\sum \sinh aj z^j$  converges. So then the radius of convergence is  $r = \frac{1}{e^a}$

**Exercise 15.**  $\sum_{j=1}^{\infty} \frac{z^j}{a^j + b^j}$ . Assume  $a < b$

$$\text{(ratio test)} \quad \frac{\frac{z^{j+1}}{b^{j+1} \left( 1 + \left( \frac{a}{b} \right)^{j+1} \right)}}{\frac{z^j}{b^j \left( 1 + \left( \frac{a}{b} \right)^j \right)}} = \frac{z}{b} \left( \frac{1 + \left( \frac{a}{b} \right)^j}{1 + \left( \frac{a}{b} \right)^{j+1}} \right) \xrightarrow{j \rightarrow \infty} \left( \frac{z}{b} \right)$$

So then  $|z| \leq b$  converges (diverges) by ratio test.

If  $a = b$ ,

$$\sum_{j=1}^{\infty} \frac{z^j}{2a^j} = \frac{1}{2} \sum_{j=1}^{\infty} \left( \frac{z}{a} \right)^j$$

By comparison test with  $x^j$ , if  $|z| \leq |a|$ , the series converges (diverges).

**Exercise 16.**  $\sum_{j=1}^{\infty} \left( \frac{a^j}{j} + \frac{b^j}{j^2} \right) z^j$  Use ratio test on each of the sums, separately.

$$\begin{aligned} \frac{(a|z|)^{j+1}}{j+1} \frac{j}{(a|z|)^j} &= \frac{a|z|}{1 + \frac{1}{j}} \xrightarrow{j \rightarrow \infty} a|z| \\ \implies |z| < \frac{1}{a} &\text{ then the series converges} \\ \frac{(b|z|)^{j+1}}{(j+1)^2} \frac{j^2}{(b|z|)^j} &= b|z| \left( \frac{1}{1 + \frac{1}{j}} \right)^2 \xrightarrow{j \rightarrow \infty} b|z| \\ \implies |z| < \frac{1}{b} \end{aligned}$$

So if  $a \geq b$ , then  $r = a; (b)$

**Exercise 17.**  $\int_0^1 f_n(x) = \int_0^1 nxe^{-nx^2} = \left. \frac{e^{-nx^2}}{-2} \right|_0^1 = \frac{e^{-n}-1}{-2} \xrightarrow{n \rightarrow \infty} \frac{1}{2}$

However,

$$\lim_{n \rightarrow \infty} nxe^{-nx^2} = 0$$

This example shows that the operations of integration and limit cannot always be interchanged. We need uniform convergence. **Exercise 18.**  $f_n(a) = \frac{\sin nx}{n} \quad \lim_{n \rightarrow \infty} \frac{\sin nx}{n} = 0$

$$f(x) = \lim_{n \rightarrow \infty} f_n(x)$$

$$f'_n(x) = \frac{n \cos nx}{n} \implies \lim_{n \rightarrow \infty} f'_n(0) = 1$$

This example shows that differentiation and limit cannot always be interchanged.

**Exercise 19.**  $\sum_{j=1}^{\infty} \frac{\sin jx}{j^2} = f(x)$

$$\begin{aligned} \frac{\sin jx}{j^2} &\leq \frac{1}{j^2} \quad \forall x \in \mathbb{R} \\ \text{by comparison test, } \sum_{j=1}^{\infty} \frac{|\sin jx|}{j^2} &\text{ converges, so } \sum_{j=1}^{\infty} \frac{\sin jx}{j^2} \text{ converges} \\ \left| \frac{\sin jx}{j^2} \right| &\leq \frac{1}{N^2} \quad \forall x \in \mathbb{R}; \quad \forall j \geq N \end{aligned}$$

$\sum N \frac{1}{N^2}$  converges, so  $\sum \frac{\sin jx}{j^2}$  uniformly converges.

Then by Thm., since  $\frac{\sin jx}{j^2}$  are continuous,  $\sum \frac{\sin jx}{j^2}$  is continuous.

Since  $\sum \frac{\sin jx}{j^2}$  uniformly converges.

$$\begin{aligned} \int_0^\pi \sum \frac{\sin jx}{j^2} &= \sum \int_0^\pi \frac{\sin jx}{j^2} = \sum \left. \frac{\cos jx}{-j^3} \right|_0^\pi = \sum \frac{(-1)^j - 1}{-j^3} = \\ &= \boxed{2 \sum_{j=1}^{\infty} \frac{1}{(2j-1)^3}} \end{aligned}$$

**Exercise 20.** It is known that  $\sum_{j=1}^{\infty} \frac{\cos jx}{j^2} = \frac{x^2}{4} - \frac{\pi x}{2} + \frac{\pi^2}{6}$  if  $0 \leq x \leq 2\pi$

(1)  $x = 2\pi$

$$\sum_{j=1}^{\infty} \frac{1}{j^2} = \frac{(2\pi)^2}{4} - \frac{2\pi^2}{2} + \frac{\pi^2}{6} = \frac{\pi^2}{6}$$

(2) As shown in Ex. 19,  $\sum \frac{\cos jx}{j^2}$  is uniformly convergent on  $\mathbb{R}$

$$\begin{aligned} \sum \int \frac{\cos jx}{j^2} &= \sum \left( \frac{\sin jx}{j^3} \right) \Big|_0^{\pi/2} = \sum \frac{(-1)^{j+1}}{(2j-1)^3} \\ \int \frac{x^2}{4} - \frac{\pi x}{2} + \frac{\pi^2}{6} &= \left( \frac{1}{3} \frac{x^3}{4} - \frac{\pi x^2}{4} + \frac{\pi^2}{6} x \right) \Big|_0^{\pi/2} = \\ &= (\pi)^3 \left( \frac{1}{12(8)} - \frac{1}{16} + \frac{1}{12} \right) = \boxed{(\pi)^3 \frac{1}{32}} \end{aligned}$$

**11.13 Exercises - Properties of functions represented by real power series, The Taylor's series generated by a function, A sufficient condition for convergence of a Taylor's series, Power-series expansions for the exponential and trigonometric functions, Bernstein's Theorem.** Sufficient Condition for convergence.

**Theorem 37** (Bernstein's Theorem). Assume  $\forall x \in [0, r], f(x), f^{(j)}(x) \geq 0 \quad \forall j \in \mathbb{N}$ .

Then if  $0 \leq x < r$

$$(25) \quad \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k \quad \text{converges}$$

*Proof.* If  $x = 0$ , we're done. Assume  $0 < x < r$ .

$$\begin{aligned} f(x) &= \sum_{k=0}^n \frac{f^{(k)}(0)}{k!} x^k + E_n(x) \\ E_n(x) &= \frac{x^{n+1}}{n!} \int_0^1 u^n f^{(n+1)}(x - xu) du \\ F_n(x) &= \frac{E_n(x)}{x^{n+1}} = \frac{1}{n!} \int_0^1 u^n f^{(n+1)}(x - xu) du \end{aligned}$$

Since  $f^{(n+1)} > 0$ ,  $f^{(n+1)}(x(1-u)) \leq f^{(n+1)}(r(1-u))$

$$\implies F_n(x) \leq F_n(r) \implies \frac{E_n(x)}{x^{n+1}} \leq \frac{E_n(r)}{r^{n+1}}$$

For  $f(x) = \sum_{j=0}^n \frac{f^{(j)}(0)}{j!} x^j + E_n(x) \implies E_n(x) \leq \left(\frac{x}{r}\right)^{n+1} E_n(r)$

$$f(r) = \sum_{j=0}^n \frac{f^{(j)}(0)}{j!} r^j + E_n(r) \geq E_n(r) \text{ since } f^{(j)}(0) \geq 0 \quad \forall j$$

So then  $0 \leq E_n(x) \leq \left(\frac{x}{r}\right)^{n+1} f(r)$

$n \rightarrow \infty$  and  $f(t)$  will be some non-infinite value, so  $E_n(x) \xrightarrow{n \rightarrow \infty} 0$ . □

**Exercise 1.**  $\sum_{j=0}^{\infty} (-1)^j x^{2j}$

Consider absolute convergence.  $\lim_{j \rightarrow \infty} (x^2)^j = 0$  If  $x^2 < 1$

If  $|x| = 1$ , then consider

$$x^{2(2j)} - x^{2(2j+1)} = x^{4j}(1-x)$$

$$\sum_{j=0}^{\infty} (1-x)x^{4j} = (1-x) \sum_{j=0}^{\infty} (x^4)^j$$

Indeed  $\sum_{j=0}^{\infty} (-1)^j x^{2j}$  converges for  $|x| \leq 1$

**Exercise 2.**  $\sum_{j=0}^{\infty} \frac{x^j}{3^{j+1}} = \frac{1}{3} \sum_{j=0}^{\infty} \left(\frac{x}{3}\right)^j$  The series converges for  $|x| < 3$

**Exercise 3.**  $\sum_{j=0}^{\infty} jx^j$

$$\int_0^x \sum_{j=0}^{\infty} jt^{j-1} = \sum_{j=0}^{\infty} x^j$$

So the series converges for  $|x| < 1$ . Note that we had used the integrability of power series.

**Exercise 4.**  $\sum_{j=0}^{\infty} (-1)^j jx^j$ .

$$jx^j j e^{j \ln x}$$

$$\lim_{j \rightarrow \infty} j e^{j \ln x} = \begin{cases} \infty & \text{if } x > 1 \\ 0 & \text{if } 0 < x < 1 \end{cases}$$

If  $x = 1$ ,

$$(2j)x^{2j} - (2j+1)x^{2j+1} = x^{2j}(2j - (2j+1)x) = \boxed{-1} \sum (-1) = \infty$$

So  $\sum (-1)^j jx^j$  converges only for  $0 \leq x < 1$ ,  $|x| < 1$ .

**Exercise 5.**  $\sum_{j=0}^{\infty} (-2)^j \frac{j+2}{j+1} x^j = \sum_{j=0}^{\infty} (-1)^j \left(\frac{j+2}{j+1}\right) (2x)^j$

$$\lim_{j \rightarrow \infty} \left(\frac{j+2}{j+1}\right) (2x)^j = \lim_{j \rightarrow \infty} (2x)^j \quad \text{if } |2x| < 1$$

So when  $|x| < \frac{1}{2}$ , the series converges.

$$\begin{aligned} \left(\frac{2j+2}{2j+1}\right) - \left(\frac{2j+3}{2j+2}\right) &= \frac{(2j+2)(2j+2) - (2j+3)(2j+1)}{(2j+1)(2j+2)} \\ \frac{4j^2 + 8j + 4 - (4j^2 + 5j + 3)}{4j^2 + 6j + 2} &= \frac{3j+1}{4j^2 + 6j + 2} = \frac{(3j+1)/2}{2j^2 + 3j + 1} = \frac{(3j+1)/2}{(2j+1)(j+1)} \\ \frac{3j+1}{4j^2 + 6j + 2} &< \frac{3j+1}{4j^2 + \frac{4}{3}} < \frac{3(j+1/3)}{4(j^2 + 1/3)} < \frac{3}{4} \left(\frac{1}{j}\right) + \frac{1}{12j^2} \end{aligned}$$

Thus, it diverges, by comparison test with  $\frac{1}{j}$  for  $x = \frac{1}{2}$ .

**Theorem 38.** Let  $f$  be represented by  $f(x) = \sum_{j=0}^{\infty} a_j(x-a)^j$  in the  $(a-r, a+r)$  interval of convergence

- (1)  $\sum_{j=1}^{\infty} j a_j(x-a)^{j-1}$  also has radius of convergence  $r$ .
- (2)  $f'(x)$  exists  $\forall x \in (a-r, a+r)$  and

$$(26) \quad f'(x) = \sum_{j=1}^{\infty} j a_j(x-a)^{j-1}$$

**Exercise 6.**  $\sum_{j=1}^{\infty} \frac{(2x)^j}{j} = \sum_{j=1}^{\infty} \frac{e^{j \ln 2x}}{j} \implies \boxed{|x| < \frac{1}{2}}$  it'll converge, since by comparison test,  $\frac{e^{j \ln 2x}}{j} < \frac{1}{j^2}$  if

$0 < x < \frac{1}{2}$ .

**Exercise 7.**  $\sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)} \left(\frac{x}{2}\right)^{2j}$ .

$$\sum_{j=0}^{\infty} \frac{x^{2j}}{2(j+1/2)2^{2j}} < \sum_{j=0}^{\infty} \frac{x^{2j}}{2^{2j+1}j} = \frac{1}{2} \sum_{j=0}^{\infty} \left(\frac{x}{2}\right)^{2j} \left(\frac{1}{j}\right) = \frac{1}{2} \sum_{j=0}^{\infty} \frac{e^{2j \ln \frac{x}{2}}}{j}$$

$$\text{since by comparison test, } \frac{e^{2j \ln \frac{x}{2}}}{j} < \frac{1}{j^2} \quad \text{if } 0 < x < 2$$

$$\begin{aligned} \text{If } x = \pm 2, \quad \frac{1}{2(2j+1)} - \frac{1}{2(2j+1)+1} &= \frac{1}{4j+1} - \frac{1}{4j+3} = \frac{2}{(4j+1)(4j+3)} \leq \\ &\leq \frac{1}{8j^2} \text{ (converges by comparison test to } \sum \frac{1}{j^2} \text{)} \end{aligned}$$

For  $|x| < 2$ ,  $\sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)} \left(\frac{x}{2}\right)^{2j}$  converges.

**Exercise 8.**  $\sum_{j=0}^{\infty} \frac{(-1)^j x^{3j}}{j!} = \sum_{j=0}^{\infty} \frac{(-x^3)^j}{j!} = e^{-x^3}$ , which converges  $\forall x \in \mathbb{R}$

**Exercise 9.**  $\sum_{j=0}^{\infty} \frac{x^j}{(j+3)!} = \frac{1}{x^3} \sum_{j=0}^{\infty} \frac{x^{j+3}}{(j+3)!} = \frac{1}{x^3} \sum_{j=3}^{\infty} \frac{x^j}{j!} = \frac{1}{x^3} (e^x - x^2/2 - x - 1)$  Thus, it converges for  $\forall x \in \mathbb{R}$ .

**Exercise 10.**  $\sum_{j=0}^{\infty} \frac{(x-1)^j}{(j+2)!} = \frac{1}{(x-1)^2} \sum_{j=0}^{\infty} \frac{(x-1)^{j+2}}{(j+2)!} = \frac{1}{(x-1)^2} \left( \sum_{j=2}^{\infty} \frac{(x-1)^j}{j!} \right) = \frac{1}{(x-1)^2} (e^{x-1} - (x-1) - 1) = \boxed{\frac{e^{x-1} - x}{(x-1)^2}}$

**Exercise 11.**  $a^x = e^{x \log a} = \sum_{j=0}^{\infty} \frac{(\log ax)^j}{j!}$

$\frac{(\log ax)^{j+1}}{(j+1)!} \frac{j!}{(\log ax)^j} = \frac{(\log ax)}{j+1} \xrightarrow{j \rightarrow \infty} 0$ . By ratio test,  $\sum_{j=0}^{\infty} \frac{(\log ax)^j}{j!}$  converges for all  $x$ .

**Exercise 12.**

$$\begin{aligned} \sinh x &= \frac{e^x - e^{-x}}{2} = \frac{1}{2} \left( \sum_{j=0}^{\infty} \frac{x^j}{j!} - \sum_{j=0}^{\infty} \frac{(-x)^j}{j!} \right) = \sum_{j=0}^{\infty} \frac{x^{2j+1}}{(2j+1)!} \\ \frac{x^{2j+3}}{(2j+3)!} \frac{(2j+1)!}{x^{2j+1}} &= \frac{x^2}{(2j+3)(2j+2)} \xrightarrow{j \rightarrow \infty} 0 \end{aligned}$$

**Exercise 13.**

$$\begin{aligned} \sin^2 x &= \frac{1 - \cos 2x}{2} = \frac{1 - \sum_{j=0}^{\infty} \frac{(2x)^{2j}}{(2j)!} (-1)^j}{2} = \sum_{j=1}^{\infty} \frac{2^{2j-1} x^{2j} (-1)^{j+1}}{(2j)!} \\ \frac{2^{2j+1} x^{2j+2}}{(2j+2)!} \frac{(2j)!}{2^{2j-1} x^{2j}} &= \frac{4x^2}{(2j+2)(2j+1)} \xrightarrow{j \rightarrow \infty} 0 \end{aligned}$$

So the series converges  $\forall x$

**Exercise 14.**  $\frac{1}{1-\frac{x}{2}} = \sum_{j=0}^{\infty} \left(\frac{x}{2}\right)^j \quad \frac{1}{2-x} = \sum_{j=0}^{\infty} \frac{x^j}{2^{j+1}}$

$$\frac{x^{j+1}}{2^{j+2}} \frac{2^{j+1}}{x^j} = \frac{x}{2} < 1 \implies \text{(the series converges for } |x| < 2, \text{ by ratio test)}$$

If  $x = 2$ , the series would diverge

$$\text{If } x = -2 \quad \frac{1}{2} \sum_{j=0}^{\infty} \frac{(-2)^j}{2^j} = \frac{1}{2} \sum_{j=0}^{\infty} (-1)^j = 0 \text{ but } \frac{1}{2 - (-2)} = \frac{1}{4}$$

**Exercise 15.**  $e^{-x^2} = \sum_{j=0}^{\infty} \frac{(-1)^j x^{2j}}{j!}$

$$\frac{x^{2j+2}}{(j+1)!} \frac{j!}{x^{2j}} = \frac{x^2}{j+1} \xrightarrow{j \rightarrow \infty} 0$$

**Exercise 16.**  $\sin 3x = \sin 2x \cos x + \sin x \cos 2x = 3 \sin x - 4 \sin^3 x$ .

$$\begin{aligned} \sin^3 x &= \frac{3 \sin x - \sin 3x}{4} = \frac{3}{4} \left( \sum_{j=0}^{\infty} \frac{x^{2j+1} (-1)^j}{(2j+1)!} - \sum_{j=0}^{\infty} \frac{(3x)^{2j+1} (-1)^j}{(2j+1)!} \right) = \\ &= \frac{3}{4} \left( \sum_{j=0}^{\infty} \frac{(-1)^j x^{2j+1} (1 - 3^{2j+1})}{(2j+1)!} \right) = \frac{3}{4} \left( \sum_{j=0}^{\infty} \frac{(-1)^{j+1} (3^{2j+1} - 1) x^{2j+1}}{(2j+1)!} \right) \end{aligned}$$

**Exercise 17.**  $\log \sqrt{\frac{1+x}{1-x}} = \frac{1}{2}(\log(1+x) - \log(1-x)) = \frac{1}{2} \left( \sum_{j=1}^{\infty} \frac{(+x)^j}{j} (-1)^{j-1} - \sum_{j=1}^{\infty} \frac{(x^j)(-1)^j}{j} \right)$

$$\begin{aligned} \ln(1+x) &= \sum_{j=0}^{\infty} \frac{(-x)^{j+1}}{j+1} (-1) = \sum_{j=1}^{\infty} \frac{(-x)^j}{j} (-1) & -\ln(1-x) &= \sum_{j=0}^{\infty} \frac{x^{j+1}}{j+1} = \sum_{j=1}^{\infty} \frac{x^j}{j} \implies -\sum_{j=1}^{\infty} \frac{((-1)^j + 1)x^j}{j} \\ &\implies \sum_{j=1}^{\infty} \boxed{\frac{x^{2j+1}}{2j+1}} \\ &\frac{x^{2j+3}}{2j+3} \frac{2j+1}{x^{2j+1}} \xrightarrow{j \rightarrow \infty} x^2 \\ |x^2| < 1, & \text{converges, with radius of convergence of 1} \end{aligned}$$

**Exercise 18.**  $\frac{3x}{1+x-2x^2} = \frac{1}{1-x} - \frac{1}{1+2x} = \sum_{j=0}^{\infty} \frac{x^j}{j} - \sum_{j=0}^{\infty} \frac{(-2x)^j}{j} = \sum_{j=0}^{\infty} \frac{x^j}{j} (1 - (-2)^j)$

$$\begin{aligned} \frac{x^{j+1} |(1 - (-2)^{j+1})|}{j+1} \frac{j}{|x^j (1 - (-2)^j)|} &= x \left| \frac{\left(\frac{1}{(-2)^j} + 2\right)}{\left(\frac{1}{(-2)^j} - 1\right)} \right| \xrightarrow{j \rightarrow \infty} 2x < 1 \\ |x| &< \frac{1}{2} \\ \text{For } x &= \frac{1}{2} \\ \frac{x^{2j}}{2j} (1 - (-2)^{2j}) + \frac{x^{2j+1}}{2j+1} (1 - (-2)^{2j+1}) &= \frac{x^{2j}((2j+1)(1-2^{2j}) + x(1+2^{2j+1})(2j))}{(2j)(2j+1)} \\ \frac{\left(\frac{1}{2}\right)^{2j} (2j+1 - (2j+1)2^{2j} + 2jx + 2^{2j+2}jx)}{(2j)(2j+1)} &\xrightarrow{j \rightarrow \infty} \frac{-(2j+1) + 4j}{(2j)(2j+1)} = \frac{2j-1}{(2j)(2j+1)} \rightarrow 0 \end{aligned}$$

So  $\frac{3x}{1+x-2x^2}$  converges for  $|x| \leq \frac{1}{2}$

**Exercise 19.**  $\frac{12-5x}{6-5x-x^2} = \sum_{j=0}^{\infty} \left(1 + \frac{(-1)^j}{6^j}\right) x^j \quad (|x| < 1).$

$$\frac{12-5x}{6-5x-x^2} = \frac{5x-12}{(x+6)(x-1)} = \frac{1}{1-x} + \frac{6}{6+x} = \sum_{j=0}^{\infty} x^j + \sum_{j=0}^{\infty} \left(\frac{-x}{6}\right)^j = \sum_{j=0}^{\infty} x^j \left(1 + \left(\frac{-1}{6}\right)^j\right)$$

$|x| < 1$  since for  $x = 1$ ,  $\lim_{j \rightarrow \infty} (1 + \left(\frac{-1}{6}\right)^j) = 1$ .

**Exercise 20.**  $\frac{1}{x^2+x+1} = \frac{2}{\sqrt{3}} \sum_{j=0}^{\infty} \sin \frac{2\pi(j+1)}{3} x^j \quad (|x| < 1)$

$$\begin{aligned} x^3 - 1 &= (x-1)(x^2+x+1) & \frac{1}{1-x^3} &= \sum_{j=0}^{\infty} (x^3)^j \\ \frac{1}{x^2+x+1} &= \frac{x-1}{x^3-1} = \frac{1-x}{1-x^3} & \frac{1-x}{1-x^3} &= \sum_{j=0}^{\infty} ((x^3)^j - x^{3j+1}) \end{aligned}$$

By induction, it could be observed that  $x^{3j}, -x^{3j+1}, 0$  appear in sequences of 3 terms.

$\frac{2}{\sqrt{3}} \sin \frac{2\pi(j+1)}{3}$  accounts for this.

$$\implies \frac{2}{\sqrt{3}} \sum_{j=0}^{\infty} \sin \frac{2\pi(j+1)}{3} x^j \quad |x| < 1$$

$\sin \frac{2\pi(j+1)}{3} x^j < x^j$  (so by comparison test to  $\sum x^j$ , the radius of convergence is 1)

**Exercise 21.**

$$\begin{aligned}\frac{1}{1-x} &= \sum_{j=0}^{\infty} x^j; & \left(\frac{1}{1-x}\right)' &= \frac{1}{(1-x)^2} = \sum_{j=1}^{\infty} jx^{j-1} = \sum_{j=0}^{\infty} (j+1)x^j \\ \frac{1}{(1-x)(1-x^2)} &= \frac{1/4}{1-x} + \frac{1/2}{(1-x)^2} + \frac{1/4}{1+x} = \frac{1}{4} \left( \sum_{j=0}^{\infty} (x^j + (-x)^j) + 2 \sum_{j=1}^{\infty} jx^{j-1} \right) \\ \frac{x}{(1-x)(1-x^2)} &= \frac{1}{4} \left( \sum_{j=0}^{\infty} x^{j+1}(1 + (-1)^j) + 2 \sum_{j=1}^{\infty} jx^j \right) = \frac{1}{4} \sum_{j=1}^{\infty} x^j (1 + (-1)^{j-1} + 2j)\end{aligned}$$

**Exercise 22.**

$$\begin{aligned}\sin\left(2x + \frac{\pi}{4}\right) &= \sum_{j=0}^{\infty} \frac{(2x)^{2j+1}}{(2j+1)!} (-1)^j; & \cos 2x &= \sum_{j=0}^{\infty} \frac{(2x)^{2j}}{(2j)!} (-1)^j \\ \sin 2x &= \sum_{j=0}^{\infty} \frac{(2x)^{2j+1}}{(2j+1)!} (-1)^j \\ \cos 2x &= \sum_{j=0}^{\infty} \frac{(2x)^{2j}}{(2j)!} (-1)^j\end{aligned}$$

$$\sum_{j=0}^{\infty} a_j x^j = \frac{\sqrt{2}}{2} (\sin 2x + \cos 2x)$$

$$\text{For } j = 98, \quad a_j = \frac{2^{98}(-1)^{49}}{98!} \left(\frac{\sqrt{2}}{2}\right) = \boxed{\frac{-2^{98}}{98!} \left(\frac{\sqrt{2}}{2}\right)}$$

**Exercise 23.**

$$\begin{aligned}f(x) &= (2+x^2)^{5/2} \\ f'(x) &= \frac{5}{2}(2+x^2)^{3/2}(2x) = 5x(2+x^2)^{3/2} \\ f''(x) &= 5(2+x^2)^{3/2} + \frac{15}{2}x(2+x^2)^{1/2}(2x) = 5(2+x^2)^{3/2} + 15x^2(2+x^2)^{1/2} \\ f'''(x) &= \frac{15}{2}(2+x^2)^{1/2}(2x) + 30x(2+x^2)^{1/2} + \frac{15x^2}{2}(2+x^2)^{-1/2}(2x) \\ f''''(x) &= 15(2+x^2)^{1/2} + \frac{15}{2}(2+x^2)^{-1/2}(2x)x + 30(2+x^2)^{1/2} + \frac{30x}{2}(2+x^2)^{1/2}(2x) \\ &\quad + 45x^2(2+x^2)^{-1/2} + 15x^3 \left(-\frac{1}{2}\right)(2+x^2)^{-3/2}(2x) \\ &= \boxed{2^{5/2} + 0x + \frac{5(2^{3/2})x^2}{2!} + \frac{0x^3}{3!} + \frac{45}{4!}\sqrt{2}x^4}\end{aligned}$$

**Exercise 24.**  $f(x) = e^{-1/x^2}$  if  $x \neq 0$  and let  $f(0) = 0$

(1)

$$\begin{aligned}f(x) &= \sum_{j=0}^{\infty} \frac{\left(\frac{-1}{x^2}\right)^j}{j!} = 1 + \frac{-1}{x^2} + \sum_{j=2}^{\infty} \frac{\left(\frac{-1}{x^2}\right)^j}{j!} = \sum_{j=0}^{\infty} \frac{(-1)^j x^{-2j}}{j!} \\ f^{(k)} &= \sum_{j=0}^{\infty} (-1)^j \frac{(-2j)(-2j-1)\dots(-2j-(k-1))}{j!} x^{-2j} = \sum_{j=0}^{\infty} (-1)^j \frac{(-2j)!}{(-2j-k)!j!} x^{-2j}\end{aligned}$$

Use *ratio test* :

$$\frac{(-2(j+1))!}{(-2j-2-k)!(j+1)!} \frac{j!(-2j-k)!}{(-2j)!} \frac{x^{-2(j+1)}}{x^{-2j}} = \frac{(-2j-k)(-2j-k-1)}{(j+1)(-2j)(-2j-1)} x^{-2} \xrightarrow{j \rightarrow \infty} 0$$

Thus, by ratio test, every order of derivative exists for every  $x$  on the real line since the series representing the derivative converges for every  $x$ .

(2)  $f(x) = \sum_{j=0}^{\infty} \frac{-x^{-2j}}{j!}$ . There are no nonzero terms of positive power, i.e. no  $x^j$ ;  $j \geq 1$ .

$$\implies f^{(j)}(0) = 0 \quad \forall j \geq 1$$



## 11.16 Exercises - Power series and differential equations, binomial series.

**Exercise 1.** For  $(1 - x^2)y'' - 2xy' + 6y = 0$ ,

$$y = \sum_{j=0}^{\infty} a_j x^j$$

$$y' = \sum_{j=1}^{\infty} j a_j x^{j-1}$$

$$f(0) = 1 \implies a_0 = 1 \quad f'(0) = 0 \implies a_1 = 0$$

$$y'' = \sum_{j=2}^{\infty} j(j-1) a_j x^{j-2} = \sum_{j=0}^{\infty} (j+2)(j+1) a_{j+2} x^j$$

$$2(1)a_2 + 3(2)a_3x + -2(1)a_1x + 6a_0 + 6a_1x + \sum_{j=2}^{\infty} ((j+2)(j+1)a_{j+2} - j(j-1)a_j - 2ja_j + 6a_j)x^j =$$

$$= 2a_2 + 6 + 6a_3x + \sum_{j=2}^{\infty} ((j+2)(j+1)a_{j+2} + a_j(j+3)(j-2))x^j$$

$$\implies \begin{matrix} a_2 = -3 \\ a_3 = 0 \end{matrix}$$

$$\boxed{a_{j+2} = \frac{(j+3)(j-2)}{(j+2)(j+1)} a_j}$$

For  $j = 2$ ,  $a_4 = 0$ , so then  $a_{j+2} = 0$  for  $j = 2, 4, \dots$ . Likewise, since  $a_3 = 0$ , then  $a_{j+2} = 0$  for  $j = 3, 5, \dots$   
 $\implies f(x) = 1 - 3x^2$

**Exercise 2.** Using the work from above, then for  $(1 - x^2)y'' - 2xy' + 12y = 0$

$$f(0) = 0 \implies a_0 = 0 \quad f'(0) = 2 \implies a_1 = 2$$

$$2(1)a_2 + 3(2)a_3x + -2(1)a_1x + 12a_0 + 12a_1x + \sum_{j=2}^{\infty} ((j+2)(j+1)a_{j+2} - j(j-1)a_j - 2ja_j + 12a_j)x^j =$$

$$= 2a_2 + 6a_3x + -4x + 0 + 24x + \sum_{j=2}^{\infty} ((j+2)(j+1)a_{j+2} - a_j(j+4)(j-3))x^j$$

$$\implies \begin{matrix} a_2 = 0 \\ a_3 = -10/3 \end{matrix}$$

$$\boxed{a_{j+2} = \frac{(j+4)(j-3)}{(j+2)(j+1)} a_j}$$

For  $j = 3$ ,  $a_5 = 0$ , so then  $a_{j+2} = 0$  for  $j = 3, 5, \dots$ . Likewise, since  $a_2 = 0$ , then  $a_{j+2} = 0$  for  $j = 2, 4, \dots$   
 $\implies f(x) = -10/3x^3 + 2$

**Exercise 3.**  $f(x) = \sum_{j=0}^{\infty} \frac{x^{4j}}{(4j)!}$ ;  $\frac{d^4 y}{dx^4} = y$

$$\begin{array}{ll} (x^4)' = 4x^3 & \left(\frac{x^{4j}}{(4j)!}\right)' = \frac{x^{4j-1}}{(4j-1)!} \\ (x^4)'' = 12x^2 & \left(\frac{x^{4j}}{(4j)!}\right)'' = \frac{x^{4j-2}}{(4j-2)!} \\ (x^4)''' = 24x & \left(\frac{x^{4j}}{(4j)!}\right)''' = \frac{x^{4j-3}}{(4j-3)!} \\ (x^4)'''' = 24 & \left(\frac{x^{4j}}{(4j)!}\right)'''' = \frac{x^{4j-4}}{(4j-4)!} \end{array} \quad \sum_{j=1}^{\infty} \frac{x^{4j-4}}{(4j-4)!} = y'''' = \boxed{\sum_{j=0}^{\infty} \frac{x^{4j}}{(4j)!}} = f(x)$$

To test convergence, use the ratio test

$$\frac{x^{4j+4}}{(4j+4)!} \frac{(4j)!}{x^{4j}} = \frac{x^4}{(4j+4)(4j+3)(4j+2)(4j+1)} \xrightarrow{j \rightarrow \infty} 0 \quad \forall x \in \mathbb{R}$$

So the series converges on  $\mathbb{R}$ .

**Exercise 4.**  $f(x) = \sum_{j=0}^{\infty} \frac{x^j}{(j!)^2}$        $xy'' + y' - y = 0$

$$y' = \sum_{j=1}^{\infty} \frac{jx^{j-1}}{(j!)^2} = \sum_{j=0}^{\infty} \frac{(j+1)x^j}{((j+1)!)^2}$$

$$y'' = \sum_{j=1}^{\infty} \frac{(j+1)jx^{j-1}}{((j+1)!)^2}$$

$$\sum_{j=1}^{\infty} \left( \frac{(j+1)j}{((j+1)!)^2} + \frac{(j+1)}{((j+1)!)^2} - \frac{1}{(j!)^2} \right) x^j + \frac{1}{1!} - 1 = \sum_{j=1}^{\infty} \left( \frac{1}{(j!)^2} - \frac{1}{(j!)^2} \right) = 0$$

**Exercise 5.**  $f(x) = 1 + \sum_{j=1}^{\infty} \frac{1 \cdot 4 \cdot 7 \dots (3j-2)}{(3j)!} x^{3j}$ ;     $y'' = x^a y + b$     (Find  $a$  and  $b$ )

$$f' = \sum_{j=1}^{\infty} \frac{1 \cdot 4 \cdot 7 \dots (3j-2)}{(3j-1)!} x^{3j-1}$$

$$f'' = \sum_{j=1}^{\infty} \frac{1 \cdot 4 \cdot 7 \dots (3j-2)}{(3j-2)!} x^{3j-2} = \sum_{j=1}^{\infty} \frac{1 \cdot 4 \cdot 7 \dots (3j-5)}{(3j-3)!} x^{3j-2} =$$

$$= x + \sum_{j=2}^{\infty} \frac{1 \cdot 4 \cdot 7 \dots (3j-5)}{(3j-3)!} x^{3j-2} = x + \sum_{j=1}^{\infty} \frac{1 \cdot 4 \cdot 7 \dots (3j-2)}{(3j)!} x^{3j+1}$$

$$x^a f = -x^a + \sum_{j=1}^{\infty} \frac{1 \cdot 4 \cdot 7 \dots (3j-2)}{(3j)!} x^{3j+a}$$

So then  $\boxed{a = 1; \quad b = 0}$ .

$$\frac{1 \cdot 4 \cdot 7 \dots (3j+1)}{(3j+3)!} x^{3j+3} \left( \frac{(3j)!}{1 \cdot 4 \cdot 7 \dots (3j-2)} \right) \frac{1}{x^{3j}} = \frac{1(3j+1)}{(3j-2)(3j+3)(3j+2)(3j+1)} x^3 \xrightarrow{j \rightarrow \infty} 0$$

So the series converges for all  $x$ .

**Exercise 6.**  $f(x) = \sum_{j=0}^{\infty} \frac{x^{2j}}{j!}$ ;     $y' = 2xy$ .

$$f' = \sum_{j=1}^{\infty} \frac{2jx^{2j-1}}{j!} = 2 \sum_{j=1}^{\infty} \frac{x^{2j-1}}{(j-1)!} = 2 \sum_{j=0}^{\infty} \frac{x^{2j+1}}{j!} = 2xf$$

$$\frac{x^{2j+2}}{(j+1)!} \frac{j!}{x^{2j}} = \frac{x^2}{j+1} \xrightarrow{j \rightarrow \infty} 0 \quad \forall x$$

By ratio test,  $f$  converges  $\forall x \in \mathbb{R}$ .

**Exercise 7.**  $f(x) = \sum_{j=2}^{\infty} \frac{x^j}{j!}$      $y' = x + y$

$$f' = \sum_{j=2}^{\infty} \frac{x^{j-1}}{(j-1)!} = \sum_{j=1}^{\infty} \frac{x^j}{j!} = x + y$$

$$\frac{x^{j+1}}{(j+1)!} \frac{j!}{x^j} = \frac{x}{j} \xrightarrow{j \rightarrow \infty} 0$$

So the series converges  $\forall x \in \mathbb{R}$  by ratio test.

**Exercise 8.**

$$f(x) = \sum_{j=0}^{\infty} \frac{(-1)^j (kx)^{2j}}{(2j)!}$$

$$f' = \sum_{j=1}^{\infty} \frac{(-1)^j (kx)^{2j-1} k}{(2j-1)!} = \sum_{j=0}^{\infty} \frac{(-1)^{j+1} (kx)^{2j+1} k}{(2j+1)!} \quad f'' - k^2 f = 0$$

$$f'' = \sum_{j=1}^{\infty} \frac{(-1)^{j+1} (kx)^{2j}}{(2j)!} k^2$$

$$\frac{(kx)^{2j+1}}{(2j+1)!} \frac{(2j)!}{(kx)^{2j}} = \frac{kx}{2k+1} \xrightarrow{j \rightarrow \infty} 0 \quad \text{by ratio test, } f \text{ converges } \forall x \in \mathbb{R}.$$

**Exercise 9.**

$$\begin{aligned}
 f'' &= \sum_{j=1}^{\infty} \frac{(3x)^{2j-1}}{(2j-1)!} 9 = \sum_{j=0}^{\infty} \frac{9(3x)^{2j+1}}{(2j+1)!} \\
 9(f-x) &= 9\left(x + \sum_{j=0}^{\infty} \frac{(3x)^{2j+1}}{(2j+1)!} - x\right) \\
 \frac{(3x)^{2j+3}}{(2j+3)!} \frac{(2j+1)!}{(3x)^{2j+1}} &= \frac{9x^2}{(2j+3)(2j+2)} \xrightarrow{j \rightarrow \infty} 0 \\
 &\text{(by ratio test, } f \text{ converges } \forall x \in \mathbb{R} \text{ )}
 \end{aligned}$$

**Exercise 10.**  $J_0(x) = \sum_{j=0}^{\infty} (-1)^j \frac{x^{2j}}{(j!)^2 2^{2j}}$        $J_1(x) = \sum_{j=0}^{\infty} (-1)^j \frac{x^{2j+1}}{j!(j+1)! 2^{2j+1}}.$

(1)

$$\begin{aligned}
 \frac{x^{2j+2}}{((j+1)!)^2 2^{2j+2}} \frac{(j!)^2 2^{2j}}{x^{2j}} &= \frac{x^2}{(j+1)^2 4} \xrightarrow{j \rightarrow \infty} 0 && \text{by ratio test, } f \text{ converges } \forall x \in \mathbb{R} \\
 \frac{x^{2j+3}}{(j+2)! 2^{2j+3}} \frac{j! 2^{2j+1}}{x^{2j+1}} &= \frac{x^2}{(j+2)(j+1)4} \xrightarrow{j \rightarrow \infty} 0 && \text{by ratio test, } f \text{ converges } \forall x \in \mathbb{R}
 \end{aligned}$$

(2)

$$J'_0(x) = \sum_{j=1}^{\infty} (-1)^j \frac{x^{2j-1}}{(j-1)!(j!)2^{2j-1}} = \sum_{j=0}^{\infty} (-1)^{j+1} \frac{x^{2j+1}}{j!(j+1)!2^{2j+1}} = -J_1(x)$$

(3)

$$\begin{aligned}
 j_0(x) &= xJ_0(x) = \sum_{j=0}^{\infty} (-1)^j \frac{x^{2j+1}}{(j!)^2 2^{2j}} && j_1(x) = xJ_1(x) = \sum_{j=0}^{\infty} (-1)^j \frac{x^{2j+2}}{j!(j+1)!2^{2j+1}} \\
 j'_1 &= \sum_{j=0}^{\infty} \frac{(-1)^j x^{2j+1}}{(j!)^2 2^{2j}} \\
 &\implies j_0 = j'_1
 \end{aligned}$$

**Exercise 11.**  $x^2 y'' + xy' + (x^2 - n^2)y = 0.$   
 $n = 0 \implies x^2 y'' + xy' + x^2 y = 0$

$$\begin{aligned}
 J_0 &= \sum_{j=0}^{\infty} (-1)^j \frac{x^{2j}}{(j!)^2 2^{2j}} = \sum_{j=1}^{\infty} (-1)^{j-1} \frac{x^{2j-2}}{((j-1)!)^2 2^{2j-2}}; \\
 J'_0 &= \sum_{j=1}^{\infty} (-1)^j \frac{x^{2j-1}}{j!(j-1)! 2^{2j-1}}; \\
 J''_0 &= \sum_{j=1}^{\infty} (-1)^j \frac{x^{2j-2}}{j!(j-1)!} \frac{(2j-1)}{2^{2j-1}} \\
 \sum_{j=1}^{\infty} (-1)^j \left( \frac{(2j-1)}{j!(j-1)! 2^{2j-1}} + \frac{1}{j!(j-1)! 2^{2j-1}} + \frac{-2j}{((j-1)!)j! 2^{2j-1}} \right) &= 0
 \end{aligned}$$

$$n = 1 \implies x^2 y'' + xy' + (x^2 - 1)y = 0$$

$$\begin{aligned}
J_1(x) &= \sum_{j=0}^{\infty} \frac{(-1)^j x^{2j+1}}{j!(j+1)!2^{2j+1}} = \frac{x}{2} + \sum_{j=1}^{\infty} \frac{(-1)^j x^{2j+1}}{j!(j+1)!2^{2j+1}} = \frac{x}{2} + \sum_{j=1}^{\infty} \frac{(-1)^{j-1} x^{2j-1}}{(j-1)!(j)!2^{2j-1}} \\
J_1' &= \frac{1}{2} + \sum_{j=1}^{\infty} \frac{(-1)^j (2j+1) x^{2j}}{j!(j+1)!2^{2j+1}} \\
J_1'' &= \sum_{j=1}^{\infty} \frac{(-1)^j (2j+1)(2j) x^{2j-1}}{(j!)(j+1)!2^{2j+1}} \\
x^2 J_1'' + x J_1' + (x^2 - 1) J_1 &= \\
&= \sum_{j=1}^{\infty} x^{2j+1} \left( \left( \frac{(-1)^j (2j+1)}{(j!)(j+1)!2^{2j+1}} \right) ((2j) + (1)) + \frac{(-1)^j (-1)}{(j-1)!j!} \left( \frac{j+1}{j+1} \right) \binom{j}{j} \left( \frac{1}{2^{2j-1}} \right) \left( \frac{2^2}{2^2} \right) - \frac{(-1)^j}{(j+1)!(j!)2^{2j+1}} \right) + \\
&\quad + \frac{x}{2} - \frac{x}{2} = \\
&= \sum_{j=1}^{\infty} \left( \frac{(-1)^j x^{2j-1}}{(j!)(j+1)!2^{2j+1}} \right) ((2j+1)(2j+1) + (-1)(2j)(2j+2) - 1) = 0
\end{aligned}$$

**Exercise 12.**  $y' = x^2 + y^2$ ;  $y = 1$  when  $x = 0$ .

$$\begin{aligned}
y &= a_0 + a_1 x + a_2 x^2 + \sum_{j=3}^{\infty} a_j x^j \\
y^2 &= a_0^2 + a_1^2 x^2 + a_2^2 x^4 + \left( \sum_{j=3}^{\infty} a_j x^j \right)^2 + \\
y'(0) &= 0 + 1^2 = 1 \\
&\quad + 2a_0 a_1 x + 2a_0 a_2 x^2 + 2a_0 \sum_{j=3}^{\infty} a_j x^j + \\
&\quad + 2a_1 a_2 x^3 + 2a_1 \sum_{j=3}^{\infty} a_j x^{j+1} + 2a_2 \sum_{j=3}^{\infty} a_j x^{j+2} \\
y' &= a_1 + 2a_2 x + \sum_{j=3}^{\infty} j a_j x^{j-1}
\end{aligned}$$

$$\boxed{a_1 = 1} \text{ since } y'(0) = 1$$

Consider the first few terms of  $x^2 + y^2$

$$\begin{aligned}
a_1 &= 1 = a_0^2 \implies \boxed{a_0 = 1} \\
a_0^2 + 2a_0 a_1 x + a_1^2 x^2 + 2a_0 a_2 x^2 + x^2 &= a_1 + 2a_2 x + 3a_3 x^2 \implies 2a_2 = 2a_0 a_1 \implies \boxed{a_2 = 1} \\
3a_3 &= a_1^2 + 2a_0 a_2 + 1 = 4 \implies \boxed{a_3 = \frac{4}{3}}
\end{aligned}$$

**Exercise 13.**  $y' = 1 + xy^2$  with  $y = 0$  when  $x = 0 \implies a_0 = 0$

$$y = a_1x + a_2x^2 + a_3x^3 + \sum_{j=4}^{\infty} a_jx^j$$

$$y^2 = a_1^2x^2 + a_2^2x^4 + a_3^2x^6 + \left( \sum_{j=4}^{\infty} a_jx^j \right)^2 +$$

$$+ 2a_1a_2x^3 + 2a_1a_3x^4 + 2a_1 \sum_{j=4}^{\infty} a_jx^{j+1} +$$

$$+ 2a_2a_3x^5 + 2a_2 \sum_{j=4}^{\infty} a_jx^{j+2} + 2a_3 \sum_{j=4}^{\infty} a_jx^{j+3}$$

$$\boxed{a_1 = 1}$$

$$x : 2a_2 = 0 \implies a_2 = 0$$

$$x^2 : 3a_3 = 0 \implies a_3 = 0$$

$$x^3 : 4a_4 = 1^2 \implies \boxed{a_4 = \frac{1}{4}}$$

$$x^4 : 5a_5 = 0 \implies a_5 = 0$$

$$x^5 : 6a_6 = 0 \implies a_6 = 0$$

$$x^6 : 7a_7 = 2a_1a_4 + 2a_2a_3 \implies \boxed{a_7 = \frac{1}{14}}$$

$$x^7 : 8a_8 = 0 + 2a_2a_4 = 0 \implies a_8 = 0$$

$$x^8 : 9a_9 = 0 \implies a_9 = 0$$

$$x^9 : 10a_{10} = \left(\frac{1}{4}\right)^2 + 2(1)\frac{1}{14} \implies \boxed{a_{10} = \frac{23}{1120}}$$

**Exercise 14.**  $y' = x + y^2$   $y = 0$  when  $x = 0 \implies a_0 = 0$   
 $y'(0) = 0 + 0 = 0 \implies a_1 = 0$

$$y = \sum_{j=2}^{\infty} a_jx^j \quad y' = \sum_{j=2}^{\infty} ja_jx^{j-1} = \sum_{j=1}^{\infty} (j+1)a_{j+1}x^j$$

$$y^2 = \left( a_2x^2 + a_3x^3 + a_4x^4 + \sum_{j=5}^{\infty} a_jx^j \right)^2 =$$

$$= a_2^2x^4 + a_3^2x^6 + a_4^2x^8 + \left( \sum_{j=5}^{\infty} a_jx^j \right)^2 +$$

$$+ 2a_2a_3x^5 + 2a_2a_4x^6 + 2a_2x^2 \sum_{j=5}^{\infty} a_jx^j + 2a_3a_4x^7 + 2a_3x^3 \sum_{j=5}^{\infty} a_jx^j + 2a_4x^4 \sum_{j=5}^{\infty} a_jx^j$$

$$y' = x + y^2$$

$$x : 2a_2 = 1 + 0 \implies \boxed{a_2 = \frac{1}{2}}$$

$$x^2 : 3a_3 = 0 \implies a_3 = 0$$

$$x^3 : 4a_4 = 0 \implies a_4 = 0$$

$$x^4 : 5a_5 = a_2^2 \implies \boxed{a_5 = \frac{1}{20}}$$

$$x^5 : 6a_6 = 0 \implies a_6 = 0 : 7a_7 = 0 \implies a_7 = 0$$

$$x^7 : 8a_8 = 2\left(\frac{1}{2}\right)\left(\frac{1}{20}\right) \implies \boxed{a_8 = \frac{1}{160}}$$

$$x^8 : 9a_9 = 0 \implies a_9 = 0$$

$$x^9 : 10a_{10} = 0 \implies a_{10} = 0$$

$$x^{10} : 11a_{11} = 2\left(\frac{1}{2}\right)\left(\frac{1}{160}\right) + \left(\frac{1}{20}\right)^2 \implies \boxed{a_{11} = \frac{7}{8800}}$$

**Exercise 15.**  $y' = \alpha y$

$$\Rightarrow \sum_{j=0}^{\infty} (j+1)a_{j+1}x^j = \alpha \sum_{j=0}^{\infty} a_j x^j$$

$$a_{j+1} = \frac{\alpha a_j}{(j+1)}$$

$$a_j = \frac{\alpha^j}{j!} a_0 \quad \text{x (by induction)}$$

**Exercise 16.**  $y'' = xy$

$$\begin{aligned} y'' &= \sum_{j=0}^{\infty} (j+2)(j+1)a_{j+2}x^j = \\ &= 2a_2 + \sum_{j=0}^{\infty} (j+3)(j+2)a_{j+3}x^{j+1} = \sum_{j=0}^{\infty} a_j x^{j+1} \end{aligned}$$

$$\Rightarrow a_2 = 0 \text{ and } a_{j+3} = \frac{a_j}{(j+3)(j+2)}$$

$$\begin{array}{lll} j=0 & a_3 = \frac{a_0}{3 \cdot 2} & j=1 \quad a_4 = \frac{a_1}{4 \cdot 3} \\ j=3 & a_6 = \frac{a_3}{6 \cdot 5} & j=4 \quad a_7 = \frac{a_4}{7 \cdot 6 \cdot 4 \cdot 3} \end{array}$$

$$a_{3j} = \frac{a_0}{(3j)!} \prod_{k=0}^{j-1} (3k+1);$$

$$a_{3j+1} = \frac{a_1}{(3j+1)!} \prod_{k=0}^{j-1} (3k+2)$$

**Exercise 17.**  $y'' + xy' + y = 0$

$$y = \sum_{j=0}^{\infty} a_j x^j$$

$$y' = \sum_{j=1}^{\infty} j a_j x^{j-1}$$

$$y'' = \sum_{j=2}^{\infty} j(j-1)a_j x^{j-2}$$

$$= \sum_{j=0}^{\infty} (j+2)(j+1)a_{j+2}x^j$$

$$y'' + xy' + y = \sum_{j=1}^{\infty} x^j ((j+2)(j+1)a_{j+2} + j a_j + a_j) = 0 \Rightarrow a_{j+2} = \frac{-a_j}{(j+2)}$$

$$\begin{array}{ll} a_2 = \frac{-a_0}{2} & a_3 = \frac{-a_1}{3} \\ a_4 = \frac{-a_2}{4} & a_5 = \frac{-a_3}{5} = \frac{a_1}{15} \end{array}$$

$$a_{2j} = \frac{(-1)^j a_0}{(2j)!!} \quad a_{2j+1} = \frac{a_1}{(2j+1)!!} (-1)^j \quad (\text{could be shown by induction})$$

**Exercise 18.** Recall that

$$\begin{aligned} y &= \sum_{j=0}^{\infty} a_j x^j \\ y' &= \sum_{j=1}^{\infty} j a_j x^{j-1} \\ &= \sum_{j=0}^{\infty} (j+1) a_{j+1} x^j \end{aligned}$$

Knowing this, we could *cleverly observe* that  $e^{-2x} = \sum_{j=0}^{\infty} (2a_j + (j+1)a_{j+1})x^j$  is actually a *differential equation!!!*

$$\Rightarrow e^{-2x} = y' + 2y$$

Solving this ODE using  $y(x) = e^{-A(x)} \left( \int_a^x Q(t) e^{A(t)} dt + y(a) \right)$  where  $A(x) = \int_a^x P(t) dt$ ,

$$y = e^{-2x} (x+1)$$

We had obtained the necessary initial conditions to solve this ODE from the information given, that  $a_0 = 1$ , so that  $y(0) = 1$ .

