Fast Integral Transforms

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Mar. 2017

This library computes common integral transforms with product kernels using the FFTLog algorithm [Tal78, Ham00]

$$G(y) = \int_0^\infty F(x)K(xy)\,\frac{\mathrm{d}x}{x} \tag{1}$$

or equivalently

$$g(y) = \int_0^\infty f(x)(xy)^q K(xy)\,\frac{\mathrm{d}x}{x} \tag{2}$$

where $K(xy)$ is the kernel function and q is a power-law tilt parameter. And with $f(x) \propto x^{-q}$, $g(y) \propto y^q$, one is free to choose q. We can tilt $f(x)$ to balance f at large and small x, in order to improve accuracy and reduce ringing.

The idea is to take advantage of the convolution theorem in $\ln x$ and $\ln y$. We approximate $f(x)$ with truncated Fourier series, and use the exact Fourier transform of the possibly oscillatory kernel. We can calculate the latter analytically via Mellin transform. This algorithm performs the best if f can be accurately approximated by a sum of finite Fourier modes and logarithmically uniform discretization, that is when $f(x)$ is smooth and spans a large range in $\ln x$.

Derivation largely follows [Ham00]. In Eq. (2) take $f(x)$ on $[x_{\min}, x_{\max}]$, impose periodicity $f(x_{\max}) = f(x_{\min})$ in $\ln x$, and then discretize it by $\ln x_n \equiv \ln x_{\min} + n\Delta$ and $\ln x_{\max}/x_{\min} = N\Delta$. The logarithmic interval $\Delta \equiv |\Delta \ln x| \equiv |\Delta \ln y|$ uniformly discretizes $\ln x$ and later also $\ln y$, i.e., $\ln y_n \equiv \ln y_{\min} + n\Delta$ and $\ln y_{\max}/y_{\min} = N\Delta$.

Define a symmetrized sum

$$\sum_n{}' a_n \equiv \sum_{m=-\lfloor N/2\rfloor}^{\lfloor N/2\rfloor} w_n a_n \tag{3}$$

with $w_n = 1$ except that $w_{-N/2} = w_{N/2} = 1/2$ if N is even. Using this and the fact that f_m and $f(x_n)$ are periodic over $m, n \in \mathbb{Z}_N$, let's rewrite the usual discrete Fourier transform

$$f(x_n) = \frac{1}{N}\sum_{m=0}^{N-1} f_m \exp\Big(\frac{i2\pi mn}{N}\Big) = \frac{1}{N}\sum_m{}' f_m \exp\Big(\frac{i2\pi mn}{N}\Big) = \frac{1}{N}\sum_m{}' f_m \exp\Big[\frac{i2\pi m \ln(x_n/x_{\min})}{N\Delta}\Big] \tag{4}$$

$$f_m = \sum_{n=0}^{N-1} f(x_n)\exp\Big(-\frac{i2\pi mn}{N}\Big) = \sum_n{}' f(x_n)\exp\Big(-\frac{i2\pi mn}{N}\Big) \tag{5}$$

For smooth $f(x)$, a good approximation is the truncated Fourier series with f_m as coefficients

$$f(x) \approx \frac{1}{N}\sum_m{}' f_m \exp\Big[\frac{i2\pi m \ln(x/x_{\min})}{N\Delta}\Big] \tag{6}$$

always real thanks to the symmetrized sum.

To evaluate Eq. (2) we can transform each Fourier mode

$$\int_0^\infty \exp\Big[\frac{i2\pi m \ln(x/x_{\min})}{N\Delta}\Big](xy)^q K(xy)\,\frac{\mathrm{d}x}{x} \tag{7}$$

$$= \int_0^\infty \exp\Big\{\frac{i2\pi m}{N\Delta}\Big[\ln(xy) - \ln\Big(\frac{y}{y_{\min}}\Big) + \ln\Big(\frac{y_{\max}}{y_{\min}}\Big) - \ln(x_{\min}y_{\max})\Big]\Big\}(xy)^q K(xy)\,\frac{\mathrm{d}x}{x} \tag{8}$$

$$= \exp\Big\{-\frac{i2\pi m}{N\Delta}\Big[\ln\Big(\frac{y}{y_{\min}}\Big) + \ln(x_{\min}y_{\max})\Big]\Big\}\int_0^\infty t^{q-1+i\frac{2\pi m}{N\Delta}} K(t)\,\mathrm{d}t \tag{9}$$

