

# Fast Integral Transforms

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This library computes common integral transforms with product kernels using the FFTLog algorithm [Tal78, Ham00]

$$G(y) = \int_0^\infty F(x)K(xy) \frac{dx}{x} \quad (1)$$

or equivalently

$$g(y) = \int_0^\infty f(x)(xy)^q K(xy) \frac{dx}{x} \quad (2)$$

where  $K(xy)$  is the kernel function and  $q$  is a power-law tilt parameter. And with  $f(x) \propto x^{-q}$ ,  $g(y) \propto y^q$ , one is free to choose  $q$ . We can tilt  $f(x)$  to balance  $f$  at large and small  $x$ , in order to improve accuracy and reduce ringing. Optimally  $f(x)$  is of the same order of magnitude at the two ends.

The idea is to take advantage of the convolution theorem in  $\ln x$  and  $\ln y$ . We approximate  $f(x)$  with truncated Fourier series, and use the exact Fourier transform of the possibly oscillatory kernel, obtained analytically via Mellin transform. This algorithm performs the best if  $f(x)$  is smooth and spans a large range in  $x$ , that is when  $f$  can be accurately approximated by a sum of finite Fourier modes, and by some discretization uniform in  $\ln x$ .

**Derivation** mostly follows [Ham00]. In Eq. (2) take  $f(x)$  on  $[x_{\min}, x_{\max}]$ , impose periodicity  $f(x_{\max}) = f(x_{\min})$  in  $\ln x$ , and then discretize it by  $\ln x_n \equiv \ln x_{\min} + n\Delta$  and  $\ln x_{\max}/x_{\min} = N\Delta$ . The logarithmic interval  $\Delta \equiv |\Delta \ln x| \equiv |\Delta \ln y|$  uniformly discretizes  $\ln x$  and later also  $\ln y$ , i.e.,  $\ln y_n \equiv \ln y_{\min} + n\Delta$  and  $\ln y_{\max}/y_{\min} = N\Delta$ .

Define a symmetrized sum

$$\sum_n' a_n \equiv \sum_{m=-\lfloor N/2 \rfloor}^{\lfloor N/2 \rfloor} w_n a_n \quad (3)$$

with  $w_n = 1$  except that  $w_{-N/2} = w_{N/2} = 1/2$  if  $N$  is even. Using this and the fact that  $f_m$  and  $f(x_n)$  are periodic over  $m, n \in \mathbb{Z}_N$ , let's rewrite the usual discrete Fourier transform

$$f(x_n) = \frac{1}{N} \sum_{m=0}^{N-1} f_m \exp\left(\frac{i2\pi mn}{N}\right) = \frac{1}{N} \sum_m' f_m \exp\left(\frac{i2\pi mn}{N}\right) = \frac{1}{N} \sum_m' f_m \exp\left[\frac{i2\pi m \ln(x_n/x_{\min})}{N\Delta}\right] \quad (4)$$

$$f_m = \sum_{n=0}^{N-1} f(x_n) \exp\left(-\frac{i2\pi mn}{N}\right) = \sum_n' f(x_n) \exp\left(-\frac{i2\pi mn}{N}\right) \quad (5)$$

For smooth  $f(x)$ , a good approximation is the truncated Fourier series with  $f_m$  as coefficients

$$f(x) \approx \frac{1}{N} \sum_m' f_m \exp\left[\frac{i2\pi m \ln(x/x_{\min})}{N\Delta}\right] \quad (6)$$

always real thanks to the symmetrized sum.

To evaluate Eq. (2) we can transform each Fourier mode

$$\int_0^\infty \exp\left[\frac{i2\pi m \ln(x/x_{\min})}{N\Delta}\right] (xy)^q K(xy) \frac{dx}{x} \quad (7)$$

$$= \int_0^\infty \exp\left\{\frac{i2\pi m}{N\Delta} \left[\ln(xy) - \ln\left(\frac{y}{y_{\min}}\right) + \ln\left(\frac{y_{\max}}{y_{\min}}\right) - \ln(x_{\min}y_{\max})\right]\right\} (xy)^q K(xy) \frac{dx}{x} \quad (8)$$

$$= \exp\left\{-\frac{i2\pi m}{N\Delta} \left[\ln\left(\frac{y}{y_{\min}}\right) + \ln(x_{\min}y_{\max})\right]\right\} \int_0^\infty t^{q-1+i\frac{2\pi m}{N\Delta}} K(t) dt \quad (9)$$