By doing some simple computation and comparison of powers with  $e^{-2x}$ , then  $a_1 = 2$ ,  $a_2 = -2$ ,  $a_3 = 4/3$

**Exercise 19.**  $\cos x = \sum_{j=0}^{\infty} a_j(j+2)x^j$  for  $f(x) = \sum_{j=0}^{\infty} a_j x^j$ .

Using  $\cos x = \sum_{j=0}^{\infty} \frac{(x)^{2j}}{(2j)!} (-1)^j$  representation, we can immediately conclude that for odd terms,  $a_{2j+1} = 0$  and by matching powers of  $x$ ,

$$a_{2j}(2j+2) = (-1)^j \frac{1}{(2j)!}$$

$$a_5 = 0$$

$$a_6(6+2) = \frac{(-1)^3}{6!} \implies \boxed{a_6 = \frac{-7}{8!}}$$

Now notice that for  $\cos x = \sum_{j=0}^{\infty} a_j(j+2)x^j = \sum_{j=1}^{\infty} j a_j x^j + 2 \sum_{j=0}^{\infty} a_j x^j$  is actually a differential equation,  $\cos x = xy' + 2y$ . We can solve this first-order ODE using

$y(x) = e^{-A(x)} \left( \int_a^x Q(t) e^{A(t)} dt + y(a) \right)$  where  $A(x) = \int_a^x P(t) dt$ . Then solving  $y' + \frac{2y}{x} = \frac{\cos x}{x}$ ,

$$y = \frac{1}{x^2} (x \sin x + \cos x - (a \sin a + \cos a) + b)$$

Plugging 0 as a good guess back into the ODE,  $\cos 0 = 1 = y(0)(2) \implies y(0) = \frac{1}{2}$  With this initial condition, we get

$$\boxed{f(x) = \frac{\sin x}{x} + \frac{\cos x - 1}{x^2} \text{ if } x \neq 0}$$

So  $f(0) = \frac{1}{2}$  and  $f(\pi) = \frac{-2}{\pi^2}$

**Exercise 20.**

(1)

$$\begin{aligned} (1-x)^{-1/2} &= \sum_{j=0}^{\infty} \binom{-1/2}{j} (-x)^j = \\ &= 1 + \frac{1}{2}x + \frac{\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)}{2}x^2 + \frac{\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)\left(\frac{-5}{2}\right)}{3!}x^3 + \frac{\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)\left(\frac{-5}{2}\right)\left(\frac{-7}{2}\right)}{4!}x^4 + \\ &\quad + \frac{\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)\left(\frac{-5}{2}\right)\left(\frac{-7}{2}\right)\left(\frac{-9}{2}\right)}{5!}x^5 + \dots = \\ &= 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{8}x^3 + \frac{35}{128}x^5 + \frac{63}{256}x^5 + \dots \end{aligned}$$

(2) To make the notation clear,  $(1-x)^{-1/2} = \sum_{j=0}^{\infty} \binom{-1/2}{j} (-x)^j = \sum_{j=0}^{\infty} b_j x^j = \sum_{j=0}^{\infty} a_j$

Now

$$\frac{\binom{\alpha}{j+1}}{\binom{\alpha}{j}} = \frac{\alpha(\alpha-1)\dots(\alpha-(j+1)+1)}{(j+1)!} \frac{j!}{\alpha(\alpha-1)\dots(\alpha-j+1)} = \frac{(\alpha-j)}{(j+1)}$$

So for  $\alpha = \frac{-1}{2}$ ,

$$\frac{a_{j+1}}{a_j} = -\left(\frac{1/2+j}{j+1}\right) (-x) < x$$

Using this, we further find that

$$\begin{aligned} b_{j+1} &< b_j \frac{1}{50} \\ b_{j+2} &< b_{j+1} \frac{1}{50} < b_j \left(\frac{1}{50}\right)^2 \end{aligned}$$

For  $x = \frac{1}{50}$ . So by induction,  $b_{n+j} < b_n \left(\frac{1}{50}\right)^j$

$$r_n = \sum_{j=1}^{\infty} a_{n+j} < \sum_{j=1}^{\infty} a_n \left(\frac{1}{50}\right)^j = a_n \frac{1/50}{1-1/50} = \frac{a_n}{49}$$

$$\boxed{r_n < \frac{a_n}{49}}$$

(3) Note that  $(1-x)^{-1/2} = \left(1 - \frac{1}{50}\right)^{-1/2} = \left(\frac{49}{50}\right)^{-1/2} = \frac{5\sqrt{2}}{7}$

$$\frac{7}{5} \left(1 - \frac{1}{50}\right)^{-1/2} = 1 + \frac{1}{100} + \frac{3}{2} \left(\frac{1}{2(50)}\right)^2 + \frac{5}{2} \left(\frac{1}{2(50)}\right)^3 + \frac{35}{8} \left(\frac{1}{2(50)}\right)^4 + \frac{63}{8} \left(\frac{1}{2(50)}\right)^5$$

$$\sqrt{2} \simeq 1.4142135624$$

**Exercise 21.**

(1)  $\frac{1732}{1000} \left(1 - \frac{176}{3000000}\right)^{-1/2} = \frac{1732}{1000} \left(\frac{3000000}{2999824}\right)^{1/2}$

Obviously,  $(3000000)^{1/2} = 1000\sqrt{3}$  so that we have  $1732 (3/2999824)^{1/2}$ .

With long multiplication, we could show easily that  $1732 * 1732 = 2999824$  (it's harder to divide). So then

$$\frac{1732}{1000} \left(1 - \frac{176}{3000000}\right)^{-1/2} = \sqrt{3}$$

(2)

**Exercise 22.**  $\arcsin x = \int \frac{1}{\sqrt{1-x^2}}$

$$(1-x^2)^{-1/2} = \sum_{j=0}^{\infty} \binom{\alpha}{j} (-x^2)^j = 1 + \sum_{j=1}^{\infty} \binom{\alpha}{j} (-x^2)^j$$

$$\implies \arcsin x = x + \sum_{j=1}^{\infty} \binom{-1/2}{j} \frac{(-1)^j}{(2j+1)} x^{2j+1}$$

$$\frac{\left(\frac{-1}{2}\right) \left(\frac{-3}{2}\right) \dots \left(\frac{-1}{2} - j + 1\right)}{j(j-1) \dots (2)(1)} = (-1)^j \frac{(1)(3) \dots (1+2j-2)}{(2j)!!} = (-1)^j \frac{(2j-1)!!}{(2j)!!}$$

$$\implies \arcsin x = x + \sum_{j=1}^{\infty} \frac{(2j-1)!!}{(2j)!!} \frac{x^{2j+1}}{2j+1}$$

**12.4 Exercises - Historical introduction, The vector space of  $n$ -tuples of real numbers, Geometric interpretation for  $n \leq 3$ .**

**Exercise 1.**

$$A = (1, 3, 6)$$

$$B = (4, -3, 3)$$

$$C = (2, 1, 5)$$

(1)  $(5, 0, 9) = A + B$

(2)  $(-3, 6, 3)$

(3)  $(3, -1, 4)$

(4)  $7A - 2B - 3C = (7, 21, 42) - (8, -6, 6) - (6, 3, 15) = (-1, 26, 36) - (6, 3, 15) = (-7, 24, 21)$

**Exercise 2.** See sketches.

**Exercise 3.** See sketches.

**Exercise 4.**

(1) See the sketch.

(2) I think it'll be a line.

(3) It'll fill a parallelogram with  $A$  and  $B$  as the sides.

(4) It'll fill the entire real, 2-dim. plane.

**Exercise 5.**

$$A = (2, 1)$$

$$B = (1, 3)$$

$$C = (c_1, c_2)$$

$$xA + yB = (2x, x) + (y, 3y) = (2x + y, x + 3y) = (c_1, c_2)$$

$$x = \frac{c_2 - 3c_1}{-5}$$

$$y = \frac{c_1 - 2c_2}{-5}$$



**Exercise 6.**

$$A = (1, 1, 1)$$

$$B = (0, 1, 1) \quad D = xA + yB + zC$$

$$C = (1, 1, 0)$$

$$(1) D = (x + z, x + y + z, x + y)$$

$$(2) 0 = x + z = x + y + z \implies y = 0$$

$$x = 0 \quad z = 0$$

$$(3) D = (1, 2, 3)$$

$$x + y + z = 2 = 3 + z \quad z = -1$$

$$x - 1 = 1 \quad x = 2 \quad y = 1$$

**Exercise 7.**

$$A = (1, 1, 1)$$

$$B = (0, 1, 1) \quad D = xA + yB + zC$$

$$C = (2, 1, 1)$$

$$(1) D = (x + 2z, x + y + z, x + y + z)$$

$$(2) x = 2, z = -1, y = -1 \text{ and } D = 0$$

$$(3) x + y + z = 2 \text{ but } x + y + z = 3$$

**Exercise 8.**

$$A = (1, 1, 1, 0)$$

$$B = (0, 1, 1, 1) \quad D = xA + yB + zC$$

$$C = (1, 1, 0, 0)$$

$$(1) D = (x + z, x + y + z, x + y, y)$$

$$(2) y = 0 \implies x = 0 \implies z = 0$$

$$(3) (1, 5, 3, 4) = D \implies y = 4, x = -1, z = 2$$

$$(4) (1, 2, 3, 4) = D \implies y = 4, x = -1, z = -1 \text{ but } x + z = 1$$

**Exercise 9.**

$$A, B \text{ parallel} \implies A = tB \quad t \neq 0$$

$$C, B \text{ parallel} \implies C = sB \quad s \neq 0$$

$$A = tB = \frac{tC}{s} \text{ and } \frac{t}{s} = k \neq 0. \text{ So } A, C \text{ are parallel.}$$

**Exercise 10. Given**

$$C = A + B$$

$$A = sD$$

$$\text{If } C \parallel D \text{ so that } C = tD$$

$$C - A = (t - s)D = B. \text{ If } t - s \neq 0 \text{ then } B \parallel C. \text{ Even if } t = s, \text{ then that meant that } C = A \text{ and } B = 0 \text{ anyways.}$$

$$\text{If } B \parallel D, \text{ so that } B = uD$$

$$c = (s + u)D \implies C \parallel D$$

**Exercise 11.**

(1) Theorem 12.1 says that vector addition is commutative and associative.

$$\text{Let } A, B \in V_n; A = (a_1, \dots, a_n), B = (b_1, \dots, b_n)$$

Use this definition,

**Definition 1.**

$$A = B \text{ iff } a_1 = b_1 \dots a_n = b_n$$

$$A + B = (a_1 + b_1, \dots, a_n + b_n)$$

$$cA = (ca_1, \dots, ca_n)$$

$$A + B = (a_1 + b_1, \dots, a_n + b_n) = (b_1 + a_1, \dots, b_n + a_n) = B + A$$

$$A + (B + C) = (a_1 + (b_1 + c_1), \dots, a_n + (b_n + c_n)) = ((a_1 + b_1) + c_1, \dots, (a_n + b_n) + c_n) = \\ = (A + B) + C$$

$$c(dA) = (c(da_1), \dots, c(da_n)) = ((cd)a_1, \dots, (cd)a_n) = (cd)A$$

$$c(A + B) = (c(a_1 + b_1), \dots, c(a_n + b_n)) = (ca_1 + cb_1, \dots, ca_n + cb_n) = cA + cB$$

$$(c + d)A = ((c + d)a_1, \dots, (c + d)a_n) = (ca_1 + da_1, \dots, ca_n + da_n) = cA + dA$$

So commutativity and associativity follows from the associativity, commutativity, and distributivity of the one-dim. reals (or even complex numbers).

(2) See sketch.

**Exercise 12.**

$$\begin{aligned} A &= B - C & A + \frac{1}{2}C - \frac{1}{2}A &= \\ C &= B - A & = \frac{1}{2}A + \frac{1}{2}C &= \frac{1}{2}(A + C) \\ B &= A + C & = \frac{1}{2}B & \end{aligned}$$

It reminds me of how the diagonals of a parallelogram bisect each other.

**12.8 Exercises - The dot product, Length or norm of a vector, Orthogonality of vectors.**

**Exercise 1.**

$$A = (1, 2, 3, 4)$$

$$B = (-1, 2, -3, 0)$$

$$C = (0, 1, 0, 1)$$

$$(1) A \cdot B = -6$$

$$(2) B \cdot C = 2$$

$$(3) A \cdot C = 6$$

$$(4) A \cdot (B + C) = A \cdot (-1, 3, -3, 1) = 0$$

$$(5) (A - B) \cdot C = (2, 0, 6, 4) \cdot C = 4$$

**Exercise 2.**

$$A = (2, 4, -7)$$

$$B = (2, 6, 3)$$

$$C = (3, 4, -5)$$

$$(1) (A \cdot B)C = 7(3, 4, -5) = (21, 28, -35)$$

$$(2) A \cdot (B + C) = A \cdot (5, 10, -2) = 64$$

$$(3) (A + B) \cdot C = (4, 10, -4) \cdot C = 72$$

$$(4) A(B \cdot C) = (30, 60, -105)$$

$$(5) A/(B \cdot C) = \left(\frac{2}{15}, \frac{4}{15}, \frac{-7}{15}\right)$$

**Exercise 3.** Given  $A \cdot B = A \cdot C$ ,  $A \neq 0$ ,

Then  $\|A\|\|B\| \cos \theta_{AB} = \|A\|\|C\| \cos \theta_{AC}$

Let  $\|B\| = \|C\|$  but  $\theta_{AB} = -\theta_{AC}$ . Then  $B \neq C$ .

**Exercise 4.** Given  $A \cdot B = 0 \quad \forall A$ ,

$\|A\|\|B\| \cos \theta_{AB} = 0$ , choose  $\|A\| \neq 0$ ,  $\cos \theta_{AB} \neq 0$

$\|B\| = 0$

**Exercise 5.**

$$A = (2, 1, -1)$$

$B = (1, -1, 2)$ ; find a nonzero  $C$  in  $V_3$  s.t.  $A \cdot C = B \cdot C = 0$

$C = (c_1, c_2, c_3)$  without loss of generality, let  $c_1 = 1$

$$2 + c_2 - c_3 = 0 \quad c_2 = -5$$

$$1 - c_2 + 2c_3 = 0 \quad \implies \quad c_3 = -3$$

**Exercise 6.**

$$A = (1, -2, 3)$$

$$B = (3, 1, 2) \quad C = xA + yB$$

$$B \cdot B = 14$$

$$C \cdot B = 0$$

$$C \cdot B = (xA + yB) \cdot B = x(A \cdot B) + yB \cdot B = 0$$

$$x(7) + y(14) = 0 \implies x + 2y = 0$$

$$y = -1, x = 2 \quad C = (-1, -5, 4)$$

**Exercise 7.**

$$A = (2, -1, 2)$$

Given

$$B = (1, 2, -2)$$

$$A = C + D$$

$$B \cdot D = 0$$

$$C \parallel B$$

$$B = tC$$

$$tC \cdot D = 0 \quad t \neq 0$$

$$A \cdot B = B \cdot C = -4$$

$$B \cdot B = tB \cdot C = 9 \quad \Rightarrow t = 9/-4$$

$$C = \frac{-4}{9}(1, 2, -2)$$

$$D = A - C = (2, -1, 2) - \left(\frac{4}{9}, \frac{8}{9}, \frac{-8}{9}\right) = \left(\frac{22}{9}, \frac{-1}{9}, \frac{10}{9}\right)$$

**Exercise 8.** Given that if

$$A = (1, 2, 3, 4, 5) \quad C, D \in V_5$$

$$B = (1, 1/2, 1/3, 1/4, 1/5)$$

$$B = C + 2D \quad D \cdot A = 0, \quad C \parallel A; \quad C = tA$$

$$\Rightarrow B = tA + 2D$$

$$A \cdot B = 5 = tA \cdot A + 2A \cdot D = t(55) \quad t = 1/11$$

$$C = \frac{1}{11}(1, 2, 3, 4, 5)$$

$$D = \frac{1}{2}\left(B - \frac{A}{11}\right) = \frac{1}{2}\left((1, 1/2, 1/3, 1/4, 1/5) - \left(\frac{1}{11}, \frac{2}{11}, \frac{3}{11}, \frac{4}{11}, \frac{5}{11}\right)\right)$$

$$D = \left(\frac{5}{11}, \frac{7}{44}, \frac{1}{33}, \frac{-5}{88}, \frac{-7}{55}\right)$$

**Exercise 9.**

$$A = (2, -1, 5) \quad \|A + B\| = \|(1, -3, 8)\| = \sqrt{74}$$

$$B = (-1, -2, 3) \quad \|A - B\| = \|(3, 1, 2)\| = \sqrt{14}$$

$$C = (1, -1, 1) \quad \|A + B - C\| = \|(0, -2, 7)\| = \sqrt{53}$$

$$\|A - B + C\| = \|(4, 0, 3)\| = 5$$

**Exercise 10.**  $B \cdot A = 0; \quad \|B\| = \|A\|$

- (1)  $B = (1, -1)$
- (2)  $A = (1, -1), B = (1, 1)$
- (3)  $A = (2, -3), B = (3, 2)$
- (4)  $A = (a, b), B = (b, -a)$

**Exercise 11.**

$$A = (1, -2, 3)$$

$$B = (3, 1, 2)$$

$$(1) A + B = (4, -1, 5) \quad \Rightarrow \quad C = \frac{(4, -1, 5)}{\sqrt{42}}$$

$$(2) A - B = (-2, -3, 1) \quad C = \frac{(-2, -3, 1)}{\sqrt{14}}$$

$$(3) A + 2B = (7, 0, 7) \quad \Rightarrow \quad C = (1/\sqrt{2}, 0, 1/\sqrt{2})$$

$$(4) 2A - B = (-1, -5, 4) \quad C = \frac{(-1, -5, 4)}{\sqrt{42}}$$

**Exercise 12.**

$$A = (4, 1, -3)$$

$$B = (1, 2, 2)$$

$$C = (1, 2, -2)$$

$$D = (2, 1, 2)$$

$$E = (2, -2, 1)$$

$$\begin{array}{l} A \cdot B = 0 \\ C \cdot D = 0 \\ C \cdot E = 0 \\ D \cdot E = 0 \end{array}$$

**Exercise 13.** The answer is easy to see geometrically. If you do the algebra, the condition of  $\|A\| = \|B\|$  fixes the length of the vector and  $A \cdot B$  orthogonality fixes the direction, up to  $\pi$  or  $180^\circ$ .

$$(1) A = (1, 2) \Rightarrow B = (-2, 1), (2, -1)$$

$$(2) A = (1, -2) \Rightarrow B = (2, 1), (-2, -1)$$

$$(3) A = (2, -1) \Rightarrow B = (1, 2), (-1, -2)$$

$$(4) A = (-2, 1) \Rightarrow B = (1, 2), (-1, -2)$$

**Exercise 14.**

$$A = (2, -1, 1)$$

$$B = (3, -4, -4)$$

$$B - A = (1, -3, -5)$$

$$A \cdot B = 6$$

$$A + B = (5, -5, -3)$$

We have the condition that we have one right angle for a right triangle:  $(A - C) \cdot (C - B) = A \cdot C - C^2 - A \cdot B + B \cdot C = (A + B) \cdot C - C^2 - 6 = 0$

$$(5, -5, -3) \cdot C = 6 + C^2$$

$$5c_1 - 5c_2 + 0 = 6 + c_1^2 + c_2^2 \quad \implies c_2 = -5$$

$$\xrightarrow{c_1=2} 10 - 5c_2 = 10 + c_2^2$$

We just need to find one  $C$ . Let  $c_3 = 0$  for  $C = (c_1, c_2, c_3)$ . Then we get the above last two statements if we also choose  $c_1 = 2$ .

So we have  $C = (2, -5, 0)$

So in summary, we get

$$B - A = (1, -3, -5) \quad \|A - C\| = \sqrt{17}$$

$$A - C = (0, 4, 1) \quad \text{and indeed } \|C - B\| = \sqrt{18} \text{ and } (A - C) \cdot (C - B) = 0, \text{ as a right triangle should.}$$

$$C - B = (-1, -1, 4) \quad \|B - A\| = \sqrt{35}$$

**Exercise 15.**

$$A = (1, -1, 2)$$

$$B = (2, 1, -1) \quad \implies C = (1, -5, -3)$$

**Exercise 16.** Given that

$$A = (1, 2) \quad A = P + Q$$

$$B = (3, 4) \quad P \parallel B \implies P = sB$$

$$Q \perp B$$

$$\implies P = \frac{11}{25}(3, 4), \quad Q = A - P = \left( \frac{-8}{25}, \frac{6}{25} \right)$$

**Exercise 17.** Given  $A = (1, 2, 3, 4)$ ,  $B = (1, 1, 1, 1)$ , then for  $A = P + Q$ ,  $P \parallel B$ ,  $Q \cdot B = 0$ ,

$$P = \frac{5}{2}(1, 1, 1, 1), \quad Q = A - P = \left( \frac{-3}{2}, \frac{-1}{2}, \frac{1}{2}, \frac{3}{2} \right)$$

**Exercise 18.** Given

$$A = (2, -1, 1) \quad D = xB + yC$$

$$B = (1, 2, -1) \quad A \cdot D = 0$$

$$C = (1, 1, -2) \quad \|D\| = 1$$

$$\text{then } D = \pm \frac{1}{\sqrt{2}}B + \mp \frac{1}{\sqrt{2}}C$$

**Exercise 19.**  $\|A + B\|^2 - \|A - B\|^2 = A^2 + 2A \cdot B + B^2 - (A^2 - 2A \cdot B + B^2) = 4A \cdot B$

**Exercise 20.**  $\|A + B\|^2 + \|A - B\|^2 = A^2 + 2A \cdot B + B^2 + (A^2 - 2A \cdot B + B^2) = 2A^2 + 2B^2$

The sum of the square of the sides of a parallelogram is equal to the sum of the square of the diagonals of a parallelogram.

**Exercise 21.**

"The sum of the squares of the sides of any quadrilateral exceeds the sum of the squares of the diagonals by four times the square of the length of the line segment which connects the midpoints of the diagonals."

$$\begin{aligned} \text{Sum of the squares of the sides} &= A^2 + (C - A)^2 + (C - B)^2 + B^2 = \\ &= A^2 + C^2 - 2AC + A^2 + C^2 - 2CB + B^2 + B^2 \end{aligned}$$

$$\begin{aligned} \text{sum of the squares of the diagonals} &= C^2 + (A - B)^2 = \\ &= C^2 + A^2 - 2AB + B^2 \end{aligned}$$

$$\text{four times the square of the length of the line segment which connects the midpoints of the diagonals} =$$

$$= 4 \left( \frac{C}{2} - \left( \frac{A+B}{2} \right) \right)^2 =$$

$$C^2 - 2C(A + B) + A^2 + 2AB + B^2$$

So then if we take the *sum of the squares of the sides* - *sum of the squares of the diagonals* =

= four times the square of the length of the line segment which connects the midpoints of the diagonals, then we get the desired equality.

**Exercise 22.** Given that  $\|A\| = 6$  and for any pair of scalars,  $x, y$ ,

$$(xA + yB) \cdot (4yA - 9xB) = 4xyA^2 - 9x^2A \cdot B + 4y^2A \cdot B - 9xyB^2 = 0 \implies 144xy + (4y^2 - 9x^2)A \cdot B = 9xyB^2$$

$x = 2, y = 3$  without loss of generality.

$$144xy = 9xyB^2 \implies 16 = B^2, \boxed{\|B\| = 4}$$

This implies that  $A \cdot B$  in order for the above statements to be true  $\forall x, y$

$$\|(2A + 3B)\|^2 = 4A^2 + 12A \cdot B + 9B^2 = 144 + 144 = (12\sqrt{2})^2 \implies \|2A + 3B\| = 12\sqrt{2}$$

**Exercise 23.** Given that

$$\begin{aligned} A &= (1, 2, 3, 4, 5) & B &= C + D \\ & & D &\perp A \\ B &= (1, 1/2, 1/3, 1/4, 1/5) & C &\parallel A \text{ or } C = tA \end{aligned}$$

$$B \cdot A = 5 = C \cdot A + D \cdot A = tA^2 = 55t$$

$$\boxed{C = \frac{1}{11}(1, 2, 3, 4, 5)}$$

$$D = B - C = (1, 1/2, 1/3, 1/4, 1/5) - \frac{1}{11}(1, 2, 3, 4, 5) = \boxed{\left(\frac{10}{11}, \frac{7}{22}, \frac{2}{33}, \frac{-5}{44}, \frac{-14}{55}\right)}$$

**Exercise 24.** Given that

$$A \cdot B \neq 0$$

$$C \parallel A \quad C = tA$$

$D \perp A$  so that  $A \cdot D = 0$  since  $\|A\| \neq 0$ , if  $t \neq 0$ , then  $C$  exists.

$$B = C + D$$

$$A \cdot B = C \cdot A + D \cdot A = tA^2 \implies t = \frac{A \cdot B}{A^2} \quad \boxed{C = \frac{A \cdot B}{A^2} A}$$

$$\boxed{D = B - C = B - \frac{A \cdot B}{A^2} A}$$

**Exercise 25.**

(1) Given that  $A \cdot B = 0$ ,  $\|A + xB\|^2 = A^2 + 2xA \cdot B + x^2B^2 = A^2 + x^2B^2 \geq A^2$  since  $x^2B^2$  is positive.

(2) Given that  $\|A + xB\| \geq \|A\|$ , then  $A^2 + 2xA \cdot B + x^2B^2 \geq A^2$ .

$$x(2A \cdot B + xB^2) \geq 0$$

If  $x > 0$

$$2A \cdot B + xB^2 \geq 0$$

$$xB^2 \geq -2A \cdot B$$

$$\text{then } A \cdot B \geq 0$$

If  $x < 0$

$$2A \cdot B + xB^2 \leq 0$$

$$xB^2 \leq -2A \cdot B$$

$$\text{then } A \cdot B \leq 0$$

$$\implies A \cdot B = 0$$

We relied heavily upon the fact that  $x$  was any number, so we could make  $x$  as small or as big as we want.

## 12.11 Exercises - Projections. Angle between vectors in $n$ -space, The unit coordinate vectors.

**Exercise 1.** Given

$$A = (1, 2, 3) \quad A \cdot B = 11$$

$$B = (1, 2, 2) \quad B \cdot B = 9$$

$$\text{So the projection of } A \text{ on } B \text{ is } \boxed{\frac{A \cdot B}{B \cdot B} B = \frac{11}{9}(1, 2, 2)}$$

**Exercise 2.** Given

$$A = (4, 3, 2, 1) \quad B \cdot B = 4$$

$$B = (1, 1, 1, 1) \quad A \cdot B = 10$$

$$\text{So the projection of } A \text{ on } B \text{ is } \boxed{\frac{5}{2}(1, 1, 1, 1)}$$

**Exercise 3.**

$$(1) A = (6, 3, -2).$$

$$\boxed{\begin{aligned} \cos a &= \frac{6}{7} \\ \cos b &= \frac{3}{7} \\ \cos c &= \frac{-2}{7} \end{aligned}}$$

$$(2) \frac{1}{7}(6, 3, -2)$$

**Exercise 4.** Given

$$A = (1, 2, 1) \quad C = (1, 4, 1)$$

$$B = (2, 1, -1) \quad D = (2, 5, 5)$$

$$\begin{aligned} |A| &= \sqrt{6} & |C| &= \sqrt{18} = 3\sqrt{2} & \frac{A \cdot B}{|A||B|} &= \frac{1}{2} \implies \theta_{AB} = \boxed{\frac{\pi}{6}} \\ |B| &= \sqrt{6} & |D| &= \sqrt{54} = 3\sqrt{6} & \frac{C \cdot D}{|C||B|} &= \frac{\sqrt{3}}{2} \implies \theta_{CD} = \boxed{\frac{\pi}{3}} \end{aligned}$$

**Exercise 5.** If we let

$A = (2, -1, 1)$	$B - A = x_1 = (-1, -2, -6)$	$ x_1  = \sqrt{41}$	$x_1 \cdot x_2 = -6$	$\theta_{12} = -6/(\sqrt{246})$ $\theta_{13} = -35/(\sqrt{1435})$ $\theta_{23} = 0$
$B = (1, -3, -5)$	$C - B = x_2 = (2, -1, 1)$	$ x_2  = \sqrt{6}$	$x_1 \cdot x_3 = -35$	
$C = (3, -4, -4)$	$A - C = x_3 = (-1, 3, 5)$	$ x_3  = \sqrt{35}$	$x_2 \cdot x_3 = 0$	

**Exercise 6.** Given that  $\|A\| = \|C\| = 5$ ,  $\|B\| = 1$ ,  $\|A - B + C\| = \|A + B + C\|$ , and  $\frac{A \cdot B}{|A||B|} = \cos \frac{\pi}{8}$ , then consider

$$(A - B + C) \cdot (A - B + C) = (A + B + C) \cdot (A + B + C)$$

$$((A + C) - B)^2 = (A + C + B)^2 \implies -2(A + C) \cdot B = 2(A + C) \cdot B$$

$$0 = A \cdot B + B \cdot C \implies -A \cdot B = B \cdot C$$

$$\implies \frac{-5(1) \cos \frac{\pi}{8}}{(1)5} = \cos \theta_{BC} \implies \theta_{BC} = \boxed{\frac{9\pi}{8}}$$

**Exercise 7.** Given that  $\frac{A \cdot C}{|A||C|} = \cos \theta_{AC} = \frac{B \cdot C}{|B||C|} = \cos \theta_{BC}$ , then  $\frac{A \cdot C}{|A|} = \frac{B \cdot C}{|B|}$  or  $|B|A \cdot C = |A|B \cdot C$ .

$$\text{So } C \cdot (|B|A - |A|B) = |B|(A \cdot C) - |A|(B \cdot C) = 0$$

**Exercise 8.** Given that

$$\begin{aligned} |A|^2 &= n \\ A &= (1, 1, \dots, 1) \\ B &= (1, 2, \dots, n) \end{aligned} \quad \text{then} \quad \begin{aligned} |B|^2 &= \sum_{j=1}^n j^2 \\ A \cdot B &= \sum_{j=1}^n j = \frac{n(n+1)}{2} \end{aligned}$$

To find  $\sum_{j=1}^n j^2$ , recall this trick:

$$\begin{aligned} S &= \sum_{j=1}^n j^3 = \sum_{j=0}^{n-1} (j+1)^3 = \sum_{j=0}^{n-1} j^3 + 3j^2 + 3j + 1 = \\ &= S - n^3 + 3 \sum_{j=1}^{n-1} j^2 + 3 \left( \frac{(n-1)n}{2} \right) + n \\ &\implies \sum_{j=1}^n j^2 = \frac{(n+1)(2n+1)n}{6} \end{aligned}$$

So now we could get a remarkable limit:

$$\cos \theta_{AB} = \frac{A \cdot B}{|A||B|} = \frac{\frac{n(n+1)}{2}}{\sqrt{n} \sqrt{\frac{(n+1)(2n+1)n}{6}}} = \frac{\sqrt{3}}{2} \frac{1 + 1/n}{\sqrt{(1 + 1/n)(1 + 1/2n)}} \xrightarrow{n \rightarrow \infty} \boxed{\frac{\sqrt{3}}{2}} \text{ or } \theta_{AB} = \pi/3$$

**Exercise 9.** Given that

$$\begin{aligned} |A|^2 &= \sum_{j=1}^n (2j)^2 \\ A &= (2, 4, 6, \dots, 2n) \\ B &= (1, 3, 5, \dots, 2n-1) \end{aligned} \quad \text{then} \quad \begin{aligned} |B|^2 &= \sum_{j=1}^n (2j-1)^2 \\ A \cdot B &= \sum_{j=1}^n 2j(2j-1) \end{aligned}$$

Doing the math,

$$\begin{aligned}
 |A|^2 &= \sum_{j=1}^n 4j^2 = 4 \sum_{j=1}^n j^2 = \frac{4(n+1)(2n+1)n}{6} \\
 |B|^2 &= \sum_{j=1}^n (2j-1)^2 = \sum_{j=1}^n (4j^2 - 4j + 1) = 4 \frac{(n+1)(2n+1)n}{6} - 4 \frac{n(n+1)}{2} + n = \\
 &= n \frac{(4n^2 + 6n + 2 - 6n - 6 + 3)}{3} = \frac{n(4n^2 - 1)}{3} \\
 A \cdot B &= \sum_{j=1}^n 2j(2j-1) = \sum_{j=1}^n 4j^2 - 2j = 4 \frac{(n+1)(2n+1)n}{6} - 2 \frac{n(n+1)}{2} = \\
 &= \frac{(2(2n^2 + 3n + 1) - 3n - 3)n}{3} = \frac{(4n^2 + 3n - 1)n}{3} \\
 \frac{A \cdot B}{|A||B|} &= \frac{n(4n^2 + 3n - 1)/3}{\sqrt{\frac{2(2n^2 + 3n + 1)n}{3} \left( \frac{n(4n^2 - 1)}{3} \right)}} = \frac{4 + 3/n - 1/n^2}{\sqrt{2(2 + 3/n + 1/n)(4 - 1/n^2)}} \xrightarrow{n \rightarrow \infty} \frac{4}{\sqrt{2(2)4}} = \boxed{1}
 \end{aligned}$$

It seems to make sense that  $\theta_{AB} \xrightarrow{n \rightarrow \infty} 0$ , as the two vector getting closer and closer.

**Exercise 10.**

$$\begin{aligned}
 (1) \quad & A = (c\theta, -s\theta) \quad c^2 + s^2 = 1 \\
 & B = (s\theta, c\theta) \quad A \cdot B = cs - cs = 0 \\
 (2) \quad & (x, y) \in V_2,
 \end{aligned}$$

$$(x, y) = xA + yB = x(C, -S) + y(S, C) = (xC + yS, -xS + yC) \implies \begin{matrix} x = xC + yS \\ y = -xS + yC \end{matrix} \implies \begin{matrix} x(1 - C) = yS \\ y(1 - C) = -xS \end{matrix}$$

If  $\theta \neq 2\pi j$  and  $y \neq 0$ , then

$$xy(1 - C) = y^2 S$$

$$x(-xS) = y^2 S$$

$$-x^2 S = y^2 S$$

If  $\theta \neq \pi j$ , then contradiction. If  $\theta = (2j - 1)\pi$ ,  $2x = 0$ ,  $y = 0$ . If  $\theta = 2j\pi$ , then  $\forall x, y \in \mathbb{R}$

**Exercise 11.** For a rhombus, with adjacent sides  $A, B$ ,  $|A| = |B|$ . The diagonals are given by  $A + B$  and  $A - B$ .

$$(A + B) \cdot (A - B) = A^2 - B^2 = 0$$

**Exercise 12.**  $(\cos a, \sin a) \cdot (\cos b, \sin b) = \cos a \cos b + \sin a \sin b$ . But  $(\cos a, \sin a) \cdot (\cos b, \sin b) = \cos a - b/(1)(1)$  because the angle between the two vectors is  $a - b$ .

**Exercise 13.**

$$\|A - B\|^2 = (A - B) \cdot (A - B) = A^2 + B^2 - 2\|A\|\|B\| \cos \theta$$

**Exercise 14.** Given  $A \cdot B = \sum_{k=1}^n |a_k b_k|$ ,

$$\begin{aligned}
 B \cdot A &= \sum_{j=1}^n |b_j a_j| = \sum_{j=1}^n |a_j b_j| = A \cdot B \\
 A \cdot (cB) &= \sum_{j=1}^n |a_j c b_j| = \sum_{j=1}^n |c a_j b_j| = (cA) \cdot B = |c| \sum_{j=1}^n |a_j b_j| = |c|(A \cdot B) \\
 A \cdot A &= \sum_{j=1}^n |a_j|^2 \geq 0, \quad A \cdot A = 0 \text{ only if } A = 0
 \end{aligned}$$

To prove the Cauchy-Schwarz inequality for this metric, recall the proof for Thm. 1.41.

$$\sum_{j=1}^n (|a_j|x + |b_j|)^2 = \sum_{j=1}^n |a_j|^2 x^2 + 2|a_j||b_j|x + |b_j|^2 = x^2 \sum_{j=1}^n |a_j|^2 + 2x \sum_{j=1}^n |a_j b_j| + \sum_{j=1}^n |b_j|^2 \geq 0$$

$$A' = \sum_{j=1}^n |a_j|^2 = A \cdot A$$

$$x^2 A' + 2xB' + C' \geq 0 \text{ where } B' = \sum_{j=1}^n |a_j b_j| = A \cdot B \implies A' \left( x + \frac{B'}{A'} \right)^2 + \frac{A'C' - B'^2}{A'} \geq 0$$

$$C' = \sum_{j=1}^n |b_j|^2 = B \cdot B$$

$$\text{Let } x = \frac{-B'}{A'}$$

$$\implies A'C' \geq B'^2 \text{ or } \boxed{\sum_{j=1}^n |a_j|^2 \sum_{j=1}^n |b_j|^2 \geq \left( \sum_{j=1}^n |a_j b_j| \right)^2}$$

**Exercise 15.** Given  $A \cdot B = 2a_1 b_1 + a_2 b_2 + a_1 b_2 + a_2 b_1$

$$B \cdot A = 2b_1 a_1 + b_2 a_2 + b_1 a_2 + b_2 a_1 = A \cdot B = 2a_1 b_1 + a_2 b_2 + a_1 b_2 + a_2 b_1$$

$$A \cdot B + A \cdot C = 2a_1 b_1 + a_2 b_2 + a_1 b_2 + a_2 b_1 + 2a_1 c_1 + a_2 c_2 + a_1 c_2 + a_2 c_1 =$$

$$= 2a_1(b_1 + c_1) + a_2(b_2 + c_2) + a_1(b_2 + c_2) + a_2(b_1 + c_1) = A \cdot (B + C)$$

$$(cA) \cdot B = 2(ca_1)b_1 + ca_2 b_2 + ca_1 b_2 + ca_2 b_1 = 2a_1(cb_1) + a_2(cb_2) + a_1(cb_2) + a_2(cb_1) =$$

$$= c(2a_1 b_1 + a_2 b_2 + a_1 b_2 + a_2 b_1) = A \cdot (cB) = c(A \cdot B)$$

$$A \cdot A = 2a_1^2 + a_2^2 + a_1 a_2 + a_2 a_1 = a_1^2 + (a_1 + a_2)^2 \geq 0, \quad A \cdot A = 0 \text{ only if } \begin{matrix} a_1 = 0 \\ a_1 + a_2 = 0 \end{matrix} \implies A = 0$$

For the Cauchy-Schwarz inequality,

$$A^2 B^2 = (a_1^2 + (a_1 + a_2)^2)(b_1^2 + (b_1 + b_2)^2)$$

$$\begin{aligned} (A \cdot B)^2 &= (a_1 b_1 + a_2(b_1 + b_2) + a_1(b_1 + b_2))^2 = (a_1 b_1 + (a_1 + a_2)(b_1 + b_2))^2 = \\ &= (a_1 b_1)^2 + 2a_1 b_1(a_1 + a_2)(b_1 + b_2) + (a_1 + a_2)^2(b_1 + b_2)^2 \end{aligned}$$

$$\text{Now } (a_1(b_1 + b_2) - b_1(a_1 + a_2))^2 \geq 0$$

$$a_1^2(b_1 + b_2)^2 - 2a_1 b_1(b_1 + b_2)(a_1 + a_2) + b_1^2(a_1 + a_2)^2 \geq 0$$

$$\text{This fact shows that } A^2 B^2 \geq (A \cdot B)^2$$

**Exercise 16.** Given  $A \cdot B = 2a_1 b_1 + a_2 b_2 + a_3 b_3 + a_1 b_3 + a_3 b_1$

$$B \cdot A = 2b_1 a_1 + b_2 a_2 + b_3 a_3 + b_1 a_3 + b_3 a_1 = A \cdot B = 2a_1 b_1 + a_2 b_2 + a_3 b_3 + a_3 b_1 + a_1 b_3$$

$$A \cdot B + A \cdot C = 2a_1 b_1 + a_2 b_2 + a_3 b_3 + a_1 b_3 + a_3 b_1 + 2a_1 c_1 + a_2 c_2 + a_3 c_3 + a_1 c_3 + a_3 c_1 =$$

$$= 2a_1(b_1 + c_1) + a_2(b_2 + c_2) + a_3(b_3 + c_3) + a_1(b_3 + c_3) + a_3(b_1 + c_1) = A \cdot (B + C)$$

$$(cA) \cdot B = 2(ca_1)b_1 + (ca_2)b_2 + (ca_3)b_3 + (ca_1)b_3 + (ca_3)b_1 = 2a_1(cb_1) + a_2(cb_2) + a_3(cb_3) + a_1(cb_3) + a_3(cb_1) =$$

$$= c(2a_1 b_1 + a_2 b_2 + a_3 b_3 + a_1 b_3 + a_3 b_1) = A \cdot (cB) = c(A \cdot B)$$

$$A \cdot A = 2a_1^2 + a_2^2 + a_3^2 + a_1 a_3 + a_3 a_1 = a_1^2 + a_2^2 + (a_1 + a_3)^2 \geq 0, \quad A \cdot A = 0 \text{ only if } \begin{matrix} a_1 = 0 \\ a_2 = 0 \\ a_1 + a_3 = 0 \end{matrix} \implies A = 0$$

For the Cauchy-Schwarz inequality,

$$A^2 B^2 = (a_1^2 + a_2^2 + (a_1 + a_3)^2)(b_1^2 + b_2^2 + (b_1 + b_3)^2)$$

$$\begin{aligned} (A \cdot B)^2 &= (a_1 b_1 + a_2 b_2 + a_3(b_1 + b_3) + a_1(b_1 + b_3))^2 = (a_1 b_1 + a_2 b_2 + (a_1 + a_3)(b_1 + b_3))^2 = \\ &= (a_1 b_1)^2 + a_2^2 b_2^2 + 2a_1 b_1 a_2 b_2 + 2(a_1 b_1 + a_2 b_2)(a_1 + a_3)(b_1 + b_3) + (a_1 + a_3)^2(b_1 + b_3)^2 \end{aligned}$$



As you could see, a direct proof has messy algebra. Instead, since this dot product satisfies all the usual dot product properties, then we can just reuse the vector method proof of the Cauchy-Schwarz inequality, without reference to a specific coordinate system, to conclude that the Cauchy-Schwarz inequality holds. *That's the trick* - think in terms of generality, of vectors in general, and not specific coordinate systems.