Finally we just need to calculate analytically the Fourier transform of the kernel including the tilt factor, as a Mellin transform

$$U_K(z) \equiv \int_0^\infty t^{z-1} K(t)\,\mathrm{d}t \tag{10}$$

which satisfies $U_K^*(z) = U_K(z^*)$, and derive

$$g(y) \approx \frac{1}{N} \sum_m{}' g_m \exp\Big[-\frac{i2\pi m \ln(y/y_{\min})}{N\Delta}\Big] \tag{11}$$

$$g(y_n) \approx \frac{1}{N} \sum_m{}' g_m \exp\Big[-\frac{i2\pi m \ln(y_n/y_{\min})}{N\Delta}\Big] = \frac{1}{N} \sum_{m=0}^{N-1} g_m \exp\Big(-\frac{i2\pi mn}{N}\Big) \tag{12}$$

where

$$g_m = f_m u_m \tag{13}$$

$$u_m = \exp\Big[-\frac{i2\pi m \ln(x_{\min} y_{\max})}{N\Delta}\Big] U_K\Big(q + i\frac{2\pi m}{N\Delta}\Big) \qquad -\lfloor N/2 \rfloor \le m \le \lfloor N/2 \rfloor \tag{14}$$

We should make u_m periodic over $m \in \mathbb{Z}_N$, however $u_{-N/2}$ and $u_{N/2}$ are not necessarily equal for even N. In practice we can impose

$$g_{\pm N/2} \to \Re(g_{N/2}) \qquad N \in 2\mathbb{Z} \tag{15}$$

which is automatically satisfied if

$$\ln(x_{\min} y_{\max}) = \frac{\Delta}{\pi} \mathrm{Arg} U_K\Big(q + i\frac{\pi}{\Delta}\Big) + \text{integer} \cdot \Delta \tag{16}$$

where the argument takes its principal value in $(-\pi, \pi]$ and we set the arbitrary integer to 0. Otherwise we can just set $x_{\min} y_{\max} = 1$ for convenience.

Algorithm Follow the steps in equations (5), (10), (16) (or (15)), (14), (13), and (12) for fast evaluation of Eq. (2). u_m from Eq. (14) should be saved for multiple transformations.

Common Integral Transforms:

Hankel transform pair

$$G(y) = \int_0^\infty F(x) J_\nu(xy)\, x\mathrm{d}x \tag{17}$$

$$f(x) \to x^{2-q} F(x) \tag{18}$$

$$g(y) \to y^q G(y) \tag{19}$$

$$K(t) \to J_\nu(t) \tag{20}$$

$$U_K(z) = 2^{z-1} \frac{\Gamma\big(\frac{\nu+z}{2}\big)}{\Gamma\big(\frac{2+\nu-z}{2}\big)} \tag{21}$$

The pair is symmetric when $q = 1$.

Spherical Bessel transform pair

$$G(y) = \sqrt{\frac{2}{\pi}} \int_0^\infty F(x) j_\nu(xy)\, x^2\mathrm{d}x \tag{22}$$

$$f(x) \to x^{3-q} F(x) \tag{23}$$

$$g(y) \to y^q G(y) \tag{24}$$

$$K(t) \to \sqrt{\frac{2}{\pi}} j_\nu(t) = \frac{J_{\nu+\frac{1}{2}}(t)}{\sqrt{t}} \tag{25}$$

$$U_K(z) = 2^{z-\frac{3}{2}} \frac{\Gamma\big(\frac{\nu+z}{2}\big)}{\Gamma\big(\frac{3+\nu-z}{2}\big)} \tag{26}$$

The pair is symmetric when $q = \frac{3}{2}$.

Fourier sine transform pair

$$G(y)=\sqrt{\frac{2}{\pi}}\int_0^\infty F(x)\sin(xy)\,\mathrm{d}x \tag{27}$$

$$f(x)\to x^{1-q}F(x) \tag{28}$$
$$g(y)\to y^qG(y) \tag{29}$$
$$K(t)\to\sqrt{\frac{2}{\pi}}\sin(t)=\sqrt{t}J_{\frac{1}{2}}(t) \tag{30}$$
$$U_K(z)=\sqrt{\frac{2}{\pi}}\Gamma(z)\sin\Big(\frac{\pi z}{2}\Big)=2^{z-\frac{1}{2}}\frac{\Gamma\big(\frac{1+z}{2}\big)}{\Gamma\big(\frac{2-z}{2}\big)} \tag{31}$$

The pair is symmetric when $q=\frac{1}{2}$.