Finally we just need to calculate analytically the Fourier transform of the kernel including the tilt factor, as a Mellin transform

$$U_K(z) \equiv \int_0^\infty t^{z-1} K(t) dt \quad (10)$$

which satisfies  $U_K^*(z) = U_K(z^*)$ , and derive

$$g(y) \approx \frac{1}{N} \sum'_m g_m \exp\left[-\frac{i2\pi m \ln(y/y_{\min})}{N\Delta}\right] \quad (11)$$

$$g(y_n) \approx \frac{1}{N} \sum'_m g_m \exp\left[-\frac{i2\pi m \ln(y_n/y_{\min})}{N\Delta}\right] = \frac{1}{N} \sum_{m=0}^{N-1} g_m \exp\left(-\frac{i2\pi mn}{N}\right) \quad (12)$$

where

$$g_m = f_m u_m \quad (13)$$

$$u_m = \exp\left[-\frac{i2\pi m \ln(x_{\min} y_{\max})}{N\Delta}\right] U_K\left(q + i\frac{2\pi m}{N\Delta}\right) \quad -\lfloor N/2 \rfloor \leq m \leq \lfloor N/2 \rfloor \quad (14)$$

We should make  $u_m$  periodic over  $m \in \mathbb{Z}_N$ , however  $u_{-N/2}$  and  $u_{N/2}$  are not necessarily equal for even  $N$ . In practice we can impose

$$g_{\pm N/2} \rightarrow \Re(g_{N/2}) \quad N \in 2\mathbb{Z} \quad (15)$$

which is automatically satisfied if

$$\ln(x_{\min} y_{\max}) = \frac{\Delta}{\pi} \text{Arg} U_K\left(q + i\frac{\pi}{\Delta}\right) + \text{integer} \cdot \Delta \quad (16)$$

where the argument takes its principal value in  $(-\pi, \pi]$  and we set the arbitrary integer to 0. Otherwise we can just set  $x_{\min} y_{\max} = 1$  for convenience.

**Algorithm** Follow the steps in equations (5), (10), (16) (or (15)), (14), (13), and (12) for fast evaluation of Eq. (2).  $u_m$  from Eq. (14) should be saved for multiple transformations.

### Common Integral Transforms:

#### Hankel transform pair

$$G(y) = \int_0^\infty F(x) J_\nu(xy) x dx \quad (17)$$

$$f(x) \rightarrow x^{2-q} F(x) \quad (18)$$

$$g(y) \rightarrow y^q G(y) \quad (19)$$

$$K(t) \rightarrow J_\nu(t) \quad (20)$$

$$U_K(z) = 2^{z-1} \frac{\Gamma(\frac{\nu+z}{2})}{\Gamma(\frac{2+\nu-z}{2})} \quad (21)$$

The pair is symmetric when  $q = 1$ .

#### Spherical Bessel transform pair

$$G(y) = \sqrt{\frac{2}{\pi}} \int_0^\infty F(x) j_\nu(xy) x^2 dx \quad (22)$$

$$f(x) \rightarrow x^{3-q} F(x) \quad (23)$$

$$g(y) \rightarrow y^q G(y) \quad (24)$$

$$K(t) \rightarrow \sqrt{\frac{2}{\pi}} j_\nu(t) = \frac{J_{\nu+\frac{1}{2}}(t)}{\sqrt{t}} \quad (25)$$

$$U_K(z) = 2^{z-\frac{3}{2}} \frac{\Gamma(\frac{\nu+z}{2})}{\Gamma(\frac{3+\nu-z}{2})} \quad (26)$$

The pair is symmetric when  $q = \frac{3}{2}$ .

### Fourier sine transform pair

$$G(y) = \sqrt{\frac{2}{\pi}} \int_0^\infty F(x) \sin(xy) \, dx \quad (27)$$

$$f(x) \rightarrow x^{1-q} F(x) \quad (28)$$

$$g(y) \rightarrow y^q G(y) \quad (29)$$

$$K(t) \rightarrow \sqrt{\frac{2}{\pi}} \sin(t) = \sqrt{t} J_{\frac{1}{2}}(t) \quad (30)$$

$$U_K(z) = \sqrt{\frac{2}{\pi}} \Gamma(z) \sin\left(\frac{\pi z}{2}\right) = 2^{z-\frac{1}{2}} \frac{\Gamma\left(\frac{1+z}{2}\right)}{\Gamma\left(\frac{2-z}{2}\right)} \quad (31)$$

The pair is symmetric when  $q = \frac{1}{2}$ .