**Exercise 17.** Given that  $\|A\| = \sum_{j=1}^n |a_j|$

(1)

$$\sum |a_j| > 0, \quad \sum |a_j| = 0 \text{ if } a_j = 0 \quad \forall j$$

$$\|cA\| = \sum |ca_j| = |c| \sum |a_j| = |c| \|A\|$$

$$\|A + B\| = \sum |a_j + b_j| \leq \sum |a_j| + |b_j| = \|A\| + \|B\|$$

(2) See sketch. Looks like a diamond.

(3) Given  $\|A\| = \left| \sum_{j=1}^n a_j \right|$

$$\|A\| > 0 \text{ if } A \neq 0 \text{ but } \|A\| = 0 \text{ for } a_j = (-1)^j \text{ and } n = 2m$$

$$\|cA\| = \left| \sum_{j=1}^n ca_j \right| = |c| \left| \sum_{j=1}^n a_j \right| = |c| \|A\|$$

$$\|A + B\| = \left| \sum_{j=1}^n (a_j + b_j) \right| \leq \left| \sum_{j=1}^n a_j \right| + \left| \sum_{j=1}^n b_j \right| = \|A\| + \|B\|$$

**Exercise 18.** Given that  $\|A\| = \max_{1 \leq j \leq n} |a_j|$

(1)  $\|A\| \geq 0$  since  $|a_j| \geq 0$  for any  $a_j \in \mathbb{R}$

If  $A \neq 0$  then  $\exists$  at least one  $a_j$  s.t.  $a_j \neq 0$   
 $|a_j| > 0$ , so  $\|A\| = 0$  only if  $A = 0$ .

$$\|cA\| = \max_{1 \leq j \leq n} |ca_j| = \max_{1 \leq j \leq n} |c| |a_j|$$

Consider  $a_J = \max_{1 \leq j \leq n} |a_j|$

$$|a_J| \geq |a_j| \quad \forall j \neq J$$

So  $|c| |a_J| \geq |c| |a_j|$  is valid.  $|c| |a_j| = |ca_j|$ . Then

$$\max_{1 \leq j \leq n} |c| |a_j| = |c| \max_{1 \leq j \leq n} |a_j|$$

$$\|A + B\| = \max_{1 \leq j \leq n} |a_j + b_j| \leq \max_{1 \leq j \leq n} |a_j| + |b_j| \leq \max_{1 \leq j \leq n} |a_j| + \max_{1 \leq k \leq n} |b_k| = \|A\| + \|B\|$$

(2) See sketch. Looks like a square.

**Exercise 19.** Given  $A = (a_1, \dots, a_n)$ ,  $\|A\|_1 = \sum_{j=1}^n |a_j|$ ,  $\|A\|_2 = \max_{1 \leq j \leq n} |a_j|$ ,  $\|A\| = \sqrt{\sum_{j=1}^n (a_j)^2}$ .

Let  $\|A\|_2 = \max_{1 \leq j \leq n} |a_j| = |a_J|$ .

$\|A\|^2 = \sum_{j=1}^n a_j^2$ ,  $\|A\|^2$  sum obvious includes  $a_J$  term, and all  $a_j^2$  terms are positive.

$$\implies \|A\|^2 \geq \|A\|_2^2 \text{ or } \|A\| \geq \|A\|_2$$

$\left( \sum_{j=1}^n |a_j| \right)^2 \geq \sum_{j=1}^n (a_j)^2$ , because of the cross terms, which are positive from the absolute value function on each individual term.

So then  $\|A\|_1 \geq \|A\|$

Note that we had used the fact that if  $b^2 > a^2$  and  $a, b > 0$ , then  $b > a$ . This is because

If  $a = b$ , then  $a^2 = b^2$ . Contradiction.

If  $a < b$ , then  $a \cdot a < a \cdot b < b^2$ , from  $ab < b^2$ . Contradiction.

$\|A\|_2 \leq \|A\| \leq \|A\|_1$  means that 2-norm will “norm” or “expand” a given coordinate pair the least, assign to it the smallest norm, so coordinates of bigger values are needed for 2-norm. In contrast, the 1-norm assigns bigger values for the norm of a given pair of coordinates. The 1-norm requires smaller values for the coordinates to get the same norm value as the 2-norm and usual spherical norm.

Notice how going from 2 norm,  $\|A\|$ , and 1 norm, we go from a square, to a circle, and finally to a diamond.

**Exercise 20.** Given that  $d(A, B) = \|A - B\|$ ,

$$d(B, A) = \|B - A\| = \|(-1)(A - B)\| = |-1|\|A - B\| = \|A - B\|$$

$$d(A, B) = \|A - B\| = 0 \text{ only if } A - B = 0 \implies A = B$$

$$d(A, B) = \|A - B\| = \|A - C + C - B\| \leq \|A - C\| + \|C - B\| = d(A, C) + d(C, B)$$

**12.15 Exercises - The linear span of a finite set of vectors, Linear independence, Bases.** **Exercise 1.**  $x(i - j) + y(i + j)$

(1)  $i; x = 1/2, y = 1/2$

(2)  $j; x = -1/2, y = 1/2$

(3)  $3i - 5j; x = 4, y = -1$

(4)  $7i + 5j; x = 1, y = 6$

**Exercise 2.**

$$A = (1, 2)$$

$$B = (2, -4) \quad C = xA + yB \implies x = 1/4, \quad y = 7/8$$

$$C = (2, -3)$$

**Exercise 3.**

$$A = (2, -1, 1)$$

$$B = (1, 2, -1) \quad C = xA + yB$$

$$C = (2, -11, 7)$$

$$\left[ \begin{array}{cc|c} 2 & 1 & 2 \\ -1 & 2 & -11 \\ 1 & -1 & 7 \end{array} \right] \implies \left[ \begin{array}{cc|c} 0 & 1 & -4 \\ -1 & 0 & -3 \\ 0 & 1 & -4 \end{array} \right]$$

$$\implies \boxed{\begin{array}{l} x = 3 \\ y = -4 \end{array}}$$

**Exercise 4.** If  $C = (2, 11, 7)$ ,

$$\left[ \begin{array}{cc|c} 2 & 1 & 2 \\ -1 & 2 & 11 \\ 1 & -1 & 7 \end{array} \right] = \left[ \begin{array}{cc|c} 0 & 5 & 24 \\ -1 & 2 & 11 \\ 0 & 3 & 29 \end{array} \right]$$

**Exercise 5.**

(1)  $A \parallel B \implies A = tB, t \neq 0$

$A - tB = 0$  and  $t \neq 0$ . So  $A, B$  linearly dependent.

(2)  $A \nparallel B$

$A - tB \neq 0$  since  $\nexists t \in \mathbb{R}$  s.t.  $A = tB$ . Then  $A, B$  linearly independent.

**Exercise 6.** Given  $(a, b), (c, d)$ , the following statements must be true.

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left( \frac{1}{ad - bc} \right) \begin{bmatrix} d & -c \\ -b & a \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

If  $ad - bc \neq 0$ , then we could apply this inverse matrix to obtain  $x = y = 0$ .

Suppose  $x = y = 0$  and  $ad - bc = 0$ . Then  $ad = bc$  and we could rewrite  $(c, d)$  to be  $(\frac{ad}{b}, \frac{bc}{a})$ . Then

$$\frac{-d}{b}(a, b) + (c, d) = 0. \text{ But we had said that } x = y = 0 \text{ were the only coefficients. Then } ad - bc \neq 0$$

**Exercise 7.** Given  $(1 + t, 1 - t), (1 - t, 1 + t)$ , and from above,

$(1 + t)(1 + t) - (1 - t)(1 - t) = ad - bc = 4t \neq 0$ . So for  $t \neq 0$ , the two vectors are linearly independent.

**Exercise 8.**  $i, j, k$  are linearly independent.  $i + j + k = L(\{i, j, k\})$ . By thm,  $\{i, j, k, i + j + k\}$  are linearly dependent (Apostol's Thm. 12.8, any set of  $k + 1$  vectors is dependent,  $S$  independent,  $|S| = k$ ).

$i, j, k$  are linearly independent and pairwise linearly independent.  $i + j + k$  is not in the span of any pair from  $i, j, k$  because  $i + j + k$  is a linear combination of one unit coordinate vector not included in the pair. Then the 3 vectors, including  $i + j + k$ , are linearly independent (if  $i + j + k$  is not in the span of two other independent vectors, then there's no linear combination of the three that would give a nontrivial representation of 0).

**Exercise 9.**

- (1)  $c_1 i + c_2(i + j) = 0$   $c_1 = -c_2$   $c_2 = 0$  Then  $c_1 = 0$
- (2)  $i + j - i = j \in L(S)$
- (3)  $3i - 4j = -4(i + j) + 5i$
- (4)  $i, i + j$  linearly independent  $|S| = 2$  by Thm. 12.10, since we have 2 linearly independent vectors in  $V_2$ ,  $S$  forms a basis for  $V_2$ .

**Exercise 10.** Given  $A = i, B = i + j, C = i + j + 3k$  in  $V_3$ 

- (1)  $xA + yB + zC = 0$ . Then  $z = 0$ . Then  $y = 0$ .  $x = 0$ .  $A, B, C$  linearly independent.
- (2)  $j = B - A$   $k = \frac{C-B}{3}$
- (3)  $2i - 3j + 5k = \frac{-14B + 5C}{3} + 5A$
- (4)  $\{A, B, C\}$  are linearly independent. Since  $|\{A, B, C\}| = 3 = n$ , for  $V_n = V_3$ , then by Thm. 12.10,  $\{A, B, C\}$  form a basis for  $V_3$ .

**Exercise 11.** Let  $A = (1, 2), B = (2, -4), C = (2, -3), D = (1, -2)$ .

Note that  $B = 2D$  ( $B, D$  are parallel).

$A, B, C, D$  are all each linearly independent, by themselves.

$$\boxed{\{A, D\}, \{A, B\}, \{A, C\}, \{B, C\}, \{D, C\}}$$

**Exercise 12.**

$$A = (1, 1, 1, 0)$$

Let  $B = (0, 1, 1, 1)$

$$C = (1, 1, 0, 0)$$

- (1)  $A, B, C$  are linearly independent since for  $xA + yB + zC = 0$ ,  $y = 0$ ,  $x = 0$  and so  $z = 0$ .
- (2)  $-(0, 1, 2, 1) = D$  would make the set dependent.
- (3)  $E = (1, 0, 0, 0)$
- (4)  $(1, 2, 3, 4) = 4B + -A - C + 3E$

**Exercise 13.**

$$y = -\sqrt{3}z$$

$$(1) \quad x(\sqrt{3}, 1, 0) + y(1, \sqrt{3}, 1) + z(0, 1, \sqrt{3}) = 0. \text{ Then } \quad x\sqrt{3} = -y \quad \implies \frac{-y}{\sqrt{3}} + \sqrt{3}y + \frac{y}{-\sqrt{3}} = 0 \quad \text{So } y = 0.$$

$$x + \sqrt{3}y + z = 0$$

Then  $x = z = 0$

- (2)  $(\sqrt{2}, 1, 0) - \sqrt{2}(1, \sqrt{2}, 1) = (0, -1, -\sqrt{2})$ . Dependent.
- (3)  $(t, 1, 0) + -t(1, t, 1) = (0, 1, t) \implies 1 - t^2 = 1$  so that  $\boxed{t = 0}$ .

**Exercise 14.**

$$(1, 0, 1, 0)$$

$$(1) \quad (1, 1, 1, 1)$$

$$(2, 0, -1, 0)$$

Note that we cannot get any bigger set, since  $(0, 1, 0, 1)$  is a linear combination of  $(1, 0, 1, 0)$  and  $(1, 1, 1, 1)$

- (2)  $(1, 1, 1, 1), (1, -1, 1, 1)$  are immediately linearly independent, since they're not parallel to each other.

$x(1, 1, 1, 1) + y(1, -1, 1, 1) = (x + y, x - y, x + y, x + y)$ , so the 3rd. and 4th. coordinate of any linear combination of these two vectors will be equal.

$$(1, 1, 1, 1)$$

Thus  $(1, -1, 1, 1)$  are linearly independent.

$$(1, -1, -1, 1)$$

$$x(1, 1, 1, 1) +$$

$$y(1, -1, 1, 1) + = (x + y + z, x - y - z, x + y - z, x + y + z)$$

$$z(1, -1, -1, 1)$$

So any linear combination of the these three vectors, will always have the 1st. and 4th. coordinates equal. Then

$(1, 1, 1, 1), (1, -1, 1, 1), (1, -1, -1, 1), (1, -1, -1, -1)$  are linearly independent.

- (1, 1, 1, 1)  
 (0, 1, 1, 1)  
 (3) (0, 0, 1, 1) are linearly independent.  
 (0, 0, 0, 1)

**Exercise 15.**

- (1)  $x(A + B) + y(B + C) + z(A + C) = (x + y + z)A + (x + y)B + (y + z)C = 0$   
 $\implies x = -z = y = -x$  since  $A, B, C$  are linearly independent. Then  $x = y = z = 0$ .  
 $x = y$
- (2)  $x(A - B) + y(B + C) + z(A + C) = (x + z)A + (-x + y)B + (y + z)C = 0$  So we have  $y = -z$   
 $x = -z$

If  $x, y \neq 0, z \neq 0$  and condition for linear independence of  $A, B, C$  is still satisfied. Thus,  $A - B, B + C, A + C$  linearly dependent.

**Exercise 16.**

- (1)  
 If  $S = \mathcal{B}_{V_3}$ ,  
 $S = \mathcal{B}_{V_3}$  means  $\forall x \in V_3, x \in L(S)$ . Since  $i, j, k \in V_3, i, j, k \in L(S)$   
 If  $i, j, k \in L(S)$ , then  $L(\{i, j, k\}) \subseteq L(S)$  (since every linear combination of  $i, j, k$  can be reexpressed as a linear combination of vectors in  $S$ , since each  $i, j, k$  is a linear combination of vectors in  $S$ ).  
 Then since  $\forall x \in V_3, x \in L(\{i, j, k\}), x \in L(S)$   
 So  $S$  forms a basis for  $V_3$ .
- (2)  
 If  $S = \mathcal{B}_{V_n}$ ,  
 $S = \mathcal{B}_{V_n}$  means  $\forall x \in V_n, x \in L(S)$ . Since  $e_1, \dots, e_n \in V_n, e_1, \dots, e_n \in L(S)$   
 If  $e_1, \dots, e_n \in L(S)$ , then  $L(\{e_1, \dots, e_n\}) \subseteq L(S)$  (since every linear combination of  $e_1, \dots, e_n$  can be reexpressed as a linear combination of vectors in  $S$ , since each  $e_1, \dots, e_n$  is a linear combination of vectors in  $S$ ).  
 Then since  $\forall x \in V_n, x \in L(\{e_1, \dots, e_n\}), x \in L(S)$   
 So  $S$  forms a basis for  $V_n$ .

**Exercise 17.**  $\{(0, 1, 1), (1, 1, 1), (0, 0, 1)\}$  and  $\{(0, 1, 1), (1, 1, 1), (0, 0, -1)\}$

**Exercise 18.**  $\{(0, 1, 1, 1), (1, 1, 1, 1), (0, 0, 0, 1), (0, 0, 1, 0)\}$  and  $\{(0, 1, 1, 1), (1, 1, 1, 1), (0, 0, 0, -1), (0, 0, -1, 0)\}$

**Exercise 19.** Given that

$$S = \{(1, 1, 1), (0, 1, 2), (1, 0, -1)\}$$

$$T = \{(2, 1, 0), (2, 0, -2)\}$$

$$U = \{(1, 2, 3), (1, 3, 5)\}$$

- (1) Consider  $x(2, 1, 0) + y(2, 0, -2) = (2x + 2y, x, -2y)$

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2(x+y) \\ x \\ -2y \end{bmatrix}$$

$$\left[ \begin{array}{ccc|c} 1 & 0 & 1 & 2(x+y) \\ 1 & 1 & 0 & x \\ 1 & 2 & -1 & -2y \end{array} \right] = \left[ \begin{array}{ccc|c} 1 & 0 & 1 & 2(x+y) \\ & 1 & -1 & -x-2y \\ & 2 & -2 & -2x-4y \end{array} \right] \implies \begin{array}{l} a + c = 2(x+y) \\ b - c = -x - 2y \end{array}$$

$$\boxed{2(x+y)(1, 1, 1) + (-x-2y)(0, 1, 2)}$$

(2)

$$\left[ \begin{array}{ccc|c} 1 & 0 & 1 & a \\ 1 & 1 & 0 & b \\ 1 & 2 & -1 & c \end{array} \right] = \left[ \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right] \implies \left[ \begin{array}{ccc|c} 1 & 0 & 1 & a \\ & 1 & -1 & b \\ & 2 & -2 & c \end{array} \right] \implies \begin{array}{l} a = -c \\ b = c \end{array} \quad \text{so } S \text{ is linearly dependent}$$

$$2(2, 1, 0) - \frac{3}{2}(2, 0, -2) = (1, 2, 3) \implies L(U) = L(T)$$

$$3(2, 1, 0) - \frac{5}{2}(2, 0, -2) = (1, 3, 5)$$

$$\begin{aligned} (1, 1, 1) + (0, 1, 2) &= (1, 2, 3) \\ (1, 1, 1) + 2(0, 1, 2) &= (1, 3, 5) \end{aligned} \implies L(S) = L(U) = L(T)$$

**Exercise 20.**  $A, B$  are finite subsets.

(1) Consider  $x_A \in L(A)$ ,  $x_A = \sum c_j A_j$ . Since  $\forall A_j \in B$ ,  $x_A \in L(B)$ . Then  $L(A) \subseteq L(B)$

(2) Consider  $c_j \in A \cap B$  so that  $c_j \in A$  and  $c_j \in B$

$$\begin{aligned} \sum c_j C_j &\in L(A) \text{ since } \forall C_j \in A \text{ and } \sum c_j C_j \in L(B) \text{ since } \forall C_j \in B \\ \implies \sum c_j C_j &\in L(A) \cap L(B) \implies L(A \cap B) \subseteq L(A) \cap L(B) \end{aligned}$$

(3) Now  $L(A \cap B) \subseteq L(A) \cap L(B)$ .

$$A = \{(1, 1, 0), (1, 0, 1)\}$$

$$\text{If } B = \{(1, 0, 0), (0, 1, 1)\}$$

then we could see that  $A \cap B = \emptyset$ , but  $L(A) \cap L(B)$  is nonempty; for instance, it'll include  $(2, 1, 1) = (1, 1, 0) + (1, 0, 1) = 2(1, 0, 0) + (0, 1, 1)$ .

### 12.16 Exercises - The vector space $V_n(\mathbb{C})$ of $n$ -tuples of complex numbers.

$$A = (1, i)$$

**Exercise 1.** Given  $B = (i, -i)$

$$C = (2i, 1)$$

$$(1) A \cdot B = -i - 1$$

$$(2) B \cdot A = \overline{A} \cdot \overline{B} = -1 + i$$

$$(3) (iA) \cdot B = i(A \cdot B) = 1 - i$$

$$(4) A \cdot (iB) = -i(A \cdot B) = -1 + i$$

$$(5) (iA) \cdot (iB) = i(-i)A \cdot B = -i - 1$$

$$(6) (i, -1) \cdot (-1, 1) = B \cdot C = 2 - i$$

$$(7) A \cdot C = -2i + i = -i$$

$$(8) (B + C) \cdot A = (3i, 1 - i) \cdot A = 3i - i - 1 = 2i - 1$$

$$(9) (A - C) \cdot B = (1 - 2i, i - 1) \cdot B = (-i - 2, -1 - i) = -3 - 2i$$

$$(10) (A - iB) \cdot (A + iB) = (A - (-1, 1)) \cdot ((1, i) + (-1, 1)) = (2, -1 + i) \cdot (0, 1 + i) = \boxed{2i}$$

**Exercise 2.** Given

$$A = (2, 1, -i)$$

$$B = (i, -1, 2i) \text{ and } V_3(\mathbb{C})$$

$$A \cdot C = (2, 1, -i) \cdot (\bar{c}_1, \bar{c}_2, \bar{c}_3) = 2\bar{c}_1 + \bar{c}_2 - i\bar{c}_3 = 0 \implies c_2 = c_1(-4 + i)$$

$$B \cdot C = (i, -1, 2i) \cdot (\bar{c}_1, \bar{c}_2, \bar{c}_3) = i\bar{c}_1 - \bar{c}_2 + 2i\bar{c}_3 = 0 \implies c_3 = c_1(-2i - 1)$$

$$\boxed{C = (1, i - 4, -1 - 2i)}$$

**Exercise 3.**

$$\begin{aligned} \|A + B\|^2 &= (A + B) \cdot (A + B) = (A + B) \cdot A + (A + B) \cdot B = \overline{A} \cdot \overline{(A + B)} + \overline{B} \cdot \overline{(A + B)} = \\ &= \overline{A} \cdot A + \overline{A} \cdot \overline{B} + \overline{B} \cdot A + \overline{B} \cdot \overline{B} = \|A\|^2 + \|B\|^2 + \overline{A} \cdot \overline{B} + \overline{B} \cdot A = \\ &= \boxed{\|A\|^2 + \|B\|^2 + \overline{A} \cdot \overline{B} + A \cdot B} \end{aligned}$$

**Exercise 4.**

$$\begin{aligned}
\|A + B\|^2 - \|A - B\|^2 &= (A + B) \cdot (A + B) - (A - B) \cdot (A - B) = \\
&= (A + B) \cdot A + (A + B) \cdot B - (A - B) \cdot A + (A - B) \cdot B = \\
&= \overline{A} \cdot (A + B) + \overline{B} \cdot (A + B) - \overline{A} \cdot (A - B) + \overline{B} \cdot (A - B) = \\
&= \overline{A} \cdot A + \overline{A} \cdot B + \overline{B} \cdot A + \overline{B} \cdot B - \overline{A} \cdot A + \overline{A} \cdot B + \overline{B} \cdot A - \overline{B} \cdot B = \\
&= \boxed{2(A \cdot B + \overline{A} \cdot \overline{B})}
\end{aligned}$$

**Exercise 5.**

$$\begin{aligned}
\|A + B\|^2 + \|A - B\|^2 &= (A + B) \cdot (A + B) + (A - B) \cdot (A - B) = \\
&= (A + B) \cdot A + (A + B) \cdot B + (A - B) \cdot A - (A - B) \cdot B = \\
&= \overline{A} \cdot (A + B) + \overline{B} \cdot (A + B) + \overline{A} \cdot (A - B) - \overline{B} \cdot (A - B) = \\
&= \overline{A} \cdot A + \overline{A} \cdot B + \overline{B} \cdot A + \overline{B} \cdot B + \overline{A} \cdot A - \overline{A} \cdot B - \overline{B} \cdot A + \overline{B} \cdot B = \\
&= \boxed{2\|A\|^2 + 2\|B\|^2}
\end{aligned}$$

**Exercise 6.**  $A, B \in V_n(\mathbb{C})$ 

- (1)  $\overline{(A \cdot B + \overline{A} \cdot \overline{B})} = \overline{A \cdot B} + \overline{\overline{A} \cdot \overline{B}} = \overline{A \cdot B} + A \cdot B = A \cdot B + \overline{A \cdot B}$  So then  $A \cdot B + \overline{A \cdot B}$  must be real.  
(2)

Now  $\|A + B\|^2 = \|A\|^2 + \|B\|^2 + A \cdot B + \overline{A \cdot B}$ , so then

$$\begin{aligned}
\left\| \frac{A}{\|A\|} + \frac{B}{\|B\|} \right\|^2 &= 1 + 1 + \frac{A \cdot B + \overline{A \cdot B}}{\|A\|\|B\|} \geq 0 \\
\implies \frac{A \cdot B + \overline{A \cdot B}}{\|A\|\|B\|} &\geq -2
\end{aligned}$$

Now  $\|A - B\|^2 = (A - B) \cdot (A - B) = (A - B) \cdot A - (A - B) \cdot B = \|A\|^2 - B \cdot A - A \cdot B + \|B\|^2 \geq 0$

$$\begin{aligned}
\text{so then } \left\| \frac{A}{\|A\|} - \frac{B}{\|B\|} \right\|^2 &= 2 - \frac{(A \cdot B + \overline{A \cdot B})}{\|A\|\|B\|} \geq 0 \\
\implies 2 &\geq \frac{A \cdot B + \overline{A \cdot B}}{\|A\|\|B\|}
\end{aligned}$$

**Exercise 7.** With this definition, that  $\theta = \arccos \frac{\frac{1}{2}(A \cdot B + \overline{A \cdot B})}{\|A\|\|B\|}$ ,

We had already showed that

$$\|A - B\|^2 = (A - B) \cdot (A - B) = (A - B) \cdot A - (A - B) \cdot B = \|A\|^2 - B \cdot A - A \cdot B + \|B\|^2 \geq 0$$

Thus  $\|A - B\|^2 = \|A\|^2 + \|B\|^2 - (A \cdot B + \overline{A \cdot B}) = \|A\|^2 + \|B\|^2 - 2\|A\|\|B\| \cos \theta$

**Exercise 8.**  $V_n(\mathbb{C})$ 

$$A = (1, 0, i, i, i)$$

$$B = (i, i, i, 0, i)$$

$$\theta = \arccos \frac{\frac{1}{2}(A \cdot B + \overline{A \cdot B})}{\|A\|\|B\|}$$

$$A \cdot B = -i + 1 + 1 = 2 - i \quad \overline{A \cdot B} = 2 + i \implies A \cdot B + \overline{A \cdot B} = 2 - i + 2 + i = 4$$

$$\|A\|^2 = (1, 0, i, i, i) \cdot (1, 0, i, i, i) = 1 + 1 + 1 + 1 = 4$$

$$\|B\|^2 = (i, i, i, 0, i) \cdot (i, i, i, 0, i) = (1 + 1 + 1 + 1) = 4$$

$$\theta = \arccos \frac{\frac{1}{2} \cdot 4}{2 \cdot 2} = \arccos 1/2 \implies \cos \theta = 1/2 \text{ or } \boxed{\theta = \pi/3}$$

**Exercise 9.**

$$A = (1, 0, 0)$$

Given that  $B = (0, i, 0)$

$$C = (1, 1, i)$$

$$(1) \quad aA + bB + cC = (x, y, z)$$

$$c = z/i = -iz \text{ Then } b = -i(y + iz) \text{ and } a = x + iz.$$

If  $x = y = z = 0$ , then  $c = 0 = b = a$ . So  $A, B, C$  does form a basis.

$$(2) \quad (5, 2 - i, 2i) = 2C + 3A - B$$

**Exercise 10.**

$$\sum_{j=1}^n c_j e_j = (x_1, \dots, x_n) = X$$

then the  $c_j$ 's simply equal the complex coordinates of  $X$   $c_j = x_j$

If  $X = 0$ , then each  $c_j = 0$ . So the  $e_j$ 's are linearly independent.  $\implies e_j$ 's form a basis for  $V_n(\mathbb{C})$

**13.5 Exercises - Introduction, Lines in  $n$ -space, Some simple properties of straight lines, Lines and vector-valued functions.**

**Exercise 1.** Recall that if  $X_0 \in L(P, A)$ , and  $P, Q \in L(P, A)$ , then  $X_0 = P + tA = P + t(Q - P)$   
 $X_0 - P = t(Q - P)$

$$Q - P = (4, 0), P = (-3, 1), Q = (1, 1)$$

- (1)  $(0, 0)$  no.
- (2)  $(0, 1)$  yes.
- (3)  $(1, 2)$  no
- (4)  $(2, 1)$  yes
- (5)  $(-2, 1)$  yes

**Exercise 2.**  $P = (2, -1); Q = (-4, 2), Q - P = (-6, 3)$

- (1)  $(0, 0)$  yes
- (2)  $(0, 1)$  no
- (3)  $(1, 2)$  no
- (4)  $(2, 1)$  no
- (5)  $(-2, 1)$  yes.

**Exercise 3.**  $P = (-3, 1, 1) \quad A = (1, -2, 3)$

- (1)  $(0, 0, 0)$  no
- (2)  $(2, -1, 4)$  no
- (3)  $(-2, -1, 4)$  yes
- (4)  $(-4, 3, -2)$  yes
- (5)  $(2, -9, 16)$  yes

**Exercise 4.**  $P = (-3, 1, 1)$  and  $Q - P = (4, 1, 6)$   
 $Q = (1, 2, 7)$

- (1)  $(-7, 0, 5)$  no
- (2)  $(-7, 0, -5)$  yes
- (3)  $(-11, 1, 11)$  no
- (4)  $(-11, -1, 11)$  no.
- (5)  $(-1, 3/2, 4)$  yes.
- (6)  $(-5/3, 4/3, 3)$  yes.
- (7)  $(-1, 3/2, -4)$  no.

**Exercise 5.**  $P = (2, 1, 1), Q = (4, 1, -1), R = (3, -1, 1)$

- (1)  $Q - P = (2, 0, -2) \quad R - P = (1, -2, 0)$ . No.
- (2)  $P = (2, 2, 3), \quad Q = (-2, 3, 1), \quad R = (-6, 4, 1)$   
 $Q - P = (-4, 1, -2), \quad R - P = (-8, 2, -2)$ . No.
- (3)  $P = (2, 1, 1); \quad Q = (-2, 3, 1), \quad R = (5, -1, 1)$   
 $Q - P = (-4, 2, 0) \quad R - P = (3, -2, 0)$ . No.

**Exercise 6.**

$$\begin{array}{llll} A = (2, 1, 1) & & D = (-2, 3, 1) & E = (1, 1, 1) \\ B = (6, -1, 1) & \text{Since } B - A = (4, -2, 0) & D - A = (-4, 2, 0) & F = (-4, 4, 1) \\ C = (-6, 5, 1) & C - A = (-8, 4, 0) & F - A = (-6, 3, 0) & G = (-13, 9, 1) \\ & & & H = (14, -6, 1) \end{array}$$

$\{A, B, C, D, F\}$

**Exercise 7.**  $P = (1, 1, 1) \quad A = (1, 2, 3)$

$$\begin{aligned} P + tA &= Q + uB & 1 + t &= 2 + 3u \\ (1, 1, 1) + t(1, 2, 3) &= (2, 1, 0) + u(3, 8, 13) & 1 + 2t &= 1 + 8u \end{aligned} \implies u = 1, \quad t = 4$$

$(5, 9, 13)$  is an intersection point.

**Exercise 8.**

$$\begin{aligned} (1) \quad X &= P + tA = Q + uB \implies P - Q = uB - tA \\ P - Q &= aA + bB \implies P + (-a)A = Q + bB \text{ and } \begin{aligned} P + (-a)A &\in L(P; A) \\ Q + bB &\in L(Q; B) \end{aligned} \\ (2) \quad L &= \{(1, 1, -1) + t(-2, 1, 3)\} L' = \{(3, -4, 1) + t(-1, 5, 2)\} \\ (1, 1, -1) + t(-2, 1, 3) &= (3, -4, 1) + u(-1, 5, 2) \end{aligned}$$

$$\begin{aligned} 1 - 2t &= 3 - u & 1 + t &= -4 + 5u & 3 &= -5 + 9u \\ u &= 8/9 \\ t &= -5/9 \end{aligned}$$

But  $t = -5/9, u = 8/9$ , doesn't work for the 3rd. coordinate.

**Exercise 9.**  $X(t) = P + tA \quad P = (1, 2, 3) \quad A = (1, -2, 2), \quad Q = (3, 3, 1)$

(1)

$$\begin{aligned} \|Q - X(t)\|^2 &= (Q - X) \cdot (Q - X) = Q^2 - 2Q \cdot X + X^2 = 19 - 2Q \cdot (P + tA) + P^2 + 2tP \cdot A + t^2A^2 = \\ &= 19 - 24 + -2t(-1) + 14 + 2t(3) + t^29 = 9 + 8t + 9t^2 \end{aligned}$$

$$(2) \quad X(t_0) \quad \|Q - X(t)\| \min. \quad \frac{d}{dt} \|Q - X(t)\|^2 = 18t + 8 = 0 \quad t = -4/9$$

$$\|Q - X(t)\| = \sqrt{9 + \frac{-32}{9} + \frac{16}{9}} = \sqrt{\frac{81-16}{9}} = \sqrt{\frac{65}{9}} = \frac{\sqrt{13 \cdot 5}}{3}$$

(3)

$$\begin{aligned} (Q - X(t_0)) \cdot A &= Q \cdot A - X \cdot A = Q \cdot A - (P + tA) \cdot A = Q \cdot A - P \cdot A - tA \cdot A = \\ &= -1 - 3 - \left(\frac{-4}{9}\right)(9) = 0 \end{aligned}$$

**Exercise 10.**  $Q \notin L(P; A)$

$$(1) \quad f(t) = \|Q - X(t)\|^2 \quad X(t) = P + tA$$

$$\begin{aligned} (Q - X) \cdot (Q - X) &= Q^2 - 2Q \cdot X + X^2 = Q^2 - 2Q \cdot (P + tA) + P^2 + 2tP \cdot A + t^2A^2 = \\ &= Q^2 - 2Q \cdot P - 2tQ \cdot A + P^2 + 2tP \cdot A + t^2A^2 \end{aligned}$$

$$\frac{d}{dt} (\|Q - X(t)\|^2) = 2tA^2 + 2(P \cdot A - Q \cdot A) = 0 \quad t = \frac{(Q-P) \cdot A}{A^2}$$

$$(2) \quad (Q - X) \cdot A = Q \cdot A - (P + tA) \cdot A = Q \cdot A - P \cdot A - tA^2 = (Q - P) \cdot A - \frac{(Q-P) \cdot A}{A^2} A^2 = 0$$

**Exercise 11.**  $L(P; A), L(Q, A)$  in  $V_n$

Consider  $x_1 = P + tA$ .

$$X_1 = Q + (P - Q) + t_1A$$

if  $P - Q = u_1A$ , then  $X_1 \in L(Q, A)$

$$\begin{aligned} \text{Likewise, if } X_2 &= Q + tA \\ X_2 &= P + (Q - P) + t_2A \end{aligned}$$

If  $P - Q = -u_2A$ , then  $X_2 \in L(P; A)$

If  $P - Q \parallel A$ , then  $X_1 \in L(Q, A)$  and  $X_2 \in L(P, A)$ , so that  $L(Q, A) = L(P, A)$

If  $P - Q \nparallel A$ , then  $X_1 \notin L(Q, A)$  and  $X_2 \notin L(P; A), \forall X_1 \in L(P, A), \forall X_2 \in L(Q, A)$

$$X_1 = Q + (P - Q) + t_1A \neq Q + (u_1 + t_1)A \implies P - Q \neq u_1A \text{ so that } X_1 \notin L(Q, A)$$

**Exercise 12.**  $L(P; A), L(Q; B), A \nparallel B$

$$X_1 = P + tA$$

$$X_1 - X_2 = 0 \text{ only if } P - Q = u_1B - t_1A$$

$$\text{Consider } X_2 = Q + uB$$

$$\text{If } P - Q = u_2B - t_2A, \text{ then } (u_1 - u_2)B = (t_1 - t_2)A$$

$$X_1 - X_2 = P - Q + tA - uB$$

$$\text{but } A \nparallel B \text{ so } t_1 = t_2, u_1 = u_2$$

(only one intersection point). Otherwise, if  $\nexists u_1, t_1$  s.t.  $P - Q = u_1B - t_1A$ , then there's no intersection.



### 13.8 Exercises - Planes in Euclidean $n$ -space, Planes and vector-valued functions.

**Exercise 1.** Let  $M = \{P + sA + tB\}$ ,  $P = (1, 2, -3)$ ,  $A = (3, 2, 1)$ ,  $B = (1, 0, 4)$ . Remember, if  $X = P + sA + tB$   
 $X - P = sA + tB$

- (1)  $(1, 2, 0)$   $X - P = (0, 0, 3)$ ; no
- (2)  $(1, 2, 1)$   $X - P = (0, 0, 4)$ ; no
- (3)  $(6, 4, 6)$   $X - P = (5, 2, 9) = A + 2B$ , yes
- (4)  $(6, 6, 6)$   $X - P = (5, 4, 9)$  no
- (5)  $(6, 6, -5)$   $X - P = (5, 4, -2) = 2A + -B$  yes.

**Exercise 2.**

- $P = (1, 1, -1)$   
 $Q = (3, 3, 2)$  so that  $Q - R = (0, 4, 4)$   
 $R = (3, -1, -2)$   $P - R = (-2, 2, 1)$
- (1)  $(2, 2, 1/2)$   $X - R = (-1, 3, 5/2)$  yes
  - (2)  $(4, 0, -1/2)$   $X - R = (1, 1, 3/2)$  yes
  - (3)  $(-3, 1, -3)$   $X - R = (-6, 2, -1)$  yes
  - (4)  $(3, 1, 3)$   $X - R = (0, 2, 5)$  no
  - (5)  $(0, 0, 0)$   $X - R = (-3, 1, 2)$  no

**Exercise 3.**

- $x = 1 + t$
- (1)  $(1, 2, 1) + s(0, 1, 0) + t(1, 1, 4) = (x, y, z) \implies y = 2 + s + t$   
 $z = 1 + 4t$
  - (2)  $(0, 1, 0) + s(1, 1, 1) + t(1, 0, 4)$

**Exercise 4.**  $(1, 2, 0) + s(1, 1, 2) + t(-2, 4, 1)$  or  $N = (-7, -5, 6)$  (by finding the cross product of the two spanning vectors), so that  $-7x - 5y + 6z = -17$ .

- (1)  $(0, 0, 0)$  no.
- (2)  $(1, 2, 0)$  yes.
- (3)  $(2, -3, -3)$  yes.

**Exercise 5.**  $M = \{P + s(Q - P) + t(R - P)\}$

- (1)  $P + s(Q - P) + t(R - P) = (1 - s - t)P + sQ + tR$   
 Since  $1 - s - t = p$ , for  $pP$ ,  $1 = p + s + t$   
 $X = P + s(Q - P) + t(R - P) =$
- (2)  $= (1 - s - t)P + sQ + tR$   
 $p = 1 - s - t, p + s + t = 1$

**Exercise 6.**  $ax + by + cz = d$

- (1)  $(2, 3, 1)$  spanned by  $(3, 2, 1)$  and  $(-1, -2, -3)$ ,

$$X = P + sA + tB$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} + s \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + t \begin{pmatrix} -1 \\ -2 \\ -3 \end{pmatrix} \implies \begin{matrix} x = 2 + 3s - t \\ y = 3 + 2s - 2t \\ z = 1 + s - 3t \end{matrix}$$

$$\implies x - 2y + z = -3$$

$$P = (-2, -1, -3)$$

- (2) If  $Q - P = (4, 4, 4)$   $(x, y, z) = (-2, -1, -3) + s(1, 1, 1) + t(3, 2, 2)$   
 $R - P = (6, 4, 4)$

$$\begin{matrix} x = -2 + s + 3t \\ y = -1 + s + 2t \\ z = -3 + s + 2t \end{matrix} \implies y + 1 = z + 3 \text{ or } y - z = 2$$

(3)  $(2, 3, 1)$  parallel to the plane through the origin, spanned by  $(2, 0, -2)$  and  $(1, 1, 1)$

$$\begin{aligned} P + sA + tB \\ \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} + s \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\ x &= 2 + 2s + t \\ \Rightarrow y &= 3 + t & \Rightarrow x + 2y + z = -3 \\ z &= 1 - 2s + t \end{aligned}$$

**Exercise 7.**  $M$  s.t.  $3x - 5y + z = 9$

$$\begin{aligned} (0, -2, -1) \\ (1) \text{ Out of } (-1, -2, 2), \begin{pmatrix} (0, -2, -1) \\ (-1, -2, 2) \end{pmatrix} \\ (3, 1, -5) \\ (2) \begin{pmatrix} (0, -2, -1) - (2, 1, 8) = (-2, -3, -9) \\ (-1, -2, 2) - (2, 1, 8) = (-3, -3, -6) \end{pmatrix} \Rightarrow (2, 1, 8) + s(2, 3, 9) + t(1, 1, 2) \end{aligned}$$

**Exercise 8.** Given two planes,  $M = \{P + sA + tB\}$ ,  $M' = \{Q + sC + tD\}$ , and the specific coordinates,

$$P = (1, 1, 1) \quad Q = (2, 3, 1)$$

$$A = (2, -1, 3) \quad C = (1, 2, 3)$$

$$B = (-1, 0, 2) \quad D = (3, 2, 1)$$

It's easiest to find the Cartesian coordinate equations for each of the planes, to find the intersection, and the easiest way to find the Cartesian coordinate equations is to find the normal to each of the planes, using the cross product on  $A, B$  and on  $C, D$ .  $A \times B = (-2, -7, -1)$  and  $C \times D = (-4, 8, -4)$ . Then

$$M : 2x + 7y + z = 10$$

$$M' : -4x + 8y - 4z = 12$$

It's easy to check that these are the correct equations because we could plug in  $P, Q$  respectively to check the equation and plug in  $A, B$  and  $C, D$ , respectively to check that we get 0 (because adding arbitrary amounts of  $A, B$  or  $C, D$ , respectively, should not affect each respective equation).

Thus, solving for both  $M$  and  $M'$ , we get  $x + 9y = 13$ . Then two distinct points include  $(4, 1, -5), (-5, 2, 6)$

**Exercise 9.**

(1) Given a plane that

$$P = (2, 3, 1), \quad A = (1, 2, 3) \quad B = (3, 2, 1) \quad \text{and } x - 2y + z = 0, \text{ so that (with a Cartesian coordinate equation, we could easily}$$

plug in numbers to find 3 points that this plane contains)

$$Q - P = (1, 1, 1)$$

$$P = (2, 1, 0) \quad R - P = (-3, -2, -1)$$

$$Q = (3, 2, 1)$$

$$R = (-1, -1, -1) \quad X = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + t \begin{pmatrix} -3 \\ -2 \\ -1 \end{pmatrix}$$

$P)$  (since  $(1, 2, 3) + (3, 2, 1) = 4(1, 1, 1)$  and  $(3, 2, 1) = -(-3, -2, -1)$ , then the two planes are parallel.)

(2) Given that

$$M' : x - 2y + z = 0$$

$$M : x + 2y + z = 0$$

then to find the intersection, solve for the two equations to get  $2x + 2z = 0 \Rightarrow x = -z$ .