Fourier cosine transform pair

$$G(y)=\sqrt{\frac{2}{\pi}}\int_0^\infty F(x)\cos(xy)\,\mathrm{d}x \tag{32}$$

$$f(x)\to x^{1-q}F(x) \tag{33}$$
$$g(y)\to y^qG(y) \tag{34}$$
$$K(t)\to\sqrt{\frac{2}{\pi}}\cos(t)=\sqrt{t}J_{-\frac{1}{2}}(t) \tag{35}$$
$$U_K(z)=\sqrt{\frac{2}{\pi}}\Gamma(z)\cos\Big(\frac{\pi z}{2}\Big)=2^{z-\frac{1}{2}}\frac{\Gamma\big(\frac{z}{2}\big)}{\Gamma\big(\frac{1-z}{2}\big)} \tag{36}$$

The pair is symmetric when $q=\frac{1}{2}$.

Applications

Top-hat smoothing of a radial function in $\mathbb{R}^d$

$$F^W(R)=\int_0^\infty\frac{k^dF(k)}{2^{d-1}\pi^{\frac{d}{2}}\Gamma\big(\frac{d}{2}\big)}W_{\mathrm{T}}(kR)\,\frac{\mathrm{d}k}{k} \tag{37}$$

where

$$W_{\mathrm{T}}(r)=\begin{cases}1 & r\le R\\ 0 & r>R\end{cases} \tag{38}$$
$$W_{\mathrm{T}}(kR)=\frac{2^{\frac{d}{2}}\Gamma\big(\frac{d}{2}+1\big)J_{\frac{d}{2}}(kR)}{(kR)^{d/2}} \tag{39}$$

$$x\to k \tag{40}$$
$$y\to R \tag{41}$$
$$f(x)\to\frac{k^{d-q}F(k)}{2^{d-1}\pi^{\frac{d}{2}}\Gamma\big(\frac{d}{2}\big)} \tag{42}$$
$$g(y)\to R^qF^W(R) \tag{43}$$
$$K(t)\to W_{\mathrm{T}}(t) \tag{44}$$
$$U_K(z)=2^{z-1}\frac{\Gamma\big(\frac{2+d}{2}\big)\Gamma\big(\frac{z}{2}\big)}{\Gamma\big(\frac{2+d-z}{2}\big)} \tag{45}$$

Gaussian smoothing of a radial function in $\mathbb{R}^d$

$$F^W(R) = \int_0^\infty \frac{k^d F(k)}{2^{d-1}\pi^{\frac{d}{2}}\Gamma\big(\frac{d}{2}\big)} W_{\mathrm{G}}(kR)\,\frac{\mathrm{d}k}{k} \tag{46}$$

where

$$W_{\mathrm{G}}(r) = \frac{1}{(2\pi R^2)^{\frac{d}{2}}} \exp\Big(-\frac{r^2}{2R^2}\Big) \tag{47}$$

$$W_{\mathrm{G}}(kR) = \exp\Big(-\frac{k^2R^2}{2}\Big) \tag{48}$$

$$x \to k \tag{49}$$

$$y \to R \tag{50}$$

$$f(x) \to \frac{k^{d-q}F(k)}{2^{d-1}\pi^{\frac{d}{2}}\Gamma\big(\frac{d}{2}\big)} \tag{51}$$

$$g(y) \to R^q F^W(R) \tag{52}$$

$$K(t) \to W_{\mathrm{G}}(t) \tag{53}$$

$$U_K(z) = 2^{\frac{z}{2}-1}\Gamma\big(\tfrac{z}{2}\big) \tag{54}$$

Cosmology applications

Power spectrum to correlation function

$$\xi_l(r) = i^l \int_0^\infty \frac{k^3 P_l(k)}{2\pi^2} j_l(kr)\,\frac{\mathrm{d}k}{k} \tag{55}$$

$$x \to k \tag{56}$$

$$y \to r \tag{57}$$

$$f(x) \to \frac{i^l k^{3-q} P_l(k)}{(2\pi)^{\frac{3}{2}}} \tag{58}$$

$$g(y) \to r^q \xi_l(r) \tag{59}$$

with the rest the same as the spherical Bessel transform.

Correlation function to power spectrum

$$P_l(k) = (-i)^l \int_0^\infty 4\pi r^3 \xi_l(r) j_l(kr)\,\frac{\mathrm{d}r}{r} \tag{60}$$

$$x \to r \tag{61}$$

$$y \to k \tag{62}$$

$$f(x) \to r^{3-q}\xi_l(r) \tag{63}$$

$$g(y) \to \frac{i^l k^q P_l(k)}{(2\pi)^{\frac{3}{2}}} \tag{64}$$

with the rest the same as the spherical Bessel transform. The 2-pt function conversion pair is symmetric when $q = \frac{3}{2}$.