### Fourier cosine transform pair

$$G(y) = \sqrt{\frac{2}{\pi}} \int_0^\infty F(x) \cos(xy) \, dx \quad (32)$$

$$f(x) \rightarrow x^{1-q} F(x) \quad (33)$$

$$g(y) \rightarrow y^q G(y) \quad (34)$$

$$K(t) \rightarrow \sqrt{\frac{2}{\pi}} \cos(t) = \sqrt{t} J_{-\frac{1}{2}}(t) \quad (35)$$

$$U_K(z) = \sqrt{\frac{2}{\pi}} \Gamma(z) \cos\left(\frac{\pi z}{2}\right) = 2^{z-\frac{1}{2}} \frac{\Gamma\left(\frac{z}{2}\right)}{\Gamma\left(\frac{1-z}{2}\right)} \quad (36)$$

The pair is symmetric when  $q = \frac{1}{2}$ .

### Applications

#### Top-hat smoothing of a radial function in $\mathbb{R}^d$

$$F^W(R) = \int_0^\infty \frac{k^d F(k)}{2^{d-1} \pi^{\frac{d}{2}} \Gamma\left(\frac{d}{2}\right)} W_T(kR) \frac{dk}{k} \quad (37)$$

where

$$W_T(r) = \begin{cases} 1 & r \leq R \\ 0 & r > R \end{cases} \quad (38)$$

$$W_T(kR) = \frac{2^{\frac{d}{2}} \Gamma\left(\frac{d}{2} + 1\right) J_{\frac{d}{2}}(kR)}{(kR)^{d/2}} \quad (39)$$

$$x \rightarrow k \quad (40)$$

$$y \rightarrow R \quad (41)$$

$$f(x) \rightarrow \frac{k^{d-q} F(k)}{2^{d-1} \pi^{\frac{d}{2}} \Gamma\left(\frac{d}{2}\right)} \quad (42)$$

$$g(y) \rightarrow R^q F^W(R) \quad (43)$$

$$K(t) \rightarrow W_T(t) \quad (44)$$

$$U_K(z) = 2^{z-1} \frac{\Gamma\left(\frac{2+d}{2}\right) \Gamma\left(\frac{z}{2}\right)}{\Gamma\left(\frac{2+d-z}{2}\right)} \quad (45)$$

## Gaussian smoothing of a radial function in $\mathbb{R}^d$

$$F^W(R) = \int_0^\infty \frac{k^d F(k)}{2^{d-1} \pi^{\frac{d}{2}} \Gamma(\frac{d}{2})} W_G(kR) \frac{dk}{k} \quad (46)$$

where

$$W_G(r) = \frac{1}{(2\pi R^2)^{\frac{d}{2}}} \exp\left(-\frac{r^2}{2R^2}\right) \quad (47)$$

$$W_G(kR) = \exp\left(-\frac{k^2 R^2}{2}\right) \quad (48)$$

$$x \rightarrow k \quad (49)$$

$$y \rightarrow R \quad (50)$$

$$f(x) \rightarrow \frac{k^{d-q} F(k)}{2^{d-1} \pi^{\frac{d}{2}} \Gamma(\frac{d}{2})} \quad (51)$$

$$g(y) \rightarrow R^q F^W(R) \quad (52)$$

$$K(t) \rightarrow W_G(t) \quad (53)$$

$$U_K(z) = 2^{\frac{z}{2}-1} \Gamma\left(\frac{z}{2}\right) \quad (54)$$

## Cosmology applications

### Power spectrum to correlation function

$$\xi_l(r) = i^l \int_0^\infty \frac{k^3 P_l(k)}{2\pi^2} j_l(kr) \frac{dk}{k} \quad (55)$$

$$x \rightarrow k \quad (56)$$

$$y \rightarrow r \quad (57)$$

$$f(x) \rightarrow \frac{i^l k^{3-q} P_l(k)}{(2\pi)^{\frac{3}{2}}} \quad (58)$$

$$g(y) \rightarrow r^q \xi_l(r) \quad (59)$$

with the rest the same as the spherical Bessel transform.

### Correlation function to power spectrum

$$P_l(k) = (-i)^l \int_0^\infty 4\pi r^3 \xi_l(r) j_l(kr) \frac{dr}{r} \quad (60)$$

$$x \rightarrow r \quad (61)$$

$$y \rightarrow k \quad (62)$$

$$f(x) \rightarrow r^{3-q} \xi_l(r) \quad (63)$$

$$g(y) \rightarrow \frac{i^l k^q P_l(k)}{(2\pi)^{\frac{3}{2}}} \quad (64)$$

with the rest the same as the spherical Bessel transform. The 2-pt function conversion pair is symmetric when  $q = \frac{3}{2}$ .