Then two points in the intersection,  $M' \cap M''$  are  $(1, 0, -1), (-1, 0, 1)$

**Exercise 10.**

$$\begin{aligned} L &= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + s \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \\ M &= \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} + t \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + u \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \end{aligned}$$

Condition for intersection:

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + s \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} + t \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + u \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

Then

$$\begin{pmatrix} 0 \\ 0 \\ -3 \end{pmatrix} = s \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} - t \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} - u \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{bmatrix} 2 & -2 & 0 \\ -1 & -1 & -1 \\ 3 & -3 & -1 \end{bmatrix} \begin{bmatrix} s \\ t \\ u \end{bmatrix}$$

$$\left[ \begin{array}{ccc|c} 2 & -2 & 0 & 0 \\ -1 & -1 & -1 & 0 \\ 3 & -3 & -1 & -3 \end{array} \right] = \left[ \begin{array}{ccc|c} 2 & 0 & 0 & -3 \\ 0 & -2 & 0 & 3 \\ 0 & 0 & 1 & 3 \end{array} \right] \Rightarrow \begin{matrix} s = -\frac{3}{2} \\ t = \frac{3}{-2} \\ u = 3 \end{matrix} \Rightarrow X = \begin{pmatrix} -2 \\ 5/2 \\ -7/2 \end{pmatrix}$$

**Exercise 11.**  $L = \{(1, 1, 1) + t(2, -1, 3)\}$

(1)  $M = \{(1, 1, -2) + s(2, 1, 3) + t(3/4, 1, 1)\}$

Suppose  $(2, -1, 3) = s(2, 1, 3) + t(3/4, 1, 1)$ . Then  $\begin{matrix} 2 = 2s + 3/4t \\ -1 = s + t \end{matrix}$ . Doing the algebra, we get  $t = -16/5, s = 11/5$ . But the third coordinate doesn't work,  $33/5 - 16/5 = 17/5$ . So  $M, L$  are not parallel since  $(2, -1, 3) \notin L(\{(2, 1, 3), (3/4, 1, 1)\})$

(2)  $(1, 1 - 2), (3, 5, 2), (2, 4, -1)$ .  $\begin{matrix} C = (2, 4, 4) \\ D = (1, 3, 1) \end{matrix}$   $sC + tD = (2, -1, 3)$ . Doing the algebra, we get  $t = -5, s = 7/2$

from the first two coordinates, but it doesn't agree with the third coordinates, so that  $7/2C + -5D \neq (2, -1, 3)$ . No, not parallel.

(3) From,  $x + 2y + 3z = -3$ , we get the spanning vectors easily by plugging in numbers that'll make the LHS equal to zero:  $(-3, 0, 1), (1, 1, -1)$ . Again, we find that  $(2, -1, 3) \neq (-1)(1, 1, -1) + (-1)(-3, 0, 1)$ . No, not parallel.

**Exercise 12.**  $P, Q \in M$ .

Consider  $M = \{P + sA + tB\}$ .  $Q \in M$ , so that  $P + sA + tB = Q$ .  $\Rightarrow Q - P = sA + tB$ .

Now  $c_1(sA + tB) + c_2A = (c_1s + c_2)A + c_1tB = 0$

$A, B$  linearly independent, so  $c_1 = 0$ , then  $c_2 = 0$ .

So  $sA + tB = Q - P$ ,  $A$  are linearly independent, and  $Q - P = sA + tB$ ,  $A$  span the same space as  $A, B$ , so by theorem,  $M = \{P + s(Q - P) + tA\}$

For  $X \in L$ ,  $X = P + s(Q - P) = P + s(Q - P) + 0A$  so  $L \subseteq M$ .

**Exercise 13.** For  $L$  containing  $(1, 2, 3)$  and  $\parallel$  to  $(1, 1, 1)$ ,  $L = \{P + sA\}$ .

Consider  $M = \{P + sA + tB\}$ , containing  $Q$ . Then  $Q = (2, 3, 5) = P + sA + tB$ . If we let  $s = t = 1$ , then

$$B = (2, 3, 5) - (1, 2, 3) - (1, 1, 1) = (0, 0, 1)$$

$(0, 0, 1)$  is linearly independent of  $(1, 1, 1)$  and so a possible  $M$  is  $M = (1, 2, 3) + s(1, 1, 1) + t(0, 0, 1)$  or  $-x + y = 1$ .

**Exercise 14.**  $L = \{Q + sA\}$  We have a plane that we want:  $M = \{Q + sA + t(P - Q)\}$ .

Consider  $M'$  s.t.  $L \subseteq M'$  and  $P \in M'$ ,  $M' = \{R + sB + tC\}$ .

Note that  $P \neq Q + sA \quad \forall s \in \mathbb{R}$

In general,  $X = Q + sA = (R + b_0B + c_0C) + s(b_1B + c_1C)$  so that

$$Q = R + b_0B + c_0C, \quad b_0, c_0 \neq 0$$

$$A = (b_1B + c_1C), \quad b_1, c_1 \neq 0$$

$$P = R + b_2B + c_2C, \quad b_2, c_2 \neq 0, \text{ since } P \in M'$$

So then  $Q - P = (b_0 - b_2)B + (c_0 - c_2)C = b_3B + c_3C$  and

$$b_3B + c_3C = b_3(A - c_1C)/b_1 + c_3C = \frac{b_3}{b_1}A + (c_3 - \frac{c_1}{b_1})C = Q - P$$

$$C = \frac{Q - P - \frac{b_3}{b_1}A}{(c_3 - \frac{c_1}{b_1})}$$

$$\begin{aligned}
X &= R + sB + tC \\
&= Q - b_0B - c_0C + sB + tC = \\
&= Q + (s - b_0)B + (t - c_0)C = \\
&= Q + (s - b_0)((Q - P) - c_3C)/b_3 + (t - c_0)C = Q + \frac{(s - b_0)}{b_3}(Q - P) + \frac{-(s - b_0)c_3 + b_3(t - c_0)}{b_3}C = \\
&= Q + \frac{(s - b_0)}{b_3}(Q - P) + \frac{-(s - b_0)c_3 + b_3(t - c_0)}{b_3} \frac{Q - P - \frac{b_3}{b_1}A}{(c_3 - \frac{c_1}{b_1})} \\
&\implies M' \subseteq M
\end{aligned}$$

Suppose if  $c_1 = 0$  (without loss of generality)

$A = b_1B(B \implies A)$ (without loss of generality)

$$P - Q = (b_2 - b_0)A + (c_2 - c_0)C \text{ since } (P \neq Q + sA, \text{ so } (c_2 - c_0) \neq 0)$$

$$\begin{aligned}
R + sB + tC &= Q - b_0A - c_0C + sA + tC = \\
&= Q + (s - b_0)A + (t - c_0) \left( \frac{P - Q - (b_2 - b_0)A}{c_2 - c_0} \right) = \\
&= Q + \left( (s - b_0) + \frac{(b_0 - b_2)(t - c_0)}{c_2 - c_0} \right) A + \left( \frac{t - c_0}{c_2 - c_0} \right) (P - Q) \in M \\
&\implies M' \subseteq M
\end{aligned}$$

$$Q + sA + t(P - Q).$$

$$\text{If } P = R + b_2B + c_2C \text{ and } \begin{matrix} Q = R + b_0B + c_0C \\ A = (b_1B + c_1C) \end{matrix}$$

$$\begin{aligned}
Q + sA + t(P - Q) &= R + b_0B + c_0C + s(b_1B + c_1C) + t(b_2B + c_2C - b_0B - c_0C) = \\
&= R + (b_0 + sb_1 + t(b_2 - b_0))B + (c_0 + sc_1 + t(c_2 - c_0))C \\
&\implies M \subseteq M'
\end{aligned}$$

So then  $M = M'$ .

### 13.11 Exercises - The cross product, The cross product expressed as a determinant.

$$A = (-1, 0, 2)$$

**Exercise 1.** Given  $B = (2, 1, -1)$

$$C = (1, 2, 2)$$

$$(1) A \times B = \begin{vmatrix} e_1 & e_2 & e_3 \\ -1 & 0 & 2 \\ 2 & 1 & -1 \end{vmatrix} = (-2, 3, -1)$$

$$(2) B \times C = (4, -5, 3)$$

$$(3) C \times A = (4, -4, 2)$$

$$(4) A \times (C \times A) = (8, 10, 4)$$

$$(5) (A \times B) \times C = (8, 3, -7)$$

$$(6) A \times (B \times C) = (10, 11, 5)$$

$$(7) (A \times C) \times B = (-2, -8, -12)$$

$$(8) (A+B) \times (A-C) = (A+B) \times A - (A+B) \times C = B \times A - A \times C - B \times C = (2, -3, 1) + (4, -4, 2) - (4, -5, 3) = (2, -2, 0)$$

$$(9) (A \times B) \times (A \times C) = (-2, 0, 4)$$

**Exercise 2.**

$$(1) \begin{matrix} A = (1, 1, 1) \\ B = (2, 3, -1) \end{matrix} \quad A \times B = (-4, 3, 1) \implies \boxed{\frac{1}{\sqrt{26}}(-4, 3, 1)}$$

$$(2) \begin{matrix} A = (2, -3, 4) \\ B = (-1, 5, 7) \end{matrix} \quad A \times B = (-41, -18, 7) \implies \boxed{\frac{\pm 1}{\sqrt{2054}}(-41, -18, 7)}$$

$$(3) \begin{matrix} A = (1, -2, 3) \\ B = (-3, 2, -1) \end{matrix} \quad A \times B = (-4, -8, -4) \implies \boxed{\frac{\pm(1, 2, 1)}{\sqrt{6}}}$$

**Exercise 3.**

$$A = (0, 2, 2)$$

$$(1) B = (2, 0, -1) \quad C - A \times B - A = (3, 2, -2) \times (2, -2, -3) = (-10, 5, -10)$$

$$C = (3, 4, 0)$$

$$\|C - A \times B - A\| = \boxed{15/2}$$

$$A = (-2, 3, 1)$$

$$(2) B = (1, -3, 4) \quad C - A \times B - A = (3, -1, 0) \times (3, -6, 3) = (-3, -9, -15)$$

$$C = (1, 2, 1)$$

$$C - A \times B - A = \sqrt{315} = \boxed{3\sqrt{35}/2}$$

$$A = (0, 0, 0)$$

$$(3) B = (0, 1, 1) \quad \|C - A \times B - A\| = \boxed{\frac{\sqrt{3}}{2}}$$

$$C = (1, 0, 1)$$

$$A = (2, 5, 3)$$

**Exercise 4.**  $B = (2, 7, 4) \quad (A - C) \times (B - A) = (-1, 2, -3) \times (0, 2, 1) = (8, 1, -2)$

$$C = (3, 3, 6)$$

**Exercise 5.**  $\|A \times B\|^2 = \|A\|^2\|B\|^2 - (A \cdot B)^2$

If  $\|A \times B\| = \|A\|\|B\|$ ,  $A \cdot B = 0$  so  $\cos \theta = 0$  or  $\theta = \pi/2$ .

If  $A \cdot B = 0$ , then  $\|A \times B\| = \|A\|\|B\|$

**Exercise 6.** Given  $A, B \in V_3$

$$C = (B \times A) - B$$

$$(1) A \cdot (B + C) = A \cdot B + A \cdot (B \times A) - A \cdot B = 0$$

$$(2)$$

$$B \cdot C = B \cdot (B \times A) - B \cdot B = -B^2$$

$$C \cdot C = ((B \times A) - B) \cdot ((B \times A) - B) = (B \times A)^2 + B^2$$

$$\begin{aligned} \frac{B \cdot C}{\|B\|\|C\|} &= \cos \theta_{BC} = \frac{-B^2}{B((B \times A)^2 + B^2)^{1/2}} = \frac{-B}{\sqrt{(BA \sin \theta_{AB})^2 + B^2}} = \\ &= \frac{-1}{\sqrt{A^2 \sin^2 \theta_{AB} + 1}} \end{aligned}$$

So  $\cos \theta_{BC}$ , since  $A^2 \sin^2 \theta_{AB} \geq 0$ , ranges from  $-1$  (if  $\sin \theta_{AB} = 0$ ), to  $0$  (if  $\|A\| \rightarrow \infty$ ). So then  $\theta_{BC}$  ranges from  $\pi/2$  to  $\pi$ .

$$(3) \|B\| = 1, \|B \times A\| = 2. C^2 = 4 + 1 = 5 \implies C = \sqrt{5}$$

**Exercise 7.**

$$(1) (A \times B)^2 = A^2 B^2 - A \cdot B = 1. A, B, A \times B \text{ normal.}$$

$$A \cdot (A \times B) = B \cdot (A \times B) = 0 \text{ orthonormal.}$$

$$c_1 A + c_2 B + c_3 (A \times B) = 0$$

$$\xrightarrow{A} c_1 A \cdot A = 0 \quad c_1 = 0$$

$$\xrightarrow{B} c_2 B^2 = 0 \quad c_2 = 0 \quad A, B, A \times B \text{ linearly independent. By Thm., } A, B, A \times B \text{ form a basis}$$

$$\xrightarrow{(A \times B)} c_3 (A \times B)^2 = 0 \quad c_3 = 0$$

for  $V_3$ .

$$(2) C = (A \times B) \times A$$

$$\|((A \times B) \times A)\| = \sqrt{\|A \times B\|^2 \|A\|^2 - ((A \times B) \cdot A)^2} = 1$$

$$A \cdot B = A_j B_j = 0 \text{ and}$$

$$(3) \text{ Remember that } A \cdot A = A_j A_j = B \cdot B = B_j B_j = 1$$

$$\begin{aligned}
(A \times (A \times B))_j &= \epsilon_{jkl} A_k (A \times B)_l = \epsilon_{jkl} A_k \epsilon_{lmn} A_m B_n = -\epsilon_{jkl} \epsilon_{nml} A_k A_m B_n = \\
&= \text{(notice how the } l \text{ index is repeated)} = -(-A_k A_j B_k) + -(A_k A_k B_j) = 0 - B_j \\
&\implies (A \times (A \times B)) = -B
\end{aligned}$$

Now  $A, B$  are interchangeable labels (the choice of name or label is not special or unique at all), so that we could reuse what we had just proven

$$\begin{aligned}
(A \times B) \times A &= B \implies (B \times A) \times B = A \\
\implies -(A \times B) \times B &= A \text{ or } (A \times B) \times B = -A
\end{aligned}$$

**Exercise 8.**

- (1) We know that  $A \cdot B = \|A\| \|B\| \cos \theta_{AB}$  and  $\|A \times B\| = \|A\| \|B\| \sin \theta_{AB}$ . Certainly, if  $A \times B$  is 0, then its magnitude is zero. Suppose  $\|A\|, \|B\| \neq 0$  and so  $\sin \theta_{AB} = 0$ . But then  $A \cdot B = 0$  and so  $\cos \theta_{AB} = 0$ . But  $\cos$  and  $\sin$  must obey  $1 = C^2 + S^2$ . So then at least one of  $A$  or  $B$  must be zero.
- (2) Given that  $A \neq 0$   
 $A \times B = A \times C$  and  $A \cdot B = A \cdot C$ , then  $A \times (B - C) = 0$  so then  $B - C = 0$  or  $B = C$   
 $A \cdot (B - C) = 0$

**Exercise 9.** Given that  $A = (2, -1, 2)$   
 $C = (3, 4, -1)$

- (1) what I did was this:  $A \times C = (-7, 8, 11) = B_p$   
 $B$  must have some of this  $A \times C$  part to twist  $A$  into  $C$ , because  $C$  is orthogonal to both  $A, B$ : ( $B$  must not have any part of  $C$ ).  
If  $B$  has any part of  $A$ ,  $A \times cA = 0$  anyways.

$$A \times \frac{B_p}{-9} = C \quad \frac{B_p}{-9} = \left( \frac{7}{9}, \frac{-8}{9}, \frac{-11}{9} \right)$$

Yes, there's more than one solution, since we could attach scalar multiples of  $A$  to  $B_p / -9$ .

- (2)  $A \cdot \frac{B_p}{-9} = \frac{14+8-22}{9} = 0$   
 $A \cdot cA = 1 \implies c = \frac{1}{9}$  So then the desired vector is  $\frac{B_p}{-9} + \frac{1}{9}A$ . Yes, there's only one vector.

**Exercise 10.** Given that  $A \neq 0$  and  $C \cdot A = 0$ . Suppose that for  $B$ , there's  $A \times B = C$  and  $A \cdot B = 1$ .

Suppose  $\exists B_1$  s.t.  $A \times B_1 = C$  and  $A \cdot B_1 = 1$ . Then

$$A \times B_1 = C = A \times B \text{ and } A \cdot B_1 = A \cdot B$$

$\implies A \times (B_1 - B) = 0$  and  $A \cdot (B_1 - B) = 0$ . Then  $B_1 - B = 0$  or  $B_1 = B$ , from Exercise 8.

**Exercise 11.** Given

$$A = (1, 0, 1)$$

$$B = (-1, 1, 1)$$

$$C = (2, -1, 2)$$

(1)

$A = (1, 0, 1)$	$A = (1, 0, 1)$	$B = (-1, 1, 1)$
$B - A = (-2, 1, 0)$	$C - A = (1, -1, 1)$	$A - B = -(-2, 1, 0)$
$C - B = (3, -2, 1)$	$B - C = -(3, -2, 1)$	$C - A = (1, -1, 1)$
$D - C = D - (2, -1, 2)$	$D - B = D - (-1, 1, 1)$	$D - C = D - (2, -1, 2)$
$A - D = (1, 0, 1) - D$	$A - D = (1, 0, 1) - D$	$B - D = (-1, 1, 1) - D$
$D - C = -(B - A)$	$D - B = -(C - A)$	$D - C = -(A - B) = (-2, 1, 0)$
$A - D = -(C - B)$	$A - D = -(B - C)$	$B - D = -(C - A) = -(1, -1, 1)$
$D = (4, -2, 2)$	$D = (-2, 2, 0)$	$D = (0, 0, 2)$

3 is the total number of possible  $D$ 's, fourth vertex (imagine permutating a 4 vertex ring).

- (2)  $B - A \times C - A = (-2, 1, 0) \times (1, -1, 1) = (1, 2, 1)$

$$\frac{\|B - A \times C - A\|}{2} = \frac{\sqrt{6}}{2}$$

**Exercise 12.** Given that  $A \cdot B = 2$ ;  $\|A\| = 1$ ,  $\|B\| = 4$ , and  $C = 2(A \times B) - 3B$

$$A \cdot (B + C) = A \cdot B + A \cdot C = A \cdot B - 3A \cdot B = \boxed{-4}$$

$$\|C\|^2 = 4(A \times B)^2 + 9B^2 = 4((1)^2 4^2 \frac{8}{4}) + 9(16) = 16(3 + 9) = 16 \cdot 12$$

$$\|C\| = \boxed{8\sqrt{3}}$$

$$\frac{B \cdot C}{\|B\|\|C\|} = \frac{-3B^2}{BC} = \boxed{\frac{-\sqrt{3}}{2}}$$

**Exercise 13.**

$$(1) \quad c_1(A + B) + c_2(A - B) = (c_1 + c_2)A + (c_1 - c_2)B = 0$$

$$c_1 + c_2 = 0 \quad c_1 - c_2 = 0, \text{ so then } c_1 = c_2 = 0.$$

$\implies A + B, A - B$  linearly independent.

So then  $A + B, A - B, A \times B$  must be linearly independent (since  $A \times B \perp A + B, A - B$ ).

$$(2) \quad c_1(A + B) + c_2(A + (A \times B)) + c_3(B + (A \times B)) =$$

$$= (c_1 + c_2)A + (c_1 + c_3)B + (c_2 + c_3)(A \times B) = 0$$

$$c_1 + c_2 = 0$$

$$c_1 + c_3 = 0 \implies c_3 = 0 = c_2 = c_1.$$

$$c_2 + c_3 = 0$$

$$(3) \quad (A + B) \times (A - B) = B \times A - A \times B = 2(B \times A). \text{ Linearly independent since we already know that } A \times B \text{ is orthogonal to } A \text{ and } B, \text{ and } A, B \text{ are linearly independent.}$$

**Exercise 14.**

$$(1) \text{ If } A, B, C \text{ lie on a line, then } C = A + t(B - A).$$

$$(B - A) \times (C - A) = (B - A) \times (t(B - A)) = 0$$

If  $(B - A) \times (C - A) = 0$ , then  $t(B - A) = C - A$  (otherwise, at least one is zero, and two points have a line through them, and so we'd be done). Then  $C = A + t(B - A)$  and so  $C$  would be part of a line joining  $A$  and  $B$ .

$$(2) \quad A \neq B,$$

$$L = \{A + t(B - A)\}$$

$$X = A + t(B - A) = B + s(A - B)$$

$$X - A = t(B - A)$$

$$X - B = s(A - B) \implies (X - A) \times (X - B) = 0$$

**Exercise 15.** Given that  $A \perp B$ ,  $\|A\| = \|B\| = 1$  and  $P \times B = A - P$ ,

$$(1) \quad P \cdot B = (A - P \times B) \cdot B = A \cdot B = 0$$

$$P^2 = A^2 - 2A \cdot (P \times B) + (P \times B)^2 = A^2 - 2A \cdot (P \times B) + P^2 B^2 = 1 - 2A \cdot (P \times B) + P^2$$

$$\implies A \cdot (P \times B) = 1/2$$

$$A \cdot (P \times B) = A \cdot (A - P) = A^2 - P \cdot A = 1/2 \implies P \cdot A = 1/2 = P \cos \theta_{PA}$$

$$A \times (P \times B) = (A \cdot B)P - (P \cdot A)B = -(P \cdot A)B = -A \times P \implies (P \cdot A)B = (A \times P)$$

$$(P \cdot A)^2 B^2 = A^2 P^2 \sin^2 \theta_{PA} = \frac{1}{4} = P^2 \sin^2 \theta_{PA}$$

$$P^2 \sin^2 \theta_{PA} + P^2 \cos^2 \theta_{PA} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2} = P^2$$

$$\implies \boxed{P = \frac{1}{\sqrt{2}}}$$

Note that we had used the cab-bac rule above, which we'll prove with tensors in the next set of exercises.

(2)  $P, B, P \times B$ .

$P \cdot B = 0$ , so  $P, B$  orthogonal.  $P, B$  orthogonal to  $P \times B$ . By thm.,  $P, B, P \times B$  form a basis (3 orthogonal vectors in  $V_3$  form a basis).

(3)

$$\begin{aligned}(B \times (B \times P))_j &= \epsilon_{jkl} B_k (B \times P)_l = \epsilon_{jkl} B_k \epsilon_{lmn} B_m P_n = \epsilon_{jkl} \epsilon_{mnl} B_k B_m P_n = \\ &= B_k B_j P_k - B_k B_k P_j = (B \cdot P) B_j - (B \cdot B) P_j = -P_j \\ &\implies \boxed{(B \times (B \times P)) = ((P \times B) \times B) = P}\end{aligned}$$

(4)

$$\begin{aligned}A \times B &= (P \times B) \times B + P \times B = -P + P \times B \\ (P \times B) + P &= A \\ P &= \frac{A - (A \times B)}{2}\end{aligned}$$

### 13.14 Exercises - The scalar triple product, Cramer's rule for solving a system of three linear equations.

**Exercise 1.**

$$A = (3, 0, 0)$$

$$(1) B = (0, 4, 0) \implies \boxed{96}$$

$$C = (0, 0, 8)$$

$$A = (2, 3, -1)$$

$$(2) B = (3, -7, 5) \quad 22 - 3 + 8 = \boxed{27}$$

$$C = (1, -5, 2)$$

$$A = (2, 1, 3)$$

$$(3) B = (-3, 0, 6) \quad -60 + 21 + 45 = \boxed{6}$$

$$C = (4, 5, -1)$$

**Exercise 2.** Given

$$(0, 1, t)$$

$$(1, t, 1) \implies t + t(1 - t^2) = 0 \text{ or } \boxed{t = 0, \pm\sqrt{2}}. \text{ Then, by theorem, since the scalar product is zero, then the vectors are } (t, 1, 0)$$

linearly dependent.

**Exercise 3.** Given

$$(1, 1, 0)$$

$$(0, 1, 1) \implies 1 + 1 = \boxed{2}$$

$$(1, 0, 1)$$

**Exercise 4.** Want  $A \times B = A \cdot (B \times e_1)e_1 + A \cdot (B \times e_2)e_2 + A \cdot (B \times e_3)e_3$

$$\text{Now } (A \times B)_j = \epsilon_{jkl} A_k B_l$$

$$\begin{aligned}A \cdot (B \times e_j) &= A_l (B \times e_j)_l = A_l \epsilon_{lmn} B_m (e_j)_n = A_l \epsilon_{lmj} B_m = \\ &= \epsilon_{jlm} A_l B_m = \epsilon_{jkl} A_k B_l\end{aligned}$$

**Exercise 5.** Prove that  $i \times (A \times i) + j \times (A \times j) + k \times (A \times k) = 2A$ .

We'll show this in two ways to be instructive.

$$i \times (A \times i) = ? = -i \times (i \times A)$$

$$\begin{aligned}\begin{vmatrix} e_1 & e_2 & e_3 \\ 1 & 0 & 0 \\ a_1 & a_2 & a_3 \end{vmatrix} &= (0, -a_3, a_2) & \begin{vmatrix} e_1 & e_2 & e_3 \\ 1 & 0 & 0 \\ 0 & -a_3 & a_2 \end{vmatrix} &= (0, -a_2, -a_3) \\ -i \times (i \times A) &= -i \times (0, -a_3, a_2) = -(-a_3 e_3 - a_2 e_2) = a_3 e_3 + a_2 e_2\end{aligned}$$

We could interchange the labels to get the results for  $j \times (A \times j)$  and  $k \times (A \times k)$ .

We could also use the tensor notation:

$$\begin{aligned}(e_j \times (A \times e_j))_l &= \epsilon_{lmn} ((e_j)_m (A \times e_j)_n) = \epsilon_{lmn} (e_j)_m \epsilon_{noj} A_o e_j = \\ &= \epsilon_{ljn} \epsilon_{noj} e_j e_j A_o = \epsilon_{jkl} \epsilon_{jko} e_j e_j A_o = \epsilon_{jkl} \epsilon_{jkl} e_j e_j A_l = \\ &= A_l \text{ for } l \neq j\end{aligned}$$

So for each  $j$ , we run through  $l \neq j$  so that we have the  $A_k e_k$  and  $A_l e_l$  components. Then we get  $2A$ .



**Exercise 6.**

$$(1) (a, b, c) \cdot ((0, 0, 1) \times (6, 3, 4)) = 3 = (a, b, c) \cdot (3, 6, 0) = -3a + 6b = 3 \implies -a + 2b = 1 \text{ or } a = 2b - 1.$$

$$\boxed{(2b - 1, b, c)} = b(2, 1, 0) + (-1, 0, c)$$

$$(2) \text{ We need to minimize } a^2 + b^2 + c^2. \text{ Since each term in this sum is positive and } c \text{ is arbitrary, let } c = 0.$$

$$a^2 + b^2 + c^2 = a^2 + b^2 = (2b - 1)^2 + b^2 = 5b^2 - 4b + 1$$

$$\implies b = 2/5 \implies a = 2b - 1 = \frac{4}{5} - 1 = -1/5$$

$$\implies \boxed{(-1/5, 2/5, 0)}$$

**Exercise 7.**

$$(1) (A + B) \cdot (A + B) \times C = D \cdot (D \times C) = 0$$

$$(2) \text{ Want: } A \cdot (B \times C) = -B \cdot (A \times C)$$

$$(A + B) \cdot (A \times C + B \times C) = 0 \implies B \cdot (A \times C) = -A \cdot (B \times C)$$

$$(3) \text{ Want: } A \cdot (B \times C) = -A \cdot (C \times B)$$

$$A \cdot (B \times C) = A \cdot (-C \times B) = -A \cdot (C \times B)$$

$$(4) A \cdot B \times C = -B \cdot A \times C = B \cdot C \times A = -C \cdot B \times A$$

**Exercise 9. Tensor notation way:**

$$(A \times (B \times C))_j = \epsilon_{jkl} A_k (B \times C)_l = \epsilon_{jkl} A_k \epsilon_{lmn} B_m C_n = \epsilon_{jkl} \epsilon_{lmn} A_k B_m C_n = \\ = \epsilon_{jkl} \epsilon_{mnl} A_k B_m C_n = A_k B_j C_k - A_k B_k C_j = (C \cdot A) B_j - (B \cdot A) C_j$$

or

$$(i \times (B \times C))_j = \epsilon_{jkl} (e_1)_k (B \times C)_l = \epsilon_{j1l} e_1 \epsilon_{lmn} B_m C_n = \\ = \epsilon_{j1l} \epsilon_{mnl} B_m C_n = B_j C_1 - B_1 C_j$$

$$i \times (B \times C) = c_1 B - b_1 C$$

more generally

$$(e_t \times (B \times C))_j = \epsilon_{jkl} (e_t)_k (B \times C)_l = \epsilon_{jtl} e_t \epsilon_{lmn} B_m C_n = \\ = \epsilon_{jtl} \epsilon_{mnl} B_m C_n = B_j C_t - c_j B_t = c_t B_j - b_t C_j$$

$$\implies e_t \times (B \times C) = c_t B - b_t C$$

$$a_1 i \times (B \times C) + a_2 j \times (B \times C) + a_3 k \times (B \times C) = (C \cdot A) B - (B \cdot A) C$$

**Exercise 10.**

$$(1) (A \times B) \times (C \times D) = (D \cdot A \times B) C - (A \times B \cdot C) D$$

$$(2)$$

$$A \times (B \times C) + B \times (C \times A) + C \times (A \times B) = \\ = (C \cdot A) B - (B \cdot A) C + (B \cdot A) C - (B \cdot C) A + (C \cdot B) A - (C \cdot A) B = 0$$

$$(3) \text{ Want: } A \times (B \times C) = (A \times B) \times C \text{ iff } B \times (C \times A) = 0$$

$$A \times (B \times C) + B \times (C \times A) + C \times (A \times B) = 0 \text{ (in general).}$$

$$\text{If } B \times (C \times A) = 0, A \times (B \times C) = -C \times (A \times B) = (A \times B) \times C.$$

$$\text{If } A \times (B \times C) = (A \times B) \times C,$$

$$A \times (B \times C) + B \times (C \times A) + -(A \times B) \times C = (A \times B) \times C - (A \times B) \times C + B \times (C \times A) = \\ = 0 + B \times (C \times A) = 0 \implies B \times (C \times A) = 0$$

$$(4) \text{ Remember, we can cyclically permute the scalar triple:}$$

$$(A \times B) \cdot (C \times D) = C \cdot (D \times (A \times B)) = C \cdot ((D \cdot B) A - (D \cdot A) B) = \\ = (B \cdot D)(A \cdot C) - (B \cdot C)(A \cdot D)$$

$$A \times C \cdot B = 5$$

$$A \times D \cdot B = 3$$

**Exercise 11. Given that  $A, B, C, D \in V_3$**

$$C + D = (1, 2, 1)$$

$$C - D = (1, 0, -1)$$

$$(A \times B) \times (C \times D) = (D \cdot (A \times B)) C - ((A \times B) \cdot C) D = (B \cdot D \times A) C + (A \times C \cdot B) D =$$

$$= (-B \cdot A \times D) C + (A \times C \cdot B) D = (-3)(1, 1, 0) + 5(0, 1, 1) = (-3, -3, 0) + (0, 5, 5) = \boxed{(-3, 2, 5)}$$

**Exercise 12.**

Consider  $(A \times B) \cdot (B \times C) \times (C \times A)$

$$\begin{aligned} (B \times C) \times (C \times A) &= (A \cdot (B \times C))C - (C \cdot (B \times C))A = (A \cdot (B \times C))C \\ \Rightarrow (A \times B) \cdot (B \times C) \times (C \times A) &= (A \cdot (B \times C))C \cdot (A \times B) = (A \cdot (B \times C))^2 \end{aligned}$$

In the last step, we cyclically permuted the scalar triple.

**Exercise 13.** Prove or disprove  $A \times (A \times (A \times B)) \cdot C = -A^2 A \cdot B \times C$

$$A \times ((A \cdot B)A - A^2 B) \cdot C = -A^2(A \times B) \cdot C = -A^2 C \cdot (A \times B) = -A^2 A \cdot (B \times C)$$

**Exercise 14.**  $A, B, C, D \in V_3$ .

- (1) If we consider  $A$  to be the origin, then consider the parallelogram with sides  $B - A, C - A, D - A \rightarrow B, C, D$ . The total volume of the parallelogram is given by  $|(B - A) \cdot (C - A) \times (D - A)| = |(D - A) \cdot (B - A) \times (C - A)|$ .

Note that we have a parallelogram from  $B, C, D$  vectors.  $B, C, D$  themselves are  $(3 + 1)$  (1 vertex from  $A$ , the “origin”) vertices of this parallelogram, and a parallelogram has 8 vertices.

We can translate  $D$  vector from the  $A$  origin to 3 other unique positions - on  $B$ , on  $C$ , and on  $B + C$  (diagonal across from  $A$ , containing  $A$ ). This will generate 3 new tetrahedrons that are congruent to the  $BCD$  tetrahedron, because of either the parallelogram has congruent opposite parallel sides or uses or shares the same vertices and vectors as the  $BCD$  tetrahedron.

Be careful because the parallelogram is *not* “filled up”, its total volume is *not* accounted for, by these 4 congruent tetrahedrons, even though each tetrahedron shares 3 of its sides each with each of the other 3 tetrahedron.

Indeed, consider a cube of sides  $a, a, a$ . Consider a tetrahedron made up of its sides. The diagonal connecting  $B$  and  $C$ ,  $B - C$ , is of length  $a\sqrt{2}$ .

$$\text{Volume of parallelogram} = 6 \times (\text{Volume of a tetrahedron})$$

$$\Rightarrow \boxed{\text{Volume of a tetrahedron} = \frac{1}{6} |(B - A) \cdot (C - A) \times (D - A)|}$$

(2)

$$\begin{aligned} \text{For } \begin{aligned} A &= (1, 1, 1) \\ B &= (0, 0, 2) \\ C &= (0, 3, 0) \\ D &= (4, 0, 0) \end{aligned} \quad \frac{1}{4} |(B - A) \cdot (C - A) \times (D - A)| &= \frac{1}{4} |(-1, -1, 1) \cdot (-1, 2, -1) \times (3, -1, -1)| = \\ &= \frac{1}{4} |(-1, -1, 1) \cdot (-3, -4, -5)| = \boxed{\frac{1}{2}} \end{aligned}$$

**Exercise 15.**

(1)

$$\left\| \frac{C - B}{\|C - B\|} \times (A - B) \right\| = \|A - B\| \sin \theta_1$$

$\theta_1$  is the angle between  $C - B / \|C - B\|$  and  $A - B$ . From the geometry, it's clear that  $\|A - B\| \sin \theta_1$  is the perpendicular distance of  $A$  from the line  $L = \{B + t(C - B)\}$ .

- (2) Given that  $A = (1, -2, -5), B = (-1, 1, 1), C = (4, 5, 1)$

$$\|(2, -3, -6) \times (5, 4, 0)\| / \|(-5, -4, 0)\| = \|(24, -30, 23)\| / \sqrt{41} = \boxed{\sqrt{2005/41}}$$

**Exercise 16.**

- (1) Recall that  $(A \times B)_j = \epsilon_{jkl} A_k B_l$  so that

$$(A \times B)^2 = \epsilon_{jkl} A_k B_l \epsilon_{jmn} A_m B_n = \epsilon_{jkl} \epsilon_{jmn} A_k A_m B_l B_n = A^2 B^2 - (A \cdot B)^2$$

and  $-2A \cdot B = \|A - B\|^2 - \|A\|^2 - \|B\|^2$  is immediately apparent by doing the algebra. Then

$$\begin{aligned} 4S^2 &= 4(\text{area of the triangle})^2 = 4\left(\frac{1}{2}\|A \times B\|\right)^2 = \|A \times B\|^2 = A^2 B^2 - (A \cdot B)^2 = \\ &= a^2 b^2 - \left(\frac{-1}{2}(c^2 - a^2 - b^2)\right)^2 = \boxed{a^2 b^2 - \frac{1}{4}(c^2 - a^2 - b^2)^2} = \\ &= \boxed{\frac{1}{4}(2ab - c^2 + a^2 + b^2)(2ab + c^2 - a^2 - b^2)} \quad (\text{just complete the square}) \end{aligned}$$

(2)

$$16S^2 = ((a+b)^2 - c^2)(-(a-b)^2 + c^2) = ((a+b)+c)((a+b)-c)(c-(a-b))(c+(a-b))$$

$$\implies S^2 = \frac{1}{16}(a+b+c)(a+b-c)(c-a+b)(c+a-b)$$

**Exercise 17.**

$$\begin{array}{l} x + 2y + 3z = 5 \\ 2x - y + 4z = 11 \\ 0 - y + z = 3 \end{array} \implies \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} x + \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} y + \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} z = \begin{pmatrix} 5 \\ 11 \\ 3 \end{pmatrix}$$

$$Ax + By + Cz = D$$

Recall that

$$x = \frac{DBC}{ABC} \quad y = \frac{DCA}{ABC} \quad z = \frac{DAB}{ABC}$$

$$A \cdot (B \times C) = (1, 2, 0) \cdot (3, -5, 11) = -7$$

$$D \cdot (B \times C) = (5, 11, 3) \cdot (3, -5, 11) = -7$$

$$D \cdot (C \times A) = -D \cdot (2, -1, -2) = -(5, 11, 3) \cdot (2, -1, -2) = 7$$

$$D \cdot (A \times B) = D \cdot (-2, 1, -5) = (5, 11, 3) \cdot (-2, 1, -5) = -14$$

$$\implies \boxed{x = 1, y = -2, z = 2}$$

**Exercise 18.**

$$\begin{array}{l} x + y + 2z = 4 \\ 3x - y - z = 2 \\ 2x + 5y + 3z = 3 \end{array} \implies \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} x + \begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix} y + \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} z = \begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix}$$

$$A \cdot (B \times C) = (1, 3, 2) \cdot (2, 7, 1) = 25$$

$$D \cdot (B \times C) = (4, 2, 3) \cdot (2, 7, 1) = 25$$

$$D \cdot (C \times A) = -D \cdot (11, 1, -7) = -(4, 2, 3) \cdot (11, 1, -7) = -25$$

$$\implies \boxed{x = 1, y = -1, z = 2}$$

**Exercise 19.**

$$\begin{array}{l} x + y = 5 \\ x + z = 2 \\ y + z = 5 \end{array} \implies \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} y + \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} z = \begin{pmatrix} 5 \\ 2 \\ 5 \end{pmatrix}$$

$$\implies \boxed{x = 1 = z, y = 4}$$

**Exercise 20.**  $P = (1, 1, 1)$   $A = (2, 1, -1)$   $X = P + tA \implies \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + t \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}.$

Note that  $X \cdot N = P \cdot N + tA \cdot N = P \cdot N$ .

$$\begin{array}{l} x - y + z = 1 \\ x + y + 3z = 5 \\ 3x + y + 7z = 11 \end{array} \implies \begin{array}{l} (1, -1, 1) \cdot X = 1 \\ (1, 1, 3) \cdot X = 5 \\ (3, 1, 7) \cdot X = 11 \end{array}$$

We could immediately then see that for all  $X = P + tA$ ,  $X$  satisfies this system of linear equations because  $X \cdot N = P \cdot N$  and  $P \cdot N$  is equal to the left hand side (LHS) of each of the equations in this system. Also,  $A \cdot N = 0$  for each of the normal vectors, as expected.

**13.17 Exercises - Normal vectors to planes, Linear Cartesian equations for planes.**

**Exercise 1.** Given  $A = (2, 3, -4)$   
 $B = (0, 1, 1)$

- (1)  $A \times B = (7, -2, 2)$
- (2)  $\implies 7x - 2y + 2z = 0$
- (3)  $7x - 2y + 2z = 9$

**Exercise 2.**  $x + 2y - 2z + 7 = 0$  or  $x + 2y - 2z = -7$ .

- (1)  $\frac{1}{3}(1, 2, -2)$

$$(-7, 0, 0)$$

$$(2) \frac{x}{-7} - \frac{2y}{7} + \frac{2z}{7} = 1 \text{ so the intercepts are } (0, -7/2, 0)$$

$$(0, 0, 7/2)$$

$$(3) \frac{N}{|N|} \cdot P = d \implies \frac{1}{3}(1, 2, -2) \cdot (0, 0, 7/2) = \boxed{7/3}$$

$$(4) \boxed{-7/9(1, 2, -2)}$$

**Exercise 3.**  $(1, 2, -3)$  is in  $3x - y + 2z = -5$  and  $(2, 0, -1)$  is in  $3x - y + 2z = 4$ .

$$\frac{N}{|N|} = \frac{(3, -1, 2)}{\sqrt{14}}$$

$$\frac{N}{|N|} \cdot P_1 = \frac{(3, -1, 2)}{\sqrt{14}} \cdot (1, 2, -3) = \frac{-5}{\sqrt{14}}$$

$$\frac{N}{|N|} \cdot P_2 = \frac{(3, -1, 2)}{\sqrt{14}} \cdot (2, 0, -1) = \frac{4}{\sqrt{14}}$$

So the distance between the two planes is  $\boxed{\frac{9}{\sqrt{14}}}$

**Exercise 4.**

(1)

$$x + 2y - 2z = 5 \quad (1, 2, -2)$$

$$3x - 6y + 3z = 2 \quad 3(1, -2, 1)$$

$$2x + y + 2z = -1 \quad (2, 1, 2)$$

$$x - 2y + z = 7 \quad (1, -2, 1)$$

So by definition, the second and fourth are parallel to each other and the first and the third are perpendicular to each other, since  $(1, 2, -2) \cdot (2, 1, 2) = 0$

(2)  $N = \frac{(1, -2, 1)}{\sqrt{6}}$ .  $(0, -1/3, 0)$  is on the second plane,  $3x - 6y + 3z = 2$ , and  $(3, 0, 4)$  is on the fourth plane,  $x - 2y + z = 7$ .

$$\begin{aligned} N \cdot (0, -1/3, 0) &= \frac{2/3}{\sqrt{6}} \\ N \cdot (3, 0, 4) &= \frac{7}{\sqrt{6}} \end{aligned} \implies 7/\sqrt{6} - \frac{2/3}{\sqrt{6}} = \frac{19/3}{\sqrt{6}}$$

**Exercise 5.** Given  $(1, 1, -1)$ ,  $(3, 3, 2)$ ,  $(3, -1, -2)$ , the spanning vectors for a plane through all three points is  $(2, 2, 3)$ ,  $(2, -2, -1)$

(1) The normal vector to this plane is  $(4, 8, -8)$

$$(2) x + 2y - 2z = 5$$

$$(3) \frac{(1, 2, -2)}{3} \cdot (1, 1, -1) = 5/3$$

**Exercise 6.** Given  $(1, 2, 3)$ ,  $(2, 3, 4)$ ,  $(-1, 7, -2)$ , the spanning vectors for a plane through all three points are  $(1, 1, 1)$ ,  $(-2, 5, -5)$

The normal vector to this plane is  $(-10, 3, 7)$ .