Variance in a Top-hat window in $\mathbb{R}^3$

$$\sigma_{\mathrm{T}}^2(R) = \int_0^\infty \frac{k^3 P(k)}{2\pi^2} W_{\mathrm{T}}^2(kR)\,\frac{\mathrm{d}k}{k} \tag{65}$$

where

$$W_{\mathrm{T}}(t) = \frac{3}{t^3}(\sin t - t\cos t) = \frac{3j_1(t)}{t} \tag{66}$$

$$x \to k \tag{67}$$
$$y \to R \tag{68}$$
$$f(x) \to \frac{k^{3-q} P(k)}{2\pi^2} \tag{69}$$
$$g(y) \to R^q \sigma_{\mathrm{T}}^2(R) \tag{70}$$
$$K(t) \to W_{\mathrm{T}}^2(t) \tag{71}$$
$$U_K(z) = \frac{9\sqrt{\pi}(z-2)}{4(z-6)} \frac{\Gamma\left(\frac{z-4}{2}\right)}{\Gamma\left(\frac{5-z}{2}\right)} \tag{72}$$

Variance in a Gaussian window in $\mathbb{R}^3$

$$\sigma_{\mathrm{G}}^2(R) = \int_0^\infty \frac{k^3 P(k)}{2\pi^2} W_{\mathrm{G}}^2(kR)\,\frac{\mathrm{d}k}{k} \tag{73}$$

where

$$W_{\mathrm{G}}(kR) = \exp\left(-\frac{k^2 R^2}{2}\right) \tag{74}$$

$$x \to k \tag{75}$$
$$y \to R \tag{76}$$
$$f(x) \to \frac{k^{3-q} P(k)}{2\pi^2} \tag{77}$$
$$g(y) \to R^q \sigma_{\mathrm{G}}^2(R) \tag{78}$$
$$K(t) \to W_{\mathrm{G}}^2(t) \tag{79}$$
$$U_K(z) = \frac{1}{2}\Gamma\left(\frac{z}{2}\right) \tag{80}$$

Excursion set in $\mathbb{R}^3$

$$\delta^W(R) = \int_0^\infty \frac{k^3 \delta(k)}{2\pi^2} W_{\mathrm{T}}(kR)\,\frac{\mathrm{d}k}{k} \tag{81}$$

same as top-hat smoothing of a radial function given previously.

$$x \to k \tag{82}$$
$$y \to R \tag{83}$$
$$f(x) \to \frac{k^{3-q} \delta(k)}{2\pi^2} \tag{84}$$
$$g(y) \to R^q \delta^W(R) \tag{85}$$
$$K(t) \to W_{\mathrm{T}}(t) \tag{86}$$
$$U_K(z) = 3\sqrt{\pi} 2^{z-3} \frac{\Gamma\left(\frac{z}{2}\right)}{\Gamma\left(\frac{5-z}{2}\right)} \tag{87}$$

Note that $\delta(k)$ is random and not smooth.

Test We test our implementation on Hankel transform in the following analytic case

$$e^{-y} = \int_0^\infty (1+x^2)^{\frac{3}{2}} J_0(xy)\, x\mathrm{d}x \tag{88}$$

$$(1+y^2)^{\frac{3}{2}} = \int_0^\infty e^{-y} J_0(xy)\, x\mathrm{d}x \tag{89}$$

Parameters

Length: N Odd N has worse ringing problem, similar to the case without low-ringing condition.

Tilt: q We can reduce ringing by balancing the large and small scale amplitudes with the tilt. The gamma function has simple poles at the non-positive integers, which can be avoided by picking q. However, one should not use a tilt too large, as it tends to shift the focus away and thus reduces the accuracy on the interested scales.

Padding: $f(x)$ is padded either with power-law extrapolations from the end segments beyond the tabulated region, or just with zeros.

Gamma function The reflection and duplication formulae are useful for playing with the gamma functions

$$\frac{\pi}{\sin \pi z} = \Gamma(1-z)\Gamma(z) = (-1)^n \Gamma(1+n-z)\Gamma(z-n) \tag{90}$$

$$\Gamma(z)\Gamma(z+\tfrac{1}{2}) = 2^{1-2z}\sqrt{\pi}\,\Gamma(2z) \tag{91}$$

References

[Ham00] A. J. S. Hamilton. Uncorrelated modes of the non-linear power spectrum. *MNRAS*, 312:257–284, February 2000.

[Tal78] J. D. Talman. Numerical Fourier and Bessel Transforms in Logarithmic Variables. *Journal of Computational Physics*, 29:35–48, October 1978.