**Variance in a Top-hat window** in  $\mathbb{R}^3$

$$\sigma_{\text{T}}^2(R) = \int_0^\infty \frac{k^3 P(k)}{2\pi^2} W_{\text{T}}^2(kR) \frac{dk}{k} \quad (65)$$

where

$$W_{\text{T}}(t) = \frac{3}{t^3}(\sin t - t \cos t) = \frac{3j_1(t)}{t} \quad (66)$$

$$x \rightarrow k \quad (67)$$

$$y \rightarrow R \quad (68)$$

$$f(x) \rightarrow \frac{k^{3-q}P(k)}{2\pi^2} \quad (69)$$

$$g(y) \rightarrow R^q \sigma_{\text{T}}^2(R) \quad (70)$$

$$K(t) \rightarrow W_{\text{T}}^2(t) \quad (71)$$

$$U_K(z) = \frac{9\sqrt{\pi}(z-2)}{4(z-6)} \frac{\Gamma(\frac{z-4}{2})}{\Gamma(\frac{5-z}{2})} \quad (72)$$

**Variance in a Gaussian window** in  $\mathbb{R}^3$

$$\sigma_{\text{G}}^2(R) = \int_0^\infty \frac{k^3 P(k)}{2\pi^2} W_{\text{G}}^2(kR) \frac{dk}{k} \quad (73)$$

where

$$W_{\text{G}}(kR) = \exp\left(-\frac{k^2 R^2}{2}\right) \quad (74)$$

$$x \rightarrow k \quad (75)$$

$$y \rightarrow R \quad (76)$$

$$f(x) \rightarrow \frac{k^{3-q}P(k)}{2\pi^2} \quad (77)$$

$$g(y) \rightarrow R^q \sigma_{\text{G}}^2(R) \quad (78)$$

$$K(t) \rightarrow W_{\text{G}}^2(t) \quad (79)$$

$$U_K(z) = \frac{1}{2} \Gamma\left(\frac{z}{2}\right) \quad (80)$$

**Excursion set** in  $\mathbb{R}^3$

$$\delta^W(R) = \int_0^\infty \frac{k^3 \delta(k)}{2\pi^2} W_{\text{T}}(kR) \frac{dk}{k} \quad (81)$$

same as top-hat smoothing of a radial function given previously.

$$x \rightarrow k \quad (82)$$

$$y \rightarrow R \quad (83)$$

$$f(x) \rightarrow \frac{k^{3-q}\delta(k)}{2\pi^2} \quad (84)$$

$$g(y) \rightarrow R^q \delta^W(R) \quad (85)$$

$$K(t) \rightarrow W_{\text{T}}(t) \quad (86)$$

$$U_K(z) = 3\sqrt{\pi} 2^{z-3} \frac{\Gamma(\frac{z}{2})}{\Gamma(\frac{5-z}{2})} \quad (87)$$

Note that  $\delta(k)$  is random and not smooth.

**Test** We test our implementation on Hankel transform in the following analytic case

$$e^{-y} = \int_0^\infty (1+x^2)^{\frac{3}{2}} J_0(xy) x dx \quad (88)$$

$$(1+y^2)^{\frac{3}{2}} = \int_0^\infty e^{-y} J_0(xy) x dx \quad (89)$$

## Parameters

**Length:**  $N$  Odd  $N$  has worse ringing problem, similar to the case without low-ringing condition.

**Tilt:**  $q$  We can reduce ringing by balancing the large and small scale amplitudes with the tilt. The gamma function has simple poles at the non-positive integers, which can be avoided by picking  $q$ . However, one should not use a tilt too large, as it tends to shift the focus away and thus reduces the accuracy on the interested scales.

**Padding:**  $f(x)$  is padded either with power-law extrapolations from the end segments beyond the tabulated region, or just with zeros.

**Gamma function** The reflection and duplication formulae are useful for playing with the gamma functions

$$\frac{\pi}{\sin \pi z} = \Gamma(1-z)\Gamma(z) = (-1)^n \Gamma(1+n-z)\Gamma(z-n) \quad (90)$$

$$\Gamma(z)\Gamma(z+\frac{1}{2}) = 2^{1-2z} \sqrt{\pi} \Gamma(2z) \quad (91)$$

## References

- [Ham00] A. J. S. Hamilton. Uncorrelated modes of the non-linear power spectrum. *MNRAS*, 312:257–284, February 2000.
- [Tal78] J. D. Talman. Numerical Fourier and Bessel Transforms in Logarithmic Variables. *Journal of Computational Physics*, 29:35–48, October 1978.