Then the Cartesian equation could easily be obtained:  $-10x + 3y + 7z = 17$

**Exercise 7.**

$$x + y = 1 \quad (1, 1, 0)$$

$$y + z = 2 \quad (0, 1, 1)$$

$$\frac{1}{\sqrt{2}\sqrt{2}} = \frac{1}{2} = \cos \theta \implies \boxed{\theta = \pi/3 \text{ or } 2\pi/3}$$

**Exercise 8.**  $(1, 2, 9) \implies x + 2y + 9z = -55$

**Exercise 9.**  $4x - 3y + z = 5 \rightarrow (4, -3, 1)$ .  $(4, -3, 1)$  is perpendicular to this plane.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} + s \begin{pmatrix} 4 \\ -3 \\ 1 \end{pmatrix}$$

**Exercise 10.**  $X(t) = (1 - t, 2 - 3t, 2t - 1)$

$$(1) X(t) = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + t \begin{pmatrix} -1 \\ -3 \\ 2 \end{pmatrix}$$

(2)  $(-1, 1, 1)$

(3) Plug in  $(1, 2, 1)$  and  $t(-1, -3, 2)$  into  $2x + 3y + 2z$  to see which  $t$  will give  $-1$  on the right hand side.  $t = 1$

(4)  $X(3) = (-2, -7, 5)$ . For a plane to be parallel to another plane, then they each have the same normal vector.  
 $\Rightarrow (2, 3, 2)$ . So the Cartesian equation is  $\boxed{2x + 3y + 2z = -15}$

(5)  $X(2) = (-1, -4, 3)$ . So then a plane perpendicular to  $L$  and through  $X(2)$  would be  
 $-x - 3y + 2z = 19$

**Exercise 11.** Try  $N = (n_1, n_2, n_3)$ .

$$\frac{N \cdot e_1}{|N|} = \frac{n_1}{\sqrt{n_1^2 + n_2^2 + n_3^2}} = \frac{1}{2} \quad \Rightarrow n_1 = n_3$$

$$\frac{n_3}{\sqrt{n_1^2 + n_2^2 + n_3^2}} = \frac{1}{2}$$

$$\frac{n_2}{\sqrt{2n_1^2 + n_2^2}} = \frac{1}{\sqrt{2}} \Rightarrow \frac{n_1}{n_2} = \frac{1}{\sqrt{2}}$$

$$N = \frac{(1, \sqrt{2}, 1)}{2}$$

$$\Rightarrow \boxed{x + \sqrt{2}y + z = 2 + \sqrt{2}}$$

**Exercise 12.**

$$x + 2y + 3z = 6$$

$$\frac{x}{6} + \frac{y}{3} + \frac{z}{2} = 1$$

$$(6, 0, 0)$$

$$(0, 3, 0) \Rightarrow \frac{36}{6} = \boxed{6}$$

$$(0, 0, 2)$$

**Exercise 13.**  $x - y + 5z = 1$  Then two linearly independent spanning vectors for this could be  
 $(1, 1, 0)$

$$(0, 5, 1)$$

We want this vector to be perpendicular to  $(1, 2, -3)$ :

$$(a(1, 1, 0) + b(0, 5, 1)) \cdot (1, 2, -3) = (a + 2(a + 5b) - 3b) = 3a + 7b = 0 \Rightarrow b = -3a/7$$

$$\text{so } a((1, 1, 0) + \frac{-3}{7}(0, 5, 1)) = a(1, \frac{-8}{7}, \frac{-3}{7}) \Rightarrow \boxed{\frac{1}{\sqrt{122}}(7, -8, -3)}$$

**Exercise 14.** Given a plane that needs to span with  $\begin{pmatrix} 1, 1, 0 \\ 0, 1, 1 \end{pmatrix}$  since it needs to be parallel to these two vectors, and intercept the intercept  $(2, 0, 0)$

$$(1, 1, 0) \times (0, 1, 1) = N = (1, -1, 1) \Rightarrow \boxed{x - y + z = 2}$$

**Exercise 15.**

$$\begin{aligned} 3x + y + z &= 5 \\ 3x + y + 5z &= 7 \\ x - y + 3z &= 3 \end{aligned} \Rightarrow \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix} x + \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} y + \begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix} z = \begin{pmatrix} 5 \\ 7 \\ 3 \end{pmatrix}$$

$$(1, 1, -1) \times (1, 5, 3) = (8, -4, 4) \Rightarrow A \cdot B \times C = 16$$

$$x = \frac{D \cdot B \times C}{A \cdot B \times C} = \frac{24}{18} = \frac{3}{2}$$

$$(1, 5, 3) \times (3, 3, 1) = (-4, 8, -12) \Rightarrow y = 0 \quad \text{then } z = 1/2$$

$$\boxed{x = 3/2, y = 0, z = 1/2}$$

**Exercise 16.** We could try to solve for a system of 3 linear equations that represents the 3 planes:

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2$$

$$a_3x + b_3y + c_3z = d_3$$

If  $a_1, a_2, a_3 \neq 0$

$$\begin{aligned} x + b_1y + c_1z &= d_1 \\ \implies x + b_2y + c_2z &= d_2 \\ x + b_3y + c_3z &= d_3 \end{aligned} \implies \begin{bmatrix} 1 & b_1 & c_1 & d_1 \\ 1 & b_2 & c_2 & d_2 \\ 1 & b_3 & c_3 & d_3 \end{bmatrix}$$

Since the normals of the planes are linearly independent, this means that

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ b_1 & b_2 & b_3 & 0 \\ c_1 & c_2 & c_3 & 0 \end{bmatrix}$$

has a unique solution and we'll be left with the identity matrix. Then for our problem above, we can do the same operations, since the equations are linear, and leave the LHS to be an identity matrix. Then we've solved for a unique point in  $\mathbb{R}^3$

**Exercise 17.** Given a line through  $(1, 2, 3)$  and parallel to the two planes given by  $x + 2y + 3z = 4$  and  $2x + 3y + 4z = 5$ , to find a direction vector for the line that is parallel to both planes, determine a vector that would make each of the two previous equations equal to 0 on the LHS.  $(1, -2, 1)$ , plugged into the LHS of each of the 2 equations, would make it equal 0. The line is through  $(1, 2, 3)$  so then  $X = (1, 2, 3) + t(1, -2, 1)$

**Exercise 18.** So given a line  $L$ , s.t.  $X = P + uA$  and plane  $M$ , s.t.  $X = Q + sB + tC$ , and that  $L \nparallel M$ , so that  $sB + tC \nparallel uA \quad \forall s, t, u \in \mathbb{R}$ ,

Consider  $L \cap M$ . Consider  $X_1 \in L \cap M$

$$X_1 = P + t_1A$$

$$X_1 - P = t_1A \implies X = P + tA = P + (X_1 - P) + (t - t_1)A = X_1 + tA \quad (\text{since } t \in \mathbb{R})$$

Since  $X_1 \in M$  as well,

$$X_1 = Q + a_1B + b_1C$$

$$X = Q + sB + tC = Q + a_1B + b_1C + (s - a_1)B + (t - b_1)C = X_1 + sB + tC$$

so we have

$$X_L = X_1 + tA$$

$$X_M = X_1 + sB + tC$$

Suppose  $X_2 \in L \cup M$

$$X_2 = X_1 + u_2A$$

$$X_2 = X_1 + s_2B + t_2C \implies u_2A = s_2B + t_2C$$

$$\text{but } sB + tC \neq uA \quad \forall s, t, u \in \mathbb{R}$$

Contradiction, so  $X_2 = X_1$

**Exercise 19.**

(1) Given  $ax + by + cz = -d$ , then the normal vector for this plane, that's normalized, is  $\frac{N}{|N|} = \frac{(a, b, c)}{\sqrt{a^2 + b^2 + c^2}}$ .

Note that the Cartesian equation for the plane is simply  $X \cdot N = P \cdot N$ .

Then the perpendicular distance from the point  $X_0$  to a point on the plane,  $P$ , is given by  $(X_0 - P) \cdot N = X_0 \cdot N - P \cdot N$ .

$$\text{dist.} = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

(2)  $5x - 14y + 2z = -9$ . The normal to this plane is  $\frac{N}{|N|} = e_n = \frac{(5, -14, 2)}{15}$ .

The perpendicular distance from  $Q$  to plane is  $\text{dist.} = \frac{(Q - P) \cdot N}{|N|} = \frac{-234 + 9}{15} = -15$

So to get this point on the plane, we simply have to go from  $Q$  and go +15 distance along the normalized normal vector  $e_n$ :

$$(-2, 15, -7) + 15e_n = (3, 1, -5)$$

**Exercise 20.** Given  $2x - y + 2z + 4 = 0$ , a plane parallel to this plane will have the same normal vector,  $\frac{N}{|N|} = e_n = \frac{(2, -1, 2)}{3}$ .

The distance from  $Q$  to this plane is

$$\frac{(Q - P) \cdot N}{|N|} = \frac{(3, 2, -1) \cdot (2, -1, 2) - (-4)}{3} = (2 + 4)/3 = 2$$

To find the other plane, simply go a distance 2 along  $e_n$  from  $Q$ , to get a point on that other plane.

$$(3, 2, -1) + 2 \left( \frac{(2, -1, 2)}{3} \right) = \frac{(9, 6, -3) + (4, -2, 4)}{3} = \frac{(13, 4, 1)}{3} = P_2$$

$$2x - y + 2z = \frac{26}{3} - \frac{4}{3} + \frac{2}{3} = P_2 \cdot N = 8 \implies \boxed{2x - y + 2z = 8}$$

**Exercise 21.**

(1) Given 3 points  $A, B, C$ , they determine a plane  $\implies M = \{A + s(B - A) + t(C - A)\}$

The normal vector, normalized, to this plane is  $\frac{N}{|N|} = e_n = \frac{(B-A) \times (C-A)}{|(B-A) \times (C-A)|}$

$$\frac{(Q - P) \cdot N}{|N|} \xrightarrow{P=A} \frac{(Q - A) \cdot (B - A) \times (C - A)}{|(B - A) \times (C - A)|}$$

(2) Given

$$Q = (1, 0, 0)$$

$$A = (0, 1, 1)$$

$$B = (1, -1, 1)$$

$$C = (2, 3, 4)$$

$$Q - A = (1, -1, -1)$$

$$B - A = (1, -2, 0)$$

$$C - A = (2, 2, 3)$$

$$\frac{(Q - A) \cdot (B - A) \times (C - A)}{|(B - A) \times (C - A)|} = \frac{(1, -1, -1) \cdot (-6, -3, 6)}{9} = \boxed{-1}$$

**Exercise 22.**  $M, M'$  not parallel.  $e_{N_m} \neq e_{N_{m'}}$ .

$$x \in M$$

$$x \in M'$$

$$x + b_1y + c_1z = d_1 \quad x + b_2y + c_2z = d_2 \implies (1, b_1, c_1) \neq (1, b_2, c_2) \text{ so that either}$$

$$b_1 \neq b_2 \text{ or } c_1 \neq c_2$$

Let's solve for  $x + b_1y + c_1z = 0$   
 $x + b_2y + c_2z = 0$

If  $b_2 - b_1 \neq 0$ ,

$$\begin{bmatrix} 1 & b_1 & c_1 \\ 1 & b_2 & c_2 \end{bmatrix} \implies \begin{bmatrix} 1 & 0 & \frac{c_1 b_2 - b_1 c_2}{b_2 - b_1} \\ 0 & b_2 - b_1 & c_2 - c_1 \end{bmatrix}$$

$$\implies (x, y, z) = z \left( \frac{b_2 c_1 - c_2 b_1}{b_1 - b_2}, \frac{-(c_2 - c_1)}{b_2 - b_1}, 1 \right)$$

Otherwise, if  $c_2 - c_1 \neq 0$

$$\begin{bmatrix} 1 & b_1 & c_1 \\ 0 & b_2 - b_1 & c_2 - c_1 \end{bmatrix} \implies \begin{bmatrix} 1 & \frac{b_1(c_2 - c_1)}{c_2 - c_1} - \left( \frac{b_2 - b_1}{c_2 - c_1} \right) c_1 & 0 \\ 0 & \frac{b_2 - b_1}{c_2 - c_1} & 1 \end{bmatrix}$$

$$\implies (x, y, z) = y \left( \frac{b_2 c_1 - b_1 c_2}{c_2 - c_1}, 1, \frac{-(b_2 - b_1)}{c_2 - c_1} \right)$$

So these two equations with RHS being 0 result in a solution that's completely determined, give a multiplicative factor. We can now find at least one point in the intersection by solving the system of equations

$$x + b_1y + c_1z = d_1$$

$$x + b_2y + c_2z = d_2 \quad \text{If } b_2 - b_1 \neq 0,$$

$$\begin{bmatrix} 1 & b_1 & c_1 & | & d_1 \\ 1 & b_2 & c_2 & | & d_2 \end{bmatrix} \implies \begin{bmatrix} 1 & 0 & \frac{c_1 b_2 - b_1 c_2}{b_2 - b_1} & | & \frac{b_2 d_1 - b_1 d_2}{b_2 - b_1} \\ 1 & 1 & \frac{c_2 - c_1}{b_2 - b_1} & | & \frac{d_2 - d_1}{b_2 - b_1} \end{bmatrix}$$

$$\implies \text{If } z = 1, (x, y, z) = \left( \frac{b_2 d_1 - b_1 d_2 - (b_2 c_1 - c_2 b_1)}{b_2 - b_1}, \frac{d_2 - d_1 - (c_2 - c_1)}{b_2 - b_1}, 1 \right)$$

Otherwise, if  $c_2 - c_1 \neq 0$

$$\begin{aligned} \left[ \begin{array}{ccc|c} 1 & b_1 & c_1 & d_1 \\ 0 & b_2 - b_1 & c_2 - c_1 & d_2 - d_1 \end{array} \right] &\Rightarrow \left[ \begin{array}{ccc|c} 1 & \frac{b_1 c_2 - c_1 b_2}{c_2 - c_1} & 0 & \frac{d_1 c_2 - c_1 d_2}{c_2 - c_1} \\ 0 & \frac{b_2 - b_1}{c_2 - c_1} & 1 & \frac{d_2 - d_1}{c_2 - c_1} \end{array} \right] \\ \Rightarrow \text{If } y = 1, (x, y, z) &= \left( \frac{d_1 c_2 - c_1 d_2 - (b_1 c_2 - c_1 b_2)}{c_2 - c_1}, 1, \frac{d_2 - d_1 - (b_2 - b_1)}{c_2 - c_1} \right) \end{aligned}$$

Thus, it's always possible to find a point in  $M \cap M'$ . This intersection is spanned by only one vector, give a multiplicative factor, which we have found above. Then the intersection  $M \cap M'$  of 2 planes is a line.

**Exercise 23.** Given  $x + 2y + 3z = 4$   
 $2x + y + z = 2$

$$\begin{aligned} \left[ \begin{array}{ccc} 1 & 2 & 3 \\ 2 & 1 & 1 \end{array} \right] &\Rightarrow \left[ \begin{array}{ccc} 1 & 0 & -1/3 \\ 0 & 1 & 5/3 \end{array} \right] \\ &\Rightarrow \left( \frac{1}{3}, \frac{-5}{3}, 1 \right) \end{aligned}$$

Let's find a point on the intersection, which is a line, for the two planes:

$$\begin{aligned} \left[ \begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 2 & 1 & 1 & 2 \end{array} \right] &\Rightarrow \left[ \begin{array}{ccc|c} 1 & 0 & -1/3 & 0 \\ 0 & 1 & 5/3 & 2 \end{array} \right] \\ &\Rightarrow k(1, -3, 3) \end{aligned}$$

The normal to this plane containing the intersection line and is parallel to  $e_2 = j$  is  
 $(0, 1, 0) \times (1/3, -5/3, 1) = (1, 0, 1/3)$

Then  $\boxed{x + \frac{1}{3}z = 2}$

**Exercise 24.** Given two equations for the two planes,

$x + y = 3$   
 $2y + 3z = 4$ , we can immediately get the spanning vector for the line that is the intersection of these planes:  $(-1, 1, -2/3)$ .

Also, by inspection, we can find a point on the intersection of these 2 planes:  $(1, 2, 0)$ .

We want our plane to be parallel to  $(3, -1, 2)$ . We can now determine the normal to this plane:

$$\begin{aligned} (3, -1, 2) \times (-1, 1, -2/3) &= \left( \frac{-4}{3}, 0, 2 \right) \\ \Rightarrow \frac{-4}{3}x + 2z &= \frac{-4}{3} \end{aligned}$$

### 13.21 Exercises - The conic sections, Eccentricity of conic sections, Polar equations for conic sections.

**Exercise 1.**  $F$  is in the positive half-plane determined by  $N$ .

$$\begin{aligned} \|X - F\| &= ed(X, L) \\ \|X - F\| &= e|(X - F) \cdot N + d| \end{aligned}$$

**Exercise 2.**

(1)

$$\begin{aligned} \|X - F\| &= ed(X, L) \\ \|X - F\| &= e|(X - F) \cdot N + d| \\ F = 0 &\Rightarrow \|X\| = e|(X \cdot N) + d|; \quad r = e|r \cos \theta + d| \\ \Rightarrow r &= e(r \cos \theta + d) \Rightarrow r = \frac{ed}{1 - e \cos \theta} \end{aligned}$$

(2) The right branch for the hyperbola is given by  $r = \frac{ed}{1 - e \cos \theta}$  because  $X \cdot N > 0$ . The left branch for  $e > 1$ ,

$$\begin{aligned} \|X - F\| &= ed(X, L) = e|(X - F) \cdot N + d| = \\ &= e|X \cdot N + d| = -e(d + r \cos \theta) = r \\ r &= \frac{-ed}{(1 + e \cos \theta)} \end{aligned}$$



**Exercise 3.** For points below the horizontal directrix,

$$\|X - F\| = ed(X, L)$$

$$F = 0 \implies \|X\| = ed(X, L) = e|(X - F) \cdot N - d| = e|X \cdot N - d| = e|r \sin \theta - d|$$

$$\text{Now Thm. 13.18 says } r = \frac{ed}{e \cos \theta + 1} \quad \text{if } 0 < e \leq 1$$

$$\implies r = e(d - r \sin \theta) \implies r = \frac{ed}{1 + e \sin \theta}$$

For the “right” or upper-half branch of a hyperbola.

$$\|X - F\| = e|(X - F) \cdot N - d| = e(r \sin \theta - d) = r$$

$$r = \frac{-ed}{1 - e \sin \theta} = \frac{ed}{e \sin \theta - 1}$$

**Exercise 4.**  $\|X - F\| = ed(X, L)$ ;  $\|X\| = e|(X - F) \cdot N - d| = e|r \cos \theta - d| = e(d - r \cos \theta) \implies r = \frac{ed}{1 + e \cos \theta}$   
 $\boxed{e = 1, d = 2}.$

**Exercise 5.**  $r = \frac{3}{1 + \frac{1}{2} \cos \theta} = \frac{6(\frac{1}{2})}{1 + \frac{1}{2} \cos \theta} \cdot e = \frac{1}{2}; d = 6$

**Exercise 6.**  $r = \frac{6}{3 + \cos \theta} = \frac{2}{1 + \frac{1}{3} \cos \theta} \cdot e = \frac{1}{3}; d = 6.$

**Exercise 7.**  $r = \frac{1}{-\frac{1}{2} + \cos \theta}.$

$$ed(X, L) = \|X - F\| = e|(X - F) \cdot N - d| = e|r \cos \theta - d| = er \cos \theta - ed = r$$

$$\frac{-ed}{1 - e \cos \theta} = r = \frac{ed}{e \cos \theta - 1}$$

So for  $r = \frac{2}{2 \cos \theta - 1}, \quad e = 2, d = 1.$

**Exercise 8.**  $r = \frac{4}{1 + 2 \cos \theta} \quad e = 2, d = 2.$

**Exercise 9.**  $r = \frac{4}{1 + \cos \theta} \quad e = 1, d = 4.$

**Exercise 10.**  $3x + 4y = 25 \implies \frac{3}{5}x + \frac{4}{5}y = 5. \quad N = (\frac{3}{5}, \frac{4}{5}).$

$L = \{x = P + tA\}, \quad N \cdot X = N \cdot P.$

To find the distance from the focus, at the origin, to the directrix,

$$dN = P + tA; \quad dN \cdot N = d = N \cdot P$$

So for this problem,  $d = 5.$

$$r = \|X - F\| = ed(X, L) = e|(X - F) \cdot N - d| = e|X \cdot N - d| = \left| \frac{3}{5}r \cos \theta + \frac{4}{5}r \sin \theta - 5 \right|$$

$$r = \frac{1}{2} \left( 5 - \frac{3}{5}r \cos \theta - \frac{4}{5}r \sin \theta \right)$$

$$r = \frac{5/2}{1 + \frac{3}{10} \cos \theta + \frac{4}{10} \sin \theta}$$

**Exercise 11.**  $e = 1, \quad 4x + 3y = 25 \quad \frac{4}{5}x + \frac{3}{5}y = 5; \quad N = (\frac{4}{5}, \frac{3}{5}).$   
 $d = 5.$

$$\|X - F\| = ed(X, L) = e|(X - F) \cdot N - d| = e \left| \frac{4}{5}r \cos \theta + \frac{3}{5}r \sin \theta - 5 \right|$$

$$r = 5 - r \left( \frac{4}{5} \cos \theta + \frac{3}{5} \sin \theta \right)$$

$$r = \frac{5}{1 + \frac{4}{5} \cos \theta + \frac{3}{5} \sin \theta}$$

**Exercise 12.**  $e = 2$ , hyperbola, so there's 2 branches.

$$\frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y = \frac{1}{\sqrt{2}} \quad L = \{x = P + tA\} \quad X \cdot N = N \cdot P$$

$$dN = P + tA; \quad dN \cdot N = d = N \cdot P = \frac{1}{\sqrt{2}}$$

Note that the sign of  $d$  here tells you what side the focus, at the origin, lies on.

$$\|X - F\| = ed(X, L) = \|X\| = e|(X - F) \cdot N - d| = e(d - \frac{1}{\sqrt{2}}r \cos \theta - \frac{1}{\sqrt{2}}r \sin \theta)$$

$$r = \frac{2/\sqrt{2}}{1 + \frac{2}{\sqrt{2}} \cos \theta + \frac{2}{\sqrt{2}} \sin \theta}$$

But for the right side branch,

$$\|X - F\| = ed(X, L) = \|X\| = e|(X - F) \cdot N - d| = -e(d - \frac{1}{\sqrt{2}}r \cos \theta - \frac{1}{\sqrt{2}}r \sin \theta)$$

$$r = \frac{-2/\sqrt{2}}{1 - \frac{2}{\sqrt{2}} \cos \theta - \frac{2}{\sqrt{2}} \sin \theta}$$

**Exercise 13.**  $e = 1$  parabola.

(1)

$$\|X - F\| = \|X\| = ed(X, L) = 1|(X - F) \cdot N - d| = d - X \cdot N = d - r \cos \frac{\pi}{3}$$

$$d = r \left( \frac{3}{2} \right) = \frac{3}{2} \times 10^8 \text{ mi}$$

$$\boxed{r = \frac{\frac{3}{2} \times 10^8 \text{ mi}}{1 + \cos \theta}} \quad \theta = 0, \quad r = \frac{3}{4} \times 10^6 \text{ mi}$$

(2) Focus is in the positive half-plane determined by  $N$ .

$$\|X - F\| = \|X\| = ed(X, L) = |(X - F) \cdot N + d| = r \cos \theta + d$$

$$d = r(1 - \cos \theta) = 10^8 \text{ mi} (1 - \cos \frac{\theta}{3}) = \frac{1}{2} \times 10^8 \text{ mi}$$

$$\boxed{r = \frac{d}{1 - \cos \theta} = \frac{\frac{1}{2} \times 10^8 \text{ mi}}{1 - \cos \theta}} \quad \boxed{r(\theta = \pi) = \frac{1}{4} \times 10^8 \text{ mi}}$$

### 13.24 Exercises - Conic sections symmetric about the origin, Cartesian equations for the conic sections.

Quick Review.

Consider symmetry about the origin.

$$\|X - F\| = ed(X, L) = e|(X - F) \cdot N - d| = e|X \cdot N - F \cdot N - d| = |eX \cdot N - e(F \cdot N + d)|$$

$$\|X - F\|^2 = \|X\|^2 - 2X \cdot F + \|F\|^2 = e^2(X \cdot N)^2 - 2aeX \cdot N + a^2$$

$$X \rightarrow -X; \quad X \cdot F = aeX \cdot N$$

$$X = (F - aeN) = 0 \implies F = aeN; \quad F \cdot N = ae; \quad a = \frac{ed}{1 - e^2} \quad F = \frac{e^2 d}{1 - e^2} N$$

$$\implies \|X\|^2 + (ae)^2 = e^2(X \cdot N)^2 + a^2$$

$$\text{if } X = \pm aN; \quad \|X\|^2 + (ae)^2 = e^2(X \cdot N)^2 + a^2 \text{ is satisfied}$$

$$\text{if } X = \pm bN'; \quad b^2 + (ae)^2 = e^2(0) + a^2 \quad b^2 = a^2(1 - e^2)$$

**Exercise 1.**  $b^2 = a^2(1 - e^2) \quad \frac{x^2}{100} + \frac{y^2}{36} = 1$

$$\sqrt{1 - \frac{b^2}{a^2}} = e \implies \boxed{e = \frac{4}{5}}.$$

$$|F| = |aeN| = 10 \left( \frac{4}{5} \right) = 8. \quad f = (\pm 8, 0). \quad (0, 0) \text{ center. Vertices } (\pm 10, 0).$$

**Exercise 2.**  $\frac{y^2}{100} + \frac{x^2}{36} = 1. \quad \frac{4}{5} = e; \quad f = (0, \pm 8).$

(0, 0) center; vertices  $(\pm 6, 0), (0, \pm 10).$

**Exercise 3.**  $\frac{(x-2)^2}{16} + \frac{(y-3)^2}{9} = 1. \text{ Center } (2, -3). |F| = ae = 4 \frac{\sqrt{7}}{4} = \sqrt{7}.$

$$\sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{9}{16}} = \frac{\sqrt{7}}{4} = e; \quad (2 + \sqrt{7}, -3), (2 - \sqrt{7}, -3) \text{ foci.}$$

Vertices  $(6, -3), (-2, -3), (2, 6), (2, -12).$

**Exercise 4.**  $\frac{x^2}{(\frac{25}{9})} + y^2 = 1. \text{ Center } x = (0, 0). |F| = ae = \left( \frac{5}{3} \right) \left( \frac{4}{5} \right) = \frac{4}{3}.$

$$e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{9}{25}} = \frac{4}{5}. \text{ Foci: } (\pm \frac{4}{3}, 0). \text{ Vertices } (\pm \frac{5}{3}, 0), (0, \pm 1).$$

**Exercise 5.**  $\frac{y^2}{(1/4)} + \frac{x^2}{(1/3)} = 1$

$$|F| = ae = \frac{1}{\sqrt{3}} \left(\frac{1}{2}\right) = \frac{1}{2\sqrt{3}}.$$

$$\sqrt{1 - \frac{1/4}{1/3}} = \frac{1}{2} = e. \text{ Foci: } \left(\pm \frac{1}{2\sqrt{3}}, 0\right). \text{ Center } (0, 0).$$

$$\text{Vertices } (\pm 1/\sqrt{3}, 0), (0, \pm 1/2).$$

**Exercise 6.** Center  $(-1, -2)$ .

$$\sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{16}{25}} = \frac{3}{5} = e; \quad |F| = ae = 5 \frac{3}{5} = 3.$$

$$\text{Foci: } (-1, -1), (-1, -5).$$

$$\text{Vertices: } (-1, 3), (-1, -7), (3, -2), (-5, -2).$$

**Exercise 7.**  $F = ae = \frac{3}{4}$ .  $a = 1$ ,  $e = \frac{3}{4}$ .  $b^2 = a^2(1 - e^2)$ ;  $b^2 = 1 \left(\frac{1}{4}\right)$ .

$$\boxed{x^2 + 4y^2 = 1}.$$

**Exercise 8.**  $2a = 4$ ,  $a^2 = 4$ .  $2b = 3$ .  $b^2 = 9/4$ .  $\implies \frac{(x+3)^2}{4} + \frac{(y-4)^2}{4} = 1$

**Exercise 9.**  $\frac{(x+3)^2}{9/4} + \frac{(y-4)^2}{4} = 1$ .

**Exercise 10.**  $2a = 6$ ,  $a = 3$ .  $\frac{(x+4)^2}{9} + \frac{(y-2)^2}{1} = 1$ .

**Exercise 11.**  $2a = 10$ ,  $a = 5$ .  $|F| = ae = 5e = 4e = 4/5$ .  $b^2 = a^2(1 - e^2) = 25 \left(1 - \frac{16}{25}\right) = 9$ .

**Exercise 12.**  $\frac{(x-2)^2}{a^2} + \frac{(y-1)^2}{b^2} = 1$ ;

$$a = 4 \text{ from } (6, 1). \quad b = 2 \text{ from } (2, 3). \implies \frac{(x-2)^2}{4^2} + \frac{(y-1)^2}{4} = 1$$

**Exercise 13.**  $b^2 = a^2(1 - e^2)$ .

$$\frac{x^2}{100} - \frac{y^2}{64} = 1; \quad b^2 = 100(1 - e^2) = -64. \quad 1 + \frac{64}{100} = e^2.$$

$$\text{Center } (0, 0). \quad e = \frac{2\sqrt{41}}{10} = \frac{\sqrt{41}}{5}.$$

$$\text{Vertices: } (\pm 10, 0). \quad F = ae = 2\sqrt{41}. \text{ Foci: } (\pm 2\sqrt{41}, 0).$$

$$\frac{x^2}{100} = \frac{y^2}{64} + 1 \xrightarrow{x, y \rightarrow \infty} y = \pm \frac{4}{5}x$$

**Exercise 14.**  $\frac{y^2}{100} - \frac{x^2}{64} = 1$ ; Center  $(0, 0)$ ,  $a^2 = 100$ ;  $b^2 = -64$ .

$$b^2 = a^2(1 - e^2). \quad e = \frac{\sqrt{41}}{5}. \text{ Vertices } (0, \pm 10). \quad F = ae = (0, \pm 2\sqrt{41}).$$

$$\frac{x^2}{64} + 1 = \frac{y^2}{100} \xrightarrow{x, y \rightarrow \infty} \pm \frac{5}{4}x = y.$$

**Exercise 15.**  $\frac{(x+3)^2}{4} - (y-3)^2 = 1$ .

$$\text{Center } (-3, 3). \quad e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{-1}{4}} = \frac{\sqrt{5}}{2}.$$

$$\text{Foci: } ae = 2 \frac{\sqrt{5}}{2} = \sqrt{5}. \quad (-3 + \sqrt{5}, 3), (-3 - \sqrt{5}, 3).$$

$$\text{Vertices: } (-3, 4), (-3, 2); \quad (1, 3), (-7, 3).$$

$$\frac{(x+3)^2}{4} = 1 + (y-3)^2 \xrightarrow{x, y \rightarrow \infty} \boxed{\frac{\pm(x+3)}{2} = y-3}$$

**Exercise 16.**  $\frac{x^2}{144/9} - \frac{y^2}{144/16} = 1 = \frac{x^2}{16} - \frac{y^2}{9}$ .

$$e = \sqrt{1 - \frac{-9}{16}} = \frac{5}{4}. \text{ Center } (0, 0). \quad |F| = ae = 5. \text{ Foci: } (5, 0), (-5, 0). \text{ Vertices } (\pm 4, 0).$$

**Exercise 17.**  $20 = 5y^2 - 4x^2$ . Center  $(0, 0)$ .  $|F| = ae = 2 \left(\frac{3}{2}\right) = 3$ . Foci:  $(0, \pm 3)$ .  $1 = \frac{y^2}{4} - \frac{x^2}{5}$ .  $e = \sqrt{1 - \frac{-5}{4}} = \frac{3}{2}$ .

$$\text{Vertices: } (0, \pm 2)$$

**Exercise 18.**  $\frac{(x-1)^2}{4} - \frac{(y+2)^2}{9} = 1$ .

$$\text{Center } (1, -2). \quad e = \sqrt{1 - \frac{-9}{4}} = \frac{\sqrt{13}}{2}; \quad |F| = 2 \frac{\sqrt{13}}{2} = \sqrt{13}. \text{ Foci: } (1 + \sqrt{13}, -2), (1 - \sqrt{13}, -2).$$

$$\text{Vertices: } (5, -2), (-3, -2).$$

**Exercise 19.**  $F = ae = 2(2) = 4$ .

$$\frac{x^2}{4} + \frac{y^2}{-12} = 1. \quad \frac{y^2}{12} + 1 = \frac{x^2}{4} \xrightarrow{x, y \rightarrow \infty} y = \pm\sqrt{3}x.$$

$$b^2 = a^2(1 - e^2) = 4(1 - 4) = -12.$$

$$\text{Exercise 20. } F = ae = \sqrt{2} = (1)e. \quad b^2 = a^2(1 - e^2) = 1(1 - 2) = -1. \implies y^2 - x^2 = 1.$$

$$\text{Exercise 21. } \frac{x^2}{4} - \frac{y^2}{16} = 1$$

$$\text{Exercise 22. } (y - 4)^2 - \frac{(x+1)^2}{-3} = 1 \text{ where}$$

$$F = ae = |-2| = ae. \quad b^2 = a^2(1 - e^2) = 1(1 - 4) = -3$$

$$\text{Exercise 23. } \pm \frac{(x-2)^2}{a^2} \mp \frac{(y+3)^2}{b^2} = 1$$

$$\begin{aligned} (3, -1) &\implies \pm \frac{1}{a^2} \mp \frac{4}{b^2} = 1 \\ (-1, 0) &\implies \pm \frac{9}{a^2} \mp \frac{9}{b^2} = 1 \end{aligned} \implies \frac{(y+3)^2}{27/8} - \frac{(x-2)^2}{(27/5)} = 1$$

$$\text{Exercise 24. } \frac{x^2-1}{3} = y^2. \quad \frac{2x}{3} = 2yy'. \quad yy' = \frac{x}{3}.$$

$$3x - 2y = C. \quad m = \frac{3}{2} \implies y_0 \frac{9}{2} = x_0. \quad \frac{81}{4}y_0^2 - 1 = 3y_0^2 \implies y_0 = \frac{\pm 2}{\sqrt{69}}.$$

$$\text{The asymptotes of } y^2 = \frac{x^2-1}{3} \text{ are } y = \pm \frac{x}{\sqrt{3}}.$$

$$3 \left( \frac{\pm 9}{\sqrt{69}} \right) - 2 \left( \pm \frac{2}{\sqrt{69}} \right) = \frac{\pm 23}{\sqrt{69}} = C$$

$$\boxed{3x \pm \sqrt{\frac{23}{3}} = 2y}$$

$$\text{Exercise 25. } \frac{\pm x^2}{a^2} + \mp \frac{y^2}{b^2} = 1 \quad \frac{\pm x^2}{a^2} \mp \frac{y^2}{4a^2} = 1.$$

$$(3, -5) \rightarrow \pm 9 \mp \frac{25}{4} = a^2; \quad a^2 = \frac{11}{4}.$$

$$\implies \boxed{\frac{x^2}{11/4} - \frac{y^2}{11} = 1}.$$

Quick Review of Parabolas.

$F$  on positive half plane to  $N$ .

$$\|X - F\| = e|(X - F) \cdot N + d|$$

$$\text{Let } N = \vec{e}_x; d = 2c; F = (c, 0); e = 1.$$

$$(x - c)^2 + y^2 = e^2((x - c) + 2c)^2 = (x - c)^2 + 4c(x - c) + 4c^2$$

$$y^2 = 4cx$$

Thus, for ellipses, the vertex is equidistant to the focus and directrix (confirming the other definition).

$$\text{Let } N = \vec{e}_y, d = 2c; F = (0, c), e = 1.$$

$$x^2 + (y - c)^2 = ((y - c) + 2c)^2 = (y - c)^2 + 4c(y - c) + 4c^2$$

$$x^2 = 4cy$$

$$\text{Exercise 26. } 4c = -8 \quad (0, 0) \text{ vertex. } y = 0 \text{ symmetry axis. } x = 5 \text{ directrix.}$$

$$\text{Exercise 27. } 4c = 3. \text{ Vertex: } (0, 0). \text{ Symmetry axis: } y = 0. \text{ Directrix: } x = -3/4.$$

$$\text{Exercise 28. } (y - 1)^2 = 12(x - \frac{1}{2}). \quad 4c = 12, c = 3. \text{ Symmetry axis: } y = 1. \text{ Directrix: } (\frac{-5}{2}, 1).$$

$$\text{Exercise 29. } x^2/6 = y. \quad 4c = \frac{1}{6} \quad c = \frac{1}{24}. \text{ Vertex: } (0, 0). \text{ Directrix: } y = -\frac{1}{24}. \text{ Symmetry axis: } x = 0.$$

$$\text{Exercise 30. } x^2 + 8y = 0. \quad 4c = \frac{-1}{8}; \quad c = \frac{-1}{32}. \quad y = \frac{1}{32} \text{ directrix; } x = 0 \text{ axis.}$$

$$\text{Exercise 31. } (x + 2)^2 = 4(y + \frac{9}{4}). \quad 4c = 4; \quad c = 1. \text{ Center } (-2, -9/4). \text{ Directrix: } y = -13/4. \text{ Axis: } x = -2.$$

$$\text{Exercise 32. } y = -x^2.$$

$$\text{Exercise 33. } x^2 = 8y.$$

$$\text{Exercise 34. } (y - 3) = -8(x + 4)^2.$$

$$\text{Exercise 35. } c = \frac{5}{4} \quad 5(x - \frac{7}{4}) = (y + 1)^2$$

**Exercise 36.**  $y = ax^2 + bx + c$

$$(0, 1) \rightarrow c = 1 \quad (1, 0) \rightarrow 0 = a + b + 1 \quad (2, 0) \rightarrow 0 = 4a + 2b + 1 \quad a = \frac{1}{2} \implies y = \frac{1}{2}x^2 - \frac{3}{2}x + 1$$

**Exercise 37.**  $4c(x-1) = (y-3)^2$ .  $4c(-2) = (-4)^2 = 16$ .  $c = -2$ .  $-8(x-1) = (y-3)^2$ .

**Exercise 38.**  $\|X - F\| = ed(X, L) = |(X - F) \cdot N - d|$

$$L = \{(x, y) | 2x + y = 10; \frac{2}{\sqrt{5}}x + \frac{y}{\sqrt{5}} = \frac{10}{\sqrt{5}}\}.$$

$$d = N = x_L \quad dN \cdot N = d = x_L \cdot N = \frac{10}{\sqrt{5}}.$$

$$\begin{aligned} F = 0 \implies \|X\|^2 &= |X \cdot N - d|^2 = \left(\frac{-2}{\sqrt{5}}x + \frac{y}{\sqrt{5}} + \frac{10}{\sqrt{5}}\right)^2 = x^2 + y^2 \\ 5x^2 + 5y^2 &= (-2x - y + 10)^2 = 4x^2 + y^2 + 100 + 4xy - 40x - 20y \\ \implies x^2 + 4y^2 - 4xy + 40x + 20y - 100 &= 0 \end{aligned}$$

### 13.25 Miscellaneous exercises on conic sections.

**Exercise 1.**

$$\begin{aligned} \frac{y^2}{b^2} &= 1 - \frac{x^2}{a^2} \quad y^2 = b^2 - \left(\frac{bx}{a}\right)^2 = b^2 \left(1 - \left(\frac{x}{a}\right)^2\right) \\ y &= 2 \int_{-a}^a b \sqrt{1 - \left(\frac{x}{a}\right)^2} dx = 2 \int_{-1}^1 ab \sqrt{1 - x^2} dx = (ab) \text{ area of a circle of radius } 1 \end{aligned}$$

**Exercise 2.**

(1) Without loss of generality, let the major axis be  $2a$  in the  $x$ -axis.  $y = b\sqrt{1 - \left(\frac{x}{a}\right)^2}$

$$V = \int_{-a}^a \pi b^2 \left(1 - \frac{x^2}{a^2}\right) dx = \pi b^2 a \int_{-1}^1 (1 - x^2) dx = \frac{4}{3} \pi (1)^3 b^2 a$$

(2) If rotated about the minor axis, suppose, without loss of generality,  $2a$  is the minor axis (just note that  $\frac{x^2}{a^2} + \frac{b^2}{a^2} = 1$  have  $x, y, a, b$  as dummy labels).

$\implies V = \frac{4}{3} \pi (1)^3 b^2 a$ , where  $2a$  is the minor axis,  $2b$  is the major axis.

**Exercise 3.**  $\frac{x^2}{(3/A)} + \frac{y^2}{(3/B)} = 1 \quad By^2 = 3 - Ax^2 \implies y^2 = \frac{3}{B} - \frac{Ax^2}{B}; \quad y = \sqrt{\frac{3}{B} - \frac{Ax^2}{B}}$ . So the area inside this ellipse is

$$2\sqrt{\frac{1}{B}} \int_{-\sqrt{3/A}}^{\sqrt{3/A}} \sqrt{3 - Ax^2} dx = 2\sqrt{\frac{3}{B}} \int_{-\sqrt{3/A}}^{\sqrt{3/A}} \sqrt{1 - \frac{x^2}{(3/A)}} dx$$

For the other ellipse equation,  $\frac{x^2}{3/(A+B)} + \frac{y^2}{3/(A-B)} = 1$ .  $y^2 = \left(\frac{3}{A-B}\right) \left(1 - \frac{x^2}{(3/(A+B))}\right); \quad y = \sqrt{\frac{3}{A-B}} \sqrt{1 - \frac{x^2}{(3/(A+B))}}$ . Thus, the area inside this ellipse is

$$2\sqrt{\frac{3}{A-B}} \int_{-\sqrt{\frac{3}{A+B}}}^{\sqrt{\frac{3}{A+B}}} \sqrt{1 - \left(\frac{x}{\sqrt{\frac{3}{A+B}}}\right)^2} dx$$

Equating the two areas after making an appropriate scale change,

$$2\sqrt{\frac{3}{B}} \sqrt{\frac{3}{A}} \int_{-1}^1 \sqrt{1 - x^2} dx = 2\sqrt{\frac{3}{A-B}} \sqrt{\frac{3}{A+B}} \int_{-1}^1 \sqrt{1 - x^2} dx$$

Thus  $A^2 - B^2 = AB \implies A^2 - BA - B^2$ . Simply try treating  $B$  as a number and solve the quadratic equation in terms of  $A$ .

$$A = \frac{B \pm \sqrt{B^2 - 4(1)(-B^2)}}{2(1)} = \frac{B \pm B\sqrt{5}}{2} = \frac{B(1 \pm \sqrt{5})}{2}$$

**Exercise 4.**  $y = -\frac{4h}{b^2}x^2$ .

$$\int_{-b/2}^{b/2} \left(\frac{4h}{-b^2}x^2 + h\right) dx = \frac{4h}{-3b^2}x^3 \Big|_{-b/2}^{b/2} + h\left(\frac{b}{2} + \frac{b}{2}\right) = \frac{2hb}{3}$$

**Exercise 5.**  $y^2 = 8x$ .  $\int_0^2 \pi 8t dt = 4\pi(2)^2 = 16\pi$

**Exercise 6.**  $y^2 = 2(x-1)$ .  $y^2 = 4(x-2)$ .

(1)

$$\begin{aligned}
A &= 2 \int_1^2 \sqrt{2(x-1)} + 2 \int_2^3 \sqrt{2(x-1)} - 2\sqrt{x-2} = \\
&= 2\sqrt{2} \left. \frac{2}{3} (x-1)^{3/2} \right|_1^2 + 2\sqrt{2} \left. \frac{2}{3} (x-1)^{3/2} \right|_2^3 - 4 \left. \frac{2}{3} (x-2)^{3/2} \right|_2^3 \\
&= 2\sqrt{2} \frac{2}{3} + \sqrt{2} \frac{4}{3} (2)^{3/2} - 2\sqrt{2} \frac{2}{3} - 4 \frac{2}{3} = 8/3
\end{aligned}$$

(2)

$$\begin{aligned}
\int_1^2 2(x-1) &= 2 \left( \frac{1}{2} x^2 - x \right) \Big|_1^2 = 2 \left( \frac{1}{2} (4-1) - (2-1) \right) = 1 \\
\int_2^3 (2(x-1) - 4(x-2)) dx &= \int_2^3 (-2x+6) dx = -x^2 \Big|_2^3 + 6x \Big|_2^3 = (-9+4+6(3-2)) = 1 \\
\implies V &= \pi \int_1^2 2(x-1) + \pi \int_2^3 (2(x-1) - 4(x-2)) = 2\pi
\end{aligned}$$

$$(3) \quad \frac{y^2}{2} + 1 = x, \quad \frac{y^2}{4} + 2 = x$$

$$\begin{aligned}
2\pi \int_0^2 \left( \left( \frac{y^2}{4} + 2 \right)^2 - \left( \frac{y^2}{2} + 1 \right)^2 \right) &= 2\pi \int_0^2 \left( \frac{-3y^4}{16} + 3 \right) dy = 2\pi \left( \frac{-3}{80} y^5 + 3y \right) \Big|_0^2 = \\
&= 2\pi \left( \frac{-3(32)}{80} + 6 \right) = 2\pi \left( \frac{-96+480}{80} \right) = 2\pi \left( \frac{384}{80} \right) = \pi \frac{48}{5}
\end{aligned}$$

**Exercise 7.** By Apostol's definition of conic sections, we are basically given the conic section definition with  $e = \frac{1}{2}$ . So just plug in the pt.  $(0, 4)$ .

$$\begin{aligned}
\frac{x^2}{a^2} + \frac{y^2}{b^2} &= 1 \xrightarrow{(0,4)} b = 4 \quad b^2 = a^2(1-e^2) = 16 = a^2 \left( 1 - \left( \frac{1}{2} \right)^2 \right) \\
\frac{x^2}{64/3} + \frac{y^2}{16} &= 1
\end{aligned}$$

**Exercise 8.**  $F = 0 \quad \|X - F\| = \|X\| = ed(X, F) = |X \cdot N + d| = \frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}} + \frac{1}{\sqrt{2}}$  because for the directrix

$$\begin{aligned}
y + x &= -1 \\
N &= \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \quad \frac{1}{\sqrt{3}} y + \frac{x}{\sqrt{2}} = -\frac{1}{\sqrt{2}} \\
X_L &= P + tA \\
X_L \cdot NN \cdot P &= -1/\sqrt{2} \quad dN = X_L \quad X_L \cdot N = d = -1/\sqrt{2}
\end{aligned}$$

So by squaring both sides of the vector equation,

$$\begin{aligned}
x^2 + y^2 &= \frac{x^2}{2} + xy + \frac{y^2}{2} + \frac{1}{2} + x + y \\
\frac{x^2}{2} + \frac{y^2}{2} - xy - x - y &= \frac{1}{2} \\
x^2 + y^2 - 2xy - 2x - 2y &= 1
\end{aligned}$$

**Exercise 9.** Center  $(1/2, 2)$  because we equate the asymptotes to see where they intersect:  $y = 2x + 1 = -2x + 3$ .

$$\begin{aligned}
\frac{(y-2)^2}{a^2} - \frac{(x-1/2)^2}{a^2/4} &= 1 \xrightarrow{(0,0)} \frac{4}{a^2} - \frac{1}{a^2} = \frac{3}{a^2} = 1 \\
\frac{(y-2)^2}{3} - \frac{(x-1/2)^2}{3/4} &= 1
\end{aligned}$$

**Exercise 10.**  $px^2 + (p+2)y^2 = p^2 + 2p. \quad \frac{x^2}{p+2} + \frac{y^2}{p} = 1.$

(1) Since  $p+2 > p$ , the foci must lie on the  $x$  axis.  $a^2 = p+2; \quad b^2 = a^2(1-e^2) = p = (p+2)(1-e^2). \quad e = \sqrt{\frac{2}{p+2}}$

$$F = ae = \sqrt{2}. \quad (\pm\sqrt{2}, 0).$$

$$(2) F = ae = \sqrt{2} = a(\sqrt{3}) \implies a = \sqrt{\frac{2}{3}}; \quad b^2 = \frac{2}{3}(1-3) = \frac{-4}{3}.$$

$$\frac{x^2}{2/3} - \frac{y^2}{4/3} = 1$$

**Exercise 11.**  $e = 1$  for an ellipse.

$$\begin{aligned}\|X - F\| &= |X \cdot N - a| = a - X \cdot N \\ \| -X - F\| &= \|X + F\| = | -X \cdot N - a| = a + X \cdot N \\ \|X - F\| + \|X + F\| &= 2a\end{aligned}$$

**Exercise 12.**

$$\begin{aligned}\|X - F\| &= e|(X - F) \cdot N - d| = e(d - (X - F) \cdot N) \\ \|X + F\| &= ed(X, L) = e|(X - F) \cdot N + d| = e(-d - (X - F) \cdot N) \\ \|X - F\| - \|X + F\| &= 2ed \\ X \rightarrow -X \quad \text{so for the other branch, } \|X + F\| - \|X - F\| &= 2ed\end{aligned}$$

**Exercise 13.**

$$\begin{aligned}(1) \quad \frac{(tx)^2}{a^2} + \frac{(by)^2}{b^2} &= 1 \quad \left(\frac{b}{t}\right)^2 = \frac{a^2(1-e^2)}{t^2} = \left(\frac{a}{t}\right)^2(1-e^2) \\ (2) \quad b_1^2 &= a_1^2(1-e^2) \quad b_2^2 = a_2^2(1-e^2).\end{aligned}$$

$$\begin{aligned}1 - \frac{b_1^2}{a_1^2} &= 1 - \frac{b_2^2}{a_2^2}; \quad \frac{b_1^2}{a_1^2} = \frac{b_2^2}{a_2^2} \\ \frac{x_1^2}{a_1^2} + \frac{y^2}{b_1^2} &= 1 = \frac{\left(\left(\frac{b_2}{b_1}\right)x\right)^2}{a_2^2} + \frac{\left(\frac{b_2}{b_1}y\right)^2}{(b_2y)^2}\end{aligned}$$

$$(3) \quad \pm \frac{(tx)^2}{a^2} \mp \frac{(ty)^2}{b^2} = 1 \quad \left(\frac{b}{t}\right)^2 = \frac{-a^2(e^2-1)}{t^2} = -\left(\frac{a}{t}\right)^2(e^2-1).$$

$$\begin{aligned}b_1^2 &= a_1^2(e^2-1) \quad b_2^2 = a_2^2(e^2-1) \\ \frac{b_1^2}{a_1^2} + 1 &= e^2 \quad \frac{b_2^2}{a_2^2} + 1 = e^2 \quad \frac{b_1^2}{a_1^2} = \frac{b_2^2}{a_2^2} \\ \pm \left(\frac{x}{a_1}\right)^2 \mp \left(\frac{y}{b_1}\right)^2 &= \pm \left(\frac{\left(\frac{b_2}{b_1}x\right)}{a_2}\right)^2 \mp \left(\frac{\frac{b_2}{b_1}y}{b_2}\right)^2 = 1\end{aligned}$$

**Exercise 14.**  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \implies \frac{x}{a^2} + \frac{y}{b^2}y' = 0$

$$\implies y' = \frac{-b^2x}{ya^2} = \frac{-a^2(1-e^2)x}{a^2y} = \frac{(e^2-1)x}{y}$$

**Exercise 15.**

$$\begin{aligned}(1) \quad y &= ax^2 + bx + c \quad ty = a(tx)^2 + btx + c \rightarrow y = atx^2 + bx + c/t = y = Ax^2 + b + C \\ (2) \quad y &= tx^2, \quad t \neq 0\end{aligned}$$

**Exercise 16.**  $x - y + 4 = 0 \quad y = 4\sqrt{x} \quad (y^2 = 16x); \quad y' = 2x^{-1/2}.$

$$y'(x=4) = 1 \quad (x, y) = (4, 8).$$

**Exercise 17.**

- (1) If we treat the two given parabolas,  $y^2 = 4p(x - a)$  and  $x^2 = 4qy$ , as two vector objects free from any specific coordinate system then we observe that we can disregard the sign of  $q$  and  $p$  and simply state that they are both positive. What matters is that we observe that  $p$  and  $q$  are the distance of the foci to the vertex for each of the respective parabolas.

Second, observe that  $a$  is not given. By diagram, if  $p, q$  are given,  $a$  must be moved along the  $x$ -axis to fit the tangency condition. Thus, in terms of doing the algebra, just eliminate  $p$  and  $q$  from the relations.

If  $(h, k)$  is the point of contact,

$$\begin{aligned} x^2 &= 4qy & y^2 &= 4p(x-a) & y &= 2\sqrt{p}\sqrt{x-a} \\ \frac{x}{2q} &= y' & y' &= \sqrt{p}\frac{1}{\sqrt{x-a}} \\ y'(h) &= \frac{h}{2q} & y'(h) &= \sqrt{p}\frac{1}{\sqrt{h-a}} \end{aligned}$$

$$\text{(Tangent condition)} \quad \left(\frac{h}{2q}\right)^2 = \frac{p}{h-a} \implies (h^2)(h-a) = (2q)^2 p$$

$$\text{(one point of contact condition)} \quad \text{with } q = \frac{h^2}{4k}, p = \frac{k^2}{4(h-a)}$$

$$\implies h^2(h-a) = \left(\frac{h^2}{2k}\right)^2 \frac{k^2}{4(h-a)} \implies (h-a)^2 = \frac{h^2}{16}$$

$$\implies h = \frac{2a \pm a/2}{15/8} = \boxed{4a/3}$$

(2)

$$\begin{aligned} \frac{h}{2q} &= \frac{\sqrt{p}}{\sqrt{h-a}} \\ \frac{2a}{3q} &= \frac{\sqrt{p}}{\sqrt{a/3}} = \frac{\sqrt{3p}}{\sqrt{a}}; \quad 2a\sqrt{a} = 3\sqrt{3pq} \\ &\implies 4a^3 = 27pq^2 \end{aligned}$$

**Exercise 18.** First hint: Vector methods triumph over algebraic manipulations of Cartesian coordinates. Think of the locus in terms of vector objects that are coordinate-free and the conic section will emerge. I mean, try evaluating  $\|P - A\|^2 = (x-2)^2 + (x-3)^2 = (x+y)^2$

$$A = (2, 3), \quad N = \frac{1}{\sqrt{2}}(1, 1), \quad X = (x, y).$$

$$\|X - A\| = x + y = \sqrt{2}(X \cdot N) = \sqrt{2}(X \cdot N - (F \cdot N - d))$$

$$\text{where } F \cdot N = d = A \cdot N = \frac{5}{\sqrt{2}}$$

$d =$  distance from focus to the directrix .

$$y = x + 1 \text{ (axis of the hyperbola)}$$

$$d = \frac{5}{\sqrt{2}} = \sqrt{(2-x)^2 + (3-y)^2} = \sqrt{2}(2-x) \quad x = -\frac{1}{2}, y = \frac{1}{2}$$

$$\left(-\frac{1}{2}, \frac{1}{2}\right) \text{ must also be the center. } y - \frac{1}{2} = -(x + \frac{1}{2}) \text{ is the directrix.}$$

$$(y - \frac{1}{2}) = \alpha(x + \frac{1}{2}) \text{ is the general form of the asymptote.}$$

Consider asymptotes in general.  $\|X - F\| = ed(X, L)$ .

$$\frac{\|X - F\|}{d(X, L)} = e = \frac{\|X - F\|}{|X \cdot N - (F \cdot N - d)|} = \frac{\|X - F\|}{(X - F) \cdot N + d}$$

For  $\|X - F\| \rightarrow \infty$ ,  $\|X - F\| > d$ . To keep ratio of  $e$ ,  $X - F$  must be ultimately directed by  $N$  by a ratio of  $e$ .

$$\implies e = \frac{\|X - F\|}{\|X - F\| \cos \phi} = \frac{1}{\cos \phi}$$

$$\text{e.g. Consider } N = \vec{e}_x. \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \implies y = \frac{b}{a}x = \sqrt{e^2 - 1}x.$$

From the vector equation,

$$(X - F) \cdot N = (x - c, y) \cdot N = \sqrt{(x - c)^2 + y^2} \cos \phi = x - c$$

$$\frac{\sqrt{(x - c)^2 + y^2}}{x - c} = \frac{1}{\cos \phi} = e; \quad \frac{(x - c)^2 + y^2}{(x - c)^2} = e^2;$$

$$\frac{y^2}{(x - c)^2} = e^2 - 1 \implies y = \sqrt{e^2 - 1}x$$



For our problem, consider the conic section approaching the asymptote. Then the conic section will look more like those linear asymptotes.

$$\begin{aligned}
& \sqrt{(x-2)^2 + (y-3)^2} = x + y \\
& \xrightarrow{y - \frac{1}{2} = \alpha(x + \frac{1}{2})} \sqrt{\left(\left(x + \frac{1}{2}\right) - \frac{5}{2}\right)^2 + \left(\alpha\left(x + \frac{1}{2}\right) - \frac{5}{2}\right)^2} = x + \alpha\left(x + \frac{1}{2}\right) + \frac{1}{2} \\
& \implies \sqrt{(1 + \alpha^2)\left(x + \frac{1}{2}\right)^2 - 5(1 + \alpha)\left(x + \frac{1}{2}\right) + \frac{25}{2}} \xrightarrow{x \rightarrow \infty} \frac{\sqrt{1 + \alpha^2}}{(1 + \alpha)} \\
& \implies \alpha = 0
\end{aligned}$$

The asymptotes are  $y = \frac{1}{2}$  and  $x = \frac{-1}{2}$ .

In the second part, each quadrant must be checked. So far, I only have that quadrant II is filled: points in quadrant III and quadrant IV cannot satisfy the given condition. To see this, consider quadrant II.

$$\|x - A\| = -x + y = \sqrt{2}(x, y) \left( \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)$$

For quadrant II,  $N = \left( \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)$ . By diagram,  $(X - F) \cdot N > 0$  and  $X \cdot N > 0$ .

$$\begin{aligned}
A \cdot N &= \frac{1}{\sqrt{2}} & -A \cdot N &= \frac{-1}{\sqrt{2}} & d &= \frac{1}{\sqrt{2}} \\
|(X - F) \cdot N + d| &= (X - F) \cdot N + d
\end{aligned}$$

The equation for the axis of the conic section is  $y = -(x - 5)$ .

By taking the asymptotic limit like above, we can show that  $\alpha = 0$  again. We only sketch the part of the hyperbola in quadrant II.

By similar procedure, I found that quadrant III, IV cannot satisfy the condition.

**Exercise 19.**

$$\begin{aligned}
\|X - F\| &= d(X, L) = |(X - F) \cdot N + d| \\
x^2 + y^2 &= (X \cdot N + d_1)^2 = y^2 + 2yd_1 + d_1^2 \\
&\xrightarrow{F=0} x^2 = 2yd_1 + d_1^2 & y'_1 &= \frac{x}{d_1}
\end{aligned}$$

$$\begin{aligned}
\|X - F\| &= |(X - F) \cdot N - d_2| = d_2 - (X - F) \cdot N \\
&\xrightarrow{F=0} \|X\| = d_2 - y \\
x^2 + y^2 &= d_2^2 - 2d_2y + y^2 \\
\implies x^2 &= d_2^2 - 2d_2y & y'_2 &= \frac{-x}{d_2}
\end{aligned}$$

$$\text{Point of intersection } x_0^2 = 2y_0d_1 + d_1^2 = d_2^2 - 2d_2y_0$$

$$\begin{aligned}
2(d_1 + d_2)y_0 &= d_2^2 - d_1^2 \implies y_0 = \frac{d_2^2 - d_1^2}{2} \\
x_0^2 &= d_2(d_2 - 2y_0) = d_2d_1 \\
y'_1 &= \frac{\pm\sqrt{d_2d_1}}{d_1} = \pm\sqrt{\frac{d_2}{d_1}} \\
\implies y'_2 &= \frac{\mp\sqrt{d_2d_1}}{d_2} = \mp\frac{1}{\sqrt{\frac{d_2}{d_1}}}
\end{aligned}$$

**Exercise 20.**

(1) Use  $X \rightarrow -X$  symmetry.

$$\|X - F\| = ed(X, L) = e|(X - F) \cdot N + d| = e|X \cdot N - F \cdot N + d| = |eX \cdot N - a|$$

$$\|X\|^2 - 2X \cdot F + \|F\|^2 = e^2(X \cdot N)^2 - 2ea(X \cdot N) + a^2$$

$$X \rightarrow -X \implies \|X\|^2 + 2X \cdot F + \|F\|^2 = e^2(X \cdot N)^2 + 2ea(X \cdot N) + a^2$$

$$\implies \|X^2\| + \|F\|^2 = e^2(X \cdot N)^2 + a^2$$

$$x^2 + y^2 + c^2 = e^2x^2 + a^2 \quad |F| = c = ae$$

$$\left(\frac{a^2 - c^2}{a^2}\right)x^2 + y^2 = a^2 - c^2 \implies \frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1$$

$$(2) \quad \frac{x}{a^2} + \frac{yy'}{a^2 - c^2} = 0 \implies y' = \frac{-(a^2 - c^2)x}{ya^2}$$

$$xy(y')^2 = \left(\frac{a^2 - c^2}{a^2}\right)^2 \frac{x^3}{y}$$

$$(x^2 - y^2 - c^2)y' = \left(x^2 + \frac{a^2 - c^2}{a^2}x^2 - a^2\right) \left(\frac{-(a^2 - c^2)x}{ya^2}\right) =$$

$$= (a^4 - (a^2 - c^2)x^2 - a^2x^2) \frac{(a^2 - c^2)x}{a^4y}$$

$$-xy = \frac{-xy^2}{y} = -\frac{x(a^2 - c^2 - \left(\frac{a^2 - c^2}{a^2}\right)x^2)}{y} =$$

$$= (a^2 - c^2)(-a^4 + a^2x^2)x/(a^4y)$$

$$\implies xyy'^2 + (x^2 - y^2 - c^2)y' - xy = 0$$

(3) For  $y'$ , consider  $-\frac{1}{y'}$  at every  $(x, y)$ .

$$xy \left(\frac{-1}{y'}\right)^2 + (x^2 - y^2 - c^2) \frac{-1}{y'} - xy = 0 = \frac{-xy}{y'} + (x^2 - y^2 - c^2) \frac{1}{y'} + xy$$

$$\xrightarrow{\text{if } y' \neq 0} -xy + (x^2 - y^2 - c^2)y' + xy(y')^2 = 0$$

Thus  $S \rightarrow S$  since the defining differential equation is invariant under the transformation of the slope.

**Exercise 21.** For a circle centered at  $C$ , then  $\|X - C\| = r_0^2$  for all points  $X$  on that circle.

For the condition of being tangent to a given line,  $L = P + tA$ , then  $(X_C - C) \cdot A = 0$  and the point lies on the circle so  $\|X_C - C\| = r_0^2$ .

Call the point that all the circles pass through  $F$ . Then  $\|C - F\| = \|C - X_C\|$ .  $\|C - X_C\|$  is by definition  $d(X, L)$ , the distance from the circle center to the line.  $\|C - F\| = \|C - X_0\|$  is by definition a parabola.

**Exercise 22.** Consider a circle that's part of the mentioned family that has its center directly below the given circle with radius  $r_0$ , and center  $Q$ .

It's given that the center is equidistant from the point of tangency and the line. This hints at a parabola because the parabola's vertex is equidistant from the focus and the directrix. Thus, we need to show that  $d(X, L)$  is equal to the distance from the circle center  $C$  to the bottom point of  $Q$ .

Let  $N$  be a unit normal vector pointing from the line towards the focus, placing the focus in the positive half-plane.

Let  $C$  be the center of an arbitrary circle in the family and  $r_1$  its radius.

Let  $X_1$  be the point of tangency between circle  $Q$  and circle  $C$ .

We want  $\|(Q + r_0N) - C\| = \|X_1 - C\|$ .

The tangency condition between circle  $Q$  and  $C$  means that

$$(X_1 - C) = -\alpha(X_1 - Q); \alpha > 0 \quad \alpha = \frac{r_1}{r_0}$$

$$Q - r_0N - C = Q - X_1 - r_0N - C + X_1$$

$$\xrightarrow{\text{take the magnitude}} \|Q - X_1\|^2 + \|X_1 - C\|^2 + r_0^2 + 2(Q - X_1)(X_1 - C) + 2(X_1 - Q)r_0N + 2(C - X)r_0N$$

$$r_0^2 + r_1^2 + r_0^2 + 2\alpha r_0^2 + 2r_0(1 + \alpha)(x_1 - Q) \cdot N$$

$$2r_0^2 + r_1^2 + 2r_1r_0 + 2(r_1 + r_0)(X_1 - Q) \cdot N$$

I had thought the key is to use *the law of cosines* to evaluate  $(X_1 - Q) \cdot N = \frac{1}{\alpha}(C - X_1) \cdot N$ .

Length  $l = d(X, L) = d(C, L)$ .

But that just gets us back to the same place.

I had found the solution by a clever construction. But to come to that conclusion it required me to be “unstuck” - if something doesn’t work, move onto the next - don’t try to make something work and go in circles. And persistence is key because there can be many **false eureka**s.

Again, consider a particular circle with its center  $C_2$  right below the given  $Q$  circle that just makes  $C_2$  tangent with the given line  $L_2$ . The directrix is not going to be  $L_2$  but  $L_1$ , a line translated below  $L_2$ , line of tangency, by  $r_0$ , so that  $\|Q - C_2\| = r_2 + r_0 = d(C_2, L_1)$ . It is a clever artificial construction.

Let’s show this for any circle  $C$  of radius  $r_1$  in the family.

Tangent to the circle  $Q$  condition:  $X_1 - C = \alpha(Q - X_1)$ .

So then  $\|Q - C\| = r_1 + r_0$

Tangent to the line  $L_2 = B_2 + tA_2$ :  $(X_2 - C) \cdot A_2 = 0$

$\|X_2 - C\| = r_1$

Consider  $L_1$ , a line translated by  $r_0$  from  $L_2$  away from  $Q$ .

If  $L_2 = B_2 + tA_2$ ,  $L_1 = B_2 - r_0N + tA_2$ .

Since  $X_2 - C = r_1(-N)$  then  $X_2 - r_0N - C = (r_1 + r_0)(-N)$  will point from  $C$  to  $L_1$ , because  $(X_2 - r_0N) = ((B_2 + tA_2) - r_0N) \in L_1$ .

$\implies \|Q - C\| = r_1 + r_0 = \|X_2 - r_0N - C\| = d(C, L_1)$ .

**Exercise 23.** Without loss of generality, use  $y^2 = 4cx$ .

The latus rectum intersect the parabola at  $(c, +2c)$ ,  $(c, -2c)$ .

Thus  $4c = \text{length of latus rectum} = 2d = 2(\text{distance from focus to directrix})$ .

$$y = 2\sqrt{cx} \quad y' = \sqrt{c}/\sqrt{x} \quad y'(c) = \pm 1$$

Tangent lines:  $y = \pm(x + c)$ .

$\xrightarrow{\text{intersection}} +(x + c) = -(x + c) \quad x = -c$  (at the directrix)

**Exercise 24.** Center of circle is given to be 0.

Collinear with center and center not between them:  $P = \alpha Q$ ;  $\alpha > 0$

$$\|P\| \|Q\| = r_0^2 = \alpha \|Q\|^2$$

For the line defined in Cartesian coordinates as  $x + 2y - 5 = 0$ , the vector form of this line is given by

$$X_L = B + tA \quad X_L \cdot N = (x, y) \cdot \left( \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right) = N \cdot B + 0 = \sqrt{5}$$

$A = \left( \frac{-2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right)$  is a vector that’s perpendicular to  $N$ ;

$B = (1, 2)$  since we can simply plug it in to satisfy the equation

$$X_L = (1, 2) + t \left( \frac{-2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right) \quad t \in \mathbb{R}$$

$$Q = B + tA \implies \|Q\|^2 = B^2 + 2tB \cdot A + t^2A^2 = 5 + t(0) + t^2 = 5 + t^2$$

$$(5 + t^2)(\alpha) = r_0^2 = 4 \quad \alpha = \frac{4}{5 + t^2}$$

$$P = \alpha Q = \frac{4}{5 + t^2} \left( (1, 2) + t \left( \frac{-2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right) \right)$$

**14.4 Exercises - Vector-valued functions of a real variable, Algebraic operations. Components; Limits, derivatives, and integrals.**

**Exercise 1.**  $F' = (1, 2t, 3t^2, 4t^3)$ .

**Exercise 2.**  $F' = (-\sin t, 2 \sin t \cos t, 2 \cos 2t, \sec^2 t)$

**Exercise 3.**  $F' = \left( \frac{1}{\sqrt{1-t^2}}, \frac{-t}{\sqrt{1-t^2}} \right)$

**Exercise 4.**  $F' = (2e^t, 3e^t)$ .

**Exercise 5.**  $F' = (\sinh t, 2 \cosh 2t, -3e^{-3t})$

**Exercise 6.**  $\left( \frac{t}{1+t^2}, \frac{1}{1+t^2}, \frac{-2t}{(1+t^2)^2} \right)$

**Exercise 7.**  $F' = \left( \frac{2}{1+t^2} + \frac{-4t^2}{(1+t^2)^2}, \frac{4t}{(1+t^2)^2}, 0 \right)$ .

$$F' \cdot F = \frac{4t(1+t^2)}{(1+t^2)^3} + \frac{-4t^2(2t)}{(1+t^2)^3} + \frac{4t-4t^3}{(1+t^2)^3} = 0$$

**Exercise 8.**  $\left( \frac{1}{2}, \frac{2}{3}, e^1 - 1 \right)$

**Exercise 9.**

$$(-\cos t, \sin t, -\ln |\cos t|)|_0^{\pi/4} = \left( -\frac{\sqrt{3}}{2} + 1, \frac{\sqrt{2}}{2}, -\ln \frac{\sqrt{2}}{2} \right)$$

**Exercise 10.**  $(\ln(1+e^t), t - \ln(1+e^t))|_0^1 = (\ln(\frac{1+e}{2}), 1 - \ln(\frac{1+e}{2}))$

**Exercise 11.**  $(te^t - e^t, t^2e^t - 2te^t + 2e^t, -te^{-t} - e^{-t})|_0^1 = (1, e - 2, -2e^{-1} + 1)$

**Exercise 12.**  $(2, -4, 1) = A$ .

$$\begin{aligned} \int_0^1 (te^{2t}, t \cosh 2t, 2te^{-2t}) dt &= \left( \frac{1}{2}te^{2t} + \frac{-1}{4}e^{2t}, \frac{t \sinh 2t}{2} - \frac{\cosh 2t}{4}, 2 \left( \frac{-1}{2}te^{-2t} - \frac{e^{-2t}}{4} \right) \right) \Big|_0^1 \\ &= \left( \frac{1}{4}e^2 + \frac{1}{4}, \frac{\sinh 2}{2} - \frac{\cosh 2}{4} + \frac{1}{4}, 2 \left( \frac{-3}{4}e^{-2} + \frac{1}{4} \right) \right). \\ A \cdot B &= \frac{1}{2}e^2 + -2 \sinh 2 + \cosh 2 + \frac{-3}{2}e^{-2}. \end{aligned}$$

**Exercise 13.**  $F'(t) = B = 1 = \|F'(t)\| |B| \cos \theta(t)$ .

Given  $\theta(t) = \theta_0$  constant,  $\|F'(t)\|$  must be a constant.

$$\|F'(t)\|^2 = F'(t) \cdot F'(t) = g \quad g' = 2F''(t) \cdot F'(t) = 0 \text{ since } \|F'\|^2 \text{ constant.}$$

$$\implies F''(t) \cdot F'(t) = 0$$

**Exercise 14.**

$$F' = 2e^{2t}A + -2e^{-2t}B$$

$$F'' = 4e^{2t}A + 4e^{-2t}B = 4(F)$$

**Exercise 15.**  $G' = F' \times F' + F \times F'' = F \times F''$

**Exercise 16.**

$$G = F \cdot (F' \times F'')$$

$$G' = F' \cdot (F' \times F'') + F \cdot (F'' \times F'' + F' \times F''') = F \cdot (F' \times F''')$$

**Exercise 17.** If  $\lim_{t \rightarrow p} F(t) = A$ ,  $\forall j$ th component,

$$\forall \sqrt{\frac{\epsilon}{n}} > 0, \exists \delta_j > 0 \quad \text{such that } |F_j(t) - A_j| < \sqrt{\frac{\epsilon}{n}} \quad \text{if } |t - p| < \delta_j$$

$$\text{Consider } \min_{j=1, \dots, n} \delta_j = \delta_0$$

$$\sum_{j=1}^n |F_j(t) - A_j|^2 < \sum_{j=1}^n \left( \sqrt{\frac{\epsilon}{n}} \right)^2 = \epsilon \quad \text{whenever } |t - p| < \delta_0$$

$$\implies \lim_{t \rightarrow p} \|F(t) - A\| = 0$$

$$\text{If } \lim_{t \rightarrow p} \|F(t) - A\| = 0, \quad \forall \epsilon > 0, \exists \delta > 0 \quad \text{such that } \sqrt{\sum_{j=1}^n (F_j(t) - A_j)^2} < \epsilon \quad \text{if } |t - p| < \delta.$$

$$\implies \sum_{j=1}^n (F_j(t) - A_j)^2 < \epsilon$$

$$\epsilon > \sum_{j=1}^n (F_j(t) - A_j)^2 > (F_k(t) - A_k)^2 > 0$$

$$\implies \epsilon > |F_k(t) - A_k| \quad \text{if } |t - p| < \delta.$$

**Exercise 18.** If  $F$  is differentiable on  $I$ , then

$$F' = \sum_{j=1}^n f'_j \vec{e}_j \quad f'_j = \lim_{h \rightarrow 0} \frac{1}{h} (f_j(t+h) - f_j(t))$$

$$F' = \sum_{j=1}^n \lim_{h \rightarrow 0} \frac{1}{h} (f_j(t+h) - f_j(t)) = \lim_{h \rightarrow 0} \frac{1}{h} \sum_{j=1}^n (f_j(t+h) - f_j(t)) e_j = \lim_{h \rightarrow 0} h \rightarrow 0 \frac{1}{h} (F(t+h) - F(t))$$

$$\text{If } F'(t) = \lim_{h \rightarrow 0} \frac{1}{h} (F(t+h) - F(t)) = \lim_{h \rightarrow 0} \frac{1}{h} \sum_{j=1}^n (f_j(t+h) - f_j(t)) e_j = \\ = \sum_{j=1}^n \lim_{h \rightarrow 0} \frac{1}{h} (f_j(t+h) - f_j(t)) e_j = \sum_{j=1}^n f'_j(t) e_j$$

So  $F'$  is differentiable.

**Exercise 19.**  $F'(t) = 0, \forall j = 1 \dots n, f'_j(t) = 0$ . By one-dimensional zero-derivative theorem,  $f_j(t) = c_j$  constant. Thus  $F(t) = \sum_{j=1}^n c_j \vec{e}_j = C$  on an open interval  $I$ .

**Exercise 20.**  $\frac{1}{6}t^3 A + \frac{1}{2}t^2 B + Ct + D$

**Exercise 21.**  $Y'(x) + p(x)Y(x) = Q(x)$ . Then  $\forall j = 1, \dots, n$

$$y'_j(x) + p(x)y_j(x) = Q_j(x)$$

Since  $p, Q$  are continuous on  $I$ , and given this initial value condition  $y_k(a) = b_k$ ,

$$y_j(x) = e^{-\int_a^x p(t)dt} \left( b_j + \int_a^x Q_j(t) e^{\int_a^t p(u)du} dt \right) \\ \implies \sum_{j=1}^n j_j(x) = Y(x) = e^{-\int_a^x p} \left( B + \int_a^x Q e^{\int_a^t p} dt \right)$$

**Exercise 22.**

$$tF' = F + tA \implies F' + tF'' = F' + A$$

$$\implies tF'' = A$$

$$F''(t) = A/t$$

$$\implies F'(t) = A \ln t + B$$

$$\implies F(t) = A(t \ln t - t) + Bt + C$$

$$F(1) = A(-1) + B + C = 2A$$

$$tF' = F + tA \implies At \ln t + Bt = A(t \ln t - t) + Bt + C + tA$$

$$C = 0, \quad B = 3A$$

$$F(t) = A(t \ln t - t) + 3At$$

$$F(3) = A(3 \ln 3 - 3) + 9A = \boxed{3A \ln 3 + 6A}$$

**Exercise 23.**

$$F'(x) = e^x A + x e^x A + -\frac{1}{x^2} \int_1^x F(t) dt + \frac{1}{x} F(x) =$$

$$= e^x A + x e^x A + e^x A - \frac{F(x)}{x} + \frac{F(x)}{x} = 2e^x A + x e^x A = (2+x)e^x A$$

$$F'(x) = (2+x)e^x A; \quad F(x) = 2e^x A + A(xe^x - e^x) + C = A x e^x + e^x A + C$$

$$\int_1^x (Ate^t + e^t A + C) dt = (A(te^t - e^t) + e^t A + Ct) \Big|_1^x =$$

$$= A(xe^x - e^x) + e^x A + C(x-1) - eA = A x e^x + C(x-1) - eA$$

$$x e^x A + A e^x + \frac{C(x-1)}{x} - \frac{eA}{x} \implies C = eA$$

**Exercise 24.**  $F'(t) = \alpha(t)F(t)$

$$\implies f'_k(t) = \alpha(t)f_k(t); \quad \ln \left( \frac{f_k(t)}{f_k(a)} \right) = \int_a^t \alpha(x) dx$$

$$f_k = f_k(a) e^{\int_a^t \alpha}$$

$$F(t) = \sum_{j=1}^n f_j(a) e^{\int_a^t \alpha} e_j = e^{\int_a^t \alpha} \sum_{j=1}^n f_j(a) e_j = u(t)A$$

# 14.7 Exercises - Applications to curves. Tangency. Applications to curvilinear motion. Velocity, speed, and acceleration.

## Exercise 1.

$$\begin{aligned}\vec{r}(t) &= ((3t - t^3), 3t^2, (3t + t^3)) \\ v &= (3 - 3t^2, 6t, 3 + 3t^2) = 3(1 - t^2, 2t, 1 + t^2) \\ a &= 3(-2t, 2, 2t) = 6(-t, 1, t) \\ s &= 3\sqrt{(1 - 2t^2 + t^4 + 4t^2 + 1 + 2t^2 + t^4)} = 3\sqrt{2}(1 + t^2)\end{aligned}$$

## Exercise 2.

$$\begin{aligned}r &= (c, s, e^t) \\ v &= (-s, c, e^t) \\ a &= (-c, -s, e^t) \\ s &= \sqrt{1 + e^{2t}}\end{aligned}$$

## Exercise 3.

$$\begin{aligned}r &= (3tc, 3ts, 4t) \\ v &= (3(c - ts), 3(s + tc), 4) = 3(c, s, 0) + (-3ts, 3tc, 4) \\ a &= 6(-s, c, 0) + (-3tc, -3ts, 0) \\ s &= \sqrt{9(c^2 - 2tsc + t^2s^2) + 9(s^2 + 2tsc + t^2c^2) + 16} = \boxed{\sqrt{9 + 9t^2 + 16}}\end{aligned}$$

## Exercise 4.

$$\begin{aligned}r(t) &= ((t - \sin t), (1 - \cos t), 4 \sin(t/2)) \\ v &= (1 - c, s, 2 \cos(t/2)) \\ a &= (s, c, -\sin(t/2)) \\ s &= \sqrt{(1 - 2c + c^2 + s^2 + 4c^2(t/2))} = \sqrt{2 - 2c + 4\left(\frac{1+c}{2}\right)} = \boxed{2}\end{aligned}$$

## Exercise 5.

$$\begin{aligned}r &= (3t^2, 2t^3, 3t) \\ v &= (6t, 6t^2, 3) \\ a &= (6, 12t, 0) \\ s &= \sqrt{36t^2 + 36t^4 + 9} = \boxed{3\sqrt{4t^2 + 4t^4 + 1}}\end{aligned}$$

## Exercise 6.

$$\begin{aligned}r &= (t, \sin t, (1 - \cos t)) \\ v &= (1, c, s) \\ a &= (0, -s, c) \\ s &= \sqrt{2}\end{aligned}$$

## Exercise 7.

$$\begin{aligned}r &= (ac(\omega t), as(\omega t), b\omega t) \\ r' &= (-\omega as(\omega t), \omega ac(\omega t), b\omega) = \omega(-as, ac, b) \\ |r'| &= \omega\sqrt{a^2 + b^2} \\ \frac{r'}{|r'|} \cdot e_z &= \frac{b\omega}{\omega\sqrt{a^2 + b^2}} = \boxed{\frac{b}{\sqrt{a^2 + b^2}}}\end{aligned}$$

## Exercise 8. $|r'| = \omega\sqrt{a^2 + b^2}$

$$\begin{aligned}a &= \omega^2(-ac, -as, 0), \quad |a| = \omega^2 a \\ \frac{\|v \times a\|}{\|v\|^3} &= \frac{1}{(\omega\sqrt{a^2 + b^2})^3} |(\omega^3(bas, -bac, a^2))| = \frac{\omega^3\sqrt{b^2a^2 + a^4}}{\omega^3(\sqrt{a^2 + b^2})^3} = \boxed{\frac{a}{a^2 + b^2}}\end{aligned}$$

**Exercise 9.**  $u = (\sin(\omega t), -\cos(\omega t), 0)$

$$v \times a = \omega^3(bas, -bac, a^2) = \omega^3bau + \omega^3a^2e_z$$

$$A = \omega^3ba$$

$$B = \omega^3a^2$$

**Exercise 10.**

$$\frac{d}{dt}(v \cdot v) = a \cdot v + v \cdot a = 2v \cdot a$$

**Exercise 11.**

(1)

$$\frac{v}{|v|} \cdot c = \cos \theta; \quad \frac{v \cdot c}{\cos \theta} = |v|$$

$$(r \cdot c)' = r' \cdot c = 2e^{2t}$$

$$\implies \frac{2e^{2t}}{\cos \theta} = |v|$$

(2)

$$\frac{4e^{4t}}{\cos^2 \theta} = |v|^2$$

$$\text{from the previous exercise, } \frac{1}{2} \frac{d}{dt} v^2 = \frac{8e^{4t}}{\cos^2 \theta} = a \cdot v$$

**Exercise 12.**  $r = (a \cosh \theta, b \sinh \theta)$ .  $a = b = 1$ .

Draw a diagram. You'll immediately see that to get sector OAP, we need to take the area of a right triangle,  $\frac{1}{2} \cosh \theta \sinh \theta = \frac{1}{2} (x\text{-coordinate})(y\text{-coordinate})$  and subtract away from it the curved piece below the graph of the hyperbola. Thus

$$\frac{1}{2} \cosh \theta \sinh \theta - \int_1^{\cosh \theta} \sqrt{x^2 - 1} dx = A(\theta) \text{ with } \frac{x^2 - y^2}{\sqrt{x^2 - 1}} = 1$$

Then

$$A'(\theta) = \frac{1}{2}(\cosh^2 \theta + \sinh^2 \theta) - \sqrt{\cosh^2 \theta - 1} \sinh \theta = \frac{1}{2}$$

$$A(\theta) = \boxed{\frac{\theta}{2}}$$

**Exercise 13.**

$$r = (a \cosh \omega t, b \sinh \omega t)$$

$$v = \omega(a \sinh \omega t, b \cosh \omega t)$$

$$a = \omega^2(a \cosh \omega t, b \sinh \omega t) = \boxed{\omega^2 r}$$

So  $a$  is centrifugal, since  $a = \omega^2 r$ .

**Exercise 14.** Parabola.

$$\|X - F\| = d(X, L) = |(X - (-2a)e_x) \cdot e_x| = X \cdot e_x + 2a$$

Let  $F = 0$ ,  $\|X - F\| = \|X\|$

Let  $X = d_1 u_1$

$$X' = d'_1 u_1 + d_1 u'_1 \quad \|X\| = d_1 = X \cdot e_x + 2a$$

$$X' \cdot u_1 = d'_1 = X' \cdot e_x \quad d'_1 = X' \cdot e_x$$

$$\xrightarrow{\frac{X'}{\|X'\|} = T} T \cdot u_1 = T \cdot e_x$$

$$\cos \theta_1 = \cos \theta_2$$

**Exercise 15.**

(1)  $|a| = 4|r| \implies a = -4r$ .

$$\xrightarrow{r(0)=4e_x; v(0)=6e_y} \begin{cases} x = 4 \cos(2t) \\ y = 3 \sin(2t) \end{cases}$$

(2)  $\boxed{\left(\frac{x}{4}\right)^2 + \left(\frac{y}{3}\right)^2 = 1}$ . Motion is counterclockwise along an ellipse.

**Exercise 16.** Given that  $x^2 + c(y - x) = 0$ , then  $y = x - \frac{x^2}{c}$

$$\begin{aligned} r &= (x, y) = \left(x, x - \frac{x^2}{c}\right) \\ r' &= (x', x' - \frac{2xx'}{c}) \\ r'' &= (x'', x'' - \frac{2}{c}(x'^2 + xx'')) \\ x'' &= x'' - \frac{2}{c}(x'^2 + xx'') \\ \implies & \quad \quad \quad xx'' + x'^2 = 0 \text{ or} \\ (xx')' &= 0 \text{ so then } \boxed{xx' = \text{constant} = k} \end{aligned}$$

$$\begin{aligned} x \frac{dx}{dt} &= k \implies x dx = k dt \\ \frac{1}{2}(x_f^2 - x_i^2) &= k(t_f - t_i) \implies \frac{1}{2}(-c^2) = k(T) \text{ so that } k = \frac{-c^2}{2T} \\ \frac{1}{2} \left( \left(\frac{c}{2}\right)^2 - c^2 \right) &= \frac{1}{2} \left( \frac{-3c^2}{4} \right) = \frac{-c^2}{2T}(t_f - 0) \\ \boxed{\frac{3T}{4} = t_f} \end{aligned}$$

**Exercise 17.** Given that  $Y(t) = X(u(t))$ ,

$$Y'(t) = X'(u)u'(t)$$

So  $Y'$  is a scalar multiple of  $X'$ ;  $Y' \parallel X'$

$$Y'' = X''(u)(u'(t))^2 + X'(u)u''(t)$$

So  $Y'' \parallel X''$  only if  $X'(u) \parallel X''(u)$  as well. This is not necessarily so.

## 14.9 Exercises - The unit tangent, the principal normal, and the osculating plane of a curve.

**Exercise 1.**  $t = 2$

(1)

$$\begin{aligned} T &= \frac{v}{s} = \frac{(1 - t^2, 2t, 1 + t^2)}{\sqrt{2}(1 + t^2)} \\ T(t = 2) &= \boxed{\frac{(-3, 4, 5)}{\sqrt{2}(5)}} \\ T' &= \frac{v'}{s} - \frac{s'v}{s^2} = \frac{(-2t, 2, 2t)}{\sqrt{2}(1 + t^2)} - \frac{\sqrt{2}(2t)(1 - t^2, 2t, 1 + t^2)}{2(1 + t^2)^2} \\ T'(t = 2) &= \frac{\sqrt{2}(-2, 1, 2)}{5} - \frac{\sqrt{2}(2)(-3, 4, 5)}{25} = \frac{\sqrt{2}}{25}((-10, 5, 10) - (-6, 8, 10)) = \frac{\sqrt{2}}{25}(-4, -3, 0) \\ N(t = 2) &= \boxed{\frac{(-4, -3, 0)}{5}} \end{aligned}$$

(2)

$$a(t = 2) = 6(-2, 1, 2) = \boxed{12\sqrt{2}T(t = 2) + 6(N)}$$

**Exercise 2.**  $t = \pi$



(1)

$$\begin{aligned}
T &= \frac{v}{s} = \frac{(-s, c, e^t)}{\sqrt{1+e^{2t}}} \\
T(t=\pi) &= \frac{(0, -1, e^\pi)}{\sqrt{1+e^{2\pi}}} \\
T' &= \frac{v'}{s} - \frac{s'v}{s^2} = \frac{(-c, -s, e^t)}{\sqrt{1+e^{2t}}} - \frac{e^{2t}}{\sqrt{1+e^{2t}}} \frac{(-s, c, e^t)}{(1+e^{2t})} \\
T'(t=\pi) &= \frac{(1, 0, e^\pi)}{\sqrt{1+e^{2\pi}}} - \frac{e^{2\pi}}{\sqrt{1+e^{2\pi}}} \frac{(0, -1, e^\pi)}{(1+e^{2\pi})} = \\
&= \frac{(1+e^{2\pi}, e^{2\pi}, e^\pi)}{(1+e^{2\pi})^{3/2}} \\
N(t=\pi) &= \frac{(1+e^{2\pi}, e^{2\pi}, e^\pi)}{\sqrt{1+3e^{2\pi}+2e^{4\pi}}}
\end{aligned}$$

(2)

$$a(t=\pi) = \frac{e^{2\pi}\sqrt{1+e^{2\pi}}T(t=\pi) + \sqrt{1+3e^{2\pi}+2e^{4\pi}}N(t=\pi)}{1+e^{2\pi}}$$

**Exercise 3.**

(1)

$$\begin{aligned}
T &= \frac{v}{s} = \frac{3(1, 0, 0) + (0, 0, 4)}{5} = \frac{(3, 0, 4)}{5} \\
T' &= \left(\frac{v}{s}\right)' = \frac{v'}{s} - \frac{s'v}{s^2} = \\
&= \frac{6(-s, c, 0) + (-3tc, 3ts, 0)}{\sqrt{9+9t^2+16}} - \frac{\frac{9t}{\sqrt{9+9t^2+16}}(3(c, s, 0) + (-3ts, 3tc, 4))}{9+9t^2+16} \\
T'(t=0) &= \frac{6}{5}(0, 1, 0) - 0 = \frac{6}{5}(0, 1, 0) \\
N'(t=0) &= (0, 1, 0)
\end{aligned}$$

(2)

$$a(t=0) = 6(0, 1, 0) = 0T + 6N'(t=0)$$

**Exercise 4.**

(1)

$$\begin{aligned}
|T'| &= \frac{1}{2}\sqrt{(1+s^2(t/2))} \\
T &= \frac{v}{s} = \frac{1}{2}(1-c, s, 2\cos(t/2)) & N &= \frac{T'}{|T'|} = \frac{(s, c, -\sin(t/2))}{\sqrt{(1+s^2(t/2))}} \\
T' &= \frac{1}{2}(s, c, -\sin(t/2)) & T(t=\pi) &= (1, 0, 0) \\
& & N(t=\pi) &= (0, -1/\sqrt{2}, -1/\sqrt{2})
\end{aligned}$$

(2)

$$\begin{aligned}
a(t=\pi) &= (0, -1, -1) \\
a(t=\pi) &= 0T + 2N(t=\pi)
\end{aligned}$$

**Exercise 5.  $t=1$** 

(1)

$$\begin{aligned}
T &= \frac{(2t, 2t^2, 1)}{\sqrt{4t^2+4t^4+1}} & T' &= \frac{v'}{s} - \frac{s'v}{s^2} = \frac{(2, 4t, 0)}{\sqrt{4t^2+4t^4+1}} - \frac{4t+8t^3}{(4t^2+4t^4+1)^{3/2}}(2t, 2t^2, 1) \\
T(t=1) &= \frac{(2, 4, 0)}{\sqrt{9}} = \frac{2}{3}(2, 4, 0) & T'(t=1) &= \frac{(2, 4, 0)}{3} - \frac{12}{27}(2, 2, 1) = \frac{1}{9}(-2, 4, -4) \\
& & N(t=1) &= \frac{1}{3}(-1, 2, -2)
\end{aligned}$$

(2)

$$a = (6, 12, 0) = 4(3T) + 2(3N) = \boxed{12T + 6N}$$

**Exercise 6.**

(1)

$$T = \frac{v}{s} = \frac{(1, c, s)}{\sqrt{2}} \quad T' = \frac{v'}{s} - \frac{s'v}{s^2} = \frac{(0, -s, c)}{\sqrt{2}}$$
$$T(t = \pi/4) = \boxed{\left(\frac{1}{\sqrt{2}}, \frac{1}{2}, \frac{1}{2}\right)} \quad N(t = \pi/4) = \boxed{(0, -1/\sqrt{2}, 1/\sqrt{2})}$$

(2)

$$a(t = \pi/4) = (0, -1/\sqrt{2}, 1/\sqrt{2}) = \boxed{N}$$

**Exercise 7.**

$$a = 0$$

$$v = v_0 = \text{const.}$$

$$\boxed{r = v_0 t + r_0}$$

**Exercise 8.**

$$\frac{\|v \times a\|}{\|v\|} = \left\| \frac{v}{\|v\|} \times a \right\| = \|T \times a\|$$

$a = s'T + sT'$ , so the normal component of  $a$  is  $s|T'|$  since  $sT' = s|T'|N$ .

$$T \times a = sT \times T'$$

$$\|T \times a\| = \|sT \times T'\| = \sqrt{|sT|^2 |T'|^2 - (sT \cdot T')^2} =$$
$$\xrightarrow{T \cdot T' = 0} |sT| |T'| = s|T'|$$

**Exercise 9.**

(1)

$$v = v_0$$

$$r = v_0 t + r_0 \quad r \in \text{line}$$

(2) Counterexample: imagine the particle moving at constant speed, but in an S-shape in one plane and then an S shape in another plane, out of the first plane.

(3)

$$a = a_0$$

$$v = a_0 t + v_0$$

$$r = \frac{a_0}{2} t^2 + v_0 t + r_0$$

$r = \frac{a_0}{2} t^2 + v_0 t + r_0$  defines a plane.

(4)  $v \cdot a = 0$

$$a = s'T + sT'$$

$$a \cdot T = s' = 0 \implies s \text{ constant}$$

$$\implies a = sT'$$
$$\implies v = kT$$

So  $s$  is constant and  $a = sT'$ . We already showed in part (b) of this exercise that there's a counterexample.

**Exercise 10.**

(1)

$$\frac{d}{dt}(r \times v) = v \times v + r \times a = r \times a = 0$$
$$\implies r \times v = \text{const.}$$

- (2) Use what you've learned in these sections: use what you've learned about  $a$  being completely in the osculating plane. The osculating plane idea is very useful.

$$r \times v = c \implies r \times a = 0$$

$$a = s'T + sT' \implies r \times (s'T + sT') = 0$$

Possibilities:

$$r = c(s'T + sT') \text{ (} r \text{ is a scalar multiple of } s'T + sT' \text{)} \implies r \in \text{osculating plane}$$

$$a = 0, \quad v = v_0, \quad r = v_0 t + r_0, \quad r \text{ is on a line}$$

$$r = 0, \quad r \text{ stationary}$$

- (3) If  $a = |a(t)|r(t)/|r(t)|$  then  $r \times a = 0$  since  $a$  is a scalar multiple of  $r$ . Then from above,  $r \times a = 0$  implies that  $r$  is on a plane.
- (4)  $r \times v$  not necessarily constant if  $r$  lies in a plane.

Counterexample:

$$r = (tc, t^2s, 0)$$

$$v = (c - ts, 2ts + t^2c, 0)$$

$$r \times v = (0, 0, 2t^2cs + t^3c^2 - t^2sc - t^3s^2) = (0, 0, t^2sc + t^3(c^2 - s^2))$$

**Exercise 11.** Given that  $r \in$  plane, the  $v$  is also in the plane, otherwise, for some time  $t$ ,  $r(t + \Delta t) = r + v\Delta t$  out of the plane.

- (1) We want  $T \times a = v \times a = 0$ .

Given that  $T \cdot c = \cos \theta_0 = k_0$ ,  $(T \cdot c)' = 0 = T' \cdot c + T \cdot c' = T' \cdot c$ .  
 $c \neq 0$ , so if  $T \cdot c \neq 1$ ,  $T' \cdot c = 0$  implies  $T' = 0$ . Then  $T \times a = 0$ .

$$\text{If } T \cdot c = 1, v = sc \quad a = v' = s'c$$

$$v \times a = sc \times s'c = ss'(c \times c) = 0$$

- (2) Example:

$$r = (c(t), s(t), t^3)$$

$$v = (-s, c, 3t^2)a = (-c, -s, 6t)$$

**Exercise 12.**

- (1) If  $y = g(t) \geq 0$ ,  $v_x \leq 0$ . counterclockwise
- (2)

$$3x^2 + y^2 = 1$$

$$r(t) = (f(t), g(t)) = (x, y) = (x, \pm\sqrt{1-3x^2})$$

$$r' = \left(x', \mp \frac{3x}{\sqrt{1-3x^2}}x'\right)$$

$$x' = v_x = -g(t) = y$$

$$\frac{\mp 3x'x}{\sqrt{1-3x^2}} = \frac{\mp 3y}{y}x = \mp 3x$$

so  $v_y = \boxed{3x}$  (since we want counterclockwise motion)

- (3)  $x' = \sqrt{1-3x^2} \implies$

$$\int_{x_0}^{x_f} \frac{dx}{\sqrt{1-3x^2}} = \int_{t_i}^{t_f} dt = T = \frac{\arcsin(\sqrt{3}x)}{\sqrt{3}} \Big|_{x_0}^{x_f} = \frac{1}{\sqrt{3}}(\pi/2 - -\pi/2) = \pi/\sqrt{3}$$

$$\int_{x_0}^{x_f} = T = \frac{\arccos(\sqrt{3}x)}{\sqrt{3}} \Big|_{-1/\sqrt{3}}^{1/\sqrt{3}} = \frac{0 - (-\pi)}{\sqrt{3}} = \pi/\sqrt{3} \quad (\text{we need } T \text{ to be positive})$$

$$\implies \boxed{2\pi/\sqrt{3}}$$

**Exercise 13.**

$$\begin{aligned} r &= (x, y) = (x(t), y(t)) & r \cdot e_x &= x = \sqrt{x^2 + y^2} \cos \theta = r \cos \theta \\ r' &= (x', y') & r' \cdot e_x &= x' = |r'| \cos \phi \end{aligned}$$

$$\tan \phi = \frac{4}{3} \cot \theta \implies \pm \frac{y'}{x'} = \frac{4}{3} \frac{x}{y}$$

$$\frac{y'}{x'} = \frac{dy}{dx} = \frac{-4x}{3y} \quad (\text{since } \frac{x}{r} > 0 \text{ for } C \text{ in first quadrant, and } \frac{dy}{dx} < 0 \text{ in first quadrant})$$

$$\implies y^2 = \frac{-4}{3} x^2 + C$$

$$\xrightarrow{(3/2, 1)} \boxed{\frac{x^2}{3} + \frac{y^2}{4} = 1}$$

**Exercise 14.**

$$T = \frac{(x', y')}{\sqrt{x'^2 + y'^2}}$$

$$N = \frac{(y', -x')}{\sqrt{x'^2 + y'^2}}$$

$$(x, y) + (\pm d)N = (x, y) + \frac{(\pm d)(y', -x')}{\sqrt{x'^2 + y'^2}} = (x_2, 0)$$

$$x + \frac{((\pm)d)y'}{\sqrt{x'^2 + y'^2}} = x_2$$

$$y + \frac{((\pm)d)(-x')}{\sqrt{x'^2 + y'^2}} = 0$$

$$|x_2 - x| = 2 = \left| x + \frac{\pm dy'}{\sqrt{x'^2 + y'^2}} - x \right| = \frac{|d||y'|}{\sqrt{x'^2 + y'^2}}$$

$$\text{Divide the above 2 by the expression for } y \implies \frac{|d||y'|}{\pm dx'} = \frac{2}{y}$$

$$\pm \frac{dy}{dx} = \frac{2}{y} \implies \pm y dy = 2 dx$$

$$\implies \pm \frac{1}{2} (y_f^2 - y_i^2) = 2(x_f - x_i)$$

$$\xrightarrow{(1, 2)} \pm (y^2 - 4) = 4(x - 1) \implies \boxed{\begin{aligned} y^2 &= 4x \\ y^2 &= -4x + 8 \end{aligned}}$$

**Exercise 15.**  $r' = A \times r$

$$(1) \quad r'' = a = A \times v \implies a \cdot A = A \cdot A \times v = 0$$

(2)

$$v = A \times r$$

$$a = A \times v$$

$$a = s'T + sT', \quad a \cdot T = s'$$

$$a \cdot T = (A \times v) \cdot T = 0 = s' \implies s \text{ constant}$$

Now note that

$$|r'| = (\sqrt{r \cdot r})' = \frac{1}{2} \frac{1}{\sqrt{r \cdot r}} (v \cdot r + r \cdot v) = \frac{v \cdot r}{\sqrt{r \cdot r}}$$

$$\text{however } v \cdot r = (A \times r) \cdot r = 0$$

$$\implies |r'| = 0$$

So the length of the position vector  $r$  is held constant. So if  $r(0) = B$ , then  $|r| = \|B\|$  for all time  $t > 0$ .

We have

$$|r'(t)| = s = |A \times r| = \|A\| \|r\| \sin \theta_{Ar}$$

Now  $s$  is constant,  $\|A\|$ ,  $\|B\|$  are constant. Then  $\sin \theta_{Ar}$  must be constant as well.  $r(0) = B$  so  $\theta_{Ar} = \theta_{AB}$ .

$$|r'(t)| = s = |A \times r| = \|A\| \|B\| \sin \theta_{AB}$$

(3) See sketch.  $r$  starts off as  $B$  and  $r$  rotates around  $A$  like a top.

#### Exercise 16.

(1)

$$\begin{aligned} Y(t) &= X[u(t)] \\ Y'(t) &= \frac{dY(t)}{dt} = \frac{d}{dt}(X(u(t))) = \frac{dX}{du}(u) \frac{du}{dt} \\ \frac{Y'(t)}{\|Y'(t)\|} &= T_Y(t) = \frac{\frac{d}{dt}(X(u(t)))}{\left\| \frac{d}{dt}(X(u(t))) \right\|} = \frac{\frac{dX(u)}{du} \frac{du}{dt}}{\left\| \frac{dX(u)}{du} \right\| \left| \frac{du}{dt} \right|} = T_X(u(t)) \frac{\frac{du}{dt}}{\left| \frac{du}{dt} \right|} \\ \text{since for } X(u(t)), \quad \frac{dX}{du}(u) &= X' \implies \frac{X'}{\|X'\|} = T_X(u) = \frac{\frac{dX(u)}{du}}{\left\| \frac{dX(u)}{du} \right\|} \end{aligned}$$

$$\text{Now } \left| \frac{\frac{du}{dt}}{\left| \frac{du}{dt} \right|} \right| = 1 \text{ always, so for}$$

$$\text{strictly increasing } u, \frac{du}{dt} > 0 \implies \frac{du}{dt} / \left| \frac{du}{dt} \right| = 1$$

$$\text{strictly decreasing } u, \frac{du}{dt} < 0 \implies \frac{du}{dt} / \left| \frac{du}{dt} \right| = -1$$

$$\begin{array}{ll} \text{strictly increasing } u, & T_Y(t) = T_X(u) \\ \text{strictly decreasing } u, & T_Y(t) = -T_X(u) \end{array}$$

(2)

$$\begin{aligned} (T_Y)' &= \left( \frac{Y'}{|Y'|} \right)' = \frac{Y''}{|Y'|} - \frac{Y'}{|Y'|^2} |Y'|' \\ Y' &= X' u' \\ Y'' &= \frac{d}{dt} \left( \frac{dX(u)}{du} u'(t) \right) = \frac{d^2 X(u)}{du^2} u'^2 + \frac{dX(u)}{du} u'' = X'' u'^2 + X' u'' \\ \frac{Y''}{|Y'|} &= \frac{X'' u'^2 + X' u''}{\sqrt{X' u' \cdot X' u'}} \\ |Y'|' &= (\sqrt{Y' \cdot Y'})' = \frac{Y' \cdot Y''}{\sqrt{Y' \cdot Y'}} = \frac{X' u' \cdot (X'' u'^2 + X' u'')}{\sqrt{X' u' \cdot X' u'}} \\ &\quad - \frac{Y'}{|Y'|^2} |Y'|' = \frac{-(X')(X' \cdot (X'' u'^2 + X' u''))}{(X')^2 \sqrt{X' u' \cdot X' u'}} \\ \frac{Y''}{|Y'|} - \frac{Y'}{|Y'|^2} |Y'|' &= \frac{X'' u'^2 + X' u''}{\sqrt{X' u' \cdot X' u'}} - \frac{(X')(X' \cdot X'' u'^2 + X'^2 u'')}{X'^2 (\sqrt{X' u' \cdot X' u'})} = \frac{X'' u'^2}{\sqrt{X' u' \cdot X' u'}} - \frac{X'(X' \cdot X'') u'^2}{X'^2 \sqrt{X' u' \cdot X' u'}} = \\ &\quad \frac{u'^2}{|u'|^2} = |u'|^2 \\ &\xrightarrow{\sqrt{u'^2} = |u'|} \left( \frac{X''}{|X'|} - \frac{X'|X'|'}{|X'|^2} \right) |u'| \end{aligned}$$

$$\begin{aligned}
\|T'_Y\|^2 &= \left( \frac{Y''}{|Y'|} - \frac{Y'}{|Y'|^2} |Y'|' \right) \cdot \left( \frac{Y''}{|Y'|} - \frac{Y'}{|Y'|^2} |Y'|' \right) = \frac{Y''^2}{Y'^2} - \frac{2Y'' \cdot Y' Y' \cdot Y''}{|Y'|^3 |Y'|} + \frac{Y'^2 (Y' \cdot Y'')^2}{|Y'|^4 Y'^2} = \\
&= \frac{Y''^2}{Y'^2} - \frac{(Y' \cdot Y'')^2}{(Y')^4} \\
&= \frac{(X''u'^2 + X'u'')^2}{(X'u')^2} - \frac{(X'u' \cdot (X''u'^2 + X'u''))^2}{(X'u')^4} = \\
&= \frac{X''^2 u'^4 + 2X'' \cdot X'u'^2 u'' + X'^2 u''^2}{(X'u')^2} - \frac{(X' \cdot X'')^2 u'^6 + 2X' \cdot X'' u'^4 X'^2 u'' + (X'^2)^2 u'^2 u''^2}{(X'u')^4} = \\
&= \left( \frac{X''^2}{X'^2} - \frac{(X' \cdot X'')^2}{X'^4} \right) u'^2 \\
\frac{T'_Y}{\|T'_Y\|} &= \frac{T'_X |u'|}{\|T'_X\| |u'|} = \frac{T'_X}{\|T'_X\|} \implies \boxed{N_Y = N_X}
\end{aligned}$$

Thus, the osculating plane is invariant under a parameter change, since the osculating plane can still be made up of  $N_Y = N_X$  and  $T_Y = \pm T_X$ .

### 14.13 Exercises - The definition of arc length, Additivity of arc length, The arc-length function.

#### Exercise 1.

$$\begin{aligned}
r(t) &= a(1 - \cos t, t - \sin t) \quad 0 \leq t \leq 2\pi, a > 0 \\
r'(t) &= a(\sin t, 1 - \cos t) \quad v^2 = a^2(\sin^2 t + 1 - 2 \cos t + \cos^2 t) = 2a^2(1 - \cos t)
\end{aligned}$$

$$\begin{aligned}
v &= \sqrt{2a} \sqrt{1 - \cos t} = 2a \sin t/2 \\
\int v dt &= 2a \int \sin t/2 = -(4a) \cos t/2 \Big|_0^{2\pi} = \boxed{8a}
\end{aligned}$$

#### Exercise 2.

$$\begin{aligned}
r(t) &= (e^t \cos t, e^t \sin t); \quad 0 \leq t \leq 2 \\
r'(t) &= (e^t(c - s), e^t(s + c)) \\
v^2 &= e^{2t}(c^2 + s^2 + s^2 + c^2) = 2e^{2t} \implies v = \sqrt{2}e^t \\
\int_0^2 \sqrt{2}e^t dt &= \boxed{\sqrt{2}(e^2 - 1)}
\end{aligned}$$

#### Exercise 3.

$$\begin{aligned}
r(t) &= (a(\cos t + t \sin t), a(\sin t - t \cos t)) = a(\cos t + t \sin t, \sin t - t \cos t) \\
r' &= a(-s + s + tc, c - c + ts) = a(tc, ts) \\
v &= at \\
\int_0^{2\pi} at &= a \frac{1}{2} 4\pi^2 = \boxed{2\pi^2 a}
\end{aligned}$$

#### Exercise 4.

$$\begin{aligned}
r(t) &= \left( \frac{c^2}{a} \cos^3 t, \frac{c^2}{b} \sin^3 t \right) = C^2 \left( \frac{c^3}{A}, \frac{s^3}{B} \right) \quad 0 \leq t \leq 2\pi \\
r' &= 3C \left( \frac{-c^2 s}{A}, \frac{s^2 c}{B} \right) \\
v^2 &= 9C^2 \left( \frac{c^4 s^2}{A^2} + \frac{s^4 c^2}{B^2} \right) = 9C^4 c^2 s^2 \left( \frac{c^2}{A^2} + \frac{s^2}{B^2} \right) \\
v &= 3C^2 |cs| \sqrt{\frac{c^2}{A^2} + \frac{s^2}{B^2}} = 3C^2 |cs| \frac{\sqrt{B^2 + C^2 s^2}}{AB}
\end{aligned}$$

$$\begin{aligned} \int cs\sqrt{B^2 + C^2 s^2} &= \frac{1}{3C^2}(B^2 + C^2 s^2)^{3/2} \\ s &= \frac{3C^2}{AB} \left( \int_0^{\pi/2} cs\sqrt{B^2 + C^2 s^2} - \int_{\pi/2}^{\pi} cs\sqrt{B^2 + C^2 s^2} + \int_{\pi}^{3\pi/2} cs\sqrt{B^2 + C^2 s^2} - \int_{3\pi/2}^{2\pi} cs\sqrt{B^2 + C^2 s^2} \right) = \\ &= \frac{3C^2}{AB} \frac{4}{3C^2} ((B^2 + C^2)^{3/2} - B^3) = \boxed{4(A^3 - B^3)/AB} \end{aligned}$$

**Exercise 5.**

$$r(t) = (a(\sinh t - t), a(\cosh t - 1))$$

$$r'(t) = a(\cosh t - 1, \sinh t)$$

$$v^2 = a^2(\cosh^2 t - 2 \cosh t + 1 + \sinh^2 t) = a^2(2)(\cosh t)(\cosh t - 1)$$

$$\text{Knowing that } \cosh(2t) = \cosh^2 t + \sinh^2 t = 2 \cosh^2 t - 1 = 1 + 2 \sinh^2 t$$

$$s = \int |v| dt = \int_0^T a\sqrt{2}\sqrt{\cosh t(\cosh t - 1)} dt = \sqrt{2}a \int_0^T \sqrt{2 \cosh^2 t/2 - 1} \sqrt{2} \sinh t/2 dt =$$

$$u = \cosh t/2 \quad \sqrt{2}u = y$$

$$\frac{du = \frac{1}{2} \sinh t/2 dt}{\longrightarrow} 2a \int \sqrt{2u^2 - 1} du \xrightarrow{du = \frac{dy}{\sqrt{2}}} \frac{4a}{\sqrt{2}} \int \sqrt{y^2 - 1} dy$$

$$\int \sqrt{y^2 - 1} dy = y\sqrt{y^2 - 1} - \int \frac{y^2}{\sqrt{y^2 - 1}} = y\sqrt{y^2 - 1} - \int \frac{y^2 - 1 + 1}{\sqrt{y^2 - 1}} = y\sqrt{y^2 - 1} - \int \sqrt{y^2 - 1} dy + \int \frac{-1}{\sqrt{y^2 - 1}}$$

$$\text{since } (\ln(y \pm \sqrt{y^2 - 1}))' = \frac{1}{y \pm \sqrt{y^2 - 1}} \left( 1 \pm \frac{y}{\sqrt{y^2 - 1}} \right) = \pm \frac{1}{\sqrt{y^2 - 1}}$$

$$\implies \int \sqrt{y^2 - 1} dy = \frac{1}{2}(y\sqrt{y^2 - 1} - \ln(y + \sqrt{y^2 - 1}))$$

$$s = \frac{4a}{\sqrt{2}} \frac{1}{2} \left( \sqrt{2} \cosh t/2 \sqrt{2 \cosh^2 t/2 - 1} - \ln(\sqrt{2} \cosh t/2 + \sqrt{2 \cosh^2 t/2 - 1}) \right) \Big|_0^T =$$

$$= \boxed{2a(\cosh T/2 \sqrt{\cosh T} - 1) + -\sqrt{2}a \ln \left( \frac{\sqrt{2} \cosh T/2 + \sqrt{\cosh T}}{\sqrt{2} + 1} \right)}$$

**Exercise 6.**

$$r(t) = (\sin t, t, (1 - \cos t)); \quad (0 \leq t \leq 2\pi)$$

$$r' = (c, 1, s)$$

$$v^2 = c^2 + 1 + s^2 = 2 \implies \int_0^{2\pi} \sqrt{2} dt = \boxed{\sqrt{2}(2\pi)}$$

**Exercise 7.**

$$r(t) = (t, 3t^2, 6t^3), \quad (0 \leq t \leq 2)$$

$$r' = (1, 6t, 18t^2)$$

$$v^2 = 1 + 36t^2 + 18^2 t^4$$

$$s = \int_0^2 \sqrt{(18t^2)^2 + 2(18)t^2 + 1} dy = \int_0^2 (18t^2 + 1) dt = (6t^3 + t) \Big|_0^2 = \boxed{50}$$

**Exercise 8.**

$$r = (t, \log(\sec t), \log(\sec t + \tan t)), \quad (0 \leq t \leq \frac{\pi}{4})$$

$$r' = (1, \tan t, \sec t)$$

$$v^2 = 1 + \tan^2 t + \sec^2 t = 2 \sec^2 t$$

$$s = \int_0^{\pi/4} \sqrt{2} \sec t dt = \sqrt{2} (\ln |\tan t + \sec t|) \Big|_0^{\pi/4} = \boxed{\sqrt{2} \ln(1 + \sqrt{2})}$$

**Exercise 9.**

$$\begin{aligned}
r(t) &= (a \cos \omega t, a \sin \omega t, b \omega t), \quad (t_0 \leq t \leq t_1) \\
r' &= (\omega)(-a \sin(\omega t), a \cos(\omega t), b) \\
v^2 &= \omega^2(a^2 + b^2) \\
s &= \int_{t_0}^{t_1} |\omega| \sqrt{a^2 + b^2} = \boxed{|\omega| \sqrt{a^2 + b^2} (t_1 - t_0)}
\end{aligned}$$

**Exercise 10.**

$$\begin{aligned}
r &= (x, y) = (g(y), y) \\
r' &= \left( \frac{dg(y(t))}{dy} \frac{dy}{dt}, \frac{dy}{dt} \right) \\
r'^2 &= \left( \left( \frac{dg}{dy} \right)^2 + 1 \right) \left( \frac{dy}{dt} \right)^2 \\
s &= \int_{t_0}^{t_f} \sqrt{1 + \left( \frac{dg(y(t))}{dy} \right)^2} \frac{dy}{dt} dt = \boxed{\int_{y(t_0)}^{y(t_f)} \sqrt{1 + \left( \frac{dg}{dy} \right)^2} dy}
\end{aligned}$$

**Exercise 11.** Given  $y^2 = x^3$ ,

$$\begin{aligned}
r &= (x, \pm x^{3/2}) \\
\frac{dy}{dx} &= \pm \frac{3}{2} x^{1/2} \\
\int_0^1 \sqrt{1 + \frac{9}{4}x} &= \frac{8}{27} (1 + \frac{9}{4}x)^{3/2} \Big|_0^1 = \boxed{\frac{8}{27} \left( \frac{26\sqrt{13}}{8} - 2 \right)} \\
r &= (y^{2/3}, y) \\
\frac{dg}{dy} &= \frac{2}{3} y^{-1/3} \\
\int_{-1}^1 \sqrt{1 + \left( \frac{2}{3} \right)^2 y^{-2/3}} dy &= \int_{-1}^1 \frac{\sqrt{y^{2/3} + \left( \frac{2}{3} \right)^2}}{|y|^{1/3}} dy = (y^{2/3} + (2/3)^2)^{3/2} \Big|_0^1 - (y^{2/3} + (2/3)^2)^{3/2} \Big|_{-1}^0 = \\
&= \left( \frac{13}{9} \right)^{3/2} - \left( \frac{2}{3} \right)^3 - \left( \left( \frac{2}{3} \right)^3 - \left( \frac{13}{9} \right)^{3/2} \right) = \boxed{2 \left( \frac{13\sqrt{13}}{27} - \frac{8}{27} \right)}
\end{aligned}$$

**Exercise 12.**

$$\begin{aligned}
r &= (\cos \theta, \sin \theta) \\
r' &= (-\sin \theta, \cos \theta) \\
v^2 &= 1 \\
s &= \int_{\theta_0}^{\theta_1} 1 d\theta = (\theta_1 - \theta_0) = 2 \left( \frac{\theta_1 - \theta_0}{2} \right)
\end{aligned}$$

So the arc  $AB$  has a length equal to twice the area of the sector.

**Exercise 13.**

(1)

$$\begin{aligned}
y &= e^x, \quad 0 \leq x \leq 1 \\
\int_0^1 \sqrt{1 + e^{2x}} dx
\end{aligned}$$



(2)

$$(x, y) = (t + \log t, t - \log t); \quad 1 \leq t \leq e$$

$$r' = \left(1 + \frac{1}{t}, 1 - \frac{1}{t}\right)$$

$$r'^2 = 2\left(1 + \left(\frac{1}{t}\right)^2\right); \implies \sqrt{2}\sqrt{1 + \left(\frac{1}{t}\right)^2}$$

$$t = e^x$$

$$\int_1^e \sqrt{2}\sqrt{1 + (1/t)^2} dt \xrightarrow{dt = e^x dx} \int_0^1 \sqrt{2}\sqrt{1 + e^{-2x}} e^x dx = \sqrt{2} \int_0^1 \sqrt{1 + e^{2x}} dx$$

**Exercise 14.**

$$y = c \cosh(x/c)$$

$$\frac{dy}{dx} = \sinh(x/c)$$

$$\int_0^a \sqrt{1 + \sinh^2(x/c)} dx = \int_0^a \cosh x/a dx = c \sinh \frac{x}{c} \Big|_0^a = c \sinh \frac{a}{c}$$

So indeed,  $\int_0^a c \cosh(x/a) dx$  gives the area underneath a curve  $\cosh x/a$  from  $x = 0$  to  $x = a$ .

**Exercise 15.** Given  $y = \cosh x$ ,  $(0, 1)$  and  $(x, \cosh x)$  is  $\sinh x$  if  $x > 0$

$$\int_0^x \sqrt{1 + \sinh^2 t} dt = \sinh t \Big|_0^x = \boxed{\sinh x}$$

**Exercise 16.**

$$A = \int_a^b f(x) dx = k \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \implies \int_a^b \left( k \sqrt{1 + \left(\frac{dy}{dx}\right)^2} - f(x) \right) dx = 0$$

$$\pm \sqrt{\left(\frac{y}{k}\right)^2 - 1} = \frac{dy}{dx}; \quad \pm(x - x_0) = \int_{y_0}^y \frac{dy}{\sqrt{\left(\frac{y}{k}\right)^2 - 1}}$$

$\int_{y_0}^y \frac{dy}{\sqrt{\left(\frac{y}{k}\right)^2 - 1}}$  could be evaluated by  $\cosh u = y/k$  substitution, and then solve for  $u$ , (inverse of  $\cosh u$ ), by  $\cosh u = \frac{e^u + e^{-u}}{2} = y/k$  and then use the quadratic equation trick to get  $e^u$ .

$$\pm(x - x_0) = k \ln \frac{\left(\frac{y}{k} + \sqrt{\frac{y^2}{k^2} - 1}\right)}{\left(\frac{y}{k} - \sqrt{\frac{y^2}{k^2} - 1}\right)} \quad \text{Let } C_0 = \frac{y_0}{k} \pm \sqrt{\frac{y_0^2}{k^2} - 1}$$

$$\left(C_0 e^{\left(\frac{\pm(x - x_0)}{k}\right)} - \frac{y}{k}\right)^2 = \frac{y^2}{k^2} - 1$$

$$C_0^2 e^{\left(\frac{\pm 2(x - x_0)}{k}\right)} + 1 = 2C_0 e^{\left(\frac{\pm(x - x_0)}{k}\right)} \frac{y}{k}$$

$$\boxed{y = \frac{k \left( C_0 e^{\left(\frac{\pm(x - x_0)}{k}\right)} + e^{\left(\frac{\mp(x - x_0)}{k}\right)} \right)}{2}}$$

If we choose  $y_0 = k$ , so that  $C_0 = 1$ , then  $\boxed{y = k \cosh \left( \frac{x - x_0}{k} \right)}$ . Also note that  $y = k$  works as well.

**Exercise 17.**

$$r = (a \sin t, b \cos t) \quad 0 < b < a$$

$$r' = (ac, -bs)$$

$$r'^2 = a^2 c^2 + b^2 s^2 = a^2(1 - s^2) + b^2 s^2 = a^2 \left( 1 - \left( \frac{a^2 - b^2}{a^2} \right) s^2 \right)$$

$$L = \int_0^{2\pi} |a| \sqrt{1 - e^2 \sin^2 t} dt = \boxed{4a \int_0^{\pi/2} \sqrt{1 - e^2 \sin^2 t} dt}$$

**Exercise 18.**

$$\begin{aligned}
 r(t) &= (a(t - \sin t), a(1 - \cos t), b \sin t/2) = a(t - \sin t, (1 - \cos t), \frac{b}{a} \sin t/2) \\
 r' &= a(1 - \cos t, \sin t, \frac{b}{2a} \cos t/2) \\
 r'^2 &= a^2(1 - 2 \cos t + 1 + \left(\frac{b}{2a}\right)^2 \cos^2 t/2) = 2a^2(1 - \cos t + \left(\frac{b}{4a}\right)^2 (1 + \cos t)) = \\
 &= 2a^2(2 \sin^2 t/2 + \left(\frac{b}{4a}\right)^2 (2 + -2 \sin^2 t/2)) = 4a^2 \left( \left(\frac{b}{4a}\right)^2 + k^2 \sin^2 t/2 \right) = 4a^2(1 - k^2 + k^2 \sin^2 t/2) = \\
 &= 4a^2(1 - k^2 \cos^2 t/2) \\
 |r'| &= 2a\sqrt{1 - k^2 \cos^2 t/2} \\
 \int_0^{2\pi} 2a\sqrt{1 - k^2 \cos^2 t/2} dt &= 4a \int_0^\pi \sqrt{1 - k^2 \cos^2 t} dt \\
 &\xrightarrow{t=\pi/2-\theta} = (-4a) \int_{\pi/2}^{-\pi/2} \sqrt{1 - k^2 \sin^2 t} dt = \boxed{8a \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 t} dt}
 \end{aligned}$$

**Exercise 19.**

$$\begin{aligned}
 A \cdot B &= \cos \pi/3 = 1/2 \\
 (A \times B)^2 &= A^2 B^2 - (A \cdot B)^2 = 1 - 1/4 = 3/4 \\
 r &= tA + t^2 B + 2 \left(\frac{2}{3}t\right)^{3/2} A \times B \\
 r' &= A + 2tB + 2 \left(\frac{2}{3}t\right)^{1/2} A \times B \\
 r'^2 &= 1 + 2t \left(\frac{1}{2}\right) + \frac{2t}{2} + (2t)^2 + 4 \left(\frac{2t}{3}\right) \frac{3}{4} = 1 + 4t + 4t^2 = (2t + 1)^2 \\
 L &= \int_0^T (2t + 1) dt = T^2 + T = 12 \implies (T + 4)(T - 3) = 0 \quad \boxed{T = 3}
 \end{aligned}$$

**Exercise 20.**

(1)

The circle rotates about its center and the circles  
rolls (moves along  $x$ -axis) without slipping, translating its origin by how far its circumference moved through.

$$\begin{aligned}
 \theta_0 &= -\pi/2 \\
 a(\cos \theta, -\sin \theta) &\quad (\text{clockwise rotation of pt. on circumference}) \\
 a(\cos(\theta - \theta_0), -\sin(\theta - \theta_0)) &= a(-\sin \theta, -\cos \theta) \\
 (\theta a, a) &\text{ is the position of the circle center.} \\
 (\theta a, a) + a(-\sin \theta, -\cos \theta) &= \boxed{a(\theta - \sin \theta, 1 - \cos \theta)} \\
 (2) \quad \frac{dy}{dx} &= \frac{\sin \theta}{1 - \cos \theta} = \frac{2 \sin \theta/2 \cos \theta/2}{2 \sin^2 \theta/2} = \boxed{\cot \theta/2} \\
 T &= \frac{(1, \cot \theta/2)}{\sqrt{1 + \cot^2 \theta/2}} = (\sin \theta/2, \cos \theta/2) \\
 T \cdot e_x &= \sin \theta/2 = \cos \phi = \cos \left( \frac{\pi}{2} - \frac{\theta}{2} \right) \\
 \phi &= \boxed{\frac{\pi - \theta}{2}}
 \end{aligned}$$

We'll show that the tangent line passes through the highest point at each place the marked point rotates through.

Marked point on the circle rotated through by  $\theta$ :  $P = a(\theta - \sin \theta, 1 - \cos \theta)$

Tangent line:  $P + sT = a(\theta - \sin \theta, 1 - \cos \theta) + s(\sin \theta/2, \cos \theta/2)$

Highest pt. on the circle:  $(a\theta, 2a)$

$$\text{equate the } y\text{-coordinates:} \quad 2a = a(1 - \cos \theta) + s \cos \theta/2 \implies s = 2a|\cos \theta/2|$$

$$\begin{aligned} a(\theta - \sin \theta) + s \sin \theta/2 &= a\theta - a \sin \theta + s \sin \theta/2 = \\ \text{equate the } x\text{-coordinates:} \quad &= a\theta - a \sin \theta + a2 \cos \theta/2 \sin \theta/2 = \\ &= \boxed{a\theta} \end{aligned}$$

So indeed, the tangent line will always intersect the highest point of the circle.

**Exercise 21.**

$$\begin{aligned} \frac{dY}{dt} &= \frac{dX}{du} \frac{du}{dt} & g(u) &= t \\ & & u &= u(t) \\ \int_c^d \|Y'(t)\| dt &= \int_c^d \|X'(u(t))\| \left| \frac{du(t)}{dt} \right| dt & \frac{du}{dt} &= 1/\frac{dg}{du}(u(t)) \end{aligned}$$

$g(u) = t$  is strictly increasing as it traces the arc, since  $c < t < d$ .

Then  $\frac{du}{dt} > 0$

$$\implies \int_c^d \|X'(u)\| \left( \frac{du}{dt} \right) dt = \int_{u(c)}^{u(d)} \|X'(u)\| du$$

**Exercise 22.** For  $t = \frac{1}{k}$ ,  $k = 2, \dots, 2n$

$$\|\pi(P)\| = \sum_{k=2}^{2n} \|r(t_k) - r(t_{k-1})\| + \|r(t=0) - r(t_{2n})\| + (\text{term to close up this polygon})$$

$$\|r(t=0) - r(t_{2n})\| = \left( \frac{1}{2n} \right) \sqrt{2}$$

$$r = (t, t \cos \left( \frac{\pi}{2t} \right))$$

$$r(t_k) - r(t_{k-1}) = \left( \frac{-1}{k(k-1)}, \frac{1}{k} \cos \left( \frac{k\pi}{2} \right) - \frac{1}{k-1} \sin \left( \frac{k\pi}{2} \right) \right)$$

$$\begin{aligned} \|r(t_k) - r(t_{k-1})\|^2 &= \left( \frac{1}{k(k-1)} \right)^2 + \frac{1}{k^2} \cos^2 \left( \frac{k\pi}{2} \right) + \frac{-2 \cos \left( \frac{k\pi}{2} \right) \sin \left( \frac{k\pi}{2} \right)}{k(k-1)} + \frac{1}{(k-1)^2} \sin^2 \left( \frac{k\pi}{2} \right) = \\ &= \left( \frac{1}{k(k-1)} \right)^2 + \frac{1}{k^2} \cos^2 \left( \frac{k\pi}{2} \right) + \frac{1}{(k-1)^2} \sin^2 \left( \frac{k\pi}{2} \right) \text{ since } k \in \mathbb{Z} \end{aligned}$$

$k = 2, \dots, 2n$

$$k = 2j \quad j = 1, \dots, n$$

$$\|r(t_k) - r(t_{k-1})\|^2 = \left( \frac{1}{k(k-1)} \right)^2 + \frac{1}{k^2}$$

$$\|r(t_k) - r(t_{k-1})\| = \frac{1}{k} \sqrt{1 + \frac{1}{(k-1)^2}} > \frac{1}{k}$$

$$k = 2j + 1, \quad j = 1, \dots, n-1$$

$$\|r(t_k) - r(t_{k-1})\|^2 = \left( \frac{1}{k(k-1)} \right)^2 + \left( \frac{1}{k-1} \right)^2 = \left( \frac{1}{k-1} \right)^2 \left( 1 + \frac{1}{k^2} \right)$$

$$\|r(t_k) - r(t_{k-1})\| = \frac{1}{k-1} \sqrt{1 + \frac{1}{k^2}} > \frac{1}{k-1} > \frac{1}{k}$$

These don't seem like very "tight" inequalities, but it'll work for the purposes of this problem.

$$\|\pi(P)\| = \sum_{k=2}^{2n} \|r(t_k) - r(t_{k-1})\| + \|r(t=0) - r(t_{2n})\| + (\text{term to close up this polygon}) >$$

$$> \sum_{k=2}^{2n} \frac{1}{k} + \frac{\sqrt{2}}{2n} + \|r(t=0) - r(t=1)\| = \sum_{k=2}^{2n} \frac{1}{k} + \frac{\sqrt{2}}{2n} + 1 > \boxed{\sum_{k=1}^{2n} \frac{1}{k}}$$

Now, by definition,  $\Lambda$ , the arclength, is the least upper bound on all possible polygonal approximations to the length of a curve.  $\sum_{k=2}^{2n} \frac{1}{k}$  is a series that is unbounded as  $n \rightarrow \infty$ . So  $\Lambda$  does not exist.

### 14.15 Exercises - Curvature of a curve.

#### Exercise 1.

(1)

$$\begin{aligned} r &= (3t - t^3, 3t^2, 3t + t^3) & a(t=2) &= 6(-2, 1, 2); & |a| &= 18 \\ v &= (3 - 3t^2, 6t, 3 + 3t^2) = 3(1 - t^2, 2t, 1 + t^2) & v(t=2) &= 3(-3, 4, 5); & |v| &= 15\sqrt{2} \\ a &= 3(-2t, 2, 2t) = 6(-t, 1, t) & a \cdot v &= 18(6 + 4 + 10) = 360 \end{aligned}$$

$$\kappa = \frac{\sqrt{18^2 15^2 2 - (18 \cdot 20)^2}}{15^3 \sqrt{2} \cdot 2} = \boxed{\frac{1}{75}}$$

(2)

$$\begin{aligned} r &= (c, s, e^t) & a(t=\pi) &= (1, 0, e^\pi) & |a| &= \sqrt{1 + e^{2\pi}} \\ v &= (-s, c, e^t) & v(t=\pi) &= (0, -1, e^\pi) & |v| &= \sqrt{1 + e^{2\pi}} \\ a &= (-c, -s, e^t) & a \cdot v &= e^{2\pi} \end{aligned}$$

$$\kappa = \frac{\sqrt{1 + 2e^{2\pi} + e^{4\pi} - e^{4\pi}}}{(1 + e^{2\pi})^{3/2}} = \boxed{\frac{\sqrt{1 + 2e^{2\pi}}}{(1 + e^{2\pi})^{3/2}}}$$

(3)

$$\begin{aligned} r &= (3tc, 3ts, 4t) & a(t=0) &= 6(0, 1, 0) & |a| &= 6 \\ v &= (3(c - ts), 3(s + tc), 4) & v(t=0) &= (3, 0, 4) & |v| &= 5 \\ a &= 6(-s, c, 0) + (-3tc, 3ts, 0) & a \cdot v &= 0 \end{aligned}$$

$$\boxed{\kappa = \frac{6}{25}}$$

(4)

$$\begin{aligned} r' &= (1 - c, s, 2 \cos t/2) & r'' \cdot r' &= s - cs + sc - 2 \cos t/2 \sin t/2 = 0 \\ r'' &= (s, c, -\sin t/2) & v &= \|(2, 0, 0)\| = 2 \\ & & a &= \|(0, -1, -1)\| = \sqrt{2} \end{aligned}$$

$$\boxed{\kappa = \frac{\sqrt{2}}{4}}$$

(5)

$$\begin{aligned} r &= (3t^2, 2t^3, 3t) & r'' \cdot r' &= 9(4t + 8t^3) \xrightarrow{t=1} 9(12) \\ r' &= (6t, 6t^2, 3) = 3(2t, 2t^2, 1) & a &= 3\sqrt{(4 + 16t^2)} \\ r'' &= 3(2, 4t, 0) & a(t=1) &= 6\sqrt{5} \\ & & v &= 3\sqrt{4t^2 + 4t^4 + 1}; \quad v(t=1) = 9 \end{aligned}$$

$$\frac{\|a \times v\|}{v^3} = \frac{\sqrt{(6\sqrt{5})^2 9^2 - (9(12))^2}}{27^2} = \boxed{\frac{2}{27}}$$

(6)

$$\begin{aligned} r(t) &= (t, \sin t, (1 - \cos t)) & r'' \cdot r' &= 0 \\ r'(t) &= (1, \cos t, \sin t) & a &= 1 \\ r'' &= (0, -s, c) & \kappa &= \frac{\|a \times v\|}{v^3} = \frac{\sqrt{2}}{2^{3/2}} = \boxed{\frac{1}{2}} \end{aligned}$$

**Exercise 2.**

$$\begin{aligned}
 r &= (ac(\omega t), as(\omega t), b\omega t) & r' \cdot r'' &= 0 \\
 r' &= \omega(-as(\omega t), ac(\omega t), b) & |r'| &= \omega\sqrt{a^2 + b^2} \\
 r'' &= \omega^2(-ac(\omega t), -as(\omega t), 0) & |r''| &= \omega^2 a \\
 \frac{\|r'' \times r'\|}{|r'|^3} &= \frac{a\sqrt{a^2 + b^2}}{(a^2 + b^2)^{3/2}} = \boxed{\frac{a}{a^2 + b^2}}
 \end{aligned}$$

**Exercise 3.** Given  $A, B$ ,  $A \cdot B = \cos \theta$ .  $A, B$  fixed.  $|A| = |B| = 1$ .

$$\begin{aligned}
 r' &= A \times r \\
 r'' &= A \times r' \implies r(0) = B, \quad \|r(t)\| = 1 \\
 r'' \times r' &= (A \times r') \times (A \times r) = (A \times (A \times R)) \times (A \times r) = \\
 &= (A \times r) \times ((A \times r) \times A) = (A \cdot (A \times r))(A \times r) - (A \times r)^2 A = -(A \times r)^2 A \\
 \frac{\|r'' \times r'\|}{\|A \times r\|^3} &= \frac{1}{\|A \times r\|} = \boxed{\frac{1}{\|B\| \sin \theta}}
 \end{aligned}$$

**Exercise 4.**

- (1)  $r(t)$  lies on a plane since a plane  $M = \{P + sA + tB\}$   
 $r(t) = 4(c, s, c) = 4c(1, 0, 1) + 4s(0, 1, 0) = 4cA + 4sB$   
 $r(t) \in M_1$  where  $M_1 = \{0 + sA + tB\} = \{sA + tB\}$

Normalize  $A, B \rightarrow A = \frac{(1, 0, 1)}{\sqrt{2}}; \quad B = (0, 1, 0)$   
 $X = 4\sqrt{2}cA + 4sB$

Recall an ellipse:  $\|X - F\| = ed(X, L) = e|(X - (F + dN)) \cdot N| = e|(X - F) \cdot N - d|$   
 Suppose  $\|F\| = f$  and suppose  $F = fA$

$$\begin{aligned}
 X - F &= (4\sqrt{2}c - f)A + 4sB \\
 \|X - F\|^2 &= 32c^2 - 8\sqrt{2}cf + f^2 + 16s^2 = 16 + 16c^2 - 8\sqrt{2}cf + f^2 \\
 e^2((X - F) \cdot N - d)^2 &= e^2(((X - F) \cdot N)^2 - 2((X - F) \cdot N)d + d^2) \\
 ((X - F) \cdot N)^2 &= 32c^2 - 8\sqrt{2}cf + f^2 \xrightarrow{e^2=1/2} 16c^2 - 4\sqrt{2}cf + \frac{1}{2}f^2 \\
 -2e^2((X - F) \cdot N)d &= -d(4\sqrt{2}c - f) \implies d = f \\
 e^2d^2 &= \frac{1}{2}f^2 \implies 16 + f^2 = 2f^2 \text{ or } f = 4
 \end{aligned}$$

$$\begin{aligned}
 \|X - F_1\| &= \|(4\sqrt{2}c - 4)A + 4sB\| = \sqrt{32c^2 - 32\sqrt{2}c + 16 + 16s^2} = \\
 \text{Indeed for} & \quad = \sqrt{16c^2 - 32\sqrt{2}c + 32} = 4(\sqrt{2} - c) \\
 \|X - F_2\| &= \|(4\sqrt{2}c + 4)A + 4sB\| = 4(\sqrt{2} + c) \\
 &\implies \|X - F_1\| + \|X - F_2\| = 8\sqrt{2}
 \end{aligned}$$

A Cartesian equation for the plane containing this ellipse would be  $\boxed{-x + z = 0}$ .

(2)

$$\begin{aligned}
 r &= 4(c, s, c) & r'^2 &= 16(1 + s^2) \\
 r' &= 4(-s, c, -s) & r''^2 &= 16(1 + c^2) \\
 r'' &= 4(-c, -s, -c) = -4(c, s, c) & r' \cdot r'' &= -16(-sc) = 16sc \\
 \frac{\|r'' \times r'\|}{|r'|^3} &= \frac{\sqrt{16^2(1 + c^2)(1 + s^2) - 16^2(sc)^2}}{|r'|^3} = \frac{\sqrt{2}}{4(1 + s^2)^{3/2}} \\
 \boxed{\frac{1}{\kappa} &= 2\sqrt{2}(1 + s^2)^{3/2}}
 \end{aligned}$$

**Exercise 5.**

$$\begin{aligned}
 r &= (e^t, e^{-t}, \sqrt{2}t) & r'' \times r' &= (\sqrt{2}e^{-t}, \sqrt{2}e^t, -1-1) = (\sqrt{2}e^{-t}, \sqrt{2}e^t, -2) \\
 r' &= (e^t, -e^{-t}, \sqrt{2}) & \|r'' \times r'\| &= \sqrt{2e^{-2t} + 2e^{2t} + 4} = \sqrt{2}(e^t + e^{-t}) \\
 r'' &= (e^t, e^{-t}, 0) & r'^2 &= e^{2t} + e^{-2t} + 2 = (e^t + e^{-t})^2
 \end{aligned}$$

$$\boxed{\frac{\|r'' \times r'\|}{|r'|^3} = \frac{\sqrt{2}}{e^t + e^{-t}}}$$

**Exercise 6.**

(1)

$$\begin{aligned}
 r(t) &= (x(t), y(t)) \\
 r' &= (x', y') \\
 r'' &= (x'', y'')
 \end{aligned}$$

$$\begin{aligned}
 \frac{\|r'' \times r'\|}{|r'|^3} &= \frac{\sqrt{(x''^2 + y''^2)(x'^2 + y'^2) - (x''x' + y''y')^2}}{(x'^2 + y'^2)^{3/2}} = \\
 &= \frac{\sqrt{x''^2x'^2 + x''^2y'^2 + y''^2x'^2 + y''^2y'^2 - (x''^2x'^2 + 2x''x'y'y' + y''^2y'^2)}}{(x'^2 + y'^2)^{3/2}} = \\
 &= \frac{\sqrt{(x''y' - y''x')^2}}{(x'^2 + y'^2)^{3/2}} = \boxed{\frac{|x''y' - y''x'|}{(x'^2 + y'^2)^{3/2}}}
 \end{aligned}$$

(2) For  $(x, y(x))$ , then we could treat the  $x$  as the parameter. Then  $x' = 1$  and  $x'' = 0$ . We can reuse our derived expression above:

$$\kappa = \frac{|x''y' - y''x'|}{(x'^2 + y'^2)^{3/2}} = \boxed{\frac{|y''|}{(1 + y'^2)^{3/2}}}$$

**Exercise 7.** A point moves so its velocity and acceleration vectors always have constant lengths.

$$\begin{aligned}
 X & & X' &= v_0 T \\
 X' & & |X'| &= v_0 & X'' &= v_0 T' = v_0^2 \kappa N \\
 X'' & & |X''| &= a_0 & \|X''\| &= v_0 \kappa = a_0
 \end{aligned}$$

since  $\kappa = \frac{\|T'\|}{v_0}$ , so then  $\boxed{\kappa = \frac{a_0}{v_0^2}}$

**Exercise 8.** If 2 plane curves with  $y = f(x)$ ,  $y = g(x)$  have the same tangent at a point  $(a, b)$  and the same curvature, prove  $|f''(a)| = |g''(a)|$ .

$$\begin{aligned}
 X_1 &= (x, f(x)) & X_2 &= (x, g(x)) \\
 X'_1 &= (1, f'(x)) & X'_2 &= (1, g'(x)) \\
 X''_1 &= (0, f''(x)) & X''_2 &= (0, g''(x))
 \end{aligned}$$

$$\begin{aligned}
 \frac{\|X''_1 \times X'_1\|}{(1 + f'^2)^{3/2}} &= \frac{\sqrt{|f''|^2(1 + f'^2) - (f''f')^2}}{(1 + f'^2)^{3/2}} = \\
 &= \frac{|f''|}{(1 + f'^2)^{3/2}}, \quad \text{likewise } \frac{\|X''_2 \times X'_2\|}{(1 + g'^2)^{3/2}} = \frac{|g''|}{(1 + g'^2)^{3/2}}
 \end{aligned}$$

For  $(a, b)$ , the 2 plane curves have the same tangent, so then  $f'(a) = g'(a)$ . So if the curvature is the same at  $(a, b)$ , then

$$\frac{|f''(a)|}{(1 + f'^2(a))^{3/2}} = \frac{|g''(a)|}{(1 + g'^2(a))^{3/2}} \implies \boxed{|g''(a)| = |f''(a)|}$$

**Exercise 9.**

(1) This problem could be done without reference to the same curvature or the same tangent line. We only need to use the fact that they intersect at the same point.

$$\begin{aligned}
 \text{Given } f &= ax(b-x), & (x+2)g &= x, \\
 & & (b+2)g &= b \\
 \text{If } x &= b, f = 0 & \text{If } b &\neq 2 \\
 & & g &= \frac{b}{b+2}
 \end{aligned}$$

Same as  $x = 0$ , so that  $f = 0$ ,  $g = 0$

So one point of intersection for any  $a, b$  is  $(0, 0)$ . Otherwise,

$$ax(b-x) = \frac{x}{x+2} \quad x \neq -2 \text{ (this is okay, plug } x = -2 \text{ into } (x+2)g = x \text{ and you'll get a contradiction)}$$

$$\implies x = \frac{-(2-b)}{2} \pm \frac{\sqrt{4+4b+b^2-4/a}}{2}$$

So we must have, for only one pt. of intersection,  $(0, 0)$ ,  $\boxed{b=2}$  and

$$\sqrt{4+4b+b^2-4/a} = \sqrt{4+8+4-4/a} = 0 \implies \boxed{a=1/4}$$

- (2) See sketch. The parabola and  $(x+2)g = x$  coincide at  $(0, 0)$ , have the same tangent line, and have the same direction at this intersection point.

**Exercise 10.** Remember that the radius of curvature is the reciprocal of the curvature.

- (1) Without loss of generality,  $y = 4cx^2$ . Then

$$\begin{aligned} y' &= 8cx \\ y'' &= 8c \end{aligned} \implies \kappa = \frac{|f''|}{(1+f'^2)^{3/2}}. \text{ Since } 1 > 0, \quad 64 + c^2x^2 > 0, \text{ and } |8c| \text{ constant, for } \kappa \text{ maximized, then } \boxed{x=0}.$$

- (2) Given 2 fixed unit vectors  $A, B$ ,  $A \cdot B = \cos \theta$ ,

$$\begin{aligned} r &= tA + t^2B & r'' \times r' &= 2B \times (A + 2tB) = 2B \times A \\ r' &= A + 2tB & |r'' \times r'| &= 2 \sin \theta \\ r'' &= 2B & r'^2 &= 1 + 4t \cos \theta + 4t^2 \\ (r'^2)' &= 8t + 4 \cos \theta = 0t & &= -\cos \theta/2 \quad (r'^2)'' = 8 > 0 \end{aligned}$$

So  $t = -\cos \theta/2$  minimizes  $r'^2$  and so maximizes  $\kappa$ . Then  $\boxed{r = -\frac{\cos \theta}{2}A + \frac{\cos^2 \theta}{4}B}$

**Exercise 11.**

- (1) We have for this plane curve,

$$\begin{aligned} r &= (x, y) \\ r' &= (x', y') \\ r'' &= (x'', y'') \end{aligned} \quad \text{Given } \sqrt{x'^2 + y'^2} = 5, \text{ and given } \begin{aligned} (x(0), y(0)) &= (0, 0) \\ (x'(0), y'(0)) &= (0, 5) \end{aligned}$$

$$\begin{aligned} a \times v &= e_z(x''y' - x'y'') \\ \kappa &= \frac{|a \times v|}{v^3} = \frac{|x''y' - x'y''|}{125} = 2t \end{aligned}$$

We want  $(x', y') \cdot e_x = 5 \cos \theta = x'$ . Then  $y' = 5 \sin \theta$ . Also, note that  $x'(0) = 0 \implies \theta_0 = \pi/2$

$$\implies 250t = |-5 \sin \theta \theta' (5 \sin \theta) - 5 \cos \theta 5 \cos \theta \theta'|$$

$$10t = |\theta'| \quad \boxed{\theta = 5t^2 + \pi/2}$$

- (2)  $v = 5(\cos(5t^2 + \pi/2), \sin(5t^2 + \pi/2))$

**Exercise 12.**

$$\text{Given that } \sqrt{x'^2 + y'^2} = 2, \quad \text{then } \begin{aligned} x' &= 2 \cos \theta \\ y' &= 2 \sin \theta \end{aligned} \quad \theta_0 = 0 \text{ since } v_0 = 2i$$

$$\kappa(t) = \frac{|x''y' - x'y''|}{v^3} = \frac{|-2 \sin \theta \theta' 2 \sin \theta - 2 \cos \theta 2 \cos \theta \theta'|}{2^3} = \frac{|\theta'|}{2} = 4t \quad \begin{aligned} \theta' &= 8t \\ \theta &= 4t^2 \end{aligned}$$

$$\boxed{v = \sqrt{2}(1, 1)}$$

## 14.19 Exercises - Velocity and acceleration in polar coordinates, Plane motion with radial acceleration, Cylindrical coordinates.

**Exercise 1.**

$$\begin{aligned}\mathbf{v} &= \frac{d\mathbf{r}}{dt} = \frac{dr}{dt}\vec{e}_r + r\frac{d\theta}{dt}\vec{e}_\theta = \vec{e}_r + r\vec{e}_\theta \quad (\theta = t) \\ \mathbf{a} &= \left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right)\vec{e}_r + \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)\vec{e}_\theta = -r\vec{e}_r + 2\vec{e}_\theta \\ \mathbf{v} &= \vec{e}_r + t\vec{e}_\theta = \cos t\vec{e}_x + \sin t\vec{e}_y + -t\sin t\vec{e}_x + t\cos t\vec{e}_y = \\ &= (\cos t - t\sin t)\vec{e}_x + (\sin t + t\cos t)\vec{e}_y \\ \mathbf{a} &= (-t\cos t - 2\sin t)\vec{e}_x + (-t\sin t + 2\cos t)\vec{e}_y\end{aligned}$$

**Exercise 2.**

$$\begin{aligned}\mathbf{v} &= \vec{e}_r + r\vec{e}_\theta + \vec{e}_z = (\cos t - t\sin t)\vec{e}_x + (\sin t + t\cos t)\vec{e}_y + \vec{e}_z \\ \mathbf{a} &= (-t\cos t - 2\sin t)\vec{e}_x + (t\sin t + 2\cos t)\vec{e}_y\end{aligned}$$

**Exercise 3.** (a).

$$\begin{aligned}r &= \sin t; \theta = t; z = \log \sec t; \theta \leq t < \frac{\pi}{2} \\ (r \cos \theta)^2 + (r \sin \theta - \frac{1}{2})^2 &= r^2 - r \sin \theta + \frac{1}{4} = \frac{1}{4}\end{aligned}$$

(b).

$$\begin{aligned}\mathbf{v} &= \frac{dt}{dt}\vec{e}_r + r\frac{d\theta}{dt}\vec{e}_\theta + \log \sec t\vec{e}_z = \cos t\vec{e}_r + r\vec{e}_\theta + \frac{\tan \theta \sec \theta}{\sec \theta}\vec{e}_z \\ v_z &= \tan \theta; v^2 = \cos^2 t + r^2 + \tan^2 \theta = \sec^2 \theta \\ \cos \phi &= \frac{\tan \theta}{\sec \theta} = \sin \theta = r = \sin t \\ \phi &= \arccos(\sin \theta)\end{aligned}$$

**Exercise 4.** Given  $r = f(\theta)$ ;  $a \leq \theta \leq b \leq a + 2\pi$

$$\begin{aligned}X &= ru_r & X'^2 &= r'^2 + r^2\theta'^2 \\ X' &= r'u_r + ru_\theta\theta' & |X'| &= \sqrt{r'^2 + r^2\theta'^2} \\ s &= \int \sqrt{r^2 + r'^2\theta'^2} dt \xrightarrow{t=\theta} s = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta\end{aligned}$$

**Exercise 5.** Given  $r = a(1 + \cos \theta)$

$$r' = -a \sin \theta$$

$$\text{so then } r^2 + r'^2 = a^2(1 + 2\cos \theta + \cos^2 \theta + \sin^2 \theta) = 4a^2 \cos^2 \theta / 2$$

$$\begin{aligned}s &= \int \sqrt{r^2 + r'^2} d\theta = 2a \int_0^{2\pi} \cos \theta / 2 d\theta = 2a \left( \int_0^\pi \cos \theta / 2 d\theta + - \int_\pi^{2\pi} \cos \theta / 2 d\theta \right) = \\ &= 4a \left( \sin \theta / 2 \Big|_0^\pi - \sin \theta / 2 \Big|_\pi^{2\pi} \right) = 4a(1 + 1) = \boxed{8a}\end{aligned}$$

**Exercise 6.**

$$A = \int R^2(\theta) d\theta = \int_0^{2\pi} \frac{1}{2} e^{2c\theta} d\theta = \frac{e^{2c\theta}}{4c} \Big|_0^{2\pi} = \frac{e^{4\pi c} - 1}{4c}$$

**Exercise 7.**

$$\int_0^\pi \frac{1}{2} \sin^4 \theta d\theta = \frac{1}{2} \int_0^\pi \sin^2 \theta (1 - \cos^2 \theta) d\theta = \frac{1}{2} \int_0^\pi \left( \left( \frac{1 - \cos 2\theta}{2} \right) - \left( \frac{1 - \cos 4\theta}{2(4)} \right) \right) d\theta = 3\pi/16$$

See sketches for exercises 8-11. **Exercise 8.** Given  $r = \theta$ ;  $0 \leq \theta \leq \pi$

$$\begin{aligned}s &= \int_0^\pi \sqrt{\theta^2 + 1} d\theta = \frac{1}{2} \left( \theta \sqrt{\theta^2 + 1} + \ln(\theta + \sqrt{\theta^2 + 1}) \right) \Big|_0^\pi = \\ &= \boxed{\frac{1}{2}(\pi \sqrt{\pi^2 + 1} + \ln(\pi + \sqrt{\pi^2 + 1}))}\end{aligned}$$



**Exercise 9.** Given  $r = e^\theta$ ;  $0 \leq \theta \leq \pi$

$$\frac{dr}{d\theta} = e^\theta$$

$$s = \int_0^\pi \sqrt{e^{2\theta} + e^{2\theta}} d\theta = \sqrt{2} \int_0^\pi e^\theta d\theta = \boxed{\sqrt{2}(e^\pi - 1)}$$

**Exercise 10.**  $r = 1 + \cos \theta$ ;  $\frac{dr}{d\theta} = -\sin \theta$ ;  $0 \leq \theta \leq \pi$

$$\int_0^\pi \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \int_0^\pi \sqrt{2} \sqrt{1 + \cos \theta} = \sqrt{2} \int_0^\pi \sqrt{2} \cos \theta/2 = 4 (\sin \theta/2)|_0^\pi = \boxed{4}$$

**Exercise 11.** Given  $r = 1 - \cos \theta$ ;  $0 \leq \theta \leq 2\pi$ , then  $\frac{dr}{d\theta} = \sin \theta$

$$\int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \int_0^{2\pi} \sqrt{2} \sqrt{1 - \cos \theta} = \sqrt{2} \int_0^{2\pi} \sqrt{2} |\sin \theta/2| = 4 - \cos \theta/2|_0^{2\pi} = \boxed{8}$$

**Exercise 12.** Given  $r = f(\theta)$ ,  $\rho = \text{radius of curvature} = \frac{1}{\kappa}$ ,  $r' = f'(\theta)$ ,  $r'' = f''(\theta)$

$$\begin{aligned} X &= ru_r \\ X' &= r'u_r + ru_\theta\theta' & u_r &= (\cos \theta, \sin \theta) \\ X'' &= r''u_r + 2r'u_\theta\theta' + -r\theta'u_r + r\theta''u_\theta = & u_\theta &= (-\sin \theta, \cos \theta) \\ &= (r'' - r\theta')u_r + \frac{1}{r}(r^2\theta')'u_{\theta} & u_r \times u_\theta &= e_z \\ X'' \times X' &= ((r'' - r\theta')u_r + \frac{1}{r}(r^2\theta')'u_\theta) \times (r'u_r + ru_\theta\theta') = \frac{1}{r}(r^2\theta')'r'(-e_z) + (r'' - r\theta')r\theta'e_z \\ \|X'' \times X'\| &= r''r\theta' - r^2\theta'^2 - 2r'\theta'^2 - rr'\theta'' \\ |X'|^2 &= r'^2 + r^2\theta'^2 \end{aligned}$$

If we let  $\theta = t$ , since the curve is described completely by  $r = f(\theta)$ , then

$$\kappa = \frac{|r^2 - r''r + 2r'^2|}{(r'^2 + r^2)^{3/2}} \implies \rho = \frac{(r'^2 + r^2)^{3/2}}{|r^2 - r''r + 2r'^2|}$$

**Exercise 13.** Using  $\rho = \frac{(r^2 + r'^2)^{3/2}}{|r^2 - rr'' + 2r'^2|}$

(1)

$$\begin{aligned} r &= \theta \\ r' &= 1 \\ r'' &= 0 \end{aligned} \quad \boxed{\rho = \frac{(\theta^2 + 1)^{3/2}}{\theta^2 + 2}}$$

(2)

$$\begin{aligned} r &= e^\theta \\ r' &= e^\theta \\ r'' &= e^\theta \end{aligned} \quad \rho = \frac{(e^{2\theta} + e^{2\theta})^{3/2}}{2e^{2\theta}} = \boxed{\sqrt{2}e^\theta}$$

(3)  $\theta = \pi/4$

$$\begin{aligned} r &= 1 + \cos \theta \\ r' &= -\sin \theta \\ r'' &= -\cos \theta \end{aligned} \quad \rho = \frac{(1 + 2c + c^2 + s^2)^{3/2}}{|1 + 2c + c^2 + (c + c^2) + 2s^2|} = \frac{(2(1 + c))^{3/2}}{3(1 + c)} = \boxed{\frac{2\sqrt{2 + \sqrt{2}}}{3}}$$

(4)  $\theta = \pi/2$

$$\begin{aligned} r &= 1 - \cos \theta \\ r' &= \sin \theta \\ r'' &= \cos \theta \end{aligned} \quad \rho = \frac{(2(1 - c))^{3/2}}{(3 + 2c)(1 - c)} = \frac{2\sqrt{2}(1 - c)^{1/2}}{3 + 2c} \xrightarrow{\pi/2} \boxed{\frac{2\sqrt{2}}{3}}$$

**Exercise 14.**

$$\begin{aligned} X &= ru_r \\ X' &= r'u_r + ru_\theta\theta' \quad \implies X \cdot X' = rr' = rv \cos \phi \text{ or } \boxed{r' = v \cos \phi} \\ X' \cdot X' &= r'^2 + r^2\theta'^2 = v^2 \end{aligned}$$

$$v^2 = v^2 \cos^2 \phi + r^2\theta'^2$$

Use  $\theta$  as the parameter for  $t$ , since the curve is invariant.  $\implies \boxed{r = v \sin \phi}$

**Exercise 15.** Place target at the center (without loss of generality). The strategy is to break up  $\mathbf{v}$  into the polar coordinate unit vectors.

$$\begin{aligned} \mathbf{r} &= r\vec{e}_r \\ \mathbf{v} &= \frac{dr}{dt}\vec{e}_r + r\frac{d\theta}{dt}\vec{e}_\theta \\ \frac{dr}{dt} &= v_r = v \cos(\pi - \alpha) = -v \cos \alpha \\ r\frac{d\theta}{dt} &= v \sin \alpha \\ \frac{v \sin \alpha}{-v \cos \alpha} &= -\tan \alpha = \frac{r \frac{d\theta}{dt}}{\frac{dr}{dt}} = r \frac{d\theta}{dr}; \quad \frac{1}{r} \frac{dr}{d\theta} = -\tan \alpha \\ r &= e^{-\tan \alpha \theta} \end{aligned}$$

**Exercise 16.** Admittedly, I had *looked* in the back of the book.

The **big trick** is that **the choice of origin is important!**. The clever choice of origin will cut out the algebra in dealing with Cartesian coordinates. The clever choice of origin will illuminate the problem entirely. So try flipping around, reflecting, and turning around the problem until the right choice of origin is selected.

Let the positive axis be from where the missile, the target missile,  $m$ , is sighted to the ground crew. **So the origin is at the target missile, not the ground crew** from which the anti-missile missile,  $a$ , is launched! It'll make the algebra much easier.

Let  $a$  go 3 miles in the **negative**  $x$  direction, to allow for the possibility that the missile is returning to the ground crew, that is 4 mi. in the positive  $x$  direction from the origin, at the time of sighting. Let  $t = 0$  be the time that missile  $a$  finishes going these 3 miles and is 1 mile in the positive  $x$  direction from the origin.

Let

$$\begin{aligned} (x_m, y_m) &= 1(t+1)(\cos \theta_{m0}, \sin \theta_{m0}) = (t+1)u_{m0} = X_m \\ (x'_m, y'_m) &= u_{m0} \\ (x_a, y_a) &= X_a = r_a u_a \\ (x'_a, y'_a) &= X'_a = r'_a u_a + r_a \theta' u_\theta \\ |X'_a| &= \sqrt{r_a'^2 + (r_a \theta')^2} = 3 \\ X_a(t=0) &= X_{a0} = (1, 0) = r_a(0)(\cos \theta(0), \sin \theta(0)) \\ \theta(0) &= 0, \quad r_a(0) = 1 \end{aligned}$$

At some point  $t_2$ ,  $(x_m, y_m) = (x_a, y_a)$ , i.e.  $X_m(t_2) = X_a(t_2)$ . Note that **we only need one**  $t_2$ ;  $\theta_{m0}$  constant, but unknown.

$$\begin{aligned} (t_2 + 1) \cos \theta_{m0} &= r_a \cos \theta \\ (t_2 + 1) \sin \theta_{m0} &= r_a \sin \theta = \\ (t_2 + 1)^2 &= r_a^2; \quad (t_2 + 1) = r_a(t_2); \quad r'_a(t_2) = ? \quad \text{well} \\ (r'_a(t_2))^2 &+ ((t_2 + 1) \frac{d\theta}{dt}(t_2))^2 = 9 \end{aligned}$$

Try  $1 = r'_a(t_2)$ .

$$\begin{aligned} 1 + (t_2 + 1)^2 \theta'^2 &= 9 \\ (t_2 + 1) \left( \frac{d\theta}{dt} \right) &= 2\sqrt{2} \implies d\theta = \frac{2\sqrt{2}}{t_2 + 1} dt \\ \implies \boxed{\theta = 2\sqrt{2} \ln(t_2 + 1)} \quad t_2 &= e \left( \frac{\theta}{2\sqrt{2}} \right) - 1 \\ \boxed{r} &= e \left( \frac{\theta}{2\sqrt{2}} \right) \end{aligned}$$

**Exercise 17.**

A first order differential equation of the form  $y' = f(x, y)$  is homogeneous if  $f(tx, ty) = f(x, y)$ . Then

$$f(r \cos \theta, r \sin \theta) = f(\cos \theta, \sin \theta) = f(\theta)$$

We find that

$$\begin{aligned}\frac{dy}{d\theta} &= \frac{dr}{d\theta} \sin \theta + r \cos \theta \\ \frac{dx}{d\theta} &= \frac{dr}{d\theta} \cos \theta - r \sin \theta\end{aligned}$$

Thus

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta} \sin \theta + r \cos \theta}{\frac{dr}{d\theta} \cos \theta - r \sin \theta} = f(\theta)$$

**Exercise 18.**

$$\begin{aligned}\mathbf{v} &= \omega \mathbf{k} \times \mathbf{r} \\ \mathbf{v} &= \frac{dr}{dt} \vec{e}_r + r \frac{d\theta}{dt} \vec{e}_\theta \\ \mathbf{v} \cdot \vec{e}_r &= 0, \text{ so } \frac{d\theta}{dt} = 0; \omega \mathbf{k} \times \mathbf{r} = r \frac{d\theta}{dt} \vec{e}_\theta = \omega r \vec{e}_\theta = r \frac{d\theta}{dt} \\ |\omega \mathbf{k} \times \mathbf{r}|^2 &= \omega^2 r^2 = r^2 \left( \frac{d\theta}{dt} \right)^2 \\ \omega &= \left| \frac{d\theta}{dt} \right|, \omega > 0\end{aligned}$$

**Exercise 19. (a)**

$$\begin{aligned}\mathbf{v} &= \frac{dr}{dt} \vec{e}_r + r \frac{d\theta}{dt} \vec{e}_\theta = r \frac{d\theta}{dt} \vec{e}_\theta; \frac{dr}{dt} = 0; \vec{e}_\theta = \vec{e}_z \times \vec{e}_r \\ \mathbf{v} &= r \frac{d\theta}{dt} \vec{e}_z \times \vec{e}_r = \frac{d\theta}{dt} \vec{e}_z \times r \vec{e}_r = \omega \times \mathbf{r}\end{aligned}$$

(b).

$$\begin{aligned}a = v' &= \omega' \times r + \omega \times r' = \alpha \times r + \omega \times r' \\ \omega \times r' &= \omega \times (\omega \times r) = (\omega \cdot r) \omega - \omega^2 r\end{aligned}$$

(c).

$$\text{Now } \omega \cdot r = 0 \implies a = -\omega^2 r$$

**Exercise 20.**

The distance  $|\mathbf{r}_p(t) - \mathbf{r}_q(t)|$  is independent of  $t$ , so  $\frac{d}{dt}|\mathbf{r}_p(t) - \mathbf{r}_q(t)| = 0$ , which implies

$$\begin{aligned}\frac{d}{dt}(r_p^2 - 2r_p \cdot r_q + r_q^2) &= 2r_p \cdot \frac{dr_p}{dt} - 2\frac{dr_p}{dt} \cdot r_q - 2r_p \cdot \frac{dr_q}{dt} + 2\frac{dr_q}{dt} \cdot r_q = 0 \\ \frac{dr_p}{dt} \cdot (r_q - r_p) &= -\frac{dr_q}{dt} \cdot (r_p - r_q)\end{aligned}$$

Suppose

$$\begin{aligned}v_p &= \omega_p \times r_p \\ v_q &= \omega_q \times r_q\end{aligned}$$

Then

$$\begin{aligned}v_p \cdot (r_q - r_p) &= -v_q \cdot (r_p - r_q) \\ \omega_p \times r_p \cdot r_q &= -\omega_q \times r_q \cdot r_p = \omega_q \times r_p \cdot r_q \\ ((\omega_p - \omega_q) \times r_p) \cdot r_q &= 0\end{aligned}$$

Thus,  $\omega_p = \omega_q$ .

### 14.21 Miscellaneous review Exercises - Applications to planetary motion.

**Exercise 1.** Use polar coordinates if Cartesian coordinates doesn't help!

$$\begin{aligned}
 y^2 &= x & r^2 \sin^2 \theta &= r \cos \theta \\
 X &= (r \cos \theta, r \sin \theta) = r u_r & r &= \frac{\cos \theta}{\sin^2 \theta} \\
 X' &= r' u_r + r \theta' u_\theta & x &= r \cos \theta \\
 |X'| &= \sqrt{r'^2 + (r \theta')^2} & y &= r \sin \theta \implies r' s^2 + 2r^2 s c \theta' = -s \theta' \\
 T &= \frac{X'}{|X'|} = \frac{r' u_r + r \theta' u_\theta}{\sqrt{r'^2 + (r \theta')^2}} & r' &= -\left(\frac{1+c^2}{s^3}\right) \theta'
 \end{aligned}$$

$$r \cdot T = \frac{r r'}{\sqrt{r'^2 + (r \theta')^2}} = r \cos \alpha \implies \frac{r'}{\sqrt{r'^2 + (r \theta')^2}} = \cos \alpha$$

$$\begin{aligned}
 X \cdot e_r &= r \cos \theta \\
 r'^2 + (r \theta')^2 &= \left(\frac{1+2c^2+c^4}{s^6}\right) \theta'^2 + \frac{c^2}{s^4} \theta'^2 = \frac{1+3c^2}{s^4} \\
 \cos \alpha &= \frac{(1+c^2)/s^3}{((1+3c^2)/s^6)^{1/2}} = \frac{1+c^2}{\sqrt{1+3c^2}} \\
 \sin^2 \alpha &= 1 - \cos^2 \alpha = \frac{1+3c^2 - (1+2c^2+c^4)}{(1+3c^2)} = \frac{c^2 s^2}{(1+3c^2)} \\
 \frac{\sin \alpha}{\cos \alpha} &= \tan \alpha = \frac{cs/\sqrt{1+3c^2}}{(1+c^2)/\sqrt{1+3c^2}} = \boxed{\frac{\tan \theta}{1 + \sec^2 \theta}}
 \end{aligned}$$

**Exercise 2.**

$$\begin{aligned}
 X &= \left(\frac{y^2}{4c}, y\right) & \frac{X'}{|X'|} &= \frac{\left(\frac{y}{2c}, 1\right)}{\sqrt{1+(y/2c)^2}} = \frac{1}{\sqrt{1+(y/2c)^2}} T \\
 X' &= y' \left(\frac{y}{2c}, 1\right) & N &= (2c, -y) \implies T \cdot N = 0
 \end{aligned}$$

**Exercise 3.**

$$\begin{aligned}
 (x, y) &= \left(\frac{y^2}{4c}, y\right) \\
 X' &= \left(\frac{y}{2c}, 1\right) y' & \frac{X'}{|X'|} &= \frac{\left(\frac{y}{2c}, 1\right)}{\sqrt{1+(y/2c)^2}}; \quad m = 1/(y/2c) = \frac{2c}{y} \\
 |X'| &= y' \sqrt{1 + \left(\frac{y}{2c}\right)^2} \\
 m &= m(x_0) = \frac{2c}{y_0}
 \end{aligned}$$

$$y = mx + b \implies y_0 = \frac{2c}{y_0} \frac{y_0^2}{4c} + b = \frac{y_0}{2} + b \text{ so that } b = \frac{y_0}{2} = c/m$$

$$\implies \boxed{y = mx + c/m} \quad \boxed{\begin{aligned} y_0 &= \frac{2c}{m} \\ x_0 &= \frac{4c^2}{m^2} \frac{1}{4c} = \frac{c}{m^2} \end{aligned}}$$

**Exercise 4.**

(1)

$$\begin{aligned}
X &= \left( \frac{(y-y_0)^2}{4c} + x_0, y \right) \\
(y-y_0)^2 &= 4c(x-x_0) \\
\frac{(y-y_0)^2}{4c} + x_0 &= x \\
\Rightarrow T &= \frac{X'}{|X'|} = \frac{y' \left( \frac{(y-y_0)}{2c}, 1 \right)}{|y'| \sqrt{1 + \left( \frac{(y-y_0)}{2c} \right)^2}} \\
m &= \frac{2c}{y-y_0} \\
m(y_1) &= \frac{2c}{y_1-y_0} \\
y &= m(y_1)(x-x_0) + b + y_0 \\
\Rightarrow y_1 &= \left( \frac{2c}{y_1-y_0} \right) \left( \frac{(y_1-y_0)^2}{4c} \right) + b + y_0 \\
y_1 - y_0 &= \frac{y_1 - y_0}{2} + b \\
(y-y_0) &= m(x-x_0) + \frac{c}{m}
\end{aligned}$$

(2)

$$\begin{aligned}
(x-x_0)^2 &= 4c(y-y_0) \\
X &= \left( x, \frac{(x-x_0)^2}{4c} + y_0 \right) \\
T &= \frac{\left( 1, \frac{x-x_0}{2c} \right)}{\sqrt{1 + \frac{(x-x_0)^2}{4c^2}}} \\
m &= \frac{x-x_0}{2c} = m(x) \\
y-y_0 &= m(x-x_0) + b \\
y_1 - y_0 &= \frac{(x_1-x_0)^2}{4c} = \left( \frac{(x_1-x_0)^2}{2c} \right) + b
\end{aligned}$$

$$\begin{aligned}
b &= \frac{-(x_1-x_0)^2}{4c} \\
\Rightarrow y-y_0 &= m(x-x_0) + -m^2c
\end{aligned}$$

$$\begin{aligned}
x^2 &= 4cy \\
X &= (x, x^2/4c) \\
T &= \frac{\left( 1, \frac{x}{2c} \right) x'}{|x'| \sqrt{1 + (x/2c)^2}} \\
m &= x/2c \\
y &= mx + b \\
y_1 &= \frac{x_1^2}{2c} + b = \frac{x_1^2}{4c} = \frac{x_1^2}{2c} + b \\
b &= \frac{-x_1^2}{4c} \\
\Rightarrow y &= mx + cm^2
\end{aligned}$$

**Exercise 5.** Given  $X = \left( \frac{y^2}{4c}, y \right)$   $\Rightarrow m = \frac{2c}{y}$   
 $X' = y' \left( \frac{y}{2c}, 1 \right)$

$$\begin{aligned}
y_1 &= \frac{2c}{y_1} x_1 + b = \frac{2c}{y_1} \frac{y_1^2}{4c} = \frac{y_1}{2} + b \Rightarrow b = \frac{y_1}{2} \\
y &= \frac{2c}{y_1} x + \frac{y_1}{2}; \Rightarrow y_1 y = 2cx + 2x_1 c = 2c(x + x_1)
\end{aligned}$$

**Exercise 6.**  $(y - y_0)^2 = 4c(x - x_0) \implies m(y) = \frac{2c}{y - y_0}$ .

$$y - y_0 = \left( \frac{2c}{y_1 - y_0} \right) (x - x_0) + b$$

$$b = (y_1 - y_0) - \frac{2c(y_1 - y_0)}{4c} = \frac{c}{m}$$

$$y - y_0 = \left( \frac{2c}{y_1 - y_0} \right) (x - x_0) + \frac{(y_1 - y_0)}{2}$$

$$(y_1 - y_0)(y - y_0) = 2c(x - x_0) + \frac{(y_1 - y_0)^2}{2} = \boxed{2c((x - x_0) + (x_1 - x_0))}$$

For  $X = (x, \frac{x^2}{4c}) \implies m(x) = \frac{x}{2c}$

$$y = \frac{x_1}{2c}x + b \implies b = \frac{x_1^2}{-4c} = -cm^2$$

$$y = \frac{x_1 x}{2c} - y_1 \implies \boxed{2c(y + y_1) = x_1 x}$$

For  $X = (x, \frac{(x - x_0)^2}{4c} + y_0) \implies m = \frac{x_1 - x_0}{2c}$

$$y - y_0 = \left( \frac{x_1 - x_0}{2c} \right) (x - x_0) + b$$

$$y - y_0 = \left( \frac{x_1 - x_0}{2c} \right) (x - x_0) + -\frac{(x_1 - x_0)^2}{4c}$$

$$\frac{(x_1 - x_0)^2}{4c} - \frac{(x_1 - x_0)^2}{2c} = \frac{-(x_1 - x_0)^2}{4c} = b$$

$$y - y_0 = \frac{(x_1 - x_0)(x - x_0)}{2c} - (y_1 - y_0)$$

$$\boxed{(y - y_0) + (y_1 - y_0) = \frac{(x_1 - x_0)(x - x_0)}{2c}}$$

**Exercise 7.**

(1) Given  $y = x^2$ , then

$$\begin{aligned} X &= (x, x^2) \\ \frac{X'}{\|X'\|} &= \frac{(1, 2x)x'}{|x'|\sqrt{1+4x^2}} \\ N &= \frac{(-2x, 1)x'}{|x'|\sqrt{1+4x^2}} \end{aligned} \implies \begin{aligned} x &= \frac{2xt}{\sqrt{1+4x^2}} \\ t &= \sqrt{\frac{1}{4} + x^2} \end{aligned} \quad \begin{aligned} tN + (x, y) &= (x, x^2) + t \frac{(-2x, 1)x'}{|x'|\sqrt{1+4x^2}} = (0, y_q) = Q \\ y &= \frac{t}{\sqrt{1+4x^2}} + x^2 = \frac{\sqrt{1+4x^2}}{2\sqrt{1+4x^2}} + x^2 \\ &= \frac{1}{2} + x^2 \end{aligned}$$

(2)

**Exercise 8.** Given  $y = x^2$

$\pm\sqrt{c} = x$  when  $y = c$

Circle condition:  $\|C - R\| = \|(0, y_0) - (\pm\sqrt{c}, c)\| = \sqrt{c + (y_0 - c)^2} = y_0$

$$\implies \boxed{\frac{1+c}{2} = y_0}$$

$c \rightarrow 0$ , then  $\boxed{y_0 = 1/2}$ , radius of circle.

**Exercise 9.** Consider the condition,  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ , pts. on an ellipse.

$$\begin{aligned} b^2 x^2 + a^2 y^2 &\geq a^2 b^2 \\ \frac{x^2}{a^2} + \frac{y^2}{b^2} &\geq 1 \iff r^2(b^2 \cos^2 \theta + a^2(1 - \cos^2 \theta)) = r^2(a^2 + (b^2 - a^2) \cos^2 \theta) = \\ &= r^2 a^2 (1 - e^2 \cos^2 \theta) \\ &\iff r^2(1 - e^2 \cos^2 \theta) \geq b^2 \end{aligned}$$

Suppose for each fixed  $\theta_1$ , we consider all pts. along a line ray from the origin (which is inside the ellipse) to infinity, radially out. Then  $r^2(1 - e^2 \cos^2 \theta_1) \geq b^2 \quad \forall r \geq r_1$

**Exercise 10.** For  $X = (x, y)$  and given that  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ , or  $b^2x^2 + a^2y^2 = a^2b^2$ , so then

$$\begin{aligned} 2b^2xx' + 2a^2yy' &= 0 \implies y' = \frac{-b^2xx'}{a^2y} \\ X = (x, y) &\implies X' = (x', y') = x'(1, \frac{-b^2x}{a^2y}) = \frac{-x'b^2}{y} \left( \frac{-y}{b^2}, \frac{x}{a^2} \right) \\ T = \left( \frac{-y}{b^2}, \frac{x}{a^2} \right) &\parallel X' \quad N = \left( \frac{x}{a^2}, \frac{y}{b^2} \right) \perp X' \text{ since } N \cdot X' = 0 \\ m &= \frac{x_0/a^2}{-y_0/b^2} = \frac{b^2x_0}{-a^2y_0} \\ (y - y_0) = m(x - x_0) &\implies y = \frac{b^2x_0}{-a^2y_0}x + \frac{b^2x_0^2}{a^2y_0} + \frac{a^2y_0}{a^2y_0}y_0 = \frac{b^2x_0}{-a^2y_0}x + \frac{b^2}{y_0} \\ \frac{x_0x}{a^2} + \frac{yy_0}{b^2} &= 1 \implies \boxed{\frac{x \cos \theta_0}{a} + \frac{y \sin \theta_0}{b} = 1} \end{aligned}$$

**Exercise 11.** Using the work of the previous exercise, Exercise 10, then

$$\frac{x \cos \theta_0}{a} + \frac{y \sin \theta_0}{b} = 1 = \frac{xa \cos \theta_0}{a^2} + \frac{yb \sin \theta_0}{b^2} = xx_0/a^2 + yy_0/b^2$$

**Exercise 12.** Use  $T = (\frac{-y}{b^2}, \frac{x}{a^2})$ ;  $N = (\frac{x}{a^2}, \frac{y}{b^2})$  from the previous problems.

Without loss of generality, center the ellipse on the origin and the major axis on the  $x$  axis. Then  $F = ae e_x$  and  $P = (x, y)$ .

$$\left| \frac{(F - P) \cdot N}{|N|} \right| = \frac{1 - \frac{ex}{a}}{\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4}}} \quad \left| \frac{(-F - P) \cdot T}{|T|} \right| = \frac{1 + \frac{xe}{a}}{\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4}}}$$

Multiply the above distances together to get

$$\left( \frac{x^2}{a^4} + \frac{y^2}{b^4} \right)^{-1/2} \left( 1 + \frac{xe}{a} \right) \left( 1 - \frac{ex}{a} \right) = \left( \frac{x^2}{a^4} + \frac{y^2}{b^4} \right)^{-1/2} \left( \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{e^2}{a^2} x^2 \right) = \boxed{b^2}$$

**Exercise 13.**  $x^2 + 4y^2 = 8$  or  $\frac{x^2}{8} + \frac{y^2}{2} = 1$ . We want to find the tangent lines parallel to  $x + 2y = 7$  or  $y = \frac{7}{2} - \frac{x}{2}$ . So the slope of this line is  $-\frac{1}{2}$ . Using  $T = (\frac{-y}{b^2}, \frac{x}{a^2}) = (\frac{-y}{2}, \frac{x}{8})$ , then

$$\frac{x/8}{-y/2} = \frac{x}{-4y} = -1/2 \text{ or } x = 2y$$

plugging back into the ellipse equation,  $(2, 1), (-2, -1)$

**Exercise 14.** By equidistant property of the circle, the focus length from origin,  $ae$ , must equal the minor axis length,  $b$ .

$$ae = b = a\sqrt{1 - e^2} \implies \boxed{e = 1/\sqrt{2}}$$

**Exercise 15.**

$$(1) \quad \boxed{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}. \text{ Recall that } b^2 = a^2(1 - e^2) = -3a^2, \text{ so that } \boxed{\frac{1}{a^2}(x^2 - \frac{y^2}{3}) = 1}.$$

(2)

$$\boxed{A = \int_a^x b \sqrt{\left(\frac{t}{a}\right)^2 - 1} dt - \frac{1}{2}(x - a)b \sqrt{\left(\frac{x}{a}\right)^2 - 1}}$$

$r$  is the length of  $VP$ :

**Exercise 16.** Given  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ , then  $y = b\sqrt{\left(\frac{x}{a}\right)^2 - 1}$

$$X' = \left( 1, \frac{b(\frac{x}{a^2})}{\sqrt{(\frac{x}{a})^2 - 1}} \right) = \left( 1, \frac{b^2(\frac{x}{a^2})}{y} \right) = \frac{b^2}{y} \left( \frac{y}{b^2}, \frac{x}{a^2} \right)$$

So  $X' \parallel T$ ; and  $X' \cdot N = 0$

**Exercise 17.**  $m(x_0) = \frac{b^2x_0}{a^2y_0}$  from the tangent vector above and so, using the ellipse equation,

$$(y - y_0) = \frac{b^2}{a^2} \frac{x_0}{y_0} (x - x_0) \implies \frac{x_0x}{a^2} - \frac{y_0y}{b^2} = 1$$

**Exercise 18.** Given  $X = (x, y)$ ,

$$\begin{aligned} X' &= (x', y'); \quad N \parallel (y', -x') \\ (x, y) + s(y', -x') &= (x_1, 0) \implies \begin{aligned} y &= sx' \\ x + sy' &= x_1 \end{aligned} \implies x_1 = x + yy'/x' \end{aligned}$$

Isosceles condition:  $\sqrt{x^2 + y^2} = \|(x_1, 0) - (x, y)\| = \sqrt{\left(\frac{yy'}{x'}\right)^2 + y^2}$

$$\frac{x}{y} = \frac{dy/dt}{dx/dt} = \frac{dy}{dx} \implies \frac{y^2}{2C} - \frac{x^2}{2C} = 1$$

Note there's some ambiguity with the signs.

**Exercise 19.**  $N \parallel (y', -x')$

$$\begin{aligned} (x, y) + s(y', -x') &= (x_1, 0) & (x, y) + t(y', -x') &= (0, y_1) \\ y &= sx' & x &= -ty' \\ x + sy' &= x_1 = x + \frac{yy'}{x'} & y - tx' &= y_1 \\ & & y + \frac{xx'}{y'} &= y_1 \\ \|(x, y) - (x_1, 0)\| &= \|(x, y) - (0, y_1)\| \\ \implies (x - x_1)^2 + y^2 &= x^2 + (y - y_1)^2 \implies \left(\frac{yy'}{x'}\right)^2 + y^2 = x^2 + \left(\frac{xx'}{y'}\right)^2 \implies \left(\frac{dy}{dx}\right)^2 = \frac{x^2}{y^2} \\ &\xrightarrow{(4,5)} y^2 - x^2 = 9 \end{aligned}$$

**Exercise 20.** Without loss of generality, consider a hyperbola given by  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$  with asymptotes of  $\frac{y}{x} = \pm \frac{b}{a}$ . The distances from a point on the hyperbola to the asymptotes are the following:

$$\begin{aligned} \frac{(X - P) \cdot N_1}{|N_1|} &= \frac{(x, y) \cdot (-b, a)}{\sqrt{a^2 + b^2}} = \frac{-bx + ay}{\sqrt{a^2 + b^2}} \\ \frac{(X - P) \cdot N_2}{|N_2|} &= \frac{(x, y) \cdot (b, a)}{\sqrt{a^2 + b^2}} = \frac{bx + ay}{\sqrt{a^2 + b^2}} \\ \left(\frac{|-bx + ay|}{\sqrt{a^2 + b^2}}\right) \left(\frac{bx + ay}{\sqrt{a^2 + b^2}}\right) &= \frac{b^2x^2 - a^2y^2}{a^2 + b^2} = \frac{a^2b^2}{a^2 + b^2} \end{aligned}$$

**Exercise 21.** Recall that for  $X = ru_r$  (useful trick to change into polar coordinates),

$$\begin{aligned} X' &= r'u_r + ru_\theta\theta' \\ |X'| &= \sqrt{r'^2 + (r\theta')^2} \implies \int \sqrt{r'^2 + (r\theta')^2} dt \\ (1) \int_{\theta_0}^{\theta} \sqrt{r'^2 + r^2} d\theta &= k(\theta - \theta_0) \\ \sqrt{r'^2 + r^2} - C &= k \\ r'^2 &= k - r^2 \implies \left(\frac{r'}{\sqrt{k}}\right)^2 + \left(\frac{r}{\sqrt{k}}\right)^2 = 1 \\ &\implies \boxed{r = \sqrt{k} \sin \theta \text{ or } r = \sqrt{k}} \\ (2) \int_{\theta_0}^{\theta} \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2} d\theta &= k(r(\theta) - r(\theta_0)) \end{aligned}$$

$$\begin{aligned} r'^2 + r^2 &= (kr')^2 \\ \implies r'^2 &= \frac{r^2}{k^2 - 1} \quad \ln \frac{r}{r_0} = \frac{\theta - \theta_0}{\sqrt{k^2 - 1}} \\ &\implies \boxed{r = Ce\left(\frac{\theta}{\sqrt{k^2 - 1}}\right)} \end{aligned}$$



$$(3) \int_{\theta_0}^{\theta} \sqrt{r'^2 + r^2} d\theta = k \int_{\theta_0}^{\theta} \frac{1}{2} r^2 d\theta$$

$$\begin{array}{lll} r'^2 + r^2 = \frac{k^2}{4} r^4 & \sec t = \frac{kr}{2} & \int \frac{\frac{2}{k} \tan t \sec t}{\frac{2}{k} |\sec t| \sqrt{\sec^2 t - 1}} = \theta - \theta_0 \\ \Rightarrow \frac{r'}{|r| \sqrt{\left(\frac{kr}{2}\right)^2 - 1}} = 1 & \tan t \sec t = \frac{k}{2} \frac{dr}{dt} & \operatorname{arcsec}\left(\frac{kr}{2}\right) = \theta - C \\ & & \boxed{r = \frac{2}{k} \sec(\theta - \theta_0)} \text{ or } r = \frac{2}{k} \end{array}$$

**Exercise 22.** Given  $r'(t) = (r(a) \times r(b))$ ,  $r'(t) = \frac{r(b) - r(a)}{b - a}$  for some  $t \in [a, b]$  (by mean-value thm. on each of the components).

$$r'(t) \cdot (r(a) \times r(b)) = 0$$

Geometrically it means that there must be some component of  $r'(t)$  that belongs in the plane containing  $r(b) - r(a)$  because the velocity vector  $r'(t)$  must “take”  $r(a)$  to  $r(b)$